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Second order perturbation theory of two-scale systems in fluid dynamics

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Abstract. In the present paper we study fast-slow systems of coupled equations from fluid dynamics, where the fast component is perturbed by additive noise. We prove that, under a suitable limit of infinite separation of scales, the slow component of the system converges in law to a solution of the initial equation perturbed by transport noise, and subject to the influence of an additional Itô–Stokes drift. The resulting limit equation is very similar to turbulent models derived heuristically. Our results apply to the Navier–Stokes equations in dimensions $d = 2, 3$; the surface quasi-geostrophic equations in dimension $d = 2$; and the primitive equations in dimensions $d = 2, 3$.

Keywords: fluid dynamics, transport noise, Itô–Stokes drift.

1. Introduction

Let $T > 0$ be fixed. In this work we consider fast-slow systems of coupled abstract equations of fluid dynamics

$$\begin{cases} du^\varepsilon_t = Au^\varepsilon_t dt + b(u^\varepsilon_t, u^\varepsilon_t)dt + b(v^\varepsilon_t, u^\varepsilon_t)dt, \\ dv^\varepsilon_t = \varepsilon^{-1} C v^\varepsilon_t dt + Av^\varepsilon dt + b(u^\varepsilon_t, v^\varepsilon_t)dt + b(v^\varepsilon_t, v^\varepsilon_t)dt + \varepsilon^{-1} Q^{1/2} dW_t, \end{cases} \tag{1.1}$$

where $t \in [0, T]$, $\varepsilon \in (0, 1)$ is a small parameter indicating separation of scales, and $Q^{1/2} dW$ is an additive Gaussian noise, white in time and coloured in space. A and C are (possibly unbounded) negative definite linear operators on a separable Hilbert space H, and should be interpreted as dissipation terms. The map $b : H \times H \to H$ is bilinear and enjoys suitable properties detailed below. In the present paper, as examples of physical equations for u^ε, v^ε that can be coupled with (1.1) and described within this formalism, we consider the Navier–Stokes equations in dimensions $d = 2, 3$; the surface quasi-geostrophic equations in dimension $d = 2$; the primitive equations in dimensions $d = 2, 3$.

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Our goal is to describe the asymptotic behaviour of the slow component u^ε as the scaling parameter ε goes to zero.

System (1.1) above aims to describe heuristically the dynamics of a coupled fast-slow system, where u^ε (resp. v^ε) corresponds to the slow-varying, large-scale (resp. fast-varying, small-scale) component of the fluid. It is not clear that such a separation of scales holds so strictly in fluids, but (1.1) can be a physically sensible, yet mathematically challenging starting point in the investigation on turbulence in fluids. Indeed, "turbulent flows contain self-sustaining velocity fluctuations in addition to the main flow" [57, Chapter 26], and additive noise in fluid dynamics equations has been widely used for decades to capture the statistical properties of turbulent fluids [12, 19, 38, 44].

Under suitable conditions, we are able to prove the following result (for a precise statement see Theorem 5.1):

Theorem. *Let* $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ *be a family of solutions to* (1.1) *in the sense of Definition* 2.2 *below. Then* $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ *admits converging-in-law subsequences as* $\varepsilon \to 0$*, and every weak accumulation point* u *solves the equation with transport noise and Itô–Stokes drift velocity* r*:*

$$du_t = Au_t dt + b(u_t, u_t)dt + b((-C)^{-1}Q^{1/2} \circ dW_t, u_t) + b(r, u_t)dt. \tag{1.2}$$

In addition, if pathwise uniqueness holds for (1.1) *and* (1.2) *then the whole sequence* u^ε *converges to* u *in probability.*

The Itô–Stokes drift velocity r, usually defined as the difference between Lagrangian and Eulerian average flows, has important consequences in wave-induced sediment transport and sandbar migration in the coastal zone, as well as transport of heat, salt and other natural or man-made tracers in the upper ocean layer [64]. Here we find a precise expression for r, related to the invariant measure μ of the linearized equation $dv_t = Cv_t dt + Q^{1/2}dW_t$ via the formula

$$r = \int (-C)^{-1}b(w, w)\, d\mu(w).$$

Equation (1.2) contains a transport noise. There have been heuristic arguments developed to derive transport noise in Navier–Stokes and Euler equations by perturbing deterministic balance laws [55], variational principles [46] or homogenization techniques [18]. Ideas of [55] have been extended and systematically used to derive stochastic fluid models in a series of work initiated in [54]. Compared to the previous ones, the models in [7] also contain the Itô–Stokes drift (see [7, (11b)]). It is remarkable that we recover exactly the same expression for this Itô–Stokes drift.

More recently, in the particular case where A equals the Laplace operator and $C = -\mathrm{Id}$, the authors of [36, 37] considered a similar problem and were able to obtain similar results for fluids in dimension 2 by proving a Wong–Zakai-type convergence for the (stochastic) characteristics associated with (1.1). In these articles, the Itô–Stokes drift was not apparent in the limit model because of their assumptions on the spatial structure of the noise. Also their arguments do not seem to be applicable to more complicated models.

The present paper adds to this picture and goes much further. The transport noise is justified rigorously for many fluid models through a very robust method, an appropriate generalization of the classical perturbed test function method to partial differential equations. Also, up to our knowledge, it is the first rigorous derivation of a stochastic fluid model containing the Itô–Stokes drift. In fact, the transport noise in the limit equation comes from a diffusion-approximation argument (instead of Wong–Zakai-type results as in [36, 37]), while the Itô–Stokes drift is due to an average constraint similar to those appearing in homogenization theory. Both phenomena are handled by the perturbed test function method.

In addition, as already mentioned, our results are very general and we are able to study many different systems, some of them rather difficult to study in the Lagrangian formulation, highlighting the fundamental nature of transport noise in fluids. More precisely, we show that our method also applies to the surface quasi-geostrophic equations and to the primitive equations. In each case, we derive stochastic fluid models containing a transport noise and an Itô–Stokes drift; they are very similar to the models obtained by E. Mémin and co-authors [7], justifying their use through rigorous mathematical arguments. We restrict to these examples but, as argued in [2, 3, 33, 51], our arguments can be used to study other climate models. Note also that the authors of [1,4] mention our work as a way to justify their models for an application in a totally different context. More generally, we believe that many two-scales partial differential equations models can and will be treated thanks to suitable adaptations of our arguments.

The perturbed test function method has been introduced in [58] and is now a standard method to study finite-dimensional approximation-diffusion problems. It has been generalized to infinite-dimensional problems recently (see for instance [22–25]). However, the partial differential equations considered there were simpler and the present work is the first application of this method to nonlinear fluid models such as the ones considered here.

Also, the fast variable is v^ε, solving a nonlinear partial differential equation. This creates difficulties since, in the context of partial differential equations, the correctors have a more complex expressions than in the classical finite-dimensional framework. They involve the fast variable and its spatial derivatives. It is well known that v^ε does not have much spatial smoothness for realistic three-dimensional models such as the ones we want to handle. We see below that in fact we can use a linearization trick and replace v^ε by an infinite-dimensional Ornstein–Uhlenbeck process solving a linear problem. Also, a central tool of the method is the generator of the fast variable. Here, this generator is an infinite-dimensional differential operator. We analyse this operator and prove that its domain contains the functions necessary for the method, prove tightness of u^ε, and take the limit in the infinite-dimensional problem thanks to the correctors. Among the various difficulties this implies, let us mention the uniform bounds on the correctors and the identification of the terms appearing in the limit, namely the transport noise, the associated Itô–Stratonovich corrector and the Itô–Stokes drift.

Besides that, in Section 6 we restrict our attention to the Navier–Stokes system and discuss the interesting problem of the limit behaviour of u^ε when the covariance operator $Q = Q_\varepsilon$ depends on the scaling parameter $\varepsilon \in (0, 1)$. Mostly inspired by [32] (cf. also the series of works [27, 28, 31]), where the authors prove the convergence, in a certain scaling limit, of the solution of Navier–Stokes equations perturbed by transport noise to their deterministic counterpart with extra dissipation, we identify conditions under which u^ε converges in law towards a process u that solves the deterministic Navier–Stokes equations with an additional dissipative term κ (Theorem 6.1):

$$du_t = Au_t dt + b(u_t, u_t)dt + \kappa(u_t)dt.$$

This is the first result showing that eddy viscosity in 3D Navier–Stokes equation can be created by additive noise. Notice that in the equation above we have neither the stochastic integral (because of the scaling) nor the Itô–Stokes drift (because of the isotropy of the noise chosen in [32]; see also Remark 5.9 (ii)).

Fluid dynamics equations perturbed by noise of transport type have been the subject of a lot of research recently:

$$du_t = Au_t dt + b(u_t, u_t)dt + b((-C)^{-1} Q^{1/2} \circ dW_t, u_t).$$

They stand out for often enjoying well-posedness [13–15, 26, 30, 32, 34, 35, 56], and manifest properties like enhanced dissipation [28, 41, 50] and mixing [27, 28], typical of turbulent fluids. Still, unlike the case of additive noise that is widely accepted as source of randomness, transport noise needs a more careful justification. For instance, in [52] the authors write:

> In principle, the turbulent velocity which advects the passive scalar should be a solution to the Navier–Stokes equations [...] with some external stirring which maintains the fluid in a turbulent state. But the analytical representation of such solutions corresponding to complex, especially turbulent flows, are typically unwieldy or unknown. We shall therefore instead utilize simplified velocity field models which exhibit some empirical features of turbulent or other flows, though these models may not be actual solutions to the Navier–Stokes equations.

More recently, passive scalars advected by solutions to the stochastic Navier–Stokes equations have been studied in a series of papers [8–11].

In view of this, and being aware of all the differences and limitations of our model, we look at (1.1) as a more accurate way to model the evolution of a fluid u^ε subject to the influence of a turbulent component v^ε and believe that our result provides a good justification of transport noise. The choice of the parameter ε^{-1} in front of both noise and dissipation is appropriate when looking at some particular systems from the point of view of a large-scale observer (see for instance [36, 37]), and we adopt the same scaling in the present paper by analogy. Also, the reader can notice a strong similarity with the theory of stochastic model reduction developed in [53], where the authors study finite-dimensional stochastic systems with quadratic nonlinearities and two widely separated time scales as an approximation of geophysical PDEs, with noise replacing the self-interaction of unresolved variables; however, their results were obtained by formal second-order perturbation theory following [49] and lack explicit error estimates, so it is not clear whether they can be extended to infinite-dimensional frameworks like ours.

1.1. Perturbed test function method

Let us describe the main technique allowing us to determine the limiting behaviour of the slow process u^ε as $\varepsilon \to 0$. The method here presented is originally due to Papanicolaou, Stroock and Varadhan [58]. The reader may find new developments and a presentation in the book [39]. It has recently been extended to infinite dimensions and partial differential equations; see for instance [22–25].

Let us consider again system (1.1), and denote $y^\varepsilon_t = \varepsilon^{1/2} v^\varepsilon_t$:

$$\begin{cases} du^\varepsilon_t = Au^\varepsilon_t dt + b(u^\varepsilon_t, u^\varepsilon_t)dt + \varepsilon^{-1/2} b(y^\varepsilon_t, u^\varepsilon_t)dt, \\ dy^\varepsilon_t = \varepsilon^{-1} C y^\varepsilon_t dt + Ay^\varepsilon_t dt + b(u^\varepsilon_t, y^\varepsilon_t)dt + \varepsilon^{-1/2} b(y^\varepsilon_t, y^\varepsilon_t)dt + \varepsilon^{-1/2} Q^{1/2} dW_t. \end{cases} \tag{1.3}$$

In order to better present the ideas, let us suppose for the moment that $(u^\varepsilon, y^\varepsilon)$ is a Markov process whose evolution is described by its infinitesimal generator $\mathscr{L}^\varepsilon$, which takes the following form when applied to a suitable test function φ:

$$\begin{aligned} \mathscr{L}^\varepsilon \varphi(u, y) = {} & \langle Au + b(u, u), D_u \varphi \rangle + \varepsilon^{-1/2} \langle b(y, u), D_u \varphi \rangle \\ & + \langle Ay + b(u, y), D_y \varphi \rangle + \varepsilon^{-1/2} \langle b(y, y), D_y \varphi \rangle \\ & + \varepsilon^{-1} \langle Cy, D_y \varphi \rangle + \frac{\varepsilon^{-1}}{2} \operatorname{Tr}(Q D^2_y \varphi). \end{aligned}$$

We shall see later that the hypothesis of $(u^\varepsilon, y^\varepsilon)$ being Markov is not strictly necessary, as one only needs the validity of a suitable Itô formula for functions of $(u^\varepsilon, y^\varepsilon)$; see Lemma 5.3. Since we are interested in the limiting behaviour of u^ε as $\varepsilon \to 0$, we add correctors to φ in order to cancel out singular terms in the expression of $\mathscr{L}^\varepsilon \varphi$, on the one hand, and simultaneously eliminate the dependence on y in the terms of order 1, on the other. The motivation behind this procedure lies in the fact that we would like to obtain in the limit a closed equation in the sole variable u^ε.

The classical method. At this point, one could carry out the previous program in the following way. Consider the *perturbed test function*

$$\varphi^\varepsilon(u, y) = \varphi(u) + \varepsilon^{1/2} \varphi_1(u, y) + \varepsilon \varphi_2(u, y),$$

where φ_1 and φ_2 are suitable correctors.

Denoting by $\mathscr{L}_y$ the operator

$$\mathscr{L}_y = \langle Cy, D_y \cdot \rangle + \tfrac{1}{2} \operatorname{Tr}(Q D^2_y \cdot),$$

it is immediately clear that terms of order ε^{-1} in the expression of $\mathscr{L}^\varepsilon\varphi^\varepsilon$ vanish: indeed, they are given by a factor ε^{-1} times $\mathscr{L}_y\varphi$, which equals zero since φ does not depend on y. Turning to terms of order $\varepsilon^{-1/2}$, they are given by a factor $\varepsilon^{-1/2}$ times the quantity

$$\langle b(y,u), D_u\varphi\rangle + \langle b(y,y), D_y\varphi\rangle + \mathscr{L}_y\varphi_1. \tag{1.4}$$

Therefore, (1.4) cancels out if φ_1 is a solution to the Poisson equation

$$\mathscr{L}_y\varphi_1(u,y) = -\langle b(y,u), D_u\varphi\rangle. \tag{1.5}$$

Finally, the terms of order 1 in the expression of $\mathscr{L}^\varepsilon\varphi^\varepsilon$ equal

$$\langle Au + b(u,u), D_u\varphi\rangle + \langle b(y,u), D_u\varphi_1\rangle + \langle b(y,y), D_y\varphi_1\rangle + \mathscr{L}_y\varphi_2.$$

As previously remarked, it is not necessary to require this quantity to be zero, but it is sufficient to just seek for φ_2 that does not depend on y, namely

$$\begin{aligned}\mathscr{L}^0\varphi(u) &= \langle Au + b(u,u), D_u\varphi\rangle + \langle b(y,u), D_u\varphi_1\rangle \\ &\quad + \langle b(y,y), D_y\varphi_1\rangle + \mathscr{L}_y\varphi_2(u,y)\end{aligned}$$

for some *effective* generator $\mathscr{L}^0$. In this way, one would formally get $\mathscr{L}^\varepsilon\varphi^\varepsilon(u^\varepsilon, y^\varepsilon) = \mathscr{L}^0\varphi(u^\varepsilon)$ and identify the limit behaviour of $\mathscr{L}^\varepsilon\varphi(u^\varepsilon)$ up to a correction that has to be shown to be infinitesimal as $\varepsilon \to 0$. Moreover, it is needed to properly justify the previous procedure for a sufficiently large class of test functions φ.

Linearization trick. The method described above is very powerful, and strictly speaking it does allow us to rigorously understand the limiting behaviour of u^ε as $\varepsilon \to 0$ for the Navier–Stokes system in dimensions $d = 2, 3$ for some sufficiently regularizing operator C; namely, it is possible to show that correctors φ_1 and φ_2 exist, and they are such that both $\varphi^\varepsilon(u^\varepsilon, y^\varepsilon) - \varphi(u^\varepsilon)$ and $\mathscr{L}^\varepsilon\varphi^\varepsilon(u^\varepsilon, y^\varepsilon) - \mathscr{L}^0\varphi(u^\varepsilon)$ are actually infinitesimal in the limit $\varepsilon \to 0$, with respect to suitable topologies. However, in general, checking these conditions requires a certain degree of regularity both for the solutions $(u^\varepsilon, y^\varepsilon)$ and the dynamics itself – namely the coefficients A, C and b cannot be too bad. In particular, the previous method does not permit studying the limiting behaviour of other equations of interest, like the surface quasi-geostrophic equations in dimension $d = 2$ and the primitive equations in dimensions $d = 2, 3$, and requires strong assumptions on C in the case of Navier–Stokes equations.

For this reason, also inspired by [6,40], here we develop a modification of the classical method that permits us to replace the process y^ε by its linear counterpart Y^ε satisfying

$$dY^\varepsilon_t = \varepsilon^{-1}CY^\varepsilon_t dt + \varepsilon^{-1/2}Q^{1/2}dW_t, \quad Y^\varepsilon_0 = 0.$$

Loosely speaking, the key observation is that one can rewrite

$$\begin{aligned}\langle b(y,u), D_u\varphi_1(y)\rangle &= \langle b(Y,u), D_u\varphi_1(Y)\rangle + \langle b(y-Y,u), D_u\varphi_1(Y)\rangle \\ &\quad + \langle b(y,u), D_u\varphi_1(y-Y)\rangle, \\ \langle b(y,y), D_y\varphi_1(u)\rangle &= \langle b(Y,Y), D_y\varphi_1(u)\rangle + \langle b(y-Y,Y), D_y\varphi_1(u)\rangle \\ &\quad + \langle b(y,y-Y), D_y\varphi_1(u)\rangle,\end{aligned}$$

and prove that the terms involving the difference $y - Y$ are infinitesimal as $\varepsilon \to 0$ when evaluated at $y = y_t^\varepsilon$, $Y = Y_t^\varepsilon$, and integrated with respect to time. Therefore, the actual terms of order 1 in the expression of $\mathscr{L}^\varepsilon \varphi^\varepsilon$ are given by

$$\langle Au + b(u,u), D_u \varphi \rangle + \langle b(Y,u), D_u \varphi_1(Y) \rangle + \langle b(Y,Y), D_y \varphi_1(u) \rangle + \mathscr{L}_y \varphi_2, \quad (1.6)$$

and thus we only need to find a corrector $\varphi_2 = \varphi_2(u, Y)$ such that the previous expression only depends on u, and control the remainders. This task is markedly easier, due to the higher space regularity of Y.

Nonhomogeneous Ornstein–Uhlenbeck generator. On top of the linearization trick just described above, we can further sharpen our results if we replace $\mathscr{L}_y$ by its inhomogeneous counterpart

$$\mathscr{L}_y^\varepsilon = \langle C_\varepsilon y, D_y \cdot \rangle + \tfrac{1}{2} \operatorname{Tr}(Q D_y^2 \cdot), \quad C_\varepsilon = C + \varepsilon A. \quad (1.7)$$

Roughly speaking, this choice is of help because it allows us to trade additional space regularity for the correctors $\varphi_1 = \varphi_1^\varepsilon$ and $\varphi_2 = \varphi_2^\varepsilon$ with multiplicative factors ε^{-1} (in our examples the operator C is less regularizing than A). As a particular instance of this trading, the reader could take a look at Proposition 4.1. A little inconvenience, however, is that using $\mathscr{L}_y^\varepsilon$ instead of $\mathscr{L}_y$ produces correctors φ_1^ε, φ_2^ε and an effective generator $\mathscr{L}^{0,\varepsilon}$ that depend on ε, although it is easy to see that this does not affect the limiting behaviour of u^ε, i.e. the equation satisfied by $u = \lim_{\varepsilon \to 0} u^\varepsilon$ remains the same.

1.2. *Structure of the paper*

The paper is structured as follows.

In Section 2 we introduce the necessary notation and preliminaries for our analysis. In particular, we introduce the abstract spaces and operators governing our system, and we identify their key properties. Working assumptions on the covariance operator Q are stated. Also, we present the notion of a bounded-energy family $\{(u^\varepsilon, y^\varepsilon)\}_{\varepsilon \in (0,1)}$ of weak martingale solutions to our system, that is, a family of solutions enjoying some uniform-in-ε bound on the energy.

In Section 3 we introduce a class of test functions ψ for which it is possible to solve the Poisson equation $\mathscr{L}_y^\varepsilon \phi = -\psi$ implicitly. The class consists of quadratic functions on H that are continuous maps from some Sobolev space H^θ to $\mathbb{R}$. We also show that, depending on the regularity of C, solutions of the Poisson equations so constructed are more regular than the datum ψ, and recover bounds on the regularity of ϕ and its derivative in terms of the regularity of ψ and C.

In Section 4 we apply abstract results on the Poisson equation to carry out the program presented in the Introduction; we identify suitable correctors φ_1^ε and φ_2^ε to cancel out divergent terms in the expression of $\mathscr{L}^\varepsilon \varphi$, and recover the limiting behaviour of the slow variable u^ε alone.

In Section 5, we prove our main Theorem 5.1 dividing the proof into three different steps: first, we prove that the family (of the laws of) $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$ is tight in a suitable space of functions; then, checking that the contribution due to the correctors φ_1^ε and φ_2^ε is actually negligible as $\varepsilon \to 0$, we prove that every weak accumulation point u is a solution of a limit closed equation; finally, we recognize the different terms in the equation solved by u as the sum of the original slow dynamics, a Stratonovich transport noise and an Itô–Stokes drift.

Section 6, whose results are applied only to the Navier–Stokes system, links our results to the ones in [32] to provide conditions under which a suitable scaling of the parameters ε, Q makes our system converge towards solutions of the deterministic Navier–Stokes equations with additional dissipation. We provide an explicit example (due to [32]) of parameters that gives in the limit a large multiple of the Laplace operator.

