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An endpoint estimate for some maximal operators

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Suppose μ is a finite positive Borel measure on $\mathbb{R}^n$. It is proved in [DR] that if the Fourier transform of μ satisfies a decay estimate

$$|\hat{\mu}(\xi)| \le C|\xi|^{-\alpha} \tag{1}$$

for some $\alpha > 0$, then the maximal operator

$$Mf(x) = \sup_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} |f(x-2^k y)|\, d\mu(y) \tag{2}$$

is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. On the other hand, Theorem 4 in [C2] states that if μ is the Lebesgue measure σ_{n-1} on the unit sphere Σ_{n-1} in $\mathbb{R}^n$, then (2) maps $H^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$. The purpose of this paper is to adapt the method of [C2] to prove an H^1-$L^{1,\infty}$ result for (2) requiring, in the spirit of [DR], only a certain decay of $\hat{\mu}$.

Theorem. *Suppose μ is a finite positive Borel measure on $\mathbb{R}^n$ with support in $[-1,1]^n$. If*

$$|\hat{\mu}(\xi)| \le C|\xi|^{-n/2},$$

then (2) maps $H^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

As indicated, our proof follows the method of proof of Theorem 4 of [C2]. Our view is that the interest of this paper lies as much in a demonstration of the flexibility of that method (see [C2, Remark 7.2]) as in our result. Although many of the details differ, the main novelty here lies in the use of the auxiliary functions φ_N to handle the control (see (7)) of

$$\left\| \Big(\sum_{\kappa(Q)=j-s} \lambda_Q \, a_Q \Big) * \mu_j \right\|_2 .$$

The proof in [C2] used the curvature of the support of σ_{n-1} in the analogous estimate. Our argument proceeds, albeit in the same spirit, with no knowledge of μ aside from the decay of $\hat{\mu}$. But we pay by requiring a higher rate of decay -$\hat{\sigma}_{n-1}(\xi)$ decays, as is well-known, like $|\xi|^{(1-n)/2}$. Still, there exist singular measures on $\mathbb{R}^n$ satisfying our hypothesis. (This was proved in [I-M] for $n = 1$ - see Lemma 1 [K, p. 165] for the extension from Fourier coefficients to Fourier transform. To get a singular measure μ on $\mathbb{R}^n$ with $|\hat{\mu}(\xi)| = O(|\xi|^{-n/2})$, let ν be the measure from [I-M] translated to have support in [1,2] and define the measure μ on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} f \, d\mu = \int_1^2 \int_{\Sigma_{n-1}} f(ry) \, d\sigma_{n-1}(y) \, r^{(n-1)/2} d\nu(r) .$$

Then asymptotic estimates for Bessel functions such as those in [SW, Lemma 3.11] combine with the decay of $\hat{\nu}$ to give $|\hat{\mu}(\xi)| = O(|\xi|^{-n/2})$.) It may be that our $n/2$ can be replaced by smaller $\alpha > 0$, thus yielding a more satisfying endpoint analog of the result of [DR]. The referee has pointed out that the paper [S] contains a point of similarity to the proof of our theorem (in its use of the Fourier transform for the L^2 estimate) and that ideas equivalent to some of those in [DR] are present in [C1]. We begin with two lemmas.

Lemma 1. *For any $\alpha > 0$ and any finite collection of dyadic cubes $Q \subseteq \mathbb{R}^n$ and associated positive scalars λ_Q, there exists a collection $\mathcal{S}$ of pairwise disjoint dyadic cubes S such that*

a) $\sum_{Q \subseteq S} \lambda_Q \le 2^n \alpha \, |S|$, *if* $S \in \mathcal{S}$,

b) $\sum |S| \le \alpha^{-1} \sum \lambda_Q$,

c) $\left\| \sum_{\substack{Q \text{ not contained} \\ \text{in any } S}} \lambda_Q \, |Q|^{-1} \chi_Q \right\|_\infty \le \alpha$.

Proof. In the proof of Lemma 4.1 of [C2], simply replace 8 by 2^n and interpret dyadic in the n-dimensional Euclidean sense (instead of the parabolic sense in $\mathbb{R}^2$).

