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Partial Differential Equations. – *Nonexistence results for elliptic equations with supercritical growth in thick planar domains*, by RICCARDO MOLLE and DONATO PASSASEO, accepted on 21 July 2025.

Dedicated to the memory of Professor Haïm Brezis.

ABSTRACT. – In this paper, we give some examples of non-star-shaped bounded domains of $\mathbb{R}^2$ where, for a class of nonlinear elliptic Dirichlet problems involving supercritical Sobolev exponents, there exists only the trivial identically zero solution (notice that a well-known result of Pohozaev concerns only the star-shaped domains).

Unlike the case of previous papers (Molle and Passaseo (2020, 2021, 2023, 2025)) where we proved nonexistence results for nontrivial solutions in thin domains sufficiently close to prescribed curves, in the present paper, the domains do not need to be thin.

KEYWORDS. – supercritical Dirichlet problems, contractible domains, nonexistence of solutions.

MATHEMATICS SUBJECT CLASSIFICATION 2020. – 35J60 (primary); 35J25 (secondary).

1. INTRODUCTION

Our aim is to study the effect of the domain shape on the existence of nontrivial solutions for nonlinear elliptic Dirichlet problems of the type

$$\operatorname{div}\left(|Du|^{p-2}Du\right) + f(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 2$, $1 < p < n$ and f is a continuous function having critical or supercritical growth from the viewpoint of the Sobolev embedding $H_0^{1,p}(\Omega) \to L^q(\Omega)$; that is, f satisfies the condition

$$0 \leq q \int_0^t f(\tau)\, d\tau \leq t f(t) \quad \forall t \in \mathbb{R} \tag{1.2}$$

for a suitable $q \geq \frac{np}{n-p}$.

For the sake of simplicity, in this paper, we consider only classical solutions $u \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ of problem (1.1). For the case of weak solutions, we refer the reader to Remark 2.4. Also notice that, when the domain Ω is unbounded, the lack of compactness is due not only to the presence of critical Sobolev exponents but also to the unboundedness of the domain. Several papers are concerned with this case. For example, when $\Omega = \mathbb{R}^n$, in [10], the pointwise asymptotic behavior is studied, with respect to suitable parameters, of the radially symmetric ground state solution of a quasilinear elliptic equation involving the p-Laplace operator; in the case $\Omega = \mathbb{R}^n \setminus \{0\}$, some nonexistence results of nontrivial solutions for p-Laplace equations with singular weights are obtained in [36].

Arguing as in [33], also for $p \neq 2$, one can easily prove that problem (1.1) has only the trivial solution $u \equiv 0$ when Ω is a bounded star-shaped domain. On the other hand, when Ω is an annulus, one can easily find positive and nodal solutions.

Thus, many questions arise, analogous to well-known questions posed by Brezis, Nirenberg, Rabinowitz, etc., concerning the effect of the domain shape on the existence of nontrivial solutions (see [1–7, 9, 11–16, 18, 22–34, 37]).

In particular, a natural question is whether or not Pohozaev's type nonexistence result for nontrivial solutions might be extended from the bounded star-shaped domains to all the bounded contractible domains.

With respect to this question, the problem presents quite different features in the case $n = 2$ (with $1 < p < 2$) and the case $n \geq 3$ (with $1 < p < n$) because for $n \geq 3$, the answer is negative, while for $n = 2$, we expect that it might be positive.

In fact, if $n \geq 3$, $1 < p < n$ and $q \geq \frac{np}{n-p}$, there exist bounded contractible domains of $\mathbb{R}^n$, even arbitrarily close to bounded star-shaped domains, where problem (1.1) has nontrivial solutions. For example, if Ω is a pierced annulus of $\mathbb{R}^n$ with $n \geq 3$, one can find many positive and nodal solutions when the size of the perforation is small enough.

On the contrary, if $n = 2$, $1 < p < 2$ and $q \geq \frac{2p}{2-p}$, some results we obtained in [17, 19–21] suggest that, for every bounded contractible domain Ω of $\mathbb{R}^2$, the trivial solution $u \equiv 0$ in Ω might be the unique solution of problem (1.1).

