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Genuine pro- p Iwahori–Hecke algebras, Gelfand–Graev representations, and some applications

To Professor Gordan Savin on his 60th birthday

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Abstract. We study the Iwahori component of the Gelfand–Graev representation of a central cover of a split linear reductive group and utilize our results for three applications. In fact, it is advantageous to begin at the pro- p level. Thus to begin we study the structure of a genuine pro- p Iwahori–Hecke algebra, establishing a Bernstein presentation. With this structure theory we first describe the pro- p part of the Gelfand–Graev representation and then the more subtle Iwahori part. For the first application we relate the Gelfand–Graev representation to the metaplectic representation of Sahi–Stokman–Venkateswaran, which conceptually realizes the Chinta–Gunnells action from the theory of Weyl group multiple Dirichlet series. For the second we compute the Whittaker dimension of the constituents of regular unramified principal series representations; for the third we do the same for unitary unramified principal series representations.

Keywords: covering groups, pro- p Iwahori–Hecke algebra, Iwahori–Hecke algebras, Gelfand–Graev representations, Chinta–Gunnells representations, Whittaker dimensions.

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1. Introduction

Let F be a p -adic local field with ring of integers O_F and residue field κ . Let \mathbf{G} be a smooth reductive linear algebraic group scheme over O_F . It is known that the generic fiber of \mathbf{G} is quasi-split and split over a finite unramified extension of F ; for simplicity, we assume that it is actually split over F . Write $G := \mathbf{G}(F)$ and $G_\kappa := \mathbf{G}(\kappa)$.

It is a well-known result of Rodier, Gelfand–Kazhdan and Shalika that every irreducible admissible representation $\pi \in \mathrm{Irr}(G)$ has at most one Whittaker model, i.e.,

$$\dim \mathrm{Hom}_G(\pi, \mathrm{Ind}_{U^-}^G \psi) \leq 1, \tag{1.1}$$

where U^- is the unipotent radical of a Borel subgroup of G and ψ is a nondegenerate character of U^- . This is a p -adic analogue of the finite field case, where it was

first shown by Gelfand–Graev that the finite field analogue $\text{Ind}_{U_\kappa^-}^{G_\kappa} \psi_\kappa$ of $\text{Ind}_{U^-}^G \psi$ has similar multiplicity-one property. This multiplicity-one property for $\text{Ind}_{U_\kappa^-}^{G_\kappa} \psi_\kappa$ follows from the commutativity of its endomorphism algebra $C(U_\kappa^-, \psi_\kappa \backslash G_\kappa / U_\kappa^-, \psi_\kappa)$, which can be proved by exhibiting a Chevalley–Steinberg involution (see [10, Section 8.1] or [49, Chapter 14]). The proof for G essentially adapts this idea. The multiplicity-one property (1.1) for Whittaker models is important for the theory of L -functions, especially the Langlands–Shahidi method and some Rankin–Selberg integrals as well.

As a refinement of (1.1), it was shown in [7, Theorem 4.3] that for a simply generic Bernstein class \mathfrak{s} with respect to ψ , there is a canonical isomorphism

$$\mathfrak{Z}^\mathfrak{s}(G) \rightarrow \text{End}_G(\text{ind}_{U^-}^G \psi)^\mathfrak{s},$$

where $\mathfrak{Z}^\mathfrak{s}(G)$ is the Bernstein center of the subcategory of $\text{Irr}(G)$ associated with \mathfrak{s} . Here $\text{ind}_{U^-}^G \psi$ is the compact induction, whose dual is just $\text{Ind}_{U^-}^G \psi^{-1}$. If one concentrates further on the unramified class $\mathfrak{s} = [T, 1]$, then further investigation of the structure of $(\text{ind}_{U^-}^G \psi)^\mathfrak{s} = (\text{ind}_{U^-}^G \psi)^I$ was given in several works, [2, 11, 43] for example. Here $I \subset G$ is the Iwahori subgroup determined by the Borel subgroup $B \subset G$. In particular, it was shown in [11] that if the conductor of ψ is \mathfrak{p}_F , then

$$(\text{ind}_{U^-}^G \psi)^I \simeq \varepsilon_W \otimes_{\mathcal{H}_W} \mathcal{H}_I, \tag{1.2}$$

where

$$\mathcal{H}_I = C_c^\infty(I \backslash G / I)$$

is the Iwahori–Hecke algebra and ε_W the sign character of the finite-dimensional subalgebra \mathcal{H}_W deformed from $\mathbb{C}[W]$. Such a result was also obtained from a more general perspective by Brubaker–Bump–Friedberg [4]. For general $\mathfrak{s} = [T, \chi]$, the structure of $(\text{ind}_{U^-}^G \psi)^\mathfrak{s}$ was determined recently by [40].

More generally, one may consider the above problems for central covers of G . Assume that F^\times contains the full group μ_n of n -th roots of unity. There are natural n -fold central covers

$$\mu_n \hookrightarrow \bar{G} \twoheadrightarrow G,$$

of which we consider only the *genuine* representations, i.e., μ_n acts via a fixed embedding $\epsilon : \mu_n \hookrightarrow \mathbb{C}^\times$. With the unipotent subgroup U^- splitting uniquely in \bar{G} , one may consider the genuine Gelfand–Graev representation

$$\mathcal{V} := \text{ind}_{\mu_n U^-}^{\bar{G}} (\epsilon \otimes \psi).$$

In this case, for fixed \bar{G} we expect the right hand side of (1.1) to be replaced by $|\bar{T}/Z(\bar{T})|^{1/2}$, where this upper bound is achieved (see [22]). In particular, multiplicity-one fails.

This multi-dimensionality of Whittaker models for genuine representations was observed decades ago. Though it gives obstacles to the classical theory of L -functions, it nevertheless motivates the theory of Weyl group multiple Dirichlet series (WMDS), which

dates back at least to the work of Kubota. These WMDS possess meromorphic continuation and functional equation, but are not Eulerian. Conjecturally, they are equal to certain Fourier coefficients of the Borel Eisenstein series of \bar{G} . See [6] for an exposition on this topic. There are various methods to construct such WMDS. One was given by Chinta–Gunnells [12] by utilizing the so-called metaplectic W -representation afforded by $\mathbb{C}(Y)$, where Y is the cocharacter lattice of G . Let $\mathcal{H}_I = C_{\epsilon,c}^\infty(I \backslash \bar{G}/I)$ be the Iwahori–Hecke algebra of \bar{G} . It was shown recently by Sahi–Stokman–Venkateswaran [45] that the metaplectic W -representation of Chinta–Gunnells arises naturally from a certain \mathcal{H}_I -module afforded on the space $\mathbb{C}[Y]$, which we call the SSV representation of \mathcal{H}_I . In fact, the SSV representation is afforded by the bigger space $\mathbb{C}[P]$, where $P \supset Y$ is the coweight lattice. In any case, it should be highlighted that the construction in [45] is algebraic and uniform and does not involve computer-assisted checking of well-definedness of the Weyl group action.

In this paper we continue the study of Whittaker spaces for tame covering groups (i.e. $\gcd(p, n) = 1$), focusing on the following problems:

- (P1) Describe the Iwahori component of the Gelfand–Graev representation, that is, determine the \mathcal{H}_I -module structure of $\mathcal{V}^I = \text{ind}_{\mu_n \bar{U}^-}^{\bar{G}}(\epsilon \otimes \psi)^I$. This will be a direct generalization of [11] to covers.
- (P2) Give a more natural and conceptual interpretation of the SSV representation in terms of \mathcal{V}^I . Relations between these two and also the metaplectic W -representation have been hinted at in several works including [12, 13, 39] via the presence of the local scattering matrices, which describe the intertwining operators between the Whittaker models of principal series representations $I(\chi)$ and $I(w\chi)$.
- (P3) Verify some speculative formulas regarding the Whittaker dimension of some Iwahori spherical representations, especially those irreducible constituents of a regular or unitary unramified principal series (see [19–21]).

A solution to (P1) would provide a valuable tool to investigate (P2) and (P3). Unfortunately, naively adapting the strategy of [11] does not work. The fundamental obstacle is that the “supports” of \mathcal{V}^I and \mathcal{H}_I do not match. More precisely, for a linear group G , the Iwahori–Hecke algebra \mathcal{H}_I is an affine Hecke algebra deformed from the extended affine Weyl group $W \ltimes Y$, and the support of $(\text{ind}_{\bar{U}^-}^{\bar{G}} \psi)^I$ corresponds to Y . In this case, the twisted Satake transform (see [25])

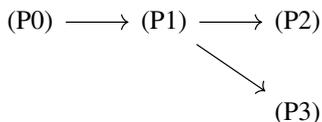
$$S_\psi : C_c^\infty(I \backslash G/I) \rightarrow (\text{ind}_{\bar{U}^-}^{\bar{G}} \psi)^I$$

of \mathcal{H}_I -modules is surjective, and realizes (1.2). However, for \bar{G} , the support of the genuine Iwahori–Hecke algebra only corresponds to $W \ltimes Y_{Q,n}$, where $Y_{Q,n} \subset Y$ is a certain sublattice; yet the support of \mathcal{V}^I still corresponds to Y . This is the support mismatch mentioned above. Consequently, S_ψ is not surjective for general covering groups.

The support mismatch disappears if I is replaced by its maximal pro- p subgroup I_1 , so we consider the algebra $\mathcal{H} := C_{\epsilon,c}^\infty(I_1 \backslash \bar{G}/I_1)$. Our first goal is then the following:

- (P0) Study the larger pro- p Hecke algebra \mathcal{H} instead of \mathcal{H}_I ; explicate the structure of \mathcal{V}^{I_1} over \mathcal{H} .

We leverage these pro- p results to investigate the Iwahori level and ultimately solve problems (P1)–(P3). The road map of the above topics and results in our paper is as follows:



1.1. Main results

To elaborate on the above, we give a brief outline of the paper and state our main results.

In Section 2, we introduce algebraic groups, their central covers, and fix some notations.

In Section 3, we first study a \mathbb{Z} -model $\mathcal{H}_{\mathbb{Z}}$ of the pro- p Iwahori–Hecke algebra \mathcal{H} and establish the braid and quadratic relations. In Section 4, based on the universal principal series (along the lines of Haines–Kottwitz–Prasad [26]) we prove the following theorem.

Theorem 1.1 (Theorem 4.18). *The pro- p Hecke algebra \mathcal{H} has an explicit Bernstein presentation.*

We note that for linear groups the pro- p Iwahori–Hecke algebra was studied by Vigneras [53] and Flicker [15].

The Bernstein presentation of \mathcal{H} gives rise to the corresponding one for \mathcal{H}_I , which we explicate in Section 4.6. This presentation already appears in the earlier work [16, 38, 46, 47]. However, by using the pro- p algebra we can easily prove Corollary 3.5 and circumvent the technical complications faced by Savin [47, Proposition 6.5].

In Section 5, we first discuss the structure of

$$\mathcal{V}_{\kappa} = (\text{Ind}_{U_{\kappa}}^{G_{\kappa}} \psi)^{U_{\kappa}},$$

the U_{κ} -fixed vectors in the Gelfand–Graev representation of G_{κ} . We show that there is a decomposition

$$\mathcal{V}_{\kappa} = \bigoplus_{\emptyset} \mathcal{V}_{\kappa, \emptyset}$$

into irreducible modules over $\mathcal{H}_{\kappa} = C(U_{\kappa} \backslash G_{\kappa} / U_{\kappa})$, where the sum is taken over all W -orbits in $\text{Hom}(T_{\kappa}, \mathbb{C}^{\times})$. There is a W -stable sublattice $Y_{Q,n} \subset Y$ which controls the noncommutativity of the covering torus \bar{T} . We write $\mathcal{X}_{Q,n} := Y / Y_{Q,n}$. The main result pertaining to (P0) and (P1) is the following:

Theorem 1.2 (Theorem 5.13, Corollary 5.15, Theorem 5.21). *There is a natural isomorphism*

$$\gamma : \mathcal{V}_{\kappa} \otimes_{\mathcal{H}_{\kappa}} \mathcal{H} \rightarrow \mathcal{V}^I$$

of \mathcal{H} -modules. It gives rise to a decomposition

$$\mathcal{V}^I = \bigoplus_{\emptyset \subset \mathcal{X}_{Q,n}} \mathcal{V}_{\emptyset}^I$$

over all W -orbits in $\mathcal{X}_{Q,n}$. Moreover, if $\mathcal{O} = \mathcal{O}_y \subset \mathcal{X}_{Q,n}$ is splitting (see Definition 5.19), then one has natural isomorphisms of \mathcal{H}_I -modules

$$\varepsilon_y \otimes_{\mathcal{H}_{I,y}} \mathcal{H}_I \rightarrow \mathcal{V}_{\kappa,\mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I \xrightarrow{\gamma} \mathcal{V}_{\mathcal{O}}^I,$$

where ε_y is the sign character of the deformed subalgebra $\mathcal{H}_{I,y} \subset \mathcal{H}_W \subset \mathcal{H}_I$ associated with the parabolic Weyl subgroup $W_y = \text{Stab}_W(y) \subset W$.

The trivial orbit of 0 and free orbits are always splitting, which immediately gives Corollaries 5.22 and 5.23. In particular, if $n = 1$, then $\bar{G} = G$ is a linear group, $\mathcal{X}_{Q,n} = \{0\}$, and in this case Corollary 5.22 implies (1.2).

From Theorem 1.2, it is thus an interesting question to determine which orbits $\mathcal{O} \subset \mathcal{X}_{Q,n}$ are splitting, and even to determine the covering groups \bar{G} for which every orbit \mathcal{O} is splitting. We give a detailed discussion of several (nested) subclasses of covering groups in Section 6. In particular, we show in Corollary 6.4 that for an oasisitic cover of an almost simple and simply-connected G , every orbit \mathcal{O} is splitting. Other examples with this splitting property include the Kazhdan–Patterson and Savin covers of GL_r , which we discuss in Example 6.5.

In the remainder of Section 6, we give a full account of the structure of $\mathcal{V}_{\mathcal{O}}^I$ for covers $\overline{\text{SL}}_2$ of SL_2 . This example is already instructive. Indeed, if the invariant $n^* = n/\text{gcd}(n, 2Q(\alpha^\vee))$ associated with $\overline{\text{SL}}_2$ is odd, then every orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$ is splitting and thus Theorem 1.2 applies. However, if n^* is even, then there is one peculiar orbit which is not splitting. This illustrates some subtleties in determining the structure of $\mathcal{V}_{\mathcal{O}}^I$ for general \mathcal{O} .

In Section 7, we identify the \mathcal{H}_I -module \mathcal{V}^I as a submodule of the metaplectic representation $(\pi, \mathbb{C}[P])$ constructed by Sahi–Stokman–Venkateswaran [45]. As mentioned above, there is a natural \mathcal{H}_I -module $(\pi, \mathbb{C}[P])$ afforded by $\mathbb{C}[P]$. The subspace $\mathbb{C}[Y]$ is invariant under π and thus gives $(\pi^0, \mathbb{C}[Y])$. Moreover, there is a decomposition

$$\pi^0 = \bigoplus_{\mathcal{O} \subset \mathcal{X}_{Q,n}} \pi_{\mathcal{O}}^0,$$

where \mathcal{O} is taken over all orbits in $\mathcal{X}_{Q,n}$. We have the following answer to (P2) above:

Theorem 1.3 (Theorem 7.3). *Let \bar{G} be an n -fold cover of a semisimple group G . Assume $(-1, \varpi)_n = 1$. Then for every W -orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$, one has*

$$\mathcal{V}_{\mathcal{O}}^I \simeq \pi_{\mathcal{O}}^0$$

as \mathcal{H}_I -modules; hence, $\mathcal{V}^I \simeq \pi^0$ as well.

In fact, for each $z \in P$, the space $\mathbb{C}[Y] \cdot x^z \subset \mathbb{C}[P]$ is also \mathcal{H}_I -invariant and thus gives a representation $(\pi^z, \mathbb{C}[Y] \cdot x^z)$ of \mathcal{H}_I . We speculate that π^z is related to $(\text{ind}_{\mu_n U}^G \epsilon \otimes {}^z \psi)^I$; see Conjecture 7.5.

In Section 8, we present a different approach to (P1), suggested by the referee who also outlined the proof. This approach is based on studying the space $\mathcal{W}^I =$

$C_{c,\varepsilon}((I_1^-, \psi) \backslash \bar{G}/I)$, which is isomorphic to \mathcal{V}^I as \mathcal{H}_I -module. One advantage of the space \mathcal{W}^I is that one can use the theory of (B, N) -pairs to efficiently derive information about how the action of \mathcal{H}_I affects the support. The main result of this section is Corollary 8.7, which complements Theorem 1.3. Corollary 8.7 is stronger than Theorem 1.3 in the sense that it removes the splitting orbits hypothesis; it is weaker in the sense that the inducing character is not explicit. Making the character explicit will be part of a future work.

We also want to mention that one step in the proof of Corollary 8.7 relies on a result about the Iwahori–Hecke algebra \mathcal{H}_I that is expected to hold, but does not seem to be in the literature. (See the remark following Corollary 8.7 for more details.) We have a strategy for proving this using the pro- p algebra \mathcal{H} . The details will appear in a future work.

The remaining part of the paper concerns (P3). We apply Theorem 1.2 to determine certain Whittaker dimensions and verify several formulas conjectured in [19, 20]. In fact, for every orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$, one can define the \mathcal{O} -Whittaker subspace of any Iwahori-spherical $\pi \in \text{Irr}(\bar{G})$ as

$$\text{Wh}_\psi(\pi)_\mathcal{O} := \text{Hom}_{\mathcal{H}_I}(\mathcal{V}_\mathcal{O}^I, \check{\pi}^I).$$

If $\pi = I(\chi)$ is an unramified principal series representation, we expect that $\text{Wh}_\psi(I(\chi))_\mathcal{O}$ is isomorphic to another naturally defined \mathcal{O} -Whittaker space $\text{Wh}_\psi(I(\chi))_\mathcal{O}^\sharp$ discussed more frequently in the literature; see Conjecture 9.7.

In any case, in the last part of Section 9, we consider regular unramified genuine principal series representations $I(\chi)$. We assume that the set $\Phi(\chi)$ of reducibility of $I(\chi)$ is a subset of simple roots. Then by Rodier’s result, the semisimplification of $I(\chi)^{\text{ss}}$ is multiplicity free and there is a natural bijection

$$\mathcal{P}(\Phi(\chi)) \rightarrow \text{JH}(I(\chi)), \quad S \mapsto \pi_S,$$

determined by the Jacquet module of π_S . Here $\mathcal{P}(\Phi(\chi))$ is the power set of $\Phi(\chi)$ and $\text{JH}(I(\chi))$ is the Jordan–Hölder set of $I(\chi)$. Every orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$ gives rise to a Weyl group permutation

$$\sigma_\mathcal{O}^\mathcal{X} : W \rightarrow \text{Perm}(\mathcal{O}), \quad \sigma_\mathcal{O}^\mathcal{X}(w)(y) := w(y).$$

We have $\sigma^\mathcal{X} = \bigoplus_{\mathcal{O} \subset \mathcal{X}_{Q,n}} \sigma_\mathcal{O}^\mathcal{X}$, which is the permutation representation of W realized on $\mathcal{X}_{Q,n}$.

Theorem 1.4 (Theorem 9.8). *Keep the notation above and let $S \subset \Phi(\chi)$. Then for every splitting orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$ one has*

$$\dim \text{Wh}_\psi(\pi_S)_\mathcal{O} = \langle \sigma_S, \sigma_\mathcal{O}^\mathcal{X} \rangle_W,$$

where $\sigma_S \in \text{Rep}(W)$ is a sum of certain Kazhdan–Lusztig representations of W naturally associated with S . Hence, for \bar{G} such that every orbit \mathcal{O} is splitting (for example, those as in Corollary 6.4 and Example 6.5) one has

$$\dim \text{Wh}_\psi(\pi_S) = \langle \sigma_S, \sigma^\mathcal{X} \rangle_W.$$

The proof of Theorem 1.4 is essentially the same as that in [19] which concerns $\text{Wh}_\psi(\pi_S)^\sharp_{\mathcal{O}}$. However, the crucial difference is that we know that the functor $\dim \text{Wh}_\psi(-)_{\mathcal{O}}$ is exact, whereas the exactness is not clear for $\text{Wh}_\psi(-)^\sharp_{\mathcal{O}}$. Thus, Theorem 1.4 verifies the analogue of [19, Conjecture 1.1] for $\text{Wh}_\psi(\pi_S)_{\mathcal{O}}$ considered in the present paper, and also for $\text{Wh}_\psi(\pi_S)^\sharp_{\mathcal{O}}$ if we assume Conjecture 9.7.

In Section 10, we consider unitary unramified genuine principal series representations $I(\chi)$ with a decomposition

$$I(\chi) = \bigoplus_{\sigma \in \text{Irr}(R_\chi)} \pi_\sigma,$$

where R_χ is the R-group of $I(\chi)$.

Theorem 1.5 (Theorem 10.2). *Let \bar{G} be a very saturated cover of an almost simple simply-connected G associated with $Q(\alpha^\vee) = 1$ for any short simple coroot α^\vee . Let $I(\chi)$ be a unitary (K, s_K) -unramified genuine principal series representation of \bar{G} . Let $\mathcal{O} \subset \mathcal{X}_{Q,n}$ be a W -orbit satisfying the S -property (see Definition 5.24). Then*

$$\dim \text{Wh}_\psi(\pi_\sigma)_{\mathcal{O}} = \langle \sigma \otimes \zeta_{\bar{\rho}}, \sigma_{\mathcal{O}}^{\mathcal{X}} \rangle_{R_\chi} \quad \text{for every } \sigma \in \text{Irr}(R_\chi).$$

In the above theorem, $\zeta_{\bar{\rho}}$ is a character of R_χ given in (10.1). The proof relies on explicating the Kazhdan–Lusztig–Reeder parametrization of π_σ in terms of a homology space. If \bar{G} is an oasisitic cover of an almost simple simply-connected group, then every W -orbit \mathcal{O} satisfies the condition in Theorem 1.5, and this gives Corollary 10.3.

At the end of Section 10, we also prove a result regarding the variation of Whittaker dimension with respect to changing the additive character from ψ of conductor \mathfrak{p}_F to ${}^\rho\psi$ of conductor O_F . This could be considered as a covering analogue of the linear case as discussed in [29, Section 4] or [17, Section 9]. It in particular settles a special case of [23, Conjecture 5.7]. See Corollary 10.4 for details.

In summary, Theorems 1.4, 1.5 and Corollary 10.3 constitute the main results for (P3) above.

2. Preliminaries

2.1. Algebraic groups

Let p be a prime number. Let F be a finite extension of \mathbb{Q}_p with ring of integers $O = O_F$ and maximal ideal $\mathfrak{p} = \mathfrak{p}_F$. Let $\varpi \in O_F$ be a fixed uniformizer, and thus $\mathfrak{p}_F = \varpi O_F$. Let q denote the size of the residue field $\kappa = O_F/\mathfrak{p}_F$.

Let \mathbf{G} be a connected reductive linear algebraic group over O_F . We assume that the generic fiber at $\text{Spec}(F)$, which is still denoted by \mathbf{G} , is split over F with maximal split torus \mathbf{T} . We write

$$X = \text{Hom}(\mathbf{T}, \mathbf{G}_m) \quad \text{and} \quad Y = \text{Hom}(\mathbf{G}_m, \mathbf{T})$$

for the lattices of characters and cocharacters of \mathbf{T} respectively. Here X and Y form a perfect pairing

$$\langle -, - \rangle : X \times Y \rightarrow \mathbb{Z} \quad \text{given by} \quad x \circ y(t) = t^{\langle x, y \rangle} \text{ for } t \in \mathbf{G}_m.$$

Let Φ and Φ^\vee denote the sets of roots and of coroots of (\mathbf{G}, \mathbf{T}) , respectively. We write $Y^{\text{sc}} \subset Y$ for the sublattice generated by the coroots.

Let $N(\mathbf{T}) \subset \mathbf{G}$ be the normalizer of \mathbf{T} in \mathbf{G} . This gives the Weyl group

$$W = N(\mathbf{T})/\mathbf{T},$$

which we identify with the Weyl group of the coroot system. Given $\alpha^\vee \in \Phi^\vee$, we write w_α for the associated reflection of $Y \otimes \mathbb{Q}$.

We fix a Borel subgroup \mathbf{B} containing \mathbf{T} with unipotent radical \mathbf{U} . This choice of \mathbf{B} identifies the set Φ_+ of positive roots (resp. Φ_+^\vee of positive coroots) and of simple roots $\Delta \subset \Phi_+$ (resp. simple coroots $\Delta^\vee \subset \Phi_+^\vee$). The choice of simple roots induces a length function

$$\ell : W \rightarrow \mathbb{Z}_{\geq 0}.$$

Let $\mathbf{B}^- = \mathbf{T}\mathbf{U}^-$ denote the opposite Borel subgroup. We fix a Chevalley–Steinberg system $\{e_\alpha : \mathbf{G}_a \rightarrow \mathbf{U}_\alpha\}_{\alpha \in \Phi}$ of pinnings for (\mathbf{G}, \mathbf{T}) . For each $\alpha \in \Phi$ we can define the map

$$w_\alpha : \mathbf{G}_m \rightarrow N(\mathbf{T}), \quad w_\alpha(a) = e_\alpha(a) \cdot e_{-\alpha}(-a^{-1}) \cdot e_\alpha(a).$$

For $\alpha \in \Phi$ we write $h_\alpha(a) := \alpha^\vee(a)$ which satisfies $h_\alpha(a) = w_\alpha(a) \cdot w_\alpha(-1)$.

For simplicity of notation, for a group \mathbf{H} over κ , we write

$$H_\kappa := \mathbf{H}(\kappa).$$

Consider the hyperspecial maximal compact subgroup $K := \mathbf{G}(O_F)$. The reduction mod \mathfrak{p} map

$$\text{red}_\mathfrak{p} : K \twoheadrightarrow G_\kappa = \mathbf{G}(\kappa)$$

is surjective (see Tits [52, Section 3.4.4]). Let

$$I := \text{red}_\mathfrak{p}^{-1}(B_\kappa) \subset K$$

be the Iwahori subgroup. Inside I one has a unique maximal pro- p normal subgroup

$$I_1 := \text{red}_\mathfrak{p}^{-1}(U_\kappa).$$

We write G, B, T, U for the F -rational points of $\mathbf{G}, \mathbf{B}, \mathbf{T}, \mathbf{U}$. For any subgroup $H \subset G$ let

$$H_1 := H \cap I_1.$$

The group U decomposes as a product of the subgroups U_α , where $\alpha \in \Phi_+$; similarly for the opposite unipotent radical U^- with respect to Φ_- . Each U_α has a filtration by the subgroups

$$U_\alpha^l = \{e_\alpha(u) : u \in \mathfrak{p}^l\}$$

indexed by $l \in \mathbb{Z}$ and satisfying $U_\alpha^l \supset U_\alpha^{l+1}$. See [52] for details.

2.2. Root system and affine root system

Consider

$$\mathcal{A} = Y \otimes \mathbb{R} \quad \text{and} \quad \mathcal{A}^{\text{sc}} = Y^{\text{sc}} \otimes \mathbb{R}.$$

The coroot system $\Phi^\vee \subset \mathcal{A}^{\text{sc}}$ decomposes into irreducible coroot systems $\Phi_j^\vee \subset \mathcal{A}_j := \text{span}_{\mathbb{R}}(\Phi_j^\vee)$ satisfying

$$\Phi^\vee = \bigcup_{j=1}^d \Phi_j^\vee, \quad \mathcal{A}^{\text{sc}} = \bigoplus_{j=1}^d \mathcal{A}_j.$$

This gives a partition of Δ^\vee into $\Delta_j^\vee = \Delta^\vee \cap \Phi_j^\vee$, the sets of simple roots for the Φ_j^\vee 's. Each Φ_j has a unique highest root α_j^\dagger with respect to Δ_j .

For any $\alpha \in \Phi$ and $k \in \mathbb{Z}$, consider the hyperplane

$$H_{\alpha,k} = \{v \in \mathcal{A} : \langle \alpha, v \rangle = k\}.$$

Let $w_{\alpha,k}$ be the reflection of \mathcal{A} fixing $H_{\alpha,k}$, i.e., $w_{\alpha,k}(v) = v - (\langle \alpha, v \rangle - k)\alpha^\vee$. The set $\{w_{\alpha,k} : \alpha \in \Phi, k \in \mathbb{Z}\}$ of reflections generates the affine Weyl group $W_{\text{aff}} = W \ltimes Y^{\text{sc}}$. Our choice of Φ_+ identifies the alcove \mathcal{C} that is contained in the positive Weyl chamber such that $0 \in \bar{\mathcal{C}}$. The set of reflections associated to the walls of \mathcal{C} ,

$$S_{\text{aff}} = \{w_\alpha : \alpha \in \Delta\} \cup \{w_{\alpha_j^\dagger, 1} : 1 \leq j \leq d\},$$

is a minimal set of generators for W_{aff} and realizes W_{aff} as a Coxeter group. The group W_{aff} acts simply transitively on the set of alcoves, which are the connected components of

$$\mathcal{A} - \bigcup_{\alpha \in \Phi, k \in \mathbb{Z}} H_{\alpha,k}.$$

Let $W_{\text{ex}} := W \ltimes Y$ be the extended affine Weyl group. One has

$$W_{\text{ex}} = W_{\text{aff}} \rtimes \Omega,$$

where $\Omega = \{w \in W_{\text{ex}} : w(\mathcal{C}) = \mathcal{C}\}$.

A *minimal expression* of the element $w \in W_{\text{aff}}$ with respect to S_{aff} is a factorization $w = w_1 \dots w_k$ with $w_j \in S_{\text{aff}}$ such that there is no factorization of w using fewer elements of S_{aff} . The (strong) *Bruhat order* $w < w'$ on the Coxeter group W_{aff} with respect to S_{aff} is defined as follows: $w < w'$ if there are minimal expressions $w = w_1 \dots w_k$ and $w' = w'_1 \dots w'_\ell$ such that the sequence w_1, \dots, w_k is a subsequence of w'_1, \dots, w'_ℓ .

2.3. Covering groups

By abuse of notation, we still use \mathbf{G} to denote the generic fiber of \mathbf{G} at the point $\text{Spec}(F)$. Consider the \mathbf{K}_2 -extension [5]

$$\mathbf{K}_2 \hookrightarrow \bar{\mathbf{G}} \xrightarrow{\wp} \mathbf{G}$$

over F which is incarnated by the pair $(D, \eta = \mathbb{1})$ (see [55] or [16, Section 2.6]). Here

$$D : Y \times Y \rightarrow \mathbb{Z}$$

is a (not necessarily symmetric) bilinear form such that the quadratic form

$$Q(y) := D(y, y)$$

is Weyl-invariant. We have a Weyl-invariant symmetric bilinear form B_Q given by

$$B_Q(y, z) = D(y, z) + D(z, y).$$

The extension $\bar{\mathbf{G}}$ splits canonically and \mathbf{G} -equivariantly over any unipotent subgroup of \mathbf{G} . Thus, we write

$$\bar{e}_\alpha : \mathbf{G}_a \rightarrow \bar{\mathbf{G}}$$

for the splitting of e_α for any $\alpha \in \Phi$. With this, one gets

$$\bar{w}_\alpha(a) := \bar{e}_\alpha(a) \cdot \bar{e}_{-\alpha}(-a^{-1}) \cdot \bar{e}_\alpha(a) \in \overline{N(\mathbf{T})}, \quad \bar{h}_\alpha(a) := \bar{w}_\alpha(a) \cdot \bar{w}_\alpha(-1) \in \bar{\mathbf{T}}.$$

Also, there is a section \mathbf{s} of the quotient map $\bar{\mathbf{T}} \twoheadrightarrow \mathbf{T}$ such that for any $a_j \in \mathbf{G}_m, y_j \in Y, j = 1, 2$, one has

$$\mathbf{s}(y_1(a_1)) \cdot \mathbf{s}(y_2(a_2)) = \{a_1, a_2\}^{D(y_1, y_2)} \cdot \mathbf{s}(y_1(a_1) \cdot y_2(a_2)), \tag{2.1}$$

where $\{a_1, a_2\} \in \mathbf{K}_2$ as in [5, Section 0.N.5]. Furthermore, since we have assumed $\eta = \mathbb{1}$, for $\alpha \in \Delta$ one has

$$\bar{h}_\alpha(a) = \mathbf{s}(h_\alpha(a)) \in \bar{\mathbf{T}}. \tag{2.2}$$

Writing \bar{w}_α for $\bar{w}_\alpha(1)$ for every $\alpha \in \Phi$, it then follows from [5, Proposition 11.3] that

$$\bar{w}_\alpha \cdot \mathbf{s}(y(a)) \cdot \bar{w}_\alpha^{-1} = \mathbf{s}(y(a)) \cdot \bar{h}_\alpha(a^{-\langle \alpha, y \rangle}) \tag{2.3}$$

for all $y \in Y$ and $a \in \mathbf{G}_m$.

Assume that F^\times contains the full group μ_n of n -th roots of unity. By push-out via the Hilbert symbol $(-, -)_n : \mathbf{K}_2(F) \rightarrow \mu_n$, one obtains from $\bar{\mathbf{G}}(F)$ a topological central extension

$$\mu_n \hookrightarrow \bar{G} \xrightarrow{\wp} G.$$

For any subset $H \subset G$, we may write $\bar{H} := \wp^{-1}(H)$. Fixing an embedding $\epsilon : \mu_n \hookrightarrow \mathbb{C}^\times$ we consider only ϵ -genuine representations of \bar{G} , i.e., when μ_n acts via ϵ .

All the properties in (2.1)–(2.3) specialize to corresponding relations on elements in \bar{G} . For example, from (2.1) we obtain

$$\mathbf{s}(y_1(a_1)) \cdot \mathbf{s}(y_2(a_2)) = (a_1, a_2)_n^{D(y_1, y_2)} \cdot \mathbf{s}(y_1(a_1) \cdot y_2(a_2)) \tag{2.4}$$

for any $y_j \in Y$ and $a_j \in F^\times$. For $y \in Y$ denote

$$\mathbf{s}_y := \mathbf{s}(y(\wp)).$$

The commutator on \bar{T} factors through T and defines a map

$$[-, -] : T \times T \rightarrow \mu_n.$$

We collect below some relations in \bar{G} which we use in our computations. Let $\alpha \in \Phi$, $t, t_j \in F^\times, u \in F, y, y_j \in Y$. Then

$$\mathbf{s}(y(t))\bar{e}_\alpha(u)\mathbf{s}(y(t))^{-1} = \bar{e}_\alpha(t^{(\alpha, y)}u), \tag{2.5a}$$

$$\bar{w}_\alpha(t)\bar{e}_\beta(u)\bar{w}_\alpha(-t) = \bar{e}_{w_\alpha\beta}(ct^{-(\beta, \alpha^\vee)}u), \tag{2.5b}$$

$$\mathbf{s}(y_1(t_1)) \cdot \mathbf{s}(y_2(t_2)) = (t_1, t_2)_n^{D(y_1, y_2)} \cdot \mathbf{s}(y_1(t_1) \cdot y_2(t_2)), \tag{2.5c}$$

$$\bar{w}_\alpha \cdot \mathbf{s}(y(t)) \cdot \bar{w}_\alpha^{-1} = \mathbf{s}(y(t)) \cdot \bar{h}_\alpha(t^{-(\alpha, y)}), \tag{2.5d}$$

$$[y_1(t_1), y_2(t_2)] = (t_1, t_2)_n^{B_Q(y_1, y_2)}, \tag{2.5e}$$

$$\bar{w}_{-\alpha}(t) = \bar{w}_\alpha(ct^{-1}), \tag{2.5f}$$

$$\bar{w}_\alpha(t_1)\bar{w}_\alpha(t_2) = (-t_1, -t_2^{-1})_n^{Q(\alpha^\vee)} \cdot \bar{h}_\alpha(-t_1t_2^{-1}), \tag{2.5g}$$

$$\bar{h}_{-\alpha}(t) = (t, c)_n^{Q(\alpha^\vee)} \cdot \bar{h}_\alpha(t^{-1}), \tag{2.5h}$$

where $c = c(\alpha, \beta) \in \{\pm 1\}$ is independent of t and u , and $c(\alpha, \beta) = c(\alpha, -\beta)$.

We provide some references for the above relations.

- Relations (2.5a) and (2.5b) follow from the analogous Chevalley relations in G and the canonical unipotent section of a BD extension (see Gan–Gao [16, Section 3.2]).
- Relation (2.5c) follows from [5, Section 0.N.5], and relation (2.5d) follows from [5, Proposition 11.3].
- For relation (2.5e), see Gan–Gao [16, Sections 2.6, 3.3].
- Relation (2.5f) follows from considering $\bar{w}_\alpha(-t^{-1})\bar{w}_{-\alpha}(t)\bar{w}_\alpha(t^{-1})$, expanding $\bar{w}_{-\alpha}(t)$, and then applying (2.5b).
- Relation (2.5g) follows from writing $\bar{w}_\alpha(t_j) = \bar{h}_\alpha(t_j)\bar{w}_\alpha(1)$ and applying (2.5d).
- Relation (2.5h) follows from writing $\bar{h}_{-\alpha}(t) = \bar{w}_{-\alpha}(t)\bar{w}_{-\alpha}(-1)$ and applying (2.5f).

2.4. Dual group

Consider the sublattice

$$Y_{Q,n} := Y \cap nY^*$$

of Y , where $Y^* \subset Y \otimes \mathbb{Q}$ is the lattice dual to Y with respect to B_Q . The lattice $Y_{Q,n}$ dictates the center $Z(\bar{T})$ of \bar{T} (see [54]). We write

$$\mathcal{X}_{Q,n} := Y/Y_{Q,n}$$

or just \mathcal{X} for simplicity. For $y \in Y$, we write $\hat{y} \in \mathcal{X}_{Q,n}$ for the coset of y , if there is a risk of confusion.