Notation. If Q is a dyadic cube in $\mathbb{R}^n$ with side-length 2^j, write $\sigma(Q)$ to stand for j. If $\sigma \in \mathbb{Z}$, let $\mathcal{R}_\sigma$ be the collection of dyadic cubes $Q \subseteq \mathbb{R}^n$ with $\sigma(Q) = \sigma$. Finally, if $Q \in \mathcal{R}_\sigma$, define $Q^* = Q + [-2^\sigma, 2^\sigma]^n$. Thus Q^* is the union of 3^n cubes in $\mathcal{R}_\sigma$.

Lemma 2. (*cf.* [C2, Lemma 5.1]) *Suppose given the following: some $\alpha > 0$, a collection $\mathcal{S}$ of pairwise disjoint dyadic cubes $S \subseteq \mathbb{R}^n$, a finite collection $\mathcal{C}$ of dyadic cubes $Q \subseteq \mathbb{R}^n$ such that each $Q \in \mathcal{C}$ is contained in some $S = S(Q) \in \mathcal{S}$, and for each $Q \in \mathcal{C}$ a positive number λ_Q. Then there exist a measurable $E \subseteq \mathbb{R}^n$ and a function $\kappa : \mathcal{C} \to \mathbb{Z}$ such that*

a) $|E| \le 3^n (\alpha^{-1} \sum \lambda_Q + \sum |S|)$,

b) $Q + [-2^j, 2^j]^n \subseteq E$, *if* $j < \kappa(Q)$ *and* $Q \in \mathcal{C}$,

c) $\sigma(S(Q)) < \kappa(Q) \qquad (Q \in \mathcal{C})$,

d) *for* $\sigma \in \mathbb{Z}$ *any* $q \in \mathcal{R}_\sigma$, $\displaystyle\sum_{\substack{Q \subseteq q \\ \kappa(Q) \le \sigma}} \lambda_Q \le \alpha\, 2^{n(\sigma+1)}$.

Proof. The proof is an adaptation of (and simpler than) that of Lemma 5.1 in [C2]. But we give the details for completeness and for the convenience of the reader.

Let $m = \min\{\sigma(Q)\}$. Find $\sigma_0 \in \mathbb{Z}$ such that

$$\sum \lambda_Q < \alpha\, 2^{n\sigma_0}, \qquad \sigma_0 > \max\{\sigma(Q)\}\,.$$

The proof is a stopping time argument on the descending parameter σ and proceeds by dividing $\mathcal{C}$ into disjoint subcollections $\mathcal{C}_1$ and $\mathcal{C}_2$. We begin with $\sigma = \sigma_0 - 1$ and define, for $q \in \mathcal{R}_\sigma$,

$$\Lambda_\sigma(q) = \sum_{Q \subseteq q} \lambda_Q\ .$$

Say that $q \in \mathcal{R}_\sigma$ is "selected at step σ" if

$$\Lambda_\sigma(q) > \alpha\, 2^{n\sigma}\ .$$

Put into $\mathcal{C}_1$ every Q such that $Q \subseteq q$ for some q selected at step σ, and for such Q define

$$\kappa(Q) = \max\{1 + \sigma, 1 + \sigma(S(Q))\}\,. \tag{3}$$

Next, put into $\mathcal{C}_2$ every $Q \in \mathcal{C} \sim \mathcal{C}_1$ such that $\sigma(Q) > \sigma$ - such a Q will actually satisfy $\sigma(Q) = \sigma + 1$ - and for such Q define

$$\kappa(Q) = 1 + \sigma(S(Q))\,. \tag{4}$$

Note that (3) and (4) guarantee that (c) holds. Now replace σ by $\sigma - 1$ and repeat the process with

$$\Lambda_\sigma(q) = \sum_{\substack{Q \subseteq q \\ Q \notin \mathcal{C}_1}} \lambda_Q = \sum_{\substack{Q \subseteq q \\ Q \notin \mathcal{C}_1 \cup \mathcal{C}_2}} \lambda_Q\,, \qquad q \in \mathcal{R}_\sigma\,.$$

(The last equality holds because $Q \in \mathcal{C}_2$ at the beginning of step σ implies $\sigma(Q) \geq \sigma + 2$.) After the step $\sigma = m$ we will have $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ and κ defined on all of $\mathcal{C}$. Next define

$$E_1 = \bigcup_{q \text{ selected}} q^*, \quad E_2 = \bigcup S^*, \quad E = E_1 \cup E_2\,.$$

Then, since distinct selected q are disjoint,

$$|E_1| \leq 3^n \sum_{q \text{ selected}} 2^{n\sigma(q)} < \frac{3^n}{\alpha} \sum_{q \text{ selected}} \Lambda_{\sigma(q)}(q) \leq \frac{3^n}{\alpha} \sum \lambda_Q\,.$$

Now a) follows since $|S^*| = 3^n|S|$.

