Rend. Lincei Mat. Appl. 36 (2025), 285–313
DOI 10.4171/RLM/1059

Number Theory. – *Effective resolution of families of norm form equations*, by Giovanni Marzenta, accepted on 10 July 2025.

Abstract. – We solve some parametric families of norm form equations using a method recently introduced by Amoroso, Masser and Zannier. We show that for all sufficiently large (in an effective way) integer values of the parameter, every integer solution arises from specializing a functional solution over $\mathbb{Z}[T]$. We find such functional solutions.

Keywords. – norm form equations, diophantine equations, Thue equations.

Mathematics Subject Classification 2020. – 11D57 (primary); 11D41, 11D59, 11D25 (secondary).

1. Introduction

In this article, we will solve some parametric families of norm form equations, i.e., diophantine equations having the shape

$$\mathrm{Norm}\left(x_0\xi_t^{(0)} + x_1\xi_t^{(1)} + \cdots + x_{d-1}\xi_t^{(d-1)}\right) = n,$$

where $\xi_t^{(0)}, \ldots, \xi_t^{(d-1)}$ are some algebraic numbers depending on an integer parameter t and the norm is from $\mathbb{Q}(\xi_t^{(0)}, \ldots, \xi_t^{(d-1)})$ to $\mathbb{Q}$. Of course, the unknown integers $x_0, \ldots, x_{d-1}$ may be required to satisfy also other relations. The study of parametric families of norm form equations, with particular focus on Thue equations, can be traced back to the work of Thomas in [22]. Since then, this topic has been explored using various techniques in many studies, including [2, 4, 7, 15, 18, 19, 23].

In our case, we set $n = 1$ and, after fixing $\xi_t^{(1)} = \xi_t$, we choose $\xi_t^{(j)} = \xi_t^j$ for all j. Our focus will be on solving uniformly these equations for all sufficiently large (in an effective way) positive integers t. Under suitable assumptions, if t is sufficiently large, all integer solutions will actually arise from "functional solutions", obtained by replacing t with a variable T and then solving the equation in $\mathbb{Z}[T]$. In every case we consider, we will be able to find such functional solutions.

We show how a new method developed by Amoroso, Masser and Zannier in [2] can be applied in order to study such equations, restricting ourselves to some specific examples, which of course will satisfy the above-mentioned "suitable assumptions".

This method is one of the various applications of the theory of heights to diophantine problems: see also Zannier's survey article [26] for an interesting overview on the topic.

In particular, we study some generalizations of the Thue equation

$$x^5 + 4t^4xy^4 - y^5 = 1,$$

which was already considered by Lombardo in [15], being perhaps the first instance of a parametric Thue equation of degree 5 solved by Skolem's method.

Firstly, we are able to drop the hypothesis $5 \mid t$ (which in principle may not be necessary, up to be willing to make many more computations, see [15, Remark 3.1]).

Most importantly, we can escape from the context of Thue equations by adding two more variables in the equation, obtaining Theorem 2.1.

We can also add a fifth variable, but then we need the variables to satisfy some polynomial relations, see Theorem 2.3.

It is fitting to remark that the equation with 4 variables (equation (1)) cannot be solved via Skolem's approach for dimensional reasons: in the language of [15, p. 3], we would have $4 + 2 = \dim X + \dim Y > \dim A = 5$. Therefore, our method can go further than Skolem's one. However, the equation with 3 variables (equation (2)) could, in principle, still be solved with Skolem's method. Nonetheless, even under the assumption $5 \mid t$, a significant number of cases would need to be treated.

A corollary of [10, Theorem 4.12], which generalizes Strassmann's lemma, can be applied alongside Skolem's method to solve the equation $x^5+4t^4xy^4-y^5=1$. Whether this corollary could also be applied to equation (2), potentially simplifying its resolution but without imposing additional assumptions on the variables or the parameter, remains unclear.

Finally, in Theorem 2.5, we solve (as usual for sufficiently large integers t) the Thue equation

$$x^4 - tx^3y - x^2y^2 + txy^3 + y^4 = 1.$$

This equation has already been studied by Pethö in [18, 19] using Baker's method, which relies on linear forms in logarithms; however, our approach could potentially handle an extra variable in such equation.

2. Main results

Let $P(T,X) = X^5 + 4T^4X - 1$. Observing that $\frac{\partial P(t,X)}{\partial X} = 5X^4 + 4t^4 \geq 0$, we deduce that $P(t,X)$ has only one real root for any choice of the integer t. We denote this root by ξ_t.

Thanks to the results shown in [2], we prove the following result.

THEOREM 2.1. *There exists an effective $t_0 \in \mathbb{N}$ such that for any integer $t \geq t_0$ the integer solutions of*

$$N_{\mathbb{Q}(\xi_t)/\mathbb{Q}}(x + y\xi_t + z\xi_t^2 + w\xi_t^3) = 1 \tag{1}$$

are precisely the following:

$$\begin{aligned}(x, y, z, w) \in \big\{&(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,-4t^4,0,0),\\ &(0,1,-4t^4,0), (0,0,1,-4t^4), (1,-8t^4,16t^8,0),\\ &(0,1,-8t^4,16t^8), (1,-12t^4,48t^8,-64t^{12}), (2t^2,\pm 2t,1,0),\\ &(0,2t^2,\pm 2t,1), (2t^2,-8t^6 \mp 2t, 1 \pm 8t^5, -4t^4),\\ &(\pm 8t^3, \mp 32t^7 + 8t^2, -32t^6 \pm 4t, 1 \mp 16t^5)\big\}.\end{aligned}$$

Since $P(t, X)$ is irreducible in $\mathbb{Q}[X]$ for any $t \neq 0$, equation (1) is a norm form equation of degree 5. Setting $w = 0$ and writing explicitly (1), we obtain the following corollary.

COROLLARY 2.2. *There exists an effective $t_0 \in \mathbb{N}$ such that for any integer $t \geq t_0$ the integer solutions of*

$$\begin{aligned}x^5 + y^5 + z^5 + 5xyz(xz - y^2)\\ + 4t^4(2x^3z^2 - 4x^2y^2z + xy^4 + 4t^4xz^4 + yz^4) = 1\end{aligned} \tag{2}$$

are precisely the following:

$$\begin{aligned}(x, y, z) \in \big\{&(1,0,0), (0,1,0), (0,0,1), (1,-4t^4,0), (0,1,-4t^4),\\ &(2t^2, \pm 2t, 1), (1, -8t^4, 16t^8)\big\}.\end{aligned}$$

In particular, setting $z = 0$ and substituting $y \mapsto -y$, we immediately recover the Thue equation previously studied by Lombardo in [15]*:*

$$x^5 + 4t^4xy^4 - y^5 = 1, \tag{3}$$

whose integer solutions (for sufficiently large $t \in \mathbb{Z}$) are precisely

$$(x, y) \in \big\{(1,0), (0,-1), (1,4t^4)\big\}.$$

In Section 7, we present an alternative proof of the resolution of equation (3), without deducing it as a corollary of Theorem 2.1. We do this because our goal is to show how the method can be applied and not only to obtain the solutions of the specific equations we are studying.

In this proof, we will deduce the functional solutions in another way, making use of the explicit writing of the norm form equation, some considerations about the degrees of the polynomial solutions and the Puiseux expansions at infinity of the roots of $P(T, X)$. The same strategy can be applied to prove directly also Corollary 2.2, but we do not display here the proof as it consists in more cumbersome computations and no substantially new ideas.

Equation (1) will be seen as the norm form equation

$$N_{\mathbb{Q}(\xi_t)/\mathbb{Q}}(x + y\xi_t + z\xi_t^2 + w\xi_t^3 + u\xi_t^4) = 1,$$

where we require the solutions to lie in the proper subvariety $\{u = 0\} \subset \mathbb{A}^5$. Also all the other examples discussed in [2] were obtained by requiring the last coordinate of the "general" norm form equation to be fixed, and in particular the number of variables in those examples is always $d - 1$, where d is the degree of the equation.

Here we provide an example of an equation where the number of variables is d.

THEOREM 2.3. *Let $a, b, c \in \mathbb{Q}$ such that $abc \neq 0$. There exists an effective $t_0 \in \mathbb{N}$ such that for any integer $t \geq t_0$ the integer solutions of*

$$N_{\mathbb{Q}(\xi_t)/\mathbb{Q}}(x + y\xi_t + z\xi_t^2 + w\xi_t^3 + u\xi_t^4) = 1 \tag{4}$$

satisfying

$$az + bw + cu = 0$$

are precisely the following:

$$(x, y, z, w, u) \in \{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (1, -4t^4, 0, 0, 0)\}.$$

The assumption $abc \neq 0$ is necessary because otherwise further solutions may arise, and these cases should be treated separately. For instance, when $a = b = 0 \neq c$, we recover Theorem 2.1.

The fact that the considered subvariety is a hyperplane is not particularly significant; thanks to Theorem 3.2, we could instead consider any proper subvariety of $\mathbb{A}^5$ defined over $\mathbb{Q}(T)$. In particular, the coefficients defining the subvariety's equations could also depend on T, and the proof would follow the same strategy.

REMARK 2.4. The resolution of equation (3) also follows directly from Theorem 2.3.