Finally, in the main body of this work we prefer to illustrate only the arguments needed for the Navier–Stokes system in dimension $d = 3$, for the sake of a clear presentation, and we devote Section 7 to discussing all the necessary changes needed to take into account other models of interest.

2. Preliminaries and assumptions

In this section we collect all the necessary notation, assumptions, preliminaries and auxiliary results useful for our analysis. As discussed in the Introduction, our theory covers at least three models of fundamental relevance in fluid dynamics: the Navier–Stokes equations in dimensions $d = 2, 3$; the surface quasi-geostrophic equations in dimension $d = 2$; and the primitive equations in dimensions $d = 2, 3$. However, each of these models requires an *ad hoc* analysis that takes into consideration peculiar features of the dynamics, and minor modifications in the arguments are needed to properly deal with each of them. Thus, for the sake of a clear and effective presentation, we present our results first for the 3D Navier–Stokes equations; in the last section, we illustrate the differences arising when considering the surface quasi-geostrophic and primitive equations, and only discuss how to adapt the arguments in the main body of the paper to the changed framework.

Throughout the paper, we use the notation $a \lesssim b$ if there exists an unimportant constant $c \in (0, \infty)$ such that $a \leq cb$.

2.1. *Coupled Navier–Stokes equations*

As a main example of application of the theory here developed, recall the Navier–Stokes system in velocity form, with additive noise and large dissipation at small scales:

$$
\begin{cases}
du_t^\varepsilon = \nu \Delta u_t^\varepsilon dt - (u_t^\varepsilon \cdot \nabla) u_t^\varepsilon dt - (v_t^\varepsilon \cdot \nabla) u_t^\varepsilon dt + \nabla p_t^\varepsilon dt, \\
\operatorname{div} u_t^\varepsilon = 0, \\
dv_t^\varepsilon = \varepsilon^{-1} C v_t^\varepsilon dt + \nu \Delta v_t^\varepsilon dt - (u_t^\varepsilon \cdot \nabla) v_t^\varepsilon dt - (v_t^\varepsilon \cdot \nabla) v_t^\varepsilon dt + \varepsilon^{-1} d\mathcal{W}_t + \nabla q_t^\varepsilon dt, \\
\operatorname{div} v_t^\varepsilon = 0,
\end{cases}
\tag{2.1}
$$

where $t \in [0, T]$, $u_t^\varepsilon, v_t^\varepsilon$ are unknown velocity fields belonging to the space $[L^2(\mathbb{T}^3)]^3$ of zero-mean square integrable velocity fields on the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$; $p_t^\varepsilon, q_t^\varepsilon$ are unknown pressure fields in $L^2(\mathbb{T}^3)$ ensuring the validity of the divergence-free conditions $\operatorname{div} u_t^\varepsilon = \operatorname{div} v_t^\varepsilon = 0$; $\mathcal{W}$ is a Wiener process on $[L^2(\mathbb{T}^3)]^3$; $\nu > 0$ is the viscosity coefficient; and $\varepsilon \in (0, 1)$ is a small scaling parameter. In dimension 2 a physically relevant choice for C is $C = -\mathrm{Id}$ (the identity operator on $[L^2(\mathbb{T}^2)]^2$), see also [36, 37] for a justification of the model; in three dimensions we cannot deal with the pure friction case and we need a more regularizing operator C (see assumption (C2) below).

Denote by $H := \{u \in [L^2(\mathbb{T}^3)]^3 : \operatorname{div} u = 0\}$ the space of periodic, zero-mean, square integrable velocity fields u with null divergence in the sense of distributions. Projecting (2.1) on H via the Helmholtz projector Π we get the equivalent system, without pressure terms:

$$\begin{cases} du_t^\varepsilon = \nu \Delta u_t^\varepsilon dt - \Pi(u_t^\varepsilon \cdot \nabla) u_t^\varepsilon dt - \Pi(v_t^\varepsilon \cdot \nabla) u_t^\varepsilon dt, \\ dv_t^\varepsilon = \varepsilon^{-1} C v_t^\varepsilon dt + \nu \Delta v_t^\varepsilon dt - \Pi(u_t^\varepsilon \cdot \nabla) v_t^\varepsilon dt - \Pi(v_t^\varepsilon \cdot \nabla) v_t^\varepsilon dt + \varepsilon^{-1} d\Pi \mathcal{W}_t. \end{cases}$$

Therefore, we can recast the system (2.1) in the more abstract setting (1.1), defining

$$Au = \nu \Delta u, \quad b(u, v) = -\Pi(u \cdot \nabla) v, \quad Q^{1/2} W = \Pi \mathcal{W}_t,$$

with W being a cylindrical Wiener process on H and Q a covariance operator. Hereafter, (1.1) will be accompanied by initial conditions $(u_0, y_0) \in H \times H$. In order to keep our analysis as simple as possible, we assume that u_0, y_0 are deterministic.

2.2. *Abstract spaces and operators*

The linear operator A and Sobolev spaces. The operator $A : H \supset D(A) \to H$ is unbounded, self-adjoint and negative definite. For $s \in \mathbb{R}$, let us define the Sobolev space H^s by $H^s := D((-A)^{s/2})$. Sobolev spaces form a Hilbert scale in the sense of Krein–Petunin [47], with respect to the operator $(-A)^{1/2}$:

$$\langle f, g \rangle_{H^s} = \langle (-A)^{s/2} f, (-A)^{s/2} g \rangle.$$

In particular, the map $(-A)^{s/2} : H^{r+s} \to H^r$ is an isomorphism for all $s, r \in \mathbb{R}$, and the following interpolation inequality holds between H^{s_1} and H^{s_2} for $s_1, s_2 \in \mathbb{R}$ and $\lambda \in (0, 1)$:

$$\|f\|_{H^{s_\lambda}} \le \|f\|_{H^{s_1}}^{\lambda} \|f\|_{H^{s_2}}^{1-\lambda}, \quad s_\lambda = \lambda s_1 + (1 - \lambda) s_2.$$

The Sobolev space H^s embeds continuously into $L^p = L^p(\mathbb{T}^d)$ provided $s > d/2$ and $p \in [1, \infty]$ or $s \in (0, d/2)$ and $p \le 2^\star = \frac{2d}{d-2s}$.

For $\alpha \in (0, 1)$, $p \ge 1$ and $s \in \mathbb{R}$, we let $W^{\alpha,p}([0, T], H^s)$ be the Sobolev–Slobodetskiĭ space of all $u \in L^p([0, T], H^s)$ such that

$$\int_0^T \int_0^T \frac{\|u_t - u_s\|_{H^s}^p}{|t - s|^{1+\alpha p}} \, dt \, ds < \infty,$$

endowed with the norm

$$\|u\|^p_{W^{\alpha,p}([0,T],H^s)} := \int_0^T \|u_t\|^p_{H^s}\,dt + \int_0^T\int_0^T \frac{\|u_t-u_s\|^p_{H^s}}{|t-s|^{1+\alpha p}}\,dt\,ds.$$

We recall the following compactness criterion, due to Simon [62].

Lemma 2.1. *For $\sigma>0$, $\alpha>1/p$ and $\beta\in(0,\sigma)$ we have the compact embeddings*

$$L^2([0,T],H^1)\cap W^{\alpha,p}([0,T],H^{-\sigma})\subset L^2([0,T],H),$$
$$L^\infty([0,T],H)\cap W^{\alpha,p}([0,T],H^{-\sigma})\subset C([0,T],H^{-\beta}).$$

Denote by $\mathscr{S}:=\bigcap_{s\in\mathbb{R}}H^s$ the class of smooth elements $h\in H$, and define

$$F:=\{\varphi:H\to\mathbb{R}:\exists h\in\mathscr{S} \text{ such that } \varphi(u)=\langle u,h\rangle\}.$$

Distributions on H are elements of the space $\mathscr{S}':=\bigcup_{s\in\mathbb{R}}H^s$. Every $\varphi\in F$ is continuous from $\mathscr{S}'$ to $\mathbb{R}$.

The linear operator C. We assume that

(C1) the operator $C:H\supset D(C)\to H$ is self-adjoint and negative definite, with principal eigenvalue $-\lambda_0<0$;

(C2) there exist $\Gamma\geq\gamma>1/4$ such that $\|x\|^2_{H^{s+\beta\gamma}}\lesssim\|(-C)^{\beta/2}x\|^2_{H^s}\lesssim\|x\|^2_{H^{s+\beta\Gamma}}$ for every $s\in\mathbb{R}$ and $\beta>0$.

Since the regularization effect produced by C is strictly stronger than that of A when $\gamma\geq1$, without loss of generality we assume $\gamma<1$ hereafter. The above assumptions imply that the operators C and $C_\varepsilon:=C+\varepsilon A$ generate C_0-semigroups on H, which we denote respectively by e^{Ct} and $e^{C_\varepsilon t}$, $t>0$. Moreover, for every $s\in\mathbb{R}$ and $\beta_1\geq0$ we have, uniformly in $t>0$ and $\varepsilon\in(0,1)$,

$$\|(-C_\varepsilon)^{\beta_1}e^{C_\varepsilon t}\|_{H^s\to H^s}\lesssim\frac{e^{-\lambda_0 t/2}}{t^{\beta_1}};$$

by interpolation, since the operators $(-C_\varepsilon)^{-1}C$ and $(-C_\varepsilon)^{-1}\varepsilon A$ are bounded, we also have, for every $\theta\in[\gamma,1]$, $\lambda=\frac{\lambda_0(1-\theta)}{2(1-\gamma)}$,

$$\|e^{C_\varepsilon t}\|_{H^s\to H^{s+2\theta\beta_1}}\lesssim\|(-C)^{\beta_1}e^{C_\varepsilon t}\|^{\frac{1-\theta}{1-\gamma}}_{H^s\to H^s}\|(-A)^{\beta_1}e^{C_\varepsilon t}\|^{\frac{\theta-\gamma}{1-\gamma}}_{H^s\to H^s}\lesssim\varepsilon^{-\beta_1\frac{\theta-\gamma}{1-\gamma}}\frac{e^{-\lambda t}}{t^{\beta_1}}.$$

In addition, for all $s\in\mathbb{R}$ and $\beta_2\in[0,1]$,

$$\|(-C_\varepsilon)^{-\beta_2}(e^{C_\varepsilon t}-1)\|_{H^s\to H^s}\lesssim t^{\beta_2},\quad \|e^{C_\varepsilon t}-1\|_{H^s\to H^{s-2\Gamma\beta_2}}\lesssim t^{\beta_2}$$

uniformly in $t>0$ and $\varepsilon\in(0,1)$, and moreover the difference of the semigroups satisfies[1] $\|e^{C_\varepsilon t}-e^{Ct}\|_{H^{s+2\beta_2}\to H^s}\lesssim\varepsilon^{\beta_2}$ uniformly in $t>0$.

[1] To see this when $\beta_2>0$, one can define $y_t:=e^{C_\varepsilon t}x-e^{Ct}x$ for $x\in H$ and notice that $y_t=\int_0^t Cy_s\,ds+\varepsilon\int_0^t Ae^{C_\varepsilon s}x\,ds$; since $y_0=0$, Duhamel's formula gives $y_t=\varepsilon\int_0^t e^{C(t-s)}Ae^{C_\varepsilon s}x\,ds$, and using $\|\varepsilon^{1-\beta_2}(-A)^{1-\beta_2}e^{C_\varepsilon t}\|_{H^s\to H^s}\lesssim e^{-\lambda_0 t/2}t^{\beta_2-1}$ produces the desired inequality.

Finally, for every $\beta_2 \in [0,1]$ the operator

$$G_\varepsilon := (-C_\varepsilon)^{-1} - (-C)^{-1} = \varepsilon(-C)^{-1}A(-C_\varepsilon)^{-1}$$

satisfies $\|G_\varepsilon\|_{H^s \to H^{s+2\gamma(1+\beta_2)-2\beta_2}} \lesssim \varepsilon^{\beta_2}$.

The bilinear operator b. Concerning the nonlinearity b of the Navier–Stokes equations in velocity form, we have the following properties[2] (take $d = 3$ below):

(B1) $b : H^s \times H^{\theta_0} \to H^s$ is bilinear and continuous for every $s \in \mathbb{R}$, $s < d/2$ and $\theta_0 > 1 + d/2$;

(B2) $b : H^s \times H^{\theta_1} \to H^s$ is bilinear and continuous for every $s \in \mathbb{R}$, $s \geq d/2$ and $\theta_1 > 1 + s$;

(B3) $b : H^s \times H^r \to H^{s+r-1-d/2}$ is bilinear and continuous if $s, r-1 \in (-d/2, d/2)$ and $s + r > 1$.

For every smooth $x_i \in \mathscr{S}$, $i = 1,2,3$, we have $\langle b(x_1,x_2), x_3\rangle = -\langle b(x_1,x_3), x_2\rangle$, by integration by parts. Therefore, if there exist $s_i \in \mathbb{R}$, $i = 1,2,3$, such that either $|\langle b(x_1,x_2),x_3\rangle| \lesssim \prod_{i=1}^3 \|x_i\|_{H^{s_i}}$ or $|\langle b(x_1,x_3),x_2\rangle| \lesssim \prod_{i=1}^3 \|x_i\|_{H^{s_i}}$, we can extend the other one by continuity preserving the same bounds. We summarize this property in

(B4) $\langle b(x_1,x_2),x_3\rangle = -\langle b(x_1,x_3),x_2\rangle$ for all $x_i \in \mathscr{S}'$, $i = 1,2,3$, such that both scalar products are well-defined.

In the following, without explicit mention, we will denote by $\theta_0, \theta_1 = \theta_1(s)$ any constants such that (B1) and (B2) hold.

The covariance operator Q. We assume that the covariance operator $Q : H \to H$ has the following properties:

(Q1) Q is symmetric, positive semidefinite and commutes with C, and the following operators on H are trace-class for every $t \geq 0$:

$$e^{Ct}Qe^{Ct}, \quad Q_\infty := \int_0^\infty e^{Ct}Qe^{Ct}\,dt = \frac{1}{2}(-C)^{-1}Q;$$

(Q2) denoting by $\mathscr{N}(0, Q_\infty)$ the Gaussian measure on H with covariance Q_∞ and $s_0 = \max\{\theta_0, 2\Gamma\}$, we have $\int_H \|w\|^2_{H^{s_0}}\, d\mathscr{N}(0,Q_\infty)(w) < \infty$.

In (Q2) above, θ_0 can be any real number such that (B1) holds and Γ is as in (C2). It is easy to see that under (Q1)–(Q2) the operators $e^{C_\varepsilon t}Qe^{C_\varepsilon t}$ and $Q^\varepsilon_\infty := \int_0^\infty e^{C_\varepsilon t}Qe^{C_\varepsilon t}\,dt$ are trace-class (although in general $Q^\varepsilon_\infty \neq \frac{1}{2}(-C_\varepsilon)^{-1}Q$ since we do not assume A and Q are commuting) and

$$\int_H \|w\|^2_{H^{\theta_0}}\, d\mathscr{N}(0,Q^\varepsilon_\infty)(w) \lesssim 1 + \int_H \|w\|^2_{H^{\theta_0}}\, d\mathscr{N}(0,Q_\infty)(w) < \infty$$

for every $\varepsilon \in (0,1)$, with implicit constant independent of ε.

[2] That will involve different regularity when dealing with the primitive equations.

2.3. Ornstein–Uhlenbeck semigroup

Assume (Q1)–(Q2). For every $\varepsilon \in (0, 1)$ and $y \in H$ there exists a unique solution $Y^y = Y^y(\varepsilon)$ of the Ornstein–Uhlenbeck equation

$$dY_t^y = C_\varepsilon Y_t^y dt + Q^{1/2} dW_t, \quad Y_0^y = y;$$

it is explicitly given by the formula

$$Y_t^y = e^{C_\varepsilon t} y + W_t^{C_\varepsilon, Q}, \quad W_t^{C_\varepsilon, Q} = \int_0^t e^{C_\varepsilon (t-s)} Q^{1/2} \, dW_s.$$

The Ornstein–Uhlenbeck semigroup $P_t^\varepsilon : C_b(H) \to C_b(H)$ is defined by

$$P_t^\varepsilon \psi(y) := \mathbb{E}[\psi(Y_t^y)], \quad \psi \in C_b(H),\ y \in H,$$

and it is a semigroup by the Markov property. It can be extended uniquely to a strongly continuous semigroup of 1-Lipschitz maps on $L^2(H, \mu^\varepsilon)$ with $\mu^\varepsilon := \mathcal{N}(0, Q_\infty^\varepsilon)$ [20, Theorem 10.1.5]. The Gaussian measure μ^ε is concentrated on $H^{\theta_0} \subset H$, and μ^ε is invariant for P_t^ε, i.e.

$$\int_H P_t^\varepsilon \psi(y) \, d\mu^\varepsilon(y) = \int_H \psi(y) \, d\mu^\varepsilon(y), \quad \forall \psi \in L^2(H, \mu^\varepsilon).$$

The domain $D(\mathscr{L}_y^\varepsilon)$ of the generator $\mathscr{L}_y^\varepsilon : D(\mathscr{L}_y^\varepsilon) \to L^2(H, \mu^\varepsilon)$ is defined as the set

$$D(\mathscr{L}_y^\varepsilon) := \left\{ \psi \in L^2(H, \mu^\varepsilon) : \exists \lim_{t \to 0+} \frac{P_t^\varepsilon \psi - \psi}{t} \in L^2(H, \mu^\varepsilon) \right\},$$

and $\mathscr{L}_y^\varepsilon$ acts on $\psi \in D(\mathscr{L}_y^\varepsilon)$ as $\mathscr{L}_y^\varepsilon \psi := \lim_{t \to 0+} \frac{P_t^\varepsilon \psi - \psi}{t}$. The generator $\mathscr{L}_y^\varepsilon$ is a closed operator on $L^2(H, \mu^\varepsilon)$.

2.4. Stochastic framework

Let us recall some basic notions and notations from stochastic analysis. For a more in-depth review of the general theory of stochastic equations we refer to [21].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a sequence $\{W^k\}_{k \in \mathbb{N}}$ of standard Wiener processes adapted to a common filtration $\{\mathcal{F}_t\}_{t \geq 0}$, assumed to be complete and right continuous. Given any complete orthonormal system $\{e_k\}_{k \in \mathbb{N}}$ of H, we define a *cylindrical* Wiener process W on H to be the formal series $W = \sum_{k \in \mathbb{N}} W^k e_k$. For every $t \geq 0$, W_t is well-defined as a random variable taking values in the space of distributions H^{-s} for some s sufficiently large so that the embedding $H \subset H^{-s}$ is Hilbert–Schmidt. We call any such $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W)$ a *stochastic basis*.

2.5. Notion of solution and energy estimates

We denote by H_w the space H endowed with the weak topology, and by $\mathcal{B}([0, T], H)$ the space of H-valued bounded functions (not necessarily continuous) endowed with the supremum norm. A similar notation is used for bounded functions on $[0, T]$ taking values in a general Banach space.

Definition 2.2. We say that the family $\{(u^\varepsilon, y^\varepsilon)\}_{\varepsilon\in(0,1)}$ is a *bounded-energy family of weak martingale solutions* to (1.3) if for every $\varepsilon \in (0, 1)$ there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$ such that

(S1) $(u^\varepsilon, y^\varepsilon) : \Omega \times [0, T] \to H \times H$ is $\{\mathcal{F}_t\}$-progressively measurable, with paths $u^\varepsilon, y^\varepsilon \in C([0, T], H_w) \cap L^2([0, T], H^1)$, $\mathbb{P}$-almost surely;

(S2) for every $h \in \mathcal{S}$, we have $\mathbb{P}$-almost surely, for every $t \in [0, T]$,

$$\langle u^\varepsilon_t, h\rangle = \langle u_0, h\rangle + \int_0^t \langle u^\varepsilon_s, Ah\rangle + \int_0^t \langle b(u^\varepsilon_s, u^\varepsilon_s), h\rangle\, ds + \varepsilon^{-1/2} \int_0^t \langle b(y^\varepsilon_s, u^\varepsilon_s), h\rangle\, ds,$$

$$\begin{aligned}\langle y^\varepsilon_t, h\rangle = \langle y_0, h\rangle + \varepsilon^{-1} \int_0^t \langle y^\varepsilon_s, Ch\rangle\, ds + \int_0^t \langle y^\varepsilon_s, Ah\rangle\, ds + \int_0^t \langle b(u^\varepsilon_s, y^\varepsilon_s), h\rangle\, ds \\ + \varepsilon^{-1/2} \int_0^t \langle b(y^\varepsilon_s, y^\varepsilon_s), h\rangle\, ds + \varepsilon^{-1/2} \langle Q^{1/2} W_t, h\rangle;\end{aligned}$$

(S3) the family $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ is uniformly bounded in

$$\mathcal{U} := L^\infty(\Omega, \mathcal{B}([0, T], H)) \cap L^\infty(\Omega, L^2([0, T], H^1));$$

(S4) for every fixed $p < \infty$, the family $\{y^\varepsilon\}_{\varepsilon\in(0,1)}$ is uniformly bounded in

$$\mathcal{Y} := \mathcal{B}([0, T], L^p(\Omega, H)) \cap L^p(\Omega, L^2([0, T], H^1)).$$

A comment on this definition is in order.

First of all, since we are working on the intersection of two fields and to avoid any confusion with the terminology, let us specify that here we are working with *analytically weak, probabilistically martingale* solutions. Solutions are analytically weak since they solve (1.3) only when tested against smooth test functions $h \in \mathcal{S}$. They are martingale solutions (sometimes also referred to as probabilistically weak solutions) since the stochastic basis in not given a priori (then we would talk about pathwise or probabilistically strong solutions). To avoid any misunderstanding we point out that hereafter the stochastic basis $\{(\Omega^\varepsilon, \mathcal{F}^\varepsilon, \{\mathcal{F}^\varepsilon_t\}_{t\geq 0}, \mathbb{P}^\varepsilon, W^\varepsilon)\}_{\varepsilon\in(0,1)}$ will always depend on ε, but we shall drop the superscripts for notational simplicity.