We also write $X_{Q,n} := \text{Hom}_{\mathbb{Z}}(Y_{Q,n}, \mathbb{Z})$. For every $\alpha \in \Phi$ we set

$$\alpha_{Q,n}^\vee = n_\alpha \alpha^\vee \quad \text{and} \quad \alpha_{Q,n} = n_\alpha^{-1} \alpha,$$

where $n_\alpha = n/\text{gcd}(n, Q(\alpha^\vee))$. This gives the modified simple roots $\Delta_{Q,n} = \{\alpha_{Q,n} :$

$\alpha \in \Delta$ and coroots $\Delta_{Q,n}^\vee$; similarly for $\Phi_{Q,n}$ and $\Phi_{Q,n}^\vee$. Let $Y_{Q,n}^{\text{sc}}$ denote the coroot lattice spanned by $\Delta_{Q,n}^{\text{sc}}$. One has $Y_{Q,n}^{\text{sc}} \subset Y_{Q,n} \subset Y$.

The tuple

$$(Y_{Q,n}, \Phi_{Q,n}^\vee; X_{Q,n}, \Phi_{Q,n})$$

forms a root datum. Let $\bar{\mathbf{G}}_{Q,n}^\vee$ be the associated reductive group over \mathbb{Z} with character lattice $Y_{Q,n}$. Write $\bar{G}_{Q,n}^\vee$ or simply \bar{G}^\vee for its complex group, which is called the *dual group* of \bar{G} .

2.5. Tame covers and splittings

The commutator of \bar{T} induces a bi-multiplicative map $[-, -] : T \times T \rightarrow \mu_n$, given explicitly on generators in (2.5e). Throughout this paper, we assume that \bar{G} is a *tame cover*, meaning $p \nmid n$. Consider the map

$$\varphi : T \rightarrow \text{Hom}(T_\kappa, \mathbb{C}^\times), \quad \varphi(t)(s) := [t, s'],$$

where $s' \in \mathbf{T}(O_F)$ is any lifting of $s \in T_\kappa$ with respect to the reduction map red_p . Since $T_1 = I_1 \cap \mathbf{T}(O_F)$, which is the kernel of red_p restricted to $\mathbf{T}(O_F)$, lies in the center of \bar{T} , we see that the map φ is a well-defined group homomorphism.

One has $\text{Ker}(\varphi) = Z(\bar{T}) \cdot \mathbf{T}(O_F)$, since $p \nmid n$; as $\mathcal{X}_{Q,n} \simeq \bar{T}/Z(\bar{T})\mathbf{T}(O_F)$, this gives a well-defined injective homomorphism

$$\varphi : \mathcal{X}_{Q,n} \hookrightarrow \text{Hom}(T_\kappa, \mathbb{C}^\times), \quad \varphi(\hat{y})(s) = [y(\varpi), s']. \tag{2.6}$$

The covering group \bar{G} *splits over a subgroup* $H \subset G$ if there is a group homomorphism $s : H \rightarrow \bar{G}$ such that $\wp \circ s = \text{id}_H$. If \bar{G} splits over a subgroup $H \subset G$, then the set of all such splittings is a torsor over $\text{Hom}(H, \mu_n)$. For tame covers, the group \bar{G} splits over $K = \mathbf{G}(O_F)$. In fact, since $\eta = \mathbf{1}$ by our assumption, the \mathbf{K}_2 -extension $\bar{\mathbf{G}}$ over F arises from a \mathbf{K}_2 -extension of \mathbf{G} over O_F (see [56, Theorem 4.3]), which also entails the splitting of K . Note that for nontame covers the groups K and I may not split. For examples of double covers of G over \mathbb{Q}_2 , see [30].

Throughout, we fix a splitting

$$s_K : K \hookrightarrow \bar{G}.$$

To simplify notation, we may write K instead of $s_K(K)$. The splitting s_K gives rise to splittings of I and I_1 by restriction. Since I_1 is a pro- p group and $p \nmid n$, it follows that $\text{Hom}(I_1, \mu_n) = \{1\}$. Thus I_1 has a unique splitting afforded by $s_K|_{I_1}$. In contrast, for I we have

$$\text{Hom}(I, \mu_n) \simeq \text{Hom}(I/I_1, \mu_n) \simeq \text{Hom}(T_\kappa, \mu_n) \simeq (\mu_n)^r,$$

where r is the rank of the \mathbb{Z} -lattice Y . The last isomorphism follows because \mathbf{T} is split.

We remark that the above contrast between the uniqueness of the splitting for I_1 and the nonuniqueness for I is partly responsible for the simpler descriptions and proofs of our results at the pro- p Iwahori–Hecke algebra level than at the usual Iwahori–Hecke level.

3. Pro- p Iwahori–Hecke algebra \mathcal{H}

In this section we establish some relations, following Vigneras [53], for the integral ϵ -genuine pro- p Hecke algebra

$$\mathcal{H}_{\mathbb{Z}} := C_{c,\epsilon}^{\infty}(I_1 \backslash \bar{G} / I_1, \mathbb{Z}[\mu_n]),$$

which consists of locally constant and compactly supported functions $f : \bar{G} \rightarrow \mathbb{Z}[\mu_n]$ such that

$$f(\gamma_1 g \gamma_2 \zeta) = \epsilon(\zeta) f(g) \quad \text{for all } g \in \bar{G}, \gamma_j \in I_1, \zeta \in \mu_n.$$

Multiplication is given by convolution, i.e., for $f_1, f_2 \in \mathcal{H}_{\mathbb{Z}}$ one has

$$f_1 * f_2(g) = \int_{\bar{G}} f_1(h) f_2(h^{-1}g) dh = \int_{\bar{G}} f_1(gh) f_2(h^{-1}) dh,$$

where dh is the Haar measure on the unimodular group \bar{G} such that $\int_{I_1} dh = 1$. We define

$$\mathcal{H} := \mathcal{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mu_n]} \mathbb{C} = C_{c,\epsilon}^{\infty}(I_1 \backslash \bar{G} / I_1, \mathbb{C}).$$

3.1. Several groups

We begin by describing a $\mathbb{Z}[\mu_n]$ -basis for $\mathcal{H}_{\mathbb{Z}}$. This basis is in bijection with the group

$$W(1) = N(T) / T_1 \simeq \overline{N(T)} / \mu_n T_1.$$

Recall that for the extended affine Weyl group one has

$$W_{\text{ex}} \simeq N(T) / \mathbf{T}(O_F),$$

where the affine reflection $w_{\alpha,k}$ corresponds to the class of $w_{\alpha}(\varpi^{-k}) \in N(T)$. This gives the exact sequence

$$T_k \hookrightarrow W(1) \xrightarrow{f} W_{\text{ex}}.$$

For more details, see [53, p. 696]. We have

$$\Omega \simeq (N_G(I) \cap N(T)) / \mathbf{T}(O_F) \simeq N_G(I) / I.$$

Since $W_{\text{aff}}, \Omega \subset W_{\text{ex}}$, we define

$$\Omega(1) = f^{-1}(\Omega), \quad W_{\text{aff}}(1) = f^{-1}(W_{\text{aff}}).$$

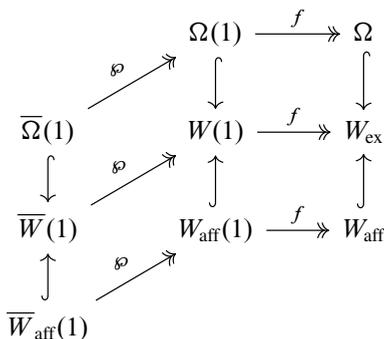
Consider $\overline{W}(1) := \overline{N(T)} / T_1$ with the natural quotient

$$\wp : \overline{W}(1) \twoheadrightarrow W(1).$$

From this we define

$$\overline{\Omega}(1) := \wp^{-1}(\Omega(1)) = (f \circ \wp)^{-1}(\Omega), \quad \overline{W}_{\text{aff}}(1) := \wp^{-1}(W_{\text{aff}}(1)) = (f \circ \wp)^{-1}(W_{\text{aff}}).$$

The various groups above are illustrated in the following diagram:



where $\text{Ker}(f) = T_\kappa$ and $\text{Ker}(\wp) = \mu_n$. Since $W_{\text{ex}} = \Omega \times W_{\text{aff}}$, one has

$$W(1) \simeq \Omega(1) \times_{T_\kappa} W_{\text{aff}}(1) := \frac{\Omega(1) \times W_{\text{aff}}(1)}{\nabla(T_\kappa)},$$

where $\nabla(t) = (t, t^{-1})$ is the anti-diagonal embedding. Similarly, there is an isomorphism

$$\overline{W}(1) \simeq \overline{\Omega}(1) \times_{\mu_n T_\kappa} \overline{W}_{\text{aff}}(1).$$

Note that the group $\overline{W}_{\text{aff}}(1)$ is generated by the set

$$(\mu_n \mathbf{T}(O_F)/T_1) \cup \{\overline{w}_\alpha(1)T_1 : \alpha \in \Delta\} \cup \{\overline{w}_{\alpha_j}(\varpi^{-1})T_1 : 1 \leq j \leq d\}.$$

Lemma 3.1. *The natural map $W(1) \rightarrow \mu_n I_1 \backslash \overline{G}/I_1$ is a bijection; similarly, one has a natural bijection $W_{\text{ex}} \rightarrow \mu_n I \backslash \overline{G}/I$.*

Proof. This follows immediately from [53, Proposition 3.35]. ■

For $g \in \overline{G}$ let $\mathcal{T}_g \in \mathcal{H}_{\mathbb{Z}}$ be the unique element such that $\text{supp}(\mathcal{T}_g) = \mu_n I_1 g I_1$ and $\mathcal{T}_g(g) = 1$. Note that \mathcal{T}_g is well-defined. To prove this it suffices to show that if

$$g = \zeta \cdot s_K(\gamma_1) g s_K(\gamma_2)$$

with $\zeta \in \mu_n$ and $\gamma_i \in I_1$, then $\zeta = 1$. Since any open subgroup of I_1 is a pro- p group and thus has a unique splitting into \overline{G} , it follows that the two splittings

$$s_K, {}^g s_K : I_1 \cap (g I_1 g^{-1}) \rightarrow \overline{G}$$

are equal. Therefore $\zeta = 1$ and thus \mathcal{T}_g is well-defined. By Lemma 3.1, any section σ of the map $\overline{N}(T) \twoheadrightarrow W(1)$ yields a $\mathbb{Z}[\mu_n]$ -basis $\{\mathcal{T}_{\sigma(w)} : w \in W(1)\}$ of $\mathcal{H}_{\mathbb{Z}}$.

3.2. Relations in \mathcal{H}

In this subsection we prove some basic relations in $\mathcal{H}_{\mathbb{Z}}$. For any $g \in \overline{G}$ we set

$$\mathfrak{q}_g = [I_1 \wp(g) I_1 : I_1].$$

By definition q_g is constant on the double cosets of $\mu_n I_1$ inside \bar{G} . By reduction to the linear case as in [47, Proposition 6.2] and arguing as in [53, Section 4.1], we establish the following:

Lemma 3.2 (Braid relations). *Let $g, g' \in \bar{G}$. If $q_g \cdot q_{g'} = q_{gg'}$, then $\mathcal{T}_g * \mathcal{T}_{g'} = \mathcal{T}_{gg'}$.*

Setting

$$\bar{N}(I_1) := \overline{N_G(I_1)},$$

a direct calculation and comparison of measures gives

$$\bar{N}(I_1) = \{g \in \bar{G} : q_g = 1\}.$$

It also follows that $\bar{\mathbf{T}}(O_F) \subset \bar{N}(I_1)$.

Let $\mathcal{H}_{\mathbb{Z},0} \subset \mathcal{H}_{\mathbb{Z}}$ be the subalgebra of functions supported on \bar{K} . The splitting s_K relates $\mathcal{H}_{\mathbb{Z},0}$ to functions on the linear group K . The next lemma follows from a direct calculation.

Lemma 3.3. *The map $s_K^{\mathcal{H}} : \mathcal{H}_{\mathbb{Z},0} \rightarrow C^\infty(I_1 \backslash K / I_1, \mathbb{Z}[\mu_n])$ defined by $f \mapsto f \circ s_K$ is an isomorphism of algebras.*

Now we prove the quadratic relations. Suppose that $\Delta = \{\alpha_1, \dots, \alpha_r\}$ is the set of simple roots and $\{\alpha_j^\dagger : 1 \leq j \leq d\}$ are the highest roots of the irreducible root systems Φ_j . Let

$$\Delta_{\text{aff}} = \Delta \cup \{-\alpha_j^\dagger : 1 \leq j \leq d\}.$$

For $\alpha \in \Phi$ and $\chi \in \text{Hom}(\kappa^\times, \mu_n)$ we define $c_\alpha(\chi) \in \mathcal{H}$ to be such that

$$(q - 1) \cdot c_\alpha(\chi) = \sum_{u \in \kappa^\times} \chi(u) \mathcal{T}_{\bar{h}_\alpha(u)} \in \mathcal{H}_{\mathbb{Z}}.$$

For every $\alpha \in \Phi$, we write $\mathcal{T}_\alpha := \mathcal{T}_{\bar{w}_\alpha(1)}$. We also write $\mathcal{T}_{\alpha^\dagger} := \mathcal{T}_{\bar{w}_{\alpha^\dagger}(\varpi^{-1})}$ for $\alpha^\dagger \in \{\alpha_1^\dagger, \dots, \alpha_d^\dagger\}$. Let $\varepsilon := (\varpi, \varpi)_n = (-1, \varpi)_n \in \{\pm 1\}$.

Proposition 3.4 (Quadratic relations). *Keep notation as above.*

(i) *For every $\alpha \in \Delta$ one has*

$$\mathcal{T}_\alpha^2 = q \mathcal{T}_{\bar{h}_\alpha(-1)} + (q - 1) c_\alpha(1) \mathcal{T}_\alpha. \tag{3.1}$$

(ii) *For every $\alpha^\dagger \in \{\alpha_1^\dagger, \dots, \alpha_d^\dagger\}$ one has*

$$\mathcal{T}_{\alpha^\dagger}^2 = q \varepsilon \mathcal{Q}(\alpha^\dagger, \vee) \mathcal{T}_{\bar{h}_{\alpha^\dagger}(-1)} + (q - 1) c_{\alpha^\dagger}((-, \varpi)_n^{\mathcal{Q}(\alpha^\dagger, \vee)}) \mathcal{T}_{\alpha^\dagger}. \tag{3.2}$$

Proof. Both assertions can be verified directly. We give the details for (3.2); for (3.1) we can reduce to the linear case by Lemma 3.3 and then apply [53, Theorem 2.2].

To begin we bound $\text{supp}(\mathcal{T}_{\alpha^\dagger}^2)$. The computation reduces to the linear case, which gives

$$\text{supp}(\mathcal{T}_{\alpha^\dagger}^2) \subseteq \mu_n \bar{h}_{\alpha^\dagger}(-1) I_1 \cup \bigcup_{u \in O_F^\times / (1+\mathfrak{p})} \mu_n I_1 \bar{h}_{\alpha^\dagger}(u) \bar{w}_{\alpha^\dagger}(\varpi^{-1}) I_1.$$

Now we evaluate $\mathcal{T}_{\alpha^\dagger}^2$ at the elements $\bar{h}_{\alpha^\dagger}(-1)$ and $\bar{w}_{\alpha^\dagger}(\varpi^{-1})\bar{h}_{\alpha^\dagger}(v)$ where $v \in O_F^\times/(1 + \mathfrak{p})$.

A preliminary calculation gives

$$\begin{aligned} \mathcal{T}_{\alpha^\dagger}^2(g) &= \int_{\bar{G}} \mathcal{T}_{\alpha^\dagger}(h) \cdot \mathcal{T}_{\alpha^\dagger}(h^{-1}g) dh \\ &= \sum_{\gamma \in \mu_n I_1 / \mu_n I_1 \cap \bar{w}_{\alpha^\dagger}(\varpi^{-1}) I_1 \bar{w}_{\alpha^\dagger}(\varpi^{-1})^{-1}} \mathcal{T}_{\alpha^\dagger}(\gamma \bar{w}_{\alpha^\dagger}(\varpi^{-1})) \mathcal{T}_{\alpha^\dagger}(\bar{w}_{\alpha^\dagger}(\varpi^{-1})^{-1} \gamma^{-1} g) \\ &= \sum_{\gamma \in I_1 / I_1 \cap w_{\alpha^\dagger}(\varpi^{-1}) I_1 w_{\alpha^\dagger}(\varpi^{-1})^{-1}} \mathcal{T}_{\alpha^\dagger}(\gamma \bar{w}_{\alpha^\dagger}(\varpi^{-1})) \mathcal{T}_{\alpha^\dagger}(\bar{w}_{\alpha^\dagger}(\varpi^{-1})^{-1} \gamma^{-1} g) \\ &= \mathcal{T}_{\alpha^\dagger}(\bar{w}_{\alpha^\dagger}(\varpi^{-1})^{-1} g) + \sum_{u \in O_F^\times/(1+\mathfrak{p})} \mathcal{T}_{\alpha^\dagger}(\bar{w}_{\alpha^\dagger}(\varpi^{-1})^{-1} \bar{e}_{-\alpha^\dagger}(\varpi u) g), \end{aligned} \tag{3.3}$$

where the last equality follows from $I_1 / I_1 \cap \bar{w}_{\alpha^\dagger}(\varpi^{-1}) I_1 \bar{w}_{\alpha^\dagger}(\varpi^{-1})^{-1} = U_{-\alpha^\dagger}^1 / U_{-\alpha^\dagger}^2$.

It follows from (2.5c) and (2.5d) that

$$\bar{w}_{\alpha^\dagger}(\varpi^{-1})^{-1} \bar{h}_{\alpha^\dagger}(-1) = \varepsilon^{Q(\alpha^\dagger, \vee)} \cdot \bar{w}_{\alpha^\dagger}(\varpi^{-1})$$

and $\bar{e}_{-\alpha^\dagger}(\varpi u) \in I_1$ for all $u \in O_F^\times$. Thus from (3.3) we have

$$\mathcal{T}_{\alpha^\dagger}^2(\bar{h}_{\alpha^\dagger}(-1)) = q \varepsilon^{Q(\alpha^\dagger, \vee)}.$$

Next, we consider the element $\bar{w}_{\alpha^\dagger}(\varpi^{-1})\bar{h}_{\alpha^\dagger}(v)$, where $v \in O_F^\times/(1 + \mathfrak{p})$. For $u \in O_F^\times$, the left and right $U_{-\alpha^\dagger}^1$ -invariance of $\mathcal{T}_{\alpha^\dagger}$ and equations (2.5b)–(2.5d) imply

$$\begin{aligned} \mathcal{T}_{\alpha^\dagger}(\bar{w}_{\alpha^\dagger}(\varpi^{-1})^{-1} \bar{e}_{-\alpha^\dagger}(\varpi u) \bar{w}_{\alpha^\dagger}(\varpi^{-1}) \bar{h}_{\alpha^\dagger}(v)) \\ = (-u, \varpi)_n^{Q(\alpha^\dagger, \vee)} \cdot \mathcal{T}_{\alpha^\dagger}(\bar{w}_{\alpha^\dagger}(\varpi^{-1}) \bar{h}_{\alpha^\dagger}(-u^{-1}v)). \end{aligned}$$

Thus by (3.3) we obtain

$$\mathcal{T}_{\alpha^\dagger}^2(\bar{w}_{\alpha^\dagger}(\varpi^{-1}) \bar{h}_{\alpha^\dagger}(v)) = (v, \varpi)_n^{Q(\alpha^\dagger, \vee)},$$

which completes the proof. ■

The braid relations and the quadratic relations imply the following.

Corollary 3.5. *For every $g \in \bar{G}$, the element $\mathcal{T}_g \in \mathcal{H}$ is invertible.*

4. Universal pro- p principal series representations and \mathcal{H}

In this section we prove that $\mathcal{H} = C_{c, \varepsilon}^\infty(I_1 \backslash \bar{G} / I_1, \mathbb{C})$ admits a Bernstein presentation. Our proof is based on the universal principal series, following Haines–Kottwitz–Prasad [26].

4.1. A first reduction

Consider the space

$$\mathcal{M} := C_{c,\epsilon}^\infty(T_1U\backslash\bar{G}/I_1, \mathbb{C})$$

consisting of locally constant functions $f : \bar{G} \rightarrow \mathbb{C}$ such that

$$f(\zeta t u g \gamma) = \epsilon(\zeta) f(g) \quad \text{for } \zeta \in \mu_n, t \in T_1, u \in U, \gamma \in I_1$$

and is also compactly supported on $U\backslash\bar{G}$. Let

$$\mathcal{R}' := \mathbb{C}[\bar{T}/T_1].$$

Here \mathcal{M} is an $(\mathcal{R}', \mathcal{H})$ -bimodule, where the right action of \mathcal{H} is given by convolution of functions. More specifically, given $F \in \mathcal{M}$ and $f \in \mathcal{H}$ we define

$$F * f(g) := \int_{\bar{G}} F(h) f(h^{-1}g) dh = \int_{\bar{G}} F(gh) f(h^{-1}) dh,$$

where dh is the Haar measure of \bar{G} normalized so that the measure of I_1 is 1. There is also a natural left action of \bar{T} on \mathcal{M} given by (for $t \in \bar{T}$ and $F \in \mathcal{M}$)

$$(t \cdot F)(g) := \delta_{\bar{B}}^{1/2}(t) F(t^{-1}g),$$

where $\delta_{\bar{B}}$ is the modular character of \bar{B} . Since T_1 is central in \bar{T} , a consequence of $\gcd(p, n) = 1$, this left action descends to \bar{T}/T_1 and can be extended \mathbb{C} -linearly to an action of \mathcal{R}' . Since the functions in \mathcal{M} are ϵ -genuine, it suffices to work with

$$\mathcal{R} := \mathcal{R}' e_\epsilon,$$

where $e_\epsilon = \sum_{\zeta \in \mu_n} \epsilon^{-1}(\zeta) \zeta \in \mathbb{C}[\mu_n]$ is the idempotent associated to ϵ . Note that \mathcal{R} has a \mathbb{C} -basis indexed by T/T_1 .

Consider the natural map $N(T) \rightarrow T_1U\backslash G/I_1$. In view of the Iwasawa decomposition $G = UTK$ and the Bruhat decomposition for K with respect to I_1 , it induces a bijection

$$W(1) = T_1\backslash N(T) \rightarrow T_1U\backslash G/I_1.$$

For any $g \in \bar{G}$, define $v_g \in \mathcal{M}$ to be the unique function such that $\text{supp}(v_g) = \mu_n T_1 U g I_1$ and $v_g(g) = 1$.

Lemma 4.1. *The function $v_g \in \mathcal{M}$ is well-defined.*

Proof. For $g = k \in s_K(K)$ this follows from a direct computation. Now for $k \in s_K(K)$ and $t \in \bar{T}$ we consider

$$v'_{tk} := t \cdot v_k.$$

The function v'_{tk} has support $\mu_n T_1 U t k I_1$ and $v'_{tk}(tk) \neq 0$. Since the set $\bar{T} s_K(K)$ contains a set of representatives for $\mu_n T_1 U \backslash \bar{G} / I_1$, the functions v_g for any $g \in \bar{G}$ are well-defined, being nonzero scalar multiples of v'_{tk} for some $t \in \bar{T}$ and $k \in s_K(K)$. ■

Fix a section σ of the map $\overline{N(T)} \rightarrow W(1)$ that sends the identity to the identity. For each $w \in W(1)$ define

$$\nu_w = \nu_{\sigma(w)} \in \mathcal{M}.$$

We see that the set $\{\nu_w : w \in W(1)\}$ constitutes a \mathbb{C} -basis for \mathcal{M} .

Lemma 4.2. *The map*

$$\mathcal{H} \rightarrow \mathcal{M}, \quad h \mapsto \nu_1 * h,$$

is an isomorphism of \mathcal{H} -modules. Furthermore, this induces an isomorphism of rings

$$\mathcal{H} \rightarrow \text{End}_{\mathcal{H}}(\mathcal{M}), \quad h \mapsto (\nu_1 * h' \mapsto \nu_1 * h * h').$$

Proof. Given a $\overline{G} \times \overline{G}$ -module A , we write ${}_U A$ for the space of left U -coinvariants and ${}^{T_1} A$ for the space of left T_1 -invariants.

The assertions follow from Haines–Kottwitz–Prasad [26, Lemma 1.6.1], since the Bruhat ordering on $W(1)$ is inflated from the underlying affine Weyl group (see [53, p. 697]).

Nevertheless, we give another proof suggested by Gordan Savin based on Bushnell–Kutzko theory. Consider $C_{c,\epsilon}^\infty(\overline{G})$ as a $\overline{G} \times \overline{G}$ -module with left and right translation. By Lemma 3.5 the elements $\mathcal{T}_t, t \in \overline{T}$, are invertible. Thus by Bushnell–Kutzko theory [8], we have an isomorphism of \overline{G} -modules

$$C_{c,\epsilon}^\infty(I_1 \backslash \overline{G}) \simeq {}^{T_1}_U C_{c,\epsilon}^\infty(\overline{G}).$$

(We note that this goes back to a result of Borel and Casselman [3, Lemma 4.7].) Integrating on the left over U gives ${}^{T_1}_U C_{c,\epsilon}^\infty(\overline{G}) \simeq C_{c,\epsilon}^\infty(UT_1 \backslash \overline{G})$. Thus we have

$$C_{c,\epsilon}^\infty(I_1 \backslash \overline{G}) \simeq C_{c,\epsilon}^\infty(UT_1 \backslash \overline{G})$$

as $\overline{T} \times \overline{G}$ -modules. Taking right I_1 -invariants, we get $\mathcal{H} \simeq \mathcal{M}$ as right \mathcal{H} -modules. Moreover, integration on the left over U sends the identity element of \mathcal{H} to $\nu_1 \in \mathcal{M}$. This completes the proof. ■

Next we record three useful identities. An element $t \in \overline{T}$ is called *dominant* if for any $l \in \mathbb{Z}$ we have

$$tU_\alpha^l t^{-1} \subseteq U_\alpha^l \quad \text{for all } \alpha \in \Phi_+.$$

Lemma 4.3. *We have the following:*

- (i) $\nu_1 * \mathcal{T}_w = \nu_w$ for any $w \in K \cap N(T)$;
- (ii) $\nu_t * \mathcal{T}_w = \nu_{tw}$ for any $t \in \overline{T}/T_1$ and $w \in K \cap N(T)$;
- (iii) $\nu_1 * \mathcal{T}_t = \nu_t$ for any $t \in \overline{T}/T_1$ dominant.

Proof. We begin with two general calculations. First, for functions f_j on the group \overline{G} , we have

$$\text{supp}(f_1 * f_2) \subseteq \text{supp}(f_1) \cdot \text{supp}(f_2). \tag{4.1}$$

Second, for $\mathcal{T}_g \in \mathcal{H}$ and $\nu \in \mathcal{M}$ we have

$$\begin{aligned} \nu * \mathcal{T}_g(g') &= \int_{\bar{G}} \nu(g'h^{-1})\mathcal{T}_g(h) dh \\ &= \int_{\mu_n I_1 g I_1} \nu(g'h^{-1})\mathcal{T}_g(h) dh \\ &= \sum_{\mu_n I_1 \setminus \mu_n I_1 g I_1} \nu(g'h^{-1})\mathcal{T}_g(h) \\ &= \sum_{h \in (I_1 \cap g^{-1} I_1 g) \setminus I_1} \nu(g'h^{-1}g^{-1})\mathcal{T}_g(gh). \end{aligned} \tag{4.2}$$

Now we prove (i). By (4.1) we have $\text{supp}(\nu_1 * \mathcal{T}_w) \subseteq \mu_n U I_1 w I_1$. We use the Iwahori factorization and $w^{-1}(I_1 \cap U^-)w \subseteq I_1$ to see that $\mu_n U I_1 w I_1 = \mu_n U w I_1$. Thus $\nu_1 * \mathcal{T}_w = c \cdot \nu_w$ for some $c \in \mathbb{C}$. Using (4.2) we compute $\nu_1 * \mathcal{T}_w(w) = 1$, thus $\nu_1 * \mathcal{T}_w = \nu_w$. Specifically,

$$\nu_1 * \mathcal{T}_w(w) = \sum_{(I_1 \cap w^{-1} I_1 w) \setminus I_1} \nu_1(wh^{-1}w^{-1})\mathcal{T}_w(wh) = 1.$$

The last equality follows because $(I_1 \cap w^{-1} I_1 w) \setminus I_1 \cong \prod_{\alpha \in \Phi_+ \cap w^{-1} \Phi_-} U_\alpha^1 \setminus U_\alpha^0$ and the factorization of $U I_1$ into $U(T \cap I_1)(U^{\text{op}} \cap I_1)$ is unique.

Item (ii) follows from item (i) and $\nu_t = \delta_B^{-1}(t)(t \cdot \nu_1)$. For (iii), by (4.1) we have

$$\text{supp}(\nu_1 * \mathcal{T}_t) \subseteq \mu_n T_1 U I_1 t I_1 = \mu_n T_1 U U^-(\mathfrak{p}) t I_1 \subseteq \mu_n T_1 U t I_1,$$

where the last containment follows because t is dominant. Thus it suffices to show that $\nu_1 * \mathcal{T}_t(t) = 1$. Using (4.2) we have

$$\nu_1 * \mathcal{T}_t(t) = \sum_{(I_1 \cap t^{-1} I_1 t) \setminus I_1} \nu_1(th^{-1}t^{-1})\mathcal{T}_t(th) = 1.$$

The last equality follows since $(I_1 \cap t^{-1} I_1 t) \setminus I_1 \simeq \prod_{\alpha \in \Phi_-} U_\alpha^{1+\ell_{t,\alpha}} \setminus U_\alpha^1$, where $\ell_{t,\alpha} \in \mathbb{Z}_{\geq 0}$. ■

Lemma 4.4. *The \mathbb{C} -algebra map $\mathcal{R} \rightarrow \text{End}_{\mathcal{H}}(\mathcal{M}) \simeq \mathcal{H}$ defined by $r \mapsto (m \mapsto r \cdot m)$ is an injection.*

Proof. Let $t \in \bar{T}$. Then $t \cdot \nu_1$ is a function supported on $\mu_n T_1 U t I_1$. Any two elements of \bar{T} give distinct supports unless they are in the same $\mu_n T_1$ -coset. ■

Using Lemmas 4.2 and 4.4, we can associate to any $t \in \bar{T}/T_1$ the element

$$\Theta_t \in \mathcal{H} \quad \text{defined by} \quad t \cdot \nu_1 = \nu_1 * \Theta_t.$$

Let

$$\mathcal{H}_\kappa := \mathcal{H}_{\mathbb{Z},0} \otimes_{\mathbb{Z}[\mu_n]} \mathbb{C} = C_{c,\epsilon}^\infty(I_1 \setminus \bar{K}/I_1)$$

be the subalgebra of \mathcal{H} consisting of functions with support contained in \bar{K} . For any

$\chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times) = \text{Hom}(T_\kappa, \mu_{q-1})$ we define

$$c(\chi) = \frac{1}{|T_\kappa|} \sum_{t \in T_\kappa} \chi(t) \mathcal{T}_t \in \mathcal{H}_\kappa.$$

The next lemma contains some useful identities, which follow from the braid relations in $\mathcal{H}_\mathbb{Z}$. Note that the Weyl group W acts on $\text{Hom}(T_\kappa, \mathbb{C}^\times)$ through its action on T_κ .

Lemma 4.5. (i) *As a \mathbb{C} -algebra, \mathcal{H}_κ is generated by the elements \mathcal{T}_α , where $\alpha \in \Delta$, and $c(\chi)$, where $\chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$.*

(ii) *Let $\alpha \in \Delta$ and $\chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$. Then*

$$\mathcal{T}_\alpha * c(\chi) = c({}^{w_\alpha} \chi) * \mathcal{T}_\alpha.$$

(iii) *Let $t \in \bar{T}$ and $\chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$. Then*

$$\Theta_t * c(\chi) = c(\varphi(t) \cdot \chi) * \Theta_t.$$

We end this section with a Bernstein decomposition.

Proposition 4.6. *Multiplication in \mathcal{H} induces a vector space isomorphism*

$$\mathcal{R} \otimes_{\mathcal{R} \cap \mathcal{H}_\kappa} \mathcal{H}_\kappa \rightarrow \mathcal{H}, \quad te_\epsilon \otimes h \mapsto \Theta_t * h.$$

Furthermore, the same is true with the factors reversed.

Proof. The result follows from [26, Lemma 1.7.1]. Consider the composite of the above map with the isomorphism $\mathcal{H} \rightarrow \mathcal{M}$ from Lemma 4.2:

$$\mathcal{R} \otimes_{\mathcal{R} \cap \mathcal{H}_\kappa} \mathcal{H}_\kappa \rightarrow \mathcal{H} \rightarrow \mathcal{M}.$$

For $t \in \bar{T}/T_1$ and $w \in N(T) \cap K$, this map sends $te_\epsilon \otimes \mathcal{T}_w$ to $v_1 * \Theta_t * \mathcal{T}_w$. By Lemma 4.3, $v_1 * \Theta_t * \mathcal{T}_w = C \cdot v_{tw}$, where $C \in \mathbb{C}^\times$. Thus the map is a \mathbb{C} -isomorphism. ■

4.2. Decomposing \mathcal{H}

In this subsection we describe the \mathbb{C} -algebra structure of \mathcal{H} as a product of matrix rings: see Proposition 4.13 below.

To begin, we decompose \mathcal{H} as a right \mathcal{H} -module using the idempotents $c(\chi) \in \mathbb{C}[T_\kappa]$. The decomposition $\mathbb{C}[T_\kappa] = \bigoplus_\chi c(\chi)\mathbb{C}[T_\kappa]$, where the direct sum is taken over $\chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$, immediately gives the following:

Lemma 4.7. *One has a decomposition*

$$\mathcal{H} \simeq \bigoplus_\chi c(\chi)\mathcal{H},$$

where χ is taken over elements in the set $\text{Hom}(T_\kappa, \mathbb{C}^\times)$.

We view \mathcal{H} as a left $\text{Hom}_{\mathcal{H}}(\mathcal{H}, \mathcal{H})$ -module. In particular, the map

$$\mathcal{H} \rightarrow \text{Hom}_{\mathcal{H}}(\mathcal{H}, \mathcal{H}), \quad h \mapsto (h' \mapsto h * h'),$$

gives an isomorphism of rings. For $\chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$ we set

$$\mathcal{H}_\chi := c(\chi)\mathcal{H}c(\chi).$$

Lemma 4.8. *Let $\chi, \chi' \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$. Then the map*

$$c(\chi')\mathcal{H}c(\chi) \rightarrow \text{Hom}_{\mathcal{H}}(c(\chi)\mathcal{H}, c(\chi')\mathcal{H}), \quad c(\chi')hc(\chi) \mapsto (f \mapsto c(\chi')hc(\chi) * f),$$

is an isomorphism of $(\mathcal{H}_{\chi'}, \mathcal{H}_\chi)$ -bimodules. Moreover, if $\chi = \chi'$, then the map is an isomorphism of rings.

Proof. There is an embedding $\text{Hom}_{\mathcal{H}}(c(\chi)\mathcal{H}, c(\chi')\mathcal{H}) \hookrightarrow \text{Hom}_{\mathcal{H}}(\mathcal{H}, \mathcal{H}) \simeq \mathcal{H}$. It is easy to check that the image of this embedding is $c(\chi')\mathcal{H}c(\chi)$. ■

There is an action of W on $\text{Hom}(T_\kappa, \mathbb{C}^\times)$ via the action of W on T_κ . There is also an action of $\mathcal{X}_{Q,n} = Y/Y_{Q,n}$ on $\text{Hom}(T_\kappa, \mathbb{C}^\times)$ via the embedding

$$\varphi : \mathcal{X}_{Q,n} \hookrightarrow \text{Hom}(T_\kappa, \mu_n). \tag{4.3}$$

These actions are compatible in the sense that they give rise to a well-defined action of $W \ltimes \mathcal{X}_{Q,n}$ on $\text{Hom}(T_\kappa, \mathbb{C}^\times)$.

Lemma 4.9. *The space $c(\chi')\mathcal{H}c(\chi)$ is nonzero if and only if χ and χ' are in the same $W \ltimes \mathcal{X}_{Q,n}$ -orbit.*

Proof. Let $\bar{w} \in \overline{N(T)}$ represent $w \in W \ltimes Y$. Let $t \in \mathbf{T}(O_F)$. The result follows from the identity

$$\mathcal{T}_{\bar{w}}\mathcal{T}_t = \mathcal{T}_{wtw^{-1}}\mathcal{T}_{\bar{w}},$$

which is due to the fact that $\mathbf{T}(O_F) \subset N_G(I_1)$. Specifically, $\mathcal{T}_{\bar{w}}c(\chi) = c({}^w\chi)\mathcal{T}_{\bar{w}}$. Here w is acting on χ through its image in $W \ltimes \mathcal{X}_{Q,n}$. ■

Corollary 4.10. *The module $c(\chi)\mathcal{H}$ is isomorphic to $c(\chi')\mathcal{H}$ if and only if χ and χ' are in the same $W \ltimes \mathcal{X}_{Q,n}$ -orbit.*

Proof. By Corollary 3.5, every \mathcal{T}_g is invertible. Thus, if $c(\chi')\mathcal{T}_g c(\chi) \neq 0$, then $c(\chi')\mathcal{T}_g c(\chi) = c(\chi')\mathcal{T}_g = \mathcal{T}_g c(\chi)$, and in fact $c(\chi)\mathcal{T}_g^{-1}c(\chi') = c(\chi)\mathcal{T}_g^{-1} = \mathcal{T}_g^{-1}c(\chi')$ is its inverse. ■

Given a ring S and $k \in \mathbb{Z}_{\geq 1}$, let $\mathbf{M}(k, S)$ denote the ring of $k \times k$ matrices with entries in S .

Proposition 4.11. *There is an isomorphism of rings*

$$\mathcal{H} \simeq \prod_{\mathcal{O}_\chi \subset \text{Hom}(T_\kappa, \mathbb{C}^\times)} \mathbf{M}(|\mathcal{O}_\chi|, \mathcal{H}_\chi).$$

The product is taken over the $W \ltimes \mathcal{X}_{Q,n}$ -orbits.