PROOF. It suffices to apply Theorem 2.3 with (a_1, b_1, c_1), (a_2, b_2, c_2) and (a_3, b_3, c_3) chosen as three linearly independent vectors in $\mathbb{Q}^3$, each having non-zero components. ■

In all the examples treated in [2], the rank of $\mathcal{O}^*_{\mathbb{Q}(\xi_t)}$ (sometimes called the "unit rank" of the equation) was 1 or 2. We will see that also equation (1) has unit rank 2.

In Section 8, we will work on a Thue equation having unit rank 3, obtaining the following result (which was already proven by Pethö in [19]).

THEOREM 2.5. *There exists an effective $t_0 \in \mathbb{N}$ such that for any integer $t \geq t_0$ the integer solutions of*

$$x^4 - tx^3y - x^2y^2 + txy^3 + y^4 = 1 \tag{5}$$

are precisely the following:

$$(x, y) \in \{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1), (\mp 1, \pm 1), (\pm t, \pm 1), (\pm 1, \mp t)\}.$$

The proof follows the same strategy as the one given in Section 7.

We can notice from the various proofs (in this article and in [2]) that the higher the unit rank, the more variables as exponents of the fundamental units will appear; so equations with higher unit rank will be hypothetically harder to solve.

Finally, one question arises naturally: what can be said about the integer values $t < t_0$? If the equation we are studying is a Thue equation, then for any fixed integer t we can solve it completely: Baker provided an effective proof of Thue's theorem in [6], establishing an explicit upper bound on the absolute values of the solutions of a given Thue equation and, consequently, an algorithm for solving these equations. However, this upper bound was too large to be "practically applied". To address this, later Tzanakis and De Weger developed in [24] a practical general method for solving any Thue equation, which combines Baker's theory with computational diophantine approximation techniques. Finally, very recently, Gherga and Siksek presented a new algorithm [11] that yields significantly improved results for solving Thue–Mahler equations, which are a generalization of Thue equations. Therefore, these results, combined with Corollary 2.2 and Theorem 2.5, enable us to completely solve equations (3) and (5).

In the case of equations with three or more variables, there are still some results that allow for an effective determination of all the solutions to a given norm form equation. However, these results currently apply only in special cases (see for example [3, 5, 12–14, 25]). Unfortunately, these results do not apply to equation (1) nor (2). By "effectively" here we mean that the solutions consist of finitely many families that can be explicitly described, along with finitely many additional solutions, outside the aforementioned families, which can be bounded by effective constants.

Concerning effectivity, Bombieri developed an alternative approach to Baker's one, through an extension of the original Thue–Siegel method: for a brief description, see [8].

As for finiteness results for norm form equations in which the number of variables is smaller than the degree – results that, in general, do not always hold – the key contributions are due to Schmidt. In [21], he proved that such equations admit at most finitely many integer solutions, provided they satisfy certain "non-degeneracy" conditions. However, his proofs rely on his Subspace Theorem and are therefore ineffective. If we are willing to settle for demonstrating the finiteness of the solutions to equation (1), then a theorem by Schlickewei concerning S-unit equations over number fields [20, Theorem 1.1] allows us to establish the following result.

THEOREM 2.6. *For any $t \in \mathbb{Z} \setminus \{0\}$, equation (1) has a finite number of integer solutions.*

In addition, Schlickewei's result provides an explicit upper bound for this number. However, this bound concerns the number of solutions, not their size, and thus does not allow for an effective resolution of the equation.

3. Notations and main tools

We recall from [2] the main results we will use in this article, and we will adopt the same notations.

Let $P \in \mathbb{Z}[T, X]$ be irreducible over $\overline{\mathbb{Q}}$ and monic of degree d in X, and let $\xi \in \overline{\mathbb{Q}(T)}$ be one of its roots. For $t \in \mathbb{Z}$ we denote with ξ_t the specialization of ξ at $T = t$. When needed, we denote[1] with $\xi = \xi^{(1)}, \xi^{(2)}, \dots, \xi^{(d)}$ the roots of $P(T, X)$, always labeling the respective specializations with the subscript t. We focus on specializing the parameter at sufficiently large integers t and we can clearly assume $t > 0$.

As explained in [2], for large $t \in \mathbb{N}$ the numbers r_1 and $2r_2$ of real and imaginary embeddings of the number field $\mathbb{Q}(\xi_t)$ are both constants, and as usual we let $r = r_1 + r_2 - 1$.

In order to proceed, we make the following assumption.

Assumption 3.1.

(1) *There exist multiplicatively independent elements* $u_1, \dots, u_r \in \mathbb{Z}[T, \xi]^*$.

(2) *Suppose* u *is in* $\mathbb{Z}[T, \xi]^*$ *and the algebraic number* v *is in the group generated by the conjugates of* u. *Then,* v *is a root of unity.*

The pillar on which this article stands is the following theorem, which in turn is a consequence of the specialization theorem [1, Theorem 1.3].

Theorem 3.2 ([2, Theorem 2.2]). *Let us assume Assumption 3.1. Let* $W \subsetneq \mathbb{A}^d$ *be a proper subvariety defined over* $\mathbb{Q}(T)$. *Then, there exists an effective* $t_0 > 0$ *such that for any integer* $t \geq t_0$ *with* $\mathbb{Z}[\xi_t]^* = (\mathbb{Z}[T, \xi]^*)_t$ *the solutions* $(x_0, \dots, x_{d-1}) \in W_t(\mathbb{Z})$ *of the norm form equation*

$$\prod_{j=1}^{d} \left(x_0 + x_1\xi_t^{(j)} + \cdots + x_{d-1}{\xi_t^{(j)}}^{d-1}\right) = 1 \tag{6}$$

are specializations of functional solutions $\mathbf{X} = (X_0, \dots, X_{d-1}) \in W(\mathbb{Z}[T])$ *of*

$$\prod_{j=1}^{d} \left(X_0 + X_1\xi^{(j)} + \cdots + X_{d-1}{\xi^{(j)}}^{d-1}\right) = 1. \tag{7}$$

[1] For notational convenience we will use both the notation with the superscript "(1)" and the one without.

Remark 3.3. Even without the assumption $\mathbb{Z}[\xi_t]^* = (\mathbb{Z}[T,\xi]^*)_t$, a result still holds – albeit more technical – that allows for a complete description of the solutions to equation (6). Indeed, an effective uniform bound for the index $[\mathbb{Z}[\xi_t]^* : (\mathbb{Z}[T,\xi]^*)_t]$ can be established (see [2, Theorem 2.3]). This leads to a slight generalization of the above theorem, namely, [2, Theorem 2.5].

We now introduce a few lemmas that are useful for verifying Assumption 3.1 and for other arguments in the following sections.

The next lemma is often useful, for example when W is a subvariety (not necessarily proper) of the hyperplane given by the vanishing of last coordinate, i.e., by the equation

$$X_{d-1} = 0.$$

It can be deduced from [16, Exercise 35, p. 65] by looking at the last element of the dual basis to $\{1, \xi^{(1)}, \dots, {\xi^{(1)}}^{d-1}\}$. However, for completeness, we provide a direct and immediate proof.

Lemma 3.4. *The following identity holds:*

$$\sum_{j=1}^{d} \alpha_j \, {\xi^{(j)}}^N = \begin{cases} 0 & \text{for } N = 0, 1, \dots, d-2, \\ 1 & \text{for } N = d-1, \end{cases}$$

where $\alpha_j = \prod_{k \neq j} \frac{1}{\xi^{(j)} - \xi^{(k)}}$.

Proof. The coefficients α_j are uniquely determined by linear system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \xi^{(1)} & \xi^{(2)} & \cdots & \xi^{(d)} \\ {\xi^{(1)}}^2 & {\xi^{(2)}}^2 & \cdots & {\xi^{(d)}}^2 \\ \vdots & \vdots & & \vdots \\ {\xi^{(1)}}^{d-1} & {\xi^{(2)}}^{d-1} & \cdots & {\xi^{(d)}}^{d-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Applying Cramer's rule to this system and noting that the matrices involved are Vandermonde matrices yields the desired formula for α_j. ■

We also recall the following result, which will be useful for verifying Assumption 3.1 (2).

Lemma 3.5 ([2, Lemma 3.5]). *Let us assume that Assumption 3.1 (1) holds. Assume further that the Puiseux series at $T = \infty$ of the roots $\xi^{(1)}, \dots, \xi^{(d)}$ of $P(T, X)$ are Laurent series with coefficients in* **k** *where* **k** *is the rational field or an imaginary quadratic field. Then, Assumption 3.1 (2) holds.*

To conclude this section, we state a version of Hensel's lemma adapted to the power series ring $\mathbf{k}[\![U]\!]$. Recall that $\mathbf{k}[\![U]\!]$ is a discrete valuation ring with maximal ideal (U) and discrete valuation $v : \mathbf{k}[\![U]\!] \to \mathbb{N} \cup \{\infty\}$ defined by

$$v\Big(\sum_{n\geq 0} a_n U^n\Big) = \inf\{n \mid a_n \neq 0\}. \tag{8}$$

In particular, $v(a) = n$ if and only if $a \in (U)^n \setminus (U)^{n+1}$, and $v(0) = \infty$.