For example, for all $n \geq 2$, $\alpha \in \,]0, \pi[$ and ρ_1, ρ_2 such that $0 < \rho_1 < \rho_2$, let us consider the domain Ω of the type

$$\Omega = \Omega^{n,\alpha}_{\rho_1,\rho_2} := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \left(x_1, \sqrt{\sum_{i=2}^{n} x_i^2} \right) \in \Omega^{2,\alpha}_{\rho_1,\rho_2} \right\}, \tag{1.3}$$

where

$$\Omega^{2,\alpha}_{\rho_1,\rho_2} := \left\{ \rho(\cos\theta, \sin\theta) \in \mathbb{R}^2 : \rho_1 \leq \rho \leq \rho_2,\ |\theta| \leq \alpha \right\}. \tag{1.4}$$

Then, if $n \geq 3$, $p = 2$, $f(t) = |t|^{q-2}t$ and $\Omega = \Omega^{n,\alpha}_{\rho_1,\rho_2}$, as a consequence of some results obtained in [12, 14–16, 22, 24, 27, 30, 31], we infer that there exists $\overline{\alpha} \in \,]\frac{\pi}{2}, \pi[$ such that problem (1.1) has nontrivial solutions for $\alpha \in \,]\overline{\alpha}, \pi[$ and their number tends to infinity as $\alpha \to \pi$.

Moreover, for all $\alpha \in]\frac{\pi}{2}, \pi[$, there exist $\bar{q} \geq \frac{2n}{n-2}$ and $\bar{\varepsilon} > 0$ such that the problem has nontrivial solutions for $q > \bar{q}$ and for $q \in]\frac{2n}{n-2}, \frac{2n}{n-2} + \bar{\varepsilon}[$, these solutions tend to zero as $q \to +\infty$ and as $q \to \frac{2n}{n-2}$, while their number tends to infinity. Notice that the domain $\Omega^{n,\alpha}_{\rho_1,\rho_2}$ is arbitrarily close to the star-shaped domain $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x| \leq \rho_2,\ x_1 \geq 0\}$ for ρ_1 small enough and α sufficiently close to $\frac{\pi}{2}$.

On the contrary, if $n = 2$, $1 < p < 2$ and condition (1.2) holds with $q > \frac{2p}{2-p}$, from some results proved in [17, 19, 20], it follows that there exists $\widehat{\varepsilon} > 0$ such that the problem has only the trivial solution $u \equiv 0$ in $\Omega^{2,\alpha}_{\rho_1,\rho_2}$ for all $\alpha \in]0, \pi[$ and ρ_1, ρ_2 such that $0 < \rho_1 < \rho_2$ and $1 - \frac{\rho_1}{\rho_2} < \widehat{\varepsilon}$ (that is when the domain $\Omega^{2,\alpha}_{\rho_1,\rho_2}$ is thin enough).

Thus, if $1 - \frac{\rho_1}{\rho_2} < \widehat{\varepsilon}$, for α sufficiently close to π, the problem has only the trivial solution $u \equiv 0$ in $\Omega^{n,\alpha}_{\rho_1,\rho_2}$ when $n = 2$, while it has many nontrivial solutions when $n \geq 3$.

The above example where $\Omega = \Omega^{n,\alpha}_{\rho_1,\rho_2}$ shows the deep difference between the case $n = 2$ and $n \geq 3$ and explains why for $n = 2$ it is natural to expect that the trivial solution $u \equiv 0$ in Ω might be the unique solution for every bounded contractible domain Ω of $\mathbb{R}^2$.

In the present paper, we develop a new method which allows us to prove that if $n = 2$, $1 < p < 2$, $\Omega = \Omega^{2,\alpha}_{\rho_1,\rho_2}$ with $\alpha \in]0, \pi[$ and $0 < \rho_1 < \rho_2$, then the problem has only the trivial solution $u \equiv 0$ in $\Omega^{2,\alpha}_{\rho_1,\rho_2}$ also when $\frac{\rho_1}{\rho_2}$ is not close to 1.

This new method is based on a careful choice of the vector field arising in the Pohozaev type integral identity we use to prove the nonexistence of nontrivial solutions for problem (1.1).

The method used in [17, 19–21] allowed us to obtain nonexistence results for elliptic problems with supercritical growth in thin bounded domains, sufficiently close to prescribed curves. These domains may be very different from the star-shaped ones, even arbitrarily close to noncontractible domains where there exist many nontrivial solutions and (for n and q large enough) also noncontractible, but they must be thin enough.