Proof. By Lemmas 4.7–4.9 and Corollary 4.10 we have

$$\begin{aligned} \mathcal{H} &\simeq \text{Hom}_{\mathcal{H}}(\mathcal{H}, \mathcal{H}) \\ &\simeq \text{Hom}_{\mathcal{H}}\left(\bigoplus_{\chi \in \mathcal{O}} c(\chi)\mathcal{H}, \bigoplus_{\chi' \in \mathcal{O}} c(\chi')\mathcal{H}\right) \\ &\simeq \prod_{\mathcal{O}_x} \text{Hom}_{\mathcal{H}}(c(\chi)\mathcal{H}^{|\mathcal{O}_x|}, c(\chi)\mathcal{H}^{|\mathcal{O}_x|}), \end{aligned}$$

where the product is taken over the $W \ltimes \mathcal{X}_{Q,n}$ -orbits in $\text{Hom}(T_\kappa, \mathbb{C}^\times)$. Finally, since

$$\text{Hom}_{\mathcal{H}}(c(\chi)\mathcal{H}^{|\mathcal{O}_x|}, c(\chi)\mathcal{H}^{|\mathcal{O}_x|}) \simeq \mathbf{M}(|\mathcal{O}|, \text{Hom}_{\mathcal{H}}(c(\chi)\mathcal{H}, c(\chi)\mathcal{H})),$$

where $\text{Hom}_{\mathcal{H}}(c(\chi)\mathcal{H}, c(\chi)\mathcal{H}) \simeq \mathcal{H}_\chi$ by Lemma 4.8, the result follows. ■

4.3. The case of covering tori

The results of the preceding subsection can be refined in the case $G = T$.

Lemma 4.12. *Let $\chi, \chi' \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$.*

- (i) *The center of \mathcal{R} denoted $Z(\mathcal{R})$ is equal to $\mathbb{C}[Z(\bar{T})/T_1]e_\epsilon$.*
- (ii) *Suppose that $y_0 \in Y$ satisfies $\chi = \varphi(y_0) \cdot \chi'$. Then*

$$c(\chi)\mathcal{R}c(\chi') = Z(\mathcal{R})c(\chi)_{\Theta_{s_{y_0}}}c(\chi').$$

In particular, if $\chi = \chi'$, then $c(\chi)\mathcal{R}c(\chi) = Z(\mathcal{R})c(\chi)$. Hence, $c(\chi)\mathcal{R}c(\chi) \simeq Z(\mathcal{R})c(\chi)$ as rings.

Proof. For (i), let $\{f\}$ be a \mathbb{C} -basis for $\mathbb{C}[\bar{T}/T_1]e_\epsilon$ given by choosing a set of representatives for the coset space $\bar{T}/\mu_n T_1$. Suppose that $\sum c_f f$ lies in the center of $\mathbb{C}[\bar{T}/T_1]e_\epsilon$. Then for any $t \in \bar{T}/T_1$ we have

$$t\left(\sum c_f f\right) = \left(\sum c_f f\right)t.$$

On the other hand, we have $t(\sum c_f f) = (\sum c_f [t, f]f)t$. Thus for any $t \in \bar{T}/T_1$ one has

$$[t, f] = 1 \quad \text{for any } f \text{ with } c_f \neq 0.$$

Thus the center of $\mathbb{C}[\bar{T}/T_1]e_\epsilon$ is equal to $\mathbb{C}[Z(\bar{T})/T_1]e_\epsilon$.

Now we prove (ii). One has

$$c(\chi)_{\Theta_{s_y}}c(\chi') \neq 0 \quad \text{if and only if } y - y_0 \in Y_{Q,n}.$$

Thus the set

$$\{c(\chi)_{\Theta_{s_{y_0}}}\Theta_{s_y}c(\chi') : y \in Y_{Q,n}\}$$

is a \mathbb{C} -basis for $c(\chi)\mathcal{R}c(\chi')$. Since $Z(\bar{T})\mathbf{T}(O_F)/\mu_n\mathbf{T}(O_F) \simeq Y_{Q,n}$, the elements in

$$\{\Theta_{s_y}c(\chi) : y \in Y_{Q,n}\}$$

form a \mathbb{C} -basis for $Z(\mathcal{R})c(\chi)$ and the result follows. ■

Using Lemma 4.12 we refine Proposition 4.11 as follows.

Proposition 4.13. *One has an isomorphism of rings*

$$\mathcal{R} \simeq \prod_{\theta_\chi} \mathbf{M}(|\mathcal{X}_{Q,n}|, Z(\mathcal{R})c(\chi)),$$

where the product is taken over the cosets in $\text{Hom}(T_\kappa, \mathbb{C}^\times)/\text{Im}(\varphi)$.

With the choice of a genuine character of $Z(\bar{T})$ we can describe $Z(\mathcal{R})c(\chi)$ as a ring.

Lemma 4.14. *Let $\sigma \in \text{Irr}_{\text{gen}}(Z(\bar{T}))$. One has an injection of \mathbb{C} -algebras*

$$\mathbb{C}[Y_{Q,n}] \hookrightarrow Z(\mathcal{R}) \quad y \mapsto \sigma(\mathfrak{s}_y)\Theta_{\mathfrak{s}_y}.$$

Furthermore, for $\chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$ the composition of this map with the projection $Z(\mathcal{R}) \twoheadrightarrow Z(\mathcal{R})c(\chi)$ defined by right multiplication by $c(\chi)$ gives an isomorphism

$$\mathbb{C}[Y_{Q,n}] \rightarrow Z(\mathcal{R})c(\chi)$$

of \mathbb{C} -algebras.

Proof. Since the set $\{\Theta_{\mathfrak{s}_y} : y \in Y_{Q,n}\}$ is \mathbb{C} -linearly independent in $Z(\mathcal{R})$, the map is an injection of \mathbb{C} -vector spaces. By relation (2.5c) and the fact that σ is genuine, we see that the map is a \mathbb{C} -algebra homomorphism. The composition of this map with the projection $Z(\mathcal{R}) \twoheadrightarrow Z(\mathcal{R})c(\chi)$ sends the \mathbb{C} -linearly independent set

$$\{\Theta_{\mathfrak{s}_y} : y \in Y_{Q,n}\} \subset Z(\mathcal{R})$$

to the \mathbb{C} -basis

$$\{\Theta_{\mathfrak{s}_y}c(\chi) : y \in Y_{Q,n}\} \subset Z(\mathcal{R})c(\chi).$$

Thus the composition is an isomorphism of \mathbb{C} -vector spaces. Again relation (2.5c) shows that the map is a \mathbb{C} -algebra homomorphism. ■

4.4. Intertwining operators

In this subsection we consider algebraic intertwining operators, following Haines–Kottwitz–Prasad. The construction of these operators mostly involves minor modifications to [26, Section 1.10]. Thus we will be brief and focus primarily on the modifications required in the context of covering groups.

To begin, we define a completion of \mathcal{M} . Since \bar{T} is in general noncommutative, it is more convenient to define our completions using $Z(\mathcal{R})$ instead of \mathcal{R} . Let J be a subset of modified coroots contained in some system of positive modified coroots, and let $\mathbb{C}[J]$ be the subalgebra of $\mathbb{C}[Y_{Q,n}]$ generated by J . We write $\mathbb{C}[J]^\wedge$ for the completion of $\mathbb{C}[J]$ with respect to the maximal ideal generated by the elements of J . We define the \mathbb{C} -algebra

$$\mathcal{R}_J = \mathbb{C}[J]^\wedge \otimes_{\mathbb{C}[J]} \mathcal{R},$$

where we view \mathcal{R} as a $\mathbb{C}[J]$ -module via the isomorphism of Lemma 4.14. Set

$$\mathcal{M}_J := \mathcal{R}_J \otimes_{\mathcal{R}} \mathcal{M} \quad \text{and} \quad \mathcal{H}_J := \mathcal{R}_J \otimes_{\mathcal{R}} \mathcal{H}.$$

The rest of [26, Section 1.10] applies without essential change.

Since our objective is to derive the Bernstein relations, we consider only the intertwining operators associated to simple reflections. For $\alpha \in \Delta$, let

$$I_\alpha = I_{s_\alpha} : \mathcal{M}_{\{-\alpha_{\mathcal{Q},n}^\vee\}} \rightarrow \mathcal{M}_{\{\alpha_{\mathcal{Q},n}^\vee\}}$$

be the intertwining operator given by

$$I_{s_\alpha}(f)(g) = \int_{U_\alpha} f(\bar{w}_\alpha(-1)ug) du.$$

The Bernstein relations follow from an analogue of Haines–Kottwitz–Prasad [26, Lemma 1.13.1 (i)]. We write $U^-(\mathfrak{p}) := U^- \cap I$.

Lemma 4.15. *Let $\alpha \in \Delta$ and $T_\alpha := \alpha^\vee(F^\times)$. Then*

$$\phi_\alpha(\mathrm{SL}_2(F)) \cap UI_1 = U_\alpha(T_\alpha \cap I_1)U_{-\alpha}^1.$$

Proof. The result follows from the Bruhat decomposition. Specifically,

$$\phi_\alpha(\mathrm{SL}_2(F)) = U_\alpha T_\alpha w_\alpha(1) \cup U_\alpha T_\alpha U_{-\alpha},$$

and $U_\alpha T_\alpha w_\alpha \cap UU^-(\mathfrak{p}) \subseteq UT w_\alpha U^- \cap UTU^- = \emptyset$. ■

Lemma 4.16. *Let $\alpha \in \Delta$ and $b \in O_F^\times$. Let $j \in \mathbb{Z}_{\geq 0}$ and $u \in F^\times$ be such that $\mathrm{val}(u) \leq 2j - 1$. The following identity of double cosets holds in \bar{G} :*

$$\begin{aligned} U \bar{w}_\alpha(-1) \bar{e}_\alpha(u) \bar{h}_\alpha(\varpi^j) \bar{h}_\alpha(b) U^-(\mathfrak{p}) \\ = (u^{-1}, \varpi^j)_n^{\mathcal{Q}(\alpha^\vee)} (u^{-1} \varpi^j, b)_n^{\mathcal{Q}(\alpha^\vee)} U \bar{h}_\alpha(u^{-1} \varpi^j b) U^-(\mathfrak{p}). \end{aligned}$$

Proof. This follows from a direct calculation by using the relations (2.5c), (2.5f), (2.5b), and (2.5d). ■

Lemma 4.17. *We have*

$$I_{s_\alpha}(v_1) = q^{-1} v_{\bar{w}_\alpha(1)} + \sum_{j \in \mathbb{Z}_{\geq 1}} q^{-j-1} \sum_{b \in \kappa^\times} (\varepsilon^j(b, \varpi^j)_n)^{\mathcal{Q}(\alpha^\vee)} v_{\bar{h}_\alpha(\varpi^j) \bar{h}_\alpha(b)}.$$

Proof. This reduces to a rank-one calculation, which we discuss in detail. First, we determine the support of the function $I_\alpha(v_1)(g)$. It suffices to consider representatives of the double cosets

$$\mu_n T_1 U \backslash \bar{G} / I_1 \cong W(1).$$

In fact, we can work in G instead of \bar{G} .

Suppose that $\bar{g} \in \bar{G}$ and $I_\alpha(v_1)(\bar{g}) \neq 0$. Let $g = \wp(\bar{g})$. We may assume that $g = wy(\varpi)t$, where $y \in Y, t \in \mathbf{T}(O_F)$ and $w \in K \cap N(T)$ represents an element of the Weyl group. We claim that w must represent 1 or w_α . Since $I_\alpha(v_1)(\bar{g}) \neq 0$, it follows that $w_\alpha(-1)Ug \cap UI_1 \neq \emptyset$ and so $w_\alpha(-1)Uw \cap UTU^- \neq \emptyset$. Thus

$$UTw_\alpha Uww_\ell U \cap UTw_\ell U \neq \emptyset.$$

Since

$$UTw_\alpha Uww_\ell U \subseteq UTw_\alpha ww_\ell U \cup UTww_\ell U,$$

the element w must represent 1 or w_α by the Bruhat decomposition. Thus we may assume that $w = 1$ or $w = w_\alpha(1)$.

Now we determine $y \in Y$ and $t \in \mathbf{T}(O_F)$. There are two cases to consider:

- If $w = w_\alpha(1)$, then

$$w_\alpha(-1)Uw_\alpha(1)y(\varpi)t \cap UI_1 \neq \emptyset$$

if and only if $U_{-\alpha}y(\varpi)t \cap UI_1 \neq \emptyset$, if and only if $y = 0$ and $t \in T_1$.

- Suppose that $w = 1$. In this case, $w_\alpha(-1)Uy(\varpi)t \cap UI_1 \neq \emptyset$, thus

$$U_{-\alpha}w_\alpha(-1)y(\varpi) \cap UK \neq \emptyset.$$

Let $t_{s_\alpha \cdot y} = y(\varpi)h_\alpha(\varpi^{-(y, \alpha)})$. Then we have

$$U_{-\alpha}w_\alpha y(\varpi) = t_{s_\alpha \cdot y} U_{-\alpha}w_\alpha,$$

and thus $U_{-\alpha} \cap Ut_{s_\alpha \cdot y}^{-1}K \neq \emptyset$. Since the elements in $U_{-\alpha}$ admit an Iwasawa decomposition in the SL_2 corresponding to α we see that $y = j\alpha^\vee$ for some $j \in \mathbb{Z}$. Furthermore, a computation in SL_2 shows that $y = j\alpha^\vee$, where $j \in \mathbb{Z}_{\geq 0}$. Similarly, we see that $t = h_\alpha(b)$, where $b \in O_F^\times$.

We have just shown that $I_\alpha(v_1)$ is determined by its values on the elements $g = \bar{w}_\alpha(1), \bar{h}_\alpha(\varpi^j)\bar{h}_\alpha(b)$, where $j \in \mathbb{Z}_{\geq 0}$ and $b \in O_F^\times$.

Now we compute $I_\alpha(v_1)(\bar{w}_\alpha(1))$. A direct computation yields

$$I_\alpha(v_1)(\bar{w}_\alpha(1)) = \int_F v_1(\bar{e}_{-\alpha}(u)) du = \text{vol}(\mathfrak{p}).$$

To compute $I_\alpha(v_1)(\bar{h}_\alpha(\varpi^j)\bar{h}_\alpha(b))$ with $j \in \mathbb{Z}_{\geq 0}$ and $b \in O_F^\times$, a calculation in SL_2 shows that if this is nonzero, then $j \geq 1$. By definition,

$$I_\alpha(v_1)(\bar{h}_\alpha(\varpi^j)\bar{h}_\alpha(b)) = \int_F v_1(\bar{w}_\alpha(-1)\bar{e}_\alpha(u)\bar{h}_\alpha(\varpi^j)\bar{h}_\alpha(b)) du.$$

We apply Lemmas 4.15 and 4.16 to get

$$\begin{aligned} \int_F v_1(\bar{w}_\alpha(-1)\bar{e}_\alpha(u)\bar{h}_\alpha(\varpi^j)\bar{h}_\alpha(b)) du &= \int_{\varpi^j b(1+\mathfrak{p})} ((u^{-1}, \varpi^j)_n (u^{-1}\varpi^j, b)_n)^{-Q(\alpha^\vee)} du \\ &= q^{-j} \text{vol}(1 + \mathfrak{p})(b\varpi^j, \varpi^j)_n^{Q(\alpha^\vee)}. \end{aligned}$$

Combining the above gives

$$I_\alpha(v_1) = q^{-1}v_{\bar{w}_\alpha(1)} + \sum_{j \in \mathbb{Z}_{\geq 1}} q^{-j-1} \sum_{b \in \kappa^\times} (b\varpi^j, \varpi^j)_n^{Q(\alpha^\vee)} v_{\bar{h}_\alpha(\varpi^j)\bar{h}_\alpha(b)}.$$

This completes the proof. ■

4.5. Bernstein presentation

For $\alpha \in \Delta$ and $j \in \mathbb{Z}$ define

$$c_\alpha(j) := \frac{1}{q-1} \sum_{b \in \kappa^\times} (b, \varpi)_n^j \Theta_{\bar{h}_\alpha(b)}.$$

The intertwining operator I_{s_α} for any $\alpha \in \Delta$ is represented by the element

$$q^{-1}\mathcal{T}_{\bar{w}_\alpha(1)} + (q-1)q^{-1} \sum_{j \in \mathbb{Z}_{\geq 1}} \varepsilon^{jQ(\alpha^\vee)} \Theta_{\bar{h}_\alpha(\varpi^j)} c_\alpha(jQ(\alpha^\vee)) \in \mathcal{H}_{\{\alpha_{Q,n}^\vee\}}.$$

Since for any $t \in \bar{T}$ the intertwining operator satisfies

$$I_{s_\alpha} \circ \Theta_t = \Theta_{w_\alpha(1) \cdot t} \circ I_{s_\alpha}, \quad \text{where } w_\alpha(1) \cdot t = w_\alpha(1)t w_\alpha(-1),$$

we get the following proposition.

Theorem 4.18 (Bernstein relations). *Let $\alpha \in \Delta$, then*

$$\begin{aligned} & \left(\mathcal{T}_{\bar{w}_\alpha(1)} + (q-1) \sum_{j \in \mathbb{Z}_{\geq 1}} \varepsilon^{jQ(\alpha^\vee)} \Theta_{\bar{h}_\alpha(\varpi^j)} c_\alpha(jQ(\alpha^\vee)) \right) * \Theta_{s_y} \\ &= \Theta_{\bar{w}_\alpha(1) \cdot s_y} * \left(\mathcal{T}_{\bar{w}_\alpha(1)} + (q-1) \sum_{j \in \mathbb{Z}_{\geq 1}} \varepsilon^{jQ(\alpha^\vee)} \Theta_{\bar{h}_\alpha(\varpi^j)} c_\alpha(jQ(\alpha^\vee)) \right) \in \mathcal{H}_{\{\alpha_{Q,n}^\vee\}}. \end{aligned}$$

Furthermore, this identity in $\mathcal{H}_{\{\alpha_{Q,n}^\vee\}}$ simplifies to an identity in \mathcal{H} as follows:

$$\begin{aligned} & \mathcal{T}_{\bar{w}_\alpha(1)} * \Theta_{s_y} + (q-1) \sum_{j \in \mathbb{Z}_{\geq 1}} \varepsilon^{jQ(\alpha^\vee)(1+\langle y, \alpha \rangle)} \Theta_{s_y \bar{h}_\alpha(\varpi^j)} c_\alpha((j + \langle y, \alpha \rangle)Q(\alpha^\vee)) \\ &= \Theta_{w_\alpha(1) \cdot s_y} * \mathcal{T}_{\bar{w}_\alpha(1)} \\ & \quad + (q-1) \sum_{j \in \mathbb{Z}_{\geq 1-\langle y, \alpha \rangle}} \varepsilon^{jQ(\alpha^\vee)(1+\langle y, \alpha \rangle)} \Theta_{s_y \bar{h}_\alpha(\varpi^j)} c_\alpha((j + \langle y, \alpha \rangle)Q(\alpha^\vee)). \end{aligned} \tag{4.4}$$

Thus, in view of Proposition 4.6 this gives a Bernstein presentation for \mathcal{H} .

For later, we record some formulas derived from Theorem 4.18. If we substitute $w_\alpha(-1) \cdot s_y$ in place of s_y and apply the identities

$$\bar{w}_\alpha(-1) \cdot s_y = \varepsilon^{B(y, \alpha^\vee)} s_y \bar{h}_\alpha(\varpi^{-\langle y, \alpha \rangle})$$

from (2.5c)–(2.5e), then formula (4.4) becomes

$$\begin{aligned} & \mathcal{T}_{\bar{w}_\alpha(1)} * \Theta_{w_\alpha(-1)\cdot s_y} \\ & + (q-1) \sum_{j \in \mathbb{Z}_{\geq 1}} \varepsilon^{jQ(\alpha^\vee)(1-\langle y, \alpha \rangle)} \Theta_{(\bar{w}_\alpha(-1)\cdot s_y)\bar{h}_\alpha(\varpi^j)} c_\alpha((j - \langle y, \alpha \rangle)Q(\alpha^\vee)) \\ & = \Theta_{s_y} * \mathcal{T}_{\bar{w}_\alpha(1)} \\ & + (q-1) \sum_{j \in \mathbb{Z}_{\geq 1+\langle y, \alpha \rangle}} \varepsilon^{jQ(\alpha^\vee)(1-\langle y, \alpha \rangle)} \Theta_{(w_\alpha(-1)\cdot s_y)\bar{h}_\alpha(\varpi^j)} c_\alpha((j - \langle y, \alpha \rangle)Q(\alpha^\vee)). \end{aligned}$$

This gives

$$\begin{aligned} & \mathcal{T}_{\bar{w}_\alpha(1)} * \Theta_{w_\alpha(-1)\cdot s_y} + (q-1) \sum_{j \in \mathbb{Z}_{\geq 1-\langle y, \alpha \rangle}} \varepsilon^{jQ(\alpha^\vee)} \Theta_{s_y \bar{h}_\alpha(\varpi^j)} c_\alpha(jQ(\alpha^\vee)) \\ & = \Theta_{s_y} * \mathcal{T}_{\bar{w}_\alpha(1)} + (q-1) \sum_{j \in \mathbb{Z}_{\geq 1}} \varepsilon^{jQ(\alpha^\vee)} \Theta_{s_y \bar{h}_\alpha(\varpi^j)} c_\alpha(jQ(\alpha^\vee)). \end{aligned} \tag{4.5}$$

Multiply (4.5) by $\mathbb{1}_I = c(\mathbb{1})$ on the right to get

$$\begin{aligned} & \mathcal{T}_{\bar{w}_\alpha(1)} * \Theta_{\bar{w}_\alpha(-1)\cdot s_y} * \mathbb{1}_I + (q-1) \sum_{\substack{j \in \mathbb{Z}_{\geq 1-\langle y, \alpha \rangle} \\ j \equiv 0 \pmod{n_\alpha}}} \varepsilon^{jQ(\alpha^\vee)} \Theta_{s_y \bar{h}_\alpha(\varpi^j)} * \mathbb{1}_I \\ & = \Theta_{s_y} * \mathcal{T}_{\bar{w}_\alpha(1)} * \mathbb{1}_I + (q-1) \sum_{\substack{j \in \mathbb{Z}_{\geq 1} \\ j \equiv 0 \pmod{n_\alpha}}} \varepsilon^{jQ(\alpha^\vee)} \Theta_{s_y \bar{h}_\alpha(\varpi^j)} * \mathbb{1}_I. \end{aligned} \tag{4.6}$$

Henceforth, we set

$$\mathcal{A} := \mathbb{1}_I * \mathcal{R} * \mathbb{1}_I.$$

Note that $j \equiv 0 \pmod{n_\alpha}$ implies $\bar{h}_\alpha(\varpi^j) \in Z(\bar{T})$ and thus

$$\Theta_{\bar{h}_\alpha(\varpi^j)} * \mathbb{1}_I \in \mathcal{A}.$$

Lemma 4.19. *One has*

$$\Theta_{s_y} * \mathcal{T}_{\bar{w}_\alpha(1)} * \mathbb{1}_I \in \mathcal{T}_{\bar{w}_\alpha(1)} * \Theta_{w_\alpha(-1)\cdot s_y} * \mathbb{1}_I + \Theta_{s_y} * \mathcal{A}.$$

Moreover, for any $\bar{w} \in K$ representing an element $w \in W$,

$$\Theta_{s_y} * \mathcal{T}_{\bar{w}} * \mathbb{1}_I \in \mathcal{T}_{\bar{w}} * \Theta_{\bar{w}^{-1}\cdot s_y} * \mathbb{1}_I + \sum_{w' < w} \mathcal{T}_{\bar{w}'} * \Theta_{(\bar{w}')^{-1}\cdot s_y} * \mathcal{A}.$$

Proof. The first claim is clear; the second follows by induction on the length of w . ■

4.6. Iwahori–Hecke algebra

Recall that

$$\mathbb{1}_I = c(\mathbb{1}) = \frac{1}{|T_\kappa|} \sum_{h \in T_\kappa} \mathcal{T}_h \in \mathcal{H}_\kappa.$$

The Iwahori–Hecke algebra for \bar{G} with respect to $(I, s_K|_I)$ is

$$\mathcal{H}_I := \mathbb{1}_I * \mathcal{H} * \mathbb{1}_I.$$

Our results for \mathcal{H} can thus be projected onto \mathcal{H}_I . For $g \in \bar{G}$ and $t \in \bar{T}$ we write

$$\mathcal{T}_g^I := \mathbb{1}_I * \mathcal{T}_g * \mathbb{1}_I, \quad \Theta_t^I := \mathbb{1}_I * \Theta_t * \mathbb{1}_I.$$

Set

$$\mathcal{H}_W := \mathbb{1}_I * \mathcal{H}_\kappa * \mathbb{1}_I \subset \mathcal{H}_\kappa.$$

Lemma 4.20. (i) *The map*

$$\mathcal{H}_W \rightarrow C^\infty(I \backslash K / I), \quad f \mapsto f|_{s(K)},$$

is a \mathbb{C} -algebra isomorphism.

(ii) *The element Θ_t^I is nonzero if and only if $t \in Z(\bar{T})$. Moreover, the choice of a genuine character $\sigma \in \text{Irr}_{\text{gen}}(Z(\bar{T}))$ induces a map*

$$\mathbb{C}[Y_{Q,n}] \rightarrow \mathcal{A}, \quad y \mapsto \sigma(\mathfrak{s}_y) \Theta_{\mathfrak{s}_y}^I,$$

which is an isomorphism of \mathbb{C} -algebras.

(iii) *The map*

$$\mathcal{H}_W \otimes_{\mathbb{C}} \mathcal{A} \rightarrow \mathcal{H}_I, \quad h \otimes a \mapsto h * a,$$

is an isomorphism of \mathbb{C} -vector spaces.

(iv) (Bernstein relation) *Let $\alpha \in \Delta$ and $y \in Y_{Q,n}$. Then*

$$\begin{aligned} \mathcal{T}_{w_\alpha(1)}^I * \Theta_{w_\alpha(-1)\mathfrak{s}_y}^I + (q-1) \sum_{\substack{j \in \mathbb{Z}_{\geq 1 - (y, \alpha)} \\ j \equiv 0 \pmod{n_\alpha}}} \varepsilon^{jQ(\alpha^\vee)} \Theta_{\mathfrak{s}_y h_\alpha(\varpi^j)}^I \\ = \Theta_{\mathfrak{s}_y}^I * \mathcal{T}_{w_\alpha(1)}^I + (q-1) \sum_{\substack{j \in \mathbb{Z}_{\geq 1} \\ j \equiv 0 \pmod{n_\alpha}}} \varepsilon^{jQ(\alpha^\vee)} \Theta_{\mathfrak{s}_y h_\alpha(\varpi^j)}^I. \end{aligned} \tag{4.7}$$

Proof. For (i) the proof is identical to that of Lemma 3.3. To prove (ii), one has

$$\Theta_t * \mathbb{1}_I = c([t, -]) * \Theta_t$$

by Lemma 4.5. Thus we see that $\mathbb{1}_I * \Theta_t * \mathbb{1}_I \neq 0$ if and only if $t \in Z(\bar{T})\mathbf{T}(O_F)$. Note that the isomorphism is the same as in Lemma 4.14 when $\chi = 1$.

The map $\mathcal{H}_W \otimes_{\mathbb{C}} \mathcal{A} \rightarrow \mathcal{H}_I$ is the restriction of the isomorphism

$$\mathcal{H}_\kappa \otimes_{\mathcal{H}_\kappa \cap \mathcal{R}} \mathcal{R} \rightarrow \mathcal{H}$$

from Proposition 4.6; thus it is injective. The map is also surjective since for any $h \in \mathcal{H}_\kappa$ we have $\mathbb{1}_I * h = h * \mathbb{1}_I$. This gives (iii).

Assertion (iv) follows by multiplying equation (4.6) on the left by $\mathbb{1}_I$. ■

5. Gelfand–Graev representation

In this section we study the Gelfand–Graev representation

$$\mathcal{V} := \text{ind}_{\mu_n U^-}^{\bar{G}}(\epsilon \otimes \psi),$$

where $\psi : U^- \rightarrow \mathbb{C}^\times$ is a nondegenerate character of conductor \mathfrak{p} and \bar{G} acts by right translation on \mathcal{V} . Our objective is to understand \mathcal{V}^I as an \mathcal{H}_I -module.

It will also be convenient to consider the space

$$\mathcal{W} := C_{c,\epsilon}^\infty((I_1^-, \psi) \backslash \bar{G}),$$

where I_1^- is the maximal pro- p subgroup of I^- , the Iwahori subgroup opposite to I , and ψ is interpreted as follows. Since ψ has conductor \mathfrak{p} , its restriction to $U^- \cap K$ factors through U_κ^- and thus can be pulled back to I_1^- under the mod- \mathfrak{p} reduction map $K \rightarrow G_\kappa$.

First we introduce bases for \mathcal{V}^{I_1} and \mathcal{W}^{I_1} . For each $w \in W$ fix a reduced word $w = w_{\alpha_{i_1}} \dots w_{\alpha_{i_k}}$ where $\alpha_{i_j} \in \Delta$. Define $\bar{w} := \bar{w}_{\alpha_{i_1}}(-1) \dots \bar{w}_{\alpha_{i_k}}(-1) \in \bar{G}$. The image of the map $W \times Y \rightarrow \bar{G}$ defined by $(w, y) \mapsto \overline{(w, y)} := \bar{w}s_{-y}$ is a complete set of distinct representatives for $\mu_n U^- \backslash \bar{G}/I$. This also gives a complete set of distinct representatives for $\mu_n I_1^- \backslash \bar{G}/I$. The next lemma determines the support of \mathcal{V}^{I_1} and \mathcal{W}^{I_1} .

- Lemma 5.1.** (i) Let $\phi \in \mathcal{V}^{I_1}$. If $\phi(\overline{(w, y)}) \neq 0$, then (w, y) is the unique element of minimal length in the coset $Wy \subset W \times Y$.
 (ii) Let $\phi \in \mathcal{W}^{I_1} = C_{c,\epsilon}^\infty((I_1^-, \psi) \backslash \bar{G}/I_1)$. If $\phi(\overline{(w, y)}) \neq 0$, then (w, y) is the unique element of minimal length in the coset $Wy \subset W \times Y$.

Proof. Let $\phi \in \mathcal{V}^{I_1}$. Let $u \in F$ and $\beta \in \Phi$. Recall the following identity:

$$\bar{w}s_{-y}\bar{e}_\beta(u) = \bar{e}_{w\cdot\beta}(\pm \varpi^{-\langle \beta, y \rangle} u) \bar{w}s_{-y}.$$

Thus if $\phi(\bar{w}s_{-y}) \neq 0$, then since ψ has conductor \mathfrak{p} , the element $(w, y) \in W \times Y$ must satisfy

$$\langle \beta, y \rangle \leq \begin{cases} -1 & \text{if } \beta \in \Phi^+ \cap w^{-1}\Delta^-, \\ 0 & \text{if } \beta \in \Phi^- \cap w^{-1}\Delta^-. \end{cases}$$

These inequalities imply that $(w, y) \cdot \mathcal{C}$ is an alcove in the Δ -dominant chamber. Thus (w, y) is the unique element of minimal length in Wy . The proof for \mathcal{W}^{I_1} is similar. ■

Now we can construct bases for \mathcal{V}^{I_1} and \mathcal{W}^{I_1} . Let $y \in Y$ and let $w_y \in W$ be such that (w_y, y) is the unique element in Wy of minimal length. We have the representative $\bar{w}_y \in \bar{G}$ of w_y . Let $\chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$. Define f_y^χ to be the unique element of \mathcal{V}^{I_1} such that

$$\text{supp}(f_y^\chi) = \mu_n U^- \bar{w}_y s_{-y} I, \quad f_y^\chi(\bar{w}_y s_{-y} t) = \chi(t) \quad \text{for all } t \in \mathbf{T}(O_F).$$

This element is well-defined because (w_y, y) is the minimal length element in Wy . Similarly, define ϕ_y^χ to be the unique element of \mathcal{W}^{I_1} such that

$$\text{supp}(\phi_y^\chi) = \mu_n I_1^- \bar{w}_y s_{-y} I, \quad \phi_y^\chi(\bar{w}_y s_{-y} t) = \chi(t) \quad \text{for all } t \in \mathbf{T}(O_F).$$

Lemma 5.2. (i) *The set $\{f_y^\chi : y \in Y, \chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)\}$ is a basis for \mathcal{V}^{I_1} .*

(ii) *The set $\{\phi_y^\chi : y \in Y, \chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)\}$ is a basis for \mathcal{W}^{I_1} .*

Proof. This is a direct consequence of Lemma 5.1. ■

Lemma 5.3. *The map $\mathcal{W}^{I_1} \rightarrow \mathcal{V}^{I_1}$ defined by $\phi \mapsto (g \mapsto \int_{U^-} \phi(ug)\psi^{-1}(u) du)$ is an \mathcal{H} -module isomorphism. Moreover, for any $y \in Y$ and $\chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$ we have $\phi_y^\chi \mapsto f_y^\chi$ in the above map. In particular, this map restricts to an isomorphism of \mathcal{W}^I with \mathcal{V}^I .*

Proof. This follows from a direct calculation. ■

In the coming sections we present two approaches to studying \mathcal{V}^I . The first is based on the Bernstein presentation of \mathcal{H} ; the second is based on the space \mathcal{W}^I and was suggested to us by the referee.

Next we turn to the finite Gelfand–Graev representation. Note that over a finite field the analogues of the spaces \mathcal{V} and \mathcal{W} are exactly equal.

5.1. Finite Gelfand–Graev representation

Since ψ has conductor \mathfrak{p} , it descends to a nondegenerate character (still denoted by)

$$\psi : U_\kappa^- \rightarrow \mathbb{C}^\times.$$

Set

$$\mathcal{V}_\kappa := \text{Ind}_{U_\kappa^-}^{G_\kappa}(\psi)^{U_\kappa},$$

the U_κ -fixed vectors in the Gelfand–Graev representation for the finite group G_κ .

By Lemma 3.3, for the finite Hecke algebra one has

$$\mathcal{H}_\kappa \simeq C_c^\infty(I_1 \backslash K / I_1) \simeq \mathbb{C}[U_\kappa \backslash G_\kappa / U_\kappa],$$

where the measure used to define convolution in $\mathbb{C}[U_\kappa \backslash G_\kappa / U_\kappa]$ is q^{-1} times the counting measure. Thus \mathcal{H}_κ acts on \mathcal{V}_κ by convolution and so does the subalgebra of functions supported on $U_\kappa T_\kappa U_\kappa = U_\kappa T_\kappa = T_\kappa U_\kappa$, which is isomorphic to $\mathbb{C}[T_\kappa]$.

Lemma 5.4. *The map $\mathcal{V}_\kappa \rightarrow \mathbb{C}[T_\kappa]$ given by $f \mapsto f|_{T_\kappa}$ is an isomorphism of $\mathbb{C}[T_\kappa]$ -modules.*

Proof. The functions in \mathcal{V}_κ have support contained in $U_\kappa^- T_\kappa U_\kappa$. By the Bruhat decomposition the map is an isomorphism of vector spaces. It is a $\mathbb{C}[T_\kappa]$ -module homomorphism because $U_\kappa T_\kappa = T_\kappa U_\kappa$. ■

Next we describe the structure of \mathcal{V}_κ as an \mathcal{H}_κ -module. For $\chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$, recall that $c(\chi) \in \mathbb{C}[T_\kappa]$ is the idempotent associated with χ . We also write $c(\chi)$ for the element of $\mathcal{V}_\kappa \simeq \mathbb{C}[T_\kappa]$, under the isomorphism of Lemma 5.4. Let

$$\mathcal{O} \subset \text{Hom}(T_\kappa, \mathbb{C}^\times)$$

be a W -orbit with respect to the action induced from that on T_κ . Let

$$\mathcal{V}_{\kappa, \mathcal{O}} := \text{Span}_{\mathbb{C}} \{c(\chi) : \chi \in \mathcal{O}\} \subseteq \mathcal{V}_\kappa.$$

For $\chi \in \text{Hom}(\kappa^\times, \mathbb{C}^\times)$ and $\Psi \in \text{Hom}(\kappa, \mathbb{C}^\times)$, we set

$$\mathbf{g}(\Psi, \chi) := \sum_{u \in \kappa^\times} \Psi(u) \cdot \chi(u). \tag{5.1}$$

For $\beta \in \Phi^+$ and $\chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$ write $\chi_\beta(u) := \chi(\bar{h}_{-\beta}(u))$ and $\psi_\beta(u) := \psi(\bar{e}_{-\beta}(u))$.

Proposition 5.5. *The finite Gelfand–Graev representation $\mathcal{V}_\kappa = \text{ind}_{U_\kappa^-}^{G_\kappa}(\psi)^{U_\kappa}$ decomposes as*

$$\mathcal{V}_\kappa = \bigoplus_{\mathcal{O}} \mathcal{V}_{\kappa, \mathcal{O}},$$

where the direct sum is taken over the W -orbits in $\text{Hom}(T_\kappa, \mathbb{C}^\times)$. Moreover, each $\mathcal{V}_{\kappa, \mathcal{O}}$ is an irreducible \mathcal{H}_κ -module.

Proof. By Lemma 5.4 we know that $\text{ind}_{U_\kappa^-}^{G_\kappa}(\psi)^{U_\kappa}$ decomposes as a $\mathbb{C}[T_\kappa]$ -module into a direct sum over all of the characters of T_κ . For $g \in G_\kappa$ we write

$$\mathcal{T}_g^\kappa := \text{ch}_{U_\kappa g U_\kappa} \in \mathbb{C}[U_\kappa \backslash G_\kappa / U_\kappa],$$

the characteristic function of $U_\kappa g U_\kappa$. For $\alpha \in \Delta$, we also set

$$\mathcal{T}_\alpha^\kappa := \mathcal{T}_{w_\alpha(1)}^\kappa.$$

For $\chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$ and $\alpha \in \Delta$ we view $c(\chi) \in \mathcal{V}_\kappa$. A direct computation gives

$$\begin{aligned} c(\chi) * \mathcal{T}_\alpha^\kappa(t) &= \int_{U w_\alpha(1) U} c(\chi)(th^{-1}) \mathcal{T}_{w_\alpha(1)}^\kappa(h) dh \\ &= \sum_{u \in \kappa^\times} c(\chi)(te_\alpha(u)w_\alpha(-1)) \\ &= \left(\sum_{u \in \kappa^\times} c(\chi)(e_\alpha(u)w_\alpha(-1)) \right) w_\alpha \chi(t) \\ &= \left(\sum_{u \in \kappa^\times} \psi^{-1}(e_{-\alpha}(-u^{-1}))c(\chi)(h_\alpha(u)) \right) w_\alpha \chi(t) \\ &= \left(\sum_{u \in \kappa^\times} \psi(e_{-\alpha}(u))\chi(h_{-\alpha}(u)) \right) w_\alpha \chi(t) \\ &= \mathbf{g}(\psi_\alpha, \chi_\alpha) \cdot w_\alpha \chi(t) \end{aligned}$$

This gives

$$c(\chi) * \mathcal{T}_\alpha^\kappa = \mathbf{g}(\psi_\alpha, \chi_\alpha) \cdot c(w_\alpha \chi). \tag{5.2}$$

The algebra \mathcal{H}_κ is generated by the elements $\mathcal{T}_\alpha^\kappa$ and \mathcal{T}_t^κ , where $\alpha \in \Delta$ and $t \in T_\kappa$. Thus

$$\mathcal{V}_\kappa = \bigoplus_{\mathcal{O}} \mathcal{V}_{\kappa, \mathcal{O}},$$

where the direct sum is taken over W -orbits in $\text{Hom}(T_\kappa, \mathbb{C}^\times)$.