The following result provides the usual root-lifting property in this context.

Lemma 3.6 ([9, Theorem 10.5]). *Let $f(X) \in \mathbf{k}[\![U]\!][X]$. If $a \in \mathbf{k}[\![U]\!]$ is such that $v(f(a)) > 0$ and $v(f'(a)) = 0$, then there exists a unique $\alpha \in \mathbf{k}[\![U]\!]$ such that*

$$f(\alpha) = 0 \quad \text{and} \quad v(\alpha - a) > 0.$$

Moreover, $v(\alpha - a) = v(f(a))$ and $v(f'(\alpha)) = 0$.

This lemma will allow us to prove that some of the considered Puiseux series are in fact Laurent series, thus verifying one of the hypotheses of Lemma 3.5.

3.1. Preliminary results

For the remainder of this article we fix the polynomial $P(T, X) = X^5 + 4T^4X - 1$.

Its homogenization $\widetilde{P}(T, X, Z)$ defines a projective curve in $\mathbb{P}^2_{\overline{\mathbb{Q}}}$, which can be readily verified to be smooth. Since smooth projective plane curves are irreducible, it follows that $\widetilde{P}(T, X, Z)$ is irreducible over $\overline{\mathbb{Q}}$, and therefore $P(T, X)$ is irreducible over $\overline{\mathbb{Q}}$.

Let $\xi_t = \xi_t^{(1)}$ denote the only real root of $P(t, X)$, and let $\xi = \xi^{(1)} \in \overline{\mathbb{Q}(T)}$ be the root of $P(T, X)$ which specialized at $T = t$ becomes ξ_t. As said, we denote with $\xi^{(2)}, \xi^{(3)}, \xi^{(4)}$ and $\xi^{(5)}$ the other roots of $P(T, X)$, always indicating the respective specializations with the subscript t.

We make use of the following result.

Theorem 3.7 ([17, Satz 3]). *Let α be a root of the polynomial $f(x) = x^5 + 4t^4x + 1$, where t is a positive integer. Then, α, $\alpha^2 - 2t\alpha + 2t^2$ and $\alpha^2 + 2t\alpha + 2t^2$ are units in $\mathbb{Z}[\alpha]$, and any two of them form a fundamental system of units of $\mathbb{Z}[\alpha]^*$.*

Applying this to our setting, we see that ξ_t and $\xi_t^2 + 2t\xi_t + 2t^2$ form a fundamental system of units for $\mathbb{Z}[\xi_t]^*$. This holds because the polynomial $-P(t, -X)$ satisfies the hypothesis of the theorem.

Since $\mathbb{Z}[\xi_t] \subset \mathbb{R}$, the only roots of unity in $\mathbb{Z}[\xi_t]^*$ are ± 1 and therefore

$$\mathbb{Z}[\xi_t]^* = \langle -1, \xi_t, \xi_t^2 + 2t\xi_t + 2t^2 \rangle. \tag{9}$$

Given that $\{\xi, \xi^2 + 2T\xi + 2T^2\} \subset \mathbb{Z}[T,\xi]^*$, we have $\mathbb{Z}[\xi_t]^* \subseteq (\mathbb{Z}[T,\xi]^*)_t$. The reverse inclusion always holds, so we conclude that

$$\big(\mathbb{Z}[T,\xi]^*\big)_t = \mathbb{Z}[\xi_t]^*.$$

Thanks to [2, Theorem 3.2], for sufficiently large t the specialization map is an isomorphism between $\mathbb{Z}[T,\xi]^*$ and $(\mathbb{Z}[T,\xi]^*)_t$, but from (9) we know the generators of the latter and thus in turn $\mathbb{Z}[T,\xi]^*$ is generated by $-1, \xi, \xi^2 + 2T\xi + 2T^2$. The two non-torsion units are multiplicatively independent, as their specializations are. Hence, Assumption 3.1 (1) is satisfied.

Let us now expand $\xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \xi^{(4)}$ and $\xi^{(5)}$ in the Puiseux series at $T = \infty$ in order to verify Assumption 3.1 (2).

By the change of variable $T = U^{-1}$, $X = \widetilde{X} U^{-1}$, the equation $P(T,X) = 0$ becomes

$$f(\widetilde{X}) := \widetilde{X}^5 + 4\widetilde{X} - U^5 = 0.$$

Note that $f'(\widetilde{X}) = 5\widetilde{X}^4 + 4$ and $f(0) = f(\pm 1 \pm i) = -U^5$.

We can apply Lemma 3.6 taking $\mathbf{k} = \mathbb{Q}$, $a_1 = 0$ and subsequently $\mathbf{k} = \mathbb{Q}(i)$, $a_j \in \{1+i, -1+i, -1-i, 1-i\}$. This yields five power series $\omega_1 \in \mathbb{Q}[\![U]\!]$ and $\omega_j \in \mathbb{Q}(i)[\![U]\!]$ for $j = 2, 3, 4, 5$, all satisfying $f(\omega_j) = 0$.

Moreover, we have

$$v(\omega_j - a_j) = v\big(f(a_j)\big) = v(-U^5) = 5 \quad \forall j,$$

where v is the valuation defined in (8).

Since $X = \widetilde{X} U^{-1}$, we obtain

$$\xi^{(1)} = \frac{\omega_1}{U} \in U^4 \mathbb{Q}[\![U]\!] = \frac{1}{T^4}\mathbb{Q}[\![T^{-1}]\!],$$

where the coefficient of T^{-4} is non-zero.

With regard to $\xi^{(2)}$, we have instead

$$\xi^{(2)} = \frac{\omega_2}{U} \in \frac{1+i}{U} + U^4 \mathbb{Q}(i)[\![U]\!] = (1+i)T + \frac{1}{T^4}\mathbb{Q}(i)[\![T^{-1}]\!]$$

and similarly

$$\xi^{(3)} \in (-1+i)T + \frac{1}{T^4}\mathbb{Q}(i)[\![T^{-1}]\!],$$

$$\xi^{(4)} \in (-1-i)T + \frac{1}{T^4}\mathbb{Q}(i)[\![T^{-1}]\!],$$

$$\xi^{(5)} \in (1-i)T + \frac{1}{T^4}\mathbb{Q}(i)[\![T^{-1}]\!].$$

Therefore, the Puiseux series at $T = \infty$ of $\xi^{(j)}$ are Laurent series with coefficients in $\mathbb{Q}(i)$, so applying Lemma 3.5 we conclude that Assumption 3.1 (2) is also satisfied.

4. Proof of Theorem 2.1

Thanks to what was proven in Section 3.1, the hypotheses of Theorem 3.2 are satisfied; therefore, it is sufficient to find all the solutions $X, Y, Z, W \in \mathbb{Z}[T]$ of

$$\prod_{j=1}^{5} \left(X + Y\xi^{(j)} + Z\xi^{(j)^2} + W\xi^{(j)^3}\right) = 1. \tag{10}$$

Since $\mathbb{Z}[T,\xi]^* = \langle -1, \xi, \xi^2 + 2T\xi + 2T^2\rangle$, $N_{\mathbb{Q}(\xi_t)/\mathbb{Q}}(-1) = -1$ and

$$N_{\mathbb{Q}(\xi_t)/\mathbb{Q}}(\xi_t) = N_{\mathbb{Q}(\xi_t)/\mathbb{Q}}(\xi_t^2 + 2t\xi_t + 2t^2) = 1,$$

we have

$$\begin{aligned} \Theta^{(j)} &:= X + Y\xi^{(j)} + Z\xi^{(j)^2} + W\xi^{(j)^3} \\ &= \xi^{(j)^m}(\xi^{(j)2} + 2T\xi^{(j)} + 2T^2)^n \end{aligned} \tag{11}$$

for some $(m,n) \in \mathbb{Z}^2$.

We now apply Lemma 3.4, obtaining

$$\sum_{j=1}^{5} \alpha_j \Theta^{(j)} = 0, \tag{12}$$

where each coefficient α_j can be explicitly computed because P is monic and the symmetric functions of the roots $\xi^{(j)}$ are determined by the coefficients of $P(T,X)$. In particular, we have

$$\alpha_j = \frac{\xi^{(j)}}{5 - 16T^4\xi^{(j)}}.$$

Let $\rho := e^{\frac{2\pi i}{5}}$. When $t = 0$, the polynomial $P(t,X)$ reduces to $P(0,X) = X^5 - 1$, so the roots $\xi^{(j)}$ specialize to the fifth roots of unity.

Since computing the Wronskian of the five functions $\Theta^{(j)}$ (similarly as done in the various examples in [2, Section 7]) would result too cumbersome in this case, we instead derive information from the vanishing of the derivatives with respect to T of the expression $\sum_{j=1}^{5} \alpha_j \Theta^{(j)}$, as expressed in equation (12). These derivatives vanish identically, and in particular at $T = 0$. The derivatives up to order 4 are precisely those that would appear in the Wronskian, while the ninth derivative somehow recovers the information we lost looking only at $T = 0$.

