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Endpoint eigenfunction bounds for the Hermite operator

Abstract. We establish the optimal L^p, $p = 2(d + 3)/(d + 1)$, eigenfunction bound for the Hermite operator $\mathcal{H} = -\Delta + |x|^2$ on $\mathbb{R}^d$. Let Π_λ denote the projection operator to the vector space spanned by the eigenfunctions of $\mathcal{H}$ with eigenvalue λ. The optimal L^2–L^p bounds on Π_λ, $2 \le p \le \infty$, have been known by the works of Karadzhov and Koch–Tataru except $p = 2(d + 3)/(d + 1)$. For $d \ge 3$, we prove the optimal bound for the missing endpoint case. Our result is built on a new phenomenon: improvement of the bound due to asymmetric localization near the sphere $\sqrt{\lambda}\mathbb{S}^{d-1}$.

Keywords: Hermite functions, spectral projection.

1. Introduction

The Hermite operator $\mathcal{H} = -\Delta + |x|^2$ on $\mathbb{R}^d$ has a discrete spectrum $\lambda \in 2\mathbb{N}_0 + d$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $\alpha \in \mathbb{N}_0^d$, we denote by Φ_α the L^2-normalized Hermite function, which is an eigenfunction of $\mathcal{H}$ with eigenvalue $2|\alpha| + d$. The set $\{\Phi_\alpha : \alpha \in \mathbb{N}_0^d\}$ forms an orthonormal basis in L^2. Let Π_λ denote the spectral projection operator to the vector space spanned by the eigenfunctions with eigenvalue λ, i.e.,

$$\Pi_\lambda f = \sum_{\alpha: d+2|\alpha|=\lambda} \langle f, \Phi_\alpha \rangle \Phi_\alpha.$$

In this paper, we are concerned with bounds on the operator norm $\|\Pi_\lambda\|_{2 \to q}$ for $2 \le q \le \infty$, where $\|T\|_{s \to r}$ denotes the norm of an operator T from L^s to L^r. The sharp bound in terms of λ has been of interest in connection to the Bochner–Riesz summability of the

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Hermite expansion. See Askey and Wainger [1], Karadzhov [10], and Thangavelu [24] (also see [2, 3, 16] for recent developments). The bounds independent of λ have applications to the strong unique continuation problem for parabolic equations [4, 5, 9, 13].

The bounds on $\|\Pi_\lambda\|_{2\to q}$ have been almost completely understood. When $d = 1$, the sharp bounds follow from those on L^p norm of the Hermite functions in $\mathbb{R}$ [23]. Let

$$q_\circ = \frac{2(d + 3)}{d + 1}.$$

In higher dimensions $d \geq 2$, by the works of Karadzhov [10] and Koch–Tataru [12], it is known that for $q \in [2, \infty] \setminus \{q_\circ\}$,

$$\|\Pi_\lambda\|_{2\to q} \sim \mathrm{B}_q(\lambda) := \max(\lambda^{(-1+d\delta(2,q))/2}, \lambda^{(d\delta(2,q)-1)/6}, \lambda^{-\delta(2,q)/2}), \tag{1.1}$$

where $\delta(r, s) = r^{-1} - s^{-1}$. (See Notation 1.4 for the precise meaning of $\sim$.) The bound for $q = 2$ is clear from Bessel's inequality, and that for $q = \infty$ is a consequence of the estimate for the kernel of Π_λ (see [23, Lemma 3.2.2]). Karadzhov [10] showed $\|\Pi_\lambda\|_{2\to 2d/(d-2)} \leq C$ for a constant C. Thangavelu [24] considered a local estimate over a compact set K and he obtained a sharp bound on $\|\chi_K \Pi_\lambda\|_{2\to 2(d+1)/(d-1)}$. A systematic study was carried out by Koch and Tataru, and they almost completely characterized L^2–L^q bounds (see [12, Corollary 3.2]) including the lower bounds $\|\Pi_\lambda\|_{2\to q} \geq C\mathrm{B}_q(\lambda)$ for some constant $C > 0$ when $2 < q < \infty$ (see [12, Section 5]).

However, prior to the present work, the optimal estimate remains unsettled for $q = q_\circ$. This contrasts with the spectral projections of other related differential operators whose optimal L^2–L^q bounds are well understood [7, 11, 18, 20]. By virtue of localized estimates over annuli (see (1.2) below), it was known [14] that

$$\|\Pi_\lambda\|_{2\to q_\circ} \leq C\lambda^{-1/(2(d+3))}(\log \lambda)^{1/q_\circ}.$$

When $d = 1$, the estimate fails without the logarithmic factor. However, when $d \geq 2$, it was conjectured in [12] that the natural bound (1.1) extends to the missing endpoint $q = q_\circ$. This case is the most significant since interpolation recovers the sharp bounds for $2 < q < 2d/(d - 2)$.

We prove the conjecture is true for every $d \geq 3$.

Theorem 1.1. *Let $d \geq 3$. Then,*

$$\|\Pi_\lambda\|_{2\to q_\circ} \sim \lambda^{-1/(2(d+3))}.$$

It is likely that the theorem continues to be true for $d = 2$ but our argument in this paper is not enough to prove this case.

Localized estimate. For $\mu \in \mathbb{D}^- := \{2^k : -k \in \mathbb{N}\}$, we set

$$A_\mu^\pm = \{x : \pm(1 - |x|) \in [\mu, 2\mu]\}, \quad A_{\lambda,\mu}^\pm = \{x : \lambda^{-1/2}x \in A_\mu^\pm\},$$

respectively.

For simplicity, we also denote

$$\chi_\mu^\pm = \chi_{A_\mu^\pm}, \quad \chi_{\lambda,\mu}^\pm = \chi_{A_{\lambda,\mu}^\pm}.$$

Of special interest is the estimate over the region near the sphere $\sqrt{\lambda}\mathbb{S}^{d-1} = \{x : |x| = \sqrt{\lambda}\}$, across which the kernel of Π_λ exhibits different behaviors. Koch and Tataru [12] considered the localized operator $\chi_{\lambda,\mu}^\pm \Pi_\lambda$. They proved the following sharp bounds:

$$\|\chi_{\lambda,\mu}^+ \Pi_\lambda\|_{2\to q} \le C \begin{cases} \lambda^{-\delta(2,q)/2} \mu^{(1-(d+3)\delta(2,q))/4}, & 2 \le q \le \dfrac{2(d+1)}{d-1}, \\ (\lambda\mu)^{(d\delta(2,q)-1)/2}, & \dfrac{2(d+1)}{d-1} \le q \le \infty \end{cases} \tag{1.2}$$

for $\lambda^{-2/3} \le \mu \le 1/4$ [12, Theorem 3].[1] Summation over μ and interpolation with the previously known bound give (1.1) except for $q = q_\circ$. Meanwhile, the estimates for $\chi_{\lambda,\mu}^- \Pi_\lambda$ are of less interest, since $\chi_{\lambda,\mu}^- \Pi_\lambda$ has much smaller bounds thanks to rapid decay of its kernel (e.g., see (2.36) below).

Let $\mu_\circ = \lambda^{-2/3}$, $c_\circ = (100d)^{-2}$. Thanks to the estimates in [12] which are mentioned above, we already have the desired L^2–$L^{q_\circ}$ bounds on the operators $\sum_{c_\circ \le \mu < 1} \chi_{\lambda,\mu}^+ \Pi_\lambda$ and $(1 - \sum_{\mu_\circ < \mu < 1} \chi_{\lambda,\mu}^+)\Pi_\lambda$. (We refer the reader forward to Section 2.7 for the detail.) Therefore, to prove Theorem 1.1 it is sufficient to consider L^2–$L^{q_\circ}$ bound on the operator

$$\Pi_\lambda' := \sum_{\mu_\circ < \mu < c_\circ} \chi_{\lambda,\mu}^+ \Pi_\lambda.$$

By duality, $\|\Pi_\lambda'\|_{2\to q_\circ}^2 = \|\Pi_\lambda'(\Pi_\lambda')^*\|_{q_\circ' \to q_\circ}$. Since the lower bound is already shown in [12], (1.1) follows if one shows

$$\|\Pi_\lambda'(\Pi_\lambda')^*\|_{q_\circ' \to q_\circ} \le C\lambda^{-\delta(q_\circ',q_\circ)/2}.$$

Note $\Pi_\lambda^* = \Pi_\lambda$ and $\Pi_\lambda^2 = \Pi_\lambda$. So, we can write

$$\Pi_\lambda'(\Pi_\lambda')^* = \sum_{\mu_\circ < \mu, \widetilde{\mu} < c_\circ} \chi_{\lambda,\mu}^+ \Pi_\lambda \chi_{\lambda,\widetilde{\mu}}^+.$$

This naturally leads us to consider L^p–L^q bounds on the operators $\chi_{\lambda,\mu}^+ \Pi_\lambda \chi_{\lambda,\widetilde{\mu}}^+$ for general exponents p, q, not necessarily restricted to the case $p = q'$. Since

$$\|\chi_{\lambda,\mu}^+ \Pi_\lambda \chi_{\lambda,\widetilde{\mu}}^+\|_{p\to q} \le \|\chi_{\lambda,\mu}^+ \Pi_\lambda\|_{2\to q} \|\chi_{\lambda,\widetilde{\mu}}^+ \Pi_\lambda\|_{2\to p'},$$

bound (1.2) yields

$$\|\chi_{\lambda,\mu}^+ \Pi_\lambda \chi_{\lambda,\widetilde{\mu}}^+\|_{p\to q} \le C\lambda^{-\delta(p,q)/2}(\mu\widetilde{\mu})^{(2-(d+3)\delta(p,q))/8} \tag{1.3}$$

for $2(d+1)/(d+3) \le p \le 2 \le q \le 2(d+1)/(d-1)$. Attempting to add those bounds with some interpolation trick does not seem feasible to recover the missing endpoint case. Due to the optimality of (1.2), (1.3) with $p = q'$ cannot be improved if $\mu \sim \widetilde{\mu}$. However, this does not exclude the possibility of an improved bound when $\mu \not\sim \widetilde{\mu}$.

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[1] This is what was proved in [12], where some care with the notation $\ell_\lambda^\infty L^p$ seems necessary.

Asymmetric localization. Our main novelty is in the following theorem which shows improvement of the bounds thanks to the *asymmetric* localization near $\sqrt{\lambda}\mathbb{S}^{d-1}$. In other words, the bound on the operator $\chi^+_{\lambda,\mu}\Pi_\lambda\chi^+_{\lambda,\tilde{\mu}}$, $\tilde{\mu} \leq \mu$, compared with (1.3), significantly improves as $\tilde{\mu}/\mu$ gets smaller.

Theorem 1.2. *Let $d \geq 3$, $\mu, \tilde{\mu} \in \mathbb{D}^-$, and $\lambda^{-2/3} \leq \tilde{\mu} \leq \mu \leq c_\circ = (100d)^{-2}$. If $2 < q \leq 2(d+1)/(d-1)$, then there are positive constants c, C, independent of μ, $\tilde{\mu}$, and λ, such that*

$$\|\chi^+_{\lambda,\mu}\Pi_\lambda\chi^+_{\lambda,\tilde{\mu}}\|_{q'\to q} \leq C\lambda^{-\delta(q',q)/2}(\mu\tilde{\mu})^{(2-(d+3)\delta(q',q))/8}\left(\frac{\tilde{\mu}}{\mu}\right)^c. \tag{1.4}$$

This is a new phenomenon which has not been observed before. Our approach in this paper provides an elementary alternative proof of estimate (1.2) which corresponds to the case $\mu \sim \tilde{\mu}$. However, as we shall see later, to obtain the improved bound (1.4) is far less trivial (Sections 3 and 4). Estimate (1.4) can also be extended to some p, q other than $p = q'$ by interpolation (see also [8]).

We briefly explain how one can obtain the missing endpoint bound from (1.4). More details are to be provided in Section 2.7. We write

$$\Pi'_\lambda(\Pi'_\lambda)^* = \sum_k \mathcal{T}_k := \sum_k \sum_{(\mu,\tilde{\mu})\in\mathfrak{D}_k} \chi^+_{\lambda,\mu}\Pi_\lambda\chi^+_{\lambda,\tilde{\mu}},$$

where $\mathfrak{D}_k = \{(\mu, \tilde{\mu}) : \tilde{\mu}/\mu \in [2^k, 2^{k+1}), \mu, \tilde{\mu} \in (\mu_\circ, c_\circ)\}$ and $\mu, \tilde{\mu} \in \mathbb{D}^-$. Considering the adjoint operators, we also have improved bounds with the additional factor $(\mu/\tilde{\mu})^c$ when $\mu < \tilde{\mu}$. Thus, applying Theorem 1.2 with $q = q_\circ$, one gets

$$\|\mathcal{T}_k\|_{q'_\circ\to q_\circ} \leq C2^{-c|k|}\lambda^{-\delta(q'_\circ,q_\circ)/2},$$

which consequently shows the desired endpoint bound $\|\Pi'_\lambda\|_{2\to q_\circ} \lesssim \lambda^{-1/(2(d+3))}$.

Weighted estimates. Through the same argument, we can obtain a more general result which contains the endpoint bound in Theorem 1.1. In fact, using Theorem 1.2, we prove the following weighted estimates which were conjectured in [12, Remark 3.1]. Let us set

$$w_\pm(x) = 1 + \lambda^{-1/3}(\lambda - |x|^2)_\pm.$$

Corollary 1.3. *Let $d \geq 3$, $2 < q \leq \infty$. Set*

$$\gamma = \gamma(q) := \min\left(\frac{d+3}{4}\delta(2,q) - \frac{1}{4}, \frac{1}{2} - \frac{d}{2}\delta(2,q)\right).$$

Then, for $N > 0$, there is a constant $C = C(N)$ such that

$$\|w^\gamma_+ w^N_- \Pi_\lambda f\|_q \leq C\lambda^{(d\delta(2,q)-1)/6}\|f\|_2. \tag{1.5}$$

In particular, if we take $q = q_\circ$, then $\gamma = 0$ and hence $w^\gamma_+ w^N_- \geq 1$. So, the endpoint bound $\|\Pi_\lambda\|_{2\to q_\circ} \leq C\lambda^{-1/(2(d+3))}$ follows from (1.5).

Organization. In Section 2, we obtain some preparatory results, and we prove Corollary 1.3 while assuming Theorem 1.2. In Section 3, we reduce the proof of Theorem 1.2 to that of an L^2-estimate, which we show in Section 4.

Notation 1.4. For nonnegative quantities A and B, the inequality $B \lesssim A$ means that there is a constant C, depending only on dimensions, such that $B \leq CA$. Likewise, $A \sim B$ if and only if $B \lesssim A$ and $A \lesssim B$. By $D = O(A)$ we denote $|D| \lesssim A$. Additionally, we denote $A \gg B$ if there is a sufficiently large constant $C > 0$ such that $A \geq CB$. By c and $\varepsilon_\circ$ we denote positive constants which are chosen to be small enough.

2. The projection operator Π_λ

In this section, we make reductions toward the proof of Theorem 1.2. We also obtain some estimates for the projection operator Π_λ, which are to be used in Sections 3 and 4. At the end of this section, we provide the proof of Corollary 1.3 while assuming Theorem 1.2.

2.1. The kernel of Π_λ

The Hermite–Schrödinger propagator $e^{-it\mathcal{H}} f$ is given by

$$e^{-it\mathcal{H}} f = \sum_{\lambda \in 2\mathbb{N}_0 + d} e^{-it\lambda} \Pi_\lambda f, \quad f \in \mathcal{S}(\mathbb{R}^d). \tag{2.1}$$

Here, $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class on $\mathbb{R}^d$. It is easy to see the series converges uniformly. Note that $\mathcal{H}^N \Phi_\alpha = (2|\alpha| + d)^N \Phi_\alpha$. Thus, integration by parts gives $\langle f, \Phi_\alpha \rangle = (d + 2|\alpha|)^{-N} \langle \mathcal{H}^N f, \Phi_\alpha \rangle$. Also note that $\|\Phi_\alpha\|_p \leq C(1 + |\alpha|)^{d/4}$ for $1 \leq p \leq \infty$ (e.g., see [23]). Thus, $|\langle f, \Phi_\alpha \rangle| \leq C(d + 2|\alpha|)^{-N+d/4}$ since $\mathcal{H}^N f \in \mathcal{S}(\mathbb{R}^d)$. Therefore, taking N large enough, we see the series converges uniformly.

Note

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it(\lambda - \lambda')/2} dt = \delta(\lambda - \lambda'), \quad \lambda, \lambda' \in 2\mathbb{N}_0 + d.$$

So, we have

$$\Pi_\lambda f = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{\lambda' \in 2\mathbb{N}_0 + d} e^{it(\lambda - \lambda')/2} \Pi_{\lambda'} f \, dt, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

since the series converges uniformly. By (2.1), it follows that

$$\Pi_\lambda f = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it(\lambda - \mathcal{H})/2} f \, dt, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Now, combining this and Mehler's formula that expresses the kernel of $e^{-it\mathcal{H}}$ (e.g., see [22, p. 11] and [17]), we get the following.

Lemma 2.1. *Let $\lambda \in 2\mathbb{N}_0 + d$. Set $\mathfrak{a}(t) = (2\pi i \sin t)^{-d/2} e^{i\pi d/4}$ and*

$$\phi_\lambda(x, y, t) = \frac{\lambda t}{2} + \frac{|x|^2 + |y|^2}{2} \cot t - \langle x, y \rangle \csc t.$$

Then, for $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\Pi_\lambda f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathfrak{a}(t) \int e^{i\phi_\lambda(x,y,t)} f(y) \mathrm{d}y \mathrm{d}t. \tag{2.2}$$

In what follows, by $T(x, y)$ we denote the kernel of an operator T. Note

$$\phi_\lambda(\mathbf{U}x, \mathbf{U}y, t) = \phi_\lambda(x, y, t), \quad \mathbf{U} \in O(d), \tag{2.3}$$

where $O(d)$ denotes the orthonormal group in $\mathbb{R}^d$. Obviously, formula (2.3) implies $\Pi_\lambda(\mathbf{U}x, \mathbf{U}y) = \Pi_\lambda(x, y)$. Kochneff [15] made the same observation by using the properties of the Hermite functions.

Dyadic decomposition. We dyadically decompose integral (2.2) away from the singularities $0, \pm\pi$ of $\mathfrak{a}$. To do so, let $\psi \in C_c^\infty([1/4, 1])$ such that $\sum_{j \in \mathbb{Z}} \psi(2^j t) = 1$ for $t > 0$, and then define ψ^0 by

$$\psi^0(t) + \sum_{j \geq 4}(\psi(2^j t) + \psi(-2^j t) + \psi(2^j(t + \pi)) + \psi(2^j(\pi - t))) = 1 \tag{2.4}$$

for $t \in (-\pi, \pi) \setminus \{0\}$. So, supp $\psi^0 \subset (-\pi, \pi) \setminus \{0\}$. For a bounded function η supported in $[-\pi, \pi]$, we consider

$$\Pi_\lambda[\eta] = \int \eta(t) e^{it(\lambda - \mathcal{H})/2} \mathrm{d}t. \tag{2.5}$$

Since $\|e^{-it\mathcal{H}} f\|_2 = \|f\|_2$ for $t \in \mathbb{R}$, we have

$$\|\Pi_\lambda[\eta]\|_{2\to2} \leq \|\eta\|_1. \tag{2.6}$$

For simplicity, let us denote

$$\psi_j^\pm(t) = \psi(\pm 2^j t), \quad \psi_j^{\pm\pi}(t) = \psi(2^j(\pi \mp t)), \quad \psi_j^0 = \begin{cases} \psi^0, & j = 4, \\ 0, & j > 4. \end{cases}$$

Then, using (2.4) we decompose

$$\Pi_\lambda f = \sum_{\kappa = 0, \pm, \pm\pi} \sum_{j \geq 4} \Pi_\lambda[\psi_j^\kappa] f. \tag{2.7}$$

Validity of the decomposition is clear since the sum on the right-hand side converges to Π_λ as a bounded operator on L^2. Indeed, (2.6) gives $\|\Pi_\lambda[\psi_j^\kappa]\|_{2\to2} \lesssim 2^{-j}, \kappa = \pm, \pm\pi$, so the convergence follows.

We obtain the estimate for Π_λ by considering each $\Pi_\lambda[\psi_j^\kappa]$. However, thanks to symmetric properties of the kernels $\Pi_\lambda[\psi_j^\kappa](x, y)$, the matter reduces to showing the

estimates for $\Pi_\lambda[\psi_j^+]$ and $\Pi_\lambda[\psi^0]$. Indeed, observe that $\phi_\lambda(x, y, -t) = -\phi_\lambda(x, y, t)$ and $\phi_\lambda(x, y, \pm(\pi - t)) = \pm\lambda\pi/2 \mp \phi_\lambda(x, -y, t)$. Thus, changes of variables give

$$\Pi_\lambda[\psi_j^-](x, y) = C_d \overline{\Pi_\lambda[\psi_j^+]}(x, y), \tag{2.8}$$

$$\Pi_\lambda[\psi_j^{\pm\pi}](x, y) = C_d' \Pi_\lambda[\psi_j^{\mp}](x, -y), \tag{2.9}$$

where C_d, C_d' are constants satisfying $|C_d| = |C_d'| = 1$. This implies

$$\Big\| \sum_j \Pi_\lambda[\psi_j^+] \Big\|_{p \to q} = \Big\| \sum_j \Pi_\lambda[\psi_j^\kappa] \Big\|_{p \to q}, \quad \kappa = -, \pm\pi.$$

Rescaled operators. Instead of Π_λ and $\Pi_\lambda[\eta]$, it is more convenient to work with the rescaled operators $\mathfrak{P}_\lambda$, $\mathfrak{P}_\lambda[\eta]$ whose kernels are given by

$$\mathfrak{P}_\lambda(x, y) = \Pi_\lambda(\sqrt{\lambda}x, \sqrt{\lambda}y),$$
$$\mathfrak{P}_\lambda[\eta](x, y) = \Pi_\lambda[\eta](\sqrt{\lambda}x, \sqrt{\lambda}y),$$

respectively. By rescaling, we have for any measurable sets E, F,

$$\|\chi_E \mathfrak{P}_\lambda[\eta]\chi_F\|_{p \to q} = \lambda^{d(1/p - 1/q - 1)/2}\|\chi_{\sqrt{\lambda}E}\Pi_\lambda[\eta]\chi_{\sqrt{\lambda}F}\|_{p \to q}. \tag{2.10}$$

To prove Theorem 1.2, we need only to consider the case

$$\lambda^{-2/3} \lesssim \widetilde{\mu}, \mu \le c_\circ, \quad \frac{\widetilde{\mu}}{\mu} \le \varepsilon_\circ \tag{2.11}$$

for a small constant $\varepsilon_\circ > 0$ since (1.4) follows from (1.2) if $\mu \sim \widetilde{\mu}$.