Second, our solutions form a *bounded-energy family* since in (S3)–(S4) we require suitable energy bounds to hold uniformly in $\varepsilon \in (0, 1)$.

In the classical theory of deterministic Navier–Stokes equations subject to external forcing $f \in L^1([0, T], H)$,

$$\begin{cases} du_t + (u_t \cdot \nabla)u_t dt = \nu \Delta u_t dt + \nabla p_t dt + f_t dt, \\ \operatorname{div} u_t = 0, \end{cases}$$

a fundamental concept is that of *Leray–Hopf* weak solutions, which are (analytically) weak solutions u enjoying the energy inequality

$$\frac{1}{2}\|u_t\|_H^2 + \int_0^t \|u_s\|_{H^1}^2\, ds \le \frac{1}{2}\|u_0\|_H^2 + \int_0^t \langle u_s, f_s\rangle\, ds.$$

In the stochastic setting the picture is more complicated since, when the external forcing $f = Q^{1/2}W$ is a stochastic process, (i) reasonable bounds can only be obtained in expected value, and (ii) formally applying the Itô formula to $\|u_t\|_H^2$ introduces an additional term $\mathrm{Tr}(Q)dt$ on the right hand side of the estimate.

In [38], the authors propose a notion of solution which encodes the energy inequality in the requirement that the process

$$E_t^p := \frac{1}{2}\|u_t\|_H^{2p} + p\int_0^t \|u_s\|_H^{2p-2}\|u_s\|_{H^1}^2\, ds - \frac{1}{2}\|u_0\|_H^{2p} \\ - \frac{p(2p-1)}{2}\mathrm{Tr}(Q)\int_0^t \|u_s\|_H^{2p-2}\, ds$$

be an *almost sure supermartingale* for every positive integer p, namely $\mathbb{E}[E_t^p] < \infty$ for all $t \in [0, T]$ and there exists a Lebesgue measurable set $\mathcal{T} \subset (0, T]$, with null Lebesgue measure, such that $\mathbb{E}[E_t^p \mathbf{1}_A] \le \mathbb{E}[E_s^p \mathbf{1}_A]$ for all $s \in \mathcal{T}$, $t \ge s$ and $A \in \mathcal{F}_s$.

However, for our purposes there are some limitations in considering solutions satisfying some kind of energy inequality, since (i) it does not seem immediate to recover uniform bounds in $\varepsilon \in (0, 1)$, and (ii) we do not need an energy inequality but just energy bounds, and recent developments in convex integration suggest that the class of weak solutions to Navier–Stokes equations with bounded energy may be strictly larger than the class of Leray–Hopf weak solutions; see [16] for a deterministic result and [45] for a stochastic one (even though the solutions constructed there are not known to satisfy H^1 bounds in the space variable).

In order to construct a bounded-energy family of weak martingale solutions to (1.3), one can make use of classical compactness arguments involving the Galerkin approximation scheme:

$$\begin{cases} du_t^{\varepsilon,n} = Au_t^{\varepsilon,n}\, dt + \Pi_n b(u_t^{\varepsilon,n}, u_t^{\varepsilon,n})\, dt + \varepsilon^{-1/2}\Pi_n b(y_t^{\varepsilon,n}, u_t^{\varepsilon,n})\, dt, \\ dy_t^{\varepsilon,n} = \varepsilon^{-1}Cy_t^{\varepsilon,n}\, dt + Ay_t^{\varepsilon,n} dt + \Pi_n b(u_t^{\varepsilon,n}, y_t^{\varepsilon,n})\, dt \\ \qquad + \varepsilon^{-1/2}\Pi_n b(y_t^{\varepsilon,n}, y_t^{\varepsilon,n})\, dt + \varepsilon^{-1/2}\Pi_n Q^{1/2}\, dW_t, \end{cases} \tag{2.2}$$

where $\{\Pi_n\}_{n\in\mathbb{N}}$ is a family of Galerkin projectors and the initial condition is $(u_0^{\varepsilon,n}, y_0^{\varepsilon,n}) = (\Pi_n u_0, \Pi_n y_0)$. Indeed, since solutions of (2.2) above are smooth in space, uniform energy estimates (S3)–(S4) can be rigorously proved for $(u^{\varepsilon,n}, y^{\varepsilon,n})$ making use of the Itô formula; then, for every fixed $\varepsilon \in (0, 1)$, one can prove via the Ascoli–Arzelà theorem that there exist $u^\varepsilon, y^\varepsilon$ such that $u^{\varepsilon,n} \to u^\varepsilon$ and $y^{\varepsilon,n} \to y^\varepsilon$ with respect to a topology that permits taking the limit in the energy estimates (S3)–(S4), on the one hand, and in the weak formulation of (S2), on the other (up to a possible change in the underlying stochastic basis, in order to gain adaptedness of the processes $u^\varepsilon, y^\varepsilon$).

Proposition 2.3. *There exists a bounded-energy family* $\{(u^\varepsilon, y^\varepsilon)\}_{\varepsilon\in(0,1)}$ *of weak martingale solutions to* (1.3).

Existence of a weak martingale solution for fixed $\varepsilon \in (0,1)$ has been known since [29]. The only difference here is uniform-in-ε energy bounds, which require suitable estimates at the level of Galerkin truncations (see Lemmas 2.5 and 2.6 below) and compactness arguments well-suited to the passage to the limit $n \to \infty$. For the sake of completeness, we present a proof of Proposition 2.3 in Appendix A. Here we limit ourselves to showing the needed energy bounds.

Remark 2.4. Notice that if (1.3) admits pathwise uniqueness then one has existence of a probabilistically strong solution, namely the stochastic basis can be taken independent of ε.

Lemma 2.5. *For every positive integer* p*,*

$$\sup_{\substack{\varepsilon\in(0,1)\\ n\in\mathbb{N}}} \int_0^T \mathbb{E}[\|y_s^{\varepsilon,n}\|_H^{2p-2}\|y_s^{\varepsilon,n}\|_{H^\gamma}^2]\,ds \lesssim 1.$$

Proof. Let $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$ be fixed, and take an arbitrary $t \in [0,T]$. Applying the Itô formula to $\frac{1}{2}\|y_t^{\varepsilon,n}\|_H^{2p}$ we get

$$\begin{aligned}\frac{1}{2}\|y_t^{\varepsilon,n}\|_H^{2p} + \varepsilon^{-1}p\int_0^t \|y_s^{\varepsilon,n}\|_H^{2p-2}\|(-C)^{1/2}y_s^{\varepsilon,n}\|_H^2\,ds + p\int_0^t \|y_s^{\varepsilon,n}\|_H^{2p-2}\|y_s^{\varepsilon,n}\|_{H^1}^2\,ds \\ = \frac{1}{2}\|\Pi_n y_0\|_H^{2p} + \varepsilon^{-1/2}p\int_0^t \|\Pi_n y_s^\varepsilon\|_H^{2p-2}\langle y_s^{\varepsilon,n}, \Pi_n Q^{1/2}dW_s\rangle \\ + \varepsilon^{-1}\frac{p(2p-1)}{2}\operatorname{Tr}(\Pi_n Q\Pi_n)\int_0^t \|y_s^{\varepsilon,n}\|_H^{2p-2}\,ds.\end{aligned}$$

Taking expectations in the expression above with $p = 1$ we obtain

$$\varepsilon^{-1}\int_0^t \mathbb{E}[\|y_s^{\varepsilon,n}\|_{H^\gamma}^2]\,ds \le \frac{1}{2M}\|y_0\|_H^2 + \varepsilon^{-1}\frac{\operatorname{Tr}(Q)}{2M}t,$$

where we have used $\|(-C)^{1/2}y_s^{\varepsilon,n}\|_H^2 \ge M\|y_s^{\varepsilon,n}\|_{H^\gamma}^2$ for some unimportant constant M; thus we deduce

$$\sup_{\substack{\varepsilon\in(0,1)\\ n\in\mathbb{N}}} \int_0^T \mathbb{E}[\|y_s^{\varepsilon,n}\|_{H^\gamma}^2]\,ds \le \frac{\varepsilon}{2M}\|y_0\|_H^2 + \frac{\operatorname{Tr}(Q)T}{2M} \lesssim 1.$$

For $p > 1$, we argue as follows: first, recalling $\|y\|_{H^\gamma}^2 \ge \nu_1^\gamma\|y\|_H^2$ for some $\nu_1 > 0$ (the principal eigenvalue of the operator $-A$), for every $t \in [0,T]$ we have

$$\begin{aligned}\int_0^t \mathbb{E}[\|y_s^{\varepsilon,n}\|_H^{2p-2}\|y_s^{\varepsilon,n}\|_{H^\gamma}^2]\,ds &\le \frac{\varepsilon}{2pM}\|y_0\|_H^{2p} + \frac{2p-1}{2M}\operatorname{Tr}(Q)\int_0^t \mathbb{E}[\|y_s^{\varepsilon,n}\|_H^{2p-2}]\,ds \\ &\le \frac{\varepsilon}{2pM}\|y_0\|_H^{2p} + \frac{2p-1}{2M}\nu_1^\gamma\operatorname{Tr}(Q)\int_0^t \mathbb{E}[\|y_s^{\varepsilon,n}\|_H^{2p-4}\|y_s^{\varepsilon,n}\|_{H^\gamma}^2]\,ds;\end{aligned}$$

then, since $p-1$ is a positive integer, by induction we get the desired inequality uniformly in $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$. ∎

Lemma 2.6. *For every $p \geq 2$,*

$$\sup_{\substack{\varepsilon\in(0,1)\\ n\in\mathbb{N}}} \sup_{t\in[0,T]} \left(\mathbb{E}[\|y_t^{\varepsilon,n}\|_H^p] + \int_0^t \mathbb{E}[\|y_s^{\varepsilon,n}\|_H^{p-2}\|y_s^{\varepsilon,n}\|_{H^1}^2]\, ds \right) \lesssim 1.$$

Proof. As in Lemma 2.5, it is sufficient to prove the result for every positive even integer p. Let us introduce the auxiliary process $Y_t^{\varepsilon,n}$ solving

$$dY_t^{\varepsilon,n} = \varepsilon^{-1} C_\varepsilon Y_t^{\varepsilon,n} dt + \varepsilon^{-1/2}\Pi_n Q^{1/2} dW_s, \quad Y_0^{\varepsilon,n} = 0,$$

so that, by the Itô formula, the difference process $\zeta_t^{\varepsilon,n} := y_t^{\varepsilon,n} - Y_t^{\varepsilon,n}$ satisfies, for all $t \in [0,T]$ and $p \geq 2$,

$$\begin{aligned}
\|\zeta_t^{\varepsilon,n}\|_H^p &+ \varepsilon^{-1} pM \int_0^t \|\zeta_s^{\varepsilon,n}\|_H^{p-2}\|\zeta_s^{\varepsilon,n}\|_{H^\gamma}^2\, ds + p\int_0^t \|\zeta_s^{\varepsilon,n}\|_H^{p-2}\|\zeta_s^{\varepsilon,n}\|_{H^1}^2\, ds \\
&\leq \|y_0\|_H^p + p\int_0^t \|\zeta_s^{\varepsilon,n}\|_H^{p-2}\langle b(u_s^{\varepsilon,n}, Y_s^{\varepsilon,n}), \zeta_s^{\varepsilon,n}\rangle\, ds \\
&\quad + \varepsilon^{-1/2} p \int_0^t \|\zeta_s^{\varepsilon,n}\|_H^{p-2}\langle b(y_s^{\varepsilon,n}, Y_s^{\varepsilon,n}), \zeta_s^{\varepsilon,n}\rangle\, ds \\
&\leq \|y_0\|^p + M_1 \int_0^t \|\zeta_s^{\varepsilon,n}\|_H^{p-1}\|u_s^{\varepsilon,n}\|_H\|Y_s^{\varepsilon,n}\|_{H^{\theta_0}}\, ds \\
&\quad + \varepsilon^{-1/2} M_1 \int_0^t \|\zeta_s^{\varepsilon,n}\|_H^{p-1}\|y_s^{\varepsilon,n}\|_H\|Y_s^{\varepsilon,n}\|_{H^{\theta_0}}\, ds, \qquad (2.3)
\end{aligned}$$

where M_1 is another unimportant constant. By the Young inequality,

$$\|Y_s^{\varepsilon,n}\|_{H^{\theta_0}}\|\zeta_s^{\varepsilon,n}\|_H^{p-1} \leq \frac{\|Y_s^{\varepsilon,n}\|_{H^{\theta_0}}^p}{p} + \frac{p-1}{p}\|\zeta_s^{\varepsilon,n}\|_H^p,$$

and for every positive constant c,

$$\begin{aligned}
\varepsilon^{-1/2}\|y_s^{\varepsilon,n}\|_H\|Y_s^{\varepsilon,n}\|_{H^{\theta_0}}\|\zeta_s^{\varepsilon,n}\|_H^{p-1} &\leq \frac{c^{-p}}{2p}\|y_s^{\varepsilon,n}\|_H^{2p} + \frac{c^{-p}}{2p}\|Y_s^{\varepsilon,n}\|_{H^{\theta_0}}^{2p} \\
&\quad + \varepsilon^{-\frac{p}{2(p-1)}}\frac{(p-1)c^{\frac{p}{p-1}}}{p}\|\zeta_s^{\varepsilon,n}\|_H^p.
\end{aligned}$$

Choosing $c = \left(\frac{p^2 M}{2(p-1)\nu_1^\gamma M_1}\right)^{\frac{p-1}{p}}$, the previous inequalities can be plugged into (2.3) to get

$$\begin{aligned}
\|\zeta_t^{\varepsilon,n}\|_H^p &+ \varepsilon^{-1}\int_0^t \|\zeta_s^{\varepsilon,n}\|_H^{p-2}\|\zeta_s^{\varepsilon,n}\|_{H^\gamma}^2\, ds + \int_0^t \|\zeta_s^{\varepsilon,n}\|_H^{p-2}\|\zeta_s^{\varepsilon,n}\|_{H^1}^2\, ds \\
&\lesssim \|y_0\|_H^p + \int_0^t \|Y_s^{\varepsilon,n}\|_{H^{\theta_0}}^p ds + \int_0^t \|y_s^{\varepsilon,n}\|_H^{2p}\, ds + \int_0^t \|Y_s^{\varepsilon,n}\|_{H^{\theta_0}}^{2p}\, ds. \qquad (2.4)
\end{aligned}$$

Since $\mathbb{E}[\|Y_s^{\varepsilon,n}\|_{H^{\theta_0}}^{2p}]$ is bounded uniformly in ε, n and s by assumption (Q2), and invoking Lemma 2.5, the previous inequality produces bounds for $\zeta^{\varepsilon,n}$:

$$\sup_{\substack{\varepsilon\in(0,1)\\ n\in\mathbb{N}}} \sup_{t\in[0,T]} \left(\mathbb{E}[\|\zeta_t^{\varepsilon,n}\|_H^p] + \int_0^t \mathbb{E}[\|\zeta_s^{\varepsilon,n}\|_H^{p-2}\|\zeta_s^{\varepsilon,n}\|_{H^1}^2]\,ds\right) \lesssim 1, \tag{2.5}$$

$$\sup_{\substack{\varepsilon\in(0,1)\\ n\in\mathbb{N}}} \varepsilon^{-1}\int_0^T \mathbb{E}[\|\zeta_s^{\varepsilon,n}\|_H^{p-2}\|\zeta_s^{\varepsilon,n}\|_{H^\gamma}^2]\,ds \lesssim 1. \tag{2.6}$$

Since $y^{\varepsilon,n} = Y^{\varepsilon,n} + \zeta^{\varepsilon,n}$, from (2.5) we deduce the assertion. ∎

Lemma 2.6 gives the uniform-in-ε bounds necessary for the proof of Proposition 2.3. As a by-product we have also obtained (2.6), which permits us to control the difference between the Galerkin approximation of the small-scale process $y^{\varepsilon,n}$ and its linearized counterpart $Y^{\varepsilon,n}$ in the Sobolev space H^γ. Recall that we have assumed $\gamma > 1/4$, and we can assume $\gamma \in (1/4, 1]$ without loss of generality. A close inspection of the proof of Proposition 2.3 in Appendix A shows that the bound (2.6) is stable under the passage to the limit $n \to \infty$; therefore, we can deduce the following result.

Proposition 2.7. *Let $\{(u^\varepsilon, y^\varepsilon)\}_{\varepsilon\in(0,1)}$ be a bounded-energy family of weak martingale solutions to* (1.3)*. For every $\varepsilon \in (0,1)$ let Y^ε be the unique strong solution of*

$$dY_t^\varepsilon = \varepsilon^{-1}C_\varepsilon Y_t^\varepsilon dt + \varepsilon^{-1/2}Q^{1/2}dW_s, \quad Y_0^\varepsilon = 0. \tag{2.7}$$

Then

$$\sup_{\varepsilon\in(0,1)} \varepsilon^{-1}\int_0^T \mathbb{E}[\|y_s^\varepsilon - Y_s^\varepsilon\|_{H^\gamma}^2]\,ds \lesssim 1.$$

This result will be fundamental in performing the linearization trick presented in Section 1.1; see also Proposition 4.2.

3. Quadratic functions and solution to the Poisson equation

Recall that, as already discussed in Section 1.1, we shall define correctors φ_1^ε, φ_2^ε as solutions to certain Poisson equations $\mathscr{L}_y^\varepsilon\phi = -\psi$. In this section we develop the technology needed to solve the Poisson equation for a class of functions ψ that is large enough for our purposes, namely the class of quadratic functions on Sobolev spaces H^θ. Moreover, we also provide partial regularity estimates for the solution ϕ so obtained in terms of analogous bounds on ψ, showing improved regularity (see Corollary 3.5). To avoid any confusion, we point out that all the estimates in the present section are uniform in $\varepsilon \in (0,1)$.

3.1. Quadratic functions

Denote by $\mathcal{E}_\theta \subset L^2(H,\mu^\varepsilon)$ for $\theta \in (-\infty, \theta_0]$ the space of quadratic functions $\psi : H^\theta \to \mathbb{R}$, that is, $\psi \in \mathcal{E}_\theta$ if there exist $a_0 \in \mathbb{R}$, $a_1 : H^\theta \to \mathbb{R}$ linear and bounded, and $a_2 : H^\theta \times H^\theta \to \mathbb{R}$ bilinear, symmetric and bounded such that $\psi(y) = a_0 + a_1(y) + a_2(y,y)$ for every $y \in H^\theta$. The inclusion in $L^2(H, \mu^\varepsilon)$ holds true by (Q2). Notice that every $\psi \in \mathcal{E}_\theta$ admits a unique decomposition $\psi = a_0 + a_1 + a_2$: indeed, $\psi(ry) = a_0 + ra_1(y) + r^2 a_2(y,y)$, and therefore

$$a_0 = \psi(0), \quad a_1(y) = \frac{d}{dr}\psi(ry)\bigg|_{r=0},$$

and since $a_2(y,y) = \psi(y) - a_1(y) - a_0$, the quadratic form $a_2(y,y)$ is uniquely defined, and hence so is the associated symmetric bilinear map, via the polarization formula.

For future purposes define

$$\begin{aligned}\|\psi\|_{\mathcal{E}_\theta} &:= |a_0| + \|a_1\|_{H^\theta \to \mathbb{R}} + \|a_2\|_{H^\theta \times H^\theta \to \mathbb{R}} \\ &= |a_0| + \sup_{\substack{y \in H^\theta \\ \|y\|_{H^\theta}=1}} |a_1(y)| + \sup_{\substack{y,y' \in H^\theta \\ \|y\|_{H^\theta} = \|y'\|_{H^\theta}=1}} |a_2(y,y')|.\end{aligned}$$

Then $\mathcal{E}_\theta$ is a Banach space when endowed with the norm $\|\cdot\|_{\mathcal{E}_\theta}$, and $\mathcal{E}_\theta \subset \mathcal{E}_{\theta'}$ with continuous embedding if $\theta \le \theta'$. As a notational convention, denote $\mathcal{E} := \mathcal{E}_0$.