Thus, by imposing the vanishing condition on the derivatives of orders $k = 0, 1, 2, 3, 4, 9$ and using the expression of $\Theta^{(j)}$ from equation (11), we obtain the following, respectively:

- $k = 0$:
$$\frac{1}{5}(1 + \rho^{m+2n+1} + \rho^{2(m+2n+1)} + \rho^{3(m+2n+1)} + \rho^{4(m+2n+1)}) = 0$$
and so necessarily
$$5 \nmid m + 2n + 1. \tag{13}$$
- $k = 1$:
$$\frac{2n}{5}\sum_{j=0}^{4}\rho^{j(m+2n)} = 0$$
and so necessarily
$$n = 0 \quad \text{or} \quad 5 \nmid m + 2n. \tag{14}$$
- $k = 2$:
$$\frac{4n^2}{5}\sum_{j=0}^{4}\rho^{j(m+2n+4)} = 0$$
and so necessarily
$$n = 0 \quad \text{or} \quad 5 \nmid m + 2n + 4. \tag{15}$$
- $k = 3$:
$$\frac{8n(n^2-1)}{5}\sum_{j=0}^{4}\rho^{j(m+2n+3)} = 0$$
and so necessarily
$$n \in \{0, 1, -1\} \quad \text{or} \quad 5 \nmid m + 2n + 3. \tag{16}$$
- $k = 4$:
$$\frac{16}{25}\cdot(5n^4 - 20n^2 + 3n - 6m + 18)\sum_{j=0}^{4}\rho^{j(m+2n+2)} = 0$$
and so necessarily
$$m = \frac{5n^4 - 20n^2 + 3n}{6} + 3 \quad \text{or} \quad 5 \nmid m + 2n + 2. \tag{17}$$
- $k = 9$:
$$\frac{512n}{25}p(m,n)\sum_{j=0}^{4}\rho^{j(m+2n+2)} = 0,$$

where

$$p(m,n) = 2268m^2 - 756mn^4 + 7560mn^2 - 2268mn - 20412m + 5n^8 - 420n^6 + 378n^5 + 6468n^4 - 3780n^3 - 29993n^2 + 10206n + 30744,$$

and so necessarily

(18) $$n = 0 \quad \text{or} \quad p(m,n) = 0 \quad \text{or} \quad 5 \nmid m + 2n + 2.$$

Since $P(T, X)$ is the minimal polynomial of ξ, we can give $\mathbb{Z}[T,\xi]$ a free $\mathbb{Z}[T]$-module structure of rank 5, where we take $\{1,\xi,\xi^2,\xi^3,\xi^4\}$ as a basis. As $\xi^5 = 1 - 4T^4\xi$, the linear isomorphism $L_T : \mathbb{Z}[T]^5 \to \mathbb{Z}[T]^5$ such that $L_T v_x = v_{\xi x}$ (where $v_x \in \mathbb{Z}[T]^5$ is the vector of the coordinates of $x \in \mathbb{Z}[T,\xi]$ with respect to the basis $\{1,\xi,\xi^2,\xi^3,\xi^4\}$) is given by

(19) $$L_T(a,b,c,d,e) = (e, a - 4T^4 e, b, c, d),$$

with inverse $L_T^{-1}(a,b,c,d,e) = (b + 4T^4 a, c, d, e, a)$. We denote by L_t the specialization of L_T at the value $T = t \in \mathbb{C}$, which is a linear isomorphism of $\mathbb{C}^5$.

Thanks to conditions (13), (14), (15), (16), (17) and (18), in order to solve equation (11), we are left with four cases to deal with.

***n* = 0**. We are seeking all integers m such that for some $X, Y, Z, W \in \mathbb{Z}[T]$ we have

$$(X,Y,Z,W,0) = L_T^{(m)}(1,0,0,0,0).$$

$U = 0$ if and only if U is identically zero on $\mathbb{C}$ and so, given a solution

$$(X,Y,Z,W,0) = \big(X_m(T), Y_m(T), Z_m(T), W_m(T), U_m(T)\big) = L_T^{(m)}(1,0,0,0,0),$$

it is necessary that $U_m(\frac{1+i}{2}) = U_m(1) = 0$.

Let us observe that

(20) $$L_{\frac{\pm(1+i)}{2}}(a,b,c,d,e) = (e, a+e, b, c, d).$$

Iterating this formula 17 times yields

$$L_{\frac{1+i}{2}}^{(17)}(1,0,0,0,0) = (1,1,1,3,3)$$

and 17 is the smallest positive integer m such that all the coordinates of $L_{\frac{1+i}{2}}^{(m)}(1,0,0,0,0)$ are positive.

By (20), it follows by induction that for $m \geq 17$ all the coordinates of $L_{\frac{1+i}{2}}^{(m)}(1,0,0,0,0)$ remain positive, and consequently for $m \geq 17$ we will not find any solution.

Let us now observe that

$$L_{\pm 1}^{-1}(a,b,c,d,e) = (b+4a,c,d,e,a). \tag{21}$$

Iterating this formula 4 times yields

$$L_1^{(-4)}(1,0,0,0,0) = (256,1,4,16,64)$$

and 4 is the smallest positive integer m such that all the coordinates of $L_1^{(-m)}(1,0,0,0,0)$ are positive.

By (21), it follows that for $m \le -4$ all the coordinates of $L_1^{(m)}(1,0,0,0,0)$ remain positive, and consequently for $m \le -4$ we will not find any solution.

For the values $m \in [-3,16]$ we find the following solutions $X,Y,Z,W \in \mathbb{Z}[T]$ of equation (10):

m	(X,Y,Z,W)
0	$(1,0,0,0)$
1	$(0,1,0,0)$
2	$(0,0,1,0)$
3	$(0,0,0,1)$
5	$(1,-4T^4,0,0)$
6	$(0,1,-4T^4,0)$
7	$(0,0,1,-4T^4)$
10	$(1,-8T^4,16T^8,0)$
11	$(0,1,-8T^4,16T^8)$
15	$(1,-12T^4,48T^8,-64T^{12})$

***n* = 1.** We are seeking all integers m such that for some $X,Y,Z,W \in \mathbb{Z}[T]$ we have

$$(X,Y,Z,W,0) = L_T^{(m)}(2T^2,2T,1,0,0). \tag{22}$$

Since

$$L_1^{(-4)}(2,2,1,0,0) = (656,2,10,41,164),$$

by (21), it follows that for $m \le -4$ all the coordinates of $L_1^{(m)}(2,2,1,0,0)$ remain positive, and consequently for $m \le -4$ we will not find any solution.

Specializing equation (22) at $T = \frac{1+i}{2}$ and choosing $m = 12$, we obtain

$$L_{\frac{1+i}{2}}^{(12)}(i,1+i,1,0,0) = (3+i,4+i,1+i,1+3i,3+3i).$$

By (20), for $m \ge 12$ none of the coordinates of $L_{\frac{1+i}{2}}^{(m)}(i,1+i,1,0,0)$ can vanish since all of them are complex numbers having positive real and imaginary parts. Consequently, for $m \ge 12$ we will not find any solution.

For the values $m \in [-3, 11]$ we find the following solutions $X, Y, Z, W \in \mathbb{Z}[T]$ of equation (10):

m	(X, Y, Z, W)
0	$(2T^2, 2T, 1, 0)$
1	$(0, 2T^2, 2T, 1)$
5	$(2T^2, -8T^6 + 2T, 1 - 8T^5, -4T^4)$

***n* = −1.** We are seeking all integers m such that for some $X, Y, Z, W \in \mathbb{Z}[T]$ we have

$$(X, Y, Z, W, 0) = L_T^{(m)}(0, 2T^2, -2T, 1, 0) \tag{23}$$

because the vector $(0, 2T^2, -2T, 1, 0)$ corresponds to the coordinates of $(\xi^2 + 2T\xi + 2T^2)^{-1}$.

Specializing equation (23) at $T = -1$ and choosing $m = -5$, we obtain

$$L_{-1}^{(-5)}(0, 2, 2, 1, 0) = (656, 2, 10, 41, 164).$$

As done previously, (21) implies that for $m \leq -5$ we will not find any solution.

Specializing equation (23) at $T = \frac{-1-i}{2}$ and choosing $m = 11$, we obtain

$$L_{\frac{-1-i}{2}}^{(11)}(0, i, 1 + i, 1, 0) = (3 + i, 4 + i, 1 + i, 1 + 3i, 3 + 3i);$$

therefore, by (20), for $m \geq 11$ we will not find any solution.

For the values $m \in [-4, 10]$ we find the following solutions $X, Y, Z, W \in \mathbb{Z}[T]$ of equation (10):

m	(X, Y, Z, W)
−1	$(2T^2, -2T, 1, 0)$
0	$(0, 2T^2, -2T, 1)$
4	$(2T^2, -8T^6 - 2T, 1 + 8T^5, -4T^4)$

***n* ≠ 0 and *n* ≠ ±1.** From conditions (13), (14), (15) and (16), it follows that $5 \mid m + 2n + 2$. Combining this with conditions (17) and (18), we deduce that necessarily

$$m = \frac{5n^4 - 20n^2 + 3n}{6} + 3 \quad \text{and} \quad p(m, n) = 0. \tag{24}$$

Substituting the expression for m in the equation $p(m, n) = 0$, we obtain

$$10(n - 1)(n + 1)(n - 2)(n + 2)(95n^4 + 55n^2 - 252) = 0.$$

The cases in which $n = \pm 1$ have already been dealt with, so by (24), we have that $(m, n) = (2, -2)$ and $(m, n) = (4, 2)$ are the only cases to be examined. In fact, both lead to a solution of equation (10), more precisely:

$$(m,n)=(2,-2) \implies (X,Y,Z,W)=(-8T^3, 32T^7+8T^2, -32T^6-4T, 16T^5+1),$$
$$(m,n)=(4,2) \implies (X,Y,Z,W)=(8T^3, -32T^7+8T^2, -32T^6+4T, 1-16T^5).$$

Therefore, the ones we have found are all solutions of equation (10) and so we can apply Theorem 3.2 taking as proper subvariety the one given by the vanishing of the coefficient of ξ^4, obtaining the desired conclusion.