If $\kappa(Q) = 1 + \sigma(S(Q))$ and if $j < \kappa(Q)$, then

$$Q + [-2^j, 2^j]^n \subseteq S^* \subseteq E_2\,.$$

If $\kappa(Q) \neq 1 + \sigma(S(Q))$, then $Q \subseteq q$ for some q selected at some step σ and $\kappa(Q) = 1 + \sigma(q)$. Thus if $j < \kappa(Q)$,

$$Q + [-2^j, 2^j]^n \subseteq E_1\,.$$

So b) is verified.

Finally, if $q \in \mathcal{R}_\sigma$ for $\sigma \geq \sigma_0 - 1$, then d) is clear from the choice of σ_0. So suppose $\sigma < \sigma_0 - 1$ and $q \in \mathcal{R}_\sigma$. Now

$$\Lambda_\sigma(q) \leq \alpha\, 2^{n(\sigma+1)}$$

or else the $q_1 \in \mathcal{R}_{\sigma+1}$ that contains q would have been selected at stage $\sigma + 1$. Since $\kappa(Q) \leq \sigma$ implies that $Q \notin \mathcal{C}_1$ at the beginning of step σ,

$$\sum_{\substack{Q \subseteq q \\ \kappa(Q) \leq \sigma}} \lambda_Q \leq \Lambda_\sigma(q)\,,$$

and so d) is proved.

Now suppose μ is a positive Borel probability measure supported on $[-1,1]^n$ and satisfying $|\hat{\mu}(\xi)| \le C|\xi|^{-n/2}$. Let $f \in H^1(\mathbb{R}^n)$ have the form of a finite sum

$$f = \sum \lambda_Q \, a_Q \ ,$$

where $\lambda_Q > 0$ and a_Q, supported in a cube Q, satisfies

$$\|a_Q\|_\infty \le |Q|^{-1}, \qquad \int_Q a_Q = 0 \, .$$

As in [C2], a device of Garnett and Jones involving auxiliary dyadic grids allows us to assume that each Q is dyadic. Fix $\alpha > 0$. It is enough to show that

$$|\{Mf > 2\alpha\}| \le \frac{C}{\alpha} \sum \lambda_Q \ , \tag{5}$$

where C depends only on μ and n.

Following [C2], let $\mathcal{S}$ be as in Lemma 1 and define

$$b = \sum_{S \in \mathcal{S}} \sum_{Q \subseteq S} \lambda_Q \, a_Q \ , \qquad g = f - b \, .$$

Then $\|g\|_\infty \le \alpha$ from Lemma 1.c) and so $|Mg| \le \alpha$ (because μ has mass 1). Thus (5) will follow from

$$|\{Mb > \alpha\}| \le \frac{C}{\alpha} \sum \lambda_Q \ .$$

Now, with $\mathcal{S}$ as above and with $\mathcal{C}$ the collection of Q's appearing in the definition of b, let κ and E be as in Lemma 2. Since $|E| \le C\alpha^{-1} \sum \lambda_Q$, it is enough to prove

$$\|Mb\|^2_{L^2(\mathbb{R}^n \sim E)} \le C\alpha \sum \lambda_Q \ . \tag{6}$$

Let μ_j be the dilate of μ defined by

$$\langle \varphi, \mu_j \rangle = \int_{\mathbb{R}^n} \varphi(2^j x) \, d\mu(x)$$

so that μ_j is supported in $[-2^j, 2^j]^n$ and

$$Mb(x) = \sup_{j \in \mathbb{Z}} |b * \mu_j(x)| .$$

If $Q \in \mathcal{C}$, then, by Lemma 2.b), $a_Q * \mu_j$ is supported in E unless $j \geq \kappa(Q)$. Thus if $x \notin E$,

$$\begin{aligned} |Mb(x)|^2 &\leq \sum_j |b * \mu_j(x)|^2 \\ &= \sum_j \Big| \Big(\sum_{\kappa(Q) \leq j} \lambda_Q a_Q \Big) * \mu_j(x) \Big|^2 \\ &= \sum_j \Big| \sum_{s=0}^{\infty} \Big(\sum_{\kappa(Q) = j-s} \lambda_Q a_Q \Big) * \mu_j(x) \Big|^2 . \end{aligned}$$

