J. Eur. Math. Soc. 28, 1457–1487 (2026) DOI 10.4171/JEMS/1499

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Complete stable minimal hypersurfaces in positively curved 4-manifolds

Received Juny 7, 2022; revised March 28, 2023

Abstract. We show that the *combination* of non-negative sectional curvature (or 2-intermediate Ricci curvature) and strict positivity of scalar curvature forces rigidity of complete (non-compact) two-sided stable minimal hypersurfaces in a 4-manifold with bounded curvature. Our work leads to new comparison results. We also construct various examples showing that rigidity of stable minimal hypersurfaces can fail under other curvature conditions.

Keywords: stable minimal hypersurfaces, complete positively curved 4-manifolds, curvature estimates.

1. Introduction

Recall that a *two-sided stable minimal hypersurface* $M^{n-1} \to (X^n, g)$ is an immersed hypersurface with vanishing mean curvature and trivial normal bundle satisfying the inequality

$$\int_M (|A_M|^2 + \mathrm{Ric}_g(\nu,\nu))\varphi^2 \le \int_M |\nabla\varphi|^2$$

for any $\varphi \in C_c^\infty(M \setminus \partial M)$. Here, A_M is the second fundamental form of the immersion, Ric_g is the ambient Ricci curvature, and ν is (any) choice of unit normal. Stable minimal hypersurfaces can be used in a similar manner to stable geodesics to probe the ambient geometry of (X^n, g). The basic results along these lines are as follows. When $M^{n-1} \to (X^n, g)$ is a *closed* (compact without boundary) two-sided stable minimal hypersurface, the following hold:

(1) If $\mathrm{Ric}_g \ge 0$ then M is totally geodesic and $\mathrm{Ric}_g(\nu,\nu) \equiv 0$ along M [61] (cf. [5]). In particular, when $\mathrm{Ric} > 0$, there are no closed two-sided stable minimal hypersurfaces.

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Mathematics Subject Classification 2020: 53A10 (primary); 53C21, 53C42 (secondary).

(2) If $R_g \geq 1$ (scalar curvature) then M also admits a metric of positive scalar curvature [54, 55]. In particular, when $n = 3$ and X is oriented, each component of M must be a 2-sphere.

The second result is a fundamental tool in the study of manifolds of positive scalar curvature.

When M is now assumed to be complete with respect to the induced metric (in particular, M has no boundary) and *non-compact*, the theory becomes considerably more complicated. As the following results show, it has been well-developed in three dimensions. Suppose $M^2 \to (X^3, g)$ is a complete two-sided stable minimal immersion.

(1) If $R_g \geq 0$, then the induced metric on M is conformal to either the plane or the cylinder [24]. In the latter case, M is totally geodesic, intrinsically flat, and $R_g \equiv 0$, $\mathrm{Ric}_g(\nu, \nu) \equiv 0$ along M (cf. [8, Proposition C.1]).

(2) If $\mathrm{Ric}_g \geq 0$, then M is totally geodesic, intrinsically flat, and $\mathrm{Ric}_g(\nu, \nu) \equiv 0$ along M [56] (cf. [20, 24, 49]).

(3) If $R_g \geq 1$, then M must be compact [57] (cf. [30]).

These results have had important applications to comparison geometry. In particular, we refer to the following (incomplete) list of results that rely specifically on (non-compact) stable minimal surfaces in 3-manifolds: [2, 30, 39, 56, 64]. For most of these applications, it is essential that no assumption is made related to properness or volume growth of M.

The first- and second- named authors have recently resolved [12] the conjecture of Schoen that a complete two-sided stable minimal immersion $M^3 \to \mathbf{R}^4$ is flat (without any additional assumptions on M); see also the subsequent proofs in [9, 13]. It is natural to ask what happens in a curved ambient background, rather than flat Euclidean space. For example, a basic question is to determine which ambient curvature conditions suffice to show that a complete stable minimal hypersurface must be compact, or cannot exist. The methods used in [12] do not seem to extend to the curved setting.[1] (See [6, 37, 52, 59] for previous results in this direction.)

In fact (as was discussed above), in a 3-manifold, $R_g \geq 1$ implies compactness and $\mathrm{Ric}_g > 0$ implies non-existence; on the other hand, in four dimensions, neither of these results can hold:

Example 1.1 (Non-compact stable minimal hypersurface in 4-manifold with $R_g \geq 1$). Fix an oriented closed 2-dimensional Riemannian manifold (X^2, g) admitting a unit speed stable geodesic

$$\sigma : \mathbf{R} \to (X^2, g).$$

Then, for $\varepsilon > 0$ sufficiently small, $(X^2, g) \times \mathbf{S}^2(\varepsilon)$ has $R_g \geq 1$, while $\sigma \times \mathbf{S}^2(\varepsilon)$ is an unbounded two-sided stable minimal immersion.

[1] In particular, the use of the L^3 Schoen–Simon–Yau inequality [52] seems troublesome in most curved ambient manifolds.

Example 1.2 (Existence of stable minimal hypersurface in a 4-manifold with positive sectional curvature). Consider a rotationally symmetric metric g on $\mathbf{R}^4$ with strictly positive sectional curvatures. There are totally geodesic copies of $\mathbf{R}^3$ in such a metric, and as long as the Ricci curvature of g decays to zero sufficiently fast, these will be (complete) two-sided stable minimal embeddings. Essentially, the point is that on flat Euclidean 3-space $(\mathbf{R}^3, \bar{g})$, any Schrödinger operator with non-negative, rapidly decaying potential[2] will be stable, thanks to a classical Hardy inequality

$$\int_{\mathbf{R}^3} 4|\bar{\nabla}\varphi|^2 \, d\bar{\mu} \geq \int_{\mathbf{R}^3} r^{-2}\varphi^2 \, d\bar{\mu}.$$

We explicitly construct such an example and check its stability in Section B.1.

1.1. Main results

It turns out that if one *combines* positivity of curvature with strict positivity of scalar curvature, such examples can be avoided. This is the main result of this article.

Theorem 1.3. *If* (X^4, g) *is a complete* 4*-manifold with weakly bounded geometry, non-negative sectional curvature, and strictly positive scalar curvature* $R_g \geq R_0 > 0$*, then any complete two-sided stable minimal immersion* $M^3 \to (X^4, g)$ *is totally geodesic and has* $\mathrm{Ric}_g(\nu, \nu) \equiv 0$ *along* M*.*

Here, *weakly bounded geometry* means that around any point, there is a *local* diffeomorphism from a ball in $\mathbf{R}^4$ of definite size so that the pullback metric is $C^{1,\alpha}$-close to Euclidean (see Definition 2.3). (Note that this allows for collapsing behavior at infinity.)

Remark 1.4. All results in this paper assume that (X, g) at least has $\mathrm{Ric}_g \geq 0$, in which case "weakly bounded geometry" could be replaced by the assumption "$\sec_g \leq K$" anywhere it occurs here. It would be interesting to understand if this condition could be removed.

Remark 1.5. Shen–Ye [59] proved that for $n \leq 4$, if (X^{n+1}, g) satisfies

$$\mathrm{biRic} \geq k > 0, \tag{1.1}$$

then any complete two-sided stable minimal immersion $M^n \to (X^{n+1}, g)$ is compact. Here biRic is the bi-Ricci curvature, defined in [59] ($\mathrm{biRic} \geq k$ implies that $R_g \geq c_n k$ for some dimensional constant c_n). (See Appendix A for precise definitions of these curvature conditions as well as Section 1.2 below.)

In particular, it follows that such a minimal immersion does not exist if (X^{n+1}, g) is assumed to be closed with positive sectional curvature. After our paper appeared,

[2]Note that a Schrödinger operator with non-negative (but not identically zero) potential cannot be stable on $\mathbf{R}^2$, thanks to the log-cutoff trick.

the Shen–Ye result was rediscovered and extended to $n = 5$ by Catino–Mastrolia–Roncoroni [9], under the additional assumption of non-negative sectional curvature (or even just non-negative 4-intermediate Ricci curvature). The approach in [9, 59] is very different from that of this paper and is based on a conformal deformation technique first introduced by Fischer-Colbrie [23]. An interesting feature of our paper (see Theorem 1.10) is that we are able to replace (when $n = 3$) the condition $\text{biRic} \geq k > 0$ by the weaker assumption on scalar curvature, $R_g \geq R_0 > 0$ in (1.1). We note that the condition considered in this paper allows for the existence of complete, *non-compact* stable minimal immersions (we prove they must be totally geodesic and have vanishing normal Ricci curvature). For example, one may consider $S^2 \times \mathbf{R} \to S^2 \times \mathbf{R} \times S^1$.

Of course, any closed manifold (X, g) has weakly bounded geometry, so we have the following result:

Corollary 1.6. *If* (X^4, g) *is a closed* 4*-manifold with non-negative sectional curvature and positive scalar curvature, then a complete two-sided stable minimal immersion* $M^3 \to (X^4, g)$ *is totally geodesic and* $\text{Ric}_g(\nu, \nu) \equiv 0$ *along* M.

This applies (for example) to the standard product metrics on S^4, $S^1 \times S^3$, $S^2 \times S^2$, and $T^2 \times S^2$. We emphasize that the latter example does not have positive bi-Ricci curvature, and thus is not covered by the results in [9, 59].

Remark 1.7. The study of complete (non-compact) stable minimal immersions in a *compact* 3-manifold has played an important role in the study of embedded minimal surfaces with bounded Morse index [7, 11, 35]. One may hope that Corollary 1.6 could lead to similar results in closed 4-manifolds.

Remark 1.8. Theorem 1.3 fails in 6-dimensional manifolds (and higher): see Appendix B.3 for a counterexample. Gromov's conjecture that a 4-manifold with uniformly positive scalar curvature has macroscopic dimension 2 suggests that the 5-dimensional version of Theorem 1.3 would be the "critical" dimension (this is related to the log-cutoff trick for minimal surfaces in 3-manifolds).

1.2. Concerning the assumption of non-negative sectional curvature

It is natural to ask whether or not there exist complete two-sided stable minimal immersions $M^3 \to (X^4, g)$ where (X^4, g) has weakly bounded geometry, $\text{Ric}_g \geq 0$ and $R_g \geq 1$. (Note that in Theorem 1.3, $\text{Ric}_g \geq 0$ is replaced by non-negative sectional curvature.)

In this connection, we have the following example indicating that this is unlikely to be true.

Example 1.9 (Stable minimal hypersurface in a 4-manifold with strictly positive Ricci curvature). There exists a closed 4-manifold (X^4, g) (so it automatically has bounded geometry) and a complete two-sided stable minimal immersion $M^3 \to (X^4, g)$ such that $\text{Ric}_g \geq 1$ (so in particular $R_g \geq 1$) in some δ-neighborhood of the image of M.

This example is constructed in detail in Section B.2, but we briefly describe it here. The minimal immersion $M^3 \to (X^4, g)$ is constructed by taking the universal cover of some embedded compact[3] minimal submanfold $M_0 \subset (X^4, g)$. We choose M_0 so that it is diffeomorphic to $(S^1 \times S^2) \# (S^1 \times S^2)$, so $\pi_1(M_0) = F_2$ is a "non-amenable" group. This implies that the universal cover has positive first eigenvalue. Hence, as long as $|A_M|^2 + \mathrm{Ric}_g(\nu, \nu)$ is sufficiently small, the universal cover will be stable.