5. Proof of Theorem 2.6

We conclude our study of equation (1) by proving Theorem 2.6.

To begin, we recall equation (12): of course, this relation still holds while specializing and so we obtain

$$\sum_{j=1}^{5} \alpha_{j,t}\Theta_t^{(j)} = 0, \tag{25}$$

where $\alpha_{j,t} = \frac{\xi_t^{(j)}}{5-16t^4\xi_t^{(j)}}$ and

$$\Theta_t^{(j)} = \xi_t^{(j)m}\left(\xi_t^{(j)2} + 2t\xi_t^{(j)} + 2t^2\right)^n$$

for some $(m,n) \in \mathbb{Z}^2$.

It is clear that $\alpha_{j,t} \neq 0$ for all j.

The assumption $t \neq 0$ ensures that $P(t,X)$ is irreducible and consequently

$$\left[\mathbb{Q}(\xi_t) : \mathbb{Q}\right] = 5.$$

We can apply [20, Theorem 1.1] with $n = 4$ and $a_j = \alpha_{j,t}$, where K is the splitting field of $P(t,X)$ and $S = M_\infty(K)$ is the set of archimedean places of K. Therefore, the equation

$$\sum_{j=1}^{5} \alpha_{j,t}x_j = 0 \tag{26}$$

admits only a finite number of projective solutions having a representative whose components are S-units and where no proper subsum vanishes.

Since $\mathbb{Z}[\xi_t^{(j)}]^* \subset \mathcal{O}_K^*$, any solution $(\Theta_t^{(1)}, \ldots, \Theta_t^{(5)})$ of equation (25) is also a solution of equation (26).

We now show that every solution of equation (26) leads to at most one solution of equation (25), and that equation (25) does not admit solutions with a vanishing proper subsum. These two facts clearly imply the desired conclusion.

Suppose that $(m, n) \neq (m', n')$ are two solutions of equation (25) that arise from the same projective solution $(x_1, \dots, x_5)$ of equation (26). Then, we have

$$x_j = \xi_t^{(j)^m} \big(\xi_t^{(j)^2} + 2t\xi_t^{(j)} + 2t^2\big)^n$$

and there exists a $\lambda \in K^*$ such that for all j

$$\lambda x_j = \xi_t^{(j)^{m'}} \big(\xi_t^{(j)^2} + 2t\xi_t^{(j)} + 2t^2\big)^{n'}.$$

Thus,

$$\lambda = \xi_t^{(j)^{m'-m}} \big(\xi_t^{(j)^2} + 2t\xi_t^{(j)} + 2t^2\big)^{n'-n} \in \mathbb{Z}[\xi_t^{(j)}]^* \setminus \{\pm 1\}$$

because at least one of $m' - m$ and $n' - n$ is non-zero. Therefore,

$$\lambda \in \bigcap_{j=1}^{5} \mathbb{Z}[\xi_t^{(j)}]^* \setminus \{\pm 1\} = \varnothing,$$

which is a contradiction.

Now, suppose that $(\Theta_t^{(1)}, \dots, \Theta_t^{(5)})$ is a solution of equation (25) having a vanishing proper subsum. Then, there exist indices $i \neq j$ such that

$$\alpha_{i,t}\Theta_t^{(i)} + \alpha_{j,t}\Theta_t^{(j)} = 0. \tag{27}$$

By [17, Satz 1], we have $G := \mathrm{Gal}(K/\mathbb{Q}) = S_5$, so the action of G on the roots of $P(t, X)$ is doubly transitive. Therefore, we can choose $\sigma \in G$ such that $\sigma(\xi_t^{(i)}) = \xi_t^{(h)}$ and $\sigma(\xi_t^{(j)}) = \xi_t^{(k)}$, where i, j, k, h are all distinct.

Consequently, we have

$$0 = \sigma\big(\alpha_{i,t}\Theta_t^{(i)} + \alpha_{j,t}\Theta_t^{(j)}\big) = \alpha_{h,t}\Theta_t^{(h)} + \alpha_{k,t}\Theta_t^{(k)}.$$

It follows that if ℓ is the index distinct from i, j, k, h, then $\alpha_{\ell,t}\Theta_t^{(\ell)} = 0$, which is a contradiction. This concludes the proof of Theorem 2.6.

Remark 5.1. As previously noted, the finiteness result we have just established is not effective. To obtain an effective result, one might consider applying [5, Theorem 1], which provides an effective upper bound for the height of a non-degenerate solution to a projective S-unit equation with five terms, under the assumption that S contains all infinite places and $|S| \leq 3$. Unfortunately, we cannot do so because the field K is "too large", leading to an excessive number of infinite places in S.

In fact, [5, Theorem 1] cannot be applied to solve any similar equation of degree $d = 5$: if the units appearing in the S-unit equation are conjugates (as in (25)), then

K must be the Galois closure of $\mathbb{Q}(\xi_t)$. To ensure that $|S| = r_1 + r_2 \leq 3$, we must have $d = r_1 + 2r_2 = [\mathbb{Q}(\xi_t) : \mathbb{Q}] \leq 6$. Thus, $K = \mathbb{Q}(\xi_t)$, but any degree 5 field that is Galois over $\mathbb{Q}$ must be totally real. This implies $|S| = r_1 = 5 > 3$, so the theorem does not apply.

6. Proof of Theorem 2.3

Let $a, b, c \in \mathbb{Q}$ be fixed, with $abc \neq 0$.

Thanks to what was proven in Section 3.1, the hypotheses of Theorem 3.2 are satisfied; therefore, it is sufficient to find all the solutions $X, Y, Z, W, U \in \mathbb{Z}[T]$ of

$$\prod_{j=1}^{5} \left(X + Y\xi^{(j)} + Z{\xi^{(j)}}^2 + W{\xi^{(j)}}^3 + U{\xi^{(j)}}^4\right) = 1 \tag{28}$$

such that

$$aZ + bW + cU = 0. \tag{29}$$

In the same way as before, now we have to find the solutions $(m, n) \in \mathbb{Z}^2$ of

$$\begin{cases} \Theta^{(j)} := X + Y\xi^{(j)} + Z{\xi^{(j)}}^2 + W{\xi^{(j)}}^3 + U{\xi^{(j)}}^4 \\ \qquad\quad = {\xi^{(j)}}^m \left({\xi^{(j)}}^2 + 2T\xi^{(j)} + 2T^2\right)^n, \\ aZ + bW + cU = 0. \end{cases} \tag{30}$$

We aim to express the polynomials Z, W and U in terms of $\xi^{(j)}$, m and n (note that $\Theta^{(j)}$ and α_j can be written in this way).

To achieve this, clearly we could solve the linear system

$$\begin{pmatrix} 1 & \xi^{(1)} & {\xi^{(1)}}^2 & {\xi^{(1)}}^3 & {\xi^{(1)}}^4 \\ 1 & \xi^{(2)} & {\xi^{(2)}}^2 & {\xi^{(2)}}^3 & {\xi^{(2)}}^4 \\ 1 & \xi^{(3)} & {\xi^{(3)}}^2 & {\xi^{(3)}}^3 & {\xi^{(3)}}^4 \\ 1 & \xi^{(4)} & {\xi^{(4)}}^2 & {\xi^{(4)}}^3 & {\xi^{(4)}}^4 \\ 1 & \xi^{(5)} & {\xi^{(5)}}^2 & {\xi^{(5)}}^3 & {\xi^{(5)}}^4 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ W \\ U \end{pmatrix} = \begin{pmatrix} {\xi^{(1)}}^m({\xi^{(1)}}^2 + 2T\xi^{(1)} + 2T^2)^n \\ {\xi^{(2)}}^m({\xi^{(2)}}^2 + 2T\xi^{(2)} + 2T^2)^n \\ {\xi^{(3)}}^m({\xi^{(3)}}^2 + 2T\xi^{(3)} + 2T^2)^n \\ {\xi^{(4)}}^m({\xi^{(4)}}^2 + 2T\xi^{(4)} + 2T^2)^n \\ {\xi^{(5)}}^m({\xi^{(5)}}^2 + 2T\xi^{(5)} + 2T^2)^n \end{pmatrix}$$

and get the job done, but here we choose instead to apply Lemma 3.4 again, obtaining

$$U = \sum_{j=1}^{5} \alpha_j \Theta^{(j)}, \tag{31}$$

where, as before, $\alpha_j = \frac{\xi^{(j)}}{5 - 16T^4\xi^{(j)}}$.