Proposition 2.2. *Let $d \ge 3$ and μ, $\widetilde{\mu}$ satisfy* (2.11). *Set*

$$B_p(\mu, \widetilde{\mu}) = \lambda^{((d-1)\delta(p,p')-d)/2}(\mu\widetilde{\mu})^{(2-(d+3)\delta(p,p'))/8}.$$

Suppose $0 < \delta(p, p') < \min(2/(d - 1), 2/3)$. Then, for a positive constant c independent of μ and $\widetilde{\mu}$, we have

$$\|\chi_\mu^+ \mathfrak{P}_\lambda \chi_{\widetilde{\mu}}^+\|_{p \to p'} \lesssim B_p(\mu, \widetilde{\mu})\Big(\frac{\widetilde{\mu}}{\mu}\Big)^c. \tag{2.12}$$

Note that $\min(2/(d - 1), 2/3) > 2/(d + 1)$ when $d \ge 3$. Thus, Theorem 1.2 follows. In fact, (1.4) holds in a slightly bigger range. The rest of the paper is devoted to the proof of Proposition 2.2.

2.2. The phase function $\mathcal{P}$

Let us set

$$\mathcal{P}(x, y, s) = \frac{s}{2} + \frac{|x|^2 + |y|^2}{2} \cot s - \langle x, y \rangle \csc s. \tag{2.13}$$

Note $\mathcal{P}(x, y, s) = \phi_1(x, y, s)$. By Lemma 2.1, we have

$$\mathfrak{P}_\lambda[\eta](x, y) = \int (\eta \mathfrak{a})(s) e^{i\lambda \mathcal{P}(x,y,s)} ds, \tag{2.14}$$

which we shall extensively make use of throughout the paper. To obtain estimates for the kernels, we take a close look at the phase function $\mathcal{P}$. A computation together with an elementary trigonometric identity gives

$$\partial_s \mathcal{P}(x, y, s) = -\frac{\mathcal{Q}(x, y, \cos s)}{2 \sin^2 s}, \tag{2.15}$$

where

$$\mathcal{Q}(x, y, \tau) := (\tau - \langle x, y \rangle)^2 - \mathcal{D}(x, y),$$
$$\mathcal{D}(x, y) := 1 + \langle x, y \rangle^2 - |x|^2 - |y|^2. \tag{2.16}$$

The stationary points of $\mathcal{P}(x, y, \cdot)$ are given by the zeros of $\mathcal{Q}(x, y, \cos \cdot)$. So, $\mathcal{D}(x, y)$, which determines the nature of those stationary points, plays a significant role in showing various estimates for the kernel $\mathfrak{P}_\lambda[\eta](x, y)$. Note

$$\mathcal{D}(x, y) = -|x|^2 |y|^2 \sin^2 \measuredangle(x, y) + (1 - |x|^2)(1 - |y|^2), \tag{2.17}$$

where $\measuredangle(x, y)$ denotes the angle between x and y. Since $(1 - |x|^2)(1 - |y|^2) \sim \mu \tilde{\mu}$ for $(x, y) \in A_\mu^+ \times A_{\tilde{\mu}}^+$, one can control $\mathcal{D}(x, y)$ by the relative size of $\measuredangle(x, y)$ to $(\mu \tilde{\mu})^{1/2}$.

2.3. Preliminary decomposition

Fix a constant $c > 0$ such that $1/(20d) \le c \le 1/(10d)$. We partition the unit sphere $\mathbb{S}^{d-1}$ into finite disjoint subsets $\{S_j\}$ of diameter less than c. Then, set $A_j = \{x \in A_\mu^+ : |x|^{-1} x \in S_j\}$ and $\tilde{A}_j = \{x \in A_{\tilde{\mu}}^+ : |x|^{-1} x \in S_j\}$, which respectively partition the annuli A_μ^+ and $A_{\tilde{\mu}}^+$ into disjoint sets of diameter $\le c$. So, we have

$$A_\mu^+ = \bigcup_k A_k, \quad A_{\tilde{\mu}}^+ = \bigcup_l \tilde{A}_l. \tag{2.18}$$

This and (2.7) give

$$\chi_\mu^+ \mathfrak{P}_\lambda \chi_{\tilde{\mu}}^+ f = \sum_{k,l} \sum_{\kappa = 0, \pm, \pm\pi} \sum_{j \ge 4} \chi_{A_k} \mathfrak{P}_\lambda[\psi_j^\kappa] \chi_{\tilde{A}_l} f. \tag{2.19}$$

Using this coarse decomposition, we can distinguish minor parts whose contributions are negligible. More precisely, we have the following.

Lemma 2.3. *Let $1 \le p \le 2$. Let $j \ge 4$, $\lambda^{-2/3} \lesssim \mu, \tilde{\mu} \le c_\circ$, and $1/(20d) \le c \le 1/(10d)$. Suppose* $\operatorname{dist}(A_k, \tilde{A}_l) \ge c$ *and* $\operatorname{dist}(A_k, -\tilde{A}_l) \ge c$. *Then,*

$$\|\chi_{A_k} \mathfrak{P}_\lambda[\psi_j^\kappa] \chi_{\tilde{A}_l} f\|_{p'} \lesssim \lambda^{-N} 2^{-Nj} (\mu \tilde{\mu})^N \|f\|_p \quad \forall N > 0 \tag{2.20}$$

for $\kappa = 0, \pm, \pm\pi$.

Moreover,

(i) *if* dist$(A_k, \widetilde{A}_l) < c$, *we have* (2.20) *for* $\kappa = \pm\pi, 0$;

(ii) *if* dist$(A_k, -\widetilde{A}_l) < c$, *we have* (2.20) *for* $\kappa = \pm, 0$.

To show Lemma 2.3, we use the following elementary lemmas.

Lemma 2.4. *Let E, F be measurable sets. If $|T(x, y)| \leq D\chi_E(x)\chi_F(y)$ for a constant D, then $\|T\|_{p \to q} \leq D|E|^{1/q}|F|^{1-1/p}$ for $1 \leq p, q \leq \infty$.*

Lemma 2.5. *Let $N \in \mathbb{N}_0$, $0 < \mu \leq 1$, and $L, \lambda > 0$. Suppose a is a smooth function supported in an interval I of length $\sim \mu$ and ϕ is smooth on I. If $|\phi'| \gtrsim L$, $|(d/ds)^{k+1}\phi| \lesssim L\mu^{-k}$, and $|(d/ds)^k a| \lesssim \mu^{-k}$, $0 \leq k \leq N$ on I, then*

$$\left| \int e^{i\lambda\phi(s)} a(s) ds \right| \leq C\mu(1 + \lambda\mu L)^{-N} \tag{2.21}$$

for a constant $C = C(N)$ independent of λ, L and μ.

One can show Lemma 2.5 by repeated integration by parts. It can also be shown by changing variables $s \to \mu s + s_0$, where $s_0 \in \operatorname{supp} a$. Setting $\widetilde{\phi}(s) = (\mu L)^{-1}\phi(\mu s + s_0)$ and $\widetilde{a}(s) = a(\mu s + s_0)$, we see that

$$|\widetilde{\phi}'| \gtrsim 1, \quad \left|\left(\frac{d}{ds}\right)^k \widetilde{a}\right| \lesssim 1, \quad \text{and} \quad \left|\left(\frac{d}{ds}\right)^{k+1}\widetilde{\phi}\right| \lesssim 1, \quad 0 \leq k \leq N,$$

on $\operatorname{supp} \widetilde{a}$. Since the integral equals $\mu \int e^{i\lambda\mu L\widetilde{\phi}(s)}\widetilde{a}(s)ds$, routine integration by parts yields (2.21).

Proof of Lemma 2.3. We first prove (2.20) for $\kappa = 0, \pm, \pm\pi$ when dist$(A_k, \widetilde{A}_l) \geq c$ and dist$(A_k, -\widetilde{A}_l) \geq c$.

Let $(x, y) \in A_k \times \widetilde{A}_l$. Note $|x|^2|y|^2 \sin^2\angle(x, y) \geq 2^{-1}c^2$ and $0 < \mu, \widetilde{\mu} \leq 2^{-4}c^2$. By (2.17) and (2.16), $-\mathcal{D}(x, y) \sim c^2$ and $\mathcal{Q}(x, y, \tau) \gtrsim c^2$ for $\tau \in \mathbb{R}$. By (2.15), we have $|\partial_s \mathcal{P}| \gtrsim 2^{2j}$ and also $|\partial_s^l \mathcal{P}| \lesssim 2^{(l+1)j}$ on $\operatorname{supp} \psi_j^\kappa$. Note $(d/ds)^m (a\psi_j^\kappa) = O(2^{(d/2+m)j})$. Thus, using Lemma 2.5, we get

$$|\mathfrak{P}_\lambda[\psi_j^\kappa](x, y)| \lesssim 2^{(d/2-1)j}(\lambda 2^j + 1)^{-M}, \quad (x, y) \in (A_k, \widetilde{A}_l) \tag{2.22}$$

for any M. Since $\lambda^{-2/3} \lesssim \mu, \widetilde{\mu} \leq c_\circ$, (2.20) follows by Lemma 2.4.

To prove (i) and (ii), it is sufficient to show (ii) only thanks to (2.9) and the change of variables $y \to -y$. Let $(x, y) \in A_k \times \widetilde{A}_l$. Since dist$(A_k, -\widetilde{A}_l) < c$, $|1 + \langle x, y \rangle| \leq 3c$ and $|\mathcal{D}(x, y)| \leq 2c$ (by (2.17)). Note $\cos s \geq -\cos 2^{-3}$ on $\operatorname{supp} \psi_j^\kappa$, $\kappa = \pm, 0$. Thus, using (2.16), we have $|\mathcal{Q}(x, y, \cos s)| \sim 1$ and $|\partial_s \mathcal{P}(x, y, s)| \gtrsim 2^{2j}$. As before, we also have $|\partial_s^l \mathcal{P}| \lesssim 2^{(l+1)j}$ for $l \geq 1$ if $s \in \operatorname{supp} \psi_j^\kappa$, $\kappa = \pm, 0$. Hence, Lemma 2.5 yields (2.22). Consequently, (2.20) follows in the same manner as above. ∎

The bounds in Lemma 2.3 are much smaller than what we need to obtain for $\mathfrak{P}_\lambda$. Recalling (2.18) and (2.19), and discarding the harmless small contributions, we need only to consider $\chi_A \mathfrak{P}_\lambda[\psi_j^\kappa]\chi_{\widetilde{A}}$ when $A \subset A_\mu^+$, $\widetilde{A} \subset A_{\widetilde{\mu}}^+$ are of diameter c and

$$\operatorname{dist}(A, \widetilde{A}) \leq c, \ \kappa = \pm, \quad \text{or} \quad \operatorname{dist}(A, -\widetilde{A}) \leq c, \ \kappa = \pm\pi.$$

By (2.9) and changing variables $y \to -y$, the estimate for the second case can be deduced from that for the first, so it suffices to consider the first case only. Moreover, the estimate for $\chi_A \mathfrak{P}_\lambda[\psi_j^-]\chi_{\widetilde{A}}$ follows from that for $\chi_A \mathfrak{P}_\lambda[\psi_j^+]\chi_{\widetilde{A}}$ thanks to (2.8). Therefore, the matter is reduced to showing

$$\Big\| \sum_{j \geq 4} \chi_A \mathfrak{P}_\lambda[\psi_j]\chi_{\widetilde{A}} \Big\|_{p \to p'} \lesssim \mathrm{B}_p(\mu, \widetilde{\mu})\Big(\frac{\widetilde{\mu}}{\mu}\Big)^c \qquad (2.23)$$

when $A \subset A_\mu^+$, $\widetilde{A} \subset A_{\widetilde{\mu}}^+$, and $\mathrm{dist}(A, \widetilde{A}) \leq c$. Henceforth, we denote $\psi_j = \psi_j^+$. By (2.3) and (2.14), it follows that

$$\mathfrak{P}_\lambda[\psi_j](x, y) = \mathfrak{P}_\lambda[\psi_j](\mathbf{U}x, \mathbf{U}y), \quad \mathbf{U} \in \mathrm{O}(d). \qquad (2.24)$$

Set $\mathbb{S} = \{e' \in \mathbb{S}^{d-1} : |e' - e_1| < 1/(25d)\}$. By rotation, we may also assume

$$A, \widetilde{A} \subset \mathbb{A}_0 := \{x : |x|^{-1}x \in \mathbb{S}\}.$$

2.4. Sectorial decomposition of annuli

We decompose $A \times \widetilde{A}$ in a way that we can conveniently control the angle between x and y. To do so, we use a Whitney type decomposition of $\mathbb{S} \times \mathbb{S}$ away from its diagonal.

Following the typical dyadic decomposition process, for each $\nu \geq 0$ we partition $\mathbb{S}$ into spherical caps Θ_k^ν such that $\Theta_k^\nu \subset \Theta_{k'}^{\nu'}$ for some k' whenever $\nu \geq \nu'$ and $c_d 2^{-\nu} \leq \mathrm{diam}(\Theta_k^\nu) \leq C_d 2^{-\nu}$ for some constants $c_d, C_d > 0$. Let $\nu_\circ = \nu_\circ(\mu, \widetilde{\mu})$ denote the integer $\nu_\circ$ such that

$$\frac{\mu \widetilde{\mu}}{2} < 2^6 C_d^2 2^{-2\nu_\circ} \leq \mu \widetilde{\mu}.$$

Then, we can write

$$\mathbb{S} \times \mathbb{S} = \bigcup_{\nu : 2^{-\nu_\circ} \leq 2^{-\nu} \lesssim 1} \bigcup_{k \sim_\nu k'} \Theta_k^\nu \times \Theta_{k'}^\nu,$$

where $k \sim_\nu k'$ means $\mathrm{dist}(\Theta_k^\nu, \Theta_{k'}^\nu) \sim 2^{-\nu}$ if $\nu > \nu_\circ$ and $\mathrm{dist}(\Theta_k^\nu, \Theta_{k'}^\nu) \lesssim 2^{-\nu}$ if $\nu = \nu_\circ$ (e.g., see [21, p. 971]). The sets $\Theta_k^{\nu_\circ}$ and $\Theta_{k'}^{\nu_\circ}$, $k \sim_{\nu_\circ} k'$, are not necessarily distanced from each other since the decomposition process terminates at $\nu = \nu_\circ$. Consequently, it follows that

$$A \times \widetilde{A} \subset \bigcup_{\nu : 2^{-\nu_\circ} \leq 2^{-\nu} \lesssim 1} \bigcup_{k \sim_\nu k'} A_k^\nu \times \widetilde{A}_{k'}^\nu,$$

where

$$A_k^\nu = \{x \in A_\mu^+ : |x|^{-1}x \in \Theta_k^\nu\}, \quad \widetilde{A}_k^\nu = \{x \in A_{\widetilde{\mu}}^+ : |x|^{-1}x \in \Theta_k^\nu\}.$$

Let $\chi_k^\nu = \chi_{A_k^\nu}$ and $\widetilde{\chi}_k^\nu = \chi_{\widetilde{A}_k^\nu}$. Estimate (2.23) follows once we obtain

$$\Big\| \sum_{j \geq 4} \sum_{\nu \leq \nu_\circ} \sum_{k \sim_\nu k'} \chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\widetilde{\chi}_{k'}^\nu \Big\|_{p \to p'} \lesssim \mathrm{B}_p(\mu, \widetilde{\mu})\Big(\frac{\widetilde{\mu}}{\mu}\Big)^c, \qquad (2.25)$$

which we prove in Section 3, for $0 < \delta(p, p') < \min(2/(d-1), 2/3)$.

We occasionally use the next elementary lemma.

Lemma 2.6. *Let $1 \le p \le q \le \infty$. Suppose $\|\chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\widetilde{\chi}_{k'}^\nu\|_{p \to q} \le B$ holds whenever $k \sim_\nu k'$. Then, for a constant C,*

$$\Big\| \sum_{k \sim_\nu k'} \chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\widetilde{\chi}_{k'}^\nu \Big\|_{p \to q} \le CB.$$

Proof. For each k, there are as many as $O(1)$ k' such that $k \sim_\nu k'$. Thus, it is clear that

$$\Big\| \sum_{k \sim_\nu k'} \chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\widetilde{\chi}_{k'}^\nu f \Big\|_q^q \lesssim \sum_{k \sim_\nu k'} \|\chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\widetilde{\chi}_{k'}^\nu f\|_q^q.$$

So, it follows that

$$\Big\| \sum_{k \sim_\nu k'} \chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\widetilde{\chi}_{k'}^{\kappa,\nu} f \Big\|_q^q \lesssim B^q \sum_{k'} \|\widetilde{\chi}_{k'}^\nu f\|_p^q.$$

The right-hand side is clearly bounded by $CB^q \|f\|_p^q$ since $p \le q$ and $\{\widetilde{A}_k^\nu\}_k$ are disjoint. ∎

2.5. The kernel of $\mathfrak{P}_\lambda[\psi_j]$

In this subsection, we obtain estimates for the kernel $\mathfrak{P}_\lambda[\psi_j](x, y)$, which we use later.

Lemma 2.7. *Let $0 < \widetilde{\mu} \le \mu \ll 2^{-\nu} \le 1/100$, and $(x, y) \in A_k^\nu \times \widetilde{A}_{k'}^\nu$, $k \sim_\nu k'$. Then, for any $N > 0$, we have*

$$|\mathfrak{P}_\lambda[\psi_j](x, y)| \lesssim 2^{(d-2)j/2}(\lambda 2^j \max(2^{-2\nu}, 2^{-4j}) + 1)^{-N}. \tag{2.26}$$

Proof. Note $\angle(x, y) \sim 2^{-\nu}$ for $(x, y) \in A_k^\nu \times \widetilde{A}_{k'}^\nu$, $k \sim_\nu k'$. Since $\mu \ll 2^{-\nu}$, it is easy to see that $|x - y| \sim 2^{-\nu}$. So, $-\mathcal{D}(x, y) \sim 2^{-2\nu}$ by (2.17). Note $|x - y|^2 + \mathcal{D}(x, y) = (1 - \langle x, y \rangle)^2$, hence $|1 - \langle x, y \rangle| \lesssim 2^{-\nu}$. Combining these observations with (2.15) and (2.16), we have

$$|\partial_s \mathcal{P}(x, y, s)| \gtrsim 2^{2j} \max(2^{-2\nu}, 2^{-4j}), \quad s \in \operatorname{supp} \psi_j.$$

By (2.15), it also follows that $|\partial_s^k \mathcal{P}(x, y, s)| \lesssim 2^{(1+k)j} \max(2^{-2\nu}, 2^{-4j})$ for $s \in \operatorname{supp} \psi_j$. Thus, using Lemma 2.5 with $L = 2^{2j} \max(2^{-2\nu}, 2^{-4j})$ and $\mu = 2^{-j}$, we get (2.26) in the same manner as in the proof of Lemma 2.3. ∎

Lemma 2.8. *Let $0 < \widetilde{\mu} \ll \mu \le 1/100$ and $(x, y) \in A_k^\nu \times \widetilde{A}_{k'}^\nu$, $k \sim_\nu k'$. Suppose $2^{-\nu_\circ} \le 2^{-\nu} \lesssim \mu$. Then, for any $N > 0$, we have*

$$|\mathfrak{P}_\lambda[\psi_j](x, y)| \lesssim \begin{cases} 2^{(d-2)j/2}(\lambda 2^j \mu^2 + 1)^{-N}, & 2^{-j} \ll \sqrt{\mu}, \\ 2^{(d-2)j/2}(\lambda 2^{-3j} + 1)^{-N}, & 2^{-j} \gg \sqrt{\mu}. \end{cases} \tag{2.27}$$

Additionally, if $2^{-j} \sim \sqrt{\mu}$, and $\mathcal{D}(x, y) \sim \mu\widetilde{\mu}$ or $\mathcal{D}(x, y) < 0$, then

$$|\mathfrak{P}_\lambda[\psi_j](x, y)| \lesssim \lambda^{-1/2} \mu^{(1-d)/4} |\mathcal{D}(x, y)|^{-1/4}. \tag{2.28}$$

Proof. We consider (2.27) first. To this end, we claim

$$|\partial_s \mathcal{P}(x, y, s)| \gtrsim \begin{cases} \mu^2 2^{2j}, & 2^{-j} \ll \sqrt{\mu}, \\ 2^{-2j}, & 2^{-j} \gg \sqrt{\mu}, \end{cases} \quad s \in \operatorname{supp} \psi_j. \quad (2.29)$$

Note that $2(1 - \langle x, y \rangle) = 1 - |x|^2 + 1 - |y|^2 + |x - y|^2$. Since $|x - y| \sim \mu$, then $|1 - \langle x, y \rangle| \sim \mu$. So, $|\partial_\tau \mathcal{Q}(x, y, \tau)| = 2|\tau - \langle x, y \rangle| \lesssim \mu$ if $\tau \in [1 - c\mu, 1]$ for some $c > 0$. If $2^{-j} \ll \sqrt{\mu}$, by the mean value theorem $|\mathcal{Q}(x, y, \cos s) - \mathcal{Q}(x, y, 1)| \ll \mu^2$ for $s \in \operatorname{supp} \psi_j$ because $|1 - \cos s| \leq s^2/2$. Observing $\mathcal{Q}(x, y, 1) = |x - y|^2 \sim \mu^2$, we see $\mathcal{Q}(x, y, \cos s) \sim \mu^2$ for $s \in \operatorname{supp} \psi_j$ if $2^{-j} \ll \sqrt{\mu}$. Thus, by (2.15) we have the first case in (2.29). For the second case, note $|\mathcal{D}(x, y)| \lesssim 2^{-2\nu} \lesssim \mu^2$ and $1 - \tau \sim 2^{-2j}$ for $\tau \in \cos(\operatorname{supp} \psi_j)$. Recalling (2.16), we have $\mathcal{Q}(x, y, \tau) = (\tau - \langle x, y \rangle)^2 - \mathcal{D}(x, y) \sim 2^{-4j}$ if $2^{-j} \gg \sqrt{\mu}$. So, by (2.15) we obtain the second case in (2.29).