On the contrary, we expect that the new method we use in the present paper in order to prove nonexistence results in bounded domains that are not necessarily thin might allow us to prove analogous nonexistence results in every bounded contractible domain of $\mathbb{R}^2$.

2. Integral identities and nonexistence results

Our main result, which we present in the following theorem, concerns the nonexistence of nontrivial solutions for the Dirichlet problem (1.1) under the assumption that the nonlinear term $f(u)$ has suitable supercritical growth and Ω is a domain of the type $\Omega = \Omega^{2,\alpha}_{\rho_1,\rho_2}$ (see (1.4)).

THEOREM 2.1. *Let us assume that* $n = 2$, $1 < p < 2$ *and* $\Omega = \Omega^{2,\alpha}_{\rho_1,\rho_2}$ *with* $0 < \alpha < \pi$ *and* $0 < \rho_1 < \rho_2$ *with* $(\rho_1/\rho_2) > 1/e$. *Moreover, assume that the function* f *satisfies condition* (1.2) *with* $q > \frac{2p}{2-p}(1 + \log\frac{\rho_1}{\rho_2})^{-1}$ *and that* $u \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ *is a solution of problem* (1.1). *Then, we have* $u \equiv 0$ *in* $\Omega^{2,\alpha}_{\rho_1,\rho_2}$.

The proof is reported below (see also Remark 2.4 for a possible generalization of this result).

REMARK 2.2. Notice that, without any loss of generality, in Theorem 2.1, we can assume $\rho_2 = 1$; otherwise, after rescaling, we obtain the equivalent Dirichlet problem in the domain $\Omega^{2,\alpha}_{\rho_1/\rho_2,1}$ (the equivalence between these two Dirichlet problems is obvious because the function f satisfies condition (1.2) with $q \geq \frac{2p}{2-p}$ if and only if the function $(\rho_2)^p f$ satisfies this condition with the same q).

In order to prove Theorem 2.1, we use the integral identity given by the following lemma which generalizes Pohozaev identity (see [33]).

LEMMA 2.3. *Let* Ω *be a piecewise smooth bounded domain of* $\mathbb{R}^2$ *and* $p > 1$. *Assume that* $u \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ *is a solution of the Dirichlet problem*

$$\operatorname{div}\left(|Du|^{p-2}Du\right) + f(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{2.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ *is a continuous function. Then, for all vector field* $v \in \mathcal{C}^1(\Omega, \mathbb{R}^2)$, *the function* u *satisfies the integral identity*

$$\begin{aligned}\left(1 - \frac{1}{p}\right)\int_{\partial\Omega} |Du|^p (v \cdot \nu)\, d\sigma \\ = \int_\Omega |Du|^{p-2}\big(dv[Du] \cdot Du\big)\, dx + \int_\Omega \operatorname{div} v\big[F(u) - \tfrac{1}{p}|Du|^p\big]dx,\end{aligned} \tag{2.2}$$

where ν *denotes the outward normal to* $\partial\Omega$, $dv[\xi] = \sum_{i=1}^2 \xi_i D_i v\ \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ *and* $F(t) = \int_0^t f(\tau)\, d\tau$.

For the proof, it suffices to apply the Gauss–Green formula to the vector field $(v \cdot Du)|Du|^{p-2}Du$ and argue as in [33]. In fact, we obtain

$$\int_{\partial\Omega} (v \cdot Du)|Du|^{p-2}(Du \cdot \nu)\, d\sigma = \int_\Omega \sum_{i=1}^2 D_i\left[\sum_{j=1}^2 v_j D_j u \cdot |Du|^{p-2} D_i u\right] dx, \tag{2.3}$$

where

$$\int_{\partial\Omega} (v \cdot Du)|Du|^{p-2}(Du \cdot \nu)\, d\sigma = \int_{\partial\Omega} |Du|^p (v \cdot \nu)\, d\sigma \tag{2.4}$$

because $u \equiv 0$ on $\partial\Omega$ and, as a consequence, $Du = (Du \cdot \nu)\nu$. Moreover, we have