In fact, each $\mathcal{V}_{\kappa, \mathcal{O}}$ is irreducible. Let $v \in \mathcal{V}_{\kappa, \mathcal{O}}$ be a nonzero vector. By Lemma 5.4, there is a character $\chi \in \text{Hom}(T_{\kappa}, \mathbb{C}^{\times})$ such that

$$v * c(\chi) = z \cdot c(\chi)$$

for some $z \in \mathbb{C}^{\times}$. Finally, (5.2) implies $v * \mathcal{H}_{\kappa} = \mathcal{V}_{\kappa, \mathcal{O}}$. Thus $\mathcal{V}_{\kappa, \mathcal{O}}$ is irreducible. ■

Now we show that each $\mathcal{V}_{\kappa, \mathcal{O}}$ is an induced representation. Let $\chi \in \text{Hom}(T_{\kappa}, \mathbb{C}^{\times})$. Then consider the algebra

$$\mathcal{H}_{\kappa, \chi} = c(\chi)\mathcal{H}_{\kappa}c(\chi).$$

Next we construct a \mathbb{C} -basis for $\mathcal{H}_{\kappa, \chi}$. Recall that for any $w \in W$ we fixed a reduced expression $w = w_{\alpha_{i_1}} \dots w_{\alpha_{i_\ell}}$. Define $\bar{w} := w_{\alpha_{i_1}}(-1) \dots w_{\alpha_{i_\ell}}(-1) \in G_{\kappa}$. Then the set

$$\{c(\chi)\mathcal{T}_{\bar{w}}^{\kappa}c(\chi) : w \in W_{\chi} := \text{Stab}_W(\chi)\}$$

is a \mathbb{C} -basis for $\mathcal{H}_{\kappa, \chi}$. The action in line (5.2) shows that $\mathcal{H}_{\kappa, \chi}$ acts on the one-dimensional subspace $\mathbb{C} \cdot c(\chi) \subset \mathcal{V}_{\kappa}$. Henceforth, we write

$$\tau_{\chi} := \mathbb{C} \cdot c(\chi)$$

for this one-dimensional representation of $\mathcal{H}_{\kappa, \chi}$.

Lemma 5.6. *Let $\chi \in \text{Hom}(T_{\kappa}, \mathbb{C}^{\times})$ and let \mathcal{O}_{χ} be the W -orbit of χ . Then the linear map*

$$\tau_{\chi} \otimes_{\mathcal{H}_{\kappa, \chi}} \mathcal{H}_{\kappa} \rightarrow \mathcal{V}_{\kappa, \mathcal{O}_{\chi}}, \quad c(\chi) \otimes h \mapsto c(\chi) * h,$$

is an \mathcal{H}_{κ} -module isomorphism.

Proof. It follows from (5.2) that the linear map $\tau_{\chi} \rightarrow \mathcal{V}_{\mathcal{O}}$ defined by $c(\chi) \mapsto c(\chi)$ is an $\mathcal{H}_{\kappa, \chi}$ -module map and induces an \mathcal{H}_{κ} -module homomorphism $\tau_{\chi} \otimes_{\mathcal{H}_{\kappa, \chi}} \mathcal{H}_{\kappa} \rightarrow \mathcal{V}_{\kappa, \mathcal{O}_{\chi}}$ defined by $c(\chi) \otimes h \mapsto c(\chi) * h$. This map is surjective by (5.2). The map is an isomorphism because both spaces have dimension $|W_{\chi} \setminus W|$. ■

Lemma 5.6 can be refined to an isomorphism

$$\tau_{\chi} \otimes_{\mathcal{H}_{\kappa, \chi}} c_{\mathcal{O}}\mathcal{H}_{\kappa}c_{\mathcal{O}} \simeq \mathcal{V}_{\kappa, \mathcal{O}},$$

where

$$c_{\mathcal{O}} := \sum_{\chi' \in \mathcal{O}} c(\chi')$$

is the central idempotent corresponding to the orbit \mathcal{O} . In particular, for the trivial orbit we have the following.

Corollary 5.7. *Let $\mathcal{O}_{\mathbb{1}}$ be the orbit of the trivial character $\mathbb{1}$ of T_{κ} . Then $\mathcal{V}_{\kappa, \mathcal{O}_{\mathbb{1}}} = \mathbb{C} \cdot c(\mathbb{1})$ and*

$$\mathcal{V}_{\kappa, \mathcal{O}_{\mathbb{1}}} \simeq \text{sign} \quad \text{as } C(B_{\kappa} \setminus G_{\kappa} / B_{\kappa})\text{-modules.}$$

Proof. By definition $\mathcal{V}_{\kappa, \mathcal{O}_{\mathbb{1}}} = \mathbb{C} \cdot c(\mathbb{1})$ and by a direct computation we see that

$$c(\mathbb{1}) * \mathcal{T}_{w_{\alpha}(1)}^{\kappa} = c(\mathbb{1}) * \mathcal{T}_{\alpha}^{\kappa} * c(\mathbb{1}) \in C(B_{\kappa} \setminus G_{\kappa} / B_{\kappa}).$$

The result then follows from (5.2). ■

5.2. *The pro- p -fixed vectors*

We follow the approach of Chan–Savin [11] to study \mathcal{V}^{I_1} . For any U -module S , we write S_U for the \bar{T} -module of U -coinvariants.

Lemma 5.8. *The natural map $\mathcal{V} \rightarrow \mathcal{V}_U$ induces an isomorphism of \bar{T} -modules*

$$\mathcal{V}^{I_1} \simeq (\mathcal{V}_U)^{T_1}.$$

Proof. This follows from the theory of Bushnell–Kutzko [8], which is applicable because I_1 possesses an Iwahori factorization and $\mathcal{T}_t \in \mathcal{H}$ is invertible for any $t \in \bar{T}$. ■

Lemma 5.9. *Let $\mathcal{V}_0 \subset \mathcal{V}$ be the subspace consisting of functions supported on $U^{-}\bar{T}U$. The inclusion $\mathcal{V}_0 \subseteq \mathcal{V}$ induces an isomorphism of \bar{T} -modules $(\mathcal{V}_0)_U \simeq \mathcal{V}_U$.*

Proof. The proof of Lemma 4.1 from Chan–Savin [11] adapts without change. ■

Lemma 5.10. *The map $\mathcal{S} : (\mathcal{V}_0)_U \rightarrow C_{c,\epsilon}^\infty(\bar{T})$ defined by*

$$\mathcal{S}(f)(t) = \delta_U(t)^{-1/2} \int_U f(tu) du$$

is an isomorphism of \bar{T} -modules.

Proof. Again this adapts directly from Chan–Savin [11, Proposition 4.2]. ■

Let $\text{ch}_t \in C_{c,\epsilon}^\infty(\bar{T}/T_1)$ be the unique function with support $t\mu_n T_1$ and $\text{ch}_t(t) = 1$. The map

$$\bar{T}/T_1 \rightarrow C_{c,\epsilon}^\infty(\bar{T}/T_1) = C_{c,\epsilon}^\infty(\bar{T})^{T_1}, \quad t \mapsto \text{ch}_t,$$

is a \bar{T}/T_1 -equivariant injection. This map extends linearly to an isomorphism of \mathcal{R} -modules $\mathcal{R} \rightarrow C_{c,\epsilon}^\infty(\bar{T}/T_1)$. Let

$$\text{ch}_{I_1}^\psi \in \mathcal{V}^{I_1}$$

be the function supported on $\mu_n U^{-}I_1$ such that $\text{ch}_{I_1}^\psi(1) = 1$. Note that $\text{ch}_{I_1}^\psi \in \mathcal{V}_0$. Since the factorization

$$\mu_n U^{-}I_1 = \mu_n U^{-}T_1(U \cap I_1)$$

is unique, a direct calculation shows that

$$\mathcal{S}(\text{ch}_{I_1}^\psi) = z \cdot \text{ch}_1 \in C_{c,\epsilon}^\infty(\bar{T}/T_1) \quad \text{with } z \in \mathbb{C}^\times.$$

Combining Lemmas 5.8–5.10, we obtain the following:

Proposition 5.11. *There is an isomorphism of \bar{T} -modules*

$$\vartheta_1 : \mathcal{V}^{I_1} \rightarrow \mathcal{R}$$

induced by the isomorphisms of Lemmas 5.8–5.10. Under this isomorphism, the functions in \mathcal{V}^{I_1} with support contained in $\mu_n U^{-}K$ are mapped to $\mathbb{C}[\mu_n, T_\kappa]$ and thus generate \mathcal{R} as an \mathcal{R} -module.

We define a map

$$\vartheta_2 : \mathcal{V}_\kappa \rightarrow \mathcal{V}^{I_1}$$

by setting

$$\vartheta_2(f)(\zeta utk) = \begin{cases} \epsilon(\zeta)\psi(u)f(tk) & \text{if } t \in T \cap K, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.12. *The map ϑ_2 is an embedding of \mathcal{H}_κ -modules. The image $\vartheta_2(\mathcal{V}_\kappa) \subset \mathcal{V}^{I_1}$ consists of the functions with support contained in $\mu_n U^- K$.*

Proof. A direct computation using the Iwasawa decomposition $\bar{G} = U\bar{T}K$ shows that the map is well-defined and is an embedding of \mathcal{H}_κ -modules. The second claim follows by the definition of $\vartheta_2(f)$. ■

Theorem 5.13. *The \mathcal{H} -module map*

$$\gamma : \mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa} \mathcal{H} \rightarrow \mathcal{V}^{I_1}$$

induced by the embedding $\vartheta_2 : \mathcal{V}_\kappa \rightarrow \mathcal{V}^{I_1}$ from Lemma 5.12 is an isomorphism.

Proof. By Frobenius reciprocity and Lemma 5.12 there is a nonzero map $\mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa} \mathcal{H} \rightarrow \mathcal{V}^{I_1}$ of \mathcal{H} -modules, which we also call ϑ_2 . It suffices to show that

$$\vartheta_1 \circ \vartheta_2 : \mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa} \mathcal{H} \rightarrow \mathcal{R}$$

is an isomorphism of \mathcal{R} -modules.

The map $\vartheta_1 \circ \vartheta_2 : \mathcal{V}_\kappa \rightarrow \mathcal{R}$ is injective with image $\mathbb{C}[\mu_n, T_\kappa] = \mathcal{H}_\kappa \cap \mathcal{R} \subset \mathcal{R}$ by Proposition 5.11 and Lemma 5.12. By Proposition 4.6 we have $\mathcal{H} \simeq \mathcal{H}_\kappa \otimes_{\mathcal{R} \cap \mathcal{H}_\kappa} \mathcal{R}$ as \mathbb{C} -vector spaces. It follows that

$$\mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa \cap \mathcal{R}} \mathcal{R} \simeq \mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa} \mathcal{H} \xrightarrow{\vartheta_1 \circ \vartheta_2} \mathcal{R}$$

is an isomorphism. This completes the proof. ■

Remark 5.14. Although our arguments in this subsection require only minor modification to those of Chan–Savin [11], the usage of the pro- p subgroup I_1 is essential. The key point is that the pro- p Hecke algebra \mathcal{H} contains functions supported on all elements of \bar{T} , while the Iwahori–Hecke algebra \mathcal{H}_I does not, a fact which is closely related to the failure of multiplicity one for Whittaker models. If one carries out the above discussion using I in place of I_1 , then the analogue of Theorem 5.13 fails. This failure is a direct consequence of the constraints on the support of functions in \mathcal{H}_I .

5.3. The Iwahori-fixed vectors

The main objective of this subsection is to describe \mathcal{V}^I as an \mathcal{H}_I -module. Our work from the last subsection provides the foundation. Let $1_I \in \mathcal{H}_I$ be the identity element.

Corollary 5.15. *The \mathcal{H} -module isomorphism $\gamma : \mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa} \mathcal{H} \rightarrow \mathcal{V}^I$ from Theorem 5.13 induces an isomorphism of \mathcal{H}_I -modules*

$$\gamma : \mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I \rightarrow \mathcal{V}^I.$$

Furthermore,

$$\mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I = \bigoplus_{\mathcal{O}} (\mathcal{V}_{\kappa, \mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I),$$

where the direct sum is taken over all the W -orbits in $\text{Hom}(T_\kappa, \mathbb{C}^\times)$.

By the above corollary, we can focus on the \mathcal{H}_I -module structure of each $\mathcal{V}_{\kappa, \mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$ individually. To begin, we first characterize the orbits \mathcal{O} such that

$$\mathcal{V}_{\kappa, \mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I \neq 0.$$

Recall that we have an injective map

$$\varphi : \mathcal{X}_{Q,n} \rightarrow \text{Hom}(T_\kappa, \mathbb{C}^\times), \quad \varphi(y)(-) = [\mathfrak{s}_y, -]|_{\text{T}(O_F)}.$$

This map is W -equivariant.

Lemma 5.16. *Let $\mathcal{O} \subset \text{Hom}(T_\kappa, \mathbb{C}^\times)$ be a W -orbit. Then $\mathcal{V}_{\kappa, \mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I \neq 0$ if and only if $\mathcal{O} \subset \text{Im}(\varphi)$.*

Proof. Let $\chi \in \text{Hom}(T_\kappa, \mathbb{C}^\times)$ and $y \in Y$. Since $\mathbb{1}_I = c(1)$ and $\Theta_{\mathfrak{s}_y} * c(1) = c([\mathfrak{s}_y, -]) * \Theta_{\mathfrak{s}_y}$, we have

$$c(\chi) \otimes_{\Theta_{\mathfrak{s}_y}} * \mathbb{1}_I = c(\chi)c(\varphi(y)) \otimes_{\Theta_{\mathfrak{s}_y}} * \mathbb{1}_I,$$

which is zero unless $\chi = \varphi(y)$. By Proposition 4.6, the elements of the form $c(\chi) \otimes_{\Theta_{\mathfrak{s}_y}} * \mathbb{1}_I$ generate $\mathcal{V}_{\mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$ as a \mathbb{C} -vector space. Thus if $\mathcal{V}_{\kappa, \mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I \neq 0$, then for every $\chi \in \mathcal{O}$ we must have $\chi = \varphi(y)$ for some $y \in Y$.

Conversely, for $y \in Y$ the element $c(\varphi(y)) \otimes_{\Theta_{\mathfrak{s}_y}} * \mathbb{1}_I \neq 0$. Indeed, by Lemma 5.12 and Theorem 5.13, $c([\mathfrak{s}_y, -]) \otimes 1 \in \mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa} \mathcal{H}$ is nonzero. Since $\Theta_{\mathfrak{s}_y}$ is invertible, the element $c(\varphi(y)) \otimes_{\Theta_{\mathfrak{s}_y}} = c(\varphi(y)) \otimes_{\Theta_{\mathfrak{s}_y}} * \mathbb{1}_I$ is nonzero. ■

Definition 5.17. For every W -orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$, we call

$$\mathcal{V}_{\mathcal{O}}^I := \gamma(\mathcal{V}_{\kappa, \mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I)$$

the \mathcal{O} -component of \mathcal{V}^I .

We have the decomposition

$$\mathcal{V}^I = \bigoplus_{\mathcal{O} \subset \mathcal{X}_{Q,n}} \mathcal{V}_{\mathcal{O}}^I$$

over all W -orbits in $\mathcal{X}_{Q,n}$.

Next we describe $\mathcal{V}_{\kappa, \mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$ (and thus also $\mathcal{V}_{\mathcal{O}}^I$) as an \mathcal{A} -module. Note that $\Theta_{s_y}^I \neq 0$ if and only if $y \in Y_{\mathcal{Q}, n}$. Thus $\mathcal{R} * \mathbb{1}_I$ is a free \mathcal{A} -module of rank $|\mathcal{X}_{\mathcal{Q}, n}|$. In particular, if $\{y_j\}_{j \in I} \subset Y$ is a set of representatives for $\mathcal{X}_{\mathcal{Q}, n}$, then $\{\Theta_{s_{y_j}} * \mathbb{1}_I\}_{j \in I}$ is an \mathcal{A} -basis for $\mathcal{R} * \mathbb{1}_I$.

Lemma 5.18. *Let $\mathcal{O} \subset \mathcal{X}_{\mathcal{Q}, n}$ be a W -orbit and let $\{y_j\}_j \subset Y$ be a set of representatives for \mathcal{O} . Then*

$$\mathcal{V}_{\kappa, \mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I \simeq \mathcal{A}^{\oplus |\mathcal{O}|}$$

as \mathcal{A} -modules with an \mathcal{A} -basis given by

$$\{c(\varphi(y_j)) \otimes \Theta_{s_{y_j}} * \mathbb{1}_I : \varphi(y_j) \in \mathcal{O}\}.$$

Also, $\mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$ is a free \mathcal{A} -module of rank $|\mathcal{X}_{\mathcal{Q}, n}|$.

Proof. By Proposition 4.6 we have

$$\mathcal{V}_{\kappa, \mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I \simeq \mathcal{V}_{\kappa, \mathcal{O}} \otimes_{\mathcal{H}_{\kappa \cap \mathcal{R}}} \mathcal{R} * \mathbb{1}_I$$

as \mathcal{A} -modules. The result follows because $\mathcal{R} * \mathbb{1}_I$ is a free \mathcal{A} -module of rank $|\mathcal{X}_{\mathcal{Q}, n}|$. Indeed, for a choice of representatives $\{y_j\}$ for $\mathcal{X}_{\mathcal{Q}, n}$, the set $\{\Theta_{s_{y_j}} * \mathbb{1}_I\}$ is an \mathcal{A} -basis for $\mathcal{R} * \mathbb{1}_I$. However, for $\chi \in \text{Hom}(T_\kappa, \mu_n)$ the element $c(\chi) \otimes \Theta_{s_{y_j}} * \mathbb{1}_I$ is nonzero if and only if $\chi = \varphi(y_j)$. ■

5.4. Further simplification

To obtain a unique and simple description of $\mathcal{V}_{\mathcal{O}}^I \simeq \mathcal{V}_{\kappa, \mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$, we will consider \mathcal{O} with a special “splitting” property as follows.

For use later, we introduce a slight generalization of the W -orbits on Y and $\mathcal{X}_{\mathcal{Q}, n}$. Let $L \subset Y$ be a W -stable sublattice of the same rank as Y with respect to the usual reflection action of W on Y . For every element z in the coweight lattice P , one considers the twisted Weyl action on Y given by

$$w[y]_z := w(y + z) - z.$$

It is well-defined since $z \in P$. This gives a group homomorphism

$$\sigma_{[z]}^{Y/L} : W \rightarrow \text{Perm}(Y/L),$$

where $\text{Perm}(Y/L)$ denotes the permutation group of Y/L . By abuse of notation and language, we also call $\sigma_{[z]}^{Y/L}$ the permutation representation of W naturally realized on the vector space $\mathbb{C}^{Y/L}$ endowed with a basis indexed by Y/L .

An orbit $\mathcal{O} \subset Y$ with respect to the action $w[-]_z$ given above is called a (W, z) -orbit. Clearly, the $(W, 0)$ -orbits are just the W -orbits. The $w[-]_z$ action is also well-defined on Y/L and the quotient map $Y \twoheadrightarrow Y/L$ is (W, z) -equivariant. Specializing to the special case $L = Y_{\mathcal{Q}, n}$, we have

Definition 5.19. A (W, z) -orbit $\mathcal{O} \subset \mathcal{X}_{Q,n} = Y/Y_{Q,n}$ is called *splitting* if there exists a section of the quotient map

$$Y \twoheadrightarrow \mathcal{X}_{Q,n}$$

which is equivariant with respect to the $w[\cdot]_z$ action on both Y and $\mathcal{X}_{Q,n}$.

The orbit of $\hat{0} \in \mathcal{X}_{Q,n}$ is always $(W, 0)$ -splitting. Also, the element $y := \rho - \rho_{Q,n}$, if it lies in Y , gives rise to a splitting $(W, -\rho)$ -orbit $\mathcal{O}_{\hat{y}} = \{\hat{y}\}$. This latter example plays a crucial role as a base point for a special form of the covering Casselman–Shalika formula (see [22, Section 5.3]). On the other hand, if $\mathcal{O}_{\hat{y}} \subset \mathcal{X}_{Q,n}$ is a free (W, z) -orbit, then it is always splitting.

For the rest of this section, we only consider the splitting W -orbits. The next lemma collects some basic consequences of a W -orbit $\mathcal{O}_{\hat{y}}$ possessing a splitting. Given $y \in Y$ we write

$$W_y := \text{Stab}_W(y) \quad \text{and} \quad W_{\hat{y}} := \text{Stab}_W(\hat{y}).$$

Lemma 5.20. *Let $\mathcal{O} \subseteq \mathcal{X}_{Q,n}$ be a splitting W -orbit. Let $s : \mathcal{O} \hookrightarrow Y$ be such a splitting.*

- (i) *For every $y \in s(\mathcal{O})$, one has $W_y = W_{\hat{y}}$, which is a parabolic subgroup of W .*
- (ii) *If $y \in Y$ and $\alpha \in \Phi$ such that $w_\alpha \in W_y$, then $\langle \alpha, y \rangle = 0$.*
- (iii) *Let $W' \subseteq W$ be a parabolic subgroup such that $w_j \in W'$ are of minimal length in $W'w_j$, $j = 1, 2$. If $w \in W'w_1$ and $w \leq w_2$, then $w_1 \leq w_2$.*

Proof. Here (i) follows from [27, Proposition 1.15], while (iii) follows from Proposition 1.10 of loc. cit. and the description of the Bruhat order in terms of subexpressions. Statement (ii) is clear by the definition of reflection. ■

In general, we let

$$\tilde{W}_{\text{aff}} := Y_{Q,n}^{\text{sc}} \rtimes W$$

be the modified affine Weyl group and also set $\tilde{W}_{\text{ex}} := Y_{Q,n} \rtimes W$. This gives

$$\tilde{W}_{\text{ex}} = \tilde{W}_{\text{aff}} \rtimes \Omega_{Q,n},$$

where $\Omega_{Q,n} \subset \tilde{W}_{\text{ex}}$ consists of the elements which fix the fundamental alcove in $Y_{Q,n} \otimes \mathbb{R}$ associated to \tilde{W}_{aff} as in Section 2.2. Thus

$$\mathcal{H}_I = \mathcal{H}_{\tilde{W}_{\text{aff}}} \otimes_{\mathbb{Z}} \mathbb{Z}[\Omega_{Q,n}],$$

where the algebra law for the right hand side is as in [28, p. 47]. For any $y \in Y$, let

$$\mathcal{H}_{I,y} := \mathcal{H}_{\tilde{W}_{\text{ex},y}} \subset \mathcal{H}_I$$

be the subalgebra of \mathcal{H}_I associated with $\tilde{W}_{\text{ex},y} := \text{Stab}_{\tilde{W}_{\text{ex}}}(y)$, i.e., $\mathcal{H}_{I,y}$ is generated by the elements in $\{\mathbb{1}_I * \mathcal{T}_{\tilde{w}} * \mathbb{1}_I : w \in \tilde{W}_{\text{ex},y}\}$. In particular, if $\tilde{W}_{\text{ex},y} \subset W$ is actually a parabolic subgroup of W , then $\mathcal{H}_{I,y} \subset \mathcal{H}_W$.

Theorem 5.21. *Let $\mathcal{O} \subseteq \mathcal{X}_{\mathcal{Q},n}$ be a splitting W -orbit with a splitting given by $s : \mathcal{O} \hookrightarrow Y$. Let $y \in Y$ be in the closure of the positive Weyl chamber with respect to Δ and $\hat{y} \in \mathcal{O}$.*

- (i) *The algebra $\mathcal{H}_{I,y}$ acts on the one-dimensional vector space $\mathbb{C} \cdot c(\varphi(y)) \otimes \Theta_{s_y} * \mathbb{1}_I$ via the sign character ε_y .*
- (ii) *The $\mathcal{H}_{I,y}$ -module map $\varepsilon_y \rightarrow \mathcal{V}_{\kappa,\mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$ induces isomorphisms of \mathcal{H}_I -modules*

$$\varepsilon_y \otimes_{\mathcal{H}_{I,y}} \mathcal{H}_I \rightarrow \mathcal{V}_{\kappa,\mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I \xrightarrow{\gamma} \mathcal{V}_{\mathcal{O}}^I,$$

where the first map is given by $1 \mapsto c(\varphi(y)) \otimes \Theta_{s_y} * \mathbb{1}_I$.

Proof. Since \mathcal{O} is splitting, we see that $\tilde{W}_{\text{aff},y} = W_y$ is a standard parabolic Weyl subgroup. Now the fact that $\mathcal{H}_{I,y}$ acts on $\mathbb{C} \cdot c(\varphi(y)) \otimes \Theta_{s_y} * \mathbb{1}_I$ follows from Lemma 5.20, the property (4.6) of the Bernstein relations, and the equality (5.2). It follows from (5.2) and Lemma 5.20 (ii) that each simple reflection in $\mathcal{H}_{I,y}$ acts by the sign character. Specifically, for $\alpha \in \Delta$ such that $w_\alpha \in \tilde{W}_{\text{aff},y}$ we have

$$\varphi(y)(h_\alpha(u)) = (\varpi, u)_n^{B_{\mathcal{Q}}(y, \alpha^\vee)} = 1,$$

where the last equality is due to the fact that $B_{\mathcal{Q}}(y, \alpha^\vee) = Q(\alpha^\vee)(\alpha, y) = 0$.

Thus, we have an \mathcal{H}_I -module homomorphism

$$f : \varepsilon_y \otimes_{\mathcal{H}_{I,y}} \mathcal{H}_I \rightarrow \mathcal{V}_{\kappa,\mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I, \quad 1 \otimes h \mapsto c(\varphi(y)) \otimes \Theta_{s_y} * h.$$

We claim that this map is an isomorphism of \mathcal{H}_I -modules. To prove this, we show that this map is upper triangular with respect to a natural choice of \mathcal{A} -bases.

For each coset $W_y w$ we choose the element of minimal length. Let $\{w_j\}_j$ be the set of these minimal elements. Let $\bar{w}_j \in K \cap \overline{N(T)}$ represent w_j . Then the Bernstein decomposition implies that

$$\{1 \otimes \mathcal{T}_{\bar{w}_j}^I : 1 \leq j \leq |W_\chi \backslash W|\}$$

is an \mathcal{A} -basis for $\varepsilon_y \otimes_{\mathcal{H}_{I,y}} \mathcal{H}_I$.

Writing $y_j := w_j^{-1} \cdot y$, the set

$$\{c(\varphi(y_j)) \otimes \Theta_{s_{y_j}} * \mathbb{1}_I : 1 \leq j \leq |W_\chi \backslash W|\}$$

is an \mathcal{A} -basis for $\mathcal{V}_{\kappa,\mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$. Under the map

$$\varepsilon_y \otimes_{\mathcal{H}_{I,y}} \mathcal{H}_I \rightarrow \mathcal{V}_{\kappa,\mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$$

we have

$$1 \otimes \mathcal{T}_{\bar{w}_j}^I \mapsto c(\varphi(y)) \otimes \Theta_{s_y} \mathcal{T}_{\bar{w}_j} * \mathbb{1}_I.$$

By Lemma 4.19, we have

$$c(\varphi(y)) \otimes \Theta_{s_y} \mathcal{T}_{\bar{w}_j} * \mathbb{1}_I \in c(\varphi(y)) \otimes \left(\mathcal{T}_{\bar{w}_j} * \Theta_{\bar{w}_j^{-1} \cdot s_y} * \mathbb{1}_I + \sum_{w' < w_j} \mathcal{T}_{\bar{w}'} * \Theta_{(\bar{w}')^{-1} \cdot s_y} \mathcal{A} \right),$$

and applying (5.2) again gives

$$\begin{aligned}
 & c(\varphi(y)) \otimes \left(\mathcal{T}_{\bar{w}_j} * \Theta_{\bar{w}_j^{-1} \cdot s_y} * \mathbb{1}_I + \sum_{w' < w_j} \mathcal{T}_{\bar{w}'} * \Theta_{(\bar{w}')^{-1} \cdot s_y} \mathcal{A} \right) \\
 &= (-1)^{\ell(w_j)} c(\varphi(y_j)) \otimes \Theta_{\bar{w}_j^{-1} \cdot s_y} * \mathbb{1}_I + \sum_{w' < w_j} c(\varphi((w')^{-1} \cdot y)) \otimes \Theta_{(\bar{w}')^{-1} \cdot s_y} \mathcal{A}. \quad (5.3)
 \end{aligned}$$

Now it follows from (5.3) and Lemma 5.20 (iii) that the map f is upper triangular. Furthermore, Lemma 5.20 (iii) also implies that the diagonal entries are nonzero. Thus f is an isomorphism of \mathcal{H}_I -modules. ■

The trivial orbit \mathcal{O}_0 and any free orbit in $\mathcal{X}_{Q,n}$ are always splitting. In this case, Theorem 5.21 gives the following.

Corollary 5.22. *Let \mathcal{O}_0 be the orbit corresponding to the trivial character of T_κ . Then we have the isomorphism*

$$\varepsilon_W \otimes_{\mathcal{H}_W} \mathcal{H}_I \simeq \mathcal{V}_{\kappa, \mathcal{O}_0} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I \simeq \mathcal{V}_{\mathcal{O}_0}^I$$

of \mathcal{H}_I -modules.

Corollary 5.23. *Let $\mathcal{O} \subset \mathcal{X}_{Q,n}$ be a free W -orbit. Then*

$$\mathcal{H}_I \simeq \mathcal{V}_{\kappa, \mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I \simeq \mathcal{V}_{\mathcal{O}}^I$$

as \mathcal{H}_I -modules.

Definition 5.24. A W -orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$ is said to have the S -property if there exists $y \in Y$ with $\hat{y} \in \mathcal{O}$ and a character $\mu_y : \mathcal{H}_{I,y} \rightarrow \mathbb{C}^\times$ such that

$$\mathcal{V}_{\mathcal{O}_y}^I \simeq \mu_y \otimes_{\mathcal{H}_{I,y}} \mathcal{H}_I.$$

Theorem 5.21 states that every splitting orbit of $\mathcal{X}_{Q,n}$ has the S -property. In fact, we will show in Corollary 8.7 that every orbit of $\mathcal{X}_{Q,n}$ has the S -property.

6. Subclasses of covers and splitting W -orbits

In this section, we give some detailed discussion of the W -orbits in $\mathcal{X}_{Q,n}$ which are splitting. In particular, we determine some covers for which every W -orbit in $\mathcal{X}_{Q,n}$ is splitting: such examples include the ‘‘oasitic’’ covers of semisimple simply-connected groups and the Kazhdan–Patterson and Savin covers.

6.1. Saturated and oasitic covers

We briefly discuss several special subclasses of covering groups which exhibit better properties. For $Q : Y \rightarrow \mathbb{Z}$ and the associated bilinear form B_Q , we write

$$\det(B_Q) = \det [B_Q(v_i, v_j)]_{1 \leq i, j \leq r} \in \mathbb{Z},$$

where $\{v_i : 1 \leq i \leq r\} \subset Y$ is any choice of \mathbb{Z} -basis. Here $\det(B_Q) \in \mathbb{Z}$ is well-defined and if G is almost simple and simply-connected, then

$$\det(B_Q) = \mathcal{I}_\Delta \cdot \prod_{\alpha \in \Delta} Q(\alpha^\vee), \tag{6.1}$$

where \mathcal{I}_Δ is the index of the root system of G .

Definition 6.1. A covering group $\bar{G}^{(n)}$ of G is called

- *saturated* (cf. [19, Definition 2.1]) if $Y^{\text{sc}} \cap Y_{Q,n} = Y_{Q,n}^{\text{sc}}$ (note that the inclusion \supset always holds);
- *aligned* if $Y_{Q,n}^{\text{sc}} = n' \cdot Y^{\text{sc}}$ for some $n' \in \mathbb{N}$;
- *very saturated* if it is both saturated and aligned;
- *oasitic* if $\gcd(n, c_\alpha^\#) = 1 = \gcd(n, \det(B_Q))$ for every $\alpha \in \Delta$, where $\alpha_\#^\vee = \sum_{\alpha \in \Delta} c_\alpha^\# \alpha^\vee$ is the highest coroot.

If G is almost simple and simply-connected, then \bar{G} is saturated if and only if its dual group \bar{G}^\vee is of adjoint type, since $Z(\bar{G}^\vee) = \text{Hom}(Y_{Q,n}/Y_{Q,n}^{\text{sc}}, \mathbb{C}^\times)$ by definition.

Lemma 6.2. *Let $\bar{G}^{(n)}$ be an n -fold cover of an almost simple simply-connected G with $Q(\alpha^\vee) = 1$ for any short α^\vee . Then $\bar{G}^{(n)}$ is very saturated if and only if one of the following holds:*

- (i) G is of type $A_r, B_r, C_r, D_r, E_6, E_7$ and $\gcd(n, \mathcal{I}_\Delta) = 1$;
- (ii) G is of type F_4 and $2 \nmid n$, or G is of type G_2 and $3 \nmid n$;
- (iii) G is of type E_8 (and n is arbitrary).

When $\bar{G}^{(n)}$ is very saturated, we have $Y_{Q,n} = Y_{Q,n}^{\text{sc}} = nY$. Moreover, except for type F_4 and G_2 , the group $\bar{G}^{(n)}$ is very saturated if and only if $\bar{G}^\vee \simeq G^\vee$.

Proof. Assertion (i) follows from [57, Section 2.7]. For (ii), we note that $\bar{F}_4^{(n)}$ (resp. \bar{G}_2) is aligned if and only if $2 \nmid n$ (resp. $3 \nmid n$). Statement (iii) is clear. The last assertion follows from loc. cit. and the fact that $\bar{E}_8^{(n)}$ is always aligned and $\bar{E}_8^\vee = E_8 = E_8^\vee$ always holds. ■

Consider a cover $\bar{G}^{(n)}$ of G as in Lemma 6.2. It is easy to check whether $\bar{G}^{(n)}$ is oasitic by using (6.1). For such G , we tabulate saturated, very saturated and oasitic covers in Tables 1 and 2.

	A_r	B_r	C_r	D_r
saturated	$\gcd(n, r + 1) = 1$	n odd, or $n \in 4\mathbb{Z} + 2$ for r odd	n odd	n odd
very saturated	$\gcd(n, r + 1) = 1$	n odd	n odd	n odd
oasitic	$\gcd(n, r + 1) = 1$	n odd	n odd	n odd

Tab. 1. For classical G

	E_6	E_7	E_8	F_4	G_2
saturated	$3 \nmid n$	$2 \nmid n$	all n	all n	all n
very saturated	$3 \nmid n$	$2 \nmid n$	all n	$2 \nmid n$	$3 \nmid n$
oasitic	$2, 3 \nmid n$	$2, 3 \nmid n$	$2, 3, 5 \nmid n$	$2, 3 \nmid n$	$2, 3 \nmid n$

Tab. 2. For exceptional G

It is clear that if G is almost simple and simply-connected, we have the following inclusions for covers of G with $Q(\alpha^\vee) = 1$:

$$\{\text{oasitic covers}\} \subseteq \{\text{very saturated covers}\} \subseteq \{\text{saturated covers}\}. \tag{6.2}$$

6.2. Splitting W -orbits

To determine the splitting W -orbits, we start with the following:

Proposition 6.3. *Let $\bar{G}^{(n)}$ be a cover of an almost simple simply-connected G with $Q(\alpha^\vee) = 1$ for any short α^\vee . Assume $\gcd(n, \mathcal{I}_\Delta) = 1$. Then every W -orbit $\mathcal{O}_{\hat{y}} \subset \mathcal{X}_{Q,n}$ such that the stabilizer $W_{\hat{y}} \subset W$ is a parabolic subgroup is splitting.*

Proof. Let $\mathcal{O}_{\hat{y}} \subset \mathcal{X}_{Q,n}$ be an orbit such that $W_{\hat{y}} = W_J$, the Weyl group generated by a certain subset $J \subset \Delta$ depending on \hat{y} .

Taking an arbitrary representative $y \in Y$ of \hat{y} , we have

$$w_\alpha(y) - y \in Y_{Q,n} = n \cdot Y.$$

Let $\{\omega_\alpha : \alpha \in \Delta\}$ be the set of coweights, which constitutes a \mathbb{Z} -basis of the coweight lattice P . If we write $y = \sum_{\alpha \in \Delta} y_\alpha \omega_\alpha$, then we have

$$n \mid y_\alpha \quad \text{for every } \alpha \in J.$$

We want to find a $y_{Q,n} \in Y_{Q,n}$ such that

$$w_\alpha(y - y_{Q,n}) = y - y_{Q,n} \quad \text{for every } \alpha \in J, \tag{6.3}$$

in which case we have $W_J \subseteq W_{y-y_{Q,n}} \subseteq W_{\hat{y}} = W_J$ and therefore $W_{y-y_{Q,n}} = W_J$ as desired.