A computation using (2.15) shows

$$|\partial_s^k \mathcal{P}(x, y, s)| \lesssim \begin{cases} \mu^2 2^{(1+k)j}, & 2^{-j} \ll \sqrt{\mu}, \\ 2^{-4j} 2^{(1+k)j}, & 2^{-j} \gg \sqrt{\mu}, \end{cases} \quad s \in \operatorname{supp} \psi_j. \quad (2.30)$$

Therefore, combining (2.29) and (2.30), we obtain (2.27) by Lemma 2.5.

We now turn to formula (2.28) and consider the case $\mathcal{D}(x, y) \sim \mu\widetilde{\mu}$ first. Since $|1 - \langle x, y \rangle| \sim \mu$, then $\mathcal{Q}(x, y, \cdot)$ has two distinct zeros $r_1 > r_2$, which are close to 1. Let $s_1, s_2 \in (0, \pi/2)$ be numbers such that $\cos s_i = r_i$, $i = 1, 2$, and $s_1 < s_2$. By (2.15) and (2.16), we have

$$\partial_s \mathcal{P}(x, y, s) = -\frac{\int_{s_1}^s \sin \tau \, d\tau \int_{s_2}^s \sin \tau \, d\tau}{2 \sin^2 s}. \quad (2.31)$$

We decompose the integral $I_j := \mathfrak{P}_\lambda[\psi_j](x, y)$ away from s_1, s_2. Let us set $\psi_l^{k,\pm}(s) = \psi(\pm 2^k(s - s_l))$, $l = 1, 2$. Let $c > 0$ be a small constant and k_0 be the smallest integer such that $2^{-k_0} \leq c\widetilde{\mu}^{1/2}$. Then, we put

$$\psi_* = 1 - \sum_k \psi_1^{k,-} - \sum_{k \geq k_0} \psi_1^{k,+} - \sum_{k \geq k_0} \psi_2^{k,-} - \sum_k \psi_2^{k,+},$$

so $\operatorname{supp} \psi_* \subset [s_1 + c_1\widetilde{\mu}^{1/2}, s_2 - c_1\widetilde{\mu}^{1/2}]$ for a constant $c_1 > 0$. Also set

$$I_l^{k,\pm} = \int (\psi_l^{k,\pm} \psi_j \mathfrak{a})(s) e^{i\lambda \mathcal{P}(x,y,s)} ds,$$

$$I_* = \int (\psi_* \psi_j \mathfrak{a})(s) e^{i\lambda \mathcal{P}(x,y,s)} ds.$$

Thus, we have $I_j = I_* + I_1^- + I_1^+ + I_2^- + I_2^+$, where

$$I_1^- = \sum_k I_1^{k,-}, \quad I_1^+ = \sum_{k \geq k_0} I_1^{k,+}, \quad I_2^- = \sum_{k \geq k_0} I_2^{k,-}, \quad I_2^+ = \sum_k I_2^{k,+}.$$

Since $\cos s_1 - \cos s_2 = 2\sqrt{\mathcal{D}} \sim (\mu\tilde{\mu})^{1/2}$ and $s_1, s_2 \sim \sqrt{\mu} \sim 2^{-j}$, then $s_2 - s_1 \sim \tilde{\mu}^{1/2}$. By (2.31), it follows that $|\partial_s \mathcal{P}(x, y, s)| \gtrsim \tilde{\mu}$ for $s \in \text{supp } \psi_*$. Since $\|(\psi_* \mathfrak{a} \psi_j)'\|_1 \lesssim 2^{dj/2}$, van der Corput's lemma (e.g., [19, p. 334]) gives

$$|I_*| \lesssim \min(\lambda^{-1} 2^{(d-4)j/2} \mathcal{D}^{-1}, 2^{(d+2)j/2} \mathcal{D}^{1/2}),$$

which yields $|I_*| \lesssim \lambda^{-1/2} \mu^{(1-d)/4} \mathcal{D}^{-1/4}$. Therefore, we have only to show that

$$|I_l^{\pm}| \lesssim \lambda^{-1/2} 2^{(d-1)j/2} \mathcal{D}^{-1/4}, \quad l = 1, 2.$$

We consider I_1^- only. The estimates for the others can be shown in a similar manner. Since $s_2 - s_1 \sim \tilde{\mu}^{1/2}$, by (2.31) we have $|\partial_s \mathcal{P}(x, y, s)| \gtrsim 2^j 2^{-k} \mathcal{D}^{1/2}$ for $s \in \text{supp } \psi_1^{k,-}$. By van der Corput's lemma, $|I_1^{k,-}| \lesssim 2^{dj/2} \min(\lambda^{-1} 2^{k-j} \mathcal{D}^{-1/2}, 2^{-k})$. Summation over k gives $|I_1^-| \lesssim \lambda^{-1/2} 2^{(d-1)j/2} \mathcal{D}^{-1/4}$ as desired.

Following the previous argument closely, we show (2.28) when $\mathcal{D} < 0$. From (2.15) and (2.16), we have

$$|\partial_s \mathcal{P}(x, y, s)| \sim ((\cos s - \langle x, y \rangle)^2 + |\mathcal{D}|)\mu^{-1}, \quad s \in \text{supp } \psi_j. \tag{2.32}$$

Let $s_* \in (0, \pi/2)$ denote the point such that $\cos s_* = \langle x, y \rangle$, and let k_* be the smallest number satisfying that $2^{-k_*} \leq |\mathcal{D}|^{1/2} \mu^{-1/2}$. We decompose the integral I_j, using the cutoff functions $\psi^k(s) = \psi(2^k(s - s_*)) + \psi(2^k(s_* - s))$, $k < k_*$, and $\psi^{k_*}(s) := 1 - \sum_{k \leq k_*} \psi^k(s)$. As before, setting

$$I^k := \int (\psi^k \psi_j \mathfrak{a})(s) e^{i\lambda \mathcal{P}(x,y,s)} ds, \quad I^{k_*} := \int (\psi^{k_*} \psi_j \mathfrak{a})(s) e^{i\lambda \mathcal{P}(x,y,s)} ds,$$

we break $I_j = \sum_{k \leq k_*} I^k$. From (2.32), we see $|\partial_s \mathcal{P}(x, y, s)| \gtrsim 2^{-2k}$ for $s \in \text{supp } \psi^k$. So, the van der Corput lemma gives $|I^k| \lesssim \mu^{-d/4} \min(\lambda^{-1} 2^{2k}, 2^{-k}) \leq \mu^{-d/4} \lambda^{-1/2} 2^{k/2}$. Taking sum over $k \leq k_*$, we get (2.28). ∎

2.6. An L^2-estimate for $\mathfrak{P}_\lambda[\eta_j]$

We denote $A_\mu^\circ = \{x : |1 - |x|| \leq 2\mu\}$ and $A_{\lambda,\mu}^\circ = \{x : \lambda^{-1/2} x \in A_\mu^\circ\}$. We also set

$$\chi_\mu^\circ = \chi_{A_\mu^\circ}, \quad \chi_{\lambda,\mu}^\circ = \chi_{A_{\lambda,\mu}^\circ}.$$

Lemma 2.9. *Let $\lambda^{-2/3} \lesssim \tilde{\mu}$, $\mu \leq 1/4$ and $2^{-j} \gtrsim (\lambda\mu)^{-1}$. Suppose $\eta_j \in C_c^\infty(-\pi, \pi)$ supported in an interval of length $\sim 2^{-j}$ satisfies $|\eta_j^{(k)}| \lesssim 2^{jk}$ for any k. Then,*

$$\|\chi_\mu^\circ \mathfrak{P}_\lambda[\eta_j] \chi_{\tilde{\mu}}^\circ\|_{2\to 2} \lesssim \lambda^{-d/2} (\mu\tilde{\mu})^{1/4}. \tag{2.33}$$

To prove (2.33), we instead show $\|\chi_{\lambda,\mu}^\circ \Pi_\lambda[\eta_j] \chi_{\lambda,\tilde{\mu}}^\circ\|_{2\to 2} \lesssim (\mu\tilde{\mu})^{1/4}$ which is equivalent to (2.33) (see (2.10)). In fact, we can show a stronger estimate

$$\|\chi_{\lambda,\mu}^e \Pi_\lambda[\eta_j] \chi_{\lambda,\tilde{\mu}}^e\|_{2\to 2} \lesssim (\mu\tilde{\mu})^{1/4} \tag{2.34}$$

for $\lambda^{-2/3} \lesssim \widetilde{\mu}, \mu \leq 1/4$, where

$$\chi^e_{\lambda,\mu} = \chi_{A^e_{\lambda,\mu}}, \quad A^e_{\lambda,\mu} = \{x : |x| \geq \lambda^{1/2}(1-\mu)\}.$$

We now recall the next estimates which follow from [12, Theorem 3]:

$$\|\chi^{\circ}_{\lambda,\lambda^{-2/3}} \Pi_\lambda \chi^{\circ}_{\lambda,\lambda^{-2/3}}\|_{p \to p'} \lesssim \lambda^{d\delta(p,p')/6-1/3}, \tag{2.35}$$

$$\|\chi^-_{\lambda,\mu} \Pi_\lambda \chi^-_{\lambda,\mu}\|_{p \to p'} \lesssim \lambda^{d\delta(p,p')/6-1/3}(\lambda^{2/3}\mu(1+\mu))^{-M}, \quad \lambda^{-2/3} \lesssim \mu, \tag{2.36}$$

for $1 \leq p \leq 2$. Estimate (2.36) holds for any $M > 0$. In particular, the L^2–L^∞-estimate in [12, Theorem 3] implies (2.36) for $p = 1$. Interpolation with L^2-estimate shows (2.36) for $1 \leq p \leq 2$. One can also show (2.35) and (2.36) in an elementary manner using estimates for the kernels of Π_λ (e.g., see [8]).

Proof of (2.34). Considering the adjoint operator, we may assume $\widetilde{\mu} \leq \mu$. We begin by showing

$$\|\chi^e_{\lambda,\mu} \Pi_\lambda \chi^e_{\lambda,\widetilde{\mu}}\|_{2 \to 2} \lesssim (\mu\widetilde{\mu})^{1/4}, \quad \lambda^{-2/3} \lesssim \widetilde{\mu} \leq \mu \leq 1. \tag{2.37}$$

Note $\|\chi^e_{\lambda,\mu} \Pi_\lambda \chi^e_{\lambda,\widetilde{\mu}}\|_{2\to2} \leq \|\chi^e_{\lambda,\mu} \Pi_\lambda\|_{2\to2}\|\Pi_\lambda \chi^e_{\lambda,\widetilde{\mu}}\|_{2\to2}$. By duality, it is enough to show

$$\|\chi^e_{\lambda,\mu} \Pi_\lambda\|_{2\to2} \lesssim \mu^{1/4}$$

for $\lambda^{-2/3} \lesssim \mu \leq 1$. To do this, recalling that $\mu, \widetilde{\mu}$ denote dyadic numbers, we decompose

$$\chi^e_{\lambda,\mu} = \sum_{\lambda^{-2/3} \leq \widetilde{\mu} \leq \mu} \chi^+_{\lambda,\widetilde{\mu}} + \chi^{\circ}_{\lambda,\lambda^{-2/3}} + \sum_{\lambda^{-2/3} \leq \widetilde{\mu} < 1} \chi^-_{\lambda,\widetilde{\mu}} + \sum_{1 \leq \widetilde{\mu}} \chi^-_{\lambda,\widetilde{\mu}}.$$

By (2.36), it follows that $\sum_{1 \leq \widetilde{\mu}} \|\chi^-_{\lambda,\widetilde{\mu}} \Pi_\lambda\|_{2\to2} \lesssim \lambda^{-N}$ for any N. Estimates (1.2), (2.35), and (2.36) yield

$$\sum_{\lambda^{-2/3} \leq \widetilde{\mu} \leq \mu} \|\chi^+_{\lambda,\widetilde{\mu}} \Pi_\lambda\|_{2\to2} \lesssim \mu^{1/4}, \quad \|\chi^{\circ}_{\lambda,\lambda^{-2/3}} \Pi_\lambda\|_{2\to2} \lesssim \lambda^{-1/6},$$

$$\sum_{\lambda^{-2/3} \leq \widetilde{\mu} \leq 1} \|\chi^-_{\lambda,\widetilde{\mu}} \Pi_\lambda\|_{2\to2} \lesssim \lambda^{-1/6},$$

respectively. Therefore, we get the desired estimate since $\mu \gtrsim \lambda^{-2/3}$.

Now, observe $\Pi_\lambda[\eta_j] = \sum_{\lambda'} \widehat{\eta}_j(2^{-1}(\lambda' - \lambda))\Pi_{\lambda'}$, which follows by (2.5) and (2.1). Since $|\widehat{\eta}_j(\tau)| \lesssim 2^{-j}(1 + 2^{-j}|\tau|)^{-N}$, we have

$$\|\chi^e_{\lambda,\mu} \Pi_\lambda[\eta_j]\chi^e_{\lambda,\widetilde{\mu}}\|_{2\to2} \lesssim 2^{-j} \sum_{\lambda'}(1 + 2^{-j}|\lambda - \lambda'|)^{-N} \|\chi^e_{\lambda,\mu} \Pi_{\lambda'} \chi^e_{\lambda,\widetilde{\mu}}\|_{2\to2}.$$

However, we can not directly apply (2.37) since $\lambda \neq \lambda'$. We get around the problem by enlarging the sets $A^e_{\lambda,\mu}$, $A^e_{\lambda,\widetilde{\mu}}$. Let

$$\ell_{\lambda'}(\rho) = \begin{cases} \left(\dfrac{\lambda}{\lambda'}\right)^{2/3} \rho, & \lambda \geq \lambda', \\ \dfrac{\lambda' - \lambda}{\lambda'} + \rho, & \lambda < \lambda'. \end{cases}$$

Note that $\ell_{\lambda'}(\rho) \gtrsim (\lambda')^{-2/3}$ for $\lambda^{-2/3} \lesssim \rho$. Since $(\lambda')^{1/2}(1 - \ell_{\lambda'}(\mu)) \le \lambda^{1/2}(1-\mu)$, i.e., $A_{\lambda,\mu}^e \subset A_{\lambda',\ell_{\lambda'}(\mu)}^e$, it follows that $\|\chi_{\lambda,\mu}^e \Pi_{\lambda'} \chi_{\lambda,\tilde{\mu}}^e\|_{2\to2} \le \|\chi_{\lambda',\ell_{\lambda'}(\mu)}^e \Pi_{\lambda'} \chi_{\lambda',\ell_{\lambda'}(\tilde{\mu})}^e\|_{2\to2}$. Using (2.37), we have

$$\|\chi_{\lambda,\mu}^e \Pi_{\lambda'} \chi_{\lambda,\tilde{\mu}}^e\|_{2\to2} \lesssim (\ell_{\lambda'}(\mu)\ell_{\lambda'}(\tilde{\mu}))^{1/4}.$$

Therefore, it suffices for (2.34) to show

$$2^{-j} \sum_{\lambda'} (1 + 2^{-j}|\lambda - \lambda'|)^{-N} (\ell_{\lambda'}(\mu)\ell_{\lambda'}(\tilde{\mu}))^{1/4} \lesssim (\mu\tilde{\mu})^{1/4}.$$

This can be shown by a simple computation because $2^{-j} \gtrsim (\lambda\mu)^{-1}$ and $\mu, \tilde{\mu} \gtrsim \lambda^{-2/3}$. We omit the detail. ∎

2.7. Proof of Corollary 1.3

Before we conclude this section, assuming Theorem 1.2, we prove Corollary 1.3. We follow the lines of argument for the endpoint bound which is sketched in the introduction.

Proof of Corollary 1.3. We first prove (1.5) for $2 < q \le q_* := 2(d+1)/(d-1)$. Let us set

$$\mathcal{W} = w_+^{((d+3)\delta(2,q)-1)/4}.$$

Note that $\mathcal{W} = w_+^\gamma$ if $2 < q \le q_*$. Recall $\mu_\circ = \lambda^{-2/3}$ and $c_\circ = (100d)^{-2}$. By the triangle inequality, $\|\mathcal{W}w_-^N \Pi_\lambda\|_{2\to q}$ is bounded by

$$\Big\| \sum_{\mu \ge \mu_\circ} \mathcal{W}\chi_{\lambda,\mu}^+ \Pi_\lambda \Big\|_{2\to q} + \|\chi_{\lambda,\mu_\circ}^\circ \Pi_\lambda\|_{2\to q} + \sum_{2^k \ge \mu_\circ} (\lambda^{2/3}2^k(2^k+1))^N \|\chi_{\lambda,2^k}^- \Pi_\lambda\|_{2\to q},$$

where $\mu \in \mathbb{D}^-$. By (2.35) and (2.36), the last two are bounded by $C\lambda^{(d\delta(2,q)-1)/6}$. It follows by (1.2) that $\sum_{\mu \ge c_\circ} \|\mathcal{W}\chi_{\lambda,\mu}^+ \Pi_\lambda\|_{2\to q} \lesssim \lambda^{(d\delta(2,q)-1)/6}$. Therefore, the matter is reduced to showing

$$\|\mathfrak{W}\|_{2\to q} \lesssim \lambda^{(d\delta(2,q)-1)/6},$$

where $\mathfrak{W} = \sum_{\mu_\circ < \mu < c_\circ} \mathcal{W}\chi_{\lambda,\mu}^+ \Pi_\lambda$. Equivalently, we need to show

$$\|\mathfrak{W}\mathfrak{W}^*\|_{q'\to q} \lesssim \lambda^{(d\delta(q',q)-2)/6}. \tag{2.38}$$

To this end, we write

$$\mathfrak{W}\mathfrak{W}^* = \sum_k (\mathfrak{W}\mathfrak{W}^*)_k := \sum_k \sum_{(\mu,\tilde{\mu})\in\mathfrak{D}_k} \mathcal{W}\chi_{\lambda,\mu}^+ \Pi_\lambda \mathcal{W}\chi_{\lambda,\tilde{\mu}}^+.$$

Recall that $\mathfrak{D}_k = \{(\mu,\tilde{\mu}) : \tilde{\mu}/\mu \in [2^k, 2^{k+1}), \mu, \tilde{\mu} \in (\mu_\circ, c_\circ)\}$ and $\mu, \tilde{\mu} \in \mathbb{D}$. Since supp $\chi_{\lambda,\tilde{\mu}}^+$ are almost disjoint, for each k we have

$$\|(\mathfrak{W}\mathfrak{W}^*)_k f\|_q \lesssim \Big(\sum_{(\mu,\tilde{\mu})\in\mathfrak{D}_k} \|\mathcal{W}\chi_{\lambda,\mu}^+ \Pi_\lambda \mathcal{W}\chi_{\lambda,\tilde{\mu}}^+ f\|_q^q \Big)^{1/q}.$$

Note $\mathcal{W}\chi^+_{\lambda,\mu} \sim \lambda^{((d+3)\delta(q',q)-2)/12}\mu^{((d+3)\delta(q',q)-2)/8}\chi^+_{\lambda,\mu}$. Using (1.4), we get

$$\|\mathcal{W}\chi^+_{\lambda,\mu}\Pi_\lambda\mathcal{W}\chi^+_{\lambda,\widetilde{\mu}}f\|_q \lesssim \lambda^{(d\delta(q',q)-2)/6}2^{-c|k|}\|\chi^+_{\lambda,\widetilde{\mu}}f\|_{q'}, \quad \frac{\widetilde{\mu}}{\mu} \sim 2^k.$$

Indeed, when $2^k > 1$, we consider the adjoint operator $T^* = \mathcal{W}\chi^+_{\lambda,\widetilde{\mu}}\Pi_\lambda\mathcal{W}\chi^+_{\lambda,\mu}$ of $T :=$ $\mathcal{W}\chi^+_{\lambda,\mu}\Pi_\lambda\mathcal{W}\chi^+_{\lambda,\widetilde{\mu}}$, and then we may use estimate (1.4) thanks to the fact that $\|T\|_{q'\to q} = \|T^*\|_{q'\to q}$. Since $\sum_{\widetilde{\mu}}\|\chi^+_{\lambda,\widetilde{\mu}}f\|^q_{q'} \lesssim \|f\|^q_{q'}$, we obtain

$$\|(\mathfrak{W}\mathfrak{W}^*)_k f\|_q \lesssim 2^{-c|k|}\lambda^{(d\delta(q',q)-2)/6}\|f\|_{q'}.$$

Summation over k gives the desired estimate (2.38).