Using the relations between the roots and the coefficients of $P(T, X)$, one readily verifies that

$$\sum_{j=1}^{5} \xi^{(j)} = \sum_{j=1}^{5} \xi^{(j)^2} = \sum_{j=1}^{5} \xi^{(j)^3} = 0 \quad \text{and} \quad \sum_{j=1}^{5} \xi^{(j)^4} = -16T^4.$$

Thus,

$$\sum_{j=1}^{5} \xi^{(j)} \Theta^{(j)} = 5U - 16T^4 W,$$

from which substituting the expression for U found in (31), we obtain the desired expression for W:

$$W = \sum_{j=1}^{5} \frac{5\alpha_j - \xi^{(j)}}{16T^4} \cdot \Theta^{(j)}. \tag{32}$$

Similarly, by computing $\sum_{j=1}^{5} \xi^{(j)^2} \Theta^{(j)} = 5W - 16T^4 Z$, we find out that

$$Z = \sum_{j=1}^{5} \frac{25\alpha_j - 5\xi^{(j)} - 16T^4 \xi^{(j)^2}}{256T^8} \Theta^{(j)}. \tag{33}$$

Remark 6.1. If the constraint (29) had also involved X and Y, the argument could still have been completed by computing the quantities $\sum_{j=1}^{5} \Theta^{(j)}$ and $\sum_{j=1}^{5} \xi^{(j)^{-1}} \Theta^{(j)}$.

We can now substitute (31), (32) and (33) inside the constraint (29). We proceed by differentiating this quantity with respect to T and by evaluating it at $T = 0$, like we did during the proof of Theorem 2.1.

Specializing at $T = 0$ and letting $\rho = e^{\frac{2\pi i}{5}}$, we obtain

$$\frac{1}{5}\left(c \sum_{j=0}^{4} \rho^{j(m+2n+1)} + b \sum_{j=0}^{4} \rho^{j(m+2n+2)} + a \sum_{j=0}^{4} \rho^{j(m+2n+3)} \right) = 0$$

and so, since a, b, c are all non-zero, necessarily

$$5 \mid m + 2n \quad \text{or} \quad 5 \mid m + 2n + 4. \tag{34}$$

By computing the first derivative and then evaluating at $T = 0$, we discover that

$$\frac{2n}{5}\left(c \sum_{j=0}^{4} \rho^{j(m+2n)} + b \sum_{j=0}^{4} \rho^{j(m+2n+1)} + a \sum_{j=0}^{4} \rho^{j(m+2n+2)} \right) = 0$$

and so necessarily

$$n = 0 \quad \text{or} \quad 5 \mid m + 2n + 3 \quad \text{or} \quad 5 \mid m + 2n + 4. \tag{35}$$

Analogously, by computing the second derivative and then evaluating at $T = 0$, we obtain

$$\frac{4n^2}{5}\left(b\sum_{j=0}^{4}\rho^{j(m+2n)} + a\sum_{j=0}^{4}\rho^{j(m+2n+1)} + c\sum_{j=0}^{4}\rho^{j(m+2n+4)}\right) = 0$$

and so necessarily

$$n = 0 \quad \text{or} \quad 5 \mid m + 2n + 2 \quad \text{or} \quad 5 \mid m + 2n + 3. \tag{36}$$

Thanks to conditions (34), (35) and (36), we can conclude that necessarily $n = 0$. Moreover, either $5 \mid m$ or $5 \mid m + 4$.

Similarly to Section 4, we are now seeking all integers m such that for some $X, Y, Z, W, U \in \mathbb{Z}[T]$ we have

$$L_T(X, Y, Z, W, U) = L_T^{(m)}(1, 0, 0, 0, 0) \quad \text{and} \quad aZ + bW + cU = 0, \tag{37}$$

where $L_T : \mathbb{Z}[T]^5 \to \mathbb{Z}[T]^5$ is the linear isomorphism defined in (19).

Let $(X_m, Y_m, Z_m, W_m, U_m) := L_T^{(m)}(1, 0, 0, 0, 0)$.

Lemma 6.2. *If $m \notin \{0, 1, 5\}$, then*[2] $\max\{\deg Z_m, \deg W_m, \deg U_m\}$ *is reached by only one of* $\{Z_m, W_m, U_m\}$. *Therefore, m is not a solution of* (37).

Proof. The last conclusion is evident, so we only need to prove that all the aforementioned values of m satisfy the condition on the degrees of $\{Z_m, W_m, U_m\}$.

Since

$$L_T^{(-4)}(1, 0, 0, 0, 0) = (256T^{16}, 1, 4T^4, 16T^8, 64T^{12})$$

and $L_T^{-1}(a, b, c, d, e) = (b + 4T^4 a, c, d, e, a)$, it can be easily shown by induction that

$$\begin{cases} \deg X_m = -4m, \\ \deg Y_m = -4m - 16, \\ \deg Z_m = -4m - 12, \qquad \forall m \leq -4. \\ \deg W_m = -4m - 8, \\ \deg U_m = -4m - 4 \end{cases}$$

Therefore, for $m \leq -4$ the condition is satisfied.

To address most of the cases $m \geq 2$, we make the following observation: if

$$\max\{\deg a, \deg b, \deg c, \deg d, \deg e\}$$

(2) By convention, we assume $\deg 0 = -\infty$.

is reached by only one of $\{a, b, c, d, e\}$ and this polynomial is c, d or e, then the same condition holds for $L_T^{(4k)}(a, b, c, d, e)$ for all $k \in \mathbb{N}$. Indeed, since

$$L_T^{(4)}(a,b,c,d,e) = (b, -4bT^4 + c, -4cT^4 + d, -4dT^4 + e, -4eT^4 + a), \tag{38}$$

the claim easily follows by induction.

The hypotheses of this observation are satisfied by $L_T^{(m)}(1,0,0,0,0)$ for $m \in \{2,3,4\}$.

Therefore, if $m \geq 2$ and $m \not\equiv 1 \pmod 4$, the condition on the degrees of $\{Z_m, W_m, U_m\}$ is satisfied.

Lastly, thanks to equation (38), we can show by induction that

$$L_T^{(4k+1)}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}(-1)^{k+1}c_1T^{4k-4}\\(-1)^k c_2 T^{4k}\\(-1)^k c_3 T^{4k-16}\\(-1)^{k+1}c_4T^{4k-12}\\(-1)^k c_5 T^{4k-8}\end{pmatrix} + \begin{pmatrix}o(T^{4k-4})\\o(T^{4k})\\o(T^{4k-16})\\o(T^{4k-12})\\o(T^{4k-8})\end{pmatrix} \quad \forall k \geq 4,$$

where the c_j's are positive integers depending on k.

This establishes the desired property for all $m \equiv 1 \pmod 4$ with $m \geq 17$.

The remaining cases $m \in \{-3, -2, -1, 9, 13\}$ can be verified directly. ■

Lemma 6.2 implies that for any choice of (a, b, c) the only possible values of m that may lead to a solution of equation (37) are $m \in \{0, 1, 5\}$.

For all these values of m we have $Z_m = W_m = U_m = 0$; thus, they always yield a solution to equation (37). These solutions are

m	(X, Y, Z, W, U)
0	$(1, 0, 0, 0, 0)$
1	$(0, 1, 0, 0, 0)$
5	$(1, -4T^4, 0, 0, 0)$

Therefore, the ones we have found are all the solutions of equation (30) and so we can apply Theorem 3.2 taking as proper subvariety the one given by equation (29), obtaining the desired conclusion.

7. An alternative approach to solving equation (3)

As promised, in this section, we present an alternative proof of the resolution of equation (3). Our strategy will roughly be to compute and compare the asymptotic behavior of some of the involved quantities, deducing then relations between the degrees of the functional solutions and the exponents of the functional units that will generate them.

We can notice from the various proofs in the previous section and in [2, Section 7] that we had to compute the derivatives of some quantities in order to find the functional solutions. Here we will not have to do so, but we will need to write explicitly the functional equation (7) and especially the first terms of the Puiseux series at $T = \infty$ of the roots $\xi^{(j)}$, so the information somehow still comes from derivatives.

Throughout this section, we will keep the same notations as in the previous ones.

To begin, let us recall a couple of definitions that may not be habitual for every reader. Given two functions[3] $f, g : \mathbb{C} \to \mathbb{C}$, we say that they

- have the *same order of magnitude* if $f = O(g)$ and $g = O(f)$. In this case, we write $f \asymp g$.
- are *asymptotically equivalent* if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. In this case, we write $f \sim g$.

As anticipated, we begin by explicitly determining the coefficients of the first terms of the Puiseux series of $\xi^{(j)}$ at $T = \infty$.

Let $\xi^{(1)} = \sum_{j=4}^{\infty} p_j T^{-j}$ with $p_4 \neq 0$, so we have

$$0 = P(T, \xi^{(1)}) \sim p_4 T^{-20} + 4T^4 p_4 T^{-4} - 1 \sim 4p_4 - 1$$

and hence necessarily $p_4 = \frac{1}{4}$.

More generally, imposing the vanishing of the coefficient of the term of degree $-d$ for $d \geq 0$, the condition $P(T, \xi^{(1)}) = 0$ yields

$$\sum_{j+k+h+\ell+m=d} p_j p_k p_h p_\ell p_m + 4p_{4+d} = \begin{cases} 1 & \text{if } d = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

From the above equation, we immediately derive that $p_5 = \cdots = p_{23} = 0$ (since the first summand is non-zero only for $d \geq 20$) and $p_{24} = -\frac{1}{4^6}$.