$$
\begin{aligned}
(2.5)\qquad \int_\Omega \sum_{i,j=1}^2 v_j D_{i,j}u|Du|^{p-2}D_i u\,dx
&= \frac{1}{2}\int_\Omega \sum_{i,j=1}^2 v_j|Du|^{p-2}D_j|D_i u|^2 dx\\
&= \frac{1}{p}\int_\Omega \sum_{j=1}^2 v_j D_j|Du|^p dx\\
&= \frac{1}{p}\int_{\partial\Omega} |Du|^p(v\cdot\nu)\,d\sigma - \frac{1}{p}\int_\Omega \operatorname{div} v\,|Du|^p dx
\end{aligned}
$$

and, since u is a solution,

$$
\begin{aligned}
(2.6)\qquad \int_\Omega \sum_{i,j=1}^2 v_j D_j u D_i\big(|Du|^{p-2}D_i u\big)\,dx &= -\int_\Omega \sum_{j=1}^2 v_j D_j u f(u)\,dx\\
&= -\int_\Omega \sum_{j=1}^2 v_j D_j F(u)\,dx\\
&= \int_\Omega \operatorname{div} v \cdot F(u)\,dx.
\end{aligned}
$$

Thus, from (2.3), (2.4), (2.5) and (2.6), one can easily obtain (2.2).

REMARK 2.4. Notice that Theorem 2.1 is concerned only with classical solutions of problem (1.1), that is, solutions $u \in \mathscr{C}^1(\overline{\Omega}) \cap \mathscr{C}^2(\Omega)$. The case of weak solutions is more complicated because for quasi-linear problems, Pohozaev identities need to be carefully justified since the function $(v \cdot Du)$, which has to be used as test function, is not an admissible function because a weak solution u is not in general $\mathscr{C}^2$ but only $\mathscr{C}^{1,\alpha}$.

However, using the result obtained in [8], concerning the regularity to be required for the solutions u in the variational identity of Pucci–Serrin [35] which generalizes Pohozaev identity [33], Theorem 2.1 can be extended to cover the case where $u \in \mathscr{C}^1(\overline{\Omega})$ is a weak solution of problem (1.1).

Now, our aim is to find a suitable vector field v to be used in the integral identity (2.2) in such a way to prove Theorem 2.1.

Taking into account Remark 2.2, it suffices to consider only the case of domains $\Omega = \Omega^{2,\alpha}_{r,1}$ with $0 < r = \frac{\rho_1}{\rho_2} < 1$.

First, in the next lemma, we prove that the domain $\Omega^{2,\alpha/k}_{r^{1/k},1}$ is star-shaped for k large enough.

LEMMA 2.5. *For all $\alpha \in]0,\pi[$ and $r \in]0,1[$ there exists $k(\alpha,r) \in \mathbb{N}$ such that, for all positive integer $k > k(\alpha,r)$, the piecewise smooth bounded domain $\Omega^{2.\alpha/k}_{r^{1/k},1}$ is star-shaped with respect to the point $x_0 = (1,0)$.*

PROOF. For all $k \in \mathbb{N}$ and for all regular point $x \in \partial\Omega^{2.\alpha/k}_{r^{1/k},1}$, let us denote by $\nu_k(x)$ the outward normal to $\partial\Omega^{2.\alpha/k}_{r^{1/k},1}$.

If we set $x = \rho(\cos\theta,\sin\theta)$, it is clear that $(x-x_0)\cdot\nu_k(x) > 0$ for $\rho = 1$ and $0 < |\theta| < \frac{\alpha}{k}$ and also for $|\theta| = \frac{\alpha}{k}$ and $\rho \in]r^{1/k},1[$ (while it is obviously zero for $x = x_0$).