We now prove (1.5) for $2(d+1)/(d-1) < q \leq \infty$. Note $\gamma = (1 - d\delta(2,q))/2$. By decomposing the operator in the same way as above, it suffices to show

$$\Big\|\sum_{\mu_\circ<\mu<c_\circ} w_+^{(1-d\delta(2,q))/2}\chi^+_{\lambda,\mu}\Pi_\lambda\Big\|_{2\to q} \lesssim \lambda^{(d\delta(2,q)-1)/6} \tag{2.39}$$

for $2(d+1)/(d-1) < q \leq \infty$. The other parts can be handled in the same way as before. By interpolation, we need only to show (2.39) for $q = \infty, 2(d+1)/(d-1)$. Thanks to disjointness of the annuli, (1.2) for $q = \infty$ gives (2.39) for $q = \infty$, while (2.38) is equivalent to (2.39) when $q = 2(d+1)/(d-1)$. ∎

3. Asymmetric improvement: Proof of Proposition 2.2

We prove Proposition 2.2 by establishing estimate (2.25). To this end, we separately consider some cases. The desired estimates can be shown by the kernel estimates (in the previous section) except for the case $2^{-j} \sim \sqrt{\mu}$, $2^{-\nu} \sim (\mu\widetilde{\mu})^{1/2}$, and $|\mathcal{D}| < \varepsilon_\circ\mu\widetilde{\mu}$, which requires a different approach. We handle this case in the next section.

To show (2.25), we distinguish the cases $\nu \in \mathcal{N}_{\mathbf{e}}$ and $\nu \in \mathcal{N}_{\mathbf{c}}$, where

$$\mathcal{N}_{\mathbf{e}} = \{\nu : 2^{-\nu} \gg (\mu\widetilde{\mu})^{1/2} \text{ or } \nu = \nu_\circ\},$$
$$\mathcal{N}_{\mathbf{c}} = \{\nu : (\mu\widetilde{\mu})^{1/2} \gtrsim 2^{-\nu} > 2^{-\nu_\circ}\}.$$

3.1. The sum over $\nu \in \mathcal{N}_{\mathbf{e}}$

In this case, the desired estimates are easier.

Proposition 3.1. Let $d \geq 2$ and $\widetilde{\mu}, \mu$ satisfy (2.11). If $2/(d+3) < \delta(p,p') < 2/(d-1)$, then for some $c > 0$, we have

$$\Big\|\sum_{\nu\in\mathcal{N}_{\mathbf{e}}}\sum_{j\geq 4}\sum_{k\sim_\nu k'} \chi^\nu_k \mathfrak{P}_\lambda[\psi_j]\widetilde{\chi}^\nu_{k'}\Big\|_{p\to p'} \lesssim \mathrm{B}_p(\mu,\widetilde{\mu})\Big(\frac{\widetilde{\mu}}{\mu}\Big)^c. \tag{3.1}$$

Proof. We further divide $\mathcal{N}_e = \mathcal{N}_e^1 \cup \mathcal{N}_e^2$, where

$$\mathcal{N}_e^1 = \{\nu : 2^{-\nu} \gg \mu\}, \quad \mathcal{N}_e^2 = \{\nu : \mu \gtrsim 2^{-\nu} \gg (\mu\tilde{\mu})^{1/2} \text{ or } \nu = \nu_\circ\}.$$

We first consider $\nu \in \mathcal{N}_e^1$. Using Lemma 2.7, we get

$$\|\chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\tilde{\chi}_{k'}^\nu\|_{p\to p'} \lesssim \begin{cases} \lambda^{((d-1)\delta(p,p')-d)/2} 2^{((d-1)\delta(p,p')/2-1)j} 2^{\nu\delta(p,p')}, & 2^{-2j} \lesssim 2^{-\nu}, \\ \lambda^{((d-1)\delta(p,p')-d)/2} 2^{((d+3)\delta(p,p')/2-1)j}, & 2^{-2j} \gg 2^{-\nu}, \end{cases}$$

for $k \sim_\nu k'$ and $1 \leq p \leq 2$. Indeed, taking $N = 1/2$ in (2.26), we get the L^1–L^∞ bounds, and then interpolation with $\|\chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\tilde{\chi}_{k'}^\nu\|_{2\to2} \lesssim \lambda^{-d/2} 2^{-j}$, which follows from (2.6) and (2.10), gives the estimates for $1 \leq p \leq 2$. By Lemma 2.6, we have the same bounds on $\|\sum_{k\sim_\nu k'} \chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\tilde{\chi}_{k'}^\nu\|_{p\to p'}$ as above. Since $2/(d+3) < \delta(p,p') < 2/(d-1)$, summation over j gives

$$\sum_j \Big\| \sum_{k\sim_\nu k'} \chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\tilde{\chi}_{k'}^\nu \Big\|_{p\to p'} \leq C\lambda^{((d-1)\delta(p,p')-d)/2} 2^{((d+3)\delta(p,p')/4-1/2)\nu}.$$

Thus, taking sum over $\{\nu : 2^{-\nu} \gg \mu\}$, we obtain

$$\sum_{\nu\in\mathcal{N}_e^1} \sum_j \Big\| \sum_{k\sim_\nu k'} \chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\tilde{\chi}_{k'}^\nu \Big\|_{p\to p'} \lesssim B_p(\mu,\tilde{\mu}) \Big(\frac{\tilde{\mu}}{\mu}\Big)^{((d+3)\delta(p,p')-2)/8}. \tag{3.2}$$

We turn to the case $\nu \in \mathcal{N}_e^2$. As above, by (2.27) and (2.28) we have

$$\|\chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\tilde{\chi}_{k'}^\nu\|_{p\to p'} \lesssim \begin{cases} \lambda_p 2^{((d-1)\delta(p,p')/2-1)j} \mu^{-\delta(p,p')}, & 2^{-j} \ll \sqrt{\mu}, \\ \lambda_p \mu^{(1-d\delta(p,p'))/4} \tilde{\mu}^{(1-\delta(p,p'))/4} 2^{\nu\delta(p,p')/2}, & 2^{-j} \sim \sqrt{\mu}, \\ \lambda_p 2^{((d+3)\delta(p,p')/2-1)j}, & 2^{-j} \gg \sqrt{\mu} \end{cases} \tag{3.3}$$

for $k \sim_\nu k'$ and $1 \leq p \leq 2$ when $\nu \in \mathcal{N}_e^2$. Here, λ_p denotes $\lambda^{((d-1)\delta(p,p')-d)/2}$. For the first and third cases, we take $N = 1/2$ in (2.27) to get the L^1–L^∞ estimates and interpolate them with $\|\chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\tilde{\chi}_{k'}^\nu\|_{2\to2} \lesssim \lambda^{-d/2} 2^{-j}$. We get the second case using $\|\chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\tilde{\chi}_{k'}^\nu\|_{2\to2} \lesssim \lambda^{-d/2}(\mu\tilde{\mu})^{1/4}$ and $\|\chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\tilde{\chi}_{k'}^\nu\|_{1\to\infty} \lesssim \lambda^{-1/2}\mu^{(1-d)/4} 2^{\nu/2}$, which follow from (2.33) and (2.28), respectively.

Since $2/(d+3) < \delta(p,p') < 2/(d-1)$, by Lemma 2.6 we get

$$\sum_j \Big\| \sum_{k\sim_\nu k'} \chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\tilde{\chi}_{k'}^\nu \Big\|_{p\to p'} \lesssim B_p(\mu,\tilde{\mu}) B_*(\mu,\tilde{\mu}), \quad \nu \in \mathcal{N}_e^2,$$

where

$$B_*(\mu,\tilde{\mu}) = \Big(\frac{\tilde{\mu}}{\mu}\Big)^{((d+3)\delta(p,p')-2)/8} + \mu^{-(d-3)\delta(p,p')/8} \tilde{\mu}^{(d+1)\delta(p,p')/8} 2^{\nu\delta(p,p')/2}.$$

Thus, $\sum_{\nu\in\mathcal{N}_e^2} \sum_j \|\sum_{k\sim_\nu k'} \chi_k^\nu \mathfrak{P}_\lambda[\psi_j]\tilde{\chi}_{k'}^\nu\|_{p\to p'}$ is bounded by

$$C B_p(\mu,\tilde{\mu}) \Big(\Big(\frac{\tilde{\mu}}{\mu}\Big)^{((d+3)\delta(p,p')-2)/8} \log\Big(\frac{\mu}{\tilde{\mu}}\Big) + \Big(\frac{\tilde{\mu}}{\mu}\Big)^{(d-1)\delta(p,p')/8}\Big).$$

Combining this and (3.2), we obtain (3.1) for some $c > 0$. ∎

3.2. The sum over $v \in \mathcal{N}_{\mathbf{c}}$

Since there are only as many as $O(1)$ v, it suffices to consider a single v.

Proposition 3.2. *Let $d \geq 3$, μ, $\tilde{\mu}$ satisfy (2.11), and $v \in \mathcal{N}_{\mathbf{c}}$. If*

$$\frac{2}{d+3} < \delta(p, p') < \min\left(\frac{2}{d-1}, \frac{2}{3}\right),$$

then for some $c > 0$ we have

$$\left\| \sum_j \sum_{k \sim_v k'} \chi_k^v \mathfrak{P}_\lambda[\psi_j] \tilde{\chi}_{k'}^v \right\|_{p \to p'} \lesssim \mathrm{B}_p(\mu, \tilde{\mu}) \left(\frac{\tilde{\mu}}{\mu}\right)^c.$$

Combining Propositions 3.1 and 3.2, we prove Proposition 2.2.

Proof of Proposition 2.2. Propositions 3.1 and 3.2 give (2.25), from which (2.12) follows for $2/(d+3) < \delta(p, p') < \min(2/(d-1), 2/3)$. Interpolating the estimate with

$$\|\chi_\mu^+ \mathfrak{P}_\lambda \chi_{\tilde{\mu}}^+\|_{2 \to 2} \lesssim \lambda^{-d/2}(\mu\tilde{\mu})^{1/4},$$

which follows from (2.37) after scaling, we obtain formula (2.12) for $0 < \delta(p, p') < \min(2/(d-1), 2/3)$. ∎

Since $v \in \mathcal{N}_{\mathbf{c}}$, the estimates in (2.27) remain valid. So, we have the first and third estimates in (3.3). By the same argument as before, we see

$$\left(\sum_{2^{-j} \ll \sqrt{\mu}} + \sum_{2^{-j} \gg \sqrt{\mu}} \right) \left\| \sum_{k \sim_v k'} \chi_k^v \mathfrak{P}_\lambda[\psi_j] \tilde{\chi}_{k'}^v \right\|_{p \to p'} \lesssim \mathrm{B}_p(\mu, \tilde{\mu}) \left(\frac{\tilde{\mu}}{\mu}\right)^c$$

for some $c > 0$ because $2/(d+3) < \delta(p, p') < 2/(d-1)$. Thus, by Lemma 2.6 the proof of Proposition 3.2 is now reduced to showing

$$\|\chi_k^v \mathfrak{P}_\lambda[\psi_j] \tilde{\chi}_{k'}^v\|_{p \to p'} \lesssim \mathrm{B}_p(\mu, \tilde{\mu}) \left(\frac{\tilde{\mu}}{\mu}\right)^c, \quad 2^{-j} \sim \sqrt{\mu}, \quad k \sim_v k'. \tag{3.4}$$

In what follows, we make further reductions to prove (3.4). By (2.24), we may assume that

$$A_k^v \subset \mathfrak{R} := \{x : |x_1 - 1| \sim \mu, \ |\bar{x}| \lesssim (\mu\tilde{\mu})^{1/2}\},$$
$$\tilde{A}_{k'}^v \subset \tilde{\mathfrak{R}} := \{y : |y_1 - 1| \sim \tilde{\mu}, \ |\bar{y}| \lesssim (\mu\tilde{\mu})^{1/2}\},$$

where $x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}$. For $x^* \in \mathbb{R}^d$ and $r > 0$, we denote

$$\mathfrak{R}_{\mu, \tilde{\mu}}(x^*, r) = \{(x_1, \bar{x}) : |x_1 - x_1^*| \leq r\mu, \ |\bar{x} - \bar{x}^*| \leq r(\mu\tilde{\mu})^{1/2}\}.$$

If $(x, y) \in \mathfrak{R} \times \tilde{\mathfrak{R}}$, unlike the previous cases, $\mathcal{D}(x, y)$ may vanish. To handle this, we further localize the value of $\mathcal{D}$ by decomposing $\mathfrak{R} \times \tilde{\mathfrak{R}}$. To this end, we break $\mathfrak{R} \times \tilde{\mathfrak{R}}$ into finitely many disjoint rectangles $\mathfrak{R}_{\mu, \tilde{\mu}}(x', \epsilon) \times \mathfrak{R}_{\tilde{\mu}, \mu}(y', \epsilon)$ so that $|\mathcal{D}| \ll \varepsilon_\circ \mu\tilde{\mu}$, or $|\mathcal{D}| \gtrsim \varepsilon_\circ \mu\tilde{\mu}$ holds on each of those rectangles for a small $\varepsilon_\circ > 0$. This particular form of decomposition shall be important later.

Lemma 3.3. *Let μ, $\tilde{\mu}$ satisfy (2.11) and $0 < \epsilon \le \varepsilon_\circ$. Let $(x^*, y^*) \in \mathfrak{R} \times \tilde{\mathfrak{R}}$. If $x' \in \mathfrak{R}_{\mu,\tilde{\mu}}(x^*, \epsilon)$ and $y' \in \mathfrak{R}_{\tilde{\mu},\mu}(y^*, \epsilon)$, then for a constant C, we have*

$$|\mathcal{D}(x', y') - \mathcal{D}(x^*, y^*)| \le C\epsilon\mu\tilde{\mu}. \tag{3.5}$$

Proof. Denote $z' := (z'_1, \ldots, z'_{2d}) = (x', y')$ and $z^* := (z^*_1, \ldots, z^*_{2d}) = (x^*, y^*)$, then set $\mathcal{D}_0 = \mathcal{D}(z')$ and $\mathcal{D}_k = \mathcal{D}(z^*_1, \ldots, z^*_k, z'_{k+1}, \ldots, z'_{2d})$, $1 \le k \le 2d$. So, $\mathcal{D}(x', y') - \mathcal{D}(x^*, y^*) = \sum_{k=0}^{2d-1} (\mathcal{D}_k - \mathcal{D}_{k+1})$.

Thus, (3.5) follows if we show

$$|\mathcal{D}_k - \mathcal{D}_{k+1}| \le C\epsilon\tilde{\mu}\mu, \quad 1 \le k \le 2d - 1.$$

Note $|x'_1 - x^*_1| \le \epsilon\mu$, $|y'_1 - y^*_1| \le \epsilon\tilde{\mu}$, and $|x'_j - x^*_j|, |y'_j - y^*_j| \le \epsilon(\mu\tilde{\mu})^{1/2}$, $j \ge 2$. By the mean value theorem, we only need to show

$$|\partial_{x_1}\mathcal{D}| \lesssim \tilde{\mu}, \quad |\partial_{y_1}\mathcal{D}| \lesssim \mu, \quad |\partial_{\bar{x}}\mathcal{D}|, |\partial_{\bar{y}}\mathcal{D}| \lesssim (\mu\tilde{\mu})^{1/2}$$

on $\mathfrak{R} \times \tilde{\mathfrak{R}}$. Since $\partial_x\mathcal{D}(x, y) = 2\langle x, y\rangle y - 2x$ and $\partial_y\mathcal{D}(x, y) = 2\langle x, y\rangle x - 2y$, it is clear that $|\partial_{\bar{x}}\mathcal{D}|, |\partial_{\bar{y}}\mathcal{D}| \lesssim (\mu\tilde{\mu})^{1/2}$. Writing $\partial_{x_1}\mathcal{D}(x, y) = 2x_1(y_1^2 - 1) + 2\langle \bar{x}, \bar{y}\rangle y_1$, we see $|\partial_{x_1}\mathcal{D}| \lesssim \tilde{\mu}$. Similarly, we get $|\partial_{y_1}\mathcal{D}| \lesssim \mu$. ∎

Assume that $\epsilon = c\varepsilon_\circ$ for a small enough $c > 0$. Let $\mathfrak{R}_{\mu,\tilde{\mu}}(x_k, \epsilon/2)$, $1 \le k \le K$, and $\mathfrak{R}_{\tilde{\mu},\mu}(y_l, \epsilon/2)$, $1 \le l \le L$, be almost disjoint rectangles which cover the rectangles $\mathfrak{R}$, $\tilde{\mathfrak{R}}$, respectively. We denote

$$\mathcal{B} = \mathfrak{R}_{\mu,\tilde{\mu}}(x_k, \epsilon), \quad \tilde{\mathcal{B}} = \mathfrak{R}_{\tilde{\mu},\mu}(y_l, \epsilon).$$

Taking $c > 0$ small enough, by Lemma 3.3 we may assume that one of the following holds:

$$|\mathcal{D}(x, y)| \gtrsim \varepsilon_\circ\mu\tilde{\mu}, \quad (x, y) \in \mathcal{B} \times \tilde{\mathcal{B}}, \tag{3.6}$$

$$|\mathcal{D}(x, y)| \ll \varepsilon_\circ\mu\tilde{\mu}, \quad (x, y) \in \mathcal{B} \times \tilde{\mathcal{B}}. \tag{3.7}$$

Let $\tilde{\chi}_{\mathcal{B}}$, $\tilde{\chi}_{\tilde{\mathcal{B}}}$ be smooth functions adapted to the rectangles $\mathcal{B}$, $\tilde{\mathcal{B}}$, respectively, i.e., $\operatorname{supp} \tilde{\chi}_{\mathcal{B}} \subset \mathcal{B}$, $\operatorname{supp} \tilde{\chi}_{\tilde{\mathcal{B}}} \subset \tilde{\mathcal{B}}$, and

$$\partial_{x_1}^{\alpha_1}\partial_{\bar{x}}^{\bar{\alpha}}\tilde{\chi}_{\mathcal{B}} = O(\mu^{-\alpha_1}(\mu\tilde{\mu})^{-|\bar{\alpha}|/2}), \quad \partial_{y_1}^{\alpha_1}\partial_{\bar{y}}^{\bar{\alpha}}\tilde{\chi}_{\tilde{\mathcal{B}}} = O(\tilde{\mu}^{-\alpha_1}(\mu\tilde{\mu})^{-|\bar{\alpha}|/2}). \tag{3.8}$$

For the proof of (3.4), it is enough to show

$$\|\tilde{\chi}_{\mathcal{B}}\mathfrak{P}_\lambda[\psi_j]\tilde{\chi}_{\tilde{\mathcal{B}}}\|_{p \to p'} \lesssim B_p(\mu, \tilde{\mu})\left(\frac{\tilde{\mu}}{\mu}\right)^c, \quad 2^{-j} \sim \sqrt{\mu}, \tag{3.9}$$

for some $c > 0$ while either (3.6) or (3.7) holds. When (3.6) holds, one can show (3.9) in the same manner as in the proof of Proposition 3.1. Indeed, since $2^{-\nu} \sim (\mu\tilde{\mu})^{1/2}$, $2^{-j} \sim \sqrt{\mu}$, and $|\mathcal{D}| \sim \mu\tilde{\mu}$ on $\mathcal{B} \times \tilde{\mathcal{B}}$, we have the second estimate in (3.3), which gives (3.9) for $1 \le p < 2$.

Therefore, to complete the proof of Proposition 3.2 it suffices to show the following.

Proposition 3.4. *Let $d \ge 3$ and μ, $\tilde{\mu}$ satisfy (2.11). Suppose that (3.7) holds. Then, if $0 < \delta(p, p') < 2/3$, we have (3.9) for a constant $c > 0$.*

3.3. Second-order derivative of $\mathcal{P}$

To prove Proposition 3.4, we shall dyadically decompose $\mathfrak{P}_\lambda[\psi_j](x, y)$ away from the zero of $\partial_s^2 \mathcal{P}$ (see (2.14)). A computation shows

$$\partial_s^2 \mathcal{P}(x, y, s) = -\langle x, y \rangle \frac{\mathcal{R}(x, y, \cos s)}{\sin^3 s},$$

where

$$\mathcal{R}(x, y, \tau) = \tau^2 - \langle x, y \rangle^{-1} (|x|^2 + |y|^2) \tau + 1.$$

For $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$, the equation $\mathcal{R}(x, y, \tau) = 0$ has two distinct zeros $\tau^\pm(x, y)$:

$$\tau^\pm(x, y) = \frac{|x|^2 + |y|^2 \pm |x + y||x - y|}{2\langle x, y \rangle}. \tag{3.10}$$

Since $\tau^+(x, y) > 1$, then $\tau^-(x, y)$ is more relevant for our purpose.

Lemma 3.5. *Define a function* $S_c \colon \mathcal{B} \times \widetilde{\mathcal{B}} \to (0, \pi/2)$ *by*

$$\cos S_c(x, y) = \tau^-(x, y). \tag{3.11}$$

Then, S_c *is smooth and* $S_c(x, y) \sim |x - y|^{1/2}$ *for* $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$.

We here record a few identities, which are to be useful later,

$$\partial_s^2 \mathcal{P}(x, y, s) = -\langle x, y \rangle \frac{(\cos S_c(x, y) - \cos s)(\tau^+(x, y) - \cos s)}{\sin^3 s}, \tag{3.12}$$

$$\tau^+ - \cos S_c = \frac{|x + y||x - y|}{\langle x, y \rangle} = \frac{\sin^2 S_c}{\cos S_c}. \tag{3.13}$$

From now on, for simplicity we occasionally omit the arguments (x, y) of S_c and related functions as long as there is no ambiguity.