We note that it is unclear if one can construct such an example where (X^4, g) is complete and satisfies $\mathrm{Ric}_g > 0$ and $R_g \geq 1$ at all points (not just near the image of M). (We discuss the possibility of doing so further in Section B.2.) However, we emphasize that this example precludes any argument concerning stable minimal immersions in such (X^4, g) that is based purely on the second variation of area.[4]

On the other hand, it is possible to weaken non-negativity of sectional curvature to an intermediate condition. Namely, we say that (X^4, g) has *non-negative 2-intermediate Ricci curvature* if the sectional curvatures satisfy $\sec_g(\Pi_1) + \sec_g(\Pi_2) \geq 0$ for any two 2-planes $\Pi_1, \Pi_2 \subset T_pX$ intersecting in a line at a right angle (see Definition 2.1). We will write $\mathrm{Ric}_2^g \geq 0$ for this condition. (Note that non-negative sectional curvature implies $\mathrm{Ric}_2^g \geq 0$ which implies $\mathrm{Ric}_g \geq 0$.) The results discussed above all hold with non-negative sectional curvature replaced by $\mathrm{Ric}_2^g \geq 0$. More precisely, we have

Theorem 1.10. *If (X^4, g) is a complete 4-manifold with weakly bounded geometry, $\mathrm{Ric}_2^g \geq 0$, and $R_g \geq R_0 > 0$, then any complete two-sided stable minimal immersion $M^3 \to (X^4, g)$ is totally geodesic and has $\mathrm{Ric}_g(\nu, \nu) \equiv 0$ along M.*

We discuss the topological implications of this result in Section 6.

We remark that $\mathrm{Ric}_g \geq 0$ (instead of $\mathrm{Ric}_2^g \geq 0$) suffices for most (but not all) steps of the proof of Theorem 1.10, so if one places additional topological assumptions on the minimal hypersurface, then the proof of Theorem 1.10 can be adapted in a straightforward way to show the following result.

Theorem 1.11. *Let (X^4, g) be a complete 4-manifold with weakly bounded geometry, $\mathrm{Ric}_2^g \geq -K$, $\mathrm{Ric}_g \geq 0$, and $R_g \geq R_0 > 0$. Let $M^3 \to (X^4, g)$ be a complete two-sided stable minimal immersion with finitely many ends and $b_1(M) < \infty$. Then M is totally geodesic and has $\mathrm{Ric}_g(\nu, \nu) \equiv 0$ along M.*

[3] Note that M_0 is necessarily unstable, since $\mathrm{Ric}_g(\nu, \nu) > 0$; the instability occurs due to the "largeness" of the universal cover.

[4] Note that Li–Wang have used the Busemann function in their analysis of *proper* ends of stable minimal hypersurfaces in non-negatively curved manifolds [37]. Hence, this argument relies on the ambient geometry outside of just a tubular neighborhood of the immersion. However, it is unclear how to extend such an argument to non-negative Ricci curvature (the Busemann function is subharmonic rather than convex so it is difficult to control the restriction to a minimal hypersurface). Along these lines, we note that the stable minimal immersion $M^3 \to (X^4, g)$ discussed in Example 1.9 has infinitely many ends.

1.3. Topological applications

Recall that Schoen–Yau used minimal surfaces to prove that a complete non-compact (X^3, g) with $\mathrm{Ric}_g > 0$ is diffeomorphic to $\mathbf{R}^3$ [56] (cf. [3, 39]). They did this by showing that $\pi_2(X) = 0$ and that X is simply connected at infinity. In this section, we observe that given Theorem 1.10, similar techniques can be used to prove new topological restrictions on (X^4, g) with $\mathrm{Ric}_2^g > 0$, $R_g \geq R_0 > 0$, and weakly bounded geometry. (See [43] for a collection of results concerning the Ric_k^g curvature condition.)

All homology groups and cohomology groups here will be taken with $\mathbb{Z}$-coefficients.

Definition 1.12. Let X be a non-compact manifold, $k \in \mathbb{N}$. Define the k-th *homology group of X at infinity* as the inverse limit

$$H_k^\infty(X) = \varprojlim_{A \Subset X} H_k(X - A).$$

We say X is *H_k-trivial at infinity* if the natural homeomorphism $H_k^\infty(X) \to H_k(X)$ is injective.

In Section 6 we prove the following result.

Theorem 1.13. *Suppose (X^4, g) is a complete, non-compact, oriented 4-manifold with weakly bounded geometry, $\mathrm{Ric}_2^g > 0$, and strictly positive scalar curvature $R_g \geq R_0 > 0$. Then*

(1) *$H_1(X)$ only has torsion elements;*

(2) *X is H_2-trivial at infinity.*

It would be interesting to see if the techniques from [3, 39] (cf. [8, 10]) could be adapted to study the $\mathrm{Ric}_2^g \geq 0$ rigidity version of this result.

Remark 1.14. If (X^n, g) has $\mathrm{Ric}_g > 0$, then $H_{n-1}(X) = 0$ by [60, 67].

In light of Examples 1.1 and 1.2, it would be interesting to know if Theorem 1.13 held under weaker curvature conditions (e.g., $\mathrm{Ric}_2^g > 0$ but without $R_g \geq R_0 > 0$).

1.4. Idea of the proof and related results

An interesting feature of Theorem 1.3 is that it combines different curvature conditions. Indeed, the different conditions come in at rather different places in the proof. As a consequence, the strategy of the proof is quite different from most previous results concerning stability of complete (non-compact) minimal surfaces (including the previous work of the first- and second-named authors on stable minimal immersions $M^3 \to \mathbf{R}^4$ [12]). A key feature of this paper is the use of μ-bubbles, a technique introduced by Gromov [26], allowing one to gain distance control (in certain settings) from positivity of scalar curvature.

The overall strategy is as follows, indicating which curvature conditions enter at which step. Suppose $M^3 \to (X^4, g)$ is a complete two-sided stable minimal hypersurface.

(1) To show that M is totally geodesic and has vanishing normal Ricci curvature, we would like to find a sequence of compactly supported functions φ_i with $\varphi_i \to 1$ and $\int_M |\nabla\varphi_i|^2 = o(1)$. Taking such functions in the stability inequality then gives the desired conclusion (this just uses $\mathrm{Ric}_g \geq 0$). Note that for minimal surfaces in a 3-manifold, this condition is usually guaranteed by proving that the surface is conformal to the plane and then using the log-cutoff trick. However, in higher dimensions, this is usually impossible (e.g., no such functions exist on flat $\mathbf{R}^3$).

(2) The non-negativity of sectional curvature (or $\mathrm{Ric}_2^g \geq 0$) ensures that M has at most one non-parabolic end (see Definition 3.3). Non-parabolic ends are more difficult to handle, since parabolic ends automatically admit test good functions as described in (1).

(3) The remaining issue[5] is to construct a good test function along the non-parabolic end. In general, this is not possible with only a non-negative sectional curvature assumption (cf. Example 1.2). This is where the positivity of scalar curvature comes in. One knows (going back to the work of Schoen–Yau [55]) that a stable minimal hypersurface in a positive scalar curvature manifold "acts" as if it itself has positive scalar curvature.

In particular, a 3-dimensional manifold with positive scalar curvature tends to be macroscopically 1-dimensional in various senses (cf. [30, 38, 40, 62]). Hence, one might imagine that a single end of M has *linear* volume growth (this is where the parabolic/non-parabolic distinction from (2) is important). Indeed, a metric tree is 1-dimensional but has exponential length growth – for instance, the universal cover of $(S^1 \times S^1) \# (S^1 \times S^2)$. Note that if this end had linear volume growth, then the standard cutoff function that is 1 in B_R and 0 in B_{2R} would have vanishing Dirichlet energy as $R \to \infty$, so this would allow us to construct the desired cutoff function.

It seems difficult (or potentially impossible)[6] to actually prove that the non-parabolic end E has linear volume growth. Instead we construct an exhaustion of E by bounded sets $\Omega_1 \subset \Omega_2 \subset \cdots$ so that $\partial\Omega_i \cap E$ has controlled diameter. To do so, we use the theory of μ-bubbles (surfaces of carefully prescribed mean curvature) first introduced by Gromov [26] (see also [27]), allowing one to study metric quantities related to scalar curvature. (This application is similar in spirit to the use of μ-bubbles in [14, 15, 28].) This is the step that uses the assumption of strictly positive scalar curvature.

(4) Finally, to construct a good test function, it remains to show that the L-neighborhood of $\partial\Omega_i \cap E$ has bounded volume (for some constant L). To upgrade diameter control (obtained in (3)) to volume control, we need to control (from below) the intrinsic

[5]The reality is somewhat more complicated, since the non-parabolic component could have infinitely many parabolic ends attached, but this captures the main idea.

[6]One can imagine that E is a cylinder $S^2 \times (0, \infty)$ connected sum at integer points with spheres S^3 with volume tending to ∞ and scalar curvature ≥ 1. It is unclear whether this could be ruled out by stability.

Ricci curvature of the minimal immersion M. This is achieved (thanks to the Gauss equations) by combining a lower bound on the sectional (or Ric_2^g) curvatures with an absolute bound on the second fundamental form of M as follows from [12]. This is where the weakly bounded geometry assumption enters.

We remark that steps (3) and (4) are somewhat reminiscent of a recent result (but not the proof) of Munteanu–Wang proving that a complete 3-manifold with $R_g \geq 1$ and $\mathrm{Ric}_g \geq 0$ has linear volume growth [46] (see also [45, 66, 68]). See our recent work [16] for an approach to this result following the methods of this paper.

1.5. Organization of the paper

In Section 2 we discuss the $\mathrm{Ric}_2 \geq 0$ and bounded geometry conditions used here. Section 3 contains some preliminary results concerning the ends of non-compact manifolds. We then study the non-parabolic ends in further detail in Section 4. We discuss the μ-bubble exhaustion result and corresponding volume growth estimates and then prove the main results in Section 5. We discuss some topological applications of the main results in Section 6. Appendix A contains various curvature conditions referred to in the paper. Appendix B contains some examples relevant to the main results. Finally, Appendix C describes how to pull back immersions along local diffeomorphisms. Finally, Appendix D relates sectional curvature bounds to the weakly bounded geometry condition.

2. Curvature preliminaries

If (X, g) is a Riemannian manifold, we denote the Riemann curvature tensor with four arguments by $R_g(\cdot,\cdot,\cdot,\cdot)$, the sectional curvature by $\sec_g$, the Ricci curvature by Ric_g, and the scalar curvature by R_g with no argument. If $M \to (X, g)$ is an immersion, we write Ric_M and R_M to denote respectively the Ricci curvature and scalar curvature of the pull-back metric on M. We follow the curvature conventions used in [48, Chapter 3]; in particular, if $\{u, v\}$ is an orthonormal basis for a 2-plane $\Pi \subset T_pX$, then $\sec_g(\Pi) = R_g(v, u, u, v)$.