If we write $y_{Q,n} = \sum_{\alpha \in \Delta} d_\alpha \omega_\alpha \in Y_{Q,n}$, then (6.3) entails that

$$d_\alpha = y_\alpha \quad \text{for every } \alpha \in J.$$

Thus, it suffices to show that there exists $d_\alpha \in \mathbb{Z}$ for $\alpha \notin J$ such that

$$\sum_{\alpha \in J} y_\alpha \omega_\alpha + \sum_{\alpha \notin J} d_\alpha \omega_\alpha \in Y_{Q,n}.$$

Since $\gcd(n, \mathcal{I}_\Delta) = 1$, there exist $a, b \in \mathbb{Z} - \{0\}$ such that

$$na - \mathcal{I}_\Delta \cdot b = 1.$$

Consider the sublattice

$$P_J = \left\{ \sum_{\alpha} y_{\alpha} \omega_{\alpha} : n \mid y_{\alpha} \text{ for every } \alpha \in J \right\} \subset P.$$

Define a \mathbb{Z} -homomorphism

$$\varphi_J : P \rightarrow P \otimes \mathbb{Z}[1/n], \quad \varphi_J(\omega_{\alpha}) = \begin{cases} w_{\alpha}/n & \text{for } \alpha \in J, \\ a \cdot \omega_{\alpha} & \text{for } \alpha \notin J. \end{cases}$$

Clearly, $\varphi_J|_{P_J} : P_J \rightarrow P$ is well-defined. Moreover, since n and \mathcal{S}_{Δ} are coprime, it is easy to see that $\varphi_J(P_J \cap Y) \subset Y$, i.e., the commutative diagram

$$\begin{array}{ccccc} P_J \cap Y & \hookrightarrow & P_J & \hookrightarrow & P \\ \varphi_J \downarrow & & \downarrow \varphi_J & & \downarrow \varphi_J \\ Y & \hookrightarrow & P & \hookrightarrow & P \otimes \mathbb{Z}[1/n] \end{array}$$

is well-defined. Now, if we take $d_{\alpha} = nay_{\alpha}$ for $\alpha \notin J$, then

$$y_{Q,n} = n \cdot \left(\sum_{\alpha \in J} (y_{\alpha}/n) \omega_{\alpha} + \sum_{\alpha \notin J} a y_{\alpha} \omega_{\alpha} \right) = n \cdot \varphi_J(y) \in n \cdot Y;$$

this completes the proof. ■

It follows from Tables 1 and 2 that the covers of G in Proposition 6.3 satisfying $\gcd(n, \mathcal{S}_{\Delta}) = 1$ are exactly the saturated covers, except the only case of $\overline{B}_r^{(n)}$ with $n \in 4\mathbb{Z} + 2$ and r odd, for which $\gcd(n, \mathcal{S}_{\Delta}) = 2$.

Corollary 6.4. *Let $\overline{G}^{(n)}$ be an oasisitic cover of an almost simple simply-connected G with $Q(\alpha^{\vee}) = 1$ for every short coroot α^{\vee} . Then every W -orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$ is splitting.*

Proof. For an oasisitic cover $\overline{G}^{(n)}$ we have

$$\gcd(n, \mathcal{S}_{\Delta}) = 1 = \gcd(n, c_{\alpha}^{\sharp})$$

for all c_{α}^{\sharp} as in Definition 6.1, and thus by [48, Proposition 4.1] for every $\hat{y} \in \mathcal{X}_{Q,n}$ the stabilizer subgroup $W_{\hat{y}}$ is conjugate to a parabolic Weyl subgroup. The result then follows from Proposition 6.3. ■

Example 6.5. Consider GL_r and its cocharacter lattice Y given with the standard \mathbb{Z} -basis $\{e_i : 1 \leq i \leq r\}$. A Brylinski–Deligne cover $\overline{GL}_r^{(n)}$ of GL_r is associated to $\mathbf{p}, \mathbf{q} \in \mathbb{Z}$ such that

$$B(e_i, e_j) = \begin{cases} 2\mathbf{p} & \text{if } i = j, \\ \mathbf{q} & \text{if } i \neq j. \end{cases}$$

We have $Q(\alpha^{\vee}) = 2\mathbf{p} - \mathbf{q}$. Every BD cover \overline{GL}_r is saturated, i.e., $Y_{Q,n} \cap Y^{\text{sc}} = Y_{Q,n}^{\text{sc}} = n_{\alpha} \cdot Y^{\text{sc}}$. The Kazhdan–Patterson covers are those with $Q(\alpha^{\vee}) = -1$, while Savin’s nice cover is the one with $\mathbf{p} = -1, \mathbf{q} = 0$.

Consider a BD cover $\overline{\text{GL}}_r$ such that

$$n_\alpha Y \subseteq Y_{Q,n}. \tag{6.4}$$

We have the natural quotient maps

$$Y \xrightarrow{f} Y/n_\alpha Y \xrightarrow{h} Y/Y_{Q,n} = \mathcal{X}_{Q,n}.$$

For every $y^* \in Y/n_\alpha Y$, it is easy to see that $\text{Stab}_W(y^*) = \text{Stab}_W(h(y^*))$, since $\overline{\text{GL}}_r$ is always saturated. Thus, every W -orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$ has a W -equivariant section of h . It is also clear that every W -orbit in $Y/n_\alpha Y$ has a W -equivariant section of f . Hence, for such $\overline{\text{GL}}_r$, every orbit of $\mathcal{X}_{Q,n}$ is splitting. The Kazhdan–Patterson covers and Savin covers both satisfy (6.4) above.

We also note that if (6.4) fails, then it is possible that a certain $\mathcal{O} \subset \mathcal{X}_{Q,n}$ is not splitting. The cover $\overline{\text{GL}}_2^{(4)}$ associated with $\mathbf{p} = 1, \mathbf{q} = 2$ is such an example.

Remark 6.6. If \overline{G} is very saturated but not oasisitic, then it is possible that certain $\mathcal{O} \subset \mathcal{X}_{Q,n}$ is not splitting. For example, consider the double cover $\overline{G}_2^{(2)}$ of the exceptional G_2 . In the Bourbaki notations, the element $\hat{\omega}_1 \in \mathcal{X}_{Q,n}$ of the coweight (also a coroot) $\omega_1 \in Y$ has stabilizer subgroup

$$W_{\hat{\omega}_1} \simeq \{1, w_{\alpha_2}\} \times \{1, w_{\omega_1}\},$$

where the right hand side is a direct product. Since $W_{\hat{\omega}_1}$ is not isomorphic to a parabolic Weyl subgroup, we see that $\mathcal{O}_{\hat{\omega}_1}$ is not splitting.

6.3. Covers of SL_2

In this subsection we focus on covers of $\text{SL}(2)$ and determine the \mathcal{H}_I -module structure of $\mathcal{V}_\mathcal{O}^I \simeq \mathcal{V}_{\kappa,\mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$ for any W -orbit $\mathcal{O} \subseteq \mathcal{X}_{Q,n}$.

Let SL_2 be the n -fold cover with $Q(\alpha^\vee) = -1$. In this case

$$Y_{Q,n} = \mathbb{Z}[\alpha^\vee_{Q,n}] = \mathbb{Z}[n^* \alpha^\vee] \quad \text{where} \quad n^* = \frac{n}{\text{gcd}(n, 2Q(\alpha^\vee))}.$$

Moreover,

$$\mathcal{X}_{Q,n} = \{j\alpha^\vee : j \in [-n^*/2, n^*/2] \cap \mathbb{Z}\}.$$

Note that if n^* is even, then $n^*/2 = -n^*/2 \in \mathcal{X}_{Q,n}$. With these representatives we can easily describe the W -orbits. We consider two cases according to the parity of n^* .

If n^* is odd, then the W -orbits of $\mathcal{X}_{Q,n}$ consist of the trivial orbit $\{0\}$ and $\lfloor n^*/2 \rfloor$ free orbits of the form

$$\{j\alpha^\vee, -j\alpha^\vee\}, \quad j \in [1, n^*/2] \cap \mathbb{Z}.$$

In this case, by Propositions 5.22 and 5.23 we have a complete description of the \mathcal{H}_I -module structure of $\mathcal{V}_\mathcal{O}^I \simeq \mathcal{V}_{\kappa,\mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$ for any W -orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$.

Now assume n^* is even. There are two trivial W -orbits in $\mathcal{X}_{Q,n}$:

$$\{0\} \quad \text{and} \quad \{n^* \alpha^\vee / 2\},$$

while

$$\{j\alpha^\vee, -j\alpha^\vee\}, \quad j \in [1, n^*/2 - 1] \cap \mathbb{Z},$$

constitute the free W -orbits in $\mathcal{X}_{Q,n}$. Again, Propositions 5.22 and 5.23 give a complete description of the \mathcal{H}_I -module $\mathcal{V}_\theta^I \simeq \mathcal{V}_{\kappa,\theta} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$ except for the orbit

$$\mathcal{O}_{n^*/2} := \{n^*\alpha^\vee/2\} \subset \mathcal{X}_{Q,n}.$$

We will describe $\mathcal{V}_{\mathcal{O}_{n^*/2}}^I \simeq \mathcal{V}_{\kappa,\mathcal{O}_{n^*/2}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$ directly. The result of our analysis uncovers a new phenomenon that does not follow from our general results proved above. Note that if n^* is even, then necessarily $2n^* = n_\alpha$.

Lemma 6.7. *The element $h := \mathcal{T}_\alpha^I \Theta_{s_{-n^*\alpha^\vee}}^I \in \mathcal{H}_I$ satisfies $h^2 = q$.*

Proof. Let $y = -n^*\alpha^\vee$. By (4.7) we have

$$\Theta_{s_y}^I \mathcal{T}_\alpha^I = \mathcal{T}_\alpha^I \Theta_{w_\alpha(-1)\cdot s_y}^I - (q-1)\Theta_{s_y \bar{h}_\alpha(\varpi^{2n^*})}^I.$$

Thus

$$h^2 = (T_\alpha^I)^2 \Theta_{(w_\alpha(-1)\cdot s_y) s_y}^I - (q-1)\mathcal{T}_\alpha^I \Theta_{s_y \bar{h}_\alpha(\varpi^{2n^*}) s_y}^I.$$

The identity $w_\alpha(-1) \cdot s_y = s_y \bar{h}_\alpha(\varpi^{2n^*})$ from (2.5d) implies that

$$h^2 = [(\mathcal{T}_\alpha^I)^2 - (q-1)\mathcal{T}_\alpha^I] \Theta_{s_y \bar{h}_\alpha(\varpi^{2n^*}) s_y}^I = q \Theta_{s_y \bar{h}_\alpha(\varpi^{2n^*}) s_y}^I = q,$$

where the last equality holds since $s_y \bar{h}_\alpha(\varpi^{2n^*}) s_y = 1 \in \bar{T}$ by (2.5c). ■

Let $\mathcal{T}' := q^{-1/2} \cdot T_\alpha^I \Theta_{s_{-n^*\alpha^\vee}}^I$ and consider the \mathbb{C} -subalgebra

$$(\mathcal{H}_I)'_0 := \langle \mathcal{T}' \rangle \subset \mathcal{H}_I$$

generated by \mathcal{T}' . Lemma 6.7 implies $(\mathcal{T}')^2 = 1$ and thus

$$(\mathcal{H}_I)'_0 \simeq \mathbb{C}[\mathbb{Z}/2\mathbb{Z}].$$

Let $\sigma : (\mathcal{H}_I)'_0 \rightarrow \mathbb{C}$ be the character given by

$$\sigma(\mathcal{T}') = q^{-1/2} \cdot \mathbf{g}(\psi_\alpha, \varphi(-n^*\alpha^\vee/2)_\alpha) \in \{\pm 1\}.$$

Proposition 6.8. *Let $y = -\frac{n^*}{2}\alpha^\vee$. The \mathbb{C} -vector space map*

$$\sigma \rightarrow \mathcal{V}_{\kappa,\mathcal{O}_{n^*/2}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I, \quad 1 \mapsto c(\varphi(y)) \otimes \Theta_{s_y} * \mathbb{1}_I,$$

is a map of $(\mathcal{H}_I)'_0$ -modules. Moreover, it induces an isomorphism

$$\sigma \otimes_{(\mathcal{H}_I)'_0} \mathcal{H}_I \rightarrow \mathcal{V}_{\kappa,\mathcal{O}_{n^*/2}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I \xrightarrow{\gamma} \mathcal{V}_{\mathcal{O}_{n^*/2}}^I$$

of \mathcal{H}_I -modules.

Proof. Using the Bernstein relation in (4.5), we have

$$c(\varphi(y)) \otimes \Theta_{s_y} * \mathbb{1}_I * \mathcal{T}_\alpha^I = \mathbf{g}(\psi_\alpha, \varphi(y)_\alpha) \cdot c(\varphi(y)) \otimes \Theta_{w_\alpha(-1) \cdot s_y} * \mathbb{1}_I.$$

Thus

$$c(\varphi(y)) \otimes \Theta_{s_y} * \mathbb{1}_I * \mathcal{T}' = \frac{\mathbf{g}(\psi_\alpha, \varphi(y)_\alpha)}{q^{1/2}} c(\varphi(y)) \otimes \Theta_{(w_\alpha(-1) \cdot s_y) s_{-n^* \alpha^\vee}} * \mathbb{1}_I.$$

Since $w_\alpha(-1) \cdot s_y = s_y h_\alpha(\varpi^{n^*})$ by (2.5d) and $h_\alpha(\varpi^{n^*}) = s(\alpha^\vee(\varpi^{n^*}))$, it follows that $(w_\alpha(-1) \cdot s_y) s(-n^* \alpha^\vee)(\varpi) = s_y$. Hence,

$$c(\varphi(y)) \otimes \Theta_{s_y} * \mathbb{1}_I * \mathcal{T}' = \frac{\mathbf{g}(\psi_\alpha, \varphi(y)_\alpha)}{q^{1/2}} c(\varphi(y)) \otimes \Theta_{s_y} * \mathbb{1}_I.$$

This shows that the map $\sigma \rightarrow \mathcal{V}_{\kappa, \mathcal{O}} \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$ defined by $1 \mapsto c(\varphi(y)) \otimes \Theta_{s_y} * \mathbb{1}_I$ is a map of $(\mathcal{H}_I)'_0$ -modules. The induced map of \mathcal{H}_I -modules is an isomorphism because it takes the \mathcal{A} -basis $\{1 \otimes \Theta_{s_{kn^* \alpha^\vee}}^I : k \in \mathbb{Z}\}$ to the \mathcal{A} -basis $\{c(\varphi(y)) \otimes \Theta_{s_y} * \mathbb{1}_I * \Theta_{s_{kn^* \alpha^\vee}}^I\}$. ■

The orbit $\mathcal{O}_{n^*/2} \subset \mathcal{X}_{Q,n}$ does not possess a W -equivariant splitting. Thus, Proposition 6.8 does not follow from Theorem 5.21. This suggests that splittings of other finite subgroups of the modified extended Weyl group $\tilde{W}_{\text{ex}} = W \ltimes Y_{Q,n}$ should also be considered, besides W . We make a few remarks in this direction and further elaborate on the example of Proposition 6.8.

For any $z \in Y \otimes \mathbb{Q}$, we set $\tilde{W}_{\text{ex},z} := \text{Stab}_{\tilde{W}_{\text{ex}}}(z)$. In particular, $\tilde{W}_{\text{ex},0} = W$. For any $y \in Y$ we have

$$W_y = \tilde{W}_{\text{ex},0} \cap \tilde{W}_{\text{ex},y}.$$

Now we may consider

$$\tilde{W}_{\text{ex},z,y} := \tilde{W}_{\text{ex},z} \cap \tilde{W}_{\text{ex},y}.$$

Let $\eta : \tilde{W}_{\text{ex}} = W \ltimes Y_{Q,n} \rightarrow W$ be the projection map. The quotient map

$$Y \twoheadrightarrow \mathcal{X}_{Q,n}$$

is equivariant with respect to $\eta : \tilde{W}_{\text{ex}} \twoheadrightarrow W$. For every $z \in Y \otimes \mathbb{Q}$, the group $\tilde{W}_{\text{ex},z}$ is finite and $\eta : \tilde{W}_{\text{ex},z} \rightarrow \eta(\tilde{W}_{\text{ex},z})$ is an isomorphism. Then an orbit $\mathcal{O}_y \subset \mathcal{X}_{Q,n}$ is said to be $\tilde{W}_{\text{ex},z}$ -splitting if

$$\eta(\tilde{W}_{\text{ex},z,y}) = \eta(\tilde{W}_{\text{ex},z}) \hat{y}. \tag{6.5}$$

As we vary z , there are more possibilities for the above equality to hold.

Consider for instance the case of $n^* \in 2\mathbb{Z}$ and $\hat{y} = n^* \alpha^\vee / 2$ for $\overline{\text{SL}}_2$ as in Proposition 6.8. If we take $z = 0$, then the two sides of (6.5) are $\{1\}$ and W respectively. On the other hand, if we take $z = -n^* \alpha^\vee / 2$, then $\tilde{W}_{\text{ex},z} \subset \tilde{W}_{\text{ex}}$ is of order 2, generated by the reflection fixing $-n^* \alpha^\vee / 2$. In this case, the two sides of (6.5) are both equal to $\eta(\tilde{W}_{\text{ex},z,y})$, and there is a $\tilde{W}_{\text{ex},z}$ -splitting for the orbit $\mathcal{O}_{n^*/2}$.

Such a $\tilde{W}_{\text{ex},z}$ -splitting is only implicitly used in the proof of Proposition 6.8. For a general \bar{G} , it is not easy to identify explicitly (with concrete relations among the generators) the finite subalgebra of \mathcal{H}_I associated to $\tilde{W}_{\text{ex},z}$. For $\overline{\text{SL}}_2$, we essentially realized this by using Lemma 6.7. We hope to return to this problem in a future work.

7. Metaplectic representation of Sahi–Stokeman–Venkateswaran

In this section we show that the representation

$$\mathcal{V}^I \simeq \mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$$

embeds naturally into the metaplectic representation constructed by Sahi–Stokeman–Venkateswaran [45] for a particular choice of what they call representation parameters (see [45, Definition 3.5]).

The proof of Lemma 5.18 shows that if $\{y_j\}_j \subset Y$ is a set of representatives for $\mathcal{X}_{Q,n}$, then

$$\{c(\varphi(y_j)) \otimes \Theta_{s_{y_j}} * \mathbb{1}_I\}_j$$

is an \mathcal{A} -basis for $\mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$. Using the Bernstein relations for \mathcal{H} we can compute the action of \mathcal{H}_I on $\mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa} \mathcal{H} * \mathbb{1}_I$ in terms of that basis. We begin with a description of the action of \mathcal{A} .

Lemma 7.1. *Let $y \in Y$ and $z \in Y_{Q,n}$. Then*

$$c(\varphi(y)) \otimes \Theta_{s_y} * \mathbb{1}_I * \Theta_{s_z} * \mathbb{1}_I = \varepsilon^{D(y,z)} c(\varphi(y)) \cdot \Theta_{s_{y+z}} * \mathbb{1}_I.$$

Proof. This follows from the definition of Θ_{s_y} in Section 4.1 and identity (2.5c). ■

Now we describe the action of $\mathcal{H}_W \subset \mathcal{H}_I$. It suffices to consider the action of the elements $\mathcal{T}_\alpha = \mathcal{T}_{\bar{w}_\alpha(1)}$, where $\alpha \in \Delta$. First we need some notation. For any $t \in \bar{T}$ such that $\wp(t) = y(\varpi)$ we write $\theta_t = c(\varphi(y)) \otimes \Theta_t \in \mathcal{V}_\kappa \otimes_{\mathcal{H}_\kappa} \mathcal{H}$.

Lemma 7.2. *Let $y \in Y$ and $\alpha \in \Delta$. Then*

$$\theta_{s_y} * \mathbb{1}_I * \mathcal{T}_\alpha * \mathbb{1}_I = \begin{cases} \mathbf{g}(\psi_\alpha, \varphi(y)\alpha) \theta_{w_\alpha(-1) \cdot s_y} * \mathbb{1}_I + (q-1) \sum_{\substack{1-(y,\alpha) \leq j \leq 0 \\ j \equiv 0 \pmod{n_\alpha}} \varepsilon^{jQ(\alpha^\vee)} \theta_{s_y h_\alpha(\varpi^j)} * \mathbb{1}_I & \text{if } \langle y, \alpha \rangle > 0, \\ \mathbf{g}(\psi_\alpha, \varphi(y)\alpha) \theta_{w_\alpha(-1) \cdot s_y} * \mathbb{1}_I & \text{if } \langle y, \alpha \rangle = 0, \\ \mathbf{g}(\psi_\alpha, \varphi(y)\alpha) \theta_{w_\alpha(-1) \cdot s_y} * \mathbb{1}_I + (q-1) \sum_{\substack{1 \leq j \leq -(y,\alpha) \\ j \equiv 0 \pmod{n_\alpha}} \varepsilon^{jQ(\alpha^\vee)} \theta_{s_y h_\alpha(\varpi^j)} * \mathbb{1}_I & \text{if } \langle y, \alpha \rangle < 0. \end{cases}$$

Proof. This result follows from (4.6). ■

7.1. The SSV representation

We briefly recall the construction of the metaplectic representation of \mathcal{H}_I in [45] using notations in our paper.

For the rest of this section, we assume that G is semisimple and almost simple. Denote by P the lattice of coweights with

$$P \supseteq Y \supseteq Y^{\text{sc}}.$$

Fix an integral-valued Weyl-invariant quadratic form $Q : Y \rightarrow \mathbb{Z}$, and extend it uniquely to a rational-valued form

$$Q : P \rightarrow \mathbb{Q}.$$

One has a sublattice $P_{Q,n} \subset P$ given in [45, Section 2.2]. The relations among these sublattices are as in

$$\begin{array}{ccc} P_{Q,n} & \hookrightarrow & P \\ \uparrow & & \uparrow \\ Y_{Q,n} & \hookrightarrow & Y \end{array}$$

In particular,

$$Y_{Q,n} \subseteq P_{Q,n} \cap Y,$$

where equality holds if $Y = Y^{\text{sc}}$, i.e., if G is simply-connected. Furthermore, if $Y = Y^{\text{sc}}$, we have an embedding

$$Y/Y_{Q,n} \hookrightarrow P/P_{Q,n}.$$

One has the extended affine Hecke algebra $\mathcal{H}(P_{Q,n})$ associated with $P_{Q,n}$, and as a \mathbb{C} -isomorphism we get

$$\mathcal{H}(P_{Q,n}) \simeq \mathcal{H}_W \otimes_{\mathbb{C}} \mathbb{C}[P_{Q,n}],$$

where \mathcal{H}_W is the finite Hecke algebra. One has an inclusion of algebras

$$\mathcal{H}_I \subset \mathcal{H}(P_{Q,n}).$$

There is a natural action of $\mathcal{H}(P_{Q,n})$ on $\mathbb{C}[P_{Q,n}]$ via transporting the $\mathcal{H}(P_{Q,n})$ -action from the left hand side of the $\mathbb{C}[P_{Q,n}]$ -module isomorphism

$$\mathcal{H}(P_{Q,n}) \otimes_{\mathcal{H}_W} \mathbb{C} \simeq \mathbb{C}[P_{Q,n}].$$

It is shown in [45, Theorem 3.7] that this action of $\mathcal{H}(P_{Q,n})$ can be extended to the bigger space $\mathbb{C}[P] \supset \mathbb{C}[P_{Q,n}]$. We let $(\pi, \mathbb{C}[P])$ denote this $\mathcal{H}(P_{Q,n})$ -module. In brief, it is constructed as follows:

- (i) Choose a W -stable subset $C \subset P$ such that $0 \in C$ and the quotient

$$P \twoheadrightarrow P/P_{Q,n}$$

is surjective when restricted to C . In [45], an explicit C is given.

(ii) Consider the $\mathcal{H}(P_{Q,n})$ -module

$$N_C := \mathcal{H}(P_{Q,n}) \otimes_{\mathcal{H}_W} V_C,$$

where V_C is the $|C|$ -dimensional space affording the \mathcal{H}_W -action “deformed” from the Weyl action (see [45, Section 3.1]). Depending on certain parameters \mathbf{c} , there is a natural $\mathbb{C}[P]$ -module surjection

$$\Psi_{\mathbf{c}}^{\mathbf{c}} : N_C \twoheadrightarrow \mathbb{C}[P].$$

(iii) By choosing \mathbf{c} properly, it is shown that $\text{Ker}(\Psi_{\mathbf{c}}^{\mathbf{c}})$ is actually $\mathcal{H}(P_{Q,n})$ -stable, and thus one has an $\mathcal{H}(P_{Q,n})$ -module structure on $\mathbb{C}[P]$ by transporting that from N_C ; this is the representation π .

In the notation of [45], the action of the generators of $\mathcal{H}(P_{Q,n})$ on $\mathbb{C}[P]$ are given as follows:

$$\begin{aligned} \pi(T_{\alpha})x^{\lambda} &= (\mathbf{k}_{\alpha} - \mathbf{k}_{\alpha}^{-1})\bar{\nu}_{\alpha}(x^{\lambda}) + \mathbf{p}_{\alpha}(\bar{\lambda})x^{s_{\alpha}(\lambda)}, \\ \pi(x^{\nu})x^{\lambda} &= x^{\lambda+\nu} \end{aligned} \tag{7.1}$$

for $\alpha \in \Delta, \lambda \in P$ and $\nu \in P_{Q,n}$.

By restricting the action of $\mathcal{H}(P_{Q,n})$ to the subalgebra \mathcal{H}_I , we get an \mathcal{H}_I -module still denoted by $(\pi, \mathbb{C}[P])$. For every $z \in P$, it follows from (7.1) that the subspace

$$\mathbb{C}[Y] \cdot x^z \subset \mathbb{C}[P]$$

is stable under the \mathcal{H}_I -action and thus gives an \mathcal{H}_I -submodule $(\pi^z, \mathbb{C}[Y] \cdot x^z)$. Moreover, one has a decomposition

$$\pi = \bigoplus_{z \in P/Y} \pi^z$$

as \mathcal{H}_I -modules.

The representation π^0 has a further decomposition. For each W -orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$, consider the vector space

$$\mathbb{C}[Y]_{\mathcal{O}} := \bigoplus_{y \in \mathcal{O}} \mathbb{C}[Y_{Q,n}] \cdot x^y,$$

which is \mathcal{H}_I -stable by (7.1). We let $(\pi_{\mathcal{O}}^0, \mathbb{C}[Y]_{\mathcal{O}})$ denote this, which gives a decomposition

$$\pi^0 = \bigoplus_{\mathcal{O} \subset \mathcal{X}_{Q,n}} \pi_{\mathcal{O}}^0.$$

Theorem 7.3. *Let \bar{G} be an n -fold cover of a semisimple group G . Assume $\varepsilon = (-1, \varpi)_n = 1$. Then we have an isomorphism*

$$\mathcal{V}_{\mathcal{O}}^I \simeq \pi_{\mathcal{O}}^0$$

of \mathcal{H}_I -modules for every orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$.

Proof. In view of Lemmas 7.1 and 7.2, checking the result essentially amounts to matching the notations used in our paper and those in [45, Theorem 3.7], i.e., (7.1) above. More precisely, the correspondence is given as follows:

- $\mathbb{1}_I * \mathcal{T}_\alpha * \mathbb{1}_I \leftrightarrow \mathbf{k}_i T_\alpha$ for a simple root α ,
- $\mathbb{1}_I * \Theta_{s_{y'}} * \mathbb{1}_I \leftrightarrow x^\nu$ where $y' \leftrightarrow \nu \in Y_{Q,n}$,
- $c(\varphi(y)) \otimes \Theta_{s_y} \leftrightarrow x^\lambda$ where $y \leftrightarrow \lambda \in Y$.

Note that the assumption $\varepsilon = 1$ is a technical assumption, enforced by the relation that the parameters \mathbf{k}_α (essentially the Gauss sums) need to satisfy. ■

One of the applications of the SSV representation π is to provide a more natural proof of the Chinta–Gunnells action [12]. This latter work assumes that $\mu_{2n} \subset F^\times$, which in particular implies that $\varepsilon = 1$. Using our formulas in Lemmas 7.1 and 7.2, the method of Sahi–Stokman–Venkateswaran can be easily adjusted to incorporate the slightly more general case without the assumption $\mu_{2n} \subset F^\times$.

Remark 7.4. It is possible to prove Theorem 5.21 by using Theorem 7.3 and refining several steps in the construction of π in [45], as outlined in (i)–(iii) above. Indeed, if $\mathcal{O} := \mathcal{O}_{\hat{y}} \subset \mathcal{X}_{Q,n}$ is a splitting W -orbit, then one may take its splitting $C_{\mathcal{O}} \subset Y$. Examining the argument in [45] will give $\pi_{\mathcal{O}}^0 \simeq \varepsilon_y \otimes_{\mathcal{H}_{I,y}} \mathcal{H}_I$, where $\mathcal{H}_{I,y} \subset \mathcal{H}_W$ is the subalgebra associated with the parabolic Weyl subgroup $W_{\hat{y}} \subset W$.

7.2. A speculation

It is natural to ask what role π^z plays for general $z \in P$. We give a speculation as follows. Consider the vector space isomorphism

$$\text{sh}_z : \mathbb{C}[Y] \rightarrow \mathbb{C}[Y] \cdot x^z, \quad f \mapsto f \cdot x^z.$$

By transport of structure, one obtains a representation

$$(\text{sh}_z^*(\pi^z), \mathbb{C}[Y])$$

of \mathcal{H}_I .

For $z \in P$, let $\varpi^z = z(\varpi)$ be the element of the adjoint torus.

Conjecture 7.5. *For every $z \in P$, consider the character $({}^z\psi)(u) = \psi(\varpi^{-z}u\varpi^z)$. Then we have an isomorphism*

$$(\text{ind}_{\mu_n U^-}^{\bar{G}} \varepsilon \otimes {}^z\psi)^I \simeq \text{sh}_z^*(\pi^z)$$

of \mathcal{H}_I -modules.

The above holds for $z = 0$. For $z = \rho$ it was also motivated from the following consideration. If $z = \rho$, then ${}^\rho\psi$ has conductor O_F . In this case, work from [20] suggests that (for oasitic covers of simply-connected G for example) one has a decomposition

$$(\text{ind}_{\mu_n U^-}^{\bar{G}} \varepsilon \otimes {}^z\psi)^I = \bigoplus_{\mathcal{O} \subset \mathcal{X}_{Q,n}} \varepsilon_{\mathcal{O}} \otimes_{\mathcal{H}_{I,\mathcal{O}}} \mathcal{H}_I = (\varepsilon_W \cdot \sigma_{[z]}^{\mathcal{X}}) \otimes_{\mathcal{H}_W} \mathcal{H}_I,$$

where \mathcal{O} is taken over the (W, z) -orbits in $\mathcal{X}_{\mathcal{Q}, n}$. Also, $\varepsilon_W \cdot \sigma_{[z]}^{\mathcal{X}}$ is an \mathcal{H}_W -module deformed from the corresponding representation of W . This seems to be compatible with $\text{sh}_z^*(\pi^\tau)$ in view of (7.1), and gives another motivation for Conjecture 7.5.

8. Structure of \mathcal{V}^I via \mathcal{W}^I

The objective of this section is to present another approach to determining the \mathcal{H}_I -module structure of \mathcal{V}^I based on studying the isomorphic module \mathcal{W}^I . (Recall Lemma 5.3.) This approach was suggested to us by the referee, who also generously outlined a proof and brought to our attention the related works [9, 37].

We assume in this section that G is semisimple and almost simple. The main result of this section is Corollary 8.7, which complements our Theorem 5.21. The corollary shows that the splitting hypothesis is not necessary, but the character σ is not explicit. The problem of making σ explicit will be taken up in a future work.

Throughout this section it is convenient to use the following left and right actions of $W_{\text{ex}} = W \rtimes Y$ on Y . Let $y' \in Y$ and $(w, y) \in W \rtimes Y$. Then $W \rtimes Y$ acts on Y on the left via $(w, y) \cdot y' = w \cdot (y + y')$. We also require the right action defined by $y' \cdot (w, y) = (w, -y)^{-1} \cdot y'$. (This is the action that naturally arises from the right action of $W \rtimes Y$ on $W \backslash W \rtimes Y \simeq Y$, where the identification with Y comes from the map $W(1, y) \mapsto y$.)

8.1. The main computation

The objective of this subsection is to compute the action of the pro- p Hecke algebra on the standard basis elements of \mathcal{W}^I .

Lemma 8.1. *Let $\alpha \in \Delta_a$, $y \in Y$, and $\chi \in \text{Hom}(T_\kappa, \mu_n)$. Then*

$$\text{supp}(\phi_y^\chi * \mathcal{T}_\alpha) \subseteq \mu_n I_1^- \overline{w_y s_{-y}} \overline{w_\alpha} I \cup \mu_n I_1^- \overline{w_y s_{-y}} I.$$

Proof. By definition, $\text{supp}(\phi_y^\chi * \mathcal{T}_\alpha) \subseteq \mu_n I_1^- \overline{w_y s_{-y}} I \overline{w_\alpha} I$. But since w_α is a simple reflection and $I_1^- = w_\ell I w_\ell$ we have

$$\mu_n I_1^- \overline{w_y s_{-y}} I \overline{w_\alpha} I \subseteq \mu_n I_1^- \overline{w_y s_{-y}} \overline{w_\alpha} I \cup \mu_n I_1^- \overline{w_y s_{-y}} I,$$

which concludes the proof. ■

In this section we will write

$$w_{\text{aff}} := w_{\alpha^\dagger, 1}, \quad \mathcal{T}_{\text{aff}} := \mathcal{T}_{\alpha^\dagger} = \mathcal{T}_{\overline{w_{\alpha^\dagger}}(\varpi^{-1})}.$$

We compute $\phi_y^\chi * \mathcal{T}_{\text{aff}}$ in detail; the cases of simple linear reflections are similar and will be omitted. By the previous lemma, to compute $\phi_y^\chi * \mathcal{T}_{\text{aff}}(g)$ it suffices to compute at the two points $g = \overline{w_y s_{-y}}$, and $\overline{w_y s_{-y}} \overline{w_{\alpha^\dagger}}(\varpi^{-1})$. The next lemma contains several identities used in the computation of $\phi_y^\chi * \mathcal{T}_{\text{aff}}(g)$.

Lemma 8.2. *Let $y \in Y$ and $u \in O_F$. Let $c \in \{\pm 1\}$ be such that (2.5f) holds for $\alpha = \alpha^\dagger$. Then*

- (1) $\overline{w}_y \mathbf{s}_{-y} \overline{e}_{\alpha^\dagger}(\varpi^{-1}u) = \overline{e}_{w_y \cdot \alpha^\dagger}(\pm \varpi^{-1 - \langle \alpha^\dagger, y \rangle} u) \overline{w}_y \mathbf{s}_{-y}$;
- (2) $\overline{w}_y \mathbf{s}_{-y} \overline{e}_{\alpha^\dagger}(\varpi^{-1}u) = \overline{e}_{-w_y \cdot \alpha^\dagger}(\pm \varpi^{1 + \langle \alpha^\dagger, y \rangle} u^{-1}) \overline{w}_y \mathbf{s}_{-y} w_{\alpha^\dagger}(\varpi^{-1}) h_{\alpha^\dagger}(cu^{-1}) \cdot e_{-\alpha^\dagger}(\varpi u^{-1})(\varpi, -cu)_n^{Q(\alpha^\dagger, \vee)}$;
- (3) $\overline{w}_y \mathbf{s}_{-y} \overline{e}_{-\alpha^\dagger}(-\varpi^{-1}u) = \overline{e}_{-w_y \cdot \alpha^\dagger}(\pm \varpi^{-1 + \langle \alpha^\dagger, y \rangle} u) \overline{w}_y \mathbf{s}_{-y}$;
- (4) $\overline{w}_y \mathbf{s}_{-y} \overline{e}_{-\alpha^\dagger}(-\varpi u) \overline{w}_{\alpha^\dagger}(-\varpi^{-1}) = \overline{e}_{w_y \cdot \alpha^\dagger}(\pm \varpi^{-1 - \langle \alpha^\dagger, y \rangle} u^{-1}) \overline{w}_y \mathbf{s}_{-y} h_{\alpha^\dagger}(u^{-1}) e_{-\alpha^\dagger}(\varpi u^{-1})(\varpi, u)_n^{Q(\alpha^\dagger, \vee)}$.

Proof. Items (1) and (3) follow from (2.5a) and (2.5b). We prove (2); item (4) is similar. We apply the identities

$$\overline{e}_{-\alpha^\dagger}(-\varpi u^{-1}) \overline{e}_{\alpha^\dagger}(\varpi^{-1}u) \overline{e}_{\alpha^\dagger}(-\varpi u^{-1}) = \overline{w}_{-\alpha^\dagger}(-\varpi u^{-1})$$

and $\overline{w}_{-\alpha^\dagger}(-\varpi u^{-1}) = \overline{w}_{\alpha^\dagger}(-c \varpi^{-1}u)$ (relation (2.5f)), and (3) to get

$$\begin{aligned} \overline{w}_y \mathbf{s}_{-y} \overline{e}_{\alpha^\dagger}(\varpi^{-1}u) &= \overline{w}_y \mathbf{s}_{-y} \overline{e}_{-\alpha^\dagger}(\varpi u^{-1}) \overline{w}_{-\alpha^\dagger}(-\varpi u^{-1}) \overline{e}_{-\alpha^\dagger}(\varpi u^{-1}) \\ &= \overline{e}_{-w_y \cdot \alpha^\dagger}(\pm \varpi^{1 + \langle \alpha^\dagger, y \rangle} u^{-1}) \overline{w}_y \mathbf{s}_{-y} \overline{w}_{\alpha^\dagger}(-c \varpi^{-1}u) \overline{e}_{-\alpha^\dagger}(\varpi u^{-1}). \end{aligned}$$

Now the result (2) follows from (2.5g):

$$\overline{w}_{\alpha^\dagger}(-\varpi^{-1}) \overline{w}_{\alpha^\dagger}(-c \varpi^{-1}u) = h_{\alpha^\dagger}(cu^{-1})(\varpi, -cu)_n^{Q(\alpha^\dagger, \vee)}. \quad \blacksquare$$

Now we come to the main computation.