Proof of Lemma 3.5. Let $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$. From (3.10), we note

$$1 - \tau^-(x, y) = \frac{|x - y|(|x + y| - |x - y|)}{2\langle x, y \rangle}. \tag{3.14}$$

So, $\tau^-(x, y) \in [1 - c_2\mu, 1 - c_1\mu]$ for some positive constants $c_2 > c_1 > 0$. Thus, it is clear that S_c is smooth on $\mathcal{B} \times \widetilde{\mathcal{B}}$. Since $1 - \cos S_c(x, y) \in [c_1\mu, c_2\mu]$, to see

$$|x - y|^{1/2} \sim S_c(x, y)$$

it suffices to observe that

$$1 - \cos S_c(x, y) = \frac{2|x - y|}{|x + y| + |x - y|}. \tag{3.15}$$

Note $|x + y|^2 - |x - y|^2 = 4\langle x, y \rangle$. Thus, (3.15) follows by (3.14). ∎

3.4. *Reduction to L^2 estimates*

From (3.15), we note

$$|\cos S_c(x,y) - \cos S_c(x^*, y^*)| \lesssim \varepsilon_\circ \mu, \quad (x,y), (x^*, y^*) \in \mathcal{B} \times \widetilde{\mathcal{B}}.$$

Consequently, $S_c(\mathcal{B} \times \widetilde{\mathcal{B}})$ is contained in an interval of length $C_1 \varepsilon_\circ \mu^{1/2}$ for a constant C_1. Using this, we make a further localization.

Let $\mathbf{c}_{\mathcal{B}}$ denote the center of the rectangle $\mathcal{B} \times \widetilde{\mathcal{B}}$, and set

$$\psi_{\mathcal{B}}(s) = \psi_\circ\Big(\frac{s - S_c(\mathbf{c}_{\mathcal{B}})}{C_1 \varepsilon_\circ \mu^{1/2}}\Big),$$

where $\psi_\circ \in C_c^\infty(-4, 4)$ such that $\psi_\circ = 1$ on $[-2, 2]$. We decompose

$$\widetilde{\chi}_{\mathcal{B}} \mathfrak{P}_\lambda[\psi_j] \widetilde{\chi}_{\widetilde{\mathcal{B}}} = \mathfrak{P}^0 + \mathfrak{P}^1 := \widetilde{\chi}_{\mathcal{B}} \mathfrak{P}_\lambda[\psi_j \psi_{\mathcal{B}}] \widetilde{\chi}_{\widetilde{\mathcal{B}}} + \widetilde{\chi}_{\mathcal{B}} \mathfrak{P}_\lambda[\psi_j(1 - \psi_{\mathcal{B}})] \widetilde{\chi}_{\widetilde{\mathcal{B}}}.$$

The operator $\mathfrak{P}^1$ is easy to handle. In fact, one can show without difficulty

$$\|\mathfrak{P}^1\|_{2\to2} \lesssim \lambda^{-d/2} (\mu\widetilde{\mu})^{1/4}, \quad \|\mathfrak{P}^1\|_{1\to\infty} \lesssim \lambda^{-1/2} \mu^{-(d+1)/4}.$$

Interpolation gives $\|\mathfrak{P}^1\|_{p\to p'} \lesssim \mathrm{B}_p(\mu, \widetilde{\mu})(\widetilde{\mu}/\mu)^c$ for $1 \le p < 2$. The L^2 bound follows from Lemma 2.9. For the L^1–L^∞ bound, we note that $|\cos s - \cos S_c(x, y)| \sim \mu$ if $s \in \mathrm{supp}(\psi_j(1 - \psi_{\mathcal{B}}))$ and $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$. We also have $\tau^+(x, y) - \cos s \gtrsim \mu$ for $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$ since $\tau^+(x, y) - 1 \sim |x - y|$. Thus, recalling (3.12), we see $|\partial_s^2 \mathcal{P}(x, y, s)| \gtrsim \mu^{1/2}$ for $s \in \mathrm{supp}(\psi_j(1 - \psi_{\mathcal{B}}))$ if $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$. Applying the van der Corput lemma to the integral $\mathfrak{P}^1(x, y)$ (see (2.14)) gives the desired estimate $|\mathfrak{P}^1(x, y)| \lesssim \lambda^{-1/2} \mu^{-(d+1)/4}$.

Proposition 3.6. *Let $d \ge 3$, and $\mu, \widetilde{\mu}$ satisfy (2.11). If $2/(d + 1) < \delta(p, p') < 2/3$, then for some $c > 0$ we have*

$$\|\mathfrak{P}^0\|_{p\to p'} \lesssim \mathrm{B}_p(\mu, \widetilde{\mu})\Big(\frac{\widetilde{\mu}}{\mu}\Big)^c. \tag{3.16}$$

Now, the proof of Proposition 3.4 is straightforward.

Proof of Proposition 3.4. Combining the estimates for $\mathfrak{P}^0$ and $\mathfrak{P}^1$, we obtain (3.9) for $2/(d + 1) < \delta(p, p') < 2/3$. Meanwhile, we have the estimate $\|\widetilde{\chi}_{\mathcal{B}} \mathfrak{P}_\lambda[\psi_j] \widetilde{\chi}_{\widetilde{\mathcal{B}}}\|_{2\to2} \lesssim \lambda^{-d/2} (\widetilde{\mu}\mu)^{1/4}$ by Lemma 2.9. Interpolation yields (3.9) for $0 < \delta(p, p') < 2/3$. ∎

If $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$, then $\mathcal{D}(x, y)$ is no longer bounded away from the zero. To get the correct order of decay in λ, i.e., $O(\lambda^{-1/2})$, we consider $\partial_s^2 \mathcal{P}$, which alone is not enough to give a favorable lower bound since it also vanishes at some point. Such difficulty is typically circumvented by considering $\partial_s \mathcal{P}$ and $\partial_s^2 \mathcal{P}$ together. However, this is not viable in our situation since the zeros of $\partial_s \mathcal{P}$ and $\partial_s^2 \mathcal{P}$ merge as $\mathcal{D}(x, y) \to 0$. This leads us to break the integral away from S_c.

Inserting the cutoff functions $\widetilde{\psi}(2^l(s - S_c))$, we decompose

$$\mathfrak{P}^0 = \sum_l \mathfrak{P}_l := \sum_l \widetilde{\chi}_{\mathcal{B}} \mathfrak{P}_\lambda[\psi_j \psi_{\mathcal{B}} \widetilde{\psi}(2^l(\cdot - S_c))] \widetilde{\chi}_{\widetilde{\mathcal{B}}}, \tag{3.17}$$

where $\widetilde{\psi} = \psi(|\cdot|)$. Clearly, we have

$$2^{-l} \leq \varepsilon_\circ \mu^{1/2}, \tag{3.18}$$

since $\mathfrak{P}_l = 0$ otherwise. As to be seen later, (3.18) makes it possible to render the minor contribution manageable if we take $\varepsilon_\circ$ small enough. We set

$$\varepsilon_1 = C \varepsilon_\circ^{1/2}$$

for a large constant $C > 0$.

We have different estimates for $\mathfrak{P}_l$ depending on l.

Lemma 3.7. *Let $d \geq 2$, and $\mu, \widetilde{\mu}$ satisfy (2.11). If $2^{-l} < \varepsilon_1 \widetilde{\mu}^{1/2}$, then*

$$\|\mathfrak{P}_l\|_{2\to 2} \lesssim \lambda^{-d/2} 2^{-l} \left(\frac{\widetilde{\mu}}{\mu}\right)^{-1/2}. \tag{3.19}$$

If $\varepsilon_1 \widetilde{\mu}^{1/2} \leq 2^{-l}$, then we have

$$\|\mathfrak{P}_l\|_{2\to 2} \lesssim \lambda^{-d/2} 2^{l/2} \mu^{1/4} \widetilde{\mu}^{1/2}. \tag{3.20}$$

We postpone the proof of Lemma 3.7 until the next section. Assuming this for the moment, we prove Proposition 3.6. To do this, we set

$$S_c^l(x, y, s) = 2^{-l} s + S_c(x, y).$$

Then, changing variables $s \to S_c^l(x, y, s)$, we have

$$\mathfrak{P}_l(x, y) = \widetilde{\chi}_{\mathcal{B}}(x) \widetilde{\chi}_{\widetilde{\mathcal{B}}}(y) 2^{dj/2} 2^{-l} \int (2^{-dj/2} \mathbf{a} \psi_j \psi_{\mathcal{B}})(S_c^l) \widetilde{\psi}(s) e^{i\lambda \mathcal{P}(x, y, S_c^l)} ds. \tag{3.21}$$

Here, we also drop the arguments of S_c^l for simplicity as before.

Proof of Proposition 3.6. Since $|\cos S_c^l - \cos S_c| \sim \mu^{1/2} 2^{-l}$ on the support of $(\mathbf{a}\psi_j \psi_{\mathcal{B}}) \circ S_c^l \widetilde{\psi}$, by (3.12) we have $|\partial_s^2(\mathcal{P}(x, y, S_c^l))| \sim 2^{-3l}$. Applying the van der Corput lemma, for l satisfying (3.18) we get

$$\|\mathfrak{P}_l\|_{1\to\infty} \lesssim \lambda^{-1/2} 2^{l/2} \mu^{-d/4}. \tag{3.22}$$

Interpolation with (3.20) gives

$$\|\mathfrak{P}_l\|_{p\to p'} \lesssim \lambda^{((d-1)\delta(p,p')-d)/2} 2^{l/2} \mu^{(1-(d+1)\delta(p,p'))/4} \widetilde{\mu}^{(1-\delta(p,p'))/2}$$

when $\varepsilon_1 \widetilde{\mu}^{1/2} \leq 2^{-l} \leq \varepsilon_\circ \mu^{1/2}$. Hence, by summation over l we have

$$\left\| \sum_{\varepsilon_1 \widetilde{\mu}^{1/2} \leq 2^{-l} \leq \varepsilon_\circ \mu^{1/2}} \mathfrak{P}_l \right\|_{p\to p'} \lesssim B_p(\mu, \widetilde{\mu}) \left(\frac{\widetilde{\mu}}{\mu}\right)^{(d-1)\delta(p,p')/8}.$$

By interpolation between (3.22) and (3.19), we get

$$\|\mathfrak{P}_l\|_{p\to p'} \lesssim \lambda^{((d-1)\delta(p,p')-d)/2} 2^{-l(1-3\delta(p,p')/2)} (\mu\widetilde{\mu})^{-d\delta(p,p')/8} \left(\frac{\widetilde{\mu}}{\mu}\right)^{(-4+(d+4)\delta(p,p'))/8}$$

for $2^{-l} < \varepsilon_1 \tilde{\mu}^{1/2}$, which yields

$$\Big\| \sum_{2^{-l} < \varepsilon_1 \tilde{\mu}^{1/2}} \mathfrak{P}_l \Big\|_{p \to p'} \lesssim B_p(\mu, \tilde{\mu}) \Big(\frac{\tilde{\mu}}{\mu}\Big)^{(-2+(d+1)\delta(p,p'))/8}$$

if $\delta(p, p') < 2/3$. Recalling (3.17), we combine the estimates above and obtain (3.16) for $2/(d+1) < \delta(p, p') < 2/3$. ∎

4. L^2-estimate: Proof of Lemma 3.7

For $a \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and a smooth function ϕ on supp a, we define

$$\mathcal{O}_\lambda[\phi, a] f(x) = \int e^{i\lambda\phi(x,y)} a(x, y) f(y) \mathrm{d}y.$$

We denote

$$\partial_x f = (\partial_{x_1} f, \ldots, \partial_{x_d} f)^\mathsf{T} \quad \text{and} \quad \partial_x^\mathsf{T} f = (\partial_{x_1} f, \ldots, \partial_{x_d} f),$$

so that $\partial_x \partial_y^\mathsf{T}$ means the matrix valued operator $(\partial_{x_i} \partial_{y_j})_{1 \le i, j \le d}$. We now recall the following lemma.

Lemma 4.1 ([6], [19, p. 377]). *Let $\lambda > 0$. If $\det(\partial_x \partial_y^\mathsf{T} \phi) \ne 0$ on supp a, then there is a constant $C = C(\phi, a)$ such that*

$$\|\mathcal{O}_\lambda[\phi, a] f\|_2 \le C \lambda^{-d/2} \|f\|_2.$$

From (3.21), we see

$$\mathfrak{P}_l f = 2^{dj/2} 2^{-l} \int \tilde{\psi}(s) \mathcal{O}_\lambda[\Phi_s, A_s] f \, \mathrm{d}s, \tag{4.1}$$

where

$$\Phi_s(x, y) = \mathcal{P}(x, y, S_c^l), \tag{4.2}$$

$$A_s(x, y) = \tilde{\chi}_{\mathcal{B}}(x) \tilde{\chi}_{\tilde{\mathcal{B}}}(y)(2^{-dj/2} \alpha \psi_j \psi_{\mathcal{B}})(S_c^l). \tag{4.3}$$

To obtain (3.19) and (3.20), one may try to use Lemma 4.1 for $\mathcal{O}_\lambda[\Phi_s, A_s]$. However, $\mathcal{O}_\lambda[\Phi_s, A_s]$ exhibits different natures depending on $\mu, \tilde{\mu}$ and l. When $\mu \sim \tilde{\mu}$, via a suitable change of variables $\mathcal{O}_\lambda[\Phi_s, A_s]$ can be handled by the estimate in Lemma 4.1. However, when $\tilde{\mu}/\mu$ gets smaller, Lemma 4.1 is not enough (see Lemma 4.10 below).

We set $\mathbb{J} = \{s : 1/4 \le |s| \le 1\}$. Since supp $\tilde{\psi} \subset \mathbb{J}$ and $2^{-j} \sim \sqrt{\mu}$, by (4.1), estimate (3.19) follows from the next lemma.

Lemma 4.2. *Let $d \ge 2$ and (2.11) hold. If $2^{-l} < \varepsilon_1 \tilde{\mu}^{1/2}$, then*

$$\|\mathcal{O}_\lambda[\Phi_s, A_s]\|_{2 \to 2} \le C \lambda^{-d/2} \mu^{d/4} \Big(\frac{\tilde{\mu}}{\mu}\Big)^{-1/2}, \quad s \in \mathbb{J}. \tag{4.4}$$

When $\varepsilon_1 \widetilde{\mu}^{1/2} \le 2^{-l} \le \varepsilon_\circ \mu^{1/2}$, in order to prove (3.20), we use a different expression of $\mathfrak{P}_l$ so that we can exploit a lower bound on $\partial_s \Phi_s$. Recalling (4.1), by integration by parts in s we have

$$\mathfrak{P}_l f(x) = 2^{dj/2} 2^{-l} \int \int e^{i\lambda \Phi_s(x,y)} \widetilde{A}_s(x,y) \mathrm{d}s f(y) \mathrm{d}y,$$

where

$$\widetilde{A}_s(x,y) = \widetilde{\psi}(s) A_s(x,y) \frac{\partial_s^2 \Phi_s(x,y)}{i\lambda(\partial_s \Phi_s(x,y))^2} - \frac{\partial_s(\widetilde{\psi}(s) A_s(x,y))}{i\lambda \partial_s \Phi_s(x,y)}.$$

Note $\partial_s \Phi_s \ne 0$ if $A_s \ne 0$. In fact, we have

$$|\partial_s \Phi_s(x,y)| \gtrsim 2^{-3l}, \quad (s,x,y) \in \mathbb{J} \times \mathcal{B} \times \widetilde{\mathcal{B}}. \tag{4.5}$$

Using (3.11) and (3.10), we see $|\langle x, y \rangle - \cos S_c(x,y)| \lesssim |\mathcal{D}(x,y)|/|x-y|$. So, by (3.7) it follows that

$$|\langle x, y \rangle - \cos S_c(x,y)| \le \varepsilon_\circ \widetilde{\mu}, \quad (x,y) \in \mathcal{B} \times \widetilde{\mathcal{B}}. \tag{4.6}$$

Note $|\cos S_c^l - \cos S_c| \sim 2^{-l} \mu^{1/2}$, so $|\langle x, y \rangle - \cos S_c^l(x,y)| \gtrsim 2^{-l} \mu^{1/2} \ge \varepsilon_1(\mu\widetilde{\mu})^{1/2}$. By (2.16), (3.7), and our choice of ε_1, we get $\mathcal{Q}(x,y,\cos S_c^l) \gtrsim 2^{-2l} \mu$. So, (4.5) follows by (2.15) since $\partial_s \Phi_s = 2^{-l} \partial_s \mathcal{P}(x,y,S_c^l)$.

Estimate (3.20) is an immediate consequence of the following.

Lemma 4.3. *Let $d \ge 2$ and let (2.11) hold. If $\varepsilon_1 \widetilde{\mu}^{1/2} \le 2^{-l} \le \varepsilon_\circ \mu^{1/2}$, then*

$$\|\mathcal{O}_\lambda[\Phi_s, \widetilde{A}_s]\|_{2 \to 2} \le C\lambda^{-d/2} \mu^{(d+1)/4} \widetilde{\mu}^{1/2} 2^{3l/2}, \quad s \in \mathbb{J}. \tag{4.7}$$

For the rest of this section, we assume (2.11) and (3.18) hold, and $(x,y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$, $s \in \mathbb{J}$, even if it is not made explicit.

4.1. Bounds on $\partial_x^\alpha \partial_y^\beta \Phi_s$ and $\partial_x^\alpha A_s$

In order to prove Lemmas 4.2 and 4.3, we need estimates for the derivatives of Φ_s and A_s. For a given multi-index $\alpha \in \mathbb{N}_0^d$, we write $\alpha = (\alpha_1, \overline{\alpha}) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1}$.

Lemma 4.4. *Let $s \in \mathbb{J}$ and $(x,y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$. Suppose (2.11) holds. Then,*

$$|\partial_x^\alpha \partial_y^\beta S_c(x,y)| \lesssim \mu^{1/2 - |\alpha| - |\beta|}, \tag{4.8}$$

$$|\partial_x^\alpha A_s(x,y)| \lesssim \mu^{-\alpha_1} (\mu\widetilde{\mu})^{-|\overline{\alpha}|/2}. \tag{4.9}$$

Proof. To show (4.8), we note from (3.15) that $S_c(x,y)$ behaves like $g(x,y) := |x-y|^{1/2}$, and that (4.8) and (4.9) are easy to show if S_c is replaced by g.

We prove (4.8) in an inductive way by making use of (3.15). In accordance to Lemma 3.5, (4.8) holds with $\alpha = \beta = 0$ since $\mathrm{dist}(\mathcal{B}, \widetilde{\mathcal{B}}) \sim \mu$. We now assume (4.8) is true

for $|\alpha| + |\beta| \le N$. Applying $\partial_x^\alpha \partial_y^\beta$ on both sides of (3.15) for $|\alpha| + |\beta| = N + 1$, it is not difficult to see

$$\sin S_c(x, y)\partial_x^\alpha \partial_y^\beta S_c(x, y) + O(\mu^{-N}) = O(\mu^{-N}) \qquad (4.10)$$

for $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$. Indeed, the right-hand side of (3.15) behaves as if it were $|x - y|$. On the left-hand side of (4.10), the terms other than the first one are given by a linear combination of the products of $\sin S_c$, $\cos S_c$, and $\prod_{i=1}^l \partial_x^{a_i} \partial_y^{b_i} S_c$ with $l \ge 2$ and $\sum_{i=1}^l (|a_i| + |b_i|) \le N + 1$. Our induction assumption shows those are $O(\mu^{-N})$, thus we get (4.10). Since $|x - y| \sim \mu$, then $S_c(x, y) \sim \sqrt{\mu}$ by Lemma 3.5. Therefore, (4.10) gives $\partial_x^\alpha \partial_y^\beta S_c(x, y) = O(\mu^{-N-1/2})$ as desired.

Once we have (4.8), (4.9) is easier to show. Recall (4.3). Since we have (3.8), it is sufficient to show

$$\partial_x^\alpha((2^{-jd/2}\mathfrak{a}\psi_j\psi_\mathcal{B}) \circ S_c^l) = O(\mu^{-|\alpha|}).$$

Note

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^k (2^{-jd/2}\mathfrak{a}\psi_j) = O(\mu^{-k/2}) \quad \text{and} \quad \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^k \psi_\mathcal{B} = O(\mu^{-k/2}).$$

By the chain rule and (4.8), we get $\partial_x^\alpha((2^{-jd/2}\mathfrak{a}\psi_j) \circ S_c^l) = O(\mu^{-|\alpha|})$ and $\partial_x^\alpha(\psi_\mathcal{B} \circ S_c^l) = O(\mu^{-|\alpha|})$. From those estimates, the desired bound follows. ∎

Lemma 4.5. *Let $s \in \mathbb{J}$ and $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$. Suppose that* (2.11) *and* (3.18) *hold. Then,*

$$\partial_x^\alpha \partial_y^\beta \Phi_s(x, y) = O(\mu^{3/2-|\alpha|-|\beta|}), \qquad |\alpha|, |\beta| \ge 1. \qquad (4.11)$$

Proof. Let us set

$$u(t) = \frac{t - \sin t}{2}, \quad v(t) = \frac{2\cos t - 2 + \sin^2 t}{2\sin t}, \quad w(t) = \frac{\cos t}{2\sin t}.$$

Recalling (4.2) and (2.13), we write

$$\Phi_s(x, y) = u(S_c^l) + 2^{-1}(1 - \langle x, y\rangle)\sin S_c^l + \langle x, y\rangle v(S_c^l) + |x - y|^2 w(S_c^l).$$

From (4.8), it is clear that

$$|\partial_x^\alpha \partial_y^\beta S_c^l(x, y)| \lesssim \mu^{1/2-|\alpha|-|\beta|}.$$

We note

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k u(t), \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k v(t) = O(t^{3-k}), \quad \text{and} \quad \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k w(t) = O(t^{-1-k}).$$

Since $S_c^l(x, y) \sim \sqrt{\mu}$, $1 - \langle x, y\rangle = O(\mu)$, and $|x - y| \sim \mu$, a routine computation yields (4.11). ∎

Remark 4.6. Under the same assumption as in Lemma 4.5, we also have $|\partial_x^\alpha \partial_y^\beta \sin^m S_c|$, $|\partial_x^\alpha \partial_y^\beta \sin^m S_c^l| \lesssim \mu^{m/2-|\alpha|-|\beta|}$ for any $m \in \mathbb{Z}$, $|\partial_x^\alpha \partial_y^\beta(1 - \cos S_c)|$, $|\partial_x^\alpha \partial_y^\beta(1 - \cos S_c^l)| \lesssim \mu^{1-|\alpha|-|\beta|}$ for any multi-indices α, β.