2.1. The $\mathrm{Ric}_2 \geq 0$ *condition*

Let (X^4, g) be a complete Riemannian 4-manifold. The following curvature condition is tailored to exploit the Gauss equation for hypersurfaces in 4-manifolds, and sits between non-negative sectional curvature and non-negative Ricci curvature.

Definition 2.1. (X^4, g) satisfies $\mathrm{Ric}_2^g \geq 0$ if

$$R_g(v, u, u, v) + R_g(w, u, u, w) \geq 0$$

for any orthonormal vectors $\{u, v, w\}$ in TX.

This condition has been called non-negativity of 2-*intermediate Ricci curvature* in the literature (cf. Appendix A). In particular, we note that there is a metric on $S^2 \times S^2$ with strictly positive 2-intermediate Ricci curvature [44] (see [42, Example 2.3]).

Lemma 2.2. *Let (X^4, g) be a 4-manifold with $\mathrm{Ric}_2^g \geq 0$. If $M^3 \to (X^4, g)$ is a minimal immersion, then*

$$\mathrm{Ric}_M \geq -|A_M|^2.$$

Proof. Pick an orthonormal basis $\{e_I\}$ at a point $p \in M$ with e_4 normal to M. The traced Gauss equation gives

$$R_g(e_2, e_1, e_1, e_2) + R_g(e_3, e_1, e_1, e_3) - \mathrm{Ric}_M(e_1, e_1) = \sum_{j=1}^{3} A_M(e_1, e_j)^2.$$

Rearranging and using $\mathrm{Ric}_2^g \geq 0$ we have

$$\mathrm{Ric}_M(e_1, e_1) \geq R_g(e_2, e_1, e_1, e_2) + R_g(e_3, e_1, e_1, e_3) - |A_M|^2 \geq -|A_M|^2.$$

Since e_1 was an arbitrary unit vector in T_pM, the conclusion follows. ∎

2.2. *Weakly bounded geometry*

Definition 2.3. In this paper, we say that a complete Riemannian manifold (X^n, g) has *Q-weakly bounded geometry* if for any $p \in X$, there is a $C^{2,\alpha}$ local diffeomorphism $\Phi : \mathbf{R}^n \supset (B(0, Q^{-1}), 0) \to (U, p) \subset X$ such that

(1) $e^{-2Q}\delta \leq \Phi^* g \leq e^{2Q}\delta$ in the sense of bilinear forms,

(2) $\|\partial_k \Phi^* g_{ij}\|_{C^\alpha} \leq Q$.

We will often say that (X, g) has weakly bounded geometry if the above holds for some Q.

Note that this condition allows for some collapsing behavior of (X, g) at infinity.[7] It is well-known that $|\mathrm{sec}_g| \leq K < \infty$ implies Q-weakly bounded geometry for $Q = Q(K)$ (cf. [50]). We outline the proof of this fact in Proposition D.1. Note also that $\mathrm{Ric}_g \geq 0$ and $\mathrm{sec}_g \leq K$ imply that $|\mathrm{sec}_g| \leq (n-2)K$, and in this paper the ambient space (X^4, g) is always assumed to have $\mathrm{Ric}_g \geq 0$ (often a stronger condition is assumed).

Alternatively, we recall that $|\mathrm{Ric}_g| \leq K$ and $\mathrm{inj}(X, g) \geq i_0 > 0$ also implies this condition (and actually one can replace “local diffeomorphism” by “diffeomorphism” in this case); this follows from [1] (cf. [48, Theorem 11.4.3]).

The first- and second-named authors have recently proven that a two-sided complete stable minimal hypersurface $M^3 \to \mathbf{R}^4$ is flat [12]. Using a standard blow-up argument, this yields curvature estimates for stable minimal hypersurfaces in (compact) 4-manifolds. Here, we observe that the same thing holds for non-compact 4-manifolds, assuming the weakly bounded geometry condition. The proof below is similar to the argument used in [50] (see also [19]) to prove curvature estimates for stable minimal (or more generally CMC) surfaces in 3-manifolds (cf. [51]).

[7] For example, a hyperbolic cusp has weakly bounded geometry but the injectivity radius tends to zero at infinity.

Lemma 2.4. *Let (X^4, g) be a complete 4-manifold with Q-weakly bounded geometry. Then there is a constant $C = C(Q) < \infty$ such that every compact two-sided stable minimal immersion $M^3 \to (X^4, g)$ satisfies*

$$\sup_{q \in M} |A_M(q)| \min\{1, d_M(q, \partial M)\} \leq C.$$

Proof. This follows by combining a standard point picking argument (cf. [12, Theorem 3]) with the lifting argument described in Appendix C. For the sake of completeness we describe the full argument below.

For contradiction, we assume that the assertion fails. Then there is a sequence of compact two-sided stable minimal immersions $M_i^3 \to (X_i^4, g_i)$ such that

$$\sup_{q \in M_i} |A_{M_i}(q)| \min\{1, d_{M_i}(q, \partial M_i)\} \to \infty$$

as $i \to \infty$. Since M_i is compact and the argument of the supremum is continuous and vanishes on ∂M_i, there is $p_i \in M_i \setminus \partial M_i$ with

$$|A_{M_i}(p_i)| \min\{1, d_{M_i}(p_i, \partial M_i)\} = \sup_{q \in M_i} |A_{M_i}(q)| \min\{1, d_{M_i}(q, \partial M_i)\} \to \infty.$$

Define $r_i = |A_{M_i}(p_i)|^{-1} \to 0$ and let x_i be the image of p_i in X_i. By the weakly bounded geometry assumption, there are local diffeomorphisms

$$\Phi_i : \mathbf{R}^4 \supset (B(0, Q^{-1}), 0) \to (X_i, x_i)$$

with $e^{-2Q}\delta \leq \Phi^* g_i \leq e^{2Q}\delta$ and $\|\partial_a \Phi_i^*(g_i)_{bc}\|_{C^\alpha} \leq Q$.

By the pullback operation described in Appendix C, we can find a sequence of pointed 3-manifolds (S_i, s_i), immersions $F_i : (S_i, s_i) \to (B(0, Q^{-1}), 0)$, and local diffeomorphisms $\Psi_i : (S_i, s_i) \to (M_i, p_i)$ such that the following diagram commutes (writing $B = B(0, Q^{-1}) \subset \mathbf{R}^4$):

$$\begin{array}{ccc} S_i & \xrightarrow{F_i} & B \\ {\scriptstyle \Psi_i}\downarrow & & \downarrow{\scriptstyle \Phi_i} \\ M_i & \longrightarrow & X_i \end{array}$$

and $F_i : S_i \to (B, \Phi_i^* g_i)$ is a two-sided stable minimal immersion. (Two-sidedness and minimality follow immediately as they are local properties, while stability follows by lifting a positive first eigenfunction for the stability operator on M_i to S_i.)

Define maps $D_i : B(0, r_i^{-1} Q^{-1}) \to B(0, Q^{-1})$, $x \mapsto r_i x$, and then consider the metrics $\tilde{g}_i := r_i^{-2} D_i^* \Phi^* g_i$ on $B(0, r_i^{-1} Q^{-1})$. The bounded geometry condition ensures that $\tilde{g}_i$ converges to the flat metric δ on $\mathbf{R}^4$ in $C^{1,\alpha}_{\text{loc}}$ (in the sense of $C^{1,\alpha}_{\text{loc}}$ convergence of the metric coefficients in Euclidean coordinates). In particular, the Christoffel symbols of $\tilde{g}_i$ with respect to the Euclidean coordinates on $B(0, r_i^{-1} Q^{-1})$ converge to 0 in $C^{0,\alpha}_{\text{loc}}$.

Consider the rescaled immersions $\tilde{F}_i := D_i^{-1} \circ F_i : S_i \to B(0, r_i^{-1} Q^{-1})$. Observe that $\tilde{F}_i$ is a minimal immersion with respect to $\tilde{g}_i$. Furthermore, by the point picking argument, if $q \in S_i$ has $d(s_i, q) \leq \rho$ (with respect to $\tilde{F}_i^* g_i$) then the second fundamental fortm of $\tilde{F}_i$ with respect to $\tilde{g}_i$ satisfies $|A_{\tilde{F}_i}(q)| \leq C = C(\rho)$.

The bounded curvature condition ensures there is $\mu > 0$ (a numerical constant) such that for any $q \in S_i$, for i sufficiently large, we can write $B_\mu^{S_i}(q)$ as a normal graph (in the Euclidean coordinates) over a subset of $T_q S_i$ of a function f_i with (Euclidean) C^2 norm $\leq \mu$ (cf. [18, Lemma 2.4], except one should use the fact that the Christoffel symbols are uniformly controlled). Geometric considerations show that the area of such a graph depends on the metric coefficients of $\tilde{g}_i$ (but not derivatives of the metric). In particular, the first variation formula (and minimality of $\tilde{F}_i$) implies that f_i satisfies a (uniformly) elliptic PDE in divergence form whose coefficients depend on the metric coefficients $\tilde{g}_i$ (but not derivatives). Thus, f_i satisfies a non-divergence form elliptic PDE, whose coefficients depend on the coefficients of $\tilde{g}_i$ and $\partial \tilde{g}_i$ (in Euclidean coordinates). These coefficients are uniformly controlled in C^α, and thus Schauder estimates yield interior $C^{2,\alpha}$ estimates for f_i. In particular, the injectivity radius of the pullback metric on $\tilde{h}_i := \tilde{F}_i^* \tilde{g}_i$ is locally uniformly bounded. The Gauss equations imply that the sectional curvature of $\tilde{h}_i$ is locally uniformly bounded.

In particular, the injectivity radius and curvature bounds imply that we can take a subsequential $C^{1,\alpha}$ limit of $(S_i, \tilde{h}_i, s_i)$ in the pointed Cheeger–Gromov sense (see [48, Theorem 11.4.7]) to a limit (S, h, s). Recall that this means that for any $s \in \Omega \Subset S$ there is $i_0(\Omega)$ so large that for $i \geq i_0(\Omega)$ there are embeddings

$$G_i : (\Omega, s) \to (S_i, s_i)$$

such that $G_i^* \tilde{h}_i \to h$ in the $C^{1,\alpha}$ topology.

We now verify that we can also pass with the immersions $\tilde{F}_i$ to a subsequential $C^{2,\beta}_{\mathrm{loc}}$ limit for any $\beta < \alpha$. To be precise, we claim that up to passing to a subsequence, for $\Omega \Subset S$ the maps $\tilde{F}_i \circ G_i : (\Omega, s) \to B(0, r_i^{-1} Q^{-1}) \subset \mathbf{R}^4$ converge in the $C^{2,\beta}$ topology. It is convenient to prove this with respect to the flat metric on $\mathbf{R}^4$; in particular, this is just a statement about $\mathbf{R}^4$-valued functions on (Ω, s) as opposed to a statement about maps between Riemannian manifolds.