It remains the case where $\rho = r^{1/k}$ and $|\theta| < \frac{\alpha}{k}$. Notice that if $\rho = r^{1/k}$ and set $\bar{x}_k = r^{1/k}(\cos\frac{\alpha}{k},\sin\frac{\alpha}{k})$, for $k \geq 2$, we have

$$(x-x_0)\cdot\nu_k(x) > 0 \quad \text{for } |\theta| < \tfrac{\alpha}{k} \iff (\bar{x}_k - x_0)\cdot(-\bar{x}_k) \geq 0 \iff r^{1/k} \leq \cos\left(\tfrac{\alpha}{k}\right)$$

as one can verify by direct computation. Therefore, it suffices to verify that $r^{1/k} < \cos(\frac{\alpha}{k})$ for k large enough, which follows easily taking into account that

$$\lim_{k\to\infty}(r^{1/k}-1)k = \log r < 0 = \lim_{k\to\infty}\left[\cos\left(\tfrac{\alpha}{k}\right)-1\right]k.$$

Thus, the proof is complete. ∎

REMARK 2.6. Notice that, more precisely, the proof of Lemma 2.5 shows that for all $k > k(\alpha,r)$, we have $(x-x_0)\cdot\nu_k(x) > 0$ for all x in the boundary of $\Omega^{2.\alpha/k}_{r^{1/k},1}$ such that x is different from $x_0 = (0,1)$ and the boundary is smooth in x.

Now, taking into account Lemma 2.5, let us choose a positive integer $k > k(\alpha,r)$, so that the domain $\Omega^{2,\alpha/k}_{r^{1/k},1}$ is star-shaped with respect to the point $x_0 = (1,0)$, and consider the map

$$\varphi_k : \Omega^{2,\alpha/k}_{r^{1/k},1} \to \Omega^{2,\alpha}_{r,1}$$

defined by

$$\varphi_k(\rho\cos\theta,\rho\sin\theta) = \left(\rho^k\cos(k\theta),\rho^k\sin(k\theta)\right) \quad \forall\rho \in [r^{1/k},1],\ \forall\theta \in \left[-\tfrac{\alpha}{k},\tfrac{\alpha}{k}\right].$$

Notice that if we use the complex notation and set $z = \rho e^{i\theta}$, then $\varphi_k(z) = z^k$, so φ_k is a biholomorphic map between $\Omega^{2,\alpha/k}_{r^{1/k},1}$ and $\Omega^{2,\alpha}_{r,1}$.

Then, in $\Omega^{2,\alpha}_{r,1}$, let us consider the vector field $v_k \in \mathcal{C}^1(\Omega^{2,\alpha}_{r,1},\mathbb{R}^2)$ defined in the following way.

For all $x \in \Omega^{2,\alpha}_{r,1}$, let us set

$$\gamma_k(x,t) = t\varphi_k^{-1}(x) + (1-t)x_0 \quad \forall t \in \mathbb{R}. \tag{2.7}$$

Notice that, for all $x \neq x_0 = (1,0)$, there exists a unique positive number $\bar{t}_k(x)$ such that $\gamma_k(x,\bar{t}_k(x)) \in \partial\Omega^{2,\alpha/k}_{r^{1/k},1}$.

Indeed, since $\gamma_k(x,1)=\varphi_k^{-1}(x)\in\Omega^{2,\alpha/k}_{r^{1/k},1}$, we have $\bar t_k(x)\ge 1$. Moreover, $\gamma_k(x,t)$ is in the interior of $\Omega^{2,\alpha/k}_{r^{1/k},1}$ for $t\in\,]0,\bar t_k(x)[$, while $\gamma_k(x,0)=x_0\in\partial\Omega^{2,\alpha/k}_{r^{1/k},1}$ $\forall k\in\mathbb{N}$.

Analogously, $\varphi_k\circ\gamma_k(x,t)$ is in the interior of $\Omega^{2,\alpha}_{r,1}$ $\forall t\in\,]0,\bar t_k(x)[$, $\varphi_k\circ\gamma_k(x,1)=x\neq x_0$, $\varphi_k\circ\gamma_k(x,\bar t_k(x))\in\partial\Omega^{2,\alpha}_{r,1}$ and $\varphi_k\circ\gamma_k(x,0)=x_0\in\partial\Omega^{2,\alpha}_{r,1}$.

Let us consider the vector field v_k defined by

$$v_k(x)=\frac{\partial(\varphi_k\circ\gamma_k)}{\partial t}(x,1)\quad\forall x\in\Omega^{2,\alpha}_{r,1}\setminus\{x_0\}. \tag{2.8}$$

One can easily verify that $v_k\in\mathcal{C}^1(\Omega^{2,\alpha}_{r,1},\mathbb{R}^2)$, so it may be used in the integral identity (2.2).