4.2. Estimate for $\partial_x \partial_y^\top \Phi_s$

For a given matrix $\mathbf{N}$, we denote by $\mathbf{N}_{i,j}$ the (i, j)-th entry of $\mathbf{N}$. We consider a $d \times d$ matrix $\mathfrak{M}^0$ which is given as follows:

$$\mathfrak{M}^0_{i,i}(x, y) = 1 - \frac{(x_i - y_i)^2}{|x - y|^2}, \qquad\qquad 1 \leq i \leq d,$$

$$\mathfrak{M}^0_{1,j}(x, y) = -\frac{x_j - y_j}{2|x - y|^2}(2x_1 - (1 + \cos S_c)y_1), \qquad j \geq 2,$$

$$\mathfrak{M}^0_{i,1}(x, y) = \mathfrak{M}^0_{1,i}(y, x), \qquad\qquad i \geq 2,$$

$$\mathfrak{M}^0_{i,j}(x, y) = -\frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2}, \qquad\qquad i, j \geq 2, \ i \neq j.$$

The following lemma shows the matrix $\partial_x \partial_y^\top \Phi_s(x, y)$ is close to $\mathfrak{M}^0$. Let

$$E_k^\alpha = \begin{cases} \varepsilon_\circ \left(\dfrac{\tilde{\mu}}{\mu}\right)^{1/k} + 2^{-2l}\mu^{-1}, & \alpha = 0, \\[12pt] \left(\left(\dfrac{\tilde{\mu}}{\mu}\right)^{1/k} + 2^{-2l}\mu^{-1}\right)\mu^{-\alpha_1}(\mu\tilde{\mu})^{-|\bar{\alpha}|/2}, & \alpha \neq 0, \end{cases} \qquad k = 1, 2.$$

Lemma 4.7. *Let $s \in \mathbb{J}$ and $(x, y) \in \mathcal{B} \times \tilde{\mathcal{B}}$. Set*

$$\mathfrak{M}(x, y) = -\sin S_c^l(x, y)\partial_x \partial_y^\top \Phi_s(x, y).$$

Suppose that (2.11) *and* (3.18) *hold. Then,* $\mathfrak{M} = \mathfrak{M}^0 + \mathcal{E}$ *and* $\mathcal{E}_{i,j}$ *satisfies*

$$|\partial_x^\alpha \mathcal{E}_{i,j}(x, y)| \lesssim \begin{cases} E_1^\alpha & \text{if } (i, j) = (1, 1) \text{ or } i, j \geq 2, \\ E_2^\alpha & \text{otherwise.} \end{cases} \qquad (4.12)$$

Note $E_1^\alpha \leq E_2^\alpha$. We postpone the proof of Lemma 4.7 until the end of this section. Instead, we deduce a couple of lemmas from it for later use. By $\tilde{\mathfrak{M}}$ we denote the $(1, 1)$ minor matrix of $\mathfrak{M}$, i.e., $\tilde{\mathfrak{M}} = (\mathfrak{M}_{i+1,j+1})_{1 \leq i,j \leq d-1}$.

Lemma 4.8. *Let $s \in \mathbb{J}$ and $(x, y) \in \mathcal{B} \times \tilde{\mathcal{B}}$. Suppose* (2.11) *and* (3.18) *hold. Then, we have*

$$\det \mathfrak{M}(x, y) \sim \frac{\tilde{\mu}}{\mu}, \quad 2^{-l} \leq \varepsilon_1 \tilde{\mu}^{1/2}, \qquad (4.13)$$

$$\det \tilde{\mathfrak{M}}(x, y) \sim 1, \quad 2^{-l} \leq \varepsilon_\circ \mu^{1/2}. \qquad (4.14)$$

Proof. Note $\mathfrak{M}^0_{i,i} = 1 + O(\tilde{\mu}/\mu)$ for $2 \leq i \leq d$ and $\mathfrak{M}^0_{i,j} = O(\tilde{\mu}/\mu)$ for $2 \leq i \neq j \leq d$. Thus, by Lemma 4.7 we have $\tilde{\mathfrak{M}} = \mathbf{I}_{d-1} + O(\varepsilon_\circ + \tilde{\mu}/\mu)$. Since $\tilde{\mu}/\mu \leq \varepsilon_\circ$, (4.14) follows if $\varepsilon_\circ$ is small enough.

For the proof of (4.13), we may assume

$$x = (r, 0, 0, \ldots, 0), \quad y = (\rho, h, 0, \ldots, 0). \qquad (4.15)$$

Indeed, observe $S_c(x, y) = S_c(\mathbf{U}x, \mathbf{U}y)$ and $\Phi_s(x, y) = \Phi_s(\mathbf{U}x, \mathbf{U}y)$ for $\mathbf{U} \in O(d)$. The second identity and a computation show $\partial_x \partial_y^\mathsf{T} \Phi_s(x, y) = \mathbf{U}^\mathsf{T} \partial_x \partial_y^\mathsf{T} \Phi_s(\mathbf{U}x, \mathbf{U}y)\mathbf{U}$, i.e., $\mathbf{U}\mathfrak{M}(x, y)\mathbf{U}^\mathsf{T} = \mathfrak{M}(\mathbf{U}x, \mathbf{U}y)$. Since $\det \mathfrak{M}(\mathbf{U}x, \mathbf{U}y) = \det \mathfrak{M}(x, y)$, we need only to choose $\mathbf{U} \in O(d)$ such that $\mathbf{U}x = (r, 0, 0, \ldots, 0)$, $\mathbf{U}y = (\rho, h, 0, \ldots, 0)$.

Let $\mathbf{M}$ be the 2×2 matrix given by

$$\mathbf{M} = \frac{1}{2|x - y|^2} \begin{pmatrix} 2h^2 & (1 + \cos S_c)\rho h - 2rh \\ 2\rho h - (1 + \cos S_c)rh & 2(r - \rho)^2 \end{pmatrix}.$$

By Lemma 4.7 and (4.15), the matrix $\mathfrak{M}$ is of the form

$$\mathfrak{M}(x, y) = \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{I}_{d-2} \end{pmatrix} + \widetilde{\mathcal{E}},$$

where $\widetilde{\mathcal{E}}_{i,j} = O((\varepsilon_\circ + \varepsilon_1)(\widetilde{\mu}/\mu))$ if $(i, j) = (1, 1)$ or $i, j \geq 2$, and $\widetilde{\mathcal{E}}_{i,j} = O((\varepsilon_\circ + \varepsilon_1)(\widetilde{\mu}/\mu)^{1/2})$ otherwise. Since $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$, we have $1 - r \sim \mu$, $1 - \rho \sim \widetilde{\mu}$, and $h \sim (\mu\widetilde{\mu})^{1/2}$. Hence, $\mathbf{M}_{1,1} \sim \widetilde{\mu}/\mu$, $\mathbf{M}_{2,2} \sim 1$, and $\mathbf{M}_{1,2}, \mathbf{M}_{1,2} = O((\widetilde{\mu}/\mu)^{1/2})$. Consequently, we see $\det \mathfrak{M}(x, y) = \det \mathbf{M} + O((\varepsilon_\circ + \varepsilon_1)\widetilde{\mu}/\mu)$. Note $r - \rho \sim \mu$ and $1 - \cos S_c \sim \mu$. So, we have

$$\det \mathbf{M} = \frac{h^2}{4|x - y|^4}(1 - \cos S_c)(2(r - \rho)^2 + r\rho(1 - \cos S_c)) \sim \frac{\widetilde{\mu}}{\mu}.$$

Therefore, (4.13) follows if $\varepsilon_\circ$ is sufficiently small. ∎

The following is a straightforward consequence of Lemma 4.7.

Lemma 4.9. *Let $s \in \mathbb{J}$ and $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$. Suppose (2.11) holds. If $2^{-l} \leq \varepsilon_1 \widetilde{\mu}^{1/2}$, then for $\alpha \in \mathbb{N}_0^d$*

$$|\partial_x^\alpha \mathfrak{M}_{i,j}(x, y)| \lesssim \begin{cases} \left(\dfrac{\widetilde{\mu}}{\mu}\right)\mu^{-\alpha_1}(\mu\widetilde{\mu})^{-|\bar{\alpha}|/2}, & i = j = 1, \\[2mm] \left(\dfrac{\widetilde{\mu}}{\mu}\right)^{1/2}\mu^{-\alpha_1}(\mu\widetilde{\mu})^{-|\bar{\alpha}|/2}, & i \text{ or } j = 1, \\[2mm] \mu^{-\alpha_1}(\mu\widetilde{\mu})^{-|\bar{\alpha}|/2}, & i, j \geq 2. \end{cases}$$

Furthermore, the last bound remains valid even if $2^{-l} \leq \varepsilon_\circ \sqrt{\mu}$.

4.3. Estimate for $\mathcal{O}_\lambda[\Phi_s, A_s]$ when $2^{-l} < \varepsilon_1 \widetilde{\mu}^{1/2}$

In order to prove (4.4), we use the following lemma, which differs from the typical one (see Lemma 4.1), in that the phase and amplitude functions depend on a parameter. We denote $\mathbb{B}_d(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$.

Lemma 4.10. *Let $0 < \omega \leq 1$. Let a be a smooth function supported in $S_\omega := \mathbb{B}_d(0, 1) \times \mathbb{B}_1(0, \omega) \times \mathbb{B}_{d-1}(0, 1)$ such that $|\partial_x^\alpha a| \lesssim 1$ for $|\alpha| \leq d + 1$. Suppose that $|\det \partial_x \partial_y^\mathsf{T} \phi| \sim 1$ and*

$$|\partial_x^\alpha \partial_y^\beta \phi| \lesssim \begin{cases} 1, & |\beta| = 1, \\ \omega^{-1 + (|\bar{\alpha}| + |\bar{\beta}|)/2}, & |\beta| = 2 \end{cases} \tag{4.16}$$

for $1 \leq |\alpha| \leq d + 1$ on S_ω. Then, we have

$$\|\mathcal{O}_\lambda[\phi, a] f\|_2 \lesssim \lambda^{-d/2} \|f\|_2. \tag{4.17}$$

Proof of Lemma 4.10. By finite decomposition and translation, we may replace S_ω by $\mathbb{B}_d(0, \epsilon_0) \times [-\epsilon_0\omega, \epsilon_0\omega] \times \mathbb{B}_{d-1}(0, \epsilon_0)$ for a small enough $\epsilon_0 > 0$.

We set

$$\Psi(x) = \phi(x, y) - \phi(x, y'), \quad A(x) = a(x, y)\overline{a}(x, y'),$$

and consider the integral

$$I_\lambda(y, y') = \int e^{i\lambda\Psi(x)} A(x) dx,$$

which is the kernel of the operator $\mathcal{O}_\lambda[\phi, a]^* \mathcal{O}_\lambda[\phi, a]$. Estimate (4.17) follows by a standard argument if we show

$$|I_\lambda(y, y')| \leq C(1 + \lambda|y - y'|)^{-d-1}. \tag{4.18}$$

Assuming, for the moment, that

$$|\nabla \Psi(x)| \gtrsim |y - y'|, \tag{4.19}$$

$$|\partial_x^\alpha \Psi(x)| \lesssim |y - y'| \quad \forall |\alpha| \geq 2, \tag{4.20}$$

we prove (4.18). Note $\partial_x^\alpha A = O(1)$ for $|\alpha| \leq d + 1$ since $|\partial_x^\alpha a| \lesssim 1$. Using the identity

$$\left(\frac{\nabla\Psi(x)}{i\lambda|\nabla\Psi(x)|^2} \cdot \nabla \right) e^{i\lambda\Psi(x)} = e^{i\lambda\Psi(x)},$$

by integration by parts $d + 1$ times we have

$$|I_\lambda(y, y')| \lesssim \sum_{\ell=0}^{d+1} \sum_{\sum |a_i| \leq d+\ell+1, |a_i| \geq 2} \int \frac{\prod_{i=1}^\ell |\partial_x^{a_i} \Psi(x)|}{\lambda^{d+1}|\nabla\Psi(x)|^{d+\ell+1}} \mathcal{A}(x) dx,$$

where $\mathcal{A}$ is a bounded function supported in $\mathbb{B}_d(0, 1)$. By (4.19) and (4.20), estimate (4.18) follows.

We now show (4.19) and (4.20). For (4.19), it is sufficient to show

$$|\nabla_x \phi(x, y) - \nabla_x \phi(x, y')| \gtrsim |y - y'|.$$

By the Taylor series expansion, $\nabla_x \phi(x, y) - \nabla_x \phi(x, y')$ is equal to

$$\partial_x \partial_y^\mathsf{T} \phi(x, y')(y - y') + \sum_{i, \beta : |\beta|=2} O(E_{i,\beta} |(y - y')^\beta|),$$

where $E_{i,\beta} = \sup_{(x,y) \in S_\omega} |\partial_{x_i} \partial_y^\beta \phi(x, y)|$. Since $|y_1 - y_1'| \leq \epsilon_0 \omega$, by (4.16) it is clear that $E_{i,\alpha}|(y - y')^\alpha| = O(\epsilon_0|y - y'|)$. We also have $|\partial_x \partial_y^\mathsf{T}\phi(x, y')(y - y')| \sim |y - y'|$ because $(\partial_x \partial_y^\mathsf{T}\phi)_{i,j} = O(1)$ and $|\det \partial_x \partial_y^\mathsf{T}\phi| \sim 1$. Thus, $|\nabla_x \phi(x, y) - \nabla_x \phi(x, y')| \gtrsim |y - y'|$. For (4.20), by the mean value theorem we only need to show $\partial_x^\alpha \partial_{y_j} \phi(x, y) = O(1)$ for $|\alpha| \geq 2$. This follows from (4.16). ∎

We prove Lemma 4.2 by combining Lemmas 4.8, 4.9, and 4.10.

Proof of Lemma 4.2. We first transform $\mathcal{O}_\lambda[\Phi_s, A_s]$ via scaling and translation so that we can apply Lemma 4.10. Recalling that $\mathbf{c}_\mathcal{B}$ denotes the center of the rectangle $\mathcal{B} \times \widetilde{\mathcal{B}}$, we set

$$L(x, y) = (L_1 x, L_2 y) + \mathbf{c}_\mathcal{B} := (\mu x_1, (\mu\widetilde{\mu})^{1/2}\overline{x}, \mu y_1, (\mu\widetilde{\mu})^{1/2}\overline{y}) + \mathbf{c}_\mathcal{B}.$$

Changing variables $(x, y) \to L(x, y)$, we have

$$\|\mathcal{O}_\lambda[\Phi_s, A_s]\|_{2\to2} = \left(\frac{\mu}{\widetilde{\mu}}\right)^{1/2}(\mu\widetilde{\mu})^{d/2}\|\mathcal{O}_{\sqrt{\mu\widetilde{\mu}}\lambda}[\widetilde{\Phi}, \widetilde{A}]\|_{2\to2}, \tag{4.21}$$

where

$$\widetilde{\Phi}(x, y) = (\sqrt{\mu\widetilde{\mu}})^{-1}\Phi_s(L(x, y)), \quad \widetilde{A}(x, y) = A_s(L(x, y)).$$

Since $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are of dimensions about $\varepsilon_\circ\mu \times \varepsilon_\circ(\mu\widetilde{\mu})^{1/2} \times \cdots \times \varepsilon_\circ(\mu\widetilde{\mu})^{1/2}$ and $\varepsilon_\circ\widetilde{\mu} \times \varepsilon_\circ(\mu\widetilde{\mu})^{1/2} \times \cdots \times \varepsilon_\circ(\mu\widetilde{\mu})^{1/2}$, respectively, $\widetilde{A}$ is supported in $\mathbb{B}_d(0, 1) \times \mathbb{B}_1(0, \widetilde{\mu}/\mu) \times \mathbb{B}_{d-1}(0, 1)$. From (4.9) it follows that $\partial_x^\alpha \widetilde{A} = O(1)$ for all α. Since $\sin S_c(x, y) \sim \sqrt{\mu}$, Lemma 4.8 gives $|\det \partial_x \partial_y^\mathsf{T}\widetilde{\Phi}| \sim 1$. Indeed, note that

$$\partial_x \partial_y^\mathsf{T}\widetilde{\Phi} = -\frac{\sqrt{\mu}}{\sin S_c^l(L(x, y))}(\mu\widetilde{\mu})^{-1}L_1^\mathsf{T}\mathfrak{M}(L(x, y))L_2.$$

Besides, since $\partial_x^\alpha(\sqrt{\mu}/\sin S_c^l) = (\mu^{-|\alpha|})$, by Lemmas 4.5 and 4.9 we also have

$$\partial_x^\alpha \partial_y^\beta \widetilde{\Phi} = \begin{cases} O(1), & |\beta| = 1, \\ O\left(\left(\frac{\widetilde{\mu}}{\mu}\right)^{-1+(|\overline{\alpha}|+|\overline{\beta}|)/2}\right), & |\beta| = 2, \end{cases}$$

for $(x, y) \in \operatorname{supp}\widetilde{A}$ and $\alpha \in \mathbb{N}_0^d$. In fact, we use Lemma 4.9 for $|\beta| = 1$ and Lemma 4.5 for $|\beta| = 2$. Therefore, we may use Lemma 4.10 with $\omega = \widetilde{\mu}/\mu$ for $\mathcal{O}_{\sqrt{\mu\widetilde{\mu}}\lambda}[\widetilde{\Phi}, \widetilde{A}]$ to get

$$\|\mathcal{O}_{\sqrt{\mu\widetilde{\mu}}\lambda}[\widetilde{\Phi}, \widetilde{A}]\|_{2\to2} \lesssim \lambda^{-d/2}\widetilde{\mu}^{-d/2}\mu^{-d/4}.$$

By (4.21), estimate (4.4) follows as desired. ∎

4.4. Estimate for $\mathcal{O}_\lambda[\Phi_s, \widetilde{A}_s]$ when $2^{-l} \geq \varepsilon_1\widetilde{\mu}^{1/2}$

In this case, as seen in Lemma 4.7, the matrix $\partial_x \partial_y^\mathsf{T}\Phi_s$ can be singular. So, we need an approach different from that used for the case $2^{-l} < \varepsilon_1\widetilde{\mu}^{1/2}$. We consider an operator given by freezing x_1, y_1 (see proof of Lemma 4.3 below) and then make use of (4.14).

To prove Lemma 4.3, we need the following.

Lemma 4.11. *Let* $2^{-l} \geq \varepsilon_1\widetilde{\mu}^{1/2}$ *and* $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$. *Then,*

$$|\partial_{\overline{x}}^{\overline{\alpha}}\widetilde{A}_s(x, y)| \lesssim (\lambda 2^{-3l})^{-1}(\mu\widetilde{\mu})^{-|\overline{\alpha}|/2}, \quad \overline{\alpha} \in \mathbb{N}_0^{d-1}. \tag{4.22}$$

Assuming this for the moment, we prove Lemma 4.3.