To prove this claim, it suffices to obtain $C^{2,\alpha}$ estimates of the functions $\tilde{x}_k := x_k \circ \tilde{F}_i \circ G_i$ for $k = 1, 2, 3, 4$ (with respect to the metrics $G_i^* \tilde{h}_i$). Indeed, this follows from the fact that $D^2_{G_i^* \tilde{F}_i^* \delta} \tilde{x}_k = (\partial_k \cdot \nu_\delta) A_{G_i \circ \tilde{F}_i, \delta}$ and that the (Euclidean) second fundamental form $A_{G_i \circ \tilde{F}_i, \delta}$ is locally uniformly in C^α (this follows in an straightforward way from the fact that the local graphical functions f_i are locally uniformly bounded in $C^{2,\alpha}$).

Thus, up to passing to a subsequence, the immersions $\tilde{F}_i$ limit to $F : (S, s) \to (\mathbf{R}^4, 0)$ in the $C^{2,\beta}_{\mathrm{loc}}$ sense as described above. This (and the $C^{1,\alpha}_{\mathrm{loc}}$ convergence of $\tilde{g}_i$ to δ) implies that F is a complete stable minimal immersion with $|A_F(s)| = 1$. By [12, Theorem 1], such an immersion must be flat. This contradiction completes the proof. ∎

Proposition D.1 shows that $|\sec_g| \leq K$ implies Q-weakly bounded geometry for $Q = Q(K)$, so we have also proven the following:

Corollary 2.5. *Let (X^4, g) be a complete 4-manifold with $|\sec_g| \le K$. Then there is a constant $C(K) < \infty$ such that every compact two-sided stable minimal immersion $M^3 \to (X^4, g)$ satisfies*

$$\sup_{q \in M} |A_M(q)| \min\{1, d_M(q, \partial M)\} \le C(K).$$

This corollary will not be used in this paper, but we have included it since it resolves the conjecture of Schoen stated in [18, Conjecture 2.13] (the analogous 3-dimensional result was proven by Schoen [51], see also [50]). As mentioned above, the result in [12, Theorem 3] established such an estimate only for closed (X^4, g) where $C = C(X, g)$ (see also [65, Section 3]).

3. Preliminary results on ends

We establish notation and collect some relevant facts about ends.

3.1. Ends adapted to geodesic balls

Let M be a complete Riemannian manifold. Fix a point $x \in M$, and take a length scale $L > 0$.

Definition 3.1. A collection $\{E_k\}_{k \in \mathbb{N}}$ of open sets is an *end of M adapted to x with length scale L* if each set E_k is an unbounded connected component of $M \setminus \overline{B}_{kL}(x)$ and satisfies $E_{k+1} \subset E_k$.

Proposition 3.2. *If M is simply connected and $\{E_k\}$ is an end adapted to x with length scale L, then both $\overline{E}_k \setminus E_{k+1}$ and ∂E_k are connected for all k.*

Proof. Suppose ∂E_k has at least two components. Since $B_{kL}(x)$ and E_k are connected, we can construct a loop in M having non-trivial intersection number with two of the components of ∂E_k, which contradicts M being simply connected. The connectedness of $\overline{E}_k \setminus E_{k+1}$ follows from the connectedness of ∂E_k by taking segments of radial geodesics from x, which must intersect ∂E_k. ■

3.2. Parabolicity and non-parabolicity

We recall the notion of parabolicity for subsets.

Definition 3.3. Let M be a complete Riemannian manifold. Let K be a compact subset of M. Let E be an unbounded component of $M \setminus K$ with smooth boundary. We say that E is *parabolic* if it does not admit a positive harmonic function f satisfying

$$f|_{\partial E} \equiv 1 \quad \text{and} \quad f|_E < 1.$$

Otherwise E is *non-parabolic*.

We first state a well-known result about parabolic sets.

Proposition 3.4. *Let M be a complete Riemannian manifold. Let K be a compact subset of M. Let E be an unbounded component of $M \setminus K$ with smooth boundary. Suppose E is parabolic. Let f_i be harmonic functions on $E \cap B_{R_i}$ satisfying*

$$f_i|_{\partial E} \equiv 1 \quad \textit{and} \quad f_i|_{\partial B_{R_i}(x)} \equiv 0,$$

where $R_i \to \infty$ and $K \subset B_{R_i}(x)$ for all i. Then f_i converges uniformly to 1 on compact subsets, and

$$\lim_{i\to\infty} \int_E |\nabla f_i|^2 = 0.$$

Proof. We follow [36, Theorem 10.1]. The maximum principle guarantees that $0 \le f_i \le 1$. Thus, up to a subsequence, f_i limits to f locally smoothly, with $\Delta f = 0$, $0 \le f \le 1$, and $f|_{\partial E} = 1$. By parabolicity, $f(x) = 1$ for some $x \in E$, so $f \equiv 1$ by the maximum principle. Thus, f_i limits locally smoothly to 1 on E. Finally, integrating by parts, we see that

$$\int_E |\nabla f_i|^2 = -\int_{\partial E} f_i \nabla_\nu f_i \to 0. \qquad \blacksquare$$

We now discuss when non-parabolicity is inherited by subsets.

Proposition 3.5. *Let M be a complete Riemannian manifold. Let $K \subset \tilde{K} \subset M$ be compact subsets of M with smooth boundary. Let E be an unbounded component of $M \setminus K$. If E is non-parabolic, then there is a non-parabolic unbounded component $\tilde{E}$ of $E \setminus \tilde{K}$.*

Proof. Since E is non-parabolic, there is a positive harmonic function f on E such that

$$f|_{\partial E} \equiv 1 \quad \text{and} \quad f|_E < 1.$$

By addition and scaling, we can assume without loss of generality that

$$\lim_{r\to\infty} \inf_{\partial B_r(x) \cap E} f = 0.$$

Since $\tilde{K}$ is compact with smooth boundary, $\partial \tilde{K}$ is a disjoint union of finitely many smooth closed surfaces. Since E is connected, the boundary of each component of $E \setminus \tilde{K}$ is a union of at least one of the components of $\partial \tilde{K}$. Since distinct components of $E \setminus \tilde{K}$ have disjoint boundaries, the number of such components is finite. Hence, there is an unbounded component $\tilde{E}$ of $E \setminus \tilde{K}$ such that

$$\lim_{r\to\infty} \inf_{\partial B_r(x) \cap \tilde{E}} f = 0.$$

We show that $\tilde{E}$ is non-parabolic.

We take a sequence of harmonic functions $\tilde{f}_i$ on $\tilde{E} \cap B_{R_i}(x)$ satisfying

$$\tilde{f}_i|_{\partial \tilde{E}} \equiv 1 \quad \text{and} \quad \tilde{f}_i|_{\partial B_{R_i}(x)} \equiv 0,$$

where $R_i \to \infty$ with $\tilde{K} \subset B_{R_i}(x)$ for all i. Then $\tilde{f}_i$ converges locally uniformly to a positive harmonic function $\tilde{f}$ on $\tilde{E}$ with

$$\tilde{f}|_{\partial \tilde{E}} \equiv 1 \quad \text{and} \quad \tilde{f}|_{\tilde{E}} \leq 1.$$

Let $c = \inf_{\partial \tilde{E}} f > 0$. Then $\tilde{f}_i \leq c^{-1} f$ on $\tilde{E} \cap B_{R_i}(x)$ for all i by the maximum principle, which implies $\tilde{f} \leq c^{-1} f$. In particular,

$$\lim_{r\to\infty} \inf_{\partial B_r(x)\cap \tilde{E}} \tilde{f} = 0,$$

so $\tilde{f} \not\equiv 1$. Hence, $\tilde{f}|_{\tilde{E}} < 1$ by the maximum principle, so $\tilde{E}$ is non-parabolic. ∎

By Proposition 3.5, it makes sense to define non-parabolicity for an end.

Definition 3.6. An end $\{E_k\}$ adapted to x with length scale L is *non-parabolic* if there exist connected open sets $F_k \subset E_k$ with smooth boundary such that

$$E_k \setminus \overline{B}_{(k+1)L}(x) \subset F_k \subset E_k$$

and F_k is non-parabolic for all k sufficiently large.

4. Non-parabolic ends of stable hypersurfaces in 4-manifolds with $\mathrm{Ric}_2 \geq 0$

In this section, we observe[8] that a minor modification of the arguments used in [37], [36, Section 11] restricts the number of non-parabolic ends of a stable hypersurface in a 4-manifold with $\mathrm{Ric}_2 \geq 0$.

Theorem 4.1. *Let (X^4, g) be a complete 4-manifold with $\mathrm{Ric}_2^g \geq 0$. Let $M^3 \to (X^4, g)$ be a complete two-sided stable minimal immersion. Let K be a compact subset of M with smooth boundary. Then $M \setminus K$ admits at most one non-parabolic unbounded component. In particular, M has at most one non-parabolic end.*

We give the proof in Section 4.3 after establishing some preliminary results.

4.1. Schoen–Yau inequality

We use the following inequality of Schoen and Yau, which is known in non-negative sectional curvature. We show that the same proof works under the weaker assumption $\mathrm{Ric}_2 \geq 0$. We include the proof for completeness.

[8] One may observe that the same result holds (with the same proof) for $M^{n-1} \to (X^n, g)$ when (X^n, g) has $\mathrm{Ric}_{n-2}^g \geq 0$ (meaning that if $\Pi_1, \ldots, \Pi_{n-2}$ is a set of planes in T_pX meeting pairwise orthogonally along a fixed line, then $\sum_{i=1}^{n-2} \sec_g(\Pi_i) \geq 0$).

Lemma 4.2 ([53]). *Let (X^4, g) be a complete 4-manifold satisfying $\mathrm{Ric}_2^g \geq 0$. Let $M^3 \to (X^4, g)$ be a complete two-sided stable minimal immersion. Let u be a harmonic function on M. Then*

$$\frac{1}{3}\int_M \phi^2|A_M|^2|\nabla u|^2 + \frac{1}{2}\int_M \phi^2\big|\nabla|\nabla u|\big|^2 \leq \int_M |\nabla\phi|^2|\nabla u|^2$$

for any compactly supported, non-negative $\phi \in W^{1,2}(M)$.

Proof. We choose a favorable test function in the stability inequality, combined with a rearrangement of the Bochner formula. Below, we assume that u is not a constant function (otherwise the desired inequality is trivial).

The favorable test function is $\psi := \phi|\nabla u|$, where ϕ is a compactly supported, non-negative function in $W^{1,2}(M)$. Since $\mathrm{Ric}_2^g \geq 0$ implies non-negative Ricci curvature, the stability inequality gives

$$\begin{aligned}\int_M \phi^2|A_M|^2|\nabla u|^2 &\leq \int_M |\nabla\phi|^2|\nabla u|^2 + 2\int_M \phi|\nabla u|\langle\nabla\phi, \nabla|\nabla u|\rangle + \int_M \phi^2\big|\nabla|\nabla u|\big|^2 \\ &= \int_M |\nabla\phi|^2|\nabla u|^2 + \frac{1}{2}\int_M \langle\nabla\phi^2, \nabla|\nabla u|^2\rangle + \int_M \phi^2\big|\nabla|\nabla u|\big|^2 \\ &= \int_M |\nabla\phi|^2|\nabla u|^2 - \int_M \phi^2\Big(\frac{1}{2}\Delta|\nabla u|^2 - \big|\nabla|\nabla u|\big|^2\Big), \qquad (4.1)\end{aligned}$$

where the divergence theorem is used in the second equality.