So, for $x \notin E$

$$|Mb(x)| \leq \sum_{s=0}^{\infty} \Big(\sum_j \Big| \Big(\sum_{\kappa(Q)=j-s} \lambda_Q a_Q \Big) * \mu_j(x) \Big|^2 \Big)^{1/2} .$$

Now (6) will follow from

$$\Big\| \Big(\sum_j \Big| \Big(\sum_{\kappa(Q)=j-s} \lambda_Q a_Q \Big) * \mu_j \Big|^2 \Big)^{1/2} \Big\|_2^2 \leq C\alpha (s+1) 2^{-s} \sum \lambda_Q$$

and so from

$$\Big\| \Big(\sum_{\kappa(Q)=j-s} \lambda_Q a_Q \Big) * \mu_j \Big\|_2^2 \leq C\alpha(s+1)2^{-s} \sum_{\kappa(Q)=j-s} \lambda_Q . \tag{7}$$

The proof of (7) requires another lemma.

Lemma 3. *For $N = 1, 2, \ldots$, there exist functions $\varphi_N \in L^1(\mathbb{R}^n)$ such that*

a) $|\hat{\varphi}_N(\xi)| \geq (1 + |\xi|)^{-n/2}/C$, *if* $|\xi| \leq N - 1$,

b) $|\hat{\varphi}_N(\xi)| \le C\,|\xi|^{-n/2}$,

and if $L_N = \varphi_N * \tilde{\varphi}_N$ $(\tilde{\varphi}_N(x) = \varphi_N(-x))$, *then*

c) $\operatorname{supp}(L_N) \subseteq [-1,1]^n$,

d) $|L_N(x) - L_N(y)| \le C\,|x-y| / \min\{|x|, |y|\}$.

PROOF. We will construct L_N first and then φ_N. Define $h_N \in C(\mathbb{R}^n)$ by

$$\hat{h}_N(\xi) = \begin{cases} 1\,, & \text{if } |\xi| \le 1\,, \\ |\xi|^{-n}\,, & \text{if } 1 < |\xi| \le N\,, \\ 0\,, & \text{if } |\xi| > N\,. \end{cases}$$

Choose a radial function $\rho \in C_C^\infty(\mathbb{R}^n)$ such that

$$\int \rho = 1\,, \qquad \operatorname{supp}(\rho) \subseteq [-1,1]^n, \qquad \hat{\rho} \ge 0\,.$$

Now let $L_N = \rho\, h_N$. Clearly c) holds. It is easy to check that

$$\hat{L}_N(\xi) \ge (1+|\xi|)^{-n}/C \quad \text{if } |\xi| \le N-1\,,$$
$$0 \le \hat{L}_N(\xi) \le C|\xi|^{-n} \quad \text{if } \xi \in \mathbb{R}^n\,.$$

So if φ_N is the inverse Fourier transform of $(\hat{L}_N)^{1/2}$, then a) and b) hold. Since

$$|L_N(x) - L_N(y)| \le |\rho(x) - \rho(y)|\,|h_N(x)| + \rho(y)\,|h_N(x) - h_N(y)|\,,$$

d) will follow from

$$|h_N(x)| \le C\Big(\log^+\Big(\frac{1}{|x|}\Big) + 1\Big)\,, \tag{8}$$

and

$$\Big|\frac{\partial}{\partial |x|} h_N(x)\Big| \le \frac{C}{|x|}\,, \qquad |x| \le 1\,. \tag{9}$$

Now

$$h_N(x) = \int_0^1 \int_{\Sigma_{n-1}} e^{irx\cdot\omega}\, d\sigma_{n-1}(\omega)\, r^{n-1} dr + \int_1^N \int_{\Sigma_{n-1}} e^{irx\cdot\omega}\, d\sigma_{n-1}(\omega)\, \frac{dr}{r}\,,$$

with the important contribution coming from the second integral. For (8) just use the well-known estimate

$$\left| \int_{\Sigma_{n-1}} e^{irx\cdot\omega}\, d\sigma_{n-1}(\omega) \right| \le \frac{C}{(1+r|x|)^{(n-1)/2}} .$$