Lemma 3.1. *Let $\psi \in \mathcal{E}$. Then for every $t \ge 0$ we have $P^\varepsilon_t \psi \in D(\mathcal{L}^\varepsilon_y)$ and $\mathcal{L}^\varepsilon_y P^\varepsilon_t \psi = P^\varepsilon_t \mathcal{L}^\varepsilon_y \psi$.*

Proof. By the Markov property, $P^\varepsilon_s P^\varepsilon_t \psi = P^\varepsilon_{t+s}\psi$. Recalling that Y^y is a strong solution of $dY^y_t = C_\varepsilon Y^y_t dt + Q^{1/2} dW_t$, $Y^y_0 = y$, by the Itô formula we have

$$\begin{aligned}P^\varepsilon_s P^\varepsilon_t \psi - P^\varepsilon_t \psi &= \mathbb{E}[\psi(Y^\cdot_{t+s})] - \mathbb{E}[\psi(Y^\cdot_t)] \\ &= \int_t^{t+s} \mathbb{E}\left[\langle C_\varepsilon Y^\cdot_r, D_y \psi(Y^\cdot_r)\rangle + \tfrac{1}{2}\operatorname{Tr}(Q D^2_y \psi(Y^\cdot_r))\right] dr.\end{aligned}$$

Let $\psi(y) = a_0 + a_1(y) + a_2(y,y)$ be the canonical decomposition of $\psi \in \mathcal{E}$. Since

$$\int_H \|C_\varepsilon y\|_H \|y\|_H \, d\mu^\varepsilon(y) < \infty$$

uniformly in ε by our assumptions on C and Q, we have

$$\begin{aligned}\bar{\psi} &:= \langle C_\varepsilon y, D_y \psi(y)\rangle + \tfrac{1}{2}\operatorname{Tr}(Q D^2_y \psi) \\ &= a_1(C_\varepsilon y) + 2a_2(C_\varepsilon y, y) + \tfrac{1}{2}\operatorname{Tr}(Q D^2_y a_2) \in L^2(H, \mu^\varepsilon).\end{aligned}$$

In addition, the semigroup $P_\cdot \bar{\psi}$ is right continuous at time t, with respect to the $L^2(H,\mu^\varepsilon)$ topology, and therefore as $s \to 0^+$,

$$\begin{aligned}\left\| \frac{P^\varepsilon_s P^\varepsilon_t \psi - P^\varepsilon_t \psi}{s} - P^\varepsilon_t \bar{\psi} \right\|_{L^2(H,\mu^\varepsilon)} &= \left\| \frac{1}{s}\int_t^{t+s} P^\varepsilon_r \bar{\psi}\, dr - P^\varepsilon_t \bar{\psi} \right\|_{L^2(H,\mu^\varepsilon)} \\ &\le \frac{1}{s}\int_t^{t+s} \|P^\varepsilon_r \bar{\psi} - P^\varepsilon_t \bar{\psi}\|_{L^2(H,\mu^\varepsilon)}\, dr \to 0.\end{aligned}$$

In particular, this means $P^\varepsilon_t \psi \in D(\mathcal{L}^\varepsilon_y)$ and $\mathcal{L}^\varepsilon_y P^\varepsilon_t \psi = P^\varepsilon_t \bar{\psi}$. Finally, taking $t = 0$ in the previous formula we deduce $\mathcal{L}^\varepsilon_y \psi = \bar{\psi}$, which yields $\mathcal{L}^\varepsilon_y P^\varepsilon_t \psi = P^\varepsilon_t \bar{\psi} = P^\varepsilon_t \mathcal{L}^\varepsilon_y \psi$. ∎

Lemma 3.2. *The semigroup P_t^ε is exponentially mixing at zero when restricted to $\mathcal{E}_\theta$ for $\theta \in (-\infty, \theta_0]$: for every $\psi \in \mathcal{E}_\theta$ and $t \geq 0$,*

$$\left| P_t^\varepsilon \psi(0) - \int_H \psi(w)\, d\mu^\varepsilon(w) \right| \lesssim \|\psi\|_{\mathcal{E}_\theta} e^{-\lambda_0 t}.$$

Proof. Let $\psi \in \mathcal{E}_\theta$ be given by $\psi(y) = a_0 + a_1(y) + a_2(y,y)$ for $y \in H^\theta$. Recall

$$\begin{aligned} P_t^\varepsilon \psi(y) &= a_0 + \mathbb{E}[a_2(W_t^{C_\varepsilon,Q}, W_t^{C_\varepsilon,Q})] + a_1(e^{C_\varepsilon t} y) + a_2(e^{C_\varepsilon t} y, e^{C_\varepsilon t} y) \\ &= P_t^\varepsilon \psi(0) + a_1(e^{C_\varepsilon t} y) + a_2(e^{C_\varepsilon t} y, e^{C_\varepsilon t} y), \end{aligned}$$

and therefore

$$\begin{aligned} |P_t^\varepsilon \psi(0) - P_t^\varepsilon \psi(y)| &\leq |a_1(e^{C_\varepsilon t} y)| + |a_2(e^{C_\varepsilon t} y, e^{C_\varepsilon t} y)| \\ &\lesssim \|\psi\|_{\mathcal{E}_\theta} e^{-\lambda_0 t} \|y\|_{H^\theta} + \|\psi\|_{\mathcal{E}_\theta} e^{-2\lambda_0 t} \|y\|_{H^\theta}^2. \end{aligned}$$

Since μ^ε is invariant for P_t^ε,

$$\begin{aligned} &\left| P_t^\varepsilon \psi(0) - \int_H \psi(w)\, d\mu^\varepsilon(w) \right| \\ &\quad = |P_t^\varepsilon \psi(0) - \int_H P_t^\varepsilon \psi(w)\, d\mu^\varepsilon(w)| \leq \int_H |P_t^\varepsilon \psi(0) - P_t^\varepsilon \psi(w)|\, d\mu^\varepsilon(w) \\ &\quad \lesssim \|\psi\|_{\mathcal{E}_\theta} e^{-\lambda_0 t} \int_H \|w\|_{H^\theta}\, d\mu^\varepsilon(w) + \|\psi\|_{\mathcal{E}_\theta} e^{-2\lambda_0 t} \int_H \|w\|_{H^\theta}^2\, d\mu^\varepsilon(w) \\ &\quad \lesssim \|\psi\|_{\mathcal{E}_\theta} e^{-\lambda_0 t}. \end{aligned}$$

∎

3.2. Solution to the Poisson equation

Recall that the operator $\mathscr{L}_y^\varepsilon$ is closed under our assumptions, and therefore the space $D(\mathscr{L}_y^\varepsilon)$ is complete when endowed with the graph norm

$$\|\psi\|_{D(\mathscr{L}_y^\varepsilon)} := \|\psi\|_{L^2(H,\mu^\varepsilon)} + \|\mathscr{L}_y^\varepsilon \psi\|_{L^2(H,\mu^\varepsilon)}.$$

Also, by Lemma 3.1 we have $\mathcal{E} \subset D(\mathscr{L}_y^\varepsilon)$ with continuous embedding, since

$$\begin{aligned} \|\psi\|_{D(\mathscr{L}_y^\varepsilon)}^2 &\lesssim \int_H |\psi(w)|^2\, d\mu^\varepsilon(w) + \int_H |\langle C_\varepsilon w, D_y \psi(w)\rangle|^2\, d\mu^\varepsilon(w) + \mathrm{Tr}(Q D_y^2 \psi)^2 \\ &\lesssim \|\psi\|_{\mathcal{E}}^2 \left(\int_H \|w\|_H^4\, d\mu^\varepsilon(w) + \int_H \|w\|_{H^{2\Gamma}}^2 (1 + \|w\|_H)^2\, d\mu^\varepsilon(w) + \mathrm{Tr}(Q)^2 \right) \\ &\lesssim \|\psi\|_{\mathcal{E}}^2 \left(\int_H \|w\|_H^2\, d\mu^\varepsilon(w) + \int_H \|w\|_{H^{2\Gamma}}^2\, d\mu^\varepsilon(w) + \mathrm{Tr}(Q) \right)^2 \\ &\lesssim \|\psi\|_{\mathcal{E}}^2. \end{aligned}$$

In the third inequality above we have used the fact that, for Gaussian measures on H, the fourth moment is controlled by the square of the second moment (up to unimportant multiplicative constants).

Lemma 3.3. *Let $\psi \in \mathcal{E}_\theta$ with $\theta \in [0, \theta_0)$ be given by $\psi(y) = a_1(y)$. Then for every $T > 0$ we have $\psi(e^{C_\varepsilon T} \cdot) \in \mathcal{E}$, $\int_{1/T}^T \psi(e^{C_\varepsilon t} \cdot)\, dt \in \mathcal{E}$, and*

$$\lim_{T\to\infty} \int_{1/T}^T \psi(e^{C_\varepsilon t} \cdot)\, dt = \psi((-C_\varepsilon)^{-1} \cdot) \quad \textit{in the } D(\mathscr{L}_y^\varepsilon) \textit{ topology.}$$

Proof. First of all, there exists a vector $\mathbf{a}_1 \in H^{-\theta}$ such that $a_1(y) = \langle y, \mathbf{a}_1\rangle$ for all $y \in H^\theta$. Hence, for every $T > 0$ we have $\psi(e^{C_\varepsilon T} \cdot) \in \mathcal{E}$ since

$$|\psi(e^{C_\varepsilon T} y)| = |\langle e^{C_\varepsilon T} y, \mathbf{a}_1\rangle| \le \|e^{C_\varepsilon T} y\|_{H^\theta} \|\mathbf{a}_1\|_{H^{-\theta}} \lesssim e^{-\lambda_0 T/(2\gamma)} T^{-\theta/2} \|y\|_H \|\mathbf{a}_1\|_{H^{-\theta}}.$$

As a consequence, $\int_{1/T}^T \psi(e^{C_\varepsilon t} \cdot)\, dt \in \mathcal{E} \subset D(\mathscr{L}_y^\varepsilon)$ as well, since

$$\left\| \int_{1/T}^T \psi(e^{C_\varepsilon t} \cdot)\, dt \right\|_{\mathcal{E}} \le \int_{1/T}^T \|\psi(e^{C_\varepsilon t} \cdot)\|_{\mathcal{E}}\, dt < \infty.$$

Let us finally check $\int_{1/T}^T \psi(e^{C_\varepsilon t} \cdot)\, dt \to \psi((-C_\varepsilon)^{-1} \cdot)$ in $D(\mathscr{L}_y^\varepsilon)$ as $T \to \infty$. For every $T > 0$ and $y \in H$ we have

$$\begin{aligned} \int_{1/T}^T \psi(e^{C_\varepsilon t} y)\, dt &= \int_{1/T}^T \langle e^{C_\varepsilon t} y, \mathbf{a}_1\rangle\, dt = \left\langle \left(\int_{1/T}^T e^{C_\varepsilon t} dt \right) y, \mathbf{a}_1 \right\rangle \\ &= \langle (e^{C_\varepsilon /T} - e^{C_\varepsilon T})(-C_\varepsilon)^{-1} y, \mathbf{a}_1\rangle, \end{aligned}$$

and therefore we only have to check that $\langle (e^{C_\varepsilon/T} - e^{C_\varepsilon T} - 1)(-C_\varepsilon)^{-1} \cdot, \mathbf{a}_1\rangle \to 0$ and $\langle (e^{C_\varepsilon/T} - e^{C_\varepsilon T} - 1)\cdot, \mathbf{a}_1\rangle \to 0$ in $L^2(H, \mu^\varepsilon)$ (recall that $D_y^2 \psi = 0$ since ψ is linear). We only prove the latter convergence, the former being easier:

$$\begin{aligned} &\int_H |\langle (e^{C_\varepsilon/T} - e^{C_\varepsilon T} - 1) w, \mathbf{a}_1\rangle|^2\, d\mu^\varepsilon(w) \\ &\lesssim \int_H |\langle (e^{C_\varepsilon/T} - 1) w, \mathbf{a}_1\rangle|^2\, d\mu^\varepsilon(w) + \int_H |\langle e^{C_\varepsilon T} w, \mathbf{a}_1\rangle|^2\, d\mu^\varepsilon(w) \\ &\le \int_H \|(e^{C_\varepsilon/T} - 1) w\|_{H^\theta}^2 \|\mathbf{a}_1\|_{H^{-\theta}}^2\, d\mu^\varepsilon(w) + \int_H \|e^{C_\varepsilon T} w\|_{H^\theta}^2 \|\mathbf{a}_1\|_{H^{-\theta}}^2\, d\mu^\varepsilon(w) \\ &\lesssim T^{(\theta-\theta_0)/\Gamma} \|\mathbf{a}_1\|_{H^{-\theta}}^2 \int_H \|w\|_{H^{\theta_0}}^2\, d\mu^\varepsilon(w) + e^{-2\lambda_0 T} \|\mathbf{a}_1\|_{H^{-\theta}}^2 \int_H \|w\|_{H^\theta}^2\, d\mu^\varepsilon(w) \to 0. \end{aligned}$$

■

Lemma 3.4. *Let $\psi \in \mathcal{E}_\theta$ with $\theta \in [0, \theta_0)$ be given by $\psi(y) = a_2(y, y)$. Then for every $T > 0$ we have $\psi(e^{C_\varepsilon T} \cdot) \in \mathcal{E}$, $\int_{1/T}^T \psi(e^{C_\varepsilon t} \cdot)\, dt \in \mathcal{E}$, and there exists $\phi \in \mathcal{E}_{\theta-\delta} \cap D(\mathscr{L}_y^\varepsilon)$ for every $\delta \in (0, \gamma)$ satisfying $\|\phi\|_{\mathcal{E}_{\theta-\delta}} \lesssim \|\psi\|_{\mathcal{E}_\theta}$ such that*

$$\lim_{T\to\infty} \int_{1/T}^T \psi(e^{C_\varepsilon t} \cdot)\, dt = \phi \quad \textit{in the } D(\mathscr{L}_y^\varepsilon) \textit{ topology.}$$

Proof. First of all, there exists a bounded linear operator $A_2 : H^\theta \to H^{-\theta}$ such that $a_2(y, v) = \langle y, A_2 v\rangle = \langle A_2 y, v\rangle$ for all $y, v \in H^\theta$, and $\|\psi\|_{H^\theta} = \|A_2\|_{H^\theta \to H^{-\theta}}$. Then for every $T > 0$ we have $\psi(e^{C_\varepsilon T}\cdot) \in \mathcal{E}$ since

$$|\psi(e^{C_\varepsilon T} y)| \le \|\psi\|_{H^\theta} \|e^{C_\varepsilon T} y\|_{H^\theta}^2 \lesssim \|\psi\|_{H^\theta} e^{-2\lambda_0 T} T^{-\theta/\gamma} \|y\|_H^2,$$

and similarly $\int_{1/T}^T \psi(e^{C_\varepsilon t}\cdot)\,dt \in \mathcal{E}$. In addition, for all $T > 0$ and $y \in H^\theta$,

$$\int_{1/T}^T \psi(e^{C_\varepsilon t} y)\,dt = \int_{1/T}^T \langle e^{C_\varepsilon t} y, A_2 e^{C_\varepsilon t} y\rangle\,dt = \Big\langle y, \Big(\int_{1/T}^T e^{C_\varepsilon t} A_2 e^{C_\varepsilon t}\,dt\Big) y\Big\rangle,$$

and since for all $\delta_1, \delta_2 \ge 0$ satisfying $\delta_1 + \delta_2 = \delta < \gamma$ we have

$$\int_0^\infty \|e^{C_\varepsilon t} A_2 e^{C_\varepsilon t}\|_{H^{\theta-2\delta_1} \to H^{2\delta_2-\theta}}\,dt \lesssim \|A_2\|_{H^\theta \to H^{-\theta}} \int_0^\infty \frac{e^{-\lambda_0 t}}{t^{\delta/\gamma}}\,dt < \infty,$$

there exists a bounded linear operator $A_2^\infty : H^{\theta-2\delta_1} \to H^{2\delta_2-\theta}$ such that

$$\int_{1/T}^T e^{C_\varepsilon t} A_2 e^{C_\varepsilon t}\,dt \to A_2^\infty$$

strongly as $T \to \infty$ when $\delta_1 + \delta_2 = \delta < \gamma$, and $\langle y, A_2^\infty v\rangle = \langle A_2^\infty y, v\rangle$ for all $y, v \in H^{\theta-\delta}$. In particular, using $\delta_1 = \delta_2 = \delta/2$, we can define $\phi \in \mathcal{E}_{\theta-\delta}$ by

$$\phi(y) = \langle y, A_2^\infty y\rangle, \quad y \in H^{\theta-\delta},$$

which of course satisfies $\|\phi\|_{\mathcal{E}_{\theta-\delta}} = \|A_2^\infty\|_{H^{\theta-\delta} \to H^{\delta-\theta}} \lesssim \|A_2\|_{H^\theta \to H^{-\theta}} = \|\psi\|_{\mathcal{E}_\theta}$. Let us now check $\phi \in D(\mathscr{L}_y^\varepsilon)$: we have, for every $y \in H^\theta$,

$$\langle C_\varepsilon y, D_y \phi(y)\rangle = 2\langle C_\varepsilon y, A_2^\infty y\rangle = -\langle y, A_2 y\rangle,$$

where the last equality comes from integration by parts. Also, given a complete orthonormal system $\{e_k\}_{k\in\mathbb{N}}$ of H and choosing $\delta \in (0, \gamma)$ such that $\theta - \delta \le \theta_0 - \gamma$, by (Q2) we have

$$\begin{aligned}\tfrac{1}{2}\operatorname{Tr}(Q D_y^2 \phi) &= \sum_{k\in\mathbb{N}} \langle Q^{1/2} e_k, A_2^\infty Q^{1/2} e_k\rangle \\ &\le \|A_2^\infty\|_{H^{\theta-\delta} \to H^{\delta-\theta}} \sum_{k\in\mathbb{N}} \|Q^{1/2} e_k\|_{H^{\theta-\delta}}^2 < \infty.\end{aligned}$$

Altogether, we get

$$\begin{aligned}\|\phi\|_{D(\mathscr{L}_y)}^2 &\lesssim \int_H |\phi(w)|^2\,d\mu^\varepsilon(w) + \int_H |\langle w, A_2 w\rangle|^2\,d\mu^\varepsilon(w) + [\operatorname{Tr}(Q D_y^2 \phi)]^2 \\ &\lesssim \|A_2^\infty\|_{H^{\theta-\delta} \to H^{\delta-\theta}}^2 \Big(\int_H \|w\|_{H^{\theta-\delta}}^2\,d\mu^\varepsilon(w)\Big)^2 \\ &\quad + \|A_2\|_{H^\theta \to H^{-\theta}}^2 \Big(\int_H \|w\|_{H^\theta}^2\,d\mu^\varepsilon(w)\Big)^2 \\ &\quad + [\operatorname{Tr}(Q D_y^2 \phi)]^2 < \infty.\end{aligned}$$

Let us finally prove $\lim_{T\to\infty}\int_{1/T}^{T}\psi(e^{C_\varepsilon t}\cdot)\,dt=\phi$ in the $D(\mathscr{L}_y^\varepsilon)$ topology. To ease the notation, denote $A_2^T=\int_{1/T}^T e^{C_\varepsilon t}A_2e^{C_\varepsilon t}\,dt$, so that $\int_{1/T}^T\psi(e^{C_\varepsilon t}y)\,dt=\langle y,A_2^Ty\rangle$ for every $y\in H$. First, we have

$$\begin{aligned}\left\|\int_{1/T}^{T}\psi(e^{C_\varepsilon t}\cdot)\,dt-\phi\right\|^2_{L^2(H,\mu^\varepsilon)}&=\int_H|\langle w,(A_2^T-A_2^\infty)w\rangle|^2\,d\mu^\varepsilon(w)\\&\lesssim\|A_2^T-A_2^\infty\|^2_{H^{\theta-\delta}\to H^{\delta-\theta}}\left(\int_H\|w\|^2_{H^{\theta-\delta}}\,d\mu^\varepsilon(w)\right)^2\to0\end{aligned}$$

as $T\to\infty$, since $A_2^T\to A_2^\infty$ strongly. Second,

$$\begin{aligned}\mathscr{L}_y^\varepsilon\left(\int_{1/T}^T\psi(e^{C_\varepsilon t}\cdot)\,dt\right)(y)&=2\langle C_\varepsilon y,A_2^Ty\rangle+\mathrm{Tr}(QA_2^T)\\&=\langle y,(e^{C_\varepsilon T}A_2e^{C_\varepsilon T}-e^{C_\varepsilon/T}A_2e^{C_\varepsilon/T})y\rangle+\mathrm{Tr}(QA_2^T)\end{aligned}$$

and

$$\mathscr{L}_y^\varepsilon\phi(y)=-\langle y,A_2y\rangle+\mathrm{Tr}(QA_2^\infty),$$

from which we get

$$\begin{aligned}&\left\|\mathscr{L}_y^\varepsilon\left(\int_{1/T}^T\psi(e^{C_\varepsilon t}\cdot)\,dt\right)-\mathscr{L}_y^\varepsilon\phi\right\|^2_{L^2(H,\mu^\varepsilon)}\\&\quad\lesssim\int_H|\langle w,(e^{C_\varepsilon T}A_2e^{C_\varepsilon T}+A_2-e^{C_\varepsilon/T}A_2e^{C_\varepsilon/T})w\rangle|^2\,d\mu^\varepsilon(w)\\&\qquad+|\mathrm{Tr}(QA_2^T)-\mathrm{Tr}(QA_2^\infty)|^2\\&\quad\lesssim\int_H|\langle w,e^{C_\varepsilon T}A_2e^{C_\varepsilon T}w\rangle|^2\,d\mu^\varepsilon(w)+\int_H|\langle w,A_2(1-e^{C_\varepsilon/T})w\rangle|^2\,d\mu^\varepsilon(w)\\&\qquad+\int_H|\langle w,(1-e^{C_\varepsilon/T})A_2e^{C_\varepsilon/T}w\rangle|^2\,d\mu^\varepsilon(w)\\&\qquad+\left|\sum_{k\in\mathbb{N}}\langle Q^{1/2}e_k,(A_2^T-A_2^\infty)Q^{1/2}e_k\rangle\right|^2\\&\quad\lesssim e^{-\lambda_0T}\|A_2\|_{H^\theta\to H^{-\theta}}\int_H\|w\|^2_{H^\theta}\,d\mu^\varepsilon(w)\\&\qquad+T^{(\theta-\theta_0)/(2\Gamma)}\|A_2\|_{H^\theta\to H^{-\theta}}(1+e^{-\lambda_0T})\int_H\|w\|_{H^\theta}\|w\|_{H^{\theta_0}}\,d\mu^\varepsilon(w)\\&\qquad+\|A_2^T-A_2^\infty\|^2_{H^{\theta-\delta}\to H^{\delta-\theta}}\left|\sum_{k\in\mathbb{N}}\|Q^{1/2}e_k\|^2_{H^{\theta-\delta}}\right|^2\to0.\end{aligned}$$

∎

Corollary 3.5. *Under the hypotheses of Lemma 3.4, let $\phi=\lim_{T\to\infty}\int_{1/T}^T\psi(e^{C_\varepsilon t}\cdot)\,dt$. Then for all $\delta_1,\delta_2\geq0$ with $\delta_1+\delta_2<\gamma$ we have $\langle D_y\phi(\cdot),v\rangle\in\mathcal{E}_{\theta-2\delta_1}$ for every $v\in H^{\theta-2\delta_2}$, with $\|\langle D_y\phi(\cdot),v\rangle\|_{\mathcal{E}_{\theta-2\delta_1}}\lesssim\|\psi\|_{\mathcal{E}_\theta}\|v\|_{H^{\theta-2\delta_2}}$.*

Proof. It is sufficient to recall the expression $\phi(y) = \langle y, A_2^\infty y\rangle$, valid for $y \in \mathscr{S}$, and notice that for every $v \in H^{\theta-2\delta_2}$ we have $\langle v, D_y\phi(y)\rangle = 2\langle v, A_2^\infty y\rangle$. To conclude, notice that $|\langle v, D_y\phi(y)\rangle| \lesssim \|y\|_{H^{\theta-2\delta_1}}\|\psi\|_{\mathcal{E}_\theta}\|v\|_{H^{\theta-2\delta_2}}$ because $A_2^\infty : H^{\theta-2\delta_1} \to H^{2\delta_2-\theta}$ is continuous with $\|A_2^\infty\|_{H^{\theta-2\delta_1}\to H^{2\delta_2-\theta}} \lesssim \|\psi\|_{\mathcal{E}_\theta}$, and therefore the identity $\langle v, D_y\phi(y)\rangle = 2\langle v, A_2^\infty y\rangle$ extends to every $y \in H^{\theta-2\delta_1}$. ■

The next proposition permits us to solve the Poisson equation $\mathscr{L}_y^\varepsilon\phi = -\psi$ in the unknown ϕ under suitable assumptions on the datum ψ. We need, in particular, ψ to be a quadratic function on some Sobolev space H^θ with $\theta \in [0, \theta_0)$ with zero average with respect to the invariant measure μ^ε, namely $\int_H \psi(w)\, d\mu^\varepsilon(w) = 0$. That this last condition is necessary is clear from invariance of μ^ε under the Ornstein–Uhlenbeck semigroup P_t^ε and

$$\left|\int_H \mathscr{L}_y^\varepsilon\phi(w)\, d\mu^\varepsilon(w)\right| \le \left|\int_H \left(\mathscr{L}_y^\varepsilon\phi(w) - \frac{1}{t}(P_t^\varepsilon\phi - \phi)(w)\right) d\mu^\varepsilon(w)\right| \to 0$$

as $t \to 0^+$; by the proposition, the zero-average condition on ψ is also sufficient, at least when we restrict ourselves to $\psi \in \mathcal{E}_\theta$. Finally, notice that the solution of the Poisson equation is more regular than the datum: if $\psi \in \mathcal{E}_\theta$, then $\phi \in \mathcal{E}_{\theta-\delta}$ for every $\delta \in (0, \gamma)$, and if $\delta_1, \delta_2 \ge 0$ and $\delta_1 + \delta_2 < \gamma$ then $\langle D_y\phi(\cdot), v\rangle \in \mathcal{E}_{\theta-2\delta_1}$ for every $v \in H^{\theta-2\delta_2}$.