We now rearrange the Bochner formula. Recall that the usual Bochner formula gives

$$\tfrac{1}{2}\Delta|\nabla u|^2 = \mathrm{Ric}_M(\nabla u, \nabla u) + |\mathrm{Hess}_M u|^2. \qquad (4.2)$$

For now, consider a point p where $\nabla u \neq 0$. Choose a local orthonormal frame around p so that $e_1 = \nabla u/|\nabla u|$. The improved Kato inequality (cf. [36, Lemma 7.2]) yields

$$|\mathrm{Hess}_M u|^2 \geq \tfrac{3}{2}\big|\nabla|\nabla u|\big|^2. \qquad (4.3)$$

Now we rearrange the Ricci curvature term. The Gauss equation and $\mathrm{Ric}_2 \geq 0$ give

$$\mathrm{Ric}_M(e_1, e_1) = \sum_{j=2}^{3} R(e_j, e_1, e_1, e_j) - \sum_{j=1}^{3} A(e_1, e_j)^2 \geq -\sum_{j=1}^{3} A(e_1, e_j)^2. \qquad (4.4)$$

On the other hand, Cauchy–Schwarz and minimality give

$$\begin{aligned}|A_M|^2 &\geq A_M(e_1, e_1)^2 + 2\sum_{j=2}^{3} A_M(e_1, e_j)^2 + \sum_{j=2}^{3} A_M(e_j, e_j)^2 \\ &\geq A_M(e_1, e_1)^2 + 2\sum_{j=2}^{3} A_M(e_1, e_j)^2 + \frac{1}{2}\Big(\sum_{j=2}^{3} A_M(e_j, e_j)\Big)^2 \\ &= A_M(e_1, e_1)^2 + 2\sum_{j=2}^{3} A_M(e_1, e_j)^2 + \frac{1}{2}A_M(e_1, e_1)^2 \geq \frac{3}{2}\sum_{j=1}^{3} A_M(e_1, e_j)^2.\end{aligned} \qquad (4.5)$$

Together, (4.4) and (4.5) give

$$\mathrm{Ric}_M(\nabla u, \nabla u) \geq -\tfrac{2}{3}|A_M|^2|\nabla u|^2. \tag{4.6}$$

Hence, (4.2), (4.3), (4.6) give the rearrangement

$$\tfrac{1}{2}\Delta|\nabla u|^2 \geq -\tfrac{2}{3}|A_M|^2|\nabla u|^2 + \tfrac{3}{2}\big|\nabla|\nabla u|\big|^2, \tag{4.7}$$

so

$$\tfrac{1}{2}\Delta|\nabla u|^2 - \big|\nabla|\nabla u|\big|^2 \geq -\tfrac{2}{3}|A_M|^2|\nabla u|^2 + \tfrac{1}{2}\big|\nabla|\nabla u|\big|^2. \tag{4.8}$$

We have derived (4.8) under the assumption $\nabla u \neq 0$ at the given point.

Now suppose that $|\nabla u|$ vanishes at p but $x \mapsto |\nabla u|(x)$ is differentiable at p. In this case, the right-hand side of (4.8) is zero (since p is a local minimum of $|\nabla u|$). On the other hand, the left-hand side is ≥ 0 since p is a local minimum of the smooth function $|\nabla u|^2$. In sum, we find that (4.8) holds whenever $|\nabla u|$ is differentiable, which is almost everywhere (by Rademacher's theorem). Thus, we can use (4.8) in (4.1) to complete the proof. ∎

4.2. Multiple non-parabolic ends

Under the assumption of multiple non-parabolic ends with respect to a fixed compact set, we produce a non-constant harmonic function.

Lemma 4.3 (cf. [36, Theorem 4.3]). *Let M be a complete Riemannian manifold. Let K be a compact subset of M with smooth boundary. Suppose $M \setminus K$ has at least two non-parabolic unbounded components. Then there exists a non-constant harmonic function with finite Dirichlet energy on M.*

Proof. Let E, F denote non-parabolic components of $M \setminus K$. There exists a harmonic function $0 \leq h \leq 1$ on $M \setminus K$ such that $h \equiv 0$ on $M \setminus (K \cup E \cup F)$, $h|_{\partial E} = 1$, $h|_{\partial F} = 0$, $\liminf_{x \in E, x \to \infty} h = 0$, and $\limsup_{x \in F, x \to \infty} h = 1$. By the argument in [36, Lemma 3.6], we can assume that h has finite Dirichlet energy on $M \setminus K$.

For $R_i \to \infty$, define f_i to be the harmonic function on $B_{R_i}(x)$ with $f_i|_{\partial B_{R_i}(x)} = h|_{\partial B_{R_i}(x)}$. Note that $0 \leq f_i \leq 1$. In particular, $f_i \leq h$ on $E \cap B_{R_i}(x)$, and $f_i \geq h$ on $F \cap B_{R_i}(x)$. Passing to a subsequence, the f_i limit locally smoothly to a harmonic function f on M with $f \leq h$ on E and $f \geq h$ on F. In particular, f is non-constant.

Finally, observe that

$$\int_{B_{R_{i+1}}(x)} |\nabla f_{i+1}|^2 \leq \int_{B_{R_i}(x)} |\nabla f_i|^2 + \int_{B_{R_{i+1}}(x)\setminus B_{R_i}(x)} |\nabla h|^2.$$

This implies that f has finite Dirichlet energy, completing the proof. ∎

4.3. Proof of Theorem 4.1

Suppose for contradiction that $\{E_k\}$ and $\{\tilde{E}_k\}$ are distinct non-parabolic ends of M adapted to x with length scale L. Let F_k and $\tilde{F}_k$ be open subsets with smooth boundary given by Definition 3.6. Since the ends are distinct, there is a $k_0 \in \mathbb{N}$ such that $E_{k_0} \cap \tilde{E}_{k_0} = \varnothing$. Taking k_0 larger if necessary, F_{k_0} and $\tilde{F}_{k_0}$ are disjoint and non-parabolic. Hence, it suffices to rule out two non-parabolic sets.

Suppose E and F are non-parabolic unbounded components of $M \setminus K$. By Lemma 4.3, there is a non-constant harmonic function u with finite Dirichlet energy on M.

We choose a nice cutoff function to use in Lemma 4.2. Let ρ_i be a smoothing of $d_M(x,\cdot)$ with $\rho_i\,|_{\partial B_{(k_0+i)L}(x)} = (k_0+i)L$, $\rho_i\,|_{\partial B_{(k_0+2i)L}(x)} = (k_0+2i)L$, and $|\nabla \rho_i| \le 2$. Then we define

$$\phi_i(y) := \begin{cases} 1, & y \in \bar{B}_{(k_0+i)L}(x), \\ \frac{(k_0+2i)L-\rho_i(y)}{iL}, & y \in \bar{B}_{(k_0+2i)L}(x) \setminus B_{(k_0+i)L}(x), \\ 0, & y \in M \setminus B_{(k_0+2i)L}(x). \end{cases}$$

Lemma 4.2 and the fact that u has finite Dirichlet energy implies

$$\begin{aligned} \int_{B_{(k_0+i)L}(x)} \Big(\frac{1}{3}|A_M|^2|\nabla u|^2 + \frac{1}{2}\big|\nabla|\nabla u|\big|^2\Big) &\le \int_M \Big(\frac{1}{3}\phi_i^2|A_M|^2|\nabla u|^2 + \frac{1}{2}\phi_i^2\big|\nabla|\nabla u|\big|^2\Big) \\ &\le \int_M |\nabla\phi_i|^2|\nabla u|^2 \le \frac{C}{i^2}. \end{aligned}$$

Taking $i \to \infty$, we conclude that

$$|A_M|^2|\nabla u|^2 \equiv 0 \quad \text{and} \quad \big|\nabla|\nabla u|\big|^2 \equiv 0.$$

Since $|\nabla u|$ is constant and u is non-constant, we have $|\nabla u| \ne 0$ everywhere. Then $|A_M| \equiv 0$, which implies $\mathrm{Ric}_M \ge 0$ by Lemma 2.2. The assumption of at least two ends implies (by Cheeger–Gromoll splitting) that M is a product $\mathbf{R} \times P$. Therefore, M has infinite volume. However, $|\nabla u|^2$ is a non-zero constant and $\int_M |\nabla u|^2 < \infty$ by Lemma 4.3, which implies that M has finite volume. Hence, we reach a contradiction. ∎

5. Stable hypersurfaces in PSC 4-manifolds with $\mathrm{Ric}_2 \ge 0$

Our aim in this section is to prove Theorem 1.10 (which immediately implies Theorem 1.3 and Corollary 1.6).

Remark 5.1. If M is parabolic, then the conclusion of Theorem 1.10 follows immediately, as parabolicity allows us to "plug 1 into the stability inequality." The hard case is therefore the non-parabolic case. In this case, Theorem 4.1 implies that M has precisely one non-parabolic end. Hence, the main difficulty of Theorem 1.10 is this troublesome end.

Remark 5.2. We can assume without loss of generality that M is simply connected. In the general case, we pass to the universal cover, which inherits minimality, stability, two-sidedness, and completeness from the original immersion. The conclusion of Theorem 1.10 for the universal cover then descends to the original immersion.

5.1. μ-bubbles

We first recall a diameter bound that uses the theory of warped μ-bubbles for stable minimal hypersurfaces in PSC 4-manifolds.

Lemma 5.3 (Warped μ-bubble diameter bound). *Let (X^4, g) be a complete 4-manifold with $R_g \geq 1$. Let $N^3 \to (X^4, g)$ be a two-sided stable minimal immersion with compact boundary. There are universal constants $L > 0$ and $c > 0$ such that if there is a $p \in N$ with $d_N(p, \partial N) > L/2$, then in N there is an open set $\Omega \subset B^N_{L/2}(\partial N)$ and a smooth surface Σ^2 such that $\partial\Omega = \Sigma \sqcup \partial N$ and each component of Σ has diameter at most c.*

Proof. This follows from Gromov's band-width estimate technique [27]. For completeness, we sketch the proof here, with references to the relevant statements from [14]. Choose $L = 20\pi$. Since N is two-sided and stable, we have

$$\int_N |\nabla\psi|^2 - \tfrac{1}{2}(R_g - R_N + |A_N|^2)\psi^2 \geq 0, \quad \forall \psi \in C^1_c(N).$$

Since $R_g \geq 1$, there exists $u \in C^\infty(N)$, $u > 0$ in $\mathring{N}$ such that

$$\Delta_N u \leq -\tfrac{1}{2}(1 - R_N)u. \tag{5.1}$$

For instance, see [24, Theorem 1].