For (9) note that

$$\int_{\Sigma_{n-1}} e^{irx\cdot\omega}\, d\sigma_{n-1}(\omega) = \int_0^1 \cos(|x|rs)\omega(s)\, ds ,$$

for some $\omega \in L^1([0,1])$. Now

$$\begin{aligned} \left| \frac{d}{dt} \int_1^N \int_0^1 \cos(trs)\, \omega(s)\, ds\, \frac{dr}{r} \right| &= \left| \int_0^1 \int_1^N \sin(trs)\, s\, dr\, \omega(s)\, ds \right| \\ &\le \int_0^1 \left| \int_s^{Ns} \sin(tu)\, du \right| \omega(s)\, ds \\ &\le \frac{C}{|t|} . \end{aligned}$$

Returning to (7) we have, because of our estimate on $\hat{\mu}$ combined with Lemma 3.a),

$$\begin{aligned} \left\| \Big(\sum_{\kappa(Q)=j-s} \lambda_Q a_Q \Big) * \mu_j \right\|_2^2 &= \int_{\mathbb{R}^n} \left| \Big(\sum_{\kappa(Q)=j-s} \lambda_Q a_Q \Big)^{\wedge} (\xi) \right|^2 |\hat{\mu}(2^j \xi)|^2 d\xi \\ &\le C \int_{\mathbb{R}^n} \left| \Big(\sum_{\kappa(Q)=j-s} \lambda_Q a_Q \Big)^{\wedge} (\xi) \right|^2 \\ &\qquad \cdot \liminf_N \left| \hat{\varphi}_N(2^j \xi) \right|^2 d\xi . \end{aligned}$$

Thus, letting $\varphi_{N,j}(x) = 2^{-nj}\varphi_N(2^{-j}x)$, (7) will follow from the estimates, uniform in N,

$$\left\| \Big(\sum_{\kappa(Q)=j-s} \lambda_Q a_Q \Big) * \varphi_{N,j} \right\|_2^2 \le C\alpha(s+1) \sum_{\kappa(Q)=j-s} \lambda_Q . \tag{10}$$

So fix N, j, and s and write φ for φ_N, φ_j for $\varphi_{N,j}$. For $q \in \mathcal{R}_{j-s}$, let

$$A_q = \sum_{\substack{\kappa(Q)=j-s \\ Q\subseteq q}} \lambda_Q a_Q, \qquad \lambda_q = \sum_{\substack{\kappa(Q)=j-s \\ Q\subseteq q}} \lambda_Q \,.$$

Then

$$\begin{aligned}\Big\| \Big(\sum_{\kappa(Q)=j-s} \lambda_Q a_Q \Big) * \varphi_j \Big\|_2^2 &\le \sum_{q,q'\in\mathcal{R}_{j-s}} \Big| \langle A_q * \varphi_j, A_{q'} * \varphi_j \rangle \Big| \\ &\le \sum_{q'} \sum_{q\subseteq (q')^*} + \sum_{q'} \sum_{q\cap (q')^*=\varnothing} \\ &= \mathrm{I} + \mathrm{II}\,.\end{aligned}$$

The inequality

$$\|a_Q * \varphi_j\|_2 \le C\, 2^{-nj/2}$$

follows easily from Lemma 3.b) and the well-known estimates

$$\Big| \hat{a}_Q(\xi) \Big| \le C\, |\xi| \operatorname{diam}(Q)\,,$$

$$\|a_Q\|_2^2 \le \frac{C}{|Q|}\,.$$

This leads, via Lemma 2.d), to

$$\begin{aligned}\mathrm{I} &\le C\, 2^{-nj} \sum_{q'} \lambda_{q'} \sum_{q\subseteq (q')^*} \lambda_q \\ &\le C\, 2^{-nj} \sum_{q'} \lambda_{q'} \sum_{\substack{Q\subseteq (q')^* \\ \kappa(Q)=j-s}} \lambda_Q \\ &\le C\, 2^{-nj} \sum_{q'} \lambda_{q'}\, \alpha\, 2^{n(j-s+1)} \\ &= C\alpha\, 2^{n(1-s)} \sum_{\kappa(Q)=j-s} \lambda_Q\,.\end{aligned} \tag{12}$$