Let now $\xi^{(2)} = \sum_{j=-1}^{\infty} q_j T^{-j}$, where by what was said above $q_{-1} = 1 + i$, $q_0 = q_1 = q_2 = q_3 = 0$ and $q_4 \neq 0$.

Equation (39) still holds when replacing all p's with q's, now for $d \geq -5$. In particular, for $d = 0$ we obtain $5q_4 q_{-1}^4 + 4q_4 = 1$, and since $q_{-1}^4 = -4$, this yields

$$q_4 = -\frac{1}{16}.$$

For $\xi^{(3)}$, $\xi^{(4)}$, $\xi^{(5)}$ an identical argument leads to the same conclusion about the coefficient of the T^{-4} term.

(3) Domain and codomain may also be subsets of $\mathbb{C}$.

In conclusion, the first terms of the Puiseux series at $T = \infty$ of the roots of $P(T, X)$ are

$$\xi^{(1)} = \frac{1/4}{T^4} - \frac{1/2^{12}}{T^{24}} + o(T^{-24}),$$
$$\xi^{(2)} = (1+i)T - \frac{1/16}{T^4} + o(T^{-4}),$$
$$\xi^{(3)} = (-1+i)T - \frac{1/16}{T^4} + o(T^{-4}),$$
$$\xi^{(4)} = (-1-i)T - \frac{1/16}{T^4} + o(T^{-4}),$$
$$\xi^{(5)} = (1-i)T - \frac{1/16}{T^4} + o(T^{-4}).$$

We now return to equation (3), which can be recognized as the explicit form of the norm form equation

$$N_{\mathbb{Q}(\xi_t)/\mathbb{Q}}(x - y\xi_t) = 1.$$

Based on Section 3.1, the hypotheses of Theorem 3.2 are satisfied; thus, it is sufficient to find all the solutions $X, Y \in \mathbb{Z}[T]$ of

$$X^5 + 4T^4XY^4 - Y^5 = 1. \tag{40}$$

As done previously, we aim to solve the equation

$$\Theta^{(j)} := X - Y\xi^{(j)} = \xi^{(j)^m}\left(\xi^{(j)^2} + 2T\xi^{(j)} + 2T^2\right)^n \tag{41}$$

for some $(m, n) \in \mathbb{Z}^2$.

From (40) it is clear that if $X = 0$, then $Y = -1$, while if $Y = 0$, then $X = 1$.

Therefore, from now on we assume that X and Y are both non-zero. Let $k := \deg X \in \mathbb{N}$ and $h := \deg Y \in \mathbb{N}$ be the degrees of X and Y, so we can write $X = \sum_{j=0}^{k} a_j T^j$ and $Y = \sum_{j=0}^{h} b_j T^j$ with $a_k \neq 0$ and $b_h \neq 0$.

From (40) we deduce

$$1 = X^5 + 4T^4XY^4 - Y^5 \sim a_k^5T^{5k} + 4a_kb_h^4T^{4+k+4h} - b_h^5T^{5h}.$$

$k = h$ is impossible as we would have $1 \asymp T^{5k+4}$.

If $k > h$, then $5k > 5h$, so necessarily $5k = 4 + k + 4h$; hence, $k = h + 1$. But then, $a_k^5 + 4a_kb_h^4 = 0$ and so $a_k = 0$, which is a contradiction.

Therefore, $k < h$, so $5k < 5h$ and then necessarily $5h = 4 + k + 4h$, which implies

$$h = k + 4.$$

Note that

$$\xi^{(2)^2} + 2T\xi^{(2)} + 2T^2 \sim (1+i)^2T^2 + 2(1+i)T^2 + 2T^2 \sim 4(1+i)T^2 \asymp T^2$$

and so since $\xi^{(2)} \asymp T$, we have

$$\Theta^{(2)} \asymp T^{m+2n}.$$

To reach an analogous conclusion for $j = 3$, we need to include the second term of the expansion of $\xi^{(3)}$, proceeding as follows:

$$\xi^{(3)^2} + 2T\xi^{(3)} + 2T^2 \sim \left[(-1+i)T - \frac{1}{16T^4}\right]^2 + 2T\left[(-1+i)T - \frac{1}{16T^4}\right] + 2T^2$$
$$\sim -2iT^2 + \frac{1-i}{8T^3} + (-2+2i)T^2 - \frac{1}{8T^3} + 2T^2 \asymp T^{-3},$$

which implies that

$$\Theta^{(3)} \asymp T^{m-3n}.$$

Since $\deg Y = k + 4$, we have

(42) $$\Theta^{(j)} = X - Y\xi^{(j)} \asymp T^{k+5} \quad \forall j \in \{2,3,4,5\}.$$

Consequently,

$$T^{m+2n} \asymp \Theta^{(2)} \asymp T^{k+5} \asymp \Theta^{(3)} \asymp T^{m-3n},$$

which implies $m + 2n = m - 3n$, and therefore, $n = 0$. Thus, equation (41) reduces to

$$\Theta^{(j)} = X - Y\xi^{(j)} = \xi^{(j)^m},$$

with (42) guaranteeing that $m \geq 5$.

Now we are ready to conclude as we did in Section 4.[4]

In this case, we are seeking all integers $m \geq 5$ such that for some $X, Y \in \mathbb{Z}[T]$ we have

$$(X, -Y, 0, 0, 0) = L_T^{(m)}(1, 0, 0, 0, 0),$$

where $L_T : \mathbb{Z}[T]^5 \to \mathbb{Z}[T]^5$ is the linear isomorphism defined in equation (19).

Note that for $m = 5$ we get $(X, Y) = (1, 4T^4)$, which is the last solution of equation (40) we were looking for: there is now only the need to prove that there are no other ones.

Given a solution

$$(X, -Y, 0, 0, 0) = \big(X_m(T), -Y_m(T), Z_m(T), W_m(T), U_m(T)\big) = L_T^{(m)}(1, 0, 0, 0, 0),$$

it is necessary that $Z_m(\frac{1+i}{2}) = W_m(\frac{1+i}{2}) = U_m(\frac{1+i}{2}) = 0$.

(4) Note that here also Lemma 6.2 would immediately get the job done.

Thanks to (20), it is immediate to see that for $m \in \{6,7,8\}$ we do not find any solution. Moreover, since $L^{(9)}_{\frac{1+i}{2}}(1,0,0,0,0) = (1,1,0,0,1)$, (20) implies that $L^{(m)}_{\frac{1+i}{2}}(1,0,0,0,0)$ has at least three positive coordinates for any $m \geq 9$, and consequently for $m \geq 9$ we will not find any solution.

Therefore, $\{(1,0),(0,-1),(1,4T^4)\}$ are all the solutions of equation (40) and so choosing as W the variety given by the vanishing of the coefficients of ξ^2, ξ^3, ξ^4, we can apply Theorem 3.2, obtaining the desired conclusion.

8. Proof of Theorem 2.5

Let $Q(T,X) := X^4 - TX^3 - X^2 + TX + 1$.

We claim that Q is irreducible over $\overline{\mathbb{Q}}$. Suppose by contradiction that it is reducible. Since $\deg_T Q = 1$, this would imply the existence of polynomials $A, B, C \in \overline{\mathbb{Q}}[X]$, with $A \neq 0$ and $C \notin \overline{\mathbb{Q}}$, such that

$$Q(T,X) = (TA + B)C.$$

Let $\gamma \in \overline{\mathbb{Q}}$ be a root of C. Then, $Q(T,\gamma) = 0$, so $Q(T,\gamma) \in \overline{\mathbb{Q}}[T]$ must be the zero polynomial. Hence, $-T\gamma^3 + T\gamma = \gamma^4 - \gamma^2 + 1 = 0$. Since the polynomials $X^3 - X$ and $X^4 - X^2 + 1$ have no common roots, we get a contradiction.

It can be easily seen (cf. [19, Section 2]) that $Q(t,X)$ has four real roots for any integer $t \geq 3$, and so in our notation, $r = 3$. Let $\xi \in \overline{\mathbb{Q}(T)}$ be a root of $Q(T,X)$, which once specialized becomes ξ_t.

According to [19, Theorem 3.1], for $t > 3$ we have that $\xi_t - 1, \xi_t, \xi_t + 1$ form a fundamental system of units for $\mathbb{Z}[\xi_t]^*$.

Analogously to what was done in Section 3.1, it can be shown that

$$\left(\mathbb{Z}[T,\xi]^*\right)_t = \mathbb{Z}[\xi_t]^* = \langle -1, \xi_t - 1, \xi_t, \xi_t + 1 \rangle$$

and

$$\mathbb{Z}[T,\xi]^* = \langle -1, \xi - 1, \xi, \xi + 1 \rangle.$$

The three non-torsion units are multiplicatively independent since their specializations at $t > 3$ are. Hence, Assumption 3.1 (1) is satisfied.

Let us now expand the four roots of $Q(T,X)$, namely, $\xi = \xi^{(1)}, \xi^{(2)}, \xi^{(3)}$ and $\xi^{(4)}$, in Puiseux series at $T = \infty$ in order to verify Assumption 3.1 (2).