Proof of Lemma 4.3. Fixing $s \in \mathbb{J}$, we set

$$\bar{\Phi}(x, y) = (\sqrt{\mu\tilde{\mu}})^{-1} \Phi_s(\bar{L}(x, y)), \quad \bar{B}(x, y) = \lambda 2^{-3l} \tilde{A}_s(\bar{L}(x, y)),$$

where

$$\bar{L}(x, y) = (\mu x_1, (\mu\tilde{\mu})^{1/2}\bar{x}, \tilde{\mu} y_1, (\mu\tilde{\mu})^{1/2}\bar{y}) + \mathbf{c}_{\mathcal{B}}.$$

Changing variables $(x, y) \to \bar{L}(x, y)$ gives

$$\|\mathcal{O}_\lambda[\Phi_s, \tilde{A}_s]\|_{2\to2} = (\lambda 2^{-3l})^{-1} (\mu\tilde{\mu})^{d/2} \|\mathcal{O}_{\sqrt{\mu\tilde{\mu}}\lambda}[\bar{\Phi}, \bar{B}]\|_{2\to2}. \tag{4.23}$$

Freezing $z_1 := (x_1, y_1)$, we define

$$\bar{\mathcal{O}}_{z_1}^\lambda h(\bar{x}) = \int e^{i\lambda\bar{\Phi}_{z_1}(\bar{x},\bar{y})} \bar{B}_{z_1}(\bar{x}, \bar{y}) h(\bar{y}) \mathrm{d}\bar{y},$$

where $\bar{\Phi}_{z_1}(\bar{x}, \bar{y}) = \bar{\Phi}(x, y)$ and $\bar{B}_{z_1}(\bar{x}, \bar{y}) = \bar{B}(x, y)$. Note $2^{-l} \geq \epsilon_1 \tilde{\mu}^{1/2}$ and $\lambda\tilde{\mu}^{3/2} \gtrsim 1$. Since $\mathcal{O}_\lambda[\bar{\Phi}, \bar{B}]f = \int \bar{\mathcal{O}}_{(x_1,y_1)}^\lambda f(y_1, \cdot)\mathrm{d}y_1$, by (4.23), (4.7) follows if we show

$$\|\mathcal{O}_{z_1}^\lambda\|_{2\to2} \leq C\lambda^{-(d-1)/2}, \quad \lambda > 0 \tag{4.24}$$

for a constant C. Note supp $\bar{B}_{z_1} \subset \mathbb{B}_{d-1}(0, 1) \times \mathbb{B}_{d-1}(0, 1)$. Moreover, by (4.11) and (4.14) we have $\partial_{\bar{x}}^\alpha \bar{B}_{z_1} = O(1)$ and $\det \partial_{\bar{x}}\partial_{\bar{y}}^{\mathsf{T}}\bar{\Phi}_{z_1} \sim 1$. Lemmas 4.5 and 4.9, as before (cf. proof of Lemma 4.2), give

$$\partial_{\bar{x}}^\alpha \partial_{\bar{y}}^\beta \bar{\Phi}_{z_1} = O(1), \quad |\alpha| \geq 1, \ |\beta| = 1, 2$$

whenever $(\bar{x}, \bar{y}) \in \mathbb{B}_{d-1}(0, 1) \times \mathbb{B}_{d-1}(0, 1)$. Therefore, applying Lemma 4.10 (with $\omega = 1$ and d replaced by $d - 1$) to $\mathcal{O}_{z_1}^\lambda$, we obtain (4.24). ∎

Proof of Lemma 4.11. We first consider the case where $\bar{\alpha} = 0$. Note that the functions $\tilde{\psi}(s)A_s(x, y)$ and $\partial_s(\tilde{\psi}(s)A_s(x, y))$ are uniformly bounded on $\mathcal{B} \times \tilde{\mathcal{B}}$. Recall now that $|\partial_s^2 \Phi_s(x, y)| \lesssim 2^{-3l}$ (see proof of Proposition 3.6). Thus, by (4.5) we get (4.22) for $\bar{\alpha} = 0$.

Let $\bar{\alpha} \neq 0$. We have (4.9), and $\partial_x^\alpha \partial_s A_s(x, y) = O((\mu\tilde{\mu})^{-|\alpha|/2})$, which follows by (4.8) since $2^{-l} \lesssim \mu^{1/2}$. Therefore, for (4.22) we need only to show

$$|\partial_x^\alpha \partial_s^2 \Phi_s(x, y)| \lesssim 2^{-3l} \mu^{-|\alpha|}, \tag{4.25}$$

$$|\partial_{\bar{x}}^{\bar{\alpha}} \partial_s \Phi_s(x, y)| \lesssim 2^{-3l} (\mu\tilde{\mu})^{-|\bar{\alpha}|/2}. \tag{4.26}$$

We verify (4.25) first. Since $\partial_s^2 \Phi_s = 2^{-2l} \partial_s^2 \mathcal{P}(x, y, S_c^l)$, it suffices to show

$$|\partial_x^\alpha (\partial_s^2 \mathcal{P}(x, y, S_c^l))| \lesssim 2^{-l} \mu^{-|\alpha|}, \quad \alpha \neq 0. \tag{4.27}$$

Using (3.12), we write

$$\frac{\partial_s^2 \mathcal{P}(x, y, S_c^l)}{\langle x, y \rangle} = \frac{(\cos S_c^l - \cos S_c)^2}{\sin^3 S_c^l} + (\cos S_c^l - \cos S_c)\frac{(\cos S_c - \tau^+)}{\sin^3 S_c^l}.$$

Note $\partial_x^\alpha(\sin S_c^l)^{-3} = O(\mu^{-3/2}\mu^{-|\alpha|})$ (see Remark 4.6). By the Leibniz rule, (4.27) follows if we show

$$|\partial_x^\alpha(\cos S_c - \tau^+)| \lesssim \mu^{1-|\alpha|}, \tag{4.28}$$

$$|\partial_x^\alpha(\cos S_c^l - \cos S_c)| \lesssim 2^{-l}\mu^{1/2-|\alpha|}. \tag{4.29}$$

Estimate (4.28) is clear from (3.13) since $\partial_x^\alpha(|x-y|) = O(\mu^{1-|\alpha|})$ for $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$. To show (4.29), we observe that $\partial_x^\alpha(\cos S_c^l - \cos S_c)$ is given by a linear combination of the terms

$$(\sin S_c^l - \sin S_c)\prod_{m=1}^{2l-1} \partial_x^{a_m} S_c, \quad (\cos S_c^l - \cos S_c)\prod_{m=1}^{2l} \partial_x^{b_m} S_c, \quad l \geq 0,$$

where $a_1, \ldots, a_{2l-1}, b_1, \ldots, b_{2l} \neq 0$; $|a_1| + \cdots + |a_{2l-1}| = |b_1| + \cdots + |b_{2l}| = |\alpha|$. Since $\sin S_c^l - \sin S_c = O(2^{-l})$ and $\cos S_c^l - \cos S_c = O(2^{-l}\mu^{1/2})$, (4.29) follows by (4.8).

We now show (4.26). Recalling $\partial_s \Phi_s = 2^{-l}\partial_s \mathcal{P}(x, y, S_c^l)$, we have

$$\partial_{\bar{x}}^{\bar{\beta}}\partial_s \Phi_s = 2^{-l}\partial_{\bar{x}}^{\bar{\beta}}\partial_s \mathcal{P}(x, y, S_c^l) + 2^l \partial_s^2 \Phi_s \partial_{\bar{x}}^{\bar{\beta}} S_c, \quad |\bar{\beta}| = 1.$$

Let $\bar{\alpha}' + \bar{\beta} = \bar{\alpha}$. By (4.25) and (4.8), we have $\partial_{\bar{x}}^{\bar{\alpha}'}(2^l \partial_s^2 \Phi_s \partial_{\bar{x}}^{\bar{\beta}} S_c) \lesssim 2^{-3l}(\mu\widetilde{\mu})^{-|\bar{\alpha}|/2}$ since $2^{-l} \geq \varepsilon_1 \widetilde{\mu}^{1/2}$. To handle the first term, from (2.15) we note

$$\partial_{x_j}\partial_s \mathcal{P}(x, y, S_c^l) = \frac{y_j \cos S_c^l - x_j}{\sin^2 S_c^l}. \tag{4.30}$$

Thus, it suffices to show

$$|\partial_{\bar{x}}^{\bar{\beta}}(y_j \cos S_c^l - x_j)| \lesssim (\mu\widetilde{\mu})^{(1-|\bar{\beta}|)/2}, \quad 2 \leq j \leq d.$$

Since $|y_j|, |x_j| \lesssim (\mu\widetilde{\mu})^{1/2}, 2 \leq j \leq d$, using (4.8) one can easily show the desired bounds (see Remark 4.6). ∎

The rest of this section is dedicated to proving Lemma 4.7.

4.5. *Proof of Lemma* 4.7

In order to prove (4.12), we start by removing an insignificant part of $\mathfrak{M}$. Differentiating (4.2), we have

$$\partial_x \partial_y^\mathsf{T} \Phi_s(x, y) = \partial_x \partial_y^\mathsf{T} \mathcal{P}(x, y, S_c^l) + \partial_x S_c \partial_y^\mathsf{T} \partial_s \mathcal{P}(x, y, S_c^l) + \partial_x \partial_s \mathcal{P}(x, y, S_c^l)\partial_y^\mathsf{T} S_c$$
$$+ \partial_s^2 \mathcal{P}(x, y, S_c^l)\partial_x S_c \partial_y^\mathsf{T} S_c + \partial_s \mathcal{P}(x, y, S_c^l)\partial_x \partial_y^\mathsf{T} S_c. \tag{4.31}$$

The last term on the right-hand side is negligible.

Lemma 4.12. *Let $s \in \mathbb{J}$ and $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$. Suppose that* (2.11) *and* (3.18) *hold. Then, for $\alpha \in \mathbb{N}_0^d$,*

$$\sin S_c^l \partial_x^\alpha(\partial_s \mathcal{P}(x, y, S_c^l)\partial_x \partial_y^\mathsf{T} S_c) = O(E_1^\alpha).$$

To prove Lemma 4.12, we make use of the following identities:

$$\partial_x S_c(x, y) = \mathfrak{G}(2x - \mathfrak{A}y), \quad \partial_y^\mathsf{T} S_c(x, y) = \mathfrak{G}(2y^\mathsf{T} - \mathfrak{A}x^\mathsf{T}), \tag{4.32}$$

where

$$\mathfrak{A}(x, y) = \frac{|x|^2 + |y|^2}{\langle x, y \rangle}, \quad \mathfrak{G}(x, y) = \frac{\cos S_c(x, y)}{\sin S_c(x, y)|x + y||x - y|}. \tag{4.33}$$

Proof of identities (4.32). Differentiating (3.15), followed by a simple computation, gives

$$\sin S_c(x, y)\partial_x S_c(x, y) = \mathrm{F}(x, y)x - 2^{-1}\mathfrak{A}(x, y)\mathrm{F}(x, y)y,$$

where

$$\mathrm{F}(x, y) = \frac{|x|^2 + |y|^2 - |x + y||x - y|}{\langle x, y \rangle |x + y||x - y|}.$$

Here, we also use the identity $|x + y|^2|x - y|^2 = (|x|^2 + |y|^2)^2 - 4\langle x, y \rangle^2$. By (3.11) and (3.10), we see $\mathrm{F}(x, y) = 2\cos S_c(x, y)/(|x - y||x + y|)$. We thus get the first identity in (4.32). Since $S_c(y, x) = S_c(x, y)$, the second one follows from the first by interchanging the roles of x, y. ∎

Proof of Lemma 4.12. Since $|\cos S_c^l - \cos S_c| \sim 2^{-l}\mu^{1/2}$, (4.6) gives

$$|\langle x, y \rangle - \cos S_c^l(x, y)| \lesssim \varepsilon_\circ(\mu\widetilde{\mu})^{1/2} + 2^{-l}\mu^{1/2}.$$

Thus, by (2.15) and (3.7) it follows that

$$\partial_s \mathcal{P}(x, y, S_c^l(x, y, s)) = O(\varepsilon_\circ\widetilde{\mu} + 2^{-2l}).$$

Combining this and (4.8), we get the desired estimate for $\alpha = 0$.

We now assume $|\alpha| \geq 1$. Thanks to (4.8), it suffices to show

$$\partial_x^\alpha(\partial_s \mathcal{P}(x, y, S_c^l)) = O((\widetilde{\mu} + 2^{-2l})\mu^{-\alpha_1}(\mu\widetilde{\mu})^{-|\overline{\alpha}|/2}). \tag{4.34}$$

Since $\partial_{x_j}(\partial_s \mathcal{P}(x, y, S_c^l)) = \partial_{x_j}\partial_s \mathcal{P}(x, y, S_c^l) + \partial_s^2\mathcal{P}(x, y, S_c^l)\partial_{x_j} S_c$, using (4.30), (3.12) and the first identity in (4.32), we write

$$\partial_{x_j}(\partial_s \mathcal{P}(x, y, S_c^l)) = \frac{\mathcal{I}_j^1}{\sin^2 S_c^l} + \frac{\cos S_c^l - \cos S_c}{\sin^3 S_c^l}\mathcal{I}_j^2,$$

where

$$\mathcal{I}_j^1 := y_j \cos S_c - x_j,$$
$$\mathcal{I}_j^2 := y_j \sin S_c^l - \mathfrak{G}\langle x, y \rangle(\cos S_c^l - \tau^+)(2x_j - \mathfrak{A}y_j).$$

We may assume $\alpha_j \neq 0$ for some j. Thus,

$$\partial_x^\alpha(\partial_s \mathcal{P}(x, y, S_c^l)) = \partial_x^\beta\left(\frac{\mathcal{I}_j^1}{\sin^2 S_c^l}\right) + \partial_x^\beta\left(\frac{\cos S_c^l - \cos S_c}{\sin^3 S_c^l}\mathcal{I}_j^2\right),$$

where $\beta = \alpha - e_j$. By the Leibniz rule, (4.29), and the bounds in Remark 4.6, the desired estimate (4.34) follows if we show

$$|\partial_x^\beta \mathcal{J}_j^1| \lesssim \begin{cases} \widetilde{\mu}\mu^{-\beta_1}(\mu\widetilde{\mu})^{-|\bar{\beta}|/2}, & j = 1, \\ (\mu\widetilde{\mu})^{1/2}\mu^{-\beta_1}(\mu\widetilde{\mu})^{-|\bar{\beta}|/2}, & j \neq 1, \end{cases} \tag{4.35}$$

$$|\partial_x^\beta \mathcal{J}_j^2| \lesssim \begin{cases} (\widetilde{\mu}^{1/2} + 2^{-l})\mu^{-\beta_1}(\mu\widetilde{\mu})^{-|\bar{\beta}|/2}, & j = 1, \\ \mu^{1/2}\mu^{-\beta_1}(\mu\widetilde{\mu})^{-|\bar{\beta}|/2}, & j \neq 1. \end{cases} \tag{4.36}$$

It should be noted that slightly weaker bounds are good enough for our purpose when $\alpha_j \neq 0$ for $j \neq 1$, i.e., $|\bar{\alpha}| - 1 = |\bar{\beta}|$, since there is an improvement of factor $(\widetilde{\mu}/\mu)^{1/2}$ thanks to the particular form of estimate (4.34).

One can easily show (4.35) for $j \neq 1$ using (4.8) since $|x_j|, |y_j| \lesssim (\mu\widetilde{\mu})^{1/2}$. To show (4.35) for $j = 1$, we write $\mathcal{J}_1^1 = x_1^{-1}(\langle x, y \rangle \cos S_c - |x|^2 + \langle \bar{x}, \bar{x} - \bar{y}\cos S_c \rangle)$. By (3.11), we have

$$\mathcal{J}_1^1 = \frac{1}{x_1}\left(\frac{2(\langle x, y \rangle^2 - |x|^2|y|^2)}{|y|^2 - |x|^2 + |x + y||x - y|} + \langle \bar{x}, \bar{x} - \bar{y}\cos S_c \rangle\right).$$

Using (4.35) with $j \neq 1$ gives

$$\partial_x^\beta(\langle \bar{x}, \bar{x} - \bar{y}\cos S_c \rangle) = O((\mu\widetilde{\mu})\mu^{-\beta_1}(\mu\widetilde{\mu})^{-|\bar{\beta}|/2}).$$

One can easily check $\partial_x^\beta(|x|^2|y|^2 - \langle x, y \rangle^2) = O(\mu\widetilde{\mu}\mu^{-\beta_1}(\mu\widetilde{\mu})^{-|\bar{\beta}|/2})$. Additionally, we note that $\partial_x^\beta(|y|^2 - |x|^2 + |x + y||x - y|) = O(\mu^{1-|\beta|})$. Since

$$|y|^2 - |x|^2 + |x + y||x - y| \sim \mu,$$

combining those estimates with the Leibniz rule, we get (4.35) with $j = 1$.

For the proof of (4.36), we first claim that we may replace $\mathcal{J}_j^2$ by

$$\widetilde{\mathcal{J}}_j^2 = y_j \sin S_c - \mathfrak{G}\langle x, y \rangle(\cos S_c - \tau^+)(2x_j - \mathfrak{A}y_j).$$

In what follows, we frequently use this type of argument to replace less favorable terms by allowing acceptable errors. To show the claim, we make use of some easy estimates. We first note

$$|\partial_x^\beta(\sin S_c^l - \sin S_c)| \lesssim 2^{-l}\mu^{-|\beta|}, \quad \beta \in \mathbb{N}^d, \tag{4.37}$$

which one can show in the same manner as (4.29). Using (4.8) and the bounds in Remark 4.6, we also have

$$\partial_x^\beta \mathfrak{G} = O(\mu^{-3/2-|\beta|}), \quad \beta \in \mathbb{N}^d. \tag{4.38}$$

Noting $\mathfrak{A} = 2 + |x - y|^2/\langle x, y \rangle$, via a routine computation we have, for $\beta \in \mathbb{N}^d$,

$$|\partial_x^\beta(2x_i - \mathfrak{A}y_i)|, \ |\partial_x^\beta(2y_i - \mathfrak{A}x_i)| \lesssim \begin{cases} \mu\mu^{-\beta_1}(\mu\widetilde{\mu})^{-|\bar{\beta}|/2}, & i = 1, \\ (\mu\widetilde{\mu})^{1/2}\mu^{-\beta_1}(\mu\widetilde{\mu})^{-|\bar{\beta}|/2}, & i \geq 2. \end{cases} \tag{4.39}$$

Indeed, to see this one needs only to write

$$2x_i - \mathfrak{A}y_i = \frac{2\langle x, y\rangle x_i - (|x|^2 + |y|^2)\mathfrak{A}y_i}{\langle x, y\rangle},$$

and $2y_i - \mathfrak{A}x_i$ can be handled similarly. Putting together estimates (4.37), (4.29), (4.28), (4.38), and (4.39), we now see

$$\partial_x^\beta (\mathcal{I}_j^2 - \tilde{\mathcal{I}}_j^2) = O(2^{-l}\mu^{-\beta_1}(\mu\tilde{\mu})^{-|\bar{\beta}|/2}), \quad \beta \in \mathbb{N}^d.$$

In view of (4.36), the difference is an acceptable error. Therefore, as claimed above, we only have to show (4.36) for $\tilde{\mathcal{I}}_j^2$ replacing $\mathcal{I}_j^2$.

To do this, using the first equality in (3.13) and (4.33), we note

$$\sin S_c \tilde{\mathcal{I}}_j^2 = \bar{\mathcal{I}}_j := y_j \sin^2 S_c + \cos S_c (2x_j - \mathfrak{A}y_j).$$

Thus, the desired bound (4.36) follows once we have

$$|\partial_x^\beta \bar{\mathcal{I}}_j| \lesssim \begin{cases} (\tilde{\mu}^{1/2} + 2^{-l})\mu^{1/2-\beta_1}(\mu\tilde{\mu})^{-|\bar{\beta}|/2}, & j = 1, \\ \mu\mu^{-\beta_1}(\mu\tilde{\mu})^{-|\bar{\beta}|/2}, & j \neq 1. \end{cases} \tag{4.40}$$

Since $|\bar{x}|, |\bar{y}| = O((\mu\tilde{\mu})^{1/2})$, (4.40) for $j \neq 1$ is easy. Indeed, it follows by (4.39) and the bounds in Remark 4.6. To show (4.40) with $j = 1$, we break

$$\bar{\mathcal{I}}_1 = \bar{\mathcal{I}}_{1,1} - \bar{\mathcal{I}}_{1,2},$$

where

$$\bar{\mathcal{I}}_{1,1} := x_1^{-1}(\langle x, y\rangle \sin^2 S_c + \cos S_c(2|x|^2 - \mathfrak{A}\langle x, y\rangle)),$$
$$\bar{\mathcal{I}}_{1,2} := x_1^{-1}(\langle \bar{x}, \bar{y}\rangle \sin^2 S_c + \cos S_c(2|\bar{x}|^2 - \mathfrak{A}\langle \bar{x}, \bar{y}\rangle)).$$

Since $|\bar{x}|, |\bar{y}| = O((\mu\tilde{\mu})^{1/2})$, one can easily see $|\partial_x^\beta \bar{\mathcal{I}}_{1,2}| \lesssim \mu\tilde{\mu}\mu^{-\beta_1}(\mu\tilde{\mu})^{-|\bar{\beta}|/2}$. To handle $\bar{\mathcal{I}}_{1,1}$, using $\mathfrak{A}\langle x, y\rangle = |x|^2 + |y|^2$, the second identity in (3.13), and then

$$|x - y||x + y| = |x|^2 + |y|^2 - 2\langle x, y\rangle \cos S_c$$

successively, we observe

$$\bar{\mathcal{I}}_{1,1} = 2x_1^{-1} \cos S_c \langle x, x - \cos S_c y\rangle.$$

By (4.35), we have $\partial_x^\beta (x - \cos S_c y) = O((\mu\tilde{\mu})^{1/2}\mu^{-\beta_1}(\mu\tilde{\mu})^{-|\bar{\beta}|/2})$. Therefore, we get (4.40) for $j = 1$, as before (see Remark 4.6). ∎

Let us set

$$\mathbf{G}(x, y) = -\sin S_c^l(\partial_x S_c \partial_y^\mathsf{T} \partial_s \mathcal{P}(x, y, S_c^l) + \partial_x \partial_s \mathcal{P}(x, y, S_c^l)\partial_y^\mathsf{T} S_c),$$
$$\mathbf{F}(x, y) = -\sin S_c^l \partial_s^2 \mathcal{P}(x, y, S_c^l)\partial_x S_c \partial_y^\mathsf{T} S_c,$$
$$\mathbf{K}(x, y) = \mathbf{I}_d + \mathbf{G}(x, y) + \mathbf{F}(x, y).$$

In order to prove Lemma 4.7, by (4.31) and Lemma 4.12 it is sufficient to show that the entries $\widetilde{\mathcal{E}}_{i,j}$ of the matrix

$$\widetilde{\mathcal{E}} := \mathbf{K} - \mathfrak{M}^0$$

satisfy (4.12) in place of $\mathcal{E}_{i,j}$. A simple computation gives

$$\partial_x \partial_s \mathcal{P}(x, y, S_c^l) = \frac{\cos S_c^l y - x}{\sin^2 S_c^l}, \quad \partial_y^\mathsf{T} \partial_s \mathcal{P}(x, y, S_c^l) = \frac{\cos S_c^l x^\mathsf{T} - y^\mathsf{T}}{\sin^2 S_c^l}.$$

Using those identities, we write

$$\mathbf{G} = -\frac{\mathfrak{G}}{\sin S_c^l}((2\cos S_c^l + \mathfrak{A})(xx^\mathsf{T} + yy^\mathsf{T}) - 4xy^\mathsf{T} - 2\mathfrak{A}\cos S_c^l yx^\mathsf{T}), \tag{4.41}$$

$$\mathbf{F} = \sin S_c^l \partial_s^2 \mathcal{P}(x, y, S_c^l)\mathfrak{G}^2(2\mathfrak{A}(xx^\mathsf{T} + yy^\mathsf{T}) - 4xy^\mathsf{T} - \mathfrak{A}^2 yx^\mathsf{T}). \tag{4.42}$$

In what follows, for $i = 1, 2$, we denote $E(x, y) = \mathcal{O}_*(E_i)$ if $\partial_x^\alpha E(x, y) = O(E_i^\alpha)$ for any α and $(x, y) \in \mathcal{B} \times \widetilde{\mathcal{B}}$. We first show (4.12) for $(i, j) = (1, 1)$, which is more involved than the others.