Remark 2.7. Notice that if in the integral identity (2.2) we set $\Omega=\Omega^{2,\alpha}_{r,1}$ and $v=v_k$, then we have $(v\cdot\nu)\ge 0$ where the boundary of Ω is smooth.

In fact, if $x\in\partial\Omega^{2,\alpha}_{r,1}\setminus\{x_0\}$ is a regular point for the boundary of $\Omega^{2,\alpha}_{r,1}$, then $\varphi_k^{-1}(x)\in\partial\Omega^{2,\alpha/k}_{r^{1/k},1}\setminus\{x_0\}$ is a regular point for the boundary of $\Omega^{2,\alpha/k}_{r^{1/k},1}$ and we have

$$\big(\varphi_k^{-1}(x)-x_0\big)\cdot\nu_k\big(\varphi_k^{-1}(x)\big)\ge 0,$$

where $\nu_k(\varphi_k^{-1}(x))$ denotes the outward normal to the boundary of $\Omega^{2,\alpha/k}_{r^{1/k},1}$ in $\varphi_k^{-1}(x)$, because $\Omega^{2,\alpha/k}_{r^{1/k},1}$ is star-shaped with respect to x_0 (see Lemma 2.5).

Therefore, if we denote by $\nu(x)$ the outward normal to $\partial\Omega^{2,\alpha}_{r,1}$ in x, taking into account the definition of $\gamma_k(x,t)$ and $v_k(x)$, since φ_k is a conformal map, we obtain $(v_k(x)\cdot\nu(x))\ge 0$.

Thus, our assertion is proved for $x\neq x_0$. For $x=x_0$, it is obviously true because $v_k(x_0)=0$.

Moreover, notice that, indeed, we have $(v\cdot\nu)>0$ almost everywhere on $\partial\Omega$, as follows from Remark 2.6.

Here, we use the complex notation with $z=\rho e^{i\theta}$. Since $\varphi_k(z)=z^k$, one can easily verify that

$$v_k(z)=k\,z^{\frac{k-1}{k}}(z^{1/k}-1).$$

Moreover, taking into account that $v_k(z)$ is a holomorphic function, it follows that

$$\operatorname{div} v_k(z)=2\operatorname{Re}v_k'(z)\quad\forall z\in\Omega^{2,\alpha}_{r,1} \tag{2.9}$$

and

$$dv_k(z)[\xi]=|\xi|^2\operatorname{Re}v_k'(z)\quad\forall z\in\Omega^{2,\alpha}_{r,1},\ \forall\xi\in\mathbb{R}^2 \tag{2.10}$$

(where $\operatorname{Re}v_k'(z)$ denotes the real part of $v_k'(z)$).

Proof of Theorem 2.1. Taking into account Remark 2.2, it suffices to consider only the case of domains of the type $\Omega=\Omega^{2,\alpha}_{r,1}$ with $1/e<r<1$ and $0<\alpha<\pi$.

Notice that condition (1.2) obviously implies $f(0) = 0$, so the function $u \equiv 0$ in Ω is obviously a solution of problem (1.1) for every domain Ω.

In order to prove that problem (1.1) has only the trivial solution $u \equiv 0$ in $\Omega^{2,\alpha}_{r,1}$, we use the integral identity (2.2) with $\Omega = \Omega^{2,\alpha}_{r,1}$ and $v = v_k$. Then, we have

$$\begin{aligned}\left(1-\frac{1}{p}\right)&\int_{\partial\Omega^{2,\alpha}_{r,1}} |Du|^p (v_k\cdot\nu)\,d\sigma\\ &= \int_{\Omega^{2,\alpha}_{r,1}} |Du|^{p-2}\big(dv_k[Du]\cdot Du\big)\,dx + \int_{\Omega^{2,\alpha}_{r,1}} \operatorname{div} v_k\big[F(u) - \tfrac{1}{p}|Du|^p\big]\,dx.\end{aligned}$$