Proposition 3.6. *Let $\psi \in \mathcal{E}_\theta$ with $\theta \in [0, \theta_0)$ be such that $\int_H \psi(w)\, d\mu^\varepsilon(w) = 0$. Then there exists $\phi \in \mathcal{E}_{\theta-\delta} \cap D(\mathscr{L}_y^\varepsilon)$ for every $\delta \in (0, \gamma)$, satisfying $\|\phi\|_{\mathcal{E}_{\theta-\delta}} \lesssim \|\psi\|_{\mathcal{E}_\theta}$ and such that*

$$\phi = \lim_{T\to\infty}\int_{1/T}^T P_t^\varepsilon\psi\, dt \quad \text{in the topology of } D(\mathscr{L}_y^\varepsilon),$$

and $\mathscr{L}_y^\varepsilon\phi = -\psi$. Moreover, if $\delta_1, \delta_2 \ge 0$ and $\delta_1 + \delta_2 < \gamma$ then $\langle D_y\phi(\cdot), v\rangle \in \mathcal{E}_{\theta-2\delta_1}$ for every $v \in H^{\theta-2\delta_2}$, with $\|\langle D_y\phi(\cdot), v\rangle\|_{\mathcal{E}_{\theta-2\delta_1}} \lesssim \|\psi\|_{\mathcal{E}_\theta}\|v\|_{H^{\theta-2\delta_2}}$.

Proof. First we prove that the limit exists. Let $\psi(y) = a_0 + a_1(y) + a_2(y, y)$. We have

$$P_t\psi(y) = P_t\psi(0) + a_1(e^{C_\varepsilon t}y) + a_2(e^{C_\varepsilon t}y, e^{C_\varepsilon t}y),$$

and by Lemmas 3.3 and 3.4, the quantity $\int_{1/T}^T(a_1(e^{C_\varepsilon t}y) + a_2(e^{C_\varepsilon t}y, e^{C_\varepsilon t}y))\, dt$ converges in the $D(\mathscr{L}_y^\varepsilon)$ topology to some $\phi_\star \in \mathcal{E}_{\theta-\delta} \cap D(\mathscr{L}_y^\varepsilon)$ for every $\delta \in (0, \gamma)$. Moreover, by Lemma 3.2,

$$|P_t^\varepsilon\psi(0)| = \left|P_t^\varepsilon\psi(0) - \int_H \psi(w)\, d\mu^\varepsilon(w)\right| \lesssim \|\psi\|_{\mathcal{E}_\theta}e^{-\lambda_0 t}$$

is integrable with respect to time, and so it converges in the $D(\mathscr{L}_y^\varepsilon)$ topology to a constant ϕ_0. Altogether,

$$\lim_{T\to\infty}\int_{1/T}^T P_t^\varepsilon\psi\, dt = \phi_0 + \phi_\star =: \phi \in \mathcal{E}_{\theta-\delta} \cap D(\mathscr{L}_y^\varepsilon).$$

Let us show that ϕ is indeed a solution of the Poisson equation $\mathscr{L}_y^\varepsilon \phi = -\psi$. Notice that $\mathscr{L}_y^\varepsilon : D(\mathscr{L}_y^\varepsilon) \to L^2(H, \mu^\varepsilon)$ is bounded, and therefore by continuity we have

$$\mathscr{L}_y^\varepsilon \phi = \mathscr{L}_y^\varepsilon \left(\lim_{T\to\infty} \int_{1/T}^T P_t^\varepsilon \psi \, dt \right) = \lim_{T\to\infty} \mathscr{L}_y^\varepsilon \left(\int_{1/T}^T P_t^\varepsilon \psi \, dt \right),$$

where the first limit is in the $D(\mathscr{L}_y^\varepsilon)$ topology, and the second in the $L^2(H, \mu^\varepsilon)$ topology. Since $\int_{1/T}^T \|P_t^\varepsilon \psi\|_{\mathcal{E}} \, dt < \infty$ for every $T > 0$, we have $\int_{1/T}^T P_t^\varepsilon \psi \, dt \in \mathcal{E} \subset D(\mathscr{L}_y^\varepsilon)$ for every $T > 0$, and by Lemma 3.1 we have

$$\mathscr{L}_y^\varepsilon \left(\int_{1/T}^T P_t^\varepsilon \psi \, dt \right) = \int_{1/T}^T \mathscr{L}_y^\varepsilon P_t^\varepsilon \psi \, dt = \int_{1/T}^T P_t^\varepsilon \mathscr{L}_y \psi \, dt = P_T^\varepsilon \psi - P_{1/T}^\varepsilon \psi.$$

In particular,

$$\mathscr{L}_y^\varepsilon \phi = \lim_{T\to\infty} (P_T^\varepsilon \psi - P_{1/T}^\varepsilon \psi).$$

Since we already know that $P_{1/T}^\varepsilon \psi \to \psi \in L^2(H, \mu^\varepsilon)$ as $T \to \infty$ by continuity of the semigroup, it remains to check $P_T^\varepsilon \psi \to 0$ in $L^2(H, \mu^\varepsilon)$ as $T \to \infty$. We have

$$\begin{aligned}
|P_T^\varepsilon \psi(y)| &\le |P_T^\varepsilon \psi(0)| + |a_1(e^{C_\varepsilon T} y)| + |a_2(e^{C_\varepsilon T} y, e^{C_\varepsilon T} y)| \\
&\le |P_T^\varepsilon \psi(0)| + \|\psi\|_{\mathcal{E}_\theta} (\|e^{C_\varepsilon T} y\|_{H^\theta} + \|e^{C_\varepsilon T} y\|_{H^\theta}^2) \\
&\lesssim \|\psi\|_{\mathcal{E}_\theta} e^{-\lambda_0 T} + \|\psi\|_{\mathcal{E}_\theta} (e^{-\lambda_0 T} \|y\|_{H^\theta} + e^{-2\lambda_0 T} \|y\|_{H^\theta}^2) \to 0
\end{aligned}$$

in $L^2(H, \mu^\varepsilon)$ as $T \to \infty$. Finally, the assertion about the derivative $D_y \phi$ follows by the explicit construction of Lemma 3.3 and Corollary 3.5. ∎

4. Perturbed test function method

Let us move back to the problem of identifying φ_1^ε, φ_2^ε in the expression of the test function φ^ε. Recall we are looking for a perturbation of $\varphi = \varphi(u)$ of the form

$$\varphi^\varepsilon(u, y) = \varphi(u) + \varepsilon^{1/2} \varphi_1^\varepsilon(u, y) + \varepsilon \varphi_2^\varepsilon(u, Y).$$

For our purposes it is sufficient to consider $\varphi \in F$, namely $\varphi(u) = \langle u, h \rangle$ for some smooth test function $h \in \mathscr{S}$. With this choice of φ, we have in particular

$$D_u \varphi = h, \quad D_u^2 \varphi = 0.$$

4.1. *Finding* φ_1

Recalling (1.5), the first corrector φ_1^ε has to solve the Poisson equation

$$\mathscr{L}_y^\varepsilon \varphi_1^\varepsilon(u, y) = -\langle b(y, u), h \rangle.$$

For every fixed $u \in H$, we can apply Proposition 3.6 to the function $\psi_u = \langle b(\cdot, u), h\rangle \in \mathcal{E}$. Indeed, $\int_H \psi_u(w)\, d\mu^\varepsilon(w) = 0$, therefore there exists $\phi_u^\varepsilon \in \mathcal{E}$ such that $\mathscr{L}_y^\varepsilon \phi_u^\varepsilon = -\psi_u$. Moreover, since ψ_u is linear in y, following the construction of Lemma 3.3 it is easy to check that

$$\phi_u^\varepsilon = \langle b((-C_\varepsilon)^{-1}\cdot, u), h\rangle.$$

Finally, we define

$$\varphi_1^\varepsilon(u, y) = \phi_u^\varepsilon(y) = \langle b((-C_\varepsilon)^{-1} y, u), h\rangle. \tag{4.1}$$

Notice that for every $v \in \mathscr{S}$,

$$\begin{aligned}\langle D_y\varphi_1^\varepsilon(u), v\rangle &= \langle b((-C_\varepsilon)^{-1} v, u), h\rangle,\\ \langle D_u\varphi_1^\varepsilon(y), v\rangle &= \langle b((-C_\varepsilon)^{-1} y, v), h\rangle.\end{aligned}$$

Proposition 4.1. *For all $u, y \in H^s$ and $s \in \mathbb{R}$, we have $D_y\varphi_1^\varepsilon(u), D_u\varphi_1^\varepsilon(y) \in H^{2\theta+s}$ for every $\theta \in [\gamma, 1]$, with*

$$\|D_y\varphi_1^\varepsilon(u)\|_{H^{2\theta+s}} \lesssim \varepsilon^{-\frac{\theta-\gamma}{1-\gamma}} \|h\|_{H^{\theta_1}} \|u\|_{H^s}, \quad \|D_u\varphi_1^\varepsilon(y)\|_{H^{2\theta+s}} \lesssim \varepsilon^{-\frac{\theta-\gamma}{1-\gamma}} \|h\|_{H^{\theta_1}} \|y\|_{H^s},$$

for some $\theta_1 = \theta_1(s)$ sufficiently large.

Proof. Take $\theta_1 = \theta_1(s)$ such that (B2) holds true. For every $v \in H^{2\gamma+s}$ we have

$$\begin{aligned}|\langle D_y\varphi_1^\varepsilon(u), (-A)^{\gamma+s/2} v\rangle| &= |\langle b((-C_\varepsilon)^{-1}(-A)^{\gamma+s/2} v, u), h\rangle|\\ &\lesssim \|(-C_\varepsilon)^{-1}(-A)^{\gamma+s/2} v\|_{H^{-s}} \|h\|_{H^{\theta_1}} \|u\|_{H^s}\\ &\lesssim \|(-C)^{-1}(-A)^{\gamma+s/2} v\|_{H^{-s}} \|h\|_{H^{\theta_1}} \|u\|_{H^s}\\ &\lesssim \|v\|_H \|h\|_{H^{\theta_1}} \|u\|_{H^s},\end{aligned}$$

and similarly for every $v \in H^{2+s}$,

$$\begin{aligned}|\langle D_y\varphi_1^\varepsilon(u), (-A)^{1+s/2} v\rangle| &= |\langle b((-C_\varepsilon)^{-1}(-A)^{1+s/2} v, u), h\rangle|\\ &\lesssim \|(-C_\varepsilon)^{-1}(-A)^{1+s/2} v\|_{H^{-s}} \|h\|_{H^{\theta_1}} \|u\|_{H^s}\\ &\lesssim \|(-\varepsilon A)^{-1}(-A)^{1+s/2} v\|_{H^{-s}} \|h\|_{H^{\theta_1}} \|u\|_{H^s}\\ &\lesssim \varepsilon^{-1} \|v\|_H \|h\|_{H^{\theta_1}} \|u\|_{H^s}.\end{aligned}$$

Since v is arbitrary, by interpolation we deduce

$$\|D_y\varphi_1^\varepsilon(u)\|_{H^{2\theta+s}} \lesssim \|D_y\varphi_1^\varepsilon(u)\|_{H^{2\gamma+s}}^{\frac{1-\theta}{1-\gamma}} \|D_y\varphi_1^\varepsilon(u)\|_{H^{2+s}}^{\frac{\theta-\gamma}{1-\gamma}} \lesssim \varepsilon^{-\frac{\theta-\gamma}{1-\gamma}} \|h\|_{H^{\theta_1}} \|u\|_{H^s}.$$

The argument is similar for the term involving $D_u\varphi_1^\varepsilon$: first, for every $v \in H^{2\gamma+s}$,

$$\begin{aligned}|\langle D_u\varphi_1^\varepsilon(y), (-A)^{\gamma+s/2} v\rangle| &\lesssim |\langle b((-C_\varepsilon)^{-1} y, (-A)^{\gamma+s/2} v), h\rangle|\\ &\lesssim \|(-C_\varepsilon)^{-1} y\|_{H^{2\gamma+s}} \|h\|_{H^{\theta_1}} \|(-A)^{\gamma+s/2} v\|_{H^{-2\gamma-s}}\\ &\lesssim \|(-C)^{-1} y\|_{H^{2\gamma+s}} \|h\|_{H^{\theta_1}} \|(-A)^{\gamma+s/2} v\|_{H^{-2\gamma-s}}\\ &\lesssim \|y\|_{H^s} \|h\|_{H^{\theta_1}} \|v\|_H,\end{aligned}$$

whereas for every $v \in H^{2+s}$,

$$\begin{aligned}|\langle D_u\varphi_1^\varepsilon(y), (-A)^{1+s/2}v\rangle| &\lesssim |\langle b((-C_\varepsilon)^{-1}y, (-A)^{1+s/2}v), h\rangle| \\ &\lesssim \|(-C_\varepsilon)^{-1}y\|_{H^{2+s}}\|h\|_{H^{\theta_1}}\|(-A)^{1+s/2}v\|_{H^{-2-s}} \\ &\lesssim \|(-\varepsilon A)^{-1}y\|_{H^{2+s}}\|h\|_{H^{\theta_1}}\|(-A)^{1+s/2}v\|_{H^{-2-s}} \\ &\lesssim \varepsilon^{-1}\|y\|_{H^s}\|h\|_{H^{\theta_1}}\|v\|_H.\end{aligned}$$

The conclusion follows by interpolation. ■

4.2. *Finding φ_2*

Let us turn to equation (1.6) for the second corrector:

$$\langle Au + b(u,u), h\rangle + \langle b(y,u), D_u\varphi_1^\varepsilon(y)\rangle + \langle b(y,y), D_y\varphi_1^\varepsilon(u)\rangle + \mathscr{L}_y^\varepsilon\varphi_2^\varepsilon(u,Y).$$

As discussed in Section 1.1, as a preliminary step it is useful to manipulate the previous expression to replace the small-scale process y with its linearized counterpart Y. We point out that at this level of generality the variables y and Y only represent variables in the Hilbert space H, but we prefer to keep the notational difference for the sake of clarity, since we intend to eventually evaluate the previous expressions at $y = y^\varepsilon$, $Y = Y^\varepsilon$.

Having said this, let $\zeta \in H$ indicate the difference $\zeta = y - Y$. We can prove the following:

Proposition 4.2. *For every $\delta \in (0, \gamma - 1/4)$, $u \in H^{1-\delta}$, $y \in H^\gamma$ and $Y \in H^{\theta_0-\gamma}$,*

$$\begin{aligned}|\langle b(y,u), D_u\varphi_1^\varepsilon(y)\rangle - \langle b(Y,u), D_u\varphi_1^\varepsilon(Y)\rangle| &\lesssim \|\zeta\|_H\|h\|_{H^{\theta_1}}\|Y\|_{H^{\theta_0-\gamma}}\|u\|_H \\ &\quad + \varepsilon^{-\frac{1-\gamma-\delta}{2(1-\gamma)}}\|\zeta\|_{H^\gamma}\|h\|_{H^{\theta_1}}\|y\|_H\|u\|_{H^{1-\delta}}, \\ |\langle b(y,y), D_y\varphi_1^\varepsilon(u)\rangle - \langle b(Y,Y), D_y\varphi_1^\varepsilon(u)\rangle| &\lesssim \|\zeta\|_H\|h\|_{H^{\theta_1}}\|Y\|_{H^{\theta_0-\gamma}}\|u\|_H \\ &\quad + \varepsilon^{-\frac{1-\gamma-\delta}{2(1-\gamma)}}\|\zeta\|_{H^\gamma}\|h\|_{H^{\theta_1}}\|y\|_H\|u\|_{H^{1-\delta}}.\end{aligned}$$

Proof. Recall

$$\begin{aligned}\langle b(y,u), D_u\varphi_1^\varepsilon(y)\rangle - \langle b(Y,u), D_u\varphi_1^\varepsilon(Y)\rangle &= \langle b(\zeta,u), D_u\varphi_1^\varepsilon(Y)\rangle + \langle b(y,u), D_u\varphi_1^\varepsilon(\zeta)\rangle, \\ \langle b(y,y), D_y\varphi_1^\varepsilon(u)\rangle - \langle b(Y,Y), D_y\varphi_1^\varepsilon(u)\rangle &= \langle b(\zeta,Y), D_y\varphi_1^\varepsilon(u)\rangle + \langle b(y,\zeta), D_y\varphi_1^\varepsilon(u)\rangle,\end{aligned}$$

and by (B3) and Proposition 4.1 with $\theta = \frac{1+\gamma-\delta}{2}$ the following estimates hold:

$$\begin{aligned}|\langle b(\zeta,u), D_u\varphi_1^\varepsilon(Y)\rangle| &\lesssim \|\zeta\|_H\|h\|_{H^{\theta_1}}\|Y\|_{H^{\theta_0-\gamma}}\|u\|_H, \\ |\langle b(y,u), D_u\varphi_1^\varepsilon(\zeta)\rangle| &\lesssim \varepsilon^{-\frac{1-\gamma-\delta}{2(1-\gamma)}}\|\zeta\|_{H^\gamma}\|h\|_{H^{\theta_1}}\|y\|_H\|u\|_{H^{1-\delta}}, \\ |\langle b(\zeta,Y), D_y\varphi_1^\varepsilon(u)\rangle| &\lesssim \|\zeta\|_H\|h\|_{H^{\theta_1}}\|Y\|_{H^{\theta_0-\gamma}}\|u\|_H, \\ |\langle b(y,\zeta), D_y\varphi_1^\varepsilon(u)\rangle| &\lesssim \varepsilon^{-\frac{1-\gamma-\delta}{2(1-\gamma)}}\|\zeta\|_{H^\gamma}\|h\|_{H^{\theta_1}}\|y\|_H\|u\|_{H^{1-\delta}},\end{aligned}$$

where we have used $b : H \times H^{2\gamma+1-\delta} \to H^{\delta-1}$ and $b : H \times H^{2+\gamma-2\delta} \to H^{-\gamma}$ continuous by (B3) and our choice of δ. ■

Remark 4.3. The previous proposition will be used in Section 5 to check rigorously that we can actually replace the small-scale process y^ε with Y^ε, up to a correction which is infinitesimal as $\varepsilon \to 0$. Indeed, since $\frac{1-\gamma-\delta}{2(1-\gamma)} < \frac{1}{2}$ we can compensate diverging factors in ε in the previous proposition with a factor $\varepsilon^{1/2}$ coming from Proposition 2.7, by taking expectation and time integral.