Proposition 8.3. *Let $y \in Y$, $\chi \in \text{Hom}(T_\kappa, \mu_n)$ and $\chi' := \chi \circ w_{\text{aff}}$. Let $c \in \{\pm 1\}$ be such that $\overline{w}_y \overline{e}_{\alpha^\dagger}(u) \overline{w}_y^{-1} = \overline{e}_{w_y \cdot \alpha^\dagger}(cu)$ for all $u \in F$. There exists $\xi \in \mathbb{C}^\times$ such that the following hold:*

(i) *If $\langle -\alpha^\dagger, y \rangle \leq 0$, then*

$$\phi_y^\chi * \mathcal{T}_{\text{aff}} = \xi \phi_{y \cdot w_{\text{aff}}}^{\chi'}$$

(ii) *If $\langle -\alpha^\dagger, y \rangle = 1$, then $w_y \cdot \alpha^\dagger \in \Delta^-$ and*

$$\phi_y^\chi * \mathcal{T}_{\text{aff}} = \chi_{\alpha^\dagger}(c) \epsilon((c, \varpi)^{Q(\alpha^\dagger, \vee)}) \mathbf{g}(\psi_{w_y \cdot \alpha^\dagger}, \chi_{\alpha^\dagger}^{-1} \cdot \epsilon((- , \varpi)_n^{-Q(\alpha^\dagger, \vee)})) \phi_y^{\chi'}$$

(iii) *If $\langle -\alpha^\dagger, y \rangle \geq 2$, then*

$$\phi_y^\chi * \mathcal{T}_{\text{aff}} = \xi q \phi_{y \cdot w_{\text{aff}}}^{\chi'} + \left(\sum_{u \in \kappa^\times} \chi(\overline{h}_{\alpha^\dagger}(u)) \epsilon((u, \varpi)_n^{Q(\alpha^\dagger, \vee)}) \right) \cdot \phi_y^{\chi'}$$

Proof. The initial steps of the computation do not depend on the value of $\langle -\alpha^\dagger, y \rangle$. By definition

$$\phi_y^\chi * \mathcal{T}_{\text{aff}}(g) = \int_{\overline{G}} \phi_y^\chi(gh^{-1}) \mathcal{T}_{\text{aff}}(h) dh. \tag{8.1}$$

In the variable h , the function $\phi_y^\chi(gh^{-1})\mathcal{T}_{\text{aff}}(h)$ is left $\mu_n I$ -invariant. Since $O_F/\varpi O_F \simeq \mu_n I \backslash I \bar{w}_{\alpha^\dagger}(\varpi^{-1})I$ via $u \mapsto \mu_n I \bar{w}_{\alpha^\dagger}(\varpi^{-1})\bar{e}_{-\alpha^\dagger}(\varpi u)$, we have

$$\int_{\bar{G}} \phi_y^\chi(gh^{-1})\mathcal{T}_{\text{aff}}(h) dh = \sum_{u \in O_F/\varpi O_F} \phi_y^\chi(g[\bar{w}_{\alpha^\dagger}(\varpi^{-1})\bar{e}_{-\alpha^\dagger}(\varpi u)]^{-1}). \tag{8.2}$$

By Lemma 8.1, it is enough to compute $\phi_y^\chi * \mathcal{T}_{\text{aff}}(g)$, where $g = \bar{w}_y s_{-y}$ or $\bar{w}_y s_{-y} \bar{w}_{\alpha^\dagger}(\varpi^{-1})$.

First let us consider $g = \bar{w}_y s_{-y} \bar{w}_{\alpha^\dagger}(\varpi^{-1})$. Then from (8.1) and (8.2) we have

$$\begin{aligned} \phi_y^\chi * \mathcal{T}_{\text{aff}}(\bar{w}_y s_{-y} \bar{w}_{\alpha^\dagger}(\varpi^{-1})) &= \sum_{u \in O_F/\varpi O_F} \phi_y^\chi(\bar{w}_y s_{-y} \bar{w}_{\alpha^\dagger}(\varpi^{-1})\bar{e}_{-\alpha^\dagger}(-\varpi u)\bar{w}_{\alpha^\dagger}(-\varpi^{-1})) \\ &= \sum_{u \in O_F/\varpi O_F} \phi_y^\chi(\bar{w}_y s_{-y} \bar{e}_{\alpha^\dagger}(\varpi^{-1}u)). \end{aligned} \tag{8.3}$$

Now the computation breaks up according to the value of $\langle -\alpha^\dagger, y \rangle$.

If $\langle -\alpha^\dagger, y \rangle \leq 0$, then $w_y \cdot \alpha^\dagger \in \Phi^+$ because (w_y, y) is of minimal length in Wy . From Lemma 8.2 (2), if $u \notin \varpi O_F$ then $\bar{w}_y s_{-y} \bar{e}_{\alpha^\dagger}(\varpi^{-1}u) \notin \text{supp}(\phi_y^\chi)$. Thus

$$\sum_{u \in O_F/\varpi O_F} \phi_y^\chi(\bar{w}_y s_{-y} \bar{e}_{\alpha^\dagger}(\varpi^{-1}u)) = 1.$$

If $\langle -\alpha^\dagger, y \rangle \geq 1$, then $w_y \cdot \alpha^\dagger \in \Phi^-$ because (w_y, y) is of minimal length in Wy . So we can apply Lemma 8.2 (1) to get

$$\sum_{u \in O_F/\varpi O_F} \phi_y^\chi(\bar{w}_y s_{-y} \bar{e}_{\alpha^\dagger}(\varpi^{-1}u)) = \sum_{u \in O_F/\varpi O_F} \psi(\bar{e}_{w_y \cdot \alpha^\dagger}(\pm \varpi^{-1-\langle \alpha^\dagger, y \rangle} u)).$$

Furthermore, if $\langle -\alpha^\dagger, y \rangle \geq 2$, then

$$\sum_{u \in O_F/\varpi O_F} \psi(\bar{e}_{w_y \cdot \alpha^\dagger}(\pm \varpi^{-1-\langle \alpha^\dagger, y \rangle} u)) = q.$$

If $\langle -\alpha^\dagger, y \rangle = 1$, then $w_y \cdot \alpha^\dagger \in \Delta^-$ and so

$$\sum_{u \in \mathcal{O}/\varpi \mathcal{O}} \psi(\bar{e}_{w_y \cdot \alpha^\dagger}(\pm \varpi^{-1-\langle \alpha^\dagger, y \rangle} u)) = 0.$$

This concludes the calculation of $\phi_y^\chi * \mathcal{T}_{\text{aff}}(\bar{w}_y s_{-y} \bar{w}_{\alpha^\dagger}(\varpi^{-1}))$.

Now let $g = \bar{w}_y s_{-y}$. The argument is similar to the previous case. By (8.1) and (8.2) we have

$$\phi_y^\chi * \mathcal{T}_{\text{aff}}(\bar{w}_y s_{-y}) = \sum_{u \in O_F/\varpi O_F} \phi_y^\chi(\bar{w}_y s_{-y} \bar{e}_{-\alpha^\dagger}(-\varpi u)\bar{w}_{\alpha^\dagger}(-\varpi^{-1})).$$

Again the computation breaks into cases depending on the value of $\langle -\alpha^\dagger, y \rangle$.

If $\langle -\alpha^\dagger, y \rangle \geq 1$, then $w_y \cdot \alpha^\dagger \in \Phi^-$. We apply Lemma 8.2 (4) to get

$$\begin{aligned} & \sum_{u \in O_F / \varpi O_F} \phi_y^\chi(\overline{w_y} s_{-y} \bar{e}_{-\alpha^\dagger}(-\varpi u) \bar{w}_{\alpha^\dagger}(-\varpi^{-1})) \\ &= \sum_{u \in (O_F / \varpi O_F)^\times} \psi(\bar{e}_{w_y \cdot \alpha^\dagger}(c \varpi^{-1 - \langle \alpha^\dagger, y \rangle} u^{-1})) \chi(h_{\alpha^\dagger}(u^{-1})) \epsilon((\varpi, u)^{\mathcal{Q}(\alpha^\dagger, \vee)}). \end{aligned} \tag{8.4}$$

Note that if $\langle -\alpha^\dagger, y \rangle = 1$, then $w_y \cdot \alpha^\dagger \in \Delta^-$. In this case, the expression simplifies to $\chi_{\alpha^\dagger}(c) \epsilon((c, \varpi)^{\mathcal{Q}(\alpha^\dagger, \vee)}) \mathbf{g}(\psi_{w_y \alpha^\dagger}, \chi_{\alpha^\dagger}^{-1} \cdot \epsilon((-, \varpi))^{-\mathcal{Q}(\alpha^\dagger, \vee)})$.

If $\langle -\alpha^\dagger, y \rangle \leq 0$, then $w_y \cdot \alpha^\dagger \in \Phi^+$. By Lemma 8.2 (3), for any $u \in \mathcal{O}$ we have $\overline{w_y} s_{-y} \bar{e}_{-\alpha^\dagger}(-\varpi u) \bar{w}_{\alpha^\dagger}(-\varpi^{-1}) \notin \text{supp}(\phi_y^\chi)$. Thus

$$\sum_{u \in O_F / \varpi O_F} \phi_y^\chi(\overline{w_y} s_{-y} \bar{e}_{-\alpha^\dagger}(-\varpi u) \bar{w}_{\alpha^\dagger}(-\varpi^{-1})) = 0.$$

This concludes our computation of $\phi_y^\chi * \mathcal{T}_{\text{aff}}$. ■

We remark that the constant ξ should be a root of unity. It encodes the difference between the elements $\overline{w_y} s_{-y} \bar{w}_{\alpha^\dagger}(\varpi^{-1})$ and $\overline{w_y \cdot w_{\text{aff}}} \cdot s_{-y \cdot w_{\text{aff}}}$. For this paper it suffices to know that ξ is nonzero.

Next we state the analogous results in the case of a simple linear reflection.

Lemma 8.4. *Let $y \in Y$, $u \in O_F$, and $\alpha \in \Delta$. Then the following equalities hold:*

- (1) $\overline{w_y} s_{-y} \bar{e}_{-\alpha}(u) = \bar{e}_{-w_y \cdot \alpha}(\pm \varpi^{-\langle -\alpha, y \rangle} u) \overline{w_y} s_{-y}$;
- (2) $\overline{w_y} s_{-y} \bar{e}_{-\alpha}(u) = \bar{e}_{w_y \cdot \alpha}(\pm \varpi^{-\langle \alpha, y \rangle} u^{-1}) \overline{w_y} s_{-y} \bar{w}_\alpha(-1) h_\alpha(u) \bar{e}_\alpha(u^{-1})$;
- (3) $\overline{w_y} s_{-y} \bar{e}_\alpha(-u) = \bar{e}_{w_y \cdot \alpha}(\pm \varpi^{-\langle \alpha, y \rangle} u) \overline{w_y} s_{-y}$;
- (4) $\overline{w_y} s_{-y} \bar{e}_\alpha(-u) \bar{w}_\alpha(-1) = \bar{e}_{-w_y \cdot \alpha}(\pm \varpi^{\langle \alpha, y \rangle} u^{-1}) \overline{w_y} s_{-y} \bar{h}_\alpha(-u) \bar{e}_\alpha(u^{-1})$.

Proof. This is similar to the proof of Lemma 8.2. ■

Proposition 8.5. *Let $y \in Y$ and $\alpha \in \Delta$. Let $\chi \in \text{Hom}(T_\kappa, \mu_n)$ and let $\chi' = \chi \circ w_\alpha$. Let $c \in \{\pm 1\}$ be such that $\overline{w_y} \bar{e}_{-\alpha}(u) \overline{w_y}^{-1} = \bar{e}_{-w_y \cdot \alpha}(cu)$ for all $u \in F$. There exists $\xi \in \mathbb{C}^\times$ such that the following hold:*

(i) *If $\langle \alpha, y \rangle \leq -1$, then*

$$\phi_y^\chi * \mathcal{T}_\alpha = \xi \phi_{y \cdot w_\alpha}^{\chi'}.$$

(ii) *If $\langle \alpha, y \rangle = 0$, then $w_y \cdot \alpha \in \Delta$ and*

$$\phi_y^\chi * \mathcal{T}_\alpha = \chi_\alpha(-c) \mathbf{g}(\psi_{-w_y \cdot \alpha}, \chi_\alpha) \phi_y^{\chi'}.$$

(iii) *If $\langle \alpha, y \rangle \geq 1$, then*

$$\phi_y^\chi * \mathcal{T}_\alpha = \xi q \phi_{y \cdot w_\alpha}^{\chi'} + \left(\sum_{u \in \kappa^\times} \chi_\alpha(u) \right) \phi_y^{\chi'}.$$

Proof. This is similar to the proof of Proposition 8.3. ■

8.2. Structure of \mathcal{W}^I

Theorem 8.6. *Let $y_0 \in Y$ be such that $-y_0 \in \bar{\mathcal{C}}_{Q,n}$, the closure of the modified fundamental alcove associated with \tilde{W}_{aff} . Let $(w, y) \in W \times Y_{Q,n}$. Then*

$$\phi_{y_0}^{\mathbb{1}} * \mathcal{T}_{wy(\varpi^{-1})}^I = C \cdot \phi_{y_0 \cdot (w,y)}^{\mathbb{1}} \tag{8.5}$$

for some $C \in \mathbb{C}^\times$.

Proof. To begin, we view $\mathcal{T}_{wy(\varpi^{-1})}^I$ as an element of the pro- p algebra \mathcal{H} . Factor $\mathcal{T}_{wy(\varpi^{-1})}^I = \mathcal{T}_{s_0} \dots \mathcal{T}_{s_k}$, where \mathcal{T}_{s_0} is an element of \mathcal{H} of length 0 and s_i are simple reflections for $i \neq 0$.

We claim that for every $0 \leq j \leq k$ one has

$$\phi_{y_0}^{\mathbb{1}} \mathcal{T}_{s_0} \dots \mathcal{T}_{s_j} = C_j \phi_{y_0 \cdot s_0 \dots s_j}^{\chi_j},$$

where $C_j \in \mathbb{C}^\times$ and $\chi_j = \varphi(-(0 \cdot s_0 \dots s_j))$. We induct on j . Note that $\phi_{y_0}^{\mathbb{1}} \mathcal{T}_{s_0} \dots \mathcal{T}_{s_j} \neq 0$ because every \mathcal{T}_{s_i} is invertible.

When $j = 0$, the element $s_0 \in W \times Y$ is of length 0 so $\bar{s}_0 \in N_{\bar{G}}(I)$. Thus

$$\text{supp}(\phi_{y_0}^{\mathbb{1}} \mathcal{T}_{s_0}) \subseteq I_1^- \overline{w_{y_0} y_0} (\varpi^{-1}) I \bar{s}_0 I = I_1^- \overline{w_{y_0} y_0} (\varpi^{-1}) \bar{s}_0 I,$$

and the result follows for $j = 0$.

Suppose the result holds for $j - 1$. We prove that it holds for j . It suffices to compute

$$\phi_{y_0 \cdot s_0 \dots s_{j-1}}^{\chi_{j-1}} * \mathcal{T}_{s_j}.$$

Let $(w', y') = s_0 \dots s_{j-1} \in W \times Y$ and $y_1 = y_0 \cdot (w', y') = (w')^{-1} \cdot y_0 + y'$. Note that since s_j is a simple reflection in $W \times Y$, we have

$$\text{supp}(\phi_{y_1}^{\chi_{j-1}} * \mathcal{T}_{s_j}) \subseteq \mu_n I_1^- \overline{w_{y_1} y_1} (\varpi^{-1}) I \cup \mu_n I_1^- \overline{w_{y_1} y_1} (\varpi^{-1}) \bar{w}_\alpha (-1) I. \tag{8.6}$$

There are two cases: $s_j = w_\alpha$, where $\alpha \in \Delta$, or $s_j = w_{\text{aff}}$. We handle the latter case; the case of linear simple reflections is similar.

Suppose $s_j = w_{\text{aff}}$. By Proposition 8.3, the result holds if $\langle -\alpha^\dagger, y_1 \rangle \leq 1$. So assume $\langle -\alpha^\dagger, y_1 \rangle \geq 2$. By Proposition 8.3, the result holds unless $\chi_{j-1} \circ \alpha^{\dagger, \vee} = \epsilon((-, \varpi)^{-\mathcal{Q}(\alpha^{\dagger, \vee})})$ as elements of $\text{Hom}(O_F^\times, \mu_n)$. We claim that

$$\chi_{j-1} \circ \alpha^{\dagger, \vee} \neq \epsilon((-, \varpi)^{-\mathcal{Q}(\alpha^{\dagger, \vee})}).$$

Since $s_0 \dots s_{j-1} w_{\text{aff}}$ is a reduced word in $W \times Y$, we know that

$$\langle \alpha^\dagger, -y' + (w')^{-1} \cdot c \rangle < 1 \tag{8.7}$$

for all $c \in \mathcal{C}$. Since $-y_0 \in \bar{\mathcal{C}}_{Q,n} \subseteq n_{\alpha^\dagger} \bar{\mathcal{C}}$ we may let c approach $\frac{-y_0}{n_{\alpha^\dagger}}$ to get

$$\left\langle \alpha^\dagger, -y' + (w')^{-1} \cdot \left(\frac{-y_0}{n_{\alpha^\dagger}} \right) \right\rangle \leq 1. \tag{8.8}$$

We add the inequalities $\langle \alpha^\dagger, y_1 \rangle \leq -2$ and (8.8) to get $\langle w' \cdot \alpha^\dagger, y_0 \rangle \leq -1$. Consequently, $w' \cdot \alpha^\dagger \in \Phi^+$.

Now the inequalities $\langle \alpha^\dagger, y_1 \rangle \leq -2$ and (8.8), $w' \cdot \frac{\alpha^\dagger}{n_{\alpha^\dagger}} \in \Phi_{Q,n}^+$, and $-y_0 \in \mathcal{C}_{Q,n}$ imply

$$n_{\alpha^\dagger} - 1 \geq \langle w' \cdot \alpha^\dagger, -y_0 \rangle - 1 \geq 1 + \langle \alpha^\dagger, y' \rangle \geq \left\langle w' \cdot \alpha^\dagger, \frac{-y_0}{n_{\alpha^\dagger}} \right\rangle \geq 0. \tag{8.9}$$

Since we are assuming that $\varphi(-y') \circ \alpha^{\dagger,\vee} = \chi_{j-1} \circ \alpha^{\dagger,\vee} = \epsilon((-,\varpi)_n^{-Q(\alpha^{\dagger,\vee})})$, it follows that

$$B(y', \alpha^{\dagger,\vee}) \equiv -Q(\alpha^{\dagger,\vee}) \pmod n.$$

This implies that n_{α^\dagger} divides $1 + \langle \alpha^\dagger, y' \rangle$. So, $1 + \langle \alpha^\dagger, y' \rangle = 0$, by the inequalities (8.9).

Finally, the conditions $1 + \langle \alpha^\dagger, y' \rangle = 0$ and $w' \cdot \alpha^\dagger \in \Phi^+$ contradict inequality (8.7). Therefore, $\chi_{j-1} \circ \alpha^{\dagger,\vee} \neq \epsilon((-,\varpi)^{-Q(\alpha^{\dagger,\vee})})$, and so $\phi_{y_0 \cdot s_0 \dots s_{j-1}}^{X_{j-1}} * \mathcal{T}_{s_j} = C_j \cdot \phi_{y_0 \cdot s_0 \dots s_j}^{X_j}$ for some $C_j \in \mathbb{C}^\times$.

This completes the induction step. ■

Let $\mathcal{O} \subset Y$ be an orbit of \tilde{W}_{ex} under the right action. (Recall that $y' \cdot (w, y) = (w, -y)^{-1} \cdot y'$.) Let $\mathcal{W}_{\mathcal{O}}^I$ be the subspace of \mathcal{W}^I spanned by the elements of the form $\phi_y^{\mathbb{1}}$ where $y \in \mathcal{O}$. By Theorem 8.6, $\mathcal{W}_{\mathcal{O}}^I$ is an \mathcal{H}_I -submodule. Our next objective is to describe the \mathcal{H}_I -module structure of $\mathcal{W}_{\mathcal{O}}^I$. We begin with some preliminaries.

Let $-y_0$ be an element of $\mathcal{O} \cap \bar{\mathcal{C}}_{Q,n}$. Let $\tilde{W}_{\text{ex},y_0}$ be the stabilizer subgroup of y_0 in $\tilde{W}_{\text{ex}} = W \ltimes Y_{Q,n}$ with respect to the right action of \tilde{W}_{ex} on Y . One has $\tilde{W}_{\text{ex}} \simeq \Omega_{Q,n} \ltimes \tilde{W}_{\text{aff}}$. Note that since $\bar{\mathcal{C}}_{Q,n}$ is a fundamental domain with respect to \tilde{W}_{aff} , it follows that $\tilde{W}_{\text{ex},y_0} \simeq (\Omega_{Q,n})_{y_0} \ltimes \tilde{W}_{\text{aff},y_0}$, where $(\Omega_{Q,n})_{y_0} = \Omega_{Q,n} \cap \tilde{W}_{\text{ex},y_0}$ and $\tilde{W}_{\text{aff},y_0} = \tilde{W}_{\text{aff}} \cap \tilde{W}_{\text{ex},y_0}$.

Let $\mathcal{H}_{I,y_0} \subset \mathcal{H}_I$ be the subalgebra of \mathcal{H}_I generated by the elements $\mathcal{T}_{\bar{w}}$, where $w \in \tilde{W}_{\text{ex},y_0}$. By Theorem 8.6, \mathcal{H}_{I,y_0} stabilizes the line $\mathbb{C} \cdot \phi_{y_0}^{\mathbb{1}}$. Thus there is a character $\sigma = \sigma_{y_0} : \mathcal{H}_{I,y_0} \rightarrow \mathbb{C}$ such that $\phi_{y_0}^{\mathbb{1}} * \mathcal{T}_{\bar{w}} = \sigma(\mathcal{T}_{\bar{w}}) \cdot \phi_{y_0}^{\mathbb{1}}$, where $w \in \tilde{W}_{\text{ex},y_0}$.

We note that the set $\{\mathcal{T}_{\bar{w}} : w \in \tilde{W}_{\text{ex},y_0}\}$ is a basis for \mathcal{H}_{I,y_0} . This follows because $\tilde{W}_{\text{aff},y_0}$ is generated by the simple reflections contained in it. (We will make an important remark about this after Corollary 8.7.) Furthermore, if we fix a set $\{w_j : j \in \tilde{W}_{\text{ex},y_0} \setminus \tilde{W}_{\text{ex}}\}$ of minimal representatives of the cosets $\tilde{W}_{\text{ex},y_0} \setminus \tilde{W}_{\text{ex}}$ with respect to the length function on \tilde{W}_{ex} , then it follows that $\{\mathcal{T}_{\bar{w}_j} : j \in \tilde{W}_{\text{ex},y_0} \setminus \tilde{W}_{\text{ex}}\}$ is a left \mathcal{H}_{I,y_0} -basis of \mathcal{H}_I .

Corollary 8.7. *Let $-y_0 \in \bar{\mathcal{C}}_{Q,n}$. We use the notation from the previous four paragraphs where $\mathcal{O} = \mathcal{O}_{y_0}$ is the orbit containing y_0 .*

*The \mathcal{H}_{I,y_0} -module map $\sigma \rightarrow \mathcal{W}^I$ defined by $c \mapsto c\phi_{y_0}^{\mathbb{1}}$ induces an \mathcal{H}_I -module isomorphism $\sigma \otimes_{\mathcal{H}_{I,y_0}} \mathcal{H}_I \rightarrow \mathcal{W}_{\mathcal{O}_{y_0}}^I$ defined by $c \otimes h \mapsto c\phi_{y_0}^{\mathbb{1}} * h$. Combining this with Lemma 5.3 yields an \mathcal{H}_I -module isomorphism $\sigma \otimes_{\mathcal{H}_{I,y_0}} \mathcal{H}_I \rightarrow \mathcal{V}_{\mathcal{O}_{y_0}}^I$ defined by $c \otimes h \mapsto c\phi_{y_0}^{\mathbb{1}} * h$. In particular, the orbit $\mathcal{O}_{\hat{y}_0} \subset \mathcal{X}_{Q,n}$ has the S-property as in Definition 5.24.*

Proof. The first assertion follows directly from Theorem 8.6 and using the fact that $\{\mathcal{T}_{\bar{w}_j} : j \in \tilde{W}_{\text{ex},y_0} \setminus \tilde{W}_{\text{ex}}\}$ is a left \mathcal{H}_{I,y_0} -basis of \mathcal{H}_I . The rest is clear. ■

Remark 8.8. Our justification of the claim that $\{\mathcal{T}_{\bar{w}} : w \in \tilde{W}_{\text{ex}, y_0}\}$ is a basis for \mathcal{H}_{I, y_0} is based on knowing that the element of \mathcal{H}_I corresponding to the simple affine reflection in \tilde{W}_{ex} is of a particular form. For example, when G is simply laced the element should be $C \cdot \mathcal{T}_{\bar{w}_{\alpha^\dagger}(\varpi^{-n} \alpha^\dagger)}$ for some $C \in \mathbb{C}^\times$. This is expected, but we are not aware of any proof in the literature. We have a strategy for proving this using the pro- p algebra \mathcal{H} . The details will appear in a future work.

9. Explicit Whittaker dimensions

The goal of this section and the next one is to apply Theorem 5.21 to compute explicit Whittaker dimensions of some Iwahori-spherical representations. We will eventually focus on irreducible constituents of regular unramified principal series representations in this section and those of unitary unramified principal series representations in the next.

For every lattice $L \subset \mathcal{A}$ closed under the action of W , one has the canonical surjection

$$\eta : L \rtimes W \twoheadrightarrow W. \tag{9.1}$$

Though we might use the same η for different L , there is no risk of confusion from the context. If L is a root lattice and W the associated Weyl group, then we write

$$W_{\text{aff}} = L \rtimes W$$

for the affine Weyl group. We also denote

$$\tilde{W}_{\text{aff}} = L' \rtimes W,$$

where $L' \subset L$ is a modified root lattice (i.e., generated by $\{k_\alpha \alpha : \alpha \text{ a simple root of } L\}$ for some $k_\alpha \in \mathbb{N}$) giving rise to the same Weyl group as L does. Again, the context makes it clear what L' refers to.

9.1. Some permutation representations

Recall from Section 5.4 that for $z \in P$ and any Weyl-invariant sublattice $L \subset Y$ of the same rank as Y , we have the well-defined action $w[y]_z := w(y + z) - z$ for $y \in Y/L$. Recall that this gives a finite-dimensional permutation representation

$$\sigma_{[z]}^{Y/L} : W \rightarrow \text{Perm}(Y/L).$$

It is easy to see that if $z - z' \in P \cap Y$, then

$$\sigma_{[z]}^{Y/L} \simeq \sigma_{[z']}^{Y/L}.$$

Thus, there is a well-defined shifting action of $P/(P \cap Y)$ on the set $\{\sigma_{[z']}^{Y/L} : z' \in P\}$ given by $z \cdot \sigma_{[z']}^{Y/L} := \sigma_{[z+z']}^{Y/L}$ for every $z \in P/(P \cap Y)$. For later purpose, we are interested in the orbit of $\sigma_{[0]}^{Y/L}$ under this action of $P/(P \cap Y)$ and also the associated stabilizer subgroup. A special case is given as follows:

Lemma 9.1. *Assume $Y = Y^{\text{sc}}$ is the root lattice of a root system R with simple roots $\Delta = \{\alpha_i : 1 \leq i \leq r\}$. Let $S \subset Y$ be a sublattice associated to a root system R' with simple roots $\Delta' = \{k_i \alpha_i : \alpha_i \in \Delta'\}$ for some $k_i \in \mathbb{N}$. Then for the above action of P/Y on $\sigma_{[0]}^{Y/S}$, one has*

$$\text{Stab}_{P/Y}(\sigma_{[0]}^{Y/S}) = \frac{P \cap (P(S) + Y)}{Y} \subseteq P/Y,$$

where $P(S) \subset Y \otimes \mathbb{Q}$ denotes the weight lattice of S .

Proof. Write $\sigma_{[z]}$ for $\sigma_{[z]}^{Y/S}$ for simplicity. Consider its character $\chi_{\sigma_{[z]}}$ which is integer-valued and satisfies $\chi_{\sigma_{[z]}}(w) \geq 0$ for every $w \in W$. We observe that if $\chi_{\sigma_{[z]}}(w) \neq 0$, then

$$\chi_{\sigma_{[z]}}(w) = \chi_{\sigma_{[0]}}(w).$$

Indeed, $\chi_{\sigma_{[z]}}(w)$ is counting the size of the set of fixed points of Y/S under the $\sigma_{[z]}$ action. This set of fixed points, if nonempty, is always a torsor over the set of fixed points of Y/S under the $\sigma_{[0]}$ action. This gives the above equality.

Thus, the following are equivalent:

- (i) $\chi_{\sigma_{[z]}}(w) \neq 0$ for every $w \in W$;
- (ii) $\chi_{\sigma_{[z]}}(w) = \chi_{\sigma_{[0]}}(w) \neq 0$ for every $w \in W$;
- (iii) $\sigma_{[z]} \simeq \sigma_{[0]}$.

One has $z \in \text{Stab}_{P/Y}(\sigma_{[0]})$ if and only if

$$W \subseteq \bigcup_{y+z \in Y+z} \eta(\text{Stab}_{S \rtimes W}(y+z)),$$

where $\eta : S \rtimes W \rightarrow W$ is the natural surjection as in (9.1). However, since the right hand side is a union of reflection subgroups of W , and the Coxeter element of W does not lie in any proper reflection subgroup (see [50, Section 5]), the above inclusion holds if and only if $Y + z$ has a special point of the affine Weyl group $S \rtimes W$, i.e., if and only if

$$z \in P(S) + Y.$$

This completes the proof. ■

Specializing to $L = Y_{Q,n}$, we have a permutation representation

$$\sigma_{[z]}^{\mathcal{X}} : W \rightarrow \text{Perm}(\mathcal{X}_{Q,n}).$$

Proposition 9.2. *Let \bar{G}^\vee be a very saturated cover of an almost simple simply-connected group G with $Q(\alpha^\vee) = 1$ for any short coroot α^\vee . Then*

$$\sigma_{[z]}^{\mathcal{X}} \simeq \sigma_{[0]}^{\mathcal{X}}$$

for every $z \in P/Y$. In fact, for every $z \in P$, there exists $y_z \in Y$ such that the bijective map

$$m_z : \mathcal{X}_{Q,n} \rightarrow \mathcal{X}_{Q,n}, \quad y \mapsto y + y_z,$$

is equivariant with respect to $w[-]_0 = w(-)$ and $w[-]_z$ on the domain and codomain respectively. In particular, \mathfrak{m}_z induces a bijection between the W -orbits and (W, z) -orbits in $\mathcal{X}_{Q,n}$.

Proof. For $G = F_4, E_8, G_2$ one has $P/Y = \{0\}$ and thus it suffices to consider G of other types. Since \bar{G} is very saturated, Lemma 6.2 gives

$$Y_{Q,n} = Y_{Q,n} = n \cdot Y$$

and also $\gcd(n, \mathcal{I}_\Delta) = 1$. Note $\mathcal{I}_\Delta = |P/Y|$. Now $P(Y_{Q,n}^{\text{sc}}) = nP$ and it follows from Lemma 9.1 that

$$\text{Stab}_{P/Y}(\sigma_{[0]}) = (P \cap (nP + Y))/Y = P/Y, \tag{9.2}$$

where the last equality follows from the fact that $\gcd(n, \mathcal{I}_\Delta) = 1$. This shows that

$$\sigma_{[z]}^{\mathcal{X}} \simeq \sigma_{[0]}^{\mathcal{X}}.$$

Now for the second assertion, \mathfrak{m}_z (depending on y_z) is equivariant if and only if

$$w[y + y_z]_z = y_z + w(y) \in \mathcal{X}_{Q,n}$$

for all $w \in W$ and $y \in \mathcal{X}_{Q,n}$. This equality is equivalent to

$$w(y_z + z) \equiv y_z + z \pmod{Y_{Q,n}^{\text{sc}}} = nY,$$

i.e., $y_z + z \in P(Y_{Q,n}^{\text{sc}}) = nP$. However, it follows from (9.2) that for every $z \in P$ there always exists $y_z \in Y$ such that $y_z + z \in P(Y_{Q,n}^{\text{sc}})$. This proves the equivariance of \mathfrak{m}_z , from which the last assertion is clear. ■

9.2. z -persistent orbits and covers

We also recall the notion of a persistent cover as follows (see [19, Definition 2.3]). In fact, we introduce a slightly more general version of z -persistency for any element $z \in P$ in the coweight lattice. These z -persistent covers will give a refinement of the chain in (6.2) since they always contain the saturated covers; see (9.3) below. It is expected that z -persistent covers exhibit better behavior if we consider ${}^z\psi$ -Whittaker space of Iwahori-spherical representations.

Consider the sublattices $Y_{Q,n}, Y_{Q,n}^{\text{sc}} \subset Y$ and

$$\mathcal{X}_{Q,n}^{\text{sc}} := Y/Y_{Q,n}^{\text{sc}}, \quad \mathcal{X}_{Q,n} = Y/Y_{Q,n},$$

on which the action $w[-]_z, z \in P/(P \cap Y)$, is well-defined. For any $y \in Y$, let y^\dagger and \hat{y} denote its image in $\mathcal{X}_{Q,n}^{\text{sc}}$ and $\mathcal{X}_{Q,n}$ respectively. With respect to the action $w[-]_z$, we have the (W, z) -orbits $\mathcal{O}_{y^\dagger} \subset \mathcal{X}_{Q,n}^{\text{sc}}$ and $\mathcal{O}_{\hat{y}} \subset \mathcal{X}_{Q,n}$, which are the images of \mathcal{O}_y in the respective quotient spaces.

Definition 9.3. A (W, z) -orbit $\mathcal{O}_y \subset Y$ is called z -persistent if

$$\text{Stab}_W(y^\dagger; \mathcal{X}_{Q,n}^{\text{sc}}) = \text{Stab}_W(\hat{y}; \mathcal{X}_{Q,n}).$$

A covering group \bar{G} is called z -persistent if every (W, z) -orbit \mathcal{O}_y is z -persistent.

If $\mathcal{O}_{\hat{y}} = \mathcal{O}_{\hat{x}} \subset \mathcal{X}_{Q,n}$, then $\mathcal{O}_y \subset Y$ is persistent if and only if $\mathcal{O}_x \subset Y$ is. The proof of this is identical to that of [19, Corollary 2.5]. Thus, checking if a covering group \bar{G} is z -persistent amounts to checking for a finite set of representative orbits of those in $\mathcal{X}_{Q,n}$.

While persistency is a slightly technical condition, we note that a saturated cover is always z -persistent for any $z \in P$. Indeed, if $w[\hat{y}]_z = \hat{y}$, then

$$w[y]_z - y \in Y_{Q,n} \cap Y^{\text{sc}};$$

but if G is saturated, then $w[y]_z - y \in Y_{Q,n}^{\text{sc}}$, i.e., w fixes $y^\dagger \in \mathcal{X}_{Q,n}^{\text{sc}}$. This shows \mathcal{O}_y is z -persistent. Thus, if G is almost simple and simply-connected, then the chain (6.2) is refined to be

$$\{\text{oasitic covers}\} \subseteq \{\text{very saturated covers}\} \subseteq \{\text{saturated covers}\} \subseteq \{z\text{-persistent covers}\}. \tag{9.3}$$

We note that the last three inclusions actually hold for arbitrary G .

Example 9.4. Every Brylinski–Deligne cover of GL_r is saturated and thus z -persistent. On the other hand, the cover $\overline{\text{SL}}_2^{(n)}$ associated with $Q(\alpha) = -1$ is saturated if and only if n is odd. For $n \in 4\mathbb{Z} + 2$, the cover $\overline{\text{SL}}_2^{(n)}$ is 0-persistent but not ρ -persistent. On the other hand, if $4 \mid n$, then $\overline{\text{SL}}_2^{(n)}$ is ρ -persistent but not 0-persistent.

We expect that saturated and z -persistent covers form subclasses of covers whose representation theory is more accessible, especially the part pertaining to Whittaker models. For instance, the ${}^\rho\psi$ -Whittaker dimension of a theta representation of $\overline{\text{SL}}_2^{(n)}$ (for $n \in 4\mathbb{Z} + 2$) depends sensitively on the choice of central characters of $Z(\bar{T})$, since it is not (W, ρ) -persistent. For odd-fold covers of SL_2 , such subtle dependence disappears. For simply-connected G , exhibiting even better properties are the very saturated covers or even the oasitic covers.

9.3. \mathcal{O} -Whittaker space

Recall that we have a decomposition

$$\mathcal{V}^I = \bigoplus_{\mathcal{O} \subset \mathcal{X}_{Q,n}} \mathcal{V}_{\mathcal{O}}^I,$$

where for every splitting W -orbit $\mathcal{O}_{\hat{y}} \subset \mathcal{X}_{Q,n}$ one has

$$\mathcal{V}_{\mathcal{O}_{\hat{y}}}^I \simeq \varepsilon_y \otimes_{\mathcal{H}_{I,y}} \mathcal{H}_I, \tag{9.4}$$

where the stabilizer subgroup $W_{\hat{y}} = W_y \subset W$ is a parabolic subgroup for a certain representative y of \hat{y} .

For an irreducible admissible Iwahori-spherical representation π , we define

$$\text{Wh}_\psi(\pi) := \text{Hom}_{\bar{G}}(\mathcal{V}, \check{\pi}) = \text{Hom}_{\mathcal{H}_I}(\mathcal{V}^I, \check{\pi}^I)$$

and

$$\text{Wh}_\psi(\pi)^\# := \text{Hom}_{\mathcal{H}_I}(\pi^I, (\text{Ind}_{U^-}^{\bar{G}} \psi^{-1})^I),$$

where $\check{\pi}$ denotes the contragredient representation of π . Since the smooth dual of $\text{ind}_{U^-}^{\bar{G}} \psi$ is just $\text{Ind}_{U^-}^{\bar{G}} \psi^{-1}$, we get a canonical isomorphism

$$\iota_\psi : \text{Wh}_\psi(\pi) \rightarrow \text{Wh}_\psi(\pi)^\#$$

of finite-dimensional vector spaces.

Definition 9.5. For every admissible Iwahori-spherical representation π and every orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$, the subspace

$$\text{Wh}_\psi(\pi)_\mathcal{O} := \text{Hom}_{\mathcal{H}_I}(\mathcal{V}_\mathcal{O}^I, \check{\pi}^I) \subset \text{Wh}_\psi(\pi)$$

is called the \mathcal{O} -Whittaker subspace of π .