By the change of variable $T = U^{-1}$, the equation $Q(T,X) = 0$ becomes

$$f(X) := UX^4 - X^3 - UX^2 + X + U = 0.$$

Note that $f'(X) = -3X^2 + 1 + U(4X^3 - 2X)$ and $f(0) = f(\pm 1) = 0$.

We can apply Lemma 3.6 taking $\mathbf{k} = \mathbb{Q}$ and, respectively, $a_1 = 1, a_2 = -1, a_3 = 0$. This yields three series $\xi^{(1)}, \xi^{(2)}, \xi^{(3)} \in \mathbb{Q}[\![U]\!]$ such that $f(\xi^{(j)}) = 0$ for all j and

$$v(\xi^{(j)} - a_j) = v\big(f(a_j)\big) = v(U) = 1.$$

Similarly to what was done in Section 7, we can compute the first terms of the Puiseux series of the $\xi^{(j)}$ recursively, obtaining

$$\begin{aligned}
\xi^{(1)} &= 1 + \frac{1/2}{T} + \frac{1/8}{T^2} + \frac{1/2}{T^3} + \frac{31/128}{T^4} + \frac{1}{T^5} + o(T^{-5}),\\
\xi^{(2)} &= -1 + \frac{1/2}{T} - \frac{1/8}{T^2} + \frac{1/2}{T^3} - \frac{31/128}{T^4} + \frac{1}{T^5} + o(T^{-5}),\\
\xi^{(3)} &= -\frac{1}{T} - \frac{1}{T^5} + o(T^{-5}).
\end{aligned}$$

Since $\sum_{j=1}^{4} \xi^{(j)} = T$, we can find also the fourth root:

$$\xi^{(4)} = T - \frac{1}{T^3} - \frac{1}{T^5} + o(T^{-5}).$$

Therefore, the Puiseux series at $T = \infty$ of $\xi^{(j)}$ are Laurent series with coefficients in $\mathbb{Q}$, so applying Lemma 3.5, we conclude that Assumption 3.1 (2) is also satisfied.

Equation (5) can be readily recognized as the explicit form of the norm form equation

$$N_{\mathbb{Q}(\xi_t)/\mathbb{Q}}(x - y\xi_t) = 1.$$

By what was said, the hypotheses of Theorem 3.2 are satisfied; therefore, it is sufficient to find all the solutions $X, Y \in \mathbb{Z}[T]$ of

$$X^4 - TX^3Y - X^2Y^2 + TXY^3 + Y^4 = 1. \tag{43}$$

If $X = 0$, then $Y = \pm 1$, while if $Y = 0$, then $X = \pm 1$, so from here on out we can assume that X and Y are both non-zero. Let $k = \deg X \in \mathbb{N}$ and $h = \deg Y \in \mathbb{N}$ be the degrees of X and Y, so we can write $X = \sum_{j=0}^{k} a_j T^j$ and $Y = \sum_{j=0}^{h} b_j T^j$ with $a_k \neq 0$ and $b_h \neq 0$.

Since we know the generators of $\mathbb{Z}[T, \xi]^*$ and considering that all of them have norm equal to 1, we need to solve the equation

$$\Theta^{(j)} := X - Y\xi^{(j)} = \pm {\xi^{(j)}}^{n} (\xi^{(j)} + 1)^{p} (\xi^{(j)} - 1)^{m} \tag{44}$$

for some $(n, p, m) \in \mathbb{Z}^3$.

From now on we will use information from both equations (43) and (44).

Equation (44) combined with the expansions of $\xi^{(j)}$ in Puiseux series at $T = \infty$ gives us the asymptotic behavior of $\Theta^{(j)}$ in terms of the integers n, p, m; namely,

$$\Theta^{(1)} \asymp T^{-m}, \quad \Theta^{(2)} \asymp T^{-p}, \quad \Theta^{(3)} \asymp T^{-n}, \quad \Theta^{(4)} \asymp T^{n+p+m}.$$

Observe that if (X, Y) is a solution of equation (43), then so are $(-X, -Y)$ and $(\pm Y, \mp X)$. Thus, without loss of generality, we may assume the positive sign in equation (44) and that $k \leq h$.

From equation (43) we deduce that $h \geq k + 2$ cannot happen; otherwise, the left-hand side would have the same order of magnitude as T^{4h}, contradicting the fact that it must equal 1. So we are left with two cases, which we will treat separately: $h = k$ and $h = k + 1$.

$\boldsymbol{h = k}$. From the definition of $\Theta^{(j)}$ we obtain $\Theta^{(3)} \asymp T^k$ and $\Theta^{(4)} \asymp T^{k+1}$. Therefore,

$$-n = k \quad \text{and} \quad n + p + m = k + 1.$$

The coefficient of the monomial of degree $4k + 1$ in the left-hand side of equation (43) is $-a_k^3 b_k + a_k b_k^3$, but it must be zero and so necessarily $a_k = \pm b_k$. Let us treat the case $a_k = b_k$, the other being analogous. In this case, the terms of degree k in the definition of $\Theta^{(2)}$ do not cancel out and hence $\Theta^{(2)} \asymp k$, so that $-p = k$.

We have thus determined that $(n, p, m) = (-k, -k, 3k + 1)$ and therefore equation (44) becomes

$$X - Y\xi^{(j)} = (\xi^{(j)} - 1)\left[\frac{(\xi^{(j)} - 1)^3}{\xi^{(j)}(\xi^{(j)} + 1)}\right]^k.$$

$k = 0$ produces the solution of equation (43) given by $(X, Y) = (-1, -1)$.

To prove that there are no further solutions for $k \geq 1$, we view $\mathbb{Z}[T, \xi]$ as a free $\mathbb{Z}[T]$-module of rank 4. With respect to the basis $\{1, \xi, \xi^2, \xi^3\}$, the multiplication by $\frac{(\xi^{(j)}-1)^3}{\xi^{(j)}(\xi^{(j)}+1)}$ is represented by the matrix

$$M := \begin{pmatrix} T-4 & 7 & -8 & 7 \\ -8T & 8T-4 & 7-8T & 7T-8 \\ 7T+8 & -8T-7 & 8T+4 & -8T \\ -7 & 8 & -7 & T+4 \end{pmatrix}.$$

It can be shown by induction that for all $k \geq 1$

$$M^k \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -T^k + o(T^k) \\ 16^k T^k + o(T^k) \\ -(16^k - 1)T^k + o(T^k) \\ (16^k - 1)T^{k-1} + o(T^{k-1}) \end{pmatrix}.$$

Thus, the third coordinate never becomes zero, and so we are done.

If $a_k = -b_k$, we can proceed in the same way, obtaining only the solution $(X, Y) = (1, -1)$, which corresponds to $(n, p, m) = (0, 1, 0)$.

***h* = *k* + 1**. We immediately obtain

$$\Theta^{(1)} \asymp \Theta^{(2)} \asymp T^{k+1} \quad \text{and} \quad \Theta^{(4)} \asymp T^{k+2};$$

hence,

$$-n = -p = k + 1 \quad \text{and} \quad n + p + m = k + 2.$$

Therefore, $(n, p, m) = (3k + 4, -k - 1, -k - 1)$ and equation (44) becomes

$$X - Y\xi^{(j)} = \frac{{\xi^{(j)}}^4}{(\xi^{(j)} - 1)(\xi^{(j)} + 1)} \left[\frac{{\xi^{(j)}}^3}{(\xi^{(j)} - 1)(\xi^{(j)} + 1)} \right]^k.$$

For $k = 0$ we obtain the solution $(X, Y) = (1, -T)$.

We need to prove that there are no other solutions for $k \geq 1$: to do so, we proceed in the same way as before. The matrix representing multiplication by $\frac{{\xi^{(j)}}^3}{(\xi^{(j)}-1)(\xi^{(j)}+1)}$ is

$$N := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & T & 1 & 0 \\ T & 0 & T & 1 \\ -1 & 0 & 0 & T \end{pmatrix}.$$

It can be shown by induction that for all $k \geq 1$

$$N^k \begin{pmatrix} 1 \\ T \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} T^k + o(T^k) \\ T^{k+1} + o(T^{k+1}) \\ kT^k + o(T^k) \\ -kT^{k-1} + o(T^{k-1}) \end{pmatrix}.$$

Thus, the third coordinate never becomes zero, and so we are done.

Altogether, we have found all the solutions of (43); namely,

$$(X, Y) \in \{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1), (\mp 1, \pm 1), (\pm T, \pm 1), (\pm 1, \mp T)\}.$$

Therefore, we can apply Theorem 3.2 taking as proper subvariety the one given by the vanishing of the coefficients of ξ^2 and ξ^3, obtaining the desired conclusion.

Acknowledgments. – I would especially like to thank Professor U. Zannier for motivating me to study this subject and to write this article, as well as for his valuable suggestions. I would also like to thank Professor D. Lombardo for his generous availability and our many conversations, which were very helpful to me. Finally, I wish to thank Professor F. Amoroso for his helpful suggestions on a preliminary version of this article, which significantly contributed to its improvement.

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Received 20 January 2025,
and in revised form 30 June 2025

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