Proof of (4.12) *for* $(i, j) = (1, 1)$. We consider $\mathbf{G}_{1,1}$ and $\mathbf{F}_{1,1}$, from which we discard some harmless parts. By (4.41),

$$\mathbf{G}_{1,1} = -\frac{\mathfrak{G}}{\sin S_c^l}((\mathfrak{A} + 2)(x_1 - y_1)^2 + 2(\cos S_c^l - 1)(x_1^2 + y_1^2 - \mathfrak{A}x_1 y_1)).$$

Using $\mathfrak{A}\langle x, y \rangle = |x|^2 + |y|^2$, we have

$$\mathbf{G}_{1,1} = -\frac{\mathfrak{G}}{\sin S_c^l}(\mathfrak{A} + 2)(x_1 - y_1)^2 + 2\mathfrak{G}\frac{(1 - \cos S_c^l)}{\sin S_c^l}(\mathfrak{A}\langle \overline{x}, \overline{y} \rangle - |\overline{x}|^2 - |\overline{y}|^2).$$

It is easy to see that the second term, which we denote by $\widetilde{\mathbf{G}}_{1,1}$, is $\mathcal{O}_*(E_1)$. Indeed, note $\partial_x^\alpha(\mathfrak{A}\langle \overline{x}, \overline{y} \rangle - |\overline{x}|^2 - |\overline{y}|^2) = O(\mu\widetilde{\mu}\mu^{-\alpha_1}(\mu\widetilde{\mu})^{-|\overline{\alpha}|/2})$. Then the bounds in Remark 4.6 and (4.38) show $\partial_x^\alpha \widetilde{\mathbf{G}}_{1,1} = O(\widetilde{\mu}\mu^{-\alpha_1}(\mu\widetilde{\mu})^{-|\overline{\alpha}|/2})$. Thus, by the first identity in (4.33) we get

$$\mathbf{G}_{1,1} = -\frac{\cos S_c(x_1 - y_1)^2}{\sin S_c \sin S_c^l|x - y|}\frac{|x + y|}{\langle x, y \rangle} + \mathcal{O}_*(E_1).$$

By writing $(x_1 - y_1)^2 = |x - y|^2 - |\overline{x} - \overline{y}|^2$ and using (3.13) (the second equality), the first term on the right-hand side equals

$$\frac{|\overline{x} - \overline{y}|^2}{|x - y|^2} + |\overline{x} - \overline{y}|^2\frac{\sin S_c - \sin S_c^l}{|x - y|^2\sin S_c^l} - \frac{|x + y||x - y|\cos S_c}{\langle x, y \rangle \sin S_c^l \sin S_c}.$$

Now, denote the second term by $\overline{\mathbf{G}}_{1,1}(x, y)$. By (4.37) and (4.8), we have

$$\partial_x^\alpha \overline{\mathbf{G}}_{1,1} = O(2^{-l}\widetilde{\mu}\mu^{-3/2}\mu^{-\alpha_1}(\mu\widetilde{\mu})^{-|\overline{\alpha}|/2}),$$

so $\bar{\mathbf{G}}_{1,1} = \mathcal{O}_*(E_1)$ by (3.18). Thus, we obtain

$$\mathbf{G}_{1,1} - \mathfrak{M}_{1,1}^0 + \mathcal{O}_*(E_1) = -\frac{|x+y||x-y|\cos S_c}{\langle x, y \rangle \sin S_c^l \sin S_c} = -\frac{(\tau^+ - \cos S_c)\cos S_c}{\sin S_c^l \sin S_c}.$$

We use (3.13) for the second equality.

From (4.42), we have

$$\mathbf{F}_{1,1} = \sin S_c^l \partial_s^2 \mathcal{P}(x, y, S_c^l)\mathfrak{G}^2(2\mathfrak{A}(|x|^2 + |y|^2) - (4 + \mathfrak{A}^2)\langle x, y \rangle)$$
$$- \sin S_c^l \partial_s^2 \mathcal{P}(x, y, S_c^l)\mathfrak{G}^2(2\mathfrak{A}(|\bar{x}|^2 + |\bar{y}|^2) - (4 + \mathfrak{A}^2)\langle \bar{x}, \bar{y} \rangle).$$

We denote by $\widetilde{\mathbf{F}}_{1,1}$ the second term in the right-hand side. By (4.27) and (4.38), we see $\partial_x^\alpha \widetilde{\mathbf{F}}_{1,1} = O(2^{-l}\widetilde{\mu}\mu^{-3/2}\mu^{-\alpha_1}(\mu\widetilde{\mu})^{-|\bar{\alpha}|/2})$, so $\widetilde{\mathbf{F}}_{1,1} = \mathcal{O}_*(E_1)$. Meanwhile, (4.33) gives

$$2\mathfrak{A}(|x|^2 + |y|^2) - (4 + \mathfrak{A}^2)\langle x, y \rangle = \frac{|x-y|^2|x+y|^2}{\langle x, y \rangle} = \mathfrak{G}^{-2}\frac{\cos^2 S_c}{\langle x, y \rangle \sin^2 S_c}.$$

Combining this with (3.12) yields

$$\mathbf{F}_{1,1} = \frac{(\tau^+ - \cos S_c^l)(\cos S_c^l - \cos S_c)\cos^2 S_c}{\sin^2 S_c^l \sin^2 S_c} + \mathcal{O}_*(E_1).$$

We now set

$$D_1 = (\tau^+ - \cos S_c^l)(\cos S_c^l - \cos S_c)\cos^2 S_c,$$
$$D_2 = (\tau^+ - \cos S_c)(\sin S_c^l - \sin S_c)\sin S_c^l \cos S_c,$$

and

$$A = \frac{D_1 + D_2}{\sin^2 S_c \sin^2 S_c^l}.$$

Since $\mathbf{K}_{1,1} = 1 + \mathbf{F}_{1,1} + \mathbf{G}_{1,1}$, using the above equalities, we obtain

$$\mathbf{K}_{1,1} - \mathfrak{M}_{1,1}^0 = A + \mathcal{O}_*(E_1).$$

Indeed, from (3.11) and (3.10) note $\sin^2 S_c = (\tau^+ - \cos S_c)\cos S_c$, which then gives $D_2 = \sin^2 S_c \sin^2 S_c^l - (\tau^+ - \cos S_c)\sin S_c \sin S_c^l \cos S_c$.

To complete the proof, it remains to show $A = \mathcal{O}_*(E_1)$. Let $\widetilde{D}_1$ denote D_1 in which the first $\cos S_c^l$ is replaced by $\cos S_c$, and by $\widetilde{D}_2$ we denote D_2 in which the second $\sin S_c^l$ is replaced by $\sin S_c$. Then, using (4.29) as before, we see

$$\partial_x^\alpha(D_1 - \widetilde{D}_1) = O(2^{-2l}\mu\mu^{-\alpha_1}(\mu\widetilde{\mu})^{-|\bar{\alpha}|/2}).$$

Similarly, $\partial_x^\alpha(D_2 - \widetilde{D}_2) = O(2^{-2l}\mu\mu^{-\alpha_1}(\mu\widetilde{\mu})^{-|\bar{\alpha}|/2})$ by (4.37) and (4.28). So,

$$\frac{D_1 - \widetilde{D}_1 + D_2 - \widetilde{D}_2}{\sin^2 S_c \sin^2 S_c^l} = \mathcal{O}_*(E_1).$$

Thus, we have

$$A = \frac{\tilde{D}_1 + \tilde{D}_2}{\sin^2 S_c \sin^2 S_c^l} + \mathcal{O}_*(E_1).$$

Note $\tilde{D}_1 + \tilde{D}_2 = 2\cos S_c(\cos S_c - \tau^+)\sin^2((S_c^l - S_c)/2)$, which follows from elementary trigonometric identities. Therefore,

$$A = 2\frac{\cos S_c(\cos S_c - \tau^+)}{\sin^2 S_c \sin^2 S_c^l}\sin^2\left(\frac{S_c^l - S_c}{2}\right) + \mathcal{O}_*(E_1).$$

Since $S_c^l - S_c = 2^{-l}s$, using (4.28) and the bounds in Remark 4.6, we conclude that $A = \mathcal{O}_*(E_1)$. ∎

Before we begin to prove (4.12) for $(i, j) \neq (1, 1)$, we show that the contribution of **F** is negligible. By (4.42), we have

$$\mathbf{F}_{i,j} = -\sin S_c^l \partial_s^2 \mathcal{P}(x, y, S_c^l)\mathfrak{G}^2(2x_i - \mathfrak{A}y_i)(2y_j - \mathfrak{A}x_j).$$

If $i, j \geq 2$, we use (4.27), (4.39), and (4.38) (also see Remark 4.6) to see that

$$\partial_x^\alpha \mathbf{F}_{i,j} = O(2^{-l}\tilde{\mu}\mu^{-3/2}\mu^{-\alpha_1}(\mu\tilde{\mu})^{-|\bar\alpha|/2}).$$

This shows $\mathbf{F}_{i,j} = \mathcal{O}_*(E_1)$ for $i, j \geq 2$. If $i = 1, j \geq 2$, or $j = 1, i \geq 2$, we similarly obtain $\partial_x^\alpha \mathbf{F}_{i,j} = O(2^{-l}\tilde{\mu}^{1/2}\mu^{-1}\mu^{-\alpha_1}(\mu\tilde{\mu})^{-|\bar\alpha|/2})$, so it follows that $\mathbf{F}_{i,j} = \mathcal{O}_*(E_2)$.

Therefore, the following completes the proof of Lemma 4.7:

$$\mathbf{G}_{i,j} - \mathfrak{M}_{i,j}^0 = \begin{cases} \mathcal{O}_*(E_1), & i, j \geq 2, \\ \mathcal{O}_*(E_2), & i = 1, j \geq 2, \text{ or } j = 1, i \geq 2. \end{cases} \tag{4.43}$$

Proof of (4.43). We consider the case $i, j \geq 2$ first. By (4.41), we have

$$\mathbf{G}_{i,j} = -\frac{\mathfrak{G}}{\sin S_c^l}((2\cos S_c^l + \mathfrak{A})(x_ix_j + y_iy_j) - 4x_iy_j - 2\mathfrak{A}\cos S_c^l y_ix_j).$$

By $\mathbf{G}'_{i,j}$ we denote $\mathbf{G}_{i,j}$ in which we replace $\mathfrak{A}$ by 2. Since $\mathfrak{A} = 2 + |x - y|^2/\langle x, y\rangle$, we have $\partial_x^\alpha(\mathfrak{A} - 2) = O(\mu^{2-|\alpha|})$. Using (4.38), we see

$$\partial_x^\alpha(\mathbf{G}_{i,j} - \mathbf{G}'_{i,j}) = O(\tilde{\mu}\mu^{1-\alpha_1}(\mu\tilde{\mu})^{-|\bar\alpha|/2})$$

because

$$x_i, x_j, y_i, y_j = O((\mu\tilde{\mu})^{1/2}), \quad i, j \geq 2.$$

Thus, it is enough to consider $\mathbf{G}'_{i,j}$. Furthermore, by (4.29) and (4.38), $\cos S_c^l$ in $\mathbf{G}'_{i,j}$ can similarly be replaced by $\cos S_c$ if we allow an error of $\mathcal{O}_*(E_1)$. Thus,

$$\mathbf{G}_{i,j} = \tilde{\mathbf{G}}_{i,j} + \bar{\mathbf{G}}_{i,j} + \mathcal{O}_*(E_1), \quad i, j \geq 2, \tag{4.44}$$

where

$$\tilde{\mathbf{G}}_{i,j} = -\frac{2\mathfrak{G}}{\sin S_c^l}(x_j - y_j)(2x_i - (1 + \cos S_c)y_i),$$

$$\bar{\mathbf{G}}_{i,j} = \frac{2\mathfrak{G}}{\sin S_c^l}(x_i - y_i)(1 - \cos S_c)x_j.$$

It is easy to see $\partial_x^\alpha \bar{\mathbf{G}}_{i,j} = O(\tilde{\mu}\mu^{-\alpha_1}(\mu\tilde{\mu})^{-|\bar{\alpha}|/2})$ for $i, j \geq 2$, thus $\bar{\mathbf{G}}_{i,j} = \mathcal{O}_*(E_1)$. Using (4.37), by $\sin S_c$ we may replace $\sin S_c^l$ in the expression of $\tilde{\mathbf{G}}_{i,j}$ with an error of $\mathcal{O}_*(E_1)$. Then, applying (4.33) and (3.13), we get

$$\mathbf{G}_{i,j} = -\frac{2\langle x, y \rangle}{|x + y|^2} \frac{(x_j - y_j)}{|x - y|^2}(2x_i - (1 + \cos S_c)y_i) + \mathcal{O}_*(E_1). \qquad (4.45)$$

As before, we may replace $2\langle x, y \rangle / |x + y|^2$ by $1/2$ since

$$\partial_x^\alpha \left(\frac{1}{2} - \frac{2\langle x, y \rangle}{|x + y|^2}\right) = O(\mu^2\mu^{-|\alpha|}),$$

$$\partial_x^\alpha (2x_i - (1 + \cos S_c)y_i) = O((\mu\tilde{\mu})^{1/2}\mu^{-\alpha_1}(\mu\tilde{\mu})^{-|\bar{\alpha}|/2}),$$

$$\partial_x^\alpha (x_j - y_j) = O((\mu\tilde{\mu})^{1/2}\mu^{-\alpha_1}(\mu\tilde{\mu})^{-|\bar{\alpha}|/2}), \quad i, j \geq 2.$$

Therefore, we have

$$\mathbf{G}_{i,j} = \mathfrak{M}_{i,j}^0 + \frac{(x_j - y_j)(\cos S_c - 1)y_i}{2|x - y|^2} + \mathcal{O}_*(E_1).$$

Note

$$\partial_x^\alpha \left(\frac{(x_j - y_j)(1 - \cos S_c)y_i}{|x - y|^2}\right) = O(\tilde{\mu}\mu^{-\alpha_1}(\mu\tilde{\mu})^{-|\bar{\alpha}|/2}).$$

Thus, (4.43) follows for $i, j \geq 2$.

We now show (4.43) when $i = 1$, $j \geq 2$, or $j = 1$, $i \geq 2$. We only consider the case $i = 1$, $j \geq 2$, since the other one can be handled in the same manner. Since $x_j, y_j = O((\mu\tilde{\mu})^{1/2})$ for $j \geq 2$, repeating the same argument used to show (4.44), we obtain

$$\mathbf{G}_{1,j} = \tilde{\mathbf{G}}_{1,j} + \bar{\mathbf{G}}_{1,j} + \mathcal{O}_*(E_2).$$

Here we use the same notations as above. Note

$$\partial_x^\alpha (2x_1 - (1 + \cos S_c)y_1) = O(\mu\mu^{-\alpha_1}(\mu\tilde{\mu})^{-|\bar{\alpha}|/2}).$$

By this and (4.38), $\partial_x^\alpha \bar{\mathbf{G}}_{1,j} = O(\mu^{1/2}\tilde{\mu}^{1/2}\mu^{-\alpha_1}(\mu\tilde{\mu})^{-|\bar{\alpha}|/2})$, thus $\mathbf{G}_{1,j} = \tilde{\mathbf{G}}_{1,j} + \mathcal{O}_*(E_2)$. Allowing an error of $\mathcal{O}_*(E_2)$, as before, we may replace $\sin S_c^l$ by $\sin S_c$ in $\tilde{\mathbf{G}}_{1,j}$. Consequently, using (4.33) and (3.13) (cf. (4.45)), we obtain

$$\tilde{\mathbf{G}}_{1,j} = -\frac{2\langle x, y \rangle}{|x + y|^2} \frac{(x_j - y_j)}{|x - y|^2}(2x_1 - (1 + \cos S_c)y_1) + \mathcal{O}_*(E_2).$$

We may also replace, as above, the factor $2\langle x, y \rangle / |x + y|^2$ by $1/2$ with an error of $\mathcal{O}_*(E_2)$. Therefore, (4.43) follows for $i = 1$, $j \geq 2$. ∎

References

[1] Askey, R., Wainger, S.: Mean convergence of expansions in Laguerre and Hermite series. Amer. J. Math. **87**, 695–708 (1965) Zbl 0125.31301 MR 182834

[2] Chen, P., Lee, S., Sikora, A., Yan, L.: Bounds on the maximal Bochner–Riesz means for elliptic operators. Trans. Amer. Math. Soc. **373**, 3793–3828 (2020) Zbl 1440.42044 MR 4105510

[3] Chen, P., Li, J., Ward, L., Yan, L.: Weak-type endpoint bounds for Bochner–Riesz means for the Hermite operator. C. R. Math. Acad. Sci. Paris **360**, 111–126 (2022) Zbl 1484.42012 MR 4384327

[4] Escauriaza, L.: Carleman inequalities and the heat operator. Duke Math. J. **104**, 113–127 (2000) Zbl 0979.35029 MR 1769727

[5] Escauriaza, L., Vega, L.: Carleman inequalities and the heat operator. II. Indiana Univ. Math. J. **50**, 1149–1169 (2001) Zbl 1029.35046 MR 1871351

[6] Hörmander, L.: Oscillatory integrals and multipliers on FL^p. Ark. Mat. **11**, 1–11 (1973) Zbl 0254.42010 MR 340924

[7] Jeong, E., Lee, S., Ryu, J.: Sharp L^p-L^q estimate for the spectral projection associated with the twisted Laplacian. Publ. Mat. **66**, 831–855 (2022) Zbl 1492.42021 MR 4443755

[8] Jeong, E., Lee, S., Ryu, J.: Bounds on the Hermite spectral projection operator. J. Funct. Anal. **286**, article no. 110175 (2024) Zbl 1530.35082 MR 4653334

[9] Jeong, E., Lee, S., Ryu, J.: Unique continuation for the heat operator with potentials in weak spaces. Anal. PDE **17**, 2257–2274 (2024)

[10] Karadzhov, G. E.: Riesz summability of multiple Hermite series in L^p spaces. C. R. Acad. Bulgare Sci. **47**, 5–8 (1994) Zbl 0829.40003 MR 1319486

[11] Koch, H., Ricci, F.: Spectral projections for the twisted Laplacian. Studia Math. **180**, 103–110 (2007) Zbl 1120.43005 MR 2314091

[12] Koch, H., Tataru, D.: L^p eigenfunction bounds for the Hermite operator. Duke Math. J. **128**, 369–392 (2005) Zbl 1075.35020 MR 2140267

[13] Koch, H., Tataru, D.: Carleman estimates and unique continuation for second order parabolic equations with nonsmooth coefficients. Comm. Partial Differential Equations **34**, 305–366 (2009) Zbl 1178.35107 MR 2530700

[14] Koch, H., Tataru, D., Zworski, M.: Semiclassical L^p estimates. Ann. Henri Poincaré **8**, 885–916 (2007) Zbl 1133.58025 MR 2342881

[15] Kochneff, E.: Rotational symmetry of the Hermite projection operators. Proc. Amer. Math. Soc. **124**, 1539–1547 (1996) Zbl 0853.33013 MR 1307540

[16] Lee, S., Ryu, J.: Bochner–Riesz means for the Hermite and special Hermite expansions. Adv. Math. **400**, article no. 108260 (2022) Zbl 1486.42044 MR 4385141

[17] Sjögren, P., Torrea, J. L.: On the boundary convergence of solutions to the Hermite–Schrödinger equation. Colloq. Math. **118**, 161–174 (2010) Zbl 1198.42019 MR 2600523

[18] Sogge, C. D.: Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds. J. Funct. Anal. **77**, 123–138 (1988) Zbl 0641.46011 MR 930395

[19] Stein, E. M.: Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals. Princeton Math. Ser. 43, Princeton University Press, Princeton, NJ (1993) Zbl 0821.42001 MR 1232192

[20] Stempak, K., Zienkiewicz, J.: Twisted convolution and Riesz means. J. Anal. Math. **76**, 93–107 (1998) Zbl 0924.42011 MR 1676932

[21] Tao, T., Vargas, A., Vega, L.: A bilinear approach to the restriction and Kakeya conjectures. J. Amer. Math. Soc. **11**, 967–1000 (1998) Zbl 0924.42008 MR 1625056

[22] Thangavelu, S.: Multipliers for Hermite expansions. Rev. Mat. Iberoam. **3**, 1–24 (1987) Zbl 0687.42015 MR 1008442

[23] Thangavelu, S.: Lectures on Hermite and Laguerre expansions. Math. Notes 42, Princeton University Press, Princeton, NJ (1993) Zbl 0791.41030 MR 1215939

[24] Thangavelu, S.: Hermite and special Hermite expansions revisited. Duke Math. J. **94**, 257–278 (1998) Zbl 0945.42014 MR 1638583