Thus, in view of the discussion above and as already described in Section 1.1, the terms of order 1 in the expression of $\mathscr{L}^\varepsilon\varphi$ are actually given by

$$\langle Au + b(u,u), h\rangle + \langle b(Y,u), D_u\varphi_1^\varepsilon(Y)\rangle + \langle b(Y,Y), D_y\varphi_1^\varepsilon(u)\rangle + \mathscr{L}_y^\varepsilon\varphi_2^\varepsilon(u,Y), \tag{4.2}$$

and thus our goal is to find $\varphi_2^\varepsilon = \varphi_2^\varepsilon(u,Y)$ such that the previous quantity is independent of Y.

Let then $u \in H$ be fixed. The idea is again to apply Proposition 3.6 to

$$\psi_u^\varepsilon = \langle b(\cdot,u), D_u\varphi_1^\varepsilon(\cdot)\rangle + \langle b(\cdot,\cdot), D_y\varphi_1^\varepsilon(u)\rangle.$$

For every $\theta > 5/4-\gamma$ we have $\psi_u^\varepsilon \in \mathcal{E}_\theta$, with $\|\psi_u^\varepsilon\|_{\mathcal{E}_\theta} \lesssim \|h\|_{H^{\theta_1}}\|u\|_H$. However, ψ_u^ε does not satisfy the hypotheses of that proposition: indeed, it may not have zero average with respect to the invariant measure μ^ε. To deal with this issue, let us consider instead

$$\Psi_u^\varepsilon = \psi_u^\varepsilon - \int_H \psi_u^\varepsilon(w)\,d\mu^\varepsilon(w).$$

With this choice of Ψ_u^ε we have $\Psi_u^\varepsilon \in \mathcal{E}_\theta$ and $\int_H \Psi_u^\varepsilon(w)\,d\mu^\varepsilon(w) = 0$, thus Proposition 3.6 applies. Given $u \in H$ and $\Phi_u^\varepsilon \in \mathcal{E}_{\theta'} \cap D(\mathscr{L}_y^\varepsilon)$ with $\theta' > 5/4 - 2\gamma$ such that $\mathscr{L}_y^\varepsilon\Phi_u^\varepsilon = -\Psi_u^\varepsilon$, we finally define

$$\varphi_2^\varepsilon(u,Y) = \Phi_u^\varepsilon(Y), \quad \|\varphi_2^\varepsilon(u,\cdot)\|_{\mathcal{E}_{\theta'}} \lesssim \|\Psi_u^\varepsilon\|_{\mathcal{E}_\theta} \lesssim \|h\|_{H^{\theta_1}}\|u\|_H. \tag{4.3}$$

With this choice of φ_2^ε, (4.2) can be rewritten as

$$\langle Au + b(u,u), h\rangle + \int_H \psi_u^\varepsilon(w)\,d\mu^\varepsilon(w) =: \mathscr{L}^{0,\varepsilon}\varphi(u), \tag{4.4}$$

which is indeed a function of the sole variable u.

In the following, specifically when computing $\mathscr{L}^\varepsilon\varphi^\varepsilon = \mathscr{L}^\varepsilon(\varphi + \varepsilon^{1/2}\varphi_1^\varepsilon + \varepsilon\varphi_2^\varepsilon)$, we will need control over the derivatives $D_u\varphi_2^\varepsilon$, $D_Y\varphi_2^\varepsilon$ to check that the corrections we impose on the test function φ do not change the underlying dynamics in the limit $\varepsilon \to 0$, i.e. $\mathscr{L}^\varepsilon\varphi^\varepsilon$ is close to $\mathscr{L}^{0,\varepsilon}\varphi$ in a suitable sense. Control over $D_Y\varphi_2^\varepsilon$ is already encoded in the statement of Proposition 3.6, as a straightforward consequence of Corollary 3.5: indeed, for all $\theta > 5/4-\gamma$, all $\delta_1,\delta_2 \geq 0$ with $\delta_1+\delta_2 < \gamma$, and all $v \in H^{\theta-2\delta_2}$ we have $\langle D_Y\varphi_2^\varepsilon(u,\cdot), v\rangle \in \mathcal{E}_{\theta-2\delta_1}$, with uniform-in-$\varepsilon$ bound

$$\|\langle D_Y\varphi_2^\varepsilon(u,\cdot), v\rangle\|_{\mathcal{E}_{\theta-2\delta_1}} \lesssim \|\Psi_u^\varepsilon\|_{\mathcal{E}_\theta}\|v\|_{H^{\theta-2\delta_2}} \lesssim \|h\|_{H^{\theta_1}}\|u\|_H\|v\|_{H^{\theta-2\delta_2}}. \tag{4.5}$$

On the other hand, to control $D_u\varphi_2^\varepsilon$ we need the following preliminary lemma:

Lemma 4.4. *For every $\theta < \theta_0 + 2\gamma - 1$ there exists θ_1 sufficiently large such that*

$$\langle v, D_u \Psi_u^\varepsilon(\cdot)\rangle \in \mathcal{E}_{\theta_0}$$

for every $v \in H^{-\theta}$, with

$$\|\langle v, D_u \Psi_u^\varepsilon(\cdot)\rangle\|_{\mathcal{E}_{\theta_0}} \lesssim \|h\|_{H^{\theta_1}} \|v\|_{H^{-\theta}}.$$

Proof. By (B4) and (B2), for every $\theta < \theta_0 + 2\gamma - 1$ we have $b : H^{\theta_0} \times H^{\theta_0+2\gamma} \to H^\theta$ and $b : H^{\theta_0} \times H^{\theta_0} \to H^{\theta-2\gamma}$ continuous, hence by Proposition 4.1, for every $v \in H^{-\theta}$,

$$|\langle b(Y, v), D_u\varphi_1^\varepsilon(Y)\rangle| \lesssim \|b(Y, D_u\varphi_1^\varepsilon(Y))\|_{H^\theta} \|v\|_{H^{-\theta}} \lesssim \|Y\|^2_{H^{\theta_0}} \|h\|_{H^{\theta_1}} \|v\|_{H^{-\theta}},$$

and

$$\begin{aligned} |\langle v, \langle b(Y, Y), D_u D_y\varphi_1^\varepsilon(u)\rangle\rangle| &= |\langle b((-C_\varepsilon)^{-1} b(Y, Y), v), h\rangle| \\ &\lesssim \|b(Y, Y)\|_{H^{\theta-2\gamma}} \|v\|_{H^{-\theta}} \|h\|_{\theta_1} \\ &\lesssim \|h\|_{\theta_1} \|Y\|^2_{H^{\theta_0}} \|v\|_{H^{-\theta}}. \end{aligned}$$

Thus, the desired result is true if we replace Ψ_u^ε by ψ_u^ε. To conclude the proof, just notice that by the same computation as above,

$$\int_H |\langle v, D_u \Psi_u^\varepsilon(w)| \, d\mu^\varepsilon(w) \lesssim \|h\|_{\theta_1} \|v\|_{H^{-\theta}} \int_H \|w\|^2_{H^{\theta_0}} \, d\mu^\varepsilon(w) \lesssim \|h\|_{\theta_1} \|v\|_{H^{-\theta}},$$

since the integral is finite by (Q2). ∎

Proposition 4.5. *For every $\theta < \theta_0 + 2\gamma - 1$ and $v \in H^{-\theta}$ we have $\langle v, D_u\varphi_2^\varepsilon(\cdot)\rangle \in \mathcal{E}_{\theta_0-\delta}$ for every $\delta \in (0, \gamma)$, with*

$$\|\langle v, D_u\varphi_2^\varepsilon(\cdot)\rangle\|_{\mathcal{E}_{\theta_0-\delta}} \lesssim \|h\|_{H^{\theta_1}} \|v\|_{H^{-\theta}}.$$

Proof. Recalling

$$\begin{aligned} \psi_u^\varepsilon(Y) &= \langle b(Y, u), D_u\varphi_1^\varepsilon(Y)\rangle + \langle b(Y, Y), D_y\varphi_1^\varepsilon(u)\rangle, \\ \Psi_u^\varepsilon(Y) &= \psi_u^\varepsilon(Y) - \int_H \psi_u^\varepsilon(w) \, d\mu^\varepsilon(w), \end{aligned}$$

we have, for every $v \in \mathscr{S}$,

$$\begin{aligned} \langle D_u\psi_u^\varepsilon(Y), v\rangle &= \langle b(Y, v), D_u\varphi_1^\varepsilon(Y)\rangle + \langle b(Y, Y), \langle D_u D_y\varphi_1^\varepsilon, v\rangle\rangle, \\ \langle D_u\Psi_u^\varepsilon(Y), v\rangle &= \langle D_u\psi_u^\varepsilon(Y), v\rangle - \int_H \langle D_u\psi_u^\varepsilon(w), v\rangle \, d\mu(w), \end{aligned}$$

and the last quantity is independent of $u \in H$. Recall that we have defined $\varphi_2^\varepsilon(u, \cdot) = \Phi_u^\varepsilon = \lim_{T\to\infty} \int_{1/T}^T P_t^\varepsilon \Psi_u^\varepsilon \, dt$, where the limit is taken in $D(\mathscr{L}_y^\varepsilon)$; we prove now that for every $v \in H$ we have

$$\langle D_u\varphi_2^\varepsilon(\cdot), v\rangle = \lim_{T\to\infty} \int_{1/T}^T P_t^\varepsilon \langle D_u\Psi_u^\varepsilon, v\rangle \, dt.$$

Denote $\varphi_2^{\varepsilon,T}(u,\cdot) = \int_{1/T}^T P_t^\varepsilon \Psi_u^\varepsilon \, dt \in \mathcal{E}$ and consider, for $r \in \mathbb{R}$,

$$\begin{aligned}
&\varphi_2^{\varepsilon,T}(u + rv, \cdot) - \varphi_2^{\varepsilon,T}(u, \cdot) \\
&\qquad = \int_{1/T}^T P_t^\varepsilon \Psi_{u+rv}^\varepsilon \, dt - \int_{1/T}^T P_t^\varepsilon \Psi_u^\varepsilon \, dt = \int_{1/T}^T P_t^\varepsilon (\Psi_{u+rv}^\varepsilon - \Psi_u^\varepsilon) \, dt \\
&\qquad = \int_{1/T}^T P_t^\varepsilon (r \langle D_u \Psi_u^\varepsilon, v \rangle) \, dt = r \int_{1/T}^T P_t^\varepsilon \langle D_u \Psi_u^\varepsilon, v \rangle \, dt,
\end{aligned}$$

where we have used linearity of P_t^ε and the fact that Ψ_u^ε is linear in u. The map $y \mapsto \langle D_u \Psi_u^\varepsilon(y), v \rangle$ satisfies the assumptions of Proposition 3.6, and therefore we can take the limit in $D(\mathscr{L}_y^\varepsilon)$ of the previous expression, as $T \to \infty$, to obtain

$$\varphi_2^\varepsilon(u + rv, \cdot) - \varphi_2^\varepsilon(u, \cdot) = r \lim_{T \to \infty} \int_{1/T}^T P_t^\varepsilon \langle D_u \Psi_u^\varepsilon, v \rangle \, dt.$$

Finally, rearranging and taking the limit as $r \to 0$ we get

$$\langle D_u \varphi_2^\varepsilon(\cdot), v \rangle = \lim_{r \to 0} \frac{1}{r} \big(\varphi_2^\varepsilon(u + rv, \cdot) - \varphi_2^\varepsilon(u, \cdot) \big) = \lim_{T \to \infty} \int_{1/T}^T P_t^\varepsilon \langle D_u \Psi_u^\varepsilon, v \rangle \, dt.$$

Since $\langle D_u \Psi_u^\varepsilon(\,\cdot), v \rangle \in \mathcal{E}_{\theta_0}$ with $\| \langle v, D_u \Psi_u^\varepsilon(\,\cdot) \rangle \|_{\mathcal{E}_{\theta_0}} \lesssim \|h\|_{H^{\theta_1}} \|v\|_{H^{-\theta}}$ for every $v \in H^{-\theta}$ and $\theta < \theta_0 + 2\gamma - 1$, by Proposition 3.6 we have $\langle D_u \varphi_2^\varepsilon(\,\cdot), v \rangle \in \mathcal{E}_{\theta_0 - \delta}$ for every $\delta \in (0, \gamma)$ with

$$\| \langle D_u \varphi_2^\varepsilon(\cdot), v \rangle \|_{\mathcal{E}_{\theta_0 - \delta}} \lesssim \| \langle v, D_u \Psi_u^\varepsilon(\cdot) \rangle \|_{\mathcal{E}_{\theta_0}} \lesssim \|h\|_{H^{\theta_1}} \|v\|_{H^{-\theta}}.$$

This bound extends to all $v \in H^{-\theta}$ by continuity, completing the proof. ∎

5. Convergence to transport noise

In this section we state and prove convergence of u^ε. We first give the precise formulation of our result:

Theorem 5.1. *Assume* (Q1)–(Q2) *and let* $u_0, y_0 \in H$ *be given. Let* $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$ *be a family of solutions to* (1.1) *in the sense of Definition* 2.2*, which exists on a family of stochastic bases* $\{(\Omega^\varepsilon, \mathcal{F}^\varepsilon, \{\mathcal{F}_t^\varepsilon\}_{t \geq 0}, \mathbb{P}^\varepsilon, W^\varepsilon)\}_{\varepsilon \in (0,1)}$ *by Proposition* 2.3*. Then for every* $\beta > 0$*, the laws of the processes* $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$ *are tight as probability measures on the space* $L^2([0,T], H) \cap C([0,T], H^{-\beta})$*, and every weak accumulation point* $(u, Q^{1/2} W)$ *of* $(u^\varepsilon, Q^{1/2} W^\varepsilon)$ *as* $\varepsilon \to 0$ *is an analytically weak solution of the equation with transport noise and Itô–Stokes drift velocity* $r = \int_H (-C)^{-1} b(w,w) \, d\mu(w)$*:*

$$du_t = Au_t dt + b(u_t, u_t) dt + b((-C)^{-1} Q^{1/2} \circ dW_t, u_t) dt + b(r, u_t) dt.$$

If in addition pathwise uniqueness holds for the limit equation then the whole sequence converges in law; moreover, convergence in $\mathbb{P}$*-probability holds true if solutions to* (1.3) *are probabilistically strong.*

"Analytically weak solution" means that the identity holds almost surely when tested against smooth functions $h \in \mathscr{S}$ and integrated with respect to time. Also, the weak convergence $Q^{1/2}W^\varepsilon \to Q^{1/2}W$ is meant as random variables taking values in $C([0,T],H)$, although many other spaces would do since $\{Q^{1/2}W^\varepsilon\}_{\varepsilon\in(0,1)}$ are identically distributed.

The proof is split into three parts. First, invoking Simon's compactness criterion (Lemma 2.1), we prove that the laws of the processes $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ are tight as probability measures on $L^2([0,T],H) \cap C([0,T],H^{-\beta})$ for every $\beta > 0$ (see Proposition 5.7 below); next, in Proposition 5.8 we show that every weak accumulation point $(u, Q^{1/2}W)$ is an analytically weak solution of the equation with effective generator $\mathscr{L}^0$ and Itô transport noise (see (5.9)); finally, we check that the generator $\mathscr{L}^0$ can be split into the sum of the deterministic Navier–Stokes dynamics, the Itô-to-Stratonovich corrector (which together with the Itô integral gives the Stratonovich transport noise) and the Itô–Stokes drift, as made explicit by (5.12)–(5.14).

The last statement of Theorem 5.1 is classical and follows from [43, Lemma 1.1]. We omit the proof of this point.

Remark 5.2. When the limit equation fails uniqueness, we do not know whether or not different subsequences can converge towards different solutions of the limit equation. It might well be that, notwithstanding the fact that the limit equation has multiple solutions, this approximating procedure only selects some special solution which enjoys additional properties. However, we are not able to answer this question at the moment; we can only provide a partial selection criterion based on the fact that every selected solution u must satisfy the same energy bounds (S3) of the approximating sequence $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ (this last property can be deduced first for Galerkin projections $\{\Pi_m u\}_{m\in\mathbb{N}}$, and then checked to be uniform in $m \in \mathbb{N}$). This is of particular interest if we start with solutions satisfying the energy inequality as in [38].

As a preliminary step towards the proof of Theorem 5.1, we need a version of the celebrated Itô formula suited for our solution processes $(u^\varepsilon, y^\varepsilon)$. Indeed, since (1.3) only holds in analytically weak sense (S2), the classical Itô formula [20, Theorem 4.32] is not a priori applicable to the process $\Phi(u^\varepsilon_t, y^\varepsilon_t)$ unless Φ only depends on a finite number of projections $\langle u^\varepsilon_t, h_i\rangle$, $\langle y^\varepsilon_t, k_i\rangle$ for some $h_i, k_i \in \mathscr{S}$. Thus, our approach consists in applying first the classical Itô formula to the Galerkin projections $\Pi_n u^\varepsilon$, $\Pi_n y^\varepsilon$, and then let $n \to \infty$, with suitable control over $D_u\Phi$, $D_y\Phi$.

Lemma 5.3 (Itô formula). *Let $\Phi : H \times H \to \mathbb{R}$ be such that, for any fixed $u, y \in H$, we have $\Phi(u,\cdot), \Phi(\cdot,y) \in \mathscr{E}$ with $\|\Phi(u,\cdot)\|_{\mathscr{E}} \lesssim \|u\|_H$ and $\|\Phi(\cdot,y)\|_{\mathscr{E}} \lesssim \|y\|_H$, and moreover*

$$\|D_u\Phi(u,y)\|_{H^1} \lesssim 1 + \|u\|_H + \|y\|_H, \quad \|D_y\Phi(u,y)\|_{H^1} \lesssim 1 + \|u\|_H + \|y\|_H. \tag{5.1}$$

Let $(u^\varepsilon, y^\varepsilon)$ be a solution to (1.3) in the sense of Definition 2.2. Then for every fixed $\varepsilon \in (0,1)$ and $t \in [0,T]$ the following Itô formula holds $\mathbb{P}$-a.s.:

$$\Phi(u^\varepsilon_t, y^\varepsilon_t) = \Phi(u_0, y_0) + \int_0^t \mathscr{L}^\varepsilon\Phi(u^\varepsilon_s, y^\varepsilon_s)\,ds + \int_0^t \langle D_y\Phi(u^\varepsilon_s, y^\varepsilon_s), Q^{1/2}dW_s\rangle.$$

Proof. Let $\{\Pi_n\}_{n\in\mathbb{N}}$ be a family of Galerkin projectors and let $h \in H$ be fixed. Since $\Pi_n h \in \mathscr{S}$ for every $n \in \mathbb{N}$, by (S2) we have, $\mathbb{P}$-a.s. for every $t \in [0, T]$,

$$\begin{aligned}
\langle u^\varepsilon_t, \Pi_n h\rangle &= \langle u_0, \Pi_n h\rangle + \int_0^t \langle u^\varepsilon_s, A\Pi_n h\rangle + \int_0^t \langle b(u^\varepsilon_s, u^\varepsilon_s), \Pi_n h\rangle\, ds \\
&\quad + \varepsilon^{-1/2} \int_0^t \langle b(y^\varepsilon_s, u^\varepsilon_s), \Pi_n h\rangle\, ds, \\
\langle y^\varepsilon_t, \Pi_n h\rangle &= \langle y_0, \Pi_n h\rangle + \varepsilon^{-1} \int_0^t \langle y^\varepsilon_s, C_\varepsilon \Pi_n h\rangle\, ds + \int_0^t \langle b(u^\varepsilon_s, y^\varepsilon_s), \Pi_n h\rangle\, ds \\
&\quad + \varepsilon^{-1/2} \int_0^t \langle b(y^\varepsilon_s, y^\varepsilon_s), \Pi_n h\rangle\, ds + \varepsilon^{-1/2}\langle Q^{1/2} W_t, \Pi_n h\rangle.
\end{aligned}$$

Letting h freely vary in H in the previous expression, we deduce that $(\Pi_n u^\varepsilon, \Pi_n y^\varepsilon)$ is an Itô process satisfying

$$\begin{aligned}
\Pi_n u^\varepsilon_t &= \Pi_n u_0 + \int_0^t \Pi_n A u^\varepsilon_s\, ds + \int_0^t \Pi_n b(u^\varepsilon_s, u^\varepsilon_s)\, ds + \varepsilon^{-1/2} \int_0^t \Pi_n b(y^\varepsilon_s, u^\varepsilon_s)\, ds, \\
\Pi_n y^\varepsilon_t &= \Pi_n y_0 + \varepsilon^{-1} \int_0^t \Pi_n C_\varepsilon y^\varepsilon_s\, ds + \int_0^t \Pi_n b(u^\varepsilon_s, y^\varepsilon_s)\, ds \\
&\quad + \varepsilon^{-1/2} \int_0^t \Pi_n b(y^\varepsilon_s, y^\varepsilon_s)\, ds + \varepsilon^{-1/2} \Pi_n Q^{1/2} W_t
\end{aligned}$$

in the strong analytical sense. In particular, by (S1) and the classical Itô formula, the following a.s. identity holds for every $t \in [0, T]$:

$$\begin{aligned}
\Phi(\Pi_n u^\varepsilon_t, \Pi_n y^\varepsilon_t) &= \Phi(\Pi_n u_0, \Pi_n y_0) + \int_0^t \mathscr{L}^{\varepsilon,n}_{u^\varepsilon_s, y^\varepsilon_s} \Phi(\Pi_n u^\varepsilon_s, \Pi_n y^\varepsilon_s)\, ds \\
&\quad + \int_0^t \langle D_y \Phi(\Pi_n u^\varepsilon_s, \Pi_n y^\varepsilon_s), \Pi_n Q^{1/2} dW_s\rangle,
\end{aligned}$$

where $\mathscr{L}^{\varepsilon,n}_{u^\varepsilon_s, y^\varepsilon_s} \Phi(\Pi_n u^\varepsilon_s, \Pi_n y^\varepsilon_s)$ is given by

$$\begin{aligned}
\mathscr{L}^{\varepsilon,n}_{u^\varepsilon_s, y^\varepsilon_s} \Phi(\Pi_n u^\varepsilon_s, \Pi_n y^\varepsilon_s) &= \langle \Pi_n A u^\varepsilon_s + \Pi_n b(u^\varepsilon_s, u^\varepsilon_s), D_u \Phi(\Pi_n u^\varepsilon_s, \Pi_n y^\varepsilon_s)\rangle \\
&\quad + \varepsilon^{-1/2} \langle \Pi_n b(y^\varepsilon_s, u^\varepsilon_s), D_u \Phi(\Pi_n u^\varepsilon_s, \Pi_n y^\varepsilon_s)\rangle \\
&\quad + \langle \Pi_n b(u^\varepsilon_s, y^\varepsilon_s), D_y \Phi(\Pi_n u^\varepsilon_s, \Pi_n y^\varepsilon_s)\rangle \\
&\quad + \varepsilon^{-1/2} \langle \Pi_n b(y^\varepsilon_s, y^\varepsilon_s), D_y \Phi(\Pi_n u^\varepsilon_s, \Pi_n y^\varepsilon_s)\rangle \\
&\quad + \varepsilon^{-1} \langle \Pi_n C_\varepsilon y^\varepsilon_s, D_y \Phi(\Pi_n u^\varepsilon_s, \Pi_n y^\varepsilon_s)\rangle \\
&\quad + \frac{\varepsilon^{-1}}{2} \mathrm{Tr}(\Pi_n Q \Pi_n D^2_y \Phi(\Pi_n u^\varepsilon_s)).
\end{aligned}$$