Take $\rho_0 \in C^\infty(M)$ to be a smoothing of $d_N(\cdot, \partial N)$, such that $|\mathrm{Lip}(\rho_0)| \leq 2$ and $\rho_0 = 0$ on ∂N. Choose $\varepsilon \in (0, 1/2)$ such that ε, $4\pi + 3/2\varepsilon$, $8\pi + 2\varepsilon$ are regular values of ρ_0. Define

$$\rho = \frac{\rho_0 - \varepsilon}{8 + \varepsilon/\pi} - \frac{\pi}{2},$$

$\Omega_1 = \{x \in N : -\pi/2 < \rho < \pi/2\}$, and $\Omega_0 = \{x \in N : -\pi/2 < \rho \leq 0\}$. Clearly $|\mathrm{Lip}(\rho)| < 1/4$.

On Ω_1, define $h(x) = -\frac{1}{2}\tan(\rho(x))$. For Caccioppoli sets Ω in Ω_1 such that $\Omega \,\triangle\, \Omega_0$ is compactly contained in Ω_1, consider

$$\mathcal{A}(\Omega) = \int_{\partial\Omega} u \; d\mathcal{H}^2 - \int_{\Omega_1} (\chi_\Omega - \chi_{\Omega_0}) h u \, d\mathcal{H}^3.$$

By [14, Proposition 12], there exists a minimizer $\tilde{\Omega}$ for $\mathcal{A}$ such that $\tilde{\Omega} \,\triangle\, \Omega_0$ is compactly contained in Ω_1. Let $\Omega = \{x \in \mathbb{N} : 0 \leq \rho_0(x) \leq \varepsilon\} \cup \tilde{\Omega}$. We claim that Ω satisfies the conclusion with $c = 4\pi/\sqrt{3}$.

Indeed, let Σ_0 be a connected component of $\Sigma = \partial\Omega \cap \Omega_1$. The non-negativity of the second variation for $\mathcal{A}$ on Σ_0 implies that (see [14, Lemma 14])

$$\begin{aligned}\int_{\Sigma_0} &|\nabla_{\Sigma_0}\psi|^2 u - \tfrac{1}{2}\big(R_N - \tfrac{1}{4} - 2K_{\Sigma_0}\big)\psi^2 u + (\Delta_N u - \Delta_{\Sigma_0} u)\psi^2 \\ &- \tfrac{1}{2}u^{-1}\langle\nabla_N u, \nu\rangle^2\psi^2 - \tfrac{1}{2}\big(\tfrac{1}{4} + h^2 + 2\langle\nabla_N h, \nu\rangle\big)\psi^2 u \geq 0, \quad \forall \psi \in C^1(\Sigma_0).\end{aligned} \tag{5.2}$$

The choice of ρ and h guarantees that $\frac{1}{4} + h^2 + 2\langle \nabla_N h, \nu \rangle \geq 0$ pointwise on Ω_1. Combined with (5.1), this implies that

$$\int_{\Sigma_0} |\nabla_{\Sigma_0} \psi|^2 u - \tfrac{1}{2}\big(\tfrac{3}{4} - 2K_{\Sigma_0}\big)\psi^2 u - (\Delta_{\Sigma_0} u)\psi^2 \geq 0, \quad \forall \psi \in C^1(\Sigma_0).$$

Thus we apply [14, Lemmas 16 and 18] to conclude that $\operatorname{diam}(\Sigma_0) \leq 4\pi/\sqrt{3}$. This completes the proof. ∎

5.2. *Almost linear volume growth*

We show that every end of a stable hypersurface in a $\mathrm{Ric}_2 \geq 0$ PSC 4-manifold has a core tube with linear volume growth.

Lemma 5.4 (Almost linear volume growth of an end). *Let (X^4, g) be a complete 4-manifold with weakly bounded geometry,* $\mathrm{Ric}_2^g \geq 0$*, and $R_g \geq 1$. Let $M^3 \to (X^4, g)$ be a simply connected complete two-sided stable minimal immersion. Let $\{E_k\}_{k\in\mathbb{N}}$ be an end of M adapted to $x \in M$ with length scale L, where L is the constant from Lemma 5.3. Let $M_k := E_k \cap B_{(k+1)L}(x)$. Then there is a constant $C > 0$ such that*

$$\mathrm{Vol}_M(M_k) \leq C \quad \textit{for all } k.$$

Proof. We apply Lemma 5.3 to E_k, which supplies an open set Ω_k in $B_{L/2}(\partial E_k) \cap E_k$ with $\partial E_k \subset \partial \Omega_k$. Note that $\Omega_k \subset M_k$ and $\partial E_{k+1} \cap \bar{\Omega}_k = \varnothing$ by construction. Let $\Sigma_k^{(i)}$ denote the components of $\partial \Omega_k \setminus \partial E_k$.

First, we claim that $\bar{M}_k$ is connected. Indeed, this follows from the connectedness of ∂E_k by taking segments of radial geodesics from x, just as in Proposition 3.2.

Second, we claim that some $\Sigma_k^{(i)}$ separates ∂E_k from ∂E_{k+1}. Let γ be any curve from ∂E_k to ∂E_{k+1} in $\bar{M}_k$, which exists by the connectedness of $\bar{M}_k$. Perturbing γ if necessary, we can guarantee that any intersection with $\bigcup_i \Sigma_k^{(i)}$ is transverse. Since $\partial E_k \subset \bar{\Omega}_k$ and $\partial E_{k+1} \cap \bar{\Omega}_k = \varnothing$, γ intersects some $\Sigma_k^{(i)}$ an odd number of times. Suppose for contradiction that $\Sigma_k^{(i)}$ does not separate ∂E_k from ∂E_{k+1}. Then there is another curve γ' from ∂E_k to ∂E_{k+1} in $\bar{M}_k \setminus \Sigma_k^{(i)}$. Since ∂E_k and ∂E_{k+1} are connected by Proposition 3.2, we have constructed a loop with non-trivial intersection number with $\Sigma_k^{(i)}$, contradicting the simple connectedness of M.

Let Σ_k denote the component of $\partial \Omega_k \setminus \partial E_k$ provided by the above claim. By Lemma 5.3, we have $\operatorname{diam}(\Sigma_k) \leq c$.

We claim that there is a constant $D > 0$ (independent of k) such that $\operatorname{diam}(M_k) \leq D$. Let y and z be any points in M_k. Take the radial geodesic from x to y. As in the proof of the connectedness of M_k, this curve intersects ∂E_k and ∂E_{k-1}. By the choice of the components $\{\Sigma_k\}$, an arc of this curve of length at most $2L$ joins y to Σ_{k-1}. The same argument applies to z. Hence, the diameter bound for Σ_{k-1} implies

$$d_M(y, z) \leq 4L + c =: D.$$

By Lemma 2.4, M has bounded second fundamental form (e.g. use the compact immersion of $B_2^M(q)$ for any $q \in M$). By Lemma 2.2, Ric_M is bounded from below. Hence, by Bishop–Gromov volume comparison, there is a constant $C > 0$ such that

$$\mathrm{Vol}(B_D^M(p)) \leq C$$

for all $p \in M$. Since $M_k \subset B_D^M(p)$ for any $p \in M_k$, the lemma follows. ∎

5.3. *Decomposition of M*

To construct a nice sequence of test functions for the stability inequality, we need to decompose M into convenient building-blocks.

Let M^3 be a complete, simply connected, Riemannian 3-manifold such that the conclusions of Theorem 4.1 hold. Let $\{E_k\}_{k\in\mathbb{N}}$ be the non-parabolic end of M adapted to $x \in M$ with length scale L (where L is the universal constant from Lemma 5.3).

We now decompose M to handle the non-parabolic end. Let $k_0, i \geq 1$.

- Set $M_k := E_k \cap B_{(k+1)L}(x)$ for all k.
- Let $\{P_{k_0}^{(\alpha)}\}_{\alpha=1}^{n_{k_0}}$ be the components of $M \setminus \bar{B}_{k_0 L}(x)$ besides E_{k_0}.
- Let $\{P_k^{(\alpha)}\}_{\alpha=1}^{n_k}$ be the components of $E_{k-1} \setminus \bar{B}_{kL}(x)$ besides E_k for $k > k_0$.

Note that we have the decomposition

$$M = \bar{B}_{(k_0+i)L}(x) \cup \bigcup_{k=k_0+i}^{k_0+2i-1} \Big(\bar{M}_k \cup \bigcup_{\alpha=1}^{n_k} \bar{P}_k^{(\alpha)}\Big) \cup (\bar{E}_{k_0+2i-1} \setminus B_{(k_0+2i)L}(x)).$$

By Theorem 4.1, $P_k^{(\alpha)}$ is either bounded or $P_k^{(\alpha)} \setminus K$ is parabolic for all compact $K \subset M$ such that $P_k^{(\alpha)} \setminus K$ has smooth boundary.

5.4. *Proof of Theorem 1.10*

Supposing it exists, let $\{E_k\}$ be the non-parabolic end of M. We use the decomposition of M from Section 5.3.

The core tube. For each k, let ρ_k be a smooth function on $\bar{M}_k$ such that $|\nabla \rho_k| \leq 2$,

$$\rho_k|_{\partial E_k} \equiv kL \quad \text{and} \quad \rho_k|_{\partial M_k \setminus \partial E_k} \equiv (k+1)L.$$

For instance, we can take ρ_k to be a smoothing of the distance function from x.

Let $\phi : [(k_0+i)L, (k_0+2i)L] \to [0,1]$ be the linear function with $\phi((k_0+i)L) = 1$ and $\phi((k_0+2i)L) = 0$.

The extraneous pieces. By Proposition 3.4, there is a compactly supported Lipschitz function $u_{k,\alpha,i}$ on $\bar{P}_k^{(\alpha)}$ such that

$$u_{k,\alpha,i}|_{\partial P_k^{(\alpha)}} \equiv 1 \quad \text{and} \quad \int_{P_k^{(\alpha)}} |\nabla u_{k,\alpha,i}|^2 < \frac{1}{i^2 n_k}.$$

Noting that if $\partial P_k^{(\alpha)}$ is not smooth, we apply Proposition 3.4 to $P_k^{(\alpha)} \setminus K$ (where K is compact and $P_k^{(\alpha)} \setminus K$ has smooth boundary) and extend the function by 1 on $P_k^{(\alpha)} \cap K$. Moreover, if $P_k^{(\alpha)}$ is bounded, we can just take $u_{k,\alpha,i} \equiv 1$.