One has a decomposition

$$\text{Wh}_\psi(\pi) = \bigoplus_{\mathcal{O} \subset \mathcal{X}_{Q,n}} \text{Wh}_\psi(\pi)_\mathcal{O}.$$

If $T \in \text{Hom}_{\bar{G}}(\pi_1, \pi_2)$ is an intertwining operator, then it induces a well-defined \mathbb{C} -homomorphism

$$T_{\psi,\mathcal{O}} : \text{Wh}_\psi(\pi_2)_\mathcal{O} \rightarrow \text{Wh}_\psi(\pi_1)_\mathcal{O}.$$

Lemma 9.6. For every orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$, the \mathcal{H}_I -module $\mathcal{V}_\mathcal{O}^I$ is projective, and hence $\text{Wh}_\psi(-)_\mathcal{O}$ is an exact functor from the category of admissible Iwahori-spherical representations of \bar{G} to the category of finite-dimensional vector spaces.

Proof. Since the functor $\text{Hom}(-, \text{Ind}_{U^-}^{\bar{G}} \psi)$ of taking the Whittaker model is exact, we find that \mathcal{V}^I is a projective \mathcal{H}_I -module. Any of its direct summands $\mathcal{V}_\mathcal{O}^I$ is thus also projective. ■

9.4. A concrete realization

Let $\pi = I(\chi)$ be an unramified genuine principal series representation (see [19, Section 3.1] for definition). Then we expect to have a more concrete description of $\text{Wh}_\psi(I(\chi))_\mathcal{O}$ which arises essentially from composing the Jacquet integral with functionals of $i(\chi)$ (see [22, 33]). We denote by $\mathfrak{f}(\psi)$ the conductor of ψ .

Let $\text{Ftn}(i(\chi))$ be the vector space of functions \mathbf{c} on \bar{T} satisfying

$$\mathbf{c}(\bar{t} \cdot \bar{z}) = \mathbf{c}(\bar{t}) \cdot \chi(\bar{z}), \quad \bar{t} \in \bar{T} \text{ and } \bar{z} \in \bar{A} = Z(\bar{T})\mathbf{T}(O_F).$$

The support of $\mathbf{c} \in \text{Ftn}(i(\chi))$ is a disjoint union of cosets in \bar{T}/\bar{A} . For every $\gamma \in \bar{T}$, let $\mathbf{c}_\gamma \in \text{Ftn}(i(\chi))$ be the unique element satisfying

$$\text{supp}(\mathbf{c}_\gamma) = \gamma \cdot \bar{A} \quad \text{and} \quad \mathbf{c}_\gamma(\gamma) = 1.$$

Clearly, $\mathbf{c}_{\gamma \cdot a} = \chi(a)^{-1} \cdot \mathbf{c}_\gamma$ for every $a \in \bar{A}$. If $\{\gamma_i\} \subset \bar{T}$ is a set of representatives of \bar{T}/\bar{A} , then $\{\mathbf{c}_{\gamma_i}\}$ forms a basis for $\text{Ftn}(i(\chi))$. Let $i(\chi)^\vee$ be the vector space of functionals of $i(\chi)$, which affords the contragredient representation of $i(\chi)$. The set $\{\gamma_i\}$ gives rise to linear functionals $l_{\gamma_i} \in i(\chi)^\vee$ such that $l_{\gamma_i}(\phi_{\gamma_j}) = \delta_{ij}$, where $\phi_{\gamma_j} \in i(\chi)$ is the unique element such that

$$\text{supp}(\phi_{\gamma_j}) = \bar{A} \cdot \gamma_j^{-1} \quad \text{and} \quad \phi_{\gamma_j}(\gamma_j^{-1}) = 1.$$

It is easy to see that for every $\gamma \in \bar{T}$ and $a \in \bar{A}$, one has

$$\phi_{\gamma a} = \chi(a) \cdot \phi_\gamma, \quad l_{\gamma a} = \chi(a)^{-1} \cdot l_\gamma.$$

Moreover, there is a natural isomorphism of vector spaces

$$\text{Ftn}(i(\chi)) \simeq i(\chi)^\vee, \quad \mathbf{c} \mapsto l_{\mathbf{c}} := \sum_{\gamma_i \in \bar{T}/\bar{A}} \mathbf{c}(\gamma_i) \cdot l_{\gamma_i}.$$

It can be checked easily that this isomorphism does not depend on the choice of representatives for \bar{T}/\bar{A} .

For any $z \in Y_{\text{ad}} = P$, consider again

$${}^z\psi(x) = \psi(z^{-1}xz).$$

There is an isomorphism between $i(\chi)^\vee$ and the space $\text{Wh}_{z\psi}(I(\chi))^\sharp$ of $(U^-, ({}^z\psi)^{-1})$ -Whittaker functionals on $I(\chi)$ given by

$$l \mapsto \lambda_l^\sharp$$

with

$$\lambda_l^\sharp : I(\chi) \rightarrow \mathbb{C}, \quad f \mapsto l \circ J_\psi^\sharp(f) \text{ where } J_\psi^\sharp(f) = \int_{U^-} f(u)\psi(u) du \in i(\chi).$$

Here $f \in I(\chi)$ is an $i(\chi)$ -valued function on \bar{G} . For any $\mathbf{c} \in \text{Ftn}(i(\chi))$, write $\lambda_{\mathbf{c}}^\sharp \in \text{Wh}_{z\psi}(I(\chi))^\sharp$ for the $(U^-, ({}^z\psi)^{-1})$ -Whittaker functional of $I(\chi)$ associated to $l_{\mathbf{c}}$. Therefore, $\mathbf{c} \mapsto \lambda_{\mathbf{c}}^\sharp$ gives an isomorphism between $\text{Ftn}(i(\chi))$ and $\text{Wh}_{z\psi}(I(\chi))^\sharp$. For $\gamma \in \bar{T}$, we will write

$$\lambda_\gamma^\sharp := \lambda_{\mathbf{c}_\gamma}^\sharp.$$

Let w_G be the long element in the Weyl group W . To describe the above $\text{Wh}_{z\psi}(I(\chi))^\sharp$ further, we relate it to the $(U, ({}^{w_G z}\psi)^{-1})$ -Whittaker model of $I(\chi)$ which we denote by $\text{Wh}_{w_G z\psi}(I(\chi))^{\text{std}}$ and which appeared more frequently in the literature. More precisely, for

$$w_G ({}^z\psi) = w_G z w_G^{-1} ({}^{w_G}\psi) : U \rightarrow \mathbb{C}^\times$$

we consider the $(U, ({}^{w_G z} \psi)^{-1})$ -functional

$$J_{w_G z \psi}^{\text{std}} : I(\chi) \rightarrow i(\chi), \quad J_{w_G z \psi}^{\text{std}}(f) = \int_U f(\bar{w}_G^{-1}u)({}^{w_G z} \psi)(u) du \in i(\chi).$$

We define $\lambda_l^{\text{std}} := l \circ J_{w_G z \psi}^{\text{std}}$ and also

$$\text{Wh}_{w_G z \psi}(I(\chi))^{\text{std}} = \{\lambda_l^{\text{std}} : l \in i(\chi)^\vee\}.$$

It is clear that

$$J_{z \psi}^\#(f) = J_{w_G z \psi}^{\text{std}}(R_{w_G}(f)), \tag{9.5}$$

where R_g means the right translation action of $I(\chi)$ by $g \in \bar{G}$.

Consider the standard intertwining map

$$T(w, \chi) : I(\chi) \rightarrow I({}^w \chi)$$

given as in McNamara [38, Section 13.7]. The map $T(w, \chi)$ has meromorphic continuation to all χ . One has a commutative diagram

$$\begin{array}{ccc} I(\chi) & \xrightarrow{T(w, \chi)} & I({}^w \chi) \\ R_{\bar{w}_G} \downarrow & & \downarrow R_{\bar{w}_G} \\ I(\chi) & \xrightarrow{T(w, \chi)} & I({}^w \chi) \end{array}$$

which induces the diagram

$$\begin{array}{ccc} \text{Wh}_{z \psi}(I(\chi))^\# & \xleftarrow{T(w, \chi)_{z \psi}} & \text{Wh}_{z \psi}(I({}^w \chi))^\# \\ R_{\bar{w}_G}^* \uparrow & & \uparrow R_{\bar{w}_G}^* \\ \text{Wh}_{w_G z \psi}(I(\chi))^{\text{std}} & \xleftarrow{T(w, \chi)_{w_G z \psi}^{\text{std}}} & \text{Wh}_{w_G z \psi}(I({}^w \chi))^{\text{std}} \end{array}$$

Now for every $(W, w_G(z))$ -orbit $\mathcal{O} \subset \mathcal{X}_{\mathcal{O}, n}$, consider

$$\text{Ftn}(i(\chi))_{\mathcal{O}} = \left\{ \mathbf{c} \in \text{Ftn}(i(\chi)) : \text{supp}(\mathbf{c}) \subset \bigcup_{y \in \mathcal{O}} \mathbf{s}_y \cdot \bar{A} \right\}.$$

This gives via the above natural isomorphisms of vector spaces

$$\text{Ftn}(i(\chi)) \rightarrow i(\chi)^\vee \rightarrow \text{Wh}_{z \psi}(I(\chi))^\#$$

a natural subspace

$$\text{Wh}_{z \psi}(I(\chi))_{\mathcal{O}}^\# \subset \text{Wh}_{z \psi}(I(\chi))^\#.$$

Similarly, one has

$$\text{Wh}_{w_G z \psi}(I(\chi))_{\mathcal{O}}^{\text{std}} \subset \text{Wh}_{w_G z \psi}(I(\chi))^{\text{std}}$$

with

$$\dim \text{Wh}_{z \psi}(I(\chi))_{\mathcal{O}}^\# = \dim \text{Wh}_{w_G z \psi}(I(\chi))_{\mathcal{O}}^{\text{std}} = |\mathcal{O}|. \tag{9.6}$$

As a consequence of (9.5), the restriction

$$R_{w_G}^* : \text{Wh}_{w_G z \psi} (I(\chi))_{\mathcal{O}}^{\text{std}} \rightarrow \text{Wh}_{z \psi} (I(\chi))_{\mathcal{O}}^{\#} \tag{9.7}$$

is a well-defined vector space isomorphism.

We want to show that

$$T(w, \chi)_{z \psi, \mathcal{O}}^{\#} : \text{Wh}_{z \psi} (I(w \chi))_{\mathcal{O}}^{\#} \rightarrow \text{Wh}_{z \psi} (I(\chi))_{\mathcal{O}}^{\#} \tag{9.8}$$

is well-defined for every $\mathcal{O} \subset \mathcal{X}_{Q,n}$. In view of (9.7), it suffices to prove that by restricting $T(w, \chi)_{w_G z \psi}^{\text{std}}$ the map

$$T(w, \chi)_{w_G z \psi, \mathcal{O}}^{\text{std}} : \text{Wh}_{w_G z \psi} (I(w \chi))_{\mathcal{O}}^{\text{std}} \rightarrow \text{Wh}_{w_G z \psi} (I(\chi))_{\mathcal{O}}^{\text{std}} \tag{9.9}$$

is well-defined.

For notational convenience, we write

$$z^* := w_G(z) \quad \text{and} \quad \psi^* = w_G z \psi = z^* (w_G \psi) : U \rightarrow \mathbb{C}^\times.$$

If we choose bases $\{\lambda_{\gamma'}^{w \chi}\}_{\gamma' \in \bar{T}/\bar{A}}$ and $\{\lambda_{\gamma}^{\chi}\}_{\gamma \in \bar{T}/\bar{A}}$ for $\text{Wh}_{\psi^*} (I(w \chi))_{\mathcal{O}}^{\text{std}}$ and $\text{Wh}_{\psi^*} (I(\chi))_{\mathcal{O}}^{\text{std}}$, then $T(w, \chi)_{\psi^*}^{\text{std}}$ is naturally represented by the so-called *scattering matrix*

$$[\tau_{\psi^*}(w, \chi, \gamma', \gamma)]_{\gamma', \gamma \in \bar{T}/\bar{A}}$$

satisfying

$$T(w, \chi)_{\psi^*}^{\text{std}} (\lambda_{\gamma'}^{w \chi}) = \sum_{\gamma \in \bar{T}/\bar{A}} \tau_{\psi^*}(w, \chi, \gamma', \gamma) \cdot \lambda_{\gamma}^{\chi}. \tag{9.10}$$

Furthermore, if one chooses a set $\mathfrak{R} \subset Y$ of representatives of the set $\mathcal{X}_{Q,n}$, it gives a natural basis for the domain and codomain of $T(w, \chi)_{\psi^*}^{\text{std}}$, which is then represented by $[\tau_{\psi^*}(w, \chi, \mathbf{s}_{y'}, \mathbf{s}_y)]_{y', y \in \mathfrak{R}}$. In any case, $\tau_{\psi^*}(w, \chi, \gamma', \gamma)$ satisfies the cocycle relation reflecting the decomposition of the intertwining operator $T(w, \chi)$ into rank-one intertwining operators. Thus, it suffices to determine $\tau_{\psi^*}(w_\alpha, \chi, \gamma', \gamma)$ for a simple reflection w_α , which is given as follows:

- $\tau_{\psi^*}(w_\alpha, \chi, \gamma', \gamma) = \tau_{\psi^*}^1(w_\alpha, \chi, \gamma', \gamma) + \tau_{\psi^*}^2(w_\alpha, \chi, \gamma', \gamma)$ with

$$\tau_{\psi^*}^i(w_\alpha, \chi, \gamma' \cdot a', \gamma \cdot a) = (w_\alpha \chi)^{-1}(a') \cdot \tau_{\psi^*}^i(w_\alpha, \chi, \gamma', \gamma) \cdot \chi(a)$$

for all $a, a' \in \bar{A}$ and $1 \leq i \leq 2$;

- $\tau_{\psi^*}^1(w_\alpha, \chi, \mathbf{s}_{y'}, \mathbf{s}_y) = 0$ unless $y' \equiv y \pmod{Y_{Q,n}}$, and

$$\tau_{\psi^*}^1(w_\alpha, \chi, \mathbf{s}_y, \mathbf{s}_y) = (1 - q^{-1}) \frac{\chi(\bar{h}_\alpha(\varpi^{n\alpha}))^{k_{z,y,\alpha}}}{1 - \chi(\bar{h}_\alpha(\varpi^{n\alpha}))},$$

where $k_{z,y,\alpha} = \lceil \frac{1+(y+z^*,\alpha)}{n\alpha} \rceil$;

- $\tau_{\psi^*}^2(w_\alpha, \chi, \mathbf{s}_{y'}, \mathbf{s}_y) = 0$ unless $y' \equiv w_\alpha[y]_{z^*} \pmod{Y_{Q,n}}$ and

$$\tau_{\psi^*}^2(w_\alpha, \chi, \mathbf{s}_{w_\alpha[y]_{z^*}}, \mathbf{s}_y) = (-1, \varpi)_n^{(y+z^*,\alpha) \cdot D(y,\alpha^\vee)} \cdot \mathbf{g}_{w_G \psi}(\langle y + z^*, \alpha \rangle_Q(\alpha^\vee)).$$

In the above, the Gauss sum $\mathbf{g}_{w_G \psi}(k)$ is given as in [22, Section 3.6], and is essentially $\mathbf{g}(\psi_\alpha, (-, \varpi)_n^k)$ in (5.1). We see that in particular if $z \in P \cap Y$, then

$$\tau_{\psi^*}^i(w_\alpha, \chi, \mathbf{s}_{y'}, \mathbf{s}_y) = \tau_{\psi^*}^i(w_\alpha, \chi, \mathbf{s}_{y'+z^*}, \mathbf{s}_{y+z^*})$$

for $1 \leq i \leq 2$. We also remark that for $z = \rho$ and thus $\mathfrak{f}(\psi^*) = O_F$, the scattering matrix was given in [18, 33, 39]. For $z = 0$, the formula for $\tau_{\psi^*}^i(w_\alpha, \chi, \mathbf{s}_{y'}, \mathbf{s}_y)$ is given in [23, Section 5.3.3].

From (9.10) and the above description of the τ -functions, we see that $T(w, \chi)_{\psi^*, \mathcal{O}}^{\text{std}}$ in (9.9) is indeed well-defined, and so is (9.8). In fact, this is the *raison d’être* for considering the (W, z^*) -orbits in $\mathcal{X}_{Q,n}$.

When specialized to $z = 0$ and thus for W -orbits \mathcal{O} , one has the natural embedding

$$\iota(\chi)_{\psi, \mathcal{O}} : \text{Wh}_\psi(I(\chi))_{\mathcal{O}} \hookrightarrow \text{Wh}_\psi(I(\chi))^\sharp.$$

Conjecture 9.7. *Keep notations as above. Then $\iota(\chi)_{\psi, \mathcal{O}}$ gives an isomorphism*

$$\iota(\chi)_{\psi, \mathcal{O}} : \text{Wh}_\psi(I(\chi))_{\mathcal{O}} \simeq \text{Wh}_\psi(I(\chi))_{\mathcal{O}}^\sharp$$

for every W -orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$. Moreover, $\iota(\chi)_{\psi, \mathcal{O}}$ is equivariant with respect to the two homomorphisms on Whittaker spaces induced from $T(w, \chi)$, i.e.,

$$\iota(w\chi)_{\psi, \mathcal{O}} \circ T(w, \chi)_{\psi, \mathcal{O}} = T(w, \chi)_{\psi, \mathcal{O}}^\sharp \circ \iota(\chi)_{\psi, \mathcal{O}}.$$

The remainder of this paper is devoted to determining $\dim \text{Wh}_\psi(\pi)_{\mathcal{O}}$ for constituents of a regular unramified principal series representation $I(\chi)$ or a unitary unramified principal series representation. We prove analogues of certain conjectural formulas for such genuine Whittaker dimensions in [19, 20] – we deal with the case $\mathfrak{f}(\psi) = \mathfrak{p}$ in this paper while loc. cit. assumes $\mathfrak{f}(\psi) = O_F$. In fact, for unitary unramified $I(\chi)$ we also partially investigate the relation between the two cases for different conductors; see Corollary 10.4.

9.5. Regular unramified $I(\chi)$

Consider an unramified χ satisfying the following:

- χ is regular, that is, its stabilizer subgroup of W is trivial,
- the set $\Phi(\chi) := \{\alpha \in \Phi : \chi(\bar{h}_\alpha(\cdot)^{n_\alpha}) = |\cdot|_F\}$ is a subset of Δ .

Such an exceptional χ (following [33]) gives a regular unramified genuine principal series representation $I(\chi)$ for which there is a decomposition

$$I(\chi)^{\text{ss}} = \bigoplus_{S \subset \Phi(\chi)} \pi_S, \tag{9.11}$$

where the left hand side denotes the semisimplification of $I(\chi)$ and $\pi_S \in \text{Irr}_{\text{gen}}(\bar{G})$. The decomposition is multiplicity-free and the irreducible constituent π_S is characterized by its Jacquet module (see [44] and [19, Section 3]). For example, if $\Phi(\chi) = \Delta$, then

$\pi_\Delta = \Theta(\chi)$ is a theta representation and π_\emptyset is a covering analogue of the Steinberg representation.

Recall that for each $S \subseteq \Phi(\chi) \subseteq \Delta$, there is a representation σ_S (possibly reducible) of the Weyl group. More precisely, for the Jacquet module one has

$$(\pi_S)_U = \bigoplus_{w \in W_S} \delta_B^{1/2} \cdot i(w^{-1}\chi),$$

where $W_S \subset W$ is a union of Kazhdan–Lusztig right cells of W . The representation σ_S is then the direct sum of the cell representations associated to W_S . In a more concrete form (see [19, Corollary 6.5]), one has

$$\sigma_S = \sum_{S': S \subseteq S' \subseteq \Phi(\chi)} (-1)^{|S'-S|} \cdot \text{Ind}_{W(S')}^W \varepsilon_{W(S')}, \tag{9.12}$$

where $W(S') \subset W$ is the Weyl subgroup generated by S' .

Also recall the permutation representation

$$\sigma_{\mathcal{O}}^{\mathcal{X}} : W \rightarrow \text{Perm}(\mathcal{O}), \quad \sigma_{\mathcal{O}}^{\mathcal{X}}(w)(y) = w(y),$$

for each $\mathcal{O} \subset \mathcal{X}_{Q,n}$.

Theorem 9.8. *Let $I(\chi)$ be a regular unramified genuine principal series representation such that $\Phi(\chi) \subset \Delta$. Let $S \subseteq \Phi(\chi)$. Then for every splitting orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$ one has*

$$\dim \text{Wh}_\psi(\pi_S)_\mathcal{O} = \langle \sigma_S, \sigma_{\mathcal{O}}^{\mathcal{X}} \rangle_W.$$

Hence, for \bar{G} such that every orbit \mathcal{O} is splitting (for example, those as in Corollary 6.4 and Example 6.5) one has

$$\dim \text{Wh}_\psi(\pi_S) = \langle \sigma_S, \sigma^{\mathcal{X}} \rangle_W.$$

Proof. The idea is the same as in [19] where $\mathfrak{f}(\psi) = O_F$. Let $\mathcal{O} = \mathcal{O}_{\mathfrak{y}}$ be a splitting orbit with $W_{\mathfrak{y}} \subset W$ a parabolic subgroup.

First, we consider the case when $S = \Phi(\chi) = \Delta$, and thus $\pi_\Delta = \Theta(\chi)$ is a theta representation of \bar{G} . Here χ is an exceptional character and thus $\Theta(\chi)$ is the Langlands quotient of $I(\chi)$, and is also the image of the intertwining operator $T(w_G, \chi)$, where $w_G \in W$ is the longest Weyl element. Its contragredient $\check{\pi}_\Delta$ is the irreducible subrepresentation of $I(\chi^{-1})$, and is also the theta representation associated to the exceptional genuine character ${}^{w_G}(\chi^{-1})$ of $Z(\bar{T})$. In any case, $\check{\pi}_\Delta$ is the unique unramified representation of $I({}^{w_G}(\chi^{-1}))$, and we find that

$$(\check{\pi}_\Delta)^I |_{\mathcal{H}_{\bar{W}_{\text{aff}}}}$$

is the trivial representation. That is, assuming \bar{W}_{aff} is generated by $\{\check{\alpha}_0^\vee\} \cup \{\check{\alpha}_i^\vee : 1 \leq i \leq r\}$,

then each $T_{\tilde{\alpha}_i^\vee} \in \mathcal{H}_I$, $0 \leq i \leq r$, acts on $(\check{\pi}_\Delta)^I$ by q . This gives

$$\begin{aligned} \dim \text{Wh}_\psi(\pi_\Delta)_\mathcal{O} &= \dim \text{Hom}_{\mathcal{H}_I}(\mathcal{V}_\mathcal{O}^I, (\check{\pi}_\Delta)^I) \\ &= \dim \text{Hom}_{\mathcal{H}_I}(\varepsilon \otimes_{\mathcal{H}_{W_{\hat{y}}}} \mathcal{H}_I, (\check{\pi}_\Delta)^I) \quad \text{by (9.4)} \\ &= \dim \text{Hom}_{\mathcal{H}_{W_{\hat{y}}}}(\varepsilon, (\check{\pi}_\Delta)^I|_{\mathcal{H}_{W_{\hat{y}}}}) \\ &= \begin{cases} 1 & \text{if } W_{\hat{y}} = \{1\}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This shows that

$$\dim \text{Wh}_\psi(\pi_\Delta)_\mathcal{O} = \langle \varepsilon_W, \sigma_\mathcal{O}^\mathcal{X} \rangle_W = \begin{cases} 1 & \text{if } \mathcal{O} \text{ is a free } W\text{-orbit,} \\ 0 & \text{otherwise.} \end{cases} \tag{9.13}$$

Second, assume in general $S \subseteq \Phi(\chi) \subseteq \Delta$. Denote by

$$\pi(w, \chi) \subset I(w, \chi)$$

the image of the intertwining operator $T(w, \chi)$. We have

$$\pi_S = \sum_{S': S \subseteq S' \subseteq \Phi(\chi)} (-1)^{|S' - S|} \cdot \pi(w_{S'}, \chi) \in \mathcal{R}(\text{Irr}_{\text{gen}}(\bar{G})),$$

where $w_{S'}$ means the longest element in the Weyl subgroup $W(S')$ generated by S' . Since $\text{Wh}_\psi(-)_\mathcal{O}$ is an exact functor, we have

$$\dim \text{Wh}_\psi(\pi_S)_\mathcal{O} = \sum_{S': S \subseteq S' \subseteq \Phi(\chi)} (-1)^{|S' - S|} \cdot \dim \text{Wh}_\psi(\pi(w_{S'}, \chi))_\mathcal{O}.$$

However, since $\pi(w_{S'}, \chi)$ is equal to the full representation parabolically induced from the theta representation $\pi_{S'}^{M_{S'}}$ of the Levi subgroup $M_{S'}$, we have

$$\text{Wh}_\psi(\pi(w_{S'}, \chi))_\mathcal{O} = \langle \varepsilon_{W(S')}, \sigma_\mathcal{O}^\mathcal{X} \rangle_{W(S')} = \langle \text{Ind}_{W(S')}^W \varepsilon_{W(S')}, \sigma_\mathcal{O}^\mathcal{X} \rangle_W,$$

where the first equality follows from (9.13). In view of (9.12), we get

$$\text{Wh}_\psi(\pi_S)_\mathcal{O} = \langle \sigma_S, \sigma_\mathcal{O}^\mathcal{X} \rangle_W$$

for every splitting orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$. This completes the proof. ■

We can verify for $\mathfrak{f}(\psi) = \mathfrak{p}$ an analogue of [19, Conjecture 6.9 (i)].

Corollary 9.9. *Let $\Theta(\chi) = \pi_\Delta$ be an unramified theta representation associated to χ with $\Phi(\chi) = \Delta$. For every splitting orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$, one has*

$$\dim \text{Wh}_\psi(\pi)_\mathcal{O} \geq \dim \text{Wh}_\psi(\Theta(\chi))_\mathcal{O}$$

for every irreducible Iwahori-spherical representation π of \bar{G} . In particular, if every orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$ is splitting, then $\dim \text{Wh}_\psi(\pi) \geq \dim \text{Wh}_\psi(\Theta(\chi))$ for every such π as well.

Proof. Let π be any Iwahori-spherical representation of \bar{G} . From (9.13), we see that $\dim \text{Wh}_\psi(\Theta(\chi))_\mathcal{O} = 1$ if and only if $\mathcal{O} \subset \mathcal{X}_{Q,n}$ is a free W -orbit. In this case,

$$\mathcal{V}_\mathcal{O}^I \simeq \mathcal{H}_I$$

as \mathcal{H}_I -modules. This gives

$$\dim \text{Wh}_\psi(\pi)_\mathcal{O} = \dim \text{Hom}_{\mathcal{H}_I}(\mathcal{H}_I, \check{\pi}^I) \geq 1$$

and thus the result follows. ■

Corollary 9.9 in particular applies to Kazhdan–Patterson covers and Savin covers by Example 6.5. In fact, for $G = \text{GL}_r$, we expect the inequality in Corollary 9.9 holds for every irreducible $\pi \in \text{Irr}_{\text{gen}}(\bar{G})$ (not necessarily Iwahori-spherical), which however may fail for general \bar{G} . For more discussion, see [19, p. 339].

Remark 9.10. First, retaining the notation $\pi(w, \chi) \subset I(w\chi)$ as in the proof of Theorem 9.8, we can define the space $\text{Wh}_\psi(\pi(w, \chi))_\mathcal{O}^\sharp$ as the image of $T(w, \chi)_{\psi, \mathcal{O}}^\sharp$. If we assume Conjecture 9.7, then $\text{Wh}_\psi(\pi(w, \chi))_\mathcal{O} \simeq \text{Wh}_\psi(\pi(w, \chi))_\mathcal{O}^\sharp$ for every orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$; in particular [19, Conjecture 4.7] holds. In this case, we have

$$\dim \text{Wh}_\psi(\pi_S)_\mathcal{O} = \dim \text{Wh}_\psi(\pi_S)_\mathcal{O}^\sharp = \langle \sigma_S, \sigma_\mathcal{O}^\mathcal{X} \rangle_W,$$

where the second equality follows from [19, Theorem 6.6]. We note that the proof in [19] actually applies to all 0-persistent covers, and thus we expect that Theorem 9.8 holds for all orbits $\mathcal{O} \subset \mathcal{X}_{Q,n}$ as long as \bar{G} is a 0-persistent cover.

Second, if we consider an orbit \mathcal{O} with the S-property (see Definition 5.24) and assume further that $\mu_{\hat{y}}|_{\mathcal{H}_{\bar{W}_{\text{aff},y}}}$ is nontrivial, then the proof of Theorem 9.8 applies in the same way for such \mathcal{O} . In view of the first remark above, it is plausible that for 0-persistent covers every orbit satisfies these two assumptions, i.e., the S-property and nontriviality of $\mu_{\hat{y}}|_{\mathcal{H}_{\bar{W}_{\text{aff},y}}}$.

10. Unitary unramified genuine principal series representations

In this section, we assume that G is almost simple and simply-connected. Let \bar{G} be the n -fold cover of G associated with

$$Q(\alpha^\vee) = 1$$

for any short coroot α^\vee . The general case of $Q(\alpha^\vee)$ imposes no extra difficulties, except for notational complications.

10.1. R-group and decomposition of $I(\chi)$

Consider a unitary unramified principal series representation $I(\chi)$. Let $R_\chi \subset W_\chi$ be the R-group associated to $I(\chi)$ satisfying

$$\mathbb{C}[R_\chi] \simeq \text{End}(I(\chi)).$$

One has a natural correspondence between $\text{Irr}(R_\chi)$ and the constituents $\Pi(\chi)$ of $I(\chi)$:

$$\text{Irr}(R_\chi) \leftrightarrow \Pi(\chi), \quad \sigma \leftrightarrow \pi_\sigma,$$

which is normalized so that π_1 is the unique unramified constituent of $I(\chi)$. It is known that R_χ is abelian (see [35]), and therefore the decomposition of $I(\chi)$ is multiplicity-free. The nontrivial R_χ is given in Table 3.

	A_r	B_r	C_r	$D_r, r \text{ even}$	$D_r, r \text{ odd}$	E_6	E_7
R_χ	$\mathbb{Z}/d\mathbb{Z}, d (r+1)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Tab. 3. Nontrivial R_χ

For every $w \in R_\chi$, we have a well-defined isomorphism

$$\mathcal{A}(w, \chi) = \gamma(w, \chi) \cdot T(w, \chi) : I(\chi) \rightarrow I(\chi),$$

where $\gamma(w, \chi)$ is the gamma-factor associated with w and χ , i.e.,

$$\gamma(w, \chi)^{-1} = \prod_{\alpha \in \Phi_w} \frac{1 - q^{-1}\chi_\alpha^{-1}}{1 - \chi_\alpha}.$$

Here $\Phi_w := \{\alpha : \alpha > 0, w(\alpha) < 0\}$, and for every $\alpha \in \Phi$ we write

$$\chi_\alpha := \chi(\bar{h}_\alpha(\varpi^{n_\alpha})).$$

(Note that this usage of χ_α is similar to but not the same as in Section 5.1. For the remainder of the paper this new usage is in effect.) More explicitly, one has

$$\mathcal{A}(w, \chi)|_{\pi_\sigma} = \sigma(w) \cdot \text{id}$$

for every irreducible constituent $\pi_\sigma \subset I(\chi)$. It induces an isomorphism

$$\mathcal{A}(w, \chi)_\psi : \text{Wh}_\psi(I(\chi)) \rightarrow \text{Wh}_\psi(I(\chi)),$$

where $\dim \text{Wh}_\psi(I(\chi)) = |\mathcal{X}_{Q,n}|$. This gives an $|\mathcal{X}_{Q,n}|$ -dimensional representation

$$\sigma^{\text{Wh}_\psi} : R_\chi \rightarrow \text{GL}(\text{Wh}_\psi(I(\chi))), \quad \sigma^{\text{Wh}_\psi}(w) := \mathcal{A}(w, \chi)_\psi.$$

Indeed, the proof in [20] for $\sigma^{\text{Wh}_{wG}^{\text{std}}}$ afforded by $\text{Wh}_{wG}^{\rho,\psi}(I(\chi))^{\text{std}}$ relies only on the fact that R_χ is abelian, thus it applies to $\text{Wh}_\psi(I(\chi))$ here.

For every W -orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$, we have from restriction

$$\mathcal{A}(w, \chi)_{\psi,\mathcal{O}} : \text{Wh}_\psi(I(w\chi))_{\mathcal{O}} \rightarrow \text{Wh}_\psi(I(\chi))_{\mathcal{O}}$$

This gives rise to the following:

- a subrepresentation

$$\sigma_{\mathcal{O}}^{\text{Wh}\psi} : R_{\chi} \rightarrow \text{GL}(\text{Wh}_{\psi}(I(\chi))_{\mathcal{O}}), \quad \sigma_{\mathcal{O}}^{\text{Wh}\psi}(w) := \mathcal{A}(w, \chi)_{\psi, \mathcal{O}},$$

such that

$$\sigma^{\text{Wh}\psi} = \bigoplus_{\mathcal{O} \subset \mathcal{X}_{Q,n}} \sigma_{\mathcal{O}}^{\text{Wh}\psi};$$

- for every $\pi_{\sigma} \subset I(\chi)$ the decomposition

$$\text{Wh}_{\psi}(\pi_{\sigma}) = \bigoplus_{\mathcal{O} \subset \mathcal{X}_{Q,n}} \text{Wh}_{\psi}(\pi_{\sigma})_{\mathcal{O}},$$

where $\text{Wh}_{\psi}(\pi_{\sigma})_{\mathcal{O}}$ is isomorphic to the subspace of $\text{Wh}_{\psi}(I(\chi))_{\mathcal{O}}$ consisting of Whittaker functionals of $I(\chi)$ restricted to π_{σ} .

It follows from Fourier inversion (see [20, Theorem 5.6]) that for every $\sigma \in \text{Irr}(R_{\chi})$ we have

$$\dim \text{Wh}_{\psi}(\pi_{\sigma})_{\mathcal{O}_y} = \langle \sigma, \sigma_{\mathcal{O}_y}^{\text{Wh}\psi} \rangle_{R_{\chi}}$$

for every orbit $\mathcal{O}_y \subset \mathcal{X}_{Q,n}$. We thus get

$$\dim \text{Wh}_{\psi}(\pi_{\sigma}) = \langle \sigma, \sigma^{\text{Wh}\psi} \rangle_{R_{\chi}}$$

for every $\pi_{\sigma} \subset I(\chi)$. The representation $\sigma^{\text{Wh}\psi}$ is in general difficult to compute. Indeed,

$$\sigma_{\mathcal{O}_y}^{\text{Wh}\psi}(w) = \gamma(w, \chi)^{-1} \cdot \text{Tr}(T(w, \chi)_{\psi, \mathcal{O}}),$$

and $T(w, \chi)_{\psi, \mathcal{O}}$ is not easily computable.

If \bar{G} is a very saturated cover of G , then it was speculated (see [20, 21]) that

$$\sigma^{\text{Wh}_{\bar{w}}^{\text{std}} G^{\rho} \psi} \simeq \sigma_{[-\rho]}^{\mathcal{X}},$$

where the right hand side is more accessible for computation. The goal of this section is to prove an analogue of that equality for ψ (see Theorem 10.2 below). Moreover, as a consequence of our discussion, we also prove a result on the variation of Whittaker dimensions of representations inside the same L -packet for a unitary unramified representation; see Corollary 10.4. This last result was also conjectured in [23, Conjecture 5.7].

10.2. Explicit Whittaker dimension of π_{σ}

Henceforth, we assume that G is almost simple and simply-connected, and that $\bar{G}^{(n)}$ is a very saturated cover of G with $Q(\alpha^{\vee}) = 1$ for any short simple coroot α^{\vee} . In this case we have

$$Y_{Q,n} = Y_{Q,n}^{\text{sc}} = n \cdot Y^{\text{sc}} \quad \text{and} \quad \bar{G}^{\vee} \simeq G^{\vee}.$$

The modified affine Weyl group is

$$\tilde{W}_{\text{aff}} = Y_{Q,n}^{\text{sc}} \rtimes W = (nY) \rtimes W,$$

which acts on $Y \subset Y \otimes \mathbb{R}$ naturally. For any $y \in Y$, we have

$$\tilde{W}_{\text{aff},y} \subset \tilde{W}_{\text{aff}},$$

the stabilizer subgroup of y .

For every root α and $k \in \mathbb{Z}$, we have

$$w_{\alpha,k} = (k\alpha^\vee, w_\alpha) \in W_{\text{aff}} = Y \rtimes W,$$

where $w_\alpha(y) = y - \langle y, \alpha \rangle \alpha^\vee$ the usual reflection. Similarly, we write

$$\tilde{w}_{\alpha,k} = (k\alpha^\vee_{Q,n}, w_\alpha) = (k \cdot n\alpha^\vee, w_\alpha) \in \tilde{W}_{\text{aff}} = Y_{Q,n} \rtimes W.$$

We have

$$w_\alpha = \tilde{w}_{\alpha,0} = w_{\alpha,0} \quad \text{for all } \alpha \in \Phi.$$

Let $\alpha^\dagger \in \Phi^+$ be the highest root of Φ . Then $\tilde{\alpha}^\dagger = \alpha^\dagger/n \in \Phi_{Q,n} = \Phi/n$ is the highest root of the modified root system. Recall the set

$$S_{\text{aff}} = \{\tilde{w}_{\alpha^\dagger,1}\} \cup \{w_{\alpha_i} : 1 \leq i \leq r\}$$

of generators of \tilde{W}_{aff} . For every $y \in Y$ there exists a proper subset $S_{\text{aff},y} \subsetneq S_{\text{aff}}$ such that

$$\tilde{W}_{\text{aff},y} \simeq W(S_{\text{aff},y}),$$

where the right hand side denotes the subgroup of \tilde{W}_{aff} generated by $S_{\text{aff},y}$. The surjective map $\eta : \tilde{W}_{\text{aff}} \twoheadrightarrow W$ restricts to give an isomorphism

$$\tilde{W}_{\text{aff},y} \simeq \eta(\tilde{W}_{\text{aff},y}).$$

If we set $\alpha_0 := -\alpha^\dagger$ and define as usual

$$\Delta_{\text{aff}} = \{\alpha_0\} \cup \Delta,$$

then it is clear that

$$\eta(\tilde{W}_{\text{aff},y}) = \langle w_\alpha : \alpha \in \Delta_{\text{aff},y} \rangle \subset W$$

for a subset $\Delta_{\text{aff},y} \subsetneq \Delta_{\text{aff}}$.

The following result will play an important role in the proof of Theorem 10.2 below.