Because by assumption $u^\varepsilon, y^\varepsilon \in C([0, T], H_w)$ for every fixed ε, and $\Phi \in \mathcal{E}$ whenever any of its two arguments is fixed, it is easy to check that for every fixed $t \in [0, T]$ the convergences $\Phi(\Pi_n u^\varepsilon_t, \Pi_n y^\varepsilon_t) \to \Phi(u^\varepsilon_t, y^\varepsilon_t)$ and $\Phi(\Pi_n u_0, \Pi_n y_0) \to \Phi(u_0, y_0)$ hold true $\mathbb{P}$-a.s. as $n \to \infty$. Indeed, with probability one $u^\varepsilon_t, y^\varepsilon_t \in H$ for every $t \in [0, T]$, and thus

$\Pi_n u_t^\varepsilon \to u_t^\varepsilon$ and $\Pi_n y_t^\varepsilon \to y_t^\varepsilon$ strongly in H. By (S3), (5.1) and the Lebesgue dominated convergence theorem, we have, up to subsequences, for every $t \in [0, T]$ fixed, $\mathbb{P}$-a.s.,

$$\int_0^t \mathscr{L}^{\varepsilon,n} \Phi(\Pi_n u_s^\varepsilon, \Pi_n y_s^\varepsilon)\, ds \to \int_0^t \mathscr{L}^\varepsilon \Phi(u_s^\varepsilon, y_s^\varepsilon)\, ds.$$

Similarly, since $\int_0^t \|\Pi_n D_y \Phi(\Pi_n u_s^\varepsilon, \Pi_n y_s^\varepsilon)\|^2\, ds \to \int_0^t \|D_y \Phi(u_s^\varepsilon, y_s^\varepsilon)\|^2\, ds$ a.s. as $n \to \infty$, the following convergence in probability holds true:

$$\int_0^t \langle D_y \Phi(\Pi_n u_s^\varepsilon, \Pi_n y_s^\varepsilon), \Pi_n Q^{1/2} dW_s\rangle \to \int_0^t \langle D_y \Phi(u_s^\varepsilon, y_s^\varepsilon), Q^{1/2} dW_s\rangle,$$

and the convergence is almost sure up to extracting a subsequence, concluding the proof. ■

Remark 5.4. (i) As a consequence of Lemma 5.3 and the definition of the corrector φ_1^ε in the previous section, we immediately deduce that φ_1^ε belongs to the domain of the generator $\mathscr{L}^\varepsilon$ and the Itô formula holds for the process $\varphi_1^\varepsilon(u^\varepsilon, y^\varepsilon)$ for every $\varepsilon > 0$. However, strictly speaking we do not actually need such a strong result. For instance, it would have been sufficient to show the existence of a *generalized corrector* $\tilde{\varphi}_1^\varepsilon$ and an adapted process H^ε such that for every $t \in [0, T]$,

$$\tilde{\varphi}_1^\varepsilon(u_t^\varepsilon, y_t^\varepsilon) = \tilde{\varphi}_1^\varepsilon(u_0, y_0) + \int_0^t H_s^\varepsilon\, ds + \int_0^t \langle D_y \tilde{\varphi}_1^\varepsilon(u_s^\varepsilon, y_s^\varepsilon), Q^{1/2} dW_s\rangle,$$

$\mathbb{P}$-a.s., with some suitable additional requirements that allow one to identify a limit equation for u^ε as done below. In particular, the generalized corrector $\tilde{\varphi}_1^\varepsilon$ need not be in the domain of $\mathscr{L}^\varepsilon$.

On the other hand, whenever the arguments of the previous sections work and produce a corrector φ_1^ε within the domain of $\mathscr{L}^\varepsilon$, it is natural to choose it to apply the perturbed function method. Moreover, we cannot avoid proving the validity of *some* Itô formula for the process $\varphi_1^\varepsilon(u^\varepsilon, y^\varepsilon)$, since it does not come directly from our notion of solution (whereas an Itô formula for $\varphi(u^\varepsilon)$ does); thus, proving Lemma 5.3 is a fully justified effort.

(ii) Since Y^ε is regular, it is possible to consider the functions $\Phi_1(u, y, Y) = \Phi_1(\Phi(u, y), Y)$ and prove an analogous Itô formula for the process $\Phi_1(u_t^\varepsilon, y_t^\varepsilon, Y_t^\varepsilon)$.

5.1. Tightness

In this subsection we prove that the laws of the processes $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ are tight as probability measures on $L^2([0, T], H) \cap C([0, T], H^{-\beta})$ for every $\beta > 0$. The idea is to apply Simon's compactness criterion of Lemma 2.1. In order to do so, we need estimates on the increments $\mathbb{E}[\|u_t^\varepsilon - u_s^\varepsilon\|_{H^{-\sigma}}^p]$, $s, t \in [0, T]$, where $p > 2$ and $\sigma > 0$ are suitable parameters. Making use of the formula

$$\|u_t^\varepsilon - u_s^\varepsilon\|_{H^{-\sigma}}^2 = \sum_{k\in\mathbb{N}} \frac{(\varphi^k(u_t^\varepsilon) - \varphi^k(u_s^\varepsilon))^2}{\nu_k^{2\sigma}}, \quad \varphi^k(u) = \langle u, e_k\rangle, \tag{5.2}$$

for $\{e_k\}_{k\in\mathbb{N}}$ a complete orthonormal system in H of eigenfunctions of $-A$ with associated eigenvalues $\{\nu_k\}_{k\in\mathbb{N}}$, we reduce the problem to providing suitable estimates for the quantity $\mathbb{E}[|\varphi^k(u_t^\varepsilon) - \varphi^k(u_s^\varepsilon)|^p]$, which can be obtained by applying the Itô formula of Lemma 5.3 to the test function $\varphi^{k,\varepsilon}(u^\varepsilon, y^\varepsilon) = \varphi^k(u^\varepsilon) + \varepsilon^{1/2}\varphi_1^{k,\varepsilon}(u^\varepsilon, y^\varepsilon)$, with $\varphi_1^{k,\varepsilon}$ being given by (4.1) with $h = e_k$.

We first prove the following auxiliary lemma, providing estimates in some negative Sobolev norms of the time increments $u_t^\varepsilon - u_s^\varepsilon$ and $y_t^\varepsilon - y_s^\varepsilon$.

Lemma 5.5. *Let $\{(u^\varepsilon, y^\varepsilon)\}_{\varepsilon\in(0,1)}$ be a bounded-energy family of weak martingale solutions to (1.3), and for every $\varepsilon \in (0,1)$ let Y^ε be the unique strong solution of (2.7). Then for every $p \geq 1$ and $\theta = \max\{\theta_0, \Gamma\}$ the following estimates hold:*

$$\mathbb{E}[\|u_t^\varepsilon - u_s^\varepsilon\|_{H^{-\theta_0}}^p] \lesssim \varepsilon^{-p/2}|t-s|^p;$$
$$\mathbb{E}[\|Y_t^\varepsilon - Y_s^\varepsilon\|_{H^{-\theta_0}}^p] \lesssim \varepsilon^{-p/2}|t-s|^{p/2};$$
$$\mathbb{E}[\|y_t^\varepsilon - y_s^\varepsilon - (Y_t^\varepsilon - Y_s^\varepsilon)\|_{H^{-\theta}}^p] \lesssim \varepsilon^{-p}|t-s|^p.$$

Proof. Let us start from the estimate on u^ε. We have, for every $h \in H^{\theta_0}$,

$$\langle u_t^\varepsilon - u_s^\varepsilon, h\rangle = \int_s^t \langle u_r^\varepsilon, Ah\rangle\, dr + \int_s^t \langle b(u_r^\varepsilon, u_r^\varepsilon), h\rangle\, dr + \varepsilon^{-1/2}\int_s^t \langle b(y_r^\varepsilon, u_r^\varepsilon), h\rangle\, dr,$$

hence, using (B1),

$$|\langle u_t^\varepsilon - u_s^\varepsilon, h\rangle| \lesssim \int_s^t \|u_r^\varepsilon\|_H \|h\|_{H^2}\, dr + \int_s^t \|u_r^\varepsilon\|_H^2 \|h\|_{H^{\theta_0}}\, dr + \varepsilon^{-1/2}\int_s^t \|y_r^\varepsilon\|_H \|u_r^\varepsilon\|_H \|h\|_{H^{\theta_0}}\, dr.$$

Therefore, taking the supremum over $h \in H^{\theta_0}$ with $\|h\|_{H^{\theta_0}} = 1$, raising to the p-th power and taking expectations we get

$$\mathbb{E}[\|u_t^\varepsilon - u_s^\varepsilon\|_{H^{-\theta_0}}^p] \lesssim \varepsilon^{-p/2}|t-s|^p.$$

In order to get an estimate on Y^ε, we first rewrite the increment $Y_t^\varepsilon - Y_s^\varepsilon$ using the mild formulation of (2.7):

$$Y_t^\varepsilon - Y_s^\varepsilon = (e^{\varepsilon^{-1}C_\varepsilon(t-s)} - 1)Y_s^\varepsilon + \varepsilon^{-1/2}\int_s^t e^{\varepsilon^{-1}C_\varepsilon(t-r)}Q^{1/2}dW_r,$$

from which we are able to deduce, on the one hand,

$$\mathbb{E}[\|(e^{\varepsilon^{-1}C_\varepsilon(t-s)} - 1)Y_s^\varepsilon\|_{H^{-\theta_0}}^p] \lesssim \varepsilon^{-p/2}|t-s|^{p/2}\mathbb{E}[\|Y_s^\varepsilon\|_{H^{\Gamma-\theta_0}}^p] \lesssim \varepsilon^{-p/2}|t-s|^{p/2},$$

and on the other hand, applying Itô isometry,

$$\mathbb{E}\left[\left\|\varepsilon^{-1/2}\int_s^t e^{\varepsilon^{-1}C_\varepsilon(t-r)}Q^{1/2}\, dW_r\right\|_{H^{-\theta_0}}^p\right] \lesssim |t-s|^{p/2}.$$

Let us turn to estimating y^ε. First, since Y^ε is a strong solution of (2.7), for every fixed $h \in H^{\theta_0}$ and $s, t \in [0, T]$, $s < t$, we have the following weak reformulation of (2.7):

$$\langle Y^\varepsilon_t - Y^\varepsilon_s, h\rangle = \varepsilon^{-1} \int_s^t \langle Y^\varepsilon_r, C_\varepsilon h\rangle \, dr + \varepsilon^{-1/2} \langle Q^{1/2}(W_t - W_s), h\rangle,$$

so that from the previous expression together with (S2) we get

$$\begin{aligned}\langle y^\varepsilon_t - y^\varepsilon_s - (Y^\varepsilon_t - Y^\varepsilon_s), h\rangle = \varepsilon^{-1} \int_s^t \langle y^\varepsilon_r - Y^\varepsilon_r, C_\varepsilon h\rangle \, dr + \int_s^t \langle b(u^\varepsilon_r, y^\varepsilon_r), h\rangle \, dr \\ + \varepsilon^{-1/2} \int_s^t \langle b(y^\varepsilon_r, y^\varepsilon_r), h\rangle \, dr.\end{aligned}$$

Hence, arguing as with $u^\varepsilon_t - u^\varepsilon_s$ we obtain

$$\mathbb{E}[\| y^\varepsilon_t - y^\varepsilon_s - (Y^\varepsilon_t - Y^\varepsilon_s) \|^p_{H^{-\theta}}] \lesssim \varepsilon^{-p} |t - s|^p. \qquad \blacksquare$$

We now turn to the main computation of this subsection.

Lemma 5.6. *There exists $\alpha > 0$ depending only on γ, Γ and θ_0 such that the following holds. For every $p > 2$ there exists $\sigma > 0$ such that, for all $s, t \in [0, T]$,*

$$\mathbb{E}[\| u^\varepsilon_t - u^\varepsilon_s \|^p_{H^{-\sigma}}] \lesssim |t - s|^{\alpha p}.$$

Proof. Take $\varphi^{k,\varepsilon} = \varphi^k + \varepsilon^{1/2} \varphi^{k,\varepsilon}_1$ as test function as above, that is, $\varphi^k(u) = \langle u, e_k\rangle$ with $\{e_k\}_{k\in\mathbb{N}}$ an eigenbasis of $-A$, and $\varphi^{k,\varepsilon}_1$ given by (4.1) with $h = e_k$. With this choice of φ^k we have $D_u \varphi^k = e_k$ and

$$\begin{aligned}\varphi^{k,\varepsilon}_1(u^\varepsilon, y^\varepsilon) &= \langle b((-C_\varepsilon)^{-1} y^\varepsilon, u^\varepsilon), e_k\rangle, \\ D_u \varphi^{k,\varepsilon}_1(y^\varepsilon) &= -b((-C_\varepsilon)^{-1} y^\varepsilon, e_k), \\ D_y \varphi^{k,\varepsilon}_1(u^\varepsilon) &= \langle b((-C_\varepsilon)^{-1} \cdot, u^\varepsilon), e_k\rangle.\end{aligned}$$

Applying the Itô formula to $\varphi^{k,\varepsilon}(u^\varepsilon, y^\varepsilon)$ we get almost surely, for any given $s, t \in [0, T]$ with $s < t$,

$$\begin{aligned}\varphi^k(u^\varepsilon_t) - \varphi^k(u^\varepsilon_s) = \varepsilon^{1/2} (\varphi^{k,\varepsilon}_1(u^\varepsilon_s, y^\varepsilon_s) - \varphi^{k,\varepsilon}_1(u^\varepsilon_t, y^\varepsilon_t)) \\ + \int_s^t \mathscr{L}^\varepsilon \varphi^{k,\varepsilon}(u^\varepsilon_r, y^\varepsilon_r) \, dr + \int_s^t \langle D_y \varphi^{k,\varepsilon}_1(u^\varepsilon_r), Q^{1/2} dW_r\rangle. \end{aligned} \tag{5.3}$$

Therefore, using (5.2) and the Hölder inequality, for every $\sigma > 0$ satisfying $\sum_{k\in\mathbb{N}} \nu_k^{-2\sigma} < \infty$ we get

$$\begin{aligned}\mathbb{E}[\| u^\varepsilon_t - u^\varepsilon_s \|^p_{H^{-\sigma}}] &= \mathbb{E}\left[\left| \sum_{k\in\mathbb{N}} \frac{(\varphi^k(u^\varepsilon_t) - \varphi^k(u^\varepsilon_s))^2}{\nu_k^{2\sigma}} \right|^{p/2} \right] \\ &\leq \left(\sum_{k\in\mathbb{N}} \frac{1}{\nu_k^{2\sigma}} \right)^{\frac{p-2}{p}} \mathbb{E}\left[\sum_{k\in\mathbb{N}} \frac{|\varphi^k(u^\varepsilon_t) - \varphi^k(u^\varepsilon_s)|^p}{\nu_k^{2\sigma}} \right]. \end{aligned} \tag{5.4}$$

Let us estimate the summands on the right hand side of (5.3) to complete the proof of the proposition. We start from the terms involving the time increment $\varphi_1^{k,\varepsilon}(u_s^\varepsilon, y_s^\varepsilon) - \varphi_1^{k,\varepsilon}(u_t^\varepsilon, y_t^\varepsilon)$: for any $s, t \in [0, T]$ with $s < t$,

$$\begin{aligned}|\varphi_1^{k,\varepsilon}(u_s^\varepsilon, y_s^\varepsilon) - \varphi_1^{k,\varepsilon}(u_t^\varepsilon, y_t^\varepsilon)| &\leq |\varphi_1^{k,\varepsilon}(u_s^\varepsilon - u_t^\varepsilon, y_s^\varepsilon)| + |\varphi_1^{k,\varepsilon}(u_t^\varepsilon, y_s^\varepsilon - y_t^\varepsilon)| \\ &= |\langle b((-C_\varepsilon)^{-1} y_s^\varepsilon, e_k), u_s^\varepsilon - u_t^\varepsilon\rangle| \\ &\quad + |\langle b((-C_\varepsilon)^{-1}(y_s^\varepsilon - y_t^\varepsilon), e_k), u_t^\varepsilon\rangle| \\ &\lesssim \|u_s^\varepsilon - u_t^\varepsilon\|_{H^{-2\gamma}} \|y_s^\varepsilon\|_H \|e_k\|_{H^{\theta_1}} \\ &\quad + \|y_s^\varepsilon - y_t^\varepsilon\|_{H^{-2\gamma}} \|u_t^\varepsilon\|_H \|e_k\|_{H^{\theta_0}}.\end{aligned}$$

We can invoke Lemma 5.5 and an interpolation inequality to estimate the $H^{-2\gamma}$ norm of the time increments $u_s^\varepsilon - u_t^\varepsilon$ and $y_s^\varepsilon - y_t^\varepsilon$, and get (without loss of generality we assume $\gamma \leq \theta_0/4$)

$$\begin{aligned}&\varepsilon^{p/2}\, \mathbb{E}[|\varphi_1^{k,\varepsilon}(u_s^\varepsilon, y_s^\varepsilon) - \varphi_1^{k,\varepsilon}(u_t^\varepsilon, y_t^\varepsilon)|^p] \\ &\quad \lesssim \varepsilon^{p/2} \|e_k\|_{H^{\theta_1}}^p\, \mathbb{E}[\|u_t^\varepsilon - u_s^\varepsilon\|_{H^{-\theta_0}}^{2\gamma p/\theta_0} \|u_t^\varepsilon - u_s^\varepsilon\|_H^{(1-2\gamma/\theta_0)p} \|y_t^\varepsilon\|_H^p] \\ &\qquad + \varepsilon^{p/2} \|e_k\|_{H^{\theta_0}}^p\, \mathbb{E}[\|y_t^\varepsilon - y_s^\varepsilon\|_{H^{-\theta}}^{2\gamma p/\theta} \|y_t^\varepsilon - y_s^\varepsilon\|_H^{(1-2\gamma/\theta)p} \|u_s^\varepsilon\|_H^p] \\ &\quad \lesssim \|e_k\|_{H^{\theta_1}}^p \varepsilon^{p/2} \varepsilon^{-2\gamma p/\theta_0} |t-s|^{2\gamma p/\theta_0} + \|e_k\|_{H^{\theta_0}}^p \varepsilon^{p/2} \varepsilon^{-2\gamma p/\theta} |t-s|^{\gamma p/\theta} \\ &\quad \lesssim \|e_k\|_{H^{\theta_1}}^p |t-s|^{2\gamma p/\theta_0} + \|e_k\|_{H^{\theta_0}}^p |t-s|^{\gamma p/\theta},\end{aligned}$$

where we recall $\theta = \max\{\theta_0, \Gamma\}$. Let us now handle the term with the time integral of $\mathscr{L}^\varepsilon \varphi^{k,\varepsilon}(u_r^\varepsilon, y_r^\varepsilon)$. We rewrite this term as $\mathscr{L}^\varepsilon \varphi^{k,\varepsilon} = \Phi^{k,\varepsilon} + \varepsilon^{1/2} \Phi_1^{k,\varepsilon}$, where for $r \in [0, T]$, $\Phi^{k,\varepsilon}$, $\Phi_1^{k,\varepsilon}$ are implicitly given by

$$\begin{aligned}\mathscr{L}^\varepsilon \varphi^{k,\varepsilon}(u_r^\varepsilon, y_r^\varepsilon) &= \langle A u_r^\varepsilon + b(u_r^\varepsilon, u_r^\varepsilon), D_u \varphi^k\rangle + \langle b(y_r^\varepsilon, u_r^\varepsilon), D_u \varphi_1^{k,\varepsilon}(y_r^\varepsilon)\rangle \\ &\quad + \langle b(y_r^\varepsilon, y_r^\varepsilon), D_y \varphi_1^{k,\varepsilon}(u_r^\varepsilon)\rangle \\ &\quad + \varepsilon^{1/2}\big(\langle A u_r^\varepsilon + b(u_r^\varepsilon, u_r^\varepsilon), D_u \varphi_1^{k,\varepsilon}(y_r^\varepsilon)\rangle + \langle b(u_r^\varepsilon, y_r^\varepsilon), D_y \varphi_1^{k,\varepsilon}(u_r^\varepsilon)\rangle\big) \\ &=: \Phi^{k,\varepsilon}(u_r^\varepsilon, y_r^\varepsilon) + \varepsilon^{1/2} \Phi_1^{k,\varepsilon}(u_r^\varepsilon, y_r^\varepsilon).\end{aligned}$$