The test function. We now construct a test function f_i for the stability inequality as follows:

$$f_i(y) := \begin{cases} 1, & y \in \bar{B}_{(k_0+i)L}(x), \\ 0, & y \in \bar{E}_{k_0+2i-1} \setminus B_{(k_0+2i)L}(x), \\ \phi(\rho_k(y)), & y \in \bar{M}_k,\ k_0+i \le k < k_0+2i, \\ \phi(kL)u_{k,\alpha,i}, & y \in \bar{P}_k^{(\alpha)},\ k_0+i \le k < k_0+2i. \end{cases}$$

By construction, f_i is compactly supported and Lipschitz. Therefore, f_i is an eligible test function for the stability inequality. Hence, the stability inequality and Lemma 5.4 give

$$\begin{aligned} \int_M (\mathrm{Ric}_g(\nu,\nu) + |A_M|^2) f_i^2 &\le \int_M |\nabla f_i|^2 \\ &= \sum_{k=k_0+i}^{k_0+2i-1} \int_{M_k} \phi'(\rho_k)^2 |\nabla \rho_k|^2 + \sum_{k=k_0+i}^{k_0+2i-1} \sum_{\alpha=1}^{n_k} \phi(kL)^2 \int_{P_k^{(\alpha)}} |\nabla u_{k,\alpha,i}|^2 \\ &\le \frac{4C}{iL^2} + \frac{1}{i} = \frac{C'}{i}. \end{aligned}$$

Since f_i converges uniformly to the constant 1 function on compact subsets, the $i \to \infty$ limit yields the desired conclusion. ∎

6. Topology of PSC 4-manifolds with $\mathrm{Ric}_2 > 0$

In this section we prove Theorem 1.13. We begin with the following lemma.

Lemma 6.1. *Suppose X^n is a non-compact oriented manifold, $\sigma \subset X$ is a closed curve and $[\sigma] \in H_1(X)$ is a non-trivial, non-torsion element. For any pre-compact open set $D \subset X$ containing a neighborhood of σ, there exists a smooth compact submanifold M_0 of dimension $n-1$, such that $\partial M_0 \subset X \setminus D$ and the algebraic intersection number of M_0 and σ is* 1.

Proof. By Poincaré duality for non-compact manifolds, $H_1(X)$ is isomorphic to $H_c^{n-1}(X)$. Thus, there exists a connected pre-compact open set A with $D \subset A$ as well as $\alpha \in H^{n-1}(X, X \setminus A)$ with $\alpha \frown \mu_A = [\sigma]$. By assumption, α is not a torsion element of $H^{n-1}(X, X \setminus A)$. Thus, by the universal coefficient theorem, we can find $\beta \in H_{n-1}(X, X \setminus A)$ with $\alpha \frown \beta = 1 \in H_0(X, X \setminus A)$. By the excision theorem and Lefschetz duality, we have

$$H_{n-1}(X, X \setminus A) \simeq H^1(A) \simeq \langle A, S^1 \rangle,$$

where $\langle A, S^1\rangle$ is the basepoint-preserving homotopy classes of maps to S^1. Take a smooth map f in $\langle A, S^1\rangle$ corresponding to β, and let $p \in S^1$ be a regular value of f. Then the smooth hypersurface $f^{-1}(p) \subset A$ represents β and hence has algebraic intersection 1 with σ. ∎

Proof of Theorem 1.13. We first prove that $H_1(X)$ consists only of torsion elements. If not, there exists a closed curve σ with $[\sigma] \in H_1(X)$ non-torsion. Take $p \in X$ and $R_0 > 0$ with $\sigma \subset B_{R_0}(p)$. By Lemma 6.1, for any integer $k > R_0$, there exists a smooth compact submanifold $\tilde{M}_k$ such that $\partial \tilde{M}_k \subset X \setminus B_k(p)$, and the algebraic intersection number of $\tilde{M}_k$ with σ is 1. Choose a pre-compact region $\Omega_k \subset X$ with smooth boundary, with $\tilde{M}_k \subset \Omega_k$. Form a metric g_k so that Ω_k has mean-convex boundary and g_k agrees with g away from $\partial\Omega_k$.

Consider the area-minimizing problem

$$\inf\{\mathcal{H}^3_{g_k}(M) : \partial M = \partial \tilde{M}_k,\ M \text{ is homologous to } \tilde{M}_k\}.$$

It is standard to show that a compact smooth two-sided embedded minimizer M_k exists. Moreover, since M_k is homologous to $\tilde{M}_k$, the algebraic intersection number of M_k with σ is 1. In particular, $M_k \cap \sigma \neq \varnothing$.

By Lemma 2.4, we have curvature estimates for $\{M_k\}$ on any compact subset of X. Thus, by passing to a subsequence (not relabeled), $\{M_k\}$ converges to a complete two-sided stable minimal immersion $M^3 \to (X, g)$. Note that M is not empty, thanks to the condition that $M_k \cap \sigma \neq \varnothing$. This contradicts Theorem 1.10.

Now we prove that X is H_2-trivial at infinity. Suppose that, on the contrary, there exists $\alpha \in H_2^\infty(X)$ whose image in $H_2(X)$ is zero. By definition, there exist nested bounded open sets $\{A_j\}_{j=1}^\infty$ such that $A_i \subset A_j$ when $i \leq j$ and $\bigcup A_j = X$, and non-trivial elements $\alpha_j \in H_2(X \setminus A_j)$, such that

(1) when $i \leq j$, $\iota_*(\alpha_j) = \alpha_i \neq 0$, where $\iota : X \setminus A_j \hookrightarrow X \setminus A_i$ is the inclusion;

(2) for each j, $\iota_*(\alpha_j) = 0$, where $\iota : X \setminus A_j \to X$ is the inclusion.

Thus, for each j, there exists a 2-cycle $\Sigma_j \subset X \setminus A_j$ such that $[\Sigma_j]$ is null homologous in X and if $\tilde{M}$ is a 3-chain with $\partial \tilde{M} = \Sigma_j$, then $M \cap A_1 \neq \varnothing$ (otherwise $[\Sigma_j]$ would be null-homologous in $X \setminus A_1$, a contradiction). Consider the area-minimizing problem (for metrics g_j as before)

$$\inf\{\mathcal{H}^3_{g_j}(\tilde{M}) : \partial \tilde{M} = \Sigma_j\}.$$

A smooth compact two-sided minimizer M_j exists. By construction, $M_j \cap A_1 \neq \varnothing$. Thus, by Lemma 2.4, a subsequence converges smoothly to a complete two-sided stable minimal immersion $M^3 \to (X^4, g)$, contradicting Theorem 1.10. This completes the proof. ∎

Appendix A. Curvature conditions

Here we review the various curvature conditions referred to in this paper. Fix a Riemannian manifold (X^{n+1}, g). We recall the convention used here: if $\Pi \subset T_pX$ is a 2-plane with orthonormal basis $\{u, v\}$ then $\sec_g(\Pi) = R_g(v, u, u, v)$.

For $k \in \{1, \dots, n\}$ and a set $\{v_0, \dots, v_k\}$ of $k+1$ orthonormal vectors, we define the *k-th intermediate Ricci curvature* by

$$\mathrm{Ric}_k(v_0, \dots, v_k) := \sum_{i=1}^{k} R(v_0, v_k, v_k, v_0).$$

Note that Ric_n is the usual Ricci curvature, while Ric_1 is the sectional curvature.

Following [59] we define the *bi-Ricci curvature* of an orthonormal set $\{u, v\}$ of vectors by

$$\mathrm{biRic}(u, v) = \mathrm{Ric}(u, u) + \mathrm{Ric}(v, v) - R(u, v, v, u).$$

If $n+1 = 3$ then $\mathrm{biRic}(u, v) = R_g/2$, but in higher dimensions $\mathrm{biRic} \geq 0$ is a stronger curvature condition than $R_g \geq 0$.

Appendix B. Examples

B.1. Positive sectional curvature

Here we give the details of Example 1.2 concerning the existence of a complete stable minimal hypersurface in ambient positive sectional curvature.

For $\alpha \in (0, 1)$ to be chosen close to 1 below, define

$$\rho(r) = \alpha r + (1-\alpha) \int_0^r e^{-s^2}\, ds.$$

Consider the metric g on $\mathbf{R}^4$ given by

$$g = dr^2 + \rho(r)^2 g_{\mathbf{S}^3}.$$

It is standard to compute (cf. [48, Section 4.2.3]) that the sectional curvatures of g_α lie between

$$-\frac{\rho''(r)}{\rho(r)} = \frac{2(1-\alpha) r e^{-r^2}}{\alpha r + (1-\alpha) \int_0^r e^{-s^2} ds}$$

and

$$\frac{1 - \rho'(r)^2}{\rho(r)^2} = \frac{1 - (\alpha + (1-\alpha) e^{-r^2})^2}{(\alpha r + (1-\alpha) \int_0^r e^{-s^2} ds)^2},$$

so $(\mathbf{R}^4, g)$ has positive sectional curvature. On the other hand, we can fix a totally geodesic $\mathbf{R}^3 \to (\mathbf{R}^4, g)$ (corresponding to $[0, \infty) \times \mathbb{S}^2 \subset [0, \infty) \times \mathbb{S}^3$ in the radial coordinates used above). We claim that this is a stable immersion (embedding), at least for α sufficiently close to 1. Indeed, for $\varphi \in C_c^\infty(\mathbf{R}^3)$, we compute (un-barred quantities denote the induced metric $dr^2 + \rho(r)^2 g_{\mathbf{S}^2}$ and barred quantities denote the Euclidean metric $dr^2 + r^2 g_{\mathbf{S}^2}$)

$$\int_{\mathbf{R}^3} |\nabla \varphi|^2\, d\mu \geq \int_{\mathbf{R}^3} \alpha^2 |\bar\nabla \varphi|^2\, d\bar\mu \geq \int_{\mathbf{R}^3} \frac{\alpha^2}{4r^2} \varphi^2\, d\bar\mu \geq \int_{\mathbf{R}^3} \frac{\alpha^2}{4r^2} \varphi^2\, d\mu.$$

In the first and third inequalities we have used $\alpha r \le \rho(r) \le r$, in the second we have used a well-known[9] Hardy inequality. Finally, we note that

$$\mathrm{Ric}_g(\nu,\nu) \le 2\left(\frac{1-\alpha^2}{\alpha^2 r^2} + \frac{1-\alpha}{\alpha} e^{-r^2}\right) \le \frac{\alpha^2}{4r^2},$$

where the second inequality holds for α sufficiently close to 1. Hence, for this choice of α, we find that $\mathbf{R}^3 \to (\mathbf{R}^4, g)$ is stable.

B.2. Positive Ricci curvature and strictly positive scalar curvature

The following example should be compared to the example of Schoen indicated in [41, Appendix]. We construct a complete two-sided stable minimal hypersurface $M^3 \to (X^4, g)$ such that along M, $R_g \ge 1, \mathrm{Ric}_g \ge 1$, and the sectional curvature is uniformly bounded below.

Remark B.1. We emphasize that (X^4, g) will not be complete, but M^3 will have a tubular neighborhood of uniform diameter. We discuss this point further below.

Remark B.2. The example below can be directly generalized to produce $M^{n-1} \to (X^n, g)$ as above, for any dimension $n \ge 4$.

Fix $M_0 = (S^1 \times S^2) \# (S^1 \times S^2)$. Because $\dim M_0 = 3 > 2$, the Schoen–Yau/Gromov–Lawson surgery construction [29, 55] and the Kazdan–Warner trichotomy theorem [32–34] (see, e.g., [34, Theorem 1.3]) imply that there is a scalar flat metric h_0 on M_0.