To estimate II, begin by fixing q, q' ($\in \mathcal{R}_{j-s}$) with $q \cap (q')^* = \varnothing$. We write

$$\langle A_q * \varphi_j, A_{q'} * \varphi_j \rangle = \int A_q(x) A_{q'} * L_j(x)\, dx\,, \tag{13}$$

where $L_j(x) = \varphi_j * \tilde{\varphi}_j(x) = 2^{-nj} L(2^{-j}x)$ and so, by Lemma 3.d),

$$|L_j(x) - L_j(y)| \le C\, 2^{-nj} |x-y| / \min\{|x|, |y|\}\,.$$

Now if $\kappa(Q) = j-s$, $Q \subseteq q'$, $x \in q$, and $y_0 \in Q$, then

$$a_Q * L_j(x) = \int a_Q(y) \big(L_j(x-y) - L_j(x-y_0) \big)\, dy\,.$$

Thus

$$|a_Q * L_j(x)| \le \frac{C\, 2^{-nj} \mathrm{diam}(Q)}{d(x,Q)} \le \frac{C\, 2^{-nj+\sigma(Q)}}{d(x,Q)} \le \frac{C\, 2^{-(n-1)j-s}}{d(x,Q)}\,,$$

since $\sigma(Q) \le \sigma(S(Q)) < \kappa(Q) = j-s$ by Lemma 2. Also, if $a_Q * L_j(x) \neq 0$, then $d(x,Q) \le C\, 2^j$ (since L_j is supported in $[-2^j, 2^j]^n$). Thus

$$|a_Q * L_j(x)| \le \frac{C\, 2^{-s}}{d(x,Q)^n}\,.$$

Now suppose $x \in q$. If $Q \subseteq q'$ and $\kappa(Q) = j-s$, then $\sigma(S(Q)) < \kappa(Q) = j-s = \sigma(q')$. Since $S(Q) \cap q' \neq \varnothing$, $S(Q) \subseteq q'$. Because $q \cap (q')^* = \varnothing$, we must have $d(x, S(Q)) \ge 2^{j-s}$. Coupled with $d(x,S(Q)) \le d(x,Q) \le C\, 2^j$ if $a_Q * L_j(x) \neq 0$, we estimate, for fixed $q \in \mathcal{R}_{s-j}$ and $x \in q$,

$$\begin{aligned}
\sum_{(q')^* \cap q = \varnothing} |A_{q'} * L_j(x)| &\le \sum_{(q')^* \cap q = \varnothing} \sum_{\substack{Q \subseteq q', \kappa(Q) = j-s \\ 2^{j-s} \le d(x,S(Q)) \le C 2^j}} \lambda_Q |a_Q * L_j(x)| \\
&\le C \sum_{(q')^* \cap q = \varnothing} \sum_{\substack{Q \subseteq q', \kappa(Q) = j-s \\ 2^{j-s} \le d(x,S(Q)) \le C 2^j}} \lambda_Q \frac{2^{-s}}{d(x,Q)^n} \\
&\le C 2^{-s} \sum_{2^{j-s} \le d(x,S) \le C 2^j} \frac{1}{d(x,S)^n} \sum_{\substack{Q \subseteq S \\ \kappa(Q) = j-s}} \lambda_Q\,.
\end{aligned}$$

By Lemma 1.a) this last term is dominated by

$$C\alpha\, 2^{-s} \sum_{2^{j-s} \leq d(x,S) \leq C2^j} \frac{|S|}{d(x,S)^n} \leq C\alpha\, 2^{-s} \int_{2^{j-s}}^{C2^j} \frac{dr}{r} \leq C\alpha\, 2^{-s}(s+1)\,.$$

That is, if $x \in q$, then

$$\sum_{(q')^* \cap q = \varnothing} |A_{q'} * L_j(x)| \leq C\alpha\, 2^{-s}(s+1))\,.$$

Thus, from (13),

$$\begin{aligned} \mathrm{II} &\leq \sum_q \int |A_q(x)| \sum_{(q')^* \cap q = \varnothing} |A_{q'} * L_j(x)|\, dx \\ &\leq C\alpha\, 2^{-s}(s+1) \sum_q \lambda_q = C\alpha\, 2^{-s}(s+1) \sum_{\kappa(Q) = j-s} \lambda_Q\,. \end{aligned}$$

With (11) and (12) this gives (10) and completes the proof of our theorem.

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