We have the inequalities

$$\begin{aligned}|\langle A u_r^\varepsilon + b(u_r^\varepsilon, u_r^\varepsilon), D_u \varphi^k\rangle| &= |\langle A u_r^\varepsilon + b(u_r^\varepsilon, u_r^\varepsilon), e_k\rangle| \\ &\lesssim \|u_r^\varepsilon\|_H \|e_k\|_{H^2} + \|u_r^\varepsilon\|_H^2 \|e_k\|_{H^{\theta_0}}, \\ |\langle b(y_r^\varepsilon, u_r^\varepsilon), D_u \varphi_1^{k,\varepsilon}(y_r^\varepsilon)\rangle| &\lesssim \|e_k\|_{H^{\theta_1}} \|y_r^\varepsilon\|_H \|y_r^\varepsilon\|_{H^{3/2-2\gamma}} \|u_r^\varepsilon\|_{H^1}, \\ |\langle b(y_r^\varepsilon, y_r^\varepsilon), D_y \varphi_1^{k,\varepsilon}(u_r^\varepsilon)\rangle| &\lesssim \|e_k\|_{H^{\theta_1}} \|y_r^\varepsilon\|_{H^1} \|y_r^\varepsilon\|_{H^{3/2-2\gamma}} \|u_r^\varepsilon\|_H,\end{aligned}$$

so that the time integral of $\Phi^{k,\varepsilon}(u_r^\varepsilon, y_r^\varepsilon)$ satisfies

$$\mathbb{E}\left[\left|\int_s^t \Phi^{k,\varepsilon}(u_r^\varepsilon, y_r^\varepsilon)\, dr\right|^p\right] \lesssim \|e_k\|_{H^{\theta_1}}^p |t-s|^{(\gamma-1/4)p}$$

uniformly in ε. Similarly,

$$|\langle Au_r^\varepsilon, D_u\varphi_1^{k,\varepsilon}(y_r^\varepsilon)\rangle| \lesssim \|e_k\|_{H^{\theta_1}}\|u_r^\varepsilon\|_{H^1}\|y_r^\varepsilon\|_{H^{1-2\gamma}},$$
$$|\langle b(u_r^\varepsilon, u_r^\varepsilon), D_u\varphi_1^{k,\varepsilon}(y_r^\varepsilon)\rangle| \lesssim \|e_k\|_{H^{\theta_1}}\|u_r^\varepsilon\|_{H^1}\|u_r^\varepsilon\|_{H^{3/2-2\gamma}}\|y_r^\varepsilon\|_{H},$$
$$|\langle b(u_r^\varepsilon, y_r^\varepsilon), D_y\varphi_1^{k,\varepsilon}(u_r^\varepsilon)\rangle| \lesssim \|e_k\|_{H^{\theta_1}}\|y_r^\varepsilon\|_{H^1}\|u_r^\varepsilon\|_{H^{3/2-2\gamma}}\|u_r^\varepsilon\|_{H},$$

and we can bound the time integral of $\Phi_1^{k,\varepsilon}(u_r^\varepsilon, y_r^\varepsilon)$ with

$$\varepsilon^{p/2}\mathbb{E}\left[\left|\int_s^t \Phi_1^{k,\varepsilon}(u_r^\varepsilon, y_r^\varepsilon)\,dr\right|^p\right] \lesssim \varepsilon^{p/2}\|e_k\|_{H^{\theta_1}}^p|t-s|^{(\gamma-1/4)p}.$$

The last term remaining is the stochastic integral; by Itô isometry we have

$$\mathbb{E}\left[\left|\int_s^t \langle D_y\varphi_1^{k,\varepsilon}(u_r^\varepsilon), Q^{1/2}dW_r\rangle\right|^p\right]$$
$$= \mathbb{E}\left[\left|\int_s^t \langle b((-C_\varepsilon)^{-1}Q^{1/2}dW_r, u_r^\varepsilon), e_k\rangle\right|^p\right] \lesssim \|e_k\|_{H^{\theta_0}}^p|t-s|^{p/2}.$$

Putting all together, we finally arrive at the following bound, uniform in ε and valid for all $k \in \mathbb{N}$, $s, t \in [0, T]$ with $s < t$, and $p > 2$:

$$\mathbb{E}[|\varphi^k(u_t^\varepsilon) - \varphi^k(u_s^\varepsilon)|^p] \lesssim \|e_k\|_{H^{\theta_1}}^p|t-s|^{\alpha p}, \quad \alpha = \min\{2\gamma/\theta_0, \gamma/\theta, \gamma - 1/4\}. \tag{5.5}$$

Recall that in order to estimate the $H^{-\sigma}$ norm of $u_t^\varepsilon - u_s^\varepsilon$ we have to sum (5.5) over all $k \in \mathbb{N}$; for this reason, we further require that σ is such that

$$\sum_{k\in\mathbb{N}} \frac{\|e_k\|_{H^{\theta_1}}^p}{\nu_k^{2\sigma}} < \infty,$$

so that by the Fubini–Tonelli theorem, (5.4) and (5.5),

$$\mathbb{E}[\|u_t^\varepsilon - u_s^\varepsilon\|_{H^{-\sigma}}^p] = \mathbb{E}\left[\left|\sum_{k\in\mathbb{N}} \frac{(\varphi^k(u_t^\varepsilon) - \varphi^k(u_s^\varepsilon))^2}{\nu_k^{2\sigma}}\right|^{p/2}\right]$$
$$\leq \left(\sum_{k\in\mathbb{N}} \frac{1}{\nu_k^{2\sigma}}\right)^{\frac{p-2}{p}} \mathbb{E}\left[\sum_{k\in\mathbb{N}} \frac{|\varphi^k(u_t^\varepsilon) - \varphi^k(u_s^\varepsilon)|^p}{\nu_k^{2\sigma}}\right] \lesssim |t-s|^{\alpha p}. \qquad \blacksquare$$

Thus, we are ready to prove the first part of Theorem 5.1:

Proposition 5.7. *For every $\beta > 0$, the laws of the processes $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ are tight as probability measures on $L^2([0, T], H) \cap C([0, T], H^{-\beta})$.*

Proof. Let α_0 be given by Lemma 5.6, and take $\alpha \in (0, \alpha_0)$, $p > 1/\alpha$ and $\sigma > 0$ such that the conclusion of the lemma holds. By the lemma, $\mathbb{E}[\|u^\varepsilon\|_{W^{\alpha,p}([0,T],H^{-\sigma})}]$ is bounded uniformly in ε; since also $\mathbb{E}[\|u^\varepsilon\|_{L^\infty([0,T],H)}]$ and $\mathbb{E}[\|u^\varepsilon\|_{L^2([0,T],H^1)}]$ are bounded uniformly in ε by assumption (S3), Simon's compactness criterion (Lemma 2.1) yields tightness of the sequence of laws of the processes $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ in the space $L^2([0, T], H) \cap C([0, T], H^{-\beta})$ for every $\beta > 0$. $\blacksquare$

5.2. Identification of the limit

Let $\varphi = \langle \cdot, h\rangle \in F$ be a test function, and denote $\varphi^\varepsilon(u, y, Y) = \varphi(u) + \varepsilon^{1/2}\varphi_1^\varepsilon(u, y) + \varepsilon\varphi_2^\varepsilon(u, Y)$, where φ_1^ε and φ_2^ε are given by (4.1) and (4.3) respectively. Let us also introduce the homogeneous corrector φ_1 via the formula

$$\varphi_1(u, y) := \langle b((-C)^{-1} y, u), h\rangle, \tag{5.6}$$

and the limiting effective generator $\mathscr{L}^0$ by

$$\mathscr{L}^0\varphi(u) = \langle Au + b(u, u), h\rangle + \int_H \psi_u(w)\, d\mu(w), \tag{5.7}$$

where $\psi_u(w) = \langle b(w, u), D_u\varphi_1(w)\rangle + \langle b(w, w), D_y\varphi_1(u)\rangle$ and $\mu = \mathcal{N}(0, Q_\infty)$.

Since $(u^\varepsilon, y^\varepsilon)$ is a weak solution of system (1.3) and Y^ε is a strong solution to (2.7), by the Itô formula (Lemma 5.3) we have, almost surely for every $t \in [0, T]$,

$$\begin{aligned}\varphi^\varepsilon(u_t^\varepsilon, y_t^\varepsilon, Y_t^\varepsilon) = \varphi^\varepsilon(u_0, y_0, 0) + \int_0^t \mathscr{L}^\varepsilon\varphi^\varepsilon(u_s^\varepsilon, y_s^\varepsilon, Y_s^\varepsilon)\, ds \\ + \varepsilon^{-1/2}\int_0^t \langle D_y\varphi^\varepsilon(u_s^\varepsilon, y_s^\varepsilon, Y_s^\varepsilon), Q^{1/2} dW_s\rangle,\end{aligned}$$

or equivalently

$$\begin{aligned}\varphi(u_t^\varepsilon) = \varphi(u_0) &+ \int_0^t \mathscr{L}^0\varphi(u_s^\varepsilon)\, ds + \int_0^t \langle b((-C)^{-1} Q^{1/2} dW_s, u_s^\varepsilon), h\rangle \\ &+ \int_0^t (\mathscr{L}^{0,\varepsilon}\varphi(u_s^\varepsilon) - \mathscr{L}^0\varphi(u_s^\varepsilon))\, ds \\ &+ \int_0^t \langle b(((-C_\varepsilon)^{-1} - (-C)^{-1}) Q^{1/2} dW_s, u_s^\varepsilon), h\rangle \\ &+ \varepsilon^{1/2}(\varphi_1^\varepsilon(u_0, y_0) - \varphi_1^\varepsilon(u_t^\varepsilon, y_t^\varepsilon)) + \varepsilon(\varphi_2^\varepsilon(u_0, 0) - \varphi_2^\varepsilon(u_t^\varepsilon, Y_t^\varepsilon)) \\ &+ \int_0^t \Phi_0^\varepsilon(u_s^\varepsilon, y_s^\varepsilon, Y_s^\varepsilon)\, ds + \varepsilon^{1/2}\int_0^t \Phi_1^\varepsilon(u_s^\varepsilon, y_s^\varepsilon, Y_s^\varepsilon)\, ds + \varepsilon\int_0^t \Phi_2^\varepsilon(u_s^\varepsilon, Y_s^\varepsilon)\, ds \\ &+ \varepsilon^{1/2}\int_0^t \langle D_Y\varphi_2^\varepsilon(u_s^\varepsilon, Y_s^\varepsilon), Q^{1/2} dW_s\rangle,\end{aligned} \tag{5.8}$$

where $\mathscr{L}^0$ is the limiting effective generator defined by (5.7), $\mathscr{L}^{0,\varepsilon}$ is the effective generator defined by (4.4), and we have set for notational simplicity

$$\begin{aligned}\Phi_0^\varepsilon(u, y, Y) &= \langle b(y, u), D_u\varphi_1^\varepsilon(y)\rangle + \langle b(y, y), D_y\varphi_1^\varepsilon(u)\rangle \\ &\quad - \langle b(Y, u), D_u\varphi_1^\varepsilon(Y)\rangle + \langle b(Y, Y), D_y\varphi_1^\varepsilon(u)\rangle, \\ \Phi_1^\varepsilon(u, y, Y) &= \langle Au + b(u, u), D_u\varphi_1^\varepsilon(y)\rangle + \langle b(u, y), D_y\varphi_1^\varepsilon(u)\rangle \\ &\quad + \langle b(y, u), D_u\varphi_2^\varepsilon(Y)\rangle, \\ \Phi_2^\varepsilon(u, Y) &= \langle Au + b(u, u), D_u\varphi_2^\varepsilon(Y)\rangle.\end{aligned}$$

Equation (5.8) clearly indicates the candidate limit dynamics (first line of the equation) and the remainder terms (lines second to sixth). Our aim is to prove, on the one hand, the convergence of the first line to the same quantity evaluated at $u^\varepsilon = u$ (for a possibly different Wiener process W; recall that at this stage the stochastic basis still depends on ε), and on the other hand the convergence of all remainders to zero.

In order to conveniently pass to the limit $\varepsilon \to 0$, we invoke a standard combination of the Prokhorov theorem and the Skorokhod theorem. Indeed, since the family of laws of the processes $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ is tight on $L^2([0,T],H) \cap C([0,T],H^{-\beta})$ for every $\beta > 0$, and $\{Q^{1/2}W = Q^{1/2}W^\varepsilon\}_{\varepsilon\in(0,1)}$ is a family of identically distributed $C([0,T],H)$-valued random variables, by the Prokhorov theorem there exists a subsequence $\varepsilon_n \to 0$ such that $(u^{\varepsilon_n}, Q^{1/2}W^{\varepsilon_n})$ converges in distribution as $n \to \infty$ to a process $(u, Q^{1/2}W)$[3] taking values in

$$\mathcal{X} := \left(L^2([0,T],H) \cap C([0,T],H^{-\beta})\right) \times C([0,T],H).$$

Then, given *any* subsequence such that $(u^{\varepsilon_n}, Q^{1/2}W^{\varepsilon_n}) \to (u, Q^{1/2}W)$ in distribution (not necessarily the one provided by the Prokhorov theorem), in virtue of the Skorokhod theorem there exists a new probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ supporting $\mathcal{X}$-valued random variables $(\tilde u, Q^{1/2}\tilde W) \sim (u, Q^{1/2}W)$ and $(\tilde u^n, Q^{1/2}\tilde W^n) \sim (u^{\varepsilon_n}, Q^{1/2}W^{\varepsilon_n})$ for every $n \in \mathbb{N}$ such that $(\tilde u^n, Q^{1/2}\tilde W^n) \to (\tilde u, Q^{1/2}\tilde W)$ $\tilde{\mathbb{P}}$-almost surely as random variables in $\mathcal{X}$. Of course, as is usually done in such situations, we drop the tildes in what follows.

Proposition 5.8. *Let $(u^n, Q^{1/2}W^n) \to (u, Q^{1/2}W)$ as above. Then for every $\varphi \in F$ we have the almost sure identity*

$$\varphi(u_t) = \varphi(u_0) + \int_0^t \mathscr{L}^0\varphi(u_s)\,ds + \int_0^t \langle b((-C)^{-1}Q^{1/2}dW_s, u_s), h\rangle, \quad \forall t \in [0,T]. \tag{5.9}$$

Proof. We divide the proof into three steps. First, we show that the remainder terms are infinitesimal in mean square as $n \to \infty$; second, we prove that the deterministic effective dynamics is a continuous function of the path $\xi \in C([0,T],H^{-\beta}) \cap L^2([0,T],H)$; finally, we invoke a martingale representation theorem to identify the limit behaviour of the martingale term in (5.9).

Step 1. Let us focus on the remainder terms on the right hand side of (5.8). They are of several kinds: (i) terms involving the differences

$$\int_0^t \mathscr{L}^{0,\varepsilon_n}\varphi(u_s^{\varepsilon_n})\,ds - \int_0^t \mathscr{L}^0\varphi(u_s^{\varepsilon_n})\,ds, \quad \int_0^t \langle b(G_{\varepsilon_n}Q^{1/2}dW_s, u_s^{\varepsilon_n}), h\rangle,$$

[3] Recall that any Wiener process with covariance operator Q can be written as $Q^{1/2}W$ for some cylindrical Wiener process W on H.

where $G_{\varepsilon_n} := (-C_{\varepsilon_n})^{-1} - (-C)^{-1} = \varepsilon_n (-C)^{-1} A (-C_\varepsilon)^{-1}$, which are controlled using the bounds $\|G_{\varepsilon_n}\|_{H^s \to H^{s+2\gamma(1+\beta)-2\beta}} \lesssim \varepsilon_n^\beta$ and $\|e^{C_{\varepsilon_n} t} - e^{Ct}\|_{H^{\theta+2\beta} \to H^\theta} \lesssim \varepsilon_n^\beta$ for every $\beta \in [0,1]$, uniformly in $t \in [0,\infty)$, and go to zero in mean square as $n \to \infty$ (and $\varepsilon_n \to 0$); (ii) terms of the form $\varepsilon_n^{1/2} \varphi_1^{\varepsilon_n}(u_t^n, y_t^n)$ or $\varepsilon_n \varphi_2^{\varepsilon_n}(u_t^n, Y_t^n)$, $t \in [0,T]$, $Y^n := Y^{\varepsilon_n}$, which can be easily shown to converge to zero in mean square as $n \to \infty$ as a consequence of energy bounds for $(u^\varepsilon, y^\varepsilon)$, the bound $\|\varphi_2^{\varepsilon_n}(u_t^n, \cdot)\|_{\mathcal{E}^{\theta'}} \lesssim \|h\|_{H^{\theta_1}} \|u_t^n\|_H$ for some $\theta' > 5/4 - 2\gamma$, and

$$
\begin{aligned}
|\varphi_1^{\varepsilon_n}(u_t^n, y_t^n)| &\lesssim \|y_t^n\|_H \|u_t^n\|_H \|h\|_{H^{\theta_0}}, \\
|\varphi_2^{\varepsilon_n}(u_t^n, Y_t^n)| &\lesssim (1 + \|Y_t^n\|_{H^{\theta'}}^2) \|h\|_{H^{\theta_1}} \|u_t^n\|_H;
\end{aligned}
$$

(iii) the term $\int_0^t \Phi_0^{\varepsilon_n}(u_s^n, y_s^n, Y_s^n)\, ds$, which is infinitesimal in mean square by Propositions 4.2 and 2.7; (iv) the terms involving the time integrals of $\Phi_1^{\varepsilon_n}(u_s^n, y_s^n, Y_s^n)$ and $\Phi_2^{\varepsilon_n}(u_s^n, Y_s^n)$, which are controlled by Propositions 4.1 and 4.5 and the estimates

$$
\begin{aligned}
|\Phi_1^{\varepsilon_n}(u_s^n, y_s^n, Y_s^n)| &\lesssim |\langle A u_s^n + b(u_s^n, u_s^n), D_u \varphi_1^{\varepsilon_n}(y_s^n) \rangle| + |\langle b(u_s^n, y_s^n), D_y \varphi_1^{\varepsilon_n}(u_s^n) \rangle| \\
&\quad + |\langle b(y_s^n, u_s^n), D_u \varphi_2^{\varepsilon_n}(Y_s^n) \rangle| \\
&\lesssim \|u_s^n\|_H (1 + \|u_s^n\|_{H^1}) \|y_s^n\|_H \|h\|_{H^{\theta_1}} \\
&\quad + \|u_s^n\|_H \|y_s^n\|_H \|Y_s^n\|_{H^{\theta_0}}^2 \|h\|_{H^{\theta_1}}, \\
|\Phi_2^{\varepsilon_n}(u_s^n, Y_s^n)| &\lesssim \|u_s^n\|_H (1 + \|u_s^n\|_H) \|Y_s^n\|_{H^{\theta_0}}^2 \|h\|_{H^{\theta_1}};
\end{aligned}
$$

and finally (v) the stochastic integral $\varepsilon_n^{1/2} \int_0^t \langle D_Y \varphi_2^{\varepsilon_n}(u_s^n, Y_s^n), Q^{1/2} dW_s^n \rangle$, which by Itô isometry satisfies

$$
\varepsilon_n \mathbb{E}\left[\left(\int_0^t \langle D_Y \varphi_2^{\varepsilon_n}(u_s^n, Y_s^n), Q^{1/2}\, dW_s^n \rangle \right)^2 \right] = \varepsilon_n \mathbb{E}\left[\int_0^t \|Q^{1/2} D_Y \varphi_2^{\varepsilon_n}(u_s^n, Y_s^n)\|_H^2\, ds \right],
$$

which tends to zero. Thus all the remainders converge to zero in mean square.

Step 2. Let us consider, on the path space $\mathcal{X}$ equipped with the Borel σ-field $\mathcal{B}$, the pushforward probability measures

$$
\mathbb{Q}^n := \mathbb{P} \circ (u^n, Q^{1/2} W^n)^{-1}, \quad \mathbb{Q} := \mathbb{P} \circ (u, Q^{1/2} W)^{-1}.
$$

Of course $\mathbb{Q}^n$ weakly converges to $\mathbb{Q}$ as $n \to \infty$. Let $\mathcal{A}$ be the $\mathbb{Q}$-completion of $\mathcal{B}$, and let $\{\mathcal{A}_t\}_{t \in [0,T]}$ be the smallest filtration of $\mathcal{A}$ that satisfies the usual conditions with respect to $\mathbb{Q}$ and such that the coordinate process (ξ, ω) on $\mathcal{X}$ is adapted. Introduce $\mathcal{A}^n$ and $\{\mathcal{A}_t^n\}_{t \in [0,T]}$ similarly. Define the process

$$
\rho_t := \varphi(\xi_t) - \varphi(\xi_0) - \int_0^t \mathcal{L}^0 \varphi(\xi_s)\, ds, \quad t \in [0,T]. \tag{5.10}
$$

Let us show that ρ_t is a continuous function of ξ.

First of all, every $\varphi \in F$ given by $\varphi(\xi) = \langle \xi, h \rangle$ for some $h \in \mathcal{S}$ is a continuous function from $H^{-\beta}$ to $\mathbb{R}$. Therefore if $\xi^n \to \xi$ in $C([0,T], H^{-\beta})$ then $\varphi(\xi^n) \to \varphi(\xi)$ in $C([0,T])$ as well. Let us now consider the term involving the effective generator $\mathcal{L}^0$.

Recall

$$\mathscr{L}^0\varphi(\xi) = \langle A\xi + b(\xi,\xi), h\rangle + \int_H \psi_\xi(w)\, d\mu(w).$$

Let us show that the map $L^2([0,T],H) \ni \xi \mapsto \int_0^\cdot \mathscr{L}^0\varphi(\xi_s)\, ds \in C([0,T])$ is sequentially continuous, or equivalently

$$\int_0^t \mathscr{L}