Write (M, h) for the universal cover of (M_0, h_0). Defining

$$\lambda_1(M,h) = \inf_{u \in C_c^\infty(M)\setminus\{0\}} \frac{\int_M |\nabla u|^2}{\int_M u^2},$$

we claim that $\lambda_1(M, h) > 0$. This is a consequence of a result of Brooks [4] (see also [21, Section 16]) which says that $\lambda_1(M, h) = 0$ if and only if $\pi_1(M_0)$ is amenable. The definition of an amenable group can be found in [4, Section 1] (among other places), but all that matters here is that $\pi_1(M_0) = F_2$, the free group on two generators, which is not amenable (see, e.g., [63, Proposition 4]).

We now fix $\varepsilon = \lambda_1(M, h)/6$.

For $\delta > 0$ to be fixed below (depending only on h), define a family of metrics h_t on M_0 by

$$h_t = h_0 + t^2(\mathrm{Ric}_{h_0} - 2\varepsilon h_0), \quad t \in (-\delta, \delta).$$

[9]For any $1 < p < n$ and $\varphi \in C_c^\infty(\mathbf{R}^n)$ we have $\int_{\mathbf{R}^n} \frac{u^p}{|x|^p}\, d\bar{\mu} \le (\frac{p}{n-p})^p \int_{\mathbf{R}^n} |\bar{\nabla}\varphi|^p\, d\bar{\mu}$. This can be deduced from the usual one-variable Hardy inequality [31, Theorem 327] via symmetrization (cf. [17, Section 2]). Alternatively, the $p = 2$ case (all we need here) can be deduced via a simple integration by parts argument (cf. [22, Section 5.8.4]).

For δ sufficiently small, h_t is a Riemannian metric. Finally, we define a metric g on $X = M_0 \times (-\delta, \delta)$ by

$$g = h_t + dt^2.$$

We make a few observations. First, since $\partial_t h_t|_{t=0} = 0$, the embedding $M_0 \times \{0\} \subset (X, g)$ is totally geodesic. Moreover, by the Riccati equation (cf. [25, Corollary 3.3]), we have

$$2\varepsilon h_0 - \mathrm{Ric}_{h_0} = -\tfrac{1}{2}\partial_t^2 h_t|_{t=0} = R_g(\cdot, \partial_t, \partial_t, \cdot). \tag{B.1}$$

Let $\{e_i\}_{i=1}^4$ be a local orthonormal frame on M, where $e_4 = \partial_t$. By the Gauss equations for $M_0 \times \{0\}$ and (B.1) we have, for $i \in \{1, 2, 3\}$,

$$\mathrm{Ric}_g(e_i, e_i) = \mathrm{Ric}_{h_0}(e_i, e_i) + R_g(e_i, e_4, e_4, e_i) = 2\varepsilon.$$

Furthermore, by (B.1), we have

$$\mathrm{Ric}_g(e_4, e_4) = \sum_{i=1}^{3} R_g(e_i, e_4, e_4, e_i) = 6\varepsilon - R_{h_0} = 6\varepsilon.$$

In particular, $\mathrm{Ric}_g \geq 2\varepsilon$ along $M_0 \times \{0\}$. If we take $\delta > 0$ even smaller, we find that $\mathrm{Ric}_g \geq \varepsilon$ on X.

Now we verify that the immersion $M^3 \to (X^4, g)$ from the universal cover is stable. Note that the potential term in the stability operator for M satisfies $|A_M|^2 + \mathrm{Ric}_g(\nu, \nu) = 6\varepsilon = \lambda_1(M)$. Thus, for any $u \in C_c^\infty(M)$, we have

$$\int_M |\nabla u|^2 \geq \lambda_1(M, h) \int_M u^2 = \int_M (|A_M|^2 + \mathrm{Ric}_g(\nu, \nu))u^2.$$

This completes the proof (after scaling so that $\mathrm{Ric}_g \geq 1$). Note that because the image of M is an embedded submanifold, the sectional curvature is uniformly bounded below along a tubular neighborhood of M.

Remark B.3. It is an interesting question whether one can find $M^3 \to (X^4, g)$ stable minimal where (X^4, g) is a *closed* (or complete) manifold with $\mathrm{Ric}_g \geq 1$ at all points. It is tempting to try to study the metric of positive Ricci curvature on $(S^2 \times S^2) \# (S^2 \times S^2)$ constructed in [58]. If the construction is carried out in the most symmetric way possible, there will be a totally geodesic submanifold diffeomorphic to $M_0 := (S^1 \times S^2) \# (S^1 \times S^2)$. If one could modify the construction so as to keep positivity of the curvature, but to ensure that $0 \leq \mathrm{Ric}_g(\nu, \nu) \to 0$ along M_0, then the proof used above would show that the universal cover of M_0 was eventually stable.

B.3. Non-negative sectional curvature and strictly positive scalar curvature in six dimensions

Choose $(\mathbf{R}^4, g)$ as in Appendix B.1 and consider $(X^6, g_X) := (\mathbf{R}^4, g) \times (S^2, g_{S^2})$. Then, (X^6, g_X) will have non-negative sectional curvature, scalar curvature $R_{g_X} \geq 2$, and strictly positive Ricci curvature. On the other hand, if we cross the totally geodesic $\mathbf{R}^3 \to (\mathbf{R}^4, g)$ by S^2, we find a two-sided minimal immersion $M^5 := \mathbf{R}^3 \times S^2 \to (X^6, g_X)$. The immersion $M \to (X, g_X)$ will be stable. To see this, consider $B_\rho(0) \times S^2 \subset M$ for $\rho > 0$. The first eigenfunction of the stability operator on this set will be S^2-invariant, so the argument used in Appendix B.1 (the Hardy inequality) implies that the first eigenvalue of the stability operator on this set is positive. Letting $\rho \to \infty$, we see that M is stable. On the other hand, $\mathrm{Ric}_{g_X}(\nu, \nu) > 0$ along M.

Appendix C. Pulling back immersions along a local diffeomorphism

Suppose that X, Y, M are smooth manifolds, $\Psi : Y \to X$ is a local diffeomorphism and $F : M \to X$ is an immersion. We describe here how to "pull back[10] F along Ψ." This construction is presumably well-known.

Below, we will use the standard notation $f \pitchfork Z$ to mean that the map f is transverse to the submanifold Z.

Consider the map

$$F \times \Psi : M \times Y \to X \times X.$$

Write $\Delta = \{(x, x) : x \in X\}$ for the diagonal in $X \times X$.

Lemma C.1. $F \times \Psi \pitchfork \Delta$.

Proof. We note that

$$(F \times \Psi)^{-1}(\Delta) = \{(m, y) \in M \times Y : F(m) = \Psi(y)\}.$$

Hence, for $(m, y) \in (F \times \Psi)^{-1}(\Delta)$, we find

$$\operatorname{image} d(F \times \Psi)_{(m,y)} = \operatorname{image} dF_m \times T_{\Psi(y)} X.$$

Hence, for $(v_1, v_2) \in T_{F(m)} X \times T_{\Psi(y)} X$, we can write

$$(v_1, v_2) = (v_1, v_1) + (0, v_2 - v_1) \in T_{(F(m), \Psi(y))} \Delta + \operatorname{image} d(F \times \Psi)_{(m,y)}.$$

This completes the proof. ∎

Thus $S := (F \times \Psi)^{-1}(\Delta)$ is a submanifold of $M \times Y$. Recall that

$$T_s S = (d(F \times \Psi)_s)^{-1}(T_{(F(s), \Psi(s))} \Delta)$$

and

$$\dim M + \dim Y - \dim S = \operatorname{codim}(S \subset Y \times M) = \operatorname{codim}(\Delta \subset X \times X) = \dim X,$$

[10] This is a pullback in the category-theoretic sense (in the category of smooth maps/manifolds). Note that the pullback of two maps need not always exist in this category, but it does when the maps are transverse.

so (because $\dim Y = \dim X$) we have

$$\dim S = \dim M.$$

Write $F_S : S \to Y$ for the restriction of the projection map $M \times Y \to Y$, and similarly for $\Psi_S : S \to M$. In particular, the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{F_S} & Y \\ {\scriptstyle \Psi_S}\downarrow & & \downarrow{\scriptstyle \Psi} \\ M & \xrightarrow{F} & X \end{array}$$

We now check that the maps Ψ_S, F_S have the desired properties.

Lemma C.2. *The map $F_S : S \to Y$ is an immersion and the map $\Psi_S : S \to M$ is a local diffeomorphism.*

Proof. We start with F_S. For $s \in S$, write $s = (m, y) \in Y \times M$. Then $(dF_S)_s$ is the restriction of the projection onto the second factor map $\pi_2 : T_m M \times T_y Y \to T_y Y$. Hence,

$$\ker (dF_S)_s = (T_m M \times 0) \cap T_s S.$$

Consequently, if $(t, 0) \in \ker (dF_S)_s$ then

$$(t, 0) \in (d(F \times \Psi)_s)^{-1}(T_{(F(s),\Psi(s))}\Delta),$$

i.e., $dF_m(t) = 0$. Because F is an immersion, we have $t = 0$. This proves that F_S is an immersion. For Ψ_S, note that $\dim S = \dim M$, so it suffices to prove that Ψ_S is an immersion. The proof is then identical to the one we just gave for F_S. ■

Appendix D. Existence of local covering maps with good regularity

Recall the definition of Q-weakly bounded geometry from Definition 2.3. The following result is well-known (cf. [50], [47, Lemma 2.1], and [48, Exercise 11.6.15]). We sketch the proof below, referring to [48, 50] for several crucial details.

Proposition D.1. *If (X^n, g) is a complete manifold with $|\sec| \le K < \infty$, then there is $Q = Q(K)$ such that (X^n, g) has Q-weakly bounded geometry.*

Proof. Fix $x \in X$ and choose an orthonormal basis for $T_x X$. We will identify $T_x X$ with $\mathbf{R}^n$ (so g_x agrees with the standard inner product on $\mathbf{R}^n$). Using Jacobi field estimates (and $|\sec| \le K < \infty$), there is $r_0 = r_0(K, n) > 0$ such that

$$\exp_x : B(0, 4r_0) \subset \mathbf{R}^n \to X$$

is a local diffeomorphism, and $\tilde{g} := \exp_x^* g$ satisfies $\frac{1}{2}\delta \le \tilde{g} \le 2\delta$ on $B(0, 4r_0) \subset \mathbf{R}^n$ as quadratic forms. By [50, Lemma 2.2] we have $\mathrm{inj}_{\tilde{g}}(v) \ge i_0 = i_0(K, n) > 0$ for $v \in B(0, r_0)$. The assertion then follows by constructing harmonic coordinates for $\tilde{g}$ in a uniformly big neighborhood of 0 as in [48, Theorem 11.4.3] (see also [1] and [50, Theorem 2.1]) ■

Acknowledgements. We thank Richard Bamler, Renato Bettiol, Neil Katz, Dan Lee, Christina Sormani and Ruobing Zhang for some helpful discussions.

Funding. O.C. was supported by an NSF grant (DMS-2016403), a Terman Fellowship, and a Sloan Fellowship. C.L. was supported by an NSF grant (DMS-2005287) and a Simons Junior Faculty Fellowship. D.S. was supported by an NDSEG Fellowship.

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