Since $(v_k \cdot \nu) \geq 0$ on $\partial\Omega^{2,\alpha}_{r,1}$, taking also into account (2.9) and (2.10), we obtain

$$\begin{aligned}0 &\leq \left(1-\frac{1}{p}\right)\int_{\partial\Omega^{2,\alpha}_{r,1}} |Du|^p v_k\cdot\nu\,d\sigma\\ &= \int_{\Omega^{2,\alpha}_{r,1}} \operatorname{Re} v_k'|Du|^p\,dx + 2\int_{\Omega^{2,\alpha}_{r,1}} \operatorname{Re} v_k'\big[F(u)-\tfrac{1}{p}|Du|^p\big]\,dx,\end{aligned}$$

where $F(u) \leq \frac{1}{q}uf(u)$ because of condition (1.2) and

$$\int_{\Omega^{2,\alpha}_{r,1}} uf(u)\,dx = \int_{\Omega^{2,\alpha}_{r,1}} |Du|^p\,dx$$

because u is a solution of problem (1.1) in $\Omega = \Omega^{2,\alpha}_{r,1}$.

Taking into account that $(1-\frac{2}{p}) < 0$ and $F(u) \geq 0$, it follows that

$$0 \leq \left[\left(1-\frac{2}{p}\right)m_k + \frac{2}{q}M_k\right]\cdot\int_{\Omega^{2,\alpha}_{r,1}} |Du|^p dx \quad \forall k > k(\alpha, r), \tag{2.11}$$

where m_k and M_k denote, respectively, the minimum and the maximum of the set

$$\big\{\operatorname{Re} v_k'(z) : z \in \Omega^{2,\alpha}_{r,1}\big\}.$$

Notice that, as $k \to \infty$, $v_k(z)$ tends to $z\log z$ in $\mathcal{C}^1(\Omega^{2,\alpha}_{r,1})$, as one can verify by direct computation. As a consequence, M_k and m_k tend, as $k \to \infty$, respectively, to the maximum and to the minimum of the set

$$\big\{\operatorname{Re}(1+\log z) : z \in \Omega^{2,\alpha}_{r,1}\big\}.$$

One can easily verify that the maximum of this set is equal to 1 and that the minimum is $(1 + \log r)$, which is a positive number because $r > 1/e$. It follows that

$$\lim_{k\to\infty}\left[\left(1-\frac{2}{p}\right)m_k + \frac{2}{q}M_k\right] = \left(1-\frac{2}{p}\right)\cdot(1+\log r) + \frac{2}{q}.$$

Since $q > \frac{2p}{2-p} \cdot (1 + \log r)^{-1}$, we have $(1 - \frac{2}{p}) \cdot (1 + \log r) + \frac{2}{q} < 0$ and, as a consequence, $(1 - \frac{2}{p})m_k + \frac{2}{q} M_k < 0$ for k large enough.

Thus, from (2.11), we infer that

$$\int_{\Omega^{2,\alpha}_{r,1}} |Du|^p \, dx = 0;$$

that is, $u \equiv 0$ in $\Omega^{2,\alpha}_{r,1}$. So, the proof is complete. ∎

REMARK 2.8. Notice that the assertion of Theorem 2.1 can be stated in the following equivalent way, as one can easily verify: *if $q > p^*$, where $p^* = \frac{2p}{2-p}$, $0 < \rho_1 < \rho_2$ and $\frac{\rho_2}{\rho_1} < e^{\frac{q-p^*}{q}}$, then problem (1.1) has only the trivial solution $u \equiv 0$ in the domain $\Omega^{2,\alpha}_{\rho_1,\rho_2}$.*

Hence, one can easily infer, in particular, that for all $q > \frac{2p}{2-p}$, there exists $\hat{\varepsilon} > 0$ such that problem (1.1) has only the trivial solution in $\Omega^{2,\alpha}_{\rho_1,\rho_2}$ when $\frac{\rho_2}{\rho_1} < 1 + \hat{\varepsilon}$.

Thus, Theorem 2.1 contains, as a particular case, also the result concerning thin domains, obtained in [17] by using a different method. However, notice that, unlike the case of [17], in Theorem 2.1, the ratio $\frac{\rho_2}{\rho_1}$ is not required to be close to 1 when q is large enough.

Finally, let us point out that in Theorem 2.1, as well as in [17], no assumption is required on the parameter $\alpha \in]0, \pi[$, so the domain $\Omega^{2,\alpha}_{\rho_1,\rho_2}$ can be arbitrarily close to the noncontractible domain $\Omega^{2,\pi}_{\rho_1,\rho_2}$, where there exist infinitely many positive and nodal solutions.

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