Proposition 10.1. *Let $\bar{G}^{(n)}$ be a very saturated cover of an almost simple simply-connected group G such that $Q(\alpha^\vee) = 1$ for any short simple coroot α^\vee . Let $y \in Y$. Assume $\alpha_0 \in \Delta_{\text{aff},y}$. Then*

$$\eta(\tilde{W}_{\text{aff},y}) \cap R_\chi = \{1\}$$

for any unitary unramified genuine character χ of $Z(\bar{T})$.

Proof. We verify this by a case by case checking. We will use the explicit representatives of the R-group elements given by C. Keys in [34, Section 3].

First, for type A_r , i.e., $G = \text{SL}_{r+1}$, the cover $\overline{\text{SL}}_{r+1}^{(n)}$ is very saturated if and only if

$$\text{gcd}(n, r + 1) = 1.$$

Identifying $W \simeq S_{r+1}$, we know that R_χ is W -conjugate to $\mathbb{Z}/d\mathbb{Z}$ generated by

$$w := (1, 2, \dots, d)(d + 1, d + 2, \dots, 2d) \dots ((k - 1)d + 1, \dots, r + 1)$$

for some d with $r + 1 = dk$ (see [34, Section 3]). Note that w is the Coxeter element of the parabolic Weyl subgroup $\prod_{i=1}^k S_d \subset W$. Since a conjugate of w lies in $W(\Delta_{\text{aff},y})$, it follows that the set $\Delta_{\text{aff},y}$ contains k connected components of the extended Dynkin diagram, each of which is of size at least $d - 1$ and one of which contains α_0 . Writing $y = \sum_{i=1}^{r+1} y_i e_i$ with $y_i \in \mathbb{Z}$ and $\sum_i y_i = 0$, together with the two equalities

$$\langle y, \tilde{\alpha}^\dagger \rangle = (y_1 - y_{r+1})/n = 1 \quad \text{and} \quad \sum_{i=1}^{r+1} y_i = 0,$$

this shows that there exist distinct $1 \leq i_j \leq r + 1$ with $1 \leq j \leq k$ such that

$$d \cdot \left(\sum_{j=1}^k y_{i_j} \right) = n.$$

Since $\text{gcd}(r + 1, n) = 1$ and $d|(r + 1)$, it follows that $d = 1$ and thus $w = 1$.

Second, for type B_r , the cover $\overline{\text{Spin}}_{2r+1}^{(n)}$ is very saturated if and only if n is odd. Write

$$y = \sum_{i=1}^r y_i e_i \in Y \quad \text{with } y_i \in \mathbb{Z} \text{ and } 2 \mid \sum_i y_i.$$

Depending on the parity of r , we have the following two cases of R_χ :

– If r is even, then $R_\chi \simeq \mathbb{Z}/2\mathbb{Z}$ is W -conjugate to the group generated by

$$w = (12)(34) \dots (r - 1, r) = w_{\alpha_1} w_{\alpha_3} \dots w_{\alpha_{r-1}}$$

(see [34, Section 3]). Suppose $w \in \eta(\tilde{W}_{\text{aff},y})$ with $\alpha_0 \in \Delta_{\text{aff},y}$. Now if $\alpha_1 \in \Delta_{\text{aff},y}$, then the equalities

$$\langle y, \tilde{\alpha}^\dagger \rangle = 1, \quad \langle y, \alpha_1 \rangle = 0$$

give $y_1 = y_2 = n/2$, which is a contradiction since n is odd. Thus, we see that $\alpha_1 \notin \Delta_{\text{aff},y}$ and this gives

$$\{\alpha_0, \alpha_3, \alpha_5, \dots, \alpha_{r-1}\} \subset \Delta_{\text{aff},y}.$$

(In fact, the above inclusion also follows from [41].) It then follows that

$$y_1 + y_2 = n \quad \text{and} \quad y_i = y_{i+1} \text{ for all odd } i \in [3, r - 1].$$

However, this contradicts the fact that $\sum_i y_i$ is even.

– If r is odd, then $R_\chi \simeq \mathbb{Z}/2\mathbb{Z}$ and R_χ is W -conjugate to the group generated by

$$w = w_{\alpha_1} w_{\alpha_3} \cdots w_{\alpha_{r-2}} w_{\alpha_r}.$$

The argument is similar to the case above and we have $w \notin \eta(\tilde{W}_{\text{aff},y})$.

Third, covers of G of type C_r are in fact simpler. More precisely, the cover $\overline{\text{Sp}}_{2r}^{(n)}$ is very saturated if and only if n is odd. We claim that there is no $y \in Y$ such that $\alpha_0 \in \Delta_{\text{aff},y}$. Indeed, in the standard coordinates, one has

$$\tilde{\alpha}^\dagger = 2e_1/n.$$

Thus, the equality $\langle y, \tilde{\alpha}^\dagger \rangle = 1$ has no solution for $y \in Y$. Hence, the desired equality holds vacuously.

Fourth, for type D_r , the cover $\overline{\text{Spin}}_{2r}^{(n)}$ is very saturated if and only if n is odd. Using the Bourbaki notations, we could write every $y \in Y$ as

$$y = \sum_{i=1}^r y_i e_i \quad \text{with } y_i \in \mathbb{Z} \text{ and } 2 \mid \sum_i y_i.$$

We have $\alpha_0 = -\alpha^\dagger = -(e_1 + e_2)$. Now according to the parity of r and the possible R_χ , we have the following cases:

– r is odd and $R_\chi \simeq \mathbb{Z}/2\mathbb{Z}$ is generated by $w := w_{\alpha_{r-1}} w_{\alpha_r}$. If w is W -conjugate to an element in $\eta(\tilde{W}_{\text{aff},y})$, then it is easy to see (cf. [14] as well) that

$$\alpha_1 \in \Delta_{\text{aff},y}.$$

Note that $\alpha_0 \in \Delta_{\text{aff},y}$ by assumption. These two imply that $y_1 = y_2 = n/2$, which is a contradiction.

– r is odd and $R_\chi \simeq \mathbb{Z}/4\mathbb{Z}$ is generated by

$$w := w_{\alpha_1} w_{\alpha_3} \cdots w_{\alpha_{r-2}} \cdot w_{\alpha_{r-1}} w_{\alpha_r}.$$

If w is W -conjugate to an element in $\eta(\tilde{W}_{\text{aff},y})$, then we get (see [14])

$$\{\alpha_0, \alpha_1, \alpha_2, \alpha_4, \alpha_6, \dots, \alpha_{r-5}, \alpha_{r-3}, \alpha_{r-1}\} \subset \Delta_{\text{aff},y}$$

or $\{\alpha_0, \alpha_1, \alpha_2, \alpha_4, \alpha_6, \dots, \alpha_{r-5}, \alpha_{r-3}, \alpha_{r-1}\} \subset \Delta_{\text{aff},y}$. In either case, we have $\{\alpha_0, \alpha_1\} \subset \Delta_{\text{aff},y}$, which gives again $y_1 = y_2 = n/2$ and is a contradiction.

– r is even and any nontrivial R_χ is isomorphic to either $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. If $R_\chi \simeq \mathbb{Z}/2\mathbb{Z}$, then R_χ is generated either by $w_{\alpha_{r-1}} w_{\alpha_r}$ or by

$$w := w_{\alpha_1} w_{\alpha_3} w_{\alpha_5} \cdots w_{\alpha_{r-3}} w_{\alpha_{r-1}}.$$

If $R_\chi \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then

$$R_\chi \simeq \langle w, w_{\alpha_{r-1}} w_{\alpha_r} \rangle.$$

In either case, the argument is similar to the case when r is odd, and there exists no $y \in Y$ satisfying both $-\alpha_0 \in \Delta_{\text{aff},y}$ and $\eta(\tilde{W}_{\text{aff},y}) \cap R_\chi \neq \{1\}$.

Fifth, a cover of $\bar{E}_6^{(n)}$ is very saturated if and only if $3 \nmid n$. The nontrivial $R_\chi \simeq \mathbb{Z}/3\mathbb{Z}$ is generated by

$$w := w_1 w_3 \cdot w_6 w_5$$

in the notations of Bourbaki. If $\eta(\tilde{W}_{\text{aff},y}) \cap R_\chi \neq \{1\}$, then (see [14, Theorem 3.4]) we have either

$$\{\alpha_0, \alpha_2, \alpha_5, \alpha_6\} \subset \Delta_{\text{aff},y} \quad \text{or} \quad \{\alpha_0, \alpha_2, \alpha_1, \alpha_3\} \subset \Delta_{\text{aff},y}.$$

In the first case, one has

$$\langle y, \alpha^\dagger \rangle = n \quad \text{and} \quad \langle y, \alpha_i \rangle = 0 \text{ for } i = 2, 5, 6.$$

In terms of the coordinates e_i , we can write $y = \sum_{i=1}^8 y_i e_i$ with $y_i \in \mathbb{Z}/2$ and $y_6 = y_7 = -y_8$. Then the above conditions give

$$3(y_3 + y_8) = 2n$$

and in particular $3 \mid n$, which is a contradiction. The second case is similar, and thus $\eta(\tilde{W}_{\text{aff},y}) \cap R_\chi = \{1\}$.

Lastly, we consider the cover $\bar{E}_7^{(n)}$ of the simply-connected E_7 , which is very saturated if and only if n is odd. Again, in the Bourbaki notations, the nontrivial $R_\chi \simeq \mathbb{Z}/2\mathbb{Z}$ is generated by

$$w := w_{\alpha_2} w_{\alpha_5} w_{\alpha_7}.$$

It then follows from [14] (see also [41]) that if w is W -conjugate to an element in $\eta(\tilde{W}_{\text{aff},y})$, then necessarily

$$\{\alpha_0, \alpha_2, \alpha_3\} \subset \Delta_{\text{aff},y}.$$

If we write $y = \sum_{i=1}^7 k_i \alpha_i^\vee$ with $k_i \in \mathbb{Z}$, then this gives

$$k_1 = n, \quad k_2 = 0, \quad k_3 = n/2,$$

which is a contradiction since n is odd.

All the above completes the proof. ■

For such \bar{G} , the unique distinguished genuine character of $Z(\bar{T})$ is used to construct a natural algebra isomorphism (see [16, Section 15.4])

$$\mathcal{H}(\bar{G}, I) \simeq \mathcal{H}(G, I).$$

We continue to write \mathcal{H}_I when there is no chance of confusion. Also, \mathcal{H}_I can be identified with the affine Hecke algebra $\mathcal{H}_{\tilde{W}_{\text{aff}}}$ associated with the affine Weyl group $W_{\text{aff}} = Y_{Q,n}^{\text{sc}} \rtimes W$ of the dual group \bar{G}^\vee . By Borel’s theorem, every Iwahori-spherical representation π of \bar{G} corresponds uniquely to a finite-dimensional irreducible \mathcal{H}_I -module.

If a linear G has connected center, then a Langlands correspondence was established by Kazhdan and Lusztig [32] between the irreducible \mathcal{H}_I -modules and certain Kazhdan–Lusztig parameters of triples (τ, u, ξ) . Such a correspondence was extended

by Reeder [43] to include the case when the center of G may not be connected. In fact, the work in loc. cit. also includes ramified $I(\chi)$ when G is assumed to have connected center. In recent works by Aubert–Baum–Plymen–Solleveld (for example [1]), a Langlands correspondence for irreducible constituents of general $I(\chi)$ was investigated and a link to the geometric side of the parameter space in terms of “extended quotients” was established.

Thus one has a local Langlands correspondence

$$\mathcal{L} : \text{Irr}_\epsilon(\bar{G})^I \rightarrow \text{Irr}(\mathcal{H}_I) \rightarrow \{(\tau, u, \xi)\}, \quad \pi \mapsto \pi^I \mapsto (\tau, u, \xi),$$

from the Iwahori-spherical representations to the set of Kazhdan–Lusztig–Reeder (KLR) parameters. Every KLR parameter that arises in the correspondence above satisfies the following:

- $\tau \in \bar{T}^\vee$ is a semisimple element in the dual torus inside \bar{G}^\vee ;
- $u \in \bar{G}^\vee$ is a nilpotent element such that $\tau u \tau^{-1} = q^{-1} \cdot u$;
- $\xi \in \text{Irr}_{\text{geom}}(\pi_0(\bar{G}_{\tau,u}^\vee))$ is a certain irreducible representation of the connected component group of the mutual centralizer subgroup $\bar{G}_{\tau,u}^\vee \subset \bar{G}^\vee$ that appears in the homology $H_*(\mathfrak{B}_{\bar{G}^\vee}^{\tau,u}, \mathbb{C})$.

Here $\mathfrak{B}_{\bar{G}^\vee}^{\tau,u}$ denotes the variety of Borel subgroups of \bar{G}^\vee containing both τ and u . In particular, the irreducible constituent $\pi_\sigma \subset I(\chi)$ has a KLR parameter

$$(\tau = s_\chi, u = 0, \xi = \sigma),$$

where $s_\chi \in \bar{G}^\vee$ is the Satake parameter associated to $I(\chi)$.

Before stating our main result on the ψ -Whittaker dimension of π_σ , we first recall a homomorphism (depending on the half sum $\tilde{\rho} := n\rho$ of positive roots of \bar{G}^\vee)

$$\zeta_{\tilde{\rho}} : R_\chi \rightarrow \mathbb{C}^\times$$

given as follows. Consider the simply-connected cover \bar{G}_{sc}^\vee of \bar{G}^\vee as in

$$Z \hookrightarrow \bar{G}_{\text{sc}}^\vee \twoheadrightarrow \bar{G}^\vee.$$

Attached to $I(\chi)$ is the Satake parameter $s_\chi \in \bar{G}^\vee$. For any $w \in R_\chi \subset W_\chi$ and any lifting $s'_\chi \in \bar{G}_{\text{sc}}^\vee$ of s_χ , one has

$$w(s'_\chi)/s'_\chi \in Z,$$

which is independent of the particular lifting s'_χ chosen. Since

$$Z = \text{Hom}(P_{Q,n}/Y_{Q,n}, \mathbb{C}^\times) \quad \text{with } P_{Q,n} = nP,$$

this gives a natural pairing

$$\phi : (P_{Q,n}/Y_{Q,n}) \times R_\chi \rightarrow \mathbb{C}^\times, \quad \phi(z, w) := \left(\frac{w(s'_\chi)}{s'_\chi} \right)(z) = (w(z) - z)(s_\chi) \in \mathbb{C}^\times.$$

Now we just set

$$\zeta_{\tilde{\rho}}(-) := \phi(\tilde{\rho}, -). \tag{10.1}$$

Since $2\tilde{\rho} \in Y_{Q,n}^{\text{sc}} \subset Y_{Q,n}$, we see that

$$\zeta_{\tilde{\rho}}^2 = \mathbf{1}, \tag{10.2}$$

that is, $\zeta_{\tilde{\rho}}$ is a (possibly trivial) quadratic character.

Theorem 10.2. *Let \bar{G} be a very saturated cover of an almost simple simply-connected G associated with $Q(\alpha^\vee) = 1$ for any short simple coroot α^\vee . Let $I(\chi)$ be a unitary (K, s_K) -unramified genuine principal series representation of \bar{G} . Let $\mathcal{O} \subset \mathcal{X}_{Q,n}$ be a W -orbit satisfying the S -property (see Definition 5.24). Then, as representations of R_χ ,*

$$\sigma_{\mathcal{O}}^{\text{Wh}_\psi} \simeq \zeta_{\tilde{\rho}}^{-1} \otimes \sigma_{\mathcal{O}}^{\mathcal{X}};$$

equivalently,

$$\dim \text{Wh}_\psi(\pi_\sigma)_\mathcal{O} = \langle \sigma \otimes \zeta_{\tilde{\rho}}, \sigma_{\mathcal{O}}^{\mathcal{X}} \rangle_{R_\chi} \text{ for every } \sigma \in \text{Irr}(R_\chi).$$

Proof. Let $\mathcal{O} = \mathcal{O}_y \subset \mathcal{X}_{Q,n}$ be a W -orbit with the S -property. It suffices to prove the second equality.

Since $Y_{Q,n} = Y_{Q,n}^{\text{sc}} = nY$ for very saturated covers, it follows from Definition 5.24 that

$$\dim \text{Wh}_\psi(\pi_\sigma)_\mathcal{O} = \dim \text{Hom}_{\mathcal{H}_I}(\mu_y \otimes \mathcal{H}_{\tilde{W}_{\text{aff},y}}, \mathcal{H}_I, \check{\pi}_\sigma^I) = \dim \text{Hom}_{\mathcal{H}_{\tilde{W}_{\text{aff},y}}}(\mu_y, \check{\pi}_\sigma^I |_{\mathcal{H}_{\tilde{W}_{\text{aff},y}}}).$$

Here $\mathcal{H}_{\tilde{W}_{\text{aff},y}} \subset \mathcal{H}_I$ is the associated/deformed subalgebra of the group $\tilde{W}_{\text{aff},y}$, and we have the $q \rightarrow 1$ operation of sending $\mathcal{H}_{\tilde{W}_{\text{aff},y}}$ -modules to $\tilde{W}_{\text{aff},y}$ -modules. Moreover, since $\tilde{W}_{\text{aff},y}$ is a finite Weyl subgroup of \tilde{W}_{aff} , this operation is an isometry (see [10, Proposition 10.11.4]). That is, we have

$$\text{Hom}_{\mathcal{H}_{\tilde{W}_{\text{aff},y}}}(\mu_y, \check{\pi}_\sigma^I |_{\mathcal{H}_{\tilde{W}_{\text{aff},y}}}) = \text{Hom}_{\tilde{W}_{\text{aff},y}}(\mu_y, (\check{\pi}_\sigma^I |_{\mathcal{H}_{\tilde{W}_{\text{aff},y}}})_{q \rightarrow 1}),$$

where we view μ_y as a character of $\tilde{W}_{\text{aff},y}$ as well. For \mathcal{H}_I -modules, one also has the operation $q \rightarrow 1$ by Lusztig’s theory (see [36], [42, Section 6] or [1, Section 12]), which commutes with restriction, and thus

$$(\check{\pi}_\sigma^I |_{\mathcal{H}_{\tilde{W}_{\text{aff},y}}})_{q \rightarrow 1} = (\check{\pi}_\sigma^I)_{q \rightarrow 1} |_{\tilde{W}_{\text{aff},y}}.$$

Consider the homology $H_*(\mathfrak{B}_{\bar{G}^\vee}^{s_\chi}, \mathbb{C})$ of the variety of Borel subgroups of \bar{G}^\vee containing s_χ . It is naturally a $\tilde{W}_{\text{aff}} \times \pi_0(Z_{\bar{G}^\vee}(s_\chi))$ -module. Identifying

$$R_\chi = \pi_0(Z_{\bar{G}^\vee}(s_\chi)),$$

we can view $H_*(\mathfrak{B}_{\bar{G}^\vee}^{s_\chi}, \mathbb{C})$ as a $\tilde{W}_{\text{aff}} \times R_\chi$ -module. It follows from [31, Proposition 6.2] that

$$H_*(\mathfrak{B}_{\bar{G}^\vee}^{s_\chi}, \mathbb{C}) = \text{Ind}_{(Y_{Q,n} \rtimes W_{s_\chi}^0) \rtimes R_\chi}^{\tilde{W}_{\text{aff}} \times R_\chi} H_*(\mathfrak{B}_{\bar{G}^\vee, s_\chi^0}, \mathbb{C}), \tag{10.3}$$

where

- $W_{s_\chi} \simeq W_\chi \subset W$ is the stabilizer subgroup of s_χ with $W_{s_\chi} = W_{s_\chi}^0 \rtimes R_\chi$;
- $\bar{G}_{s_\chi}^\vee \subset \bar{G}^\vee$ is the stabilizer subgroup of s_χ , its connected component is denoted by $\bar{G}_{s_\chi}^{\vee,0}$ whose Weyl group is $W_{s_\chi}^0$;
- the inclusion $(Y_{Q,n} \rtimes W_{s_\chi}^0) \rtimes R_\chi \hookrightarrow \tilde{W}_{\text{aff}} \times R_\chi$ is given by

$$(y, w, r) \mapsto ((y, w, r), r),$$

where we note the canonical $(Y_{Q,n} \rtimes W_{s_\chi}^0) \rtimes R_\chi = Y_{Q,n} \rtimes W_\chi \subset \tilde{W}_{\text{aff}}$;

- the action of $Y_{Q,n} \rtimes (W_{s_\chi}^0 \rtimes R_\chi)$ on $H_*(\mathfrak{B}_{\bar{G}_{s_\chi}^{\vee,0}}, \mathbb{C})$ is given by $(s_\chi \otimes \rho_{\text{Spr}}) \otimes \mathbb{1}$, where s_χ is viewed as a character of $Y_{Q,n}$, and $W_{s_\chi}^0$ acts on $H_*(\mathfrak{B}_{\bar{G}_{s_\chi}^{\vee,0}}, \mathbb{C})$ by the Springer correspondence ρ_{Spr} , and R_χ acts trivially (see [31, p. 199]).

It follows from [1, Theorem 9.1] and its proof that as $\tilde{W}_{\text{aff}} = Y_{Q,n} \rtimes W$ -modules,

$$(\tilde{\pi}_\sigma^I)_{q \rightarrow 1} = \text{Hom}_{R_\chi}(\sigma, H_*(\mathfrak{B}_{\bar{G}_{s_\chi}^{\vee,0}}, \mathbb{C}))^\vee. \tag{10.4}$$

It is a classical result of Borel that

$$H_*(\mathfrak{B}_{\bar{G}_{s_\chi}^{\vee,0}}, \mathbb{C}) \simeq \mathbb{C}[W_{s_\chi}^0] \tag{10.5}$$

as $W_{s_\chi}^0$ -representations. Now (10.3)–(10.5) together give

$$(\tilde{\pi}_\sigma^I)_{q \rightarrow 1} = (\text{Ind}_{Y_{Q,n} \rtimes R_\chi}^{Y_{Q,n} \rtimes W} (s_\chi \rtimes \sigma^{-1}))^\vee = \text{Ind}_{Y_{Q,n} \rtimes R_\chi}^{Y_{Q,n} \rtimes W} (s_\chi^{-1} \rtimes \sigma).$$

It follows that

$$(\tilde{\pi}_\sigma^I |_{\mathcal{H}_{\tilde{W}_{\text{aff},y}}})_{q \rightarrow 1} = \text{Res}_{\tilde{W}_{\text{aff},y}} \text{Ind}_{Y_{Q,n} \rtimes R_\chi}^{Y_{Q,n} \rtimes W} (s_\chi^{-1} \rtimes \sigma).$$

Since we need to consider the restriction to $\tilde{W}_{\text{aff},y}$, which may not be a parabolic subgroup, we “pull back” all the representations to be those of subgroups W as in [42, p. 54].

For any $w \in W$ and $\varsigma := {}^w\chi$, we have the following diagram:

$$\begin{array}{ccc}
 \eta(\tilde{W}_{\text{aff},y}) & \xrightarrow{\eta^{-1}} & \tilde{W}_{\text{aff},y} \\
 \uparrow & & \uparrow \\
 R_\varsigma \cap \eta(\tilde{W}_{\text{aff},y}) & \xrightarrow{\eta^{-1}} & (Y_{Q,n} \rtimes R_\varsigma) \cap \tilde{W}_{\text{aff},y} \hookrightarrow (Y_{Q,n} \rtimes R_\varsigma) \xrightarrow{s_\varsigma^{-1}} \mathbb{C}^\times \\
 & \searrow & \text{---} s_\varsigma^y \text{---} \\
 & & \mathbb{C}^\times
 \end{array}$$

Define

$$s_\varsigma^y = s_\varsigma \circ \eta^{-1} : R_\varsigma \cap \eta(\tilde{W}_{\text{aff},y}) \rightarrow \mathbb{C}^\times.$$

For simplicity of notation we write, for the rest of the proof,

$$W_{\text{aff},y}^\eta := \eta(\tilde{W}_{\text{aff},y}) \quad \text{and} \quad W_{y,\varsigma} := W_{\text{aff},y}^\eta \cap R_\varsigma.$$

Mackey’s formula shows that, as representations of $\tilde{W}_{\text{aff},y}^\eta$,

$$\begin{aligned} \eta_*((\check{\pi}_\sigma^I | \mathcal{H}_{\tilde{W}_{\text{aff},y}})_{q \rightarrow 1}) &= \bigoplus_{w \in W_{\text{aff},y}^\eta \backslash W/R_\chi} \eta_* (\text{Ind}_{\tilde{W}_{\text{aff},y} \cap R_{w_\chi}}^{\tilde{W}_{\text{aff},y}} w(s_\chi^{-1}) \otimes w\sigma) \\ &= \bigoplus_{w \in W_{\text{aff},y}^\eta \backslash W/R_\chi} \text{Ind}_{W_{y,w_\chi}}^{W_{\text{aff},y}^\eta} (s_{w_\chi}^y \otimes w\sigma). \end{aligned}$$

This gives

$$\begin{aligned} \dim \text{Hom}_{\tilde{W}_{\text{aff},y}}(\mu_y, (\check{\pi}_\sigma^I | \mathcal{H}_{\tilde{W}_{\text{aff},y}})_{q \rightarrow 1}) &= \langle \eta_*(\mu_y), \eta_*((\check{\pi}_\sigma^I | \mathcal{H}_{\tilde{W}_{\text{aff},y}})_{q \rightarrow 1}) \rangle_{W_{\text{aff},y}^\eta} \\ &= \bigoplus_{w \in W_{\text{aff},y}^\eta \backslash W/R_\chi} \langle \eta_*(\mu_y), \text{Ind}_{W_{y,w_\chi}}^{W_{\text{aff},y}^\eta} (s_{w_\chi}^y \otimes w\sigma) \rangle_{W_{\text{aff},y}^\eta} \\ &= \bigoplus_{w \in W_{\text{aff},y}^\eta \backslash W/R_\chi} \langle \eta_*(\mu_y), s_{w_\chi}^y \otimes w\sigma \rangle_{W_{y,w_\chi}} \end{aligned}$$

On the other hand, since $\sigma_{\emptyset}^{\mathcal{X}} = \text{Ind}_{W_{\text{aff},y}^\eta}^W \mathbf{1}$, one has

$$\begin{aligned} \langle \sigma \otimes \zeta_{\tilde{\rho}}, \sigma_{\emptyset}^{\mathcal{X}} \rangle_{R_\chi} &= \bigoplus_{w \in R_\chi \backslash W/W_{\text{aff},y}^\eta} \langle \mathbf{1}, \zeta_{\tilde{\rho}} \otimes \sigma \rangle_{R_\chi \cap {}^w(W_{\text{aff},y}^\eta)} \\ &= \bigoplus_{w \in W_{\text{aff},y}^\eta \backslash W/R_\chi} \langle \mathbf{1}, {}^w\zeta_{\tilde{\rho}} \otimes {}^w\sigma \rangle_{W_{y,w_\chi}}. \end{aligned}$$

We will verify the equality

$$\eta_*(\mu_y)^{-1} \otimes s_{w_\chi}^y = {}^w\zeta_{\tilde{\rho}} \quad \text{on } W_{y,w_\chi} \text{ for every } w \in W_{\text{aff},y}^\eta \backslash W/R_\chi, \tag{10.6}$$

which clearly suffices to give the desired result. We discuss the two cases according to whether $\alpha_0 \in \Delta_{\text{aff},y}$ or not.

First, if $\alpha_0 \in \Delta_{\text{aff},y}$, then it follows from Proposition 10.1 that

$$W_{y,w_\chi} = \{1\} \quad \text{for every } w,$$

and thus equality (10.6) holds trivially.

Second, we assume $\alpha_0 \notin \Delta_{\text{aff},y}$ and thus $\tilde{W}_{\text{aff},y} \subset W$ is isomorphic to a parabolic subgroup. Hence the orbit $\mathcal{O}_y \subset \mathcal{X}_{Q,n}$ is splitting. In this case, we simply write $W_y := \tilde{W}_{\text{aff},y}$. It follows that $\mu_y = \varepsilon_{W_y}$ and η is the identity map on W_y . Also, in this case $s_{w_\chi}^y$ is trivial on W_{y,w_χ} for all w , and (10.6) is equivalent to

$$\varepsilon_{W_y} = {}^w\zeta_{\tilde{\rho}} \quad \text{on } W_{y,w_\chi}. \tag{10.7}$$

Now we take ${}^w w_1 \in W_{y,w_\chi} = W_y \cap R_{w_\chi} = W_y \cap {}^w R_\chi$ with $w_1 \in R_\chi$ being the standard representative as in [34]. We have

$$\varepsilon_{W_y}({}^w w_1) = \varepsilon_W({}^w w_1) = \varepsilon_W(w_1).$$

On the other hand,

$${}^w\zeta_{\tilde{\rho}}({}^w w_1) = \zeta_{\tilde{\rho}}(w_1) = s_{\chi}(w_1(\tilde{\rho}) - \tilde{\rho}).$$

Let $w_1 = \prod_{j=1}^k w_{\alpha_{i_j}}$ and set $\Delta_{w_1} = \{\alpha_{i_j} : 1 \leq j \leq k\}$. We divide the computation of the above into two situations according to whether Δ_{w_1} is totally disconnected in the Dynkin diagram of G or not. Recall that for every $\alpha \in \Phi$, we denote

$$\chi_{\alpha} := \chi(\bar{h}_{\alpha}(\varpi^{n_{\alpha}})).$$

First, if Δ_{w_1} is totally disconnected, then by [34, Section 3] or [21], we see that $l(w_1) = |\Delta_{w_1}| = k$ and $\chi_{\alpha_{i_j}} = -1$ for every j . Since in this case

$$w_1(\tilde{\rho}) - \tilde{\rho} = -\sum_j \alpha_{i_j}^{\vee},$$

it is clear that

$${}^w\zeta_{\tilde{\rho}}({}^w w_1) = \varepsilon_{W_y}({}^w w_1) = (-1)^{l(w_1)}.$$

This applies to the cases when $R_{\chi} \simeq \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, in particular, to all covers of type B_r, C_r, E_7 and some covers of type A_r and D_r .

Second, we deal with the remaining cases. Again, we use the standard notations and coordinates from Bourbaki. We discuss case by case.

- For A_r with $R_{\chi} \simeq \mathbb{Z}/d\mathbb{Z}, r + 1 = dk$ we have

$$w_1 = (1, \dots, d)(d + 1, \dots, 2d) \dots (dk - d + 1, \dots, dk),$$

where $\chi_{\alpha_i} = \xi$, a primitive d -th root of unity. We see that

$$s_{\chi}(w_1(\tilde{\rho}) - \tilde{\rho}) = \xi^{-d(d-1)/2} = (-1)^{d-1} = \varepsilon_W(w_1).$$

- For D_r with $R_{\chi} \simeq \mathbb{Z}/4\mathbb{Z}$, we find that r is odd, and a generator for R_{χ} is

$$w_1 = w_{\alpha_1} w_{\alpha_3} \dots w_{\alpha_{r-4}} \cdot w_{\alpha_{r-2}} w_{\alpha_{r-1}} w_{\alpha_r},$$

and

$$\chi_{\alpha_i} = -1 \text{ for } i \in \{1, 3, 5, \dots, r - 4, r - 2\}, \quad \chi_{\alpha_{r-1}}^2 = -1, \quad \chi_{\alpha_r} = \chi_{\alpha_{r-1}}^{-1}.$$

A straightforward computation gives

$$s_{\chi}(w_1(\tilde{\rho}) - \tilde{\rho}) = (-1)^{(r-1)/2},$$

which is equal to $\varepsilon_W(w_1) = (-1)^{(r+3)/2}$.

- The last case is for E_6 with $R_{\chi} \simeq \mathbb{Z}/3\mathbb{Z}$ with a generator

$$w_1 = w_{\alpha_1} w_{\alpha_3} \cdot w_{\alpha_6} w_{\alpha_5},$$

and

$$\chi_{\alpha_1} = \chi_{\alpha_3} = \chi_{\alpha_5}^{-1} = \chi_{\alpha_6}^{-1} = \xi,$$

where ξ is a primitive third root of unity. A direct computation gives

$$s_\chi(w_1(\tilde{\rho}) - \tilde{\rho}) = 1 = \varepsilon_W(w_1),$$

where the first equality already follows from (10.2).

This completes the proof of the equality (10.7) and of the theorem. ■

For \mathcal{O} with the S-property as in Theorem 10.2, we have

$$\sigma_{\mathcal{O}}^{\text{Wh}\psi} \simeq \zeta_{\tilde{\rho}}^{-1} \otimes \sigma_{\mathcal{O}}^{\mathcal{X}} \simeq \zeta_{\tilde{\rho}} \otimes \sigma_{\mathcal{O}}^{\mathcal{X}}.$$

Corollary 10.3. *Let \bar{G} be an oasisitic cover of a simply-connected semisimple group G associated with $Q(\alpha^\vee) = 1$ for any short simple coroot α^\vee . Then*

$$\dim \text{Wh}_\psi(\pi_\sigma)_\mathcal{O} = \langle \sigma \otimes \zeta_{\tilde{\rho}}, \sigma_{\mathcal{O}}^{\mathcal{X}} \rangle_{R_\chi} = \langle \sigma \otimes \zeta_{\tilde{\rho}}^{-1}, \sigma_{\mathcal{O}}^{\mathcal{X}} \rangle_{R_\chi}$$

for every W -orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$ and $\sigma \in \text{Irr}(R_\chi)$. Hence,

$$\dim \text{Wh}_\psi(\pi_\sigma) = \langle \sigma \otimes \zeta_{\tilde{\rho}}, \sigma^{\mathcal{X}} \rangle_{R_\chi} = \langle \sigma \otimes \zeta_{\tilde{\rho}}^{-1}, \sigma^{\mathcal{X}} \rangle_{R_\chi}.$$

10.3. Conductor of ψ varied

Note that for any z one clearly has a decomposition

$$\text{Wh}_z(I(\chi))^\sharp = \bigoplus_{\sigma \in \text{Irr}(R_\chi)} \text{Wh}_z(\pi_\sigma)^\sharp.$$

We want to compare $\dim \text{Wh}_\psi(\pi_\sigma)^\sharp$ with $\dim \text{Wh}_{\rho_\psi}(\pi_\sigma)^\sharp$ when $\mathfrak{f}(\rho_\psi) = O_F$, and verify some speculations given in [23, Conjecture 5.7].

Corollary 10.4. *Let \bar{G} be an oasisitic cover of an almost simple simply-connected G with $Q(\alpha^\vee) = 1$. As in Proposition 9.2, let $y_{-\rho} \in Y$ be such that*

$$\mathfrak{m}_{-\rho} : \mathcal{X}_{Q,n} \rightarrow \mathcal{X}_{Q,n}, \quad y \mapsto y + y_{-\rho},$$

induces a bijective correspondence between W -orbits and $(W, -\rho)$ -orbits of the domain and codomain of $\mathfrak{m}_{-\rho}$ respectively. Let $I(\chi)$ be a unitary unramified principal series representation. Assume that Conjecture 9.7 holds. Also assume $R_\chi \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then for every $\pi_\sigma \subset I(\chi)$ and every W -orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$, one has

$$\dim \text{Wh}_\psi(\pi_\sigma)_\mathcal{O}^\sharp = \dim \text{Wh}_{\rho_\psi}(\pi_{\sigma \otimes \zeta_{\tilde{\rho}}^{-1}})_{\mathfrak{m}_{-\rho}(\mathcal{O})}^\sharp;$$

hence also

$$\dim \text{Wh}_\psi(\pi_\sigma)^\sharp = \dim \text{Wh}_{\rho_\psi}(\pi_{\sigma \otimes \zeta_{\tilde{\rho}}^{-1}})^\sharp.$$

Proof. Assuming Conjecture 9.7, one has

$$\dim \text{Wh}_\psi(\pi_\sigma)_\mathcal{O}^\sharp = \dim \text{Wh}_\psi(\pi_\sigma)_\mathcal{O} = \langle \sigma \otimes \zeta_{\tilde{\rho}}^{-1}, \sigma_{[0]}^\mathcal{O} \rangle_{R_\chi},$$

where the second equality follows from Corollary 10.3. (Here we use the notation $\sigma_{[0]}^\mathcal{O}$ for $\sigma_\mathcal{O}^\mathcal{X}$ to highlight the twisting parameter.)

On the other hand, from (9.7) we have

$$\dim \text{Wh}_{\rho,\psi}(\pi_\sigma)_{\mathcal{O}'}^\sharp = \dim \text{Wh}_{-\rho(w_G\psi)}(\pi_\sigma)_{\mathcal{O}'}^{\text{std}}$$

for every $(W, -\rho)$ -orbit $\mathcal{O}' \subset \mathcal{X}_{Q,n}$. Note that every \mathcal{O}' is $\mathfrak{m}_{-\rho}(\mathcal{O})$ for a unique W -orbit $\mathcal{O} \subset \mathcal{X}_{Q,n}$. Since $\mathfrak{f}(-\rho(w_G\psi)) = O_F$, for $R_\chi \simeq (\mathbb{Z}/2\mathbb{Z})^i, i = 1, 2$, it is shown in [21, Theorem 1.4] that

$$\dim \text{Wh}_{-\rho(w_G\psi)}(\pi_{\sigma \otimes \xi_{\bar{\rho}}^{-1}})_{\mathfrak{m}_{-\rho}(\mathcal{O})}^{\text{std}} = \langle \sigma_{[-\rho]}^{\mathfrak{m}_{-\rho}(\mathcal{O})}, \sigma \otimes \xi_{\bar{\rho}}^{-1} \rangle_{R_\chi}. \tag{10.8}$$

However, it follows from Proposition 9.2 that for such covers \bar{G} , one has

$$\sigma_{[-\rho]}^{\mathfrak{m}_{-\rho}(\mathcal{O})} \simeq \sigma_{[0]}^\mathcal{O}.$$

The desired first equality then follows from comparing all the above. The second equality is clear from this. ■

Remark 10.5. The equality (10.8) is expected to hold for general R_χ (see [20, 21]) and if proven will give a stronger version of Corollary 10.4 for all R_χ . Moreover, for very saturated covers, one can compute directly the two sides of the equality in Corollary 10.4 as in [21]. Indeed, the result in loc. cit. was obtained by considering the scattering matrix τ_{ψ^*} in Section 9.4 with $\psi^* = z^*(w_G\psi)$ and $z^* = -\rho$, while the method equally applies to $\tau_{w_G\psi}$. In fact, even if χ is ramified, similar results are expected to hold following the work [24, 51] on the local coefficient matrices and scattering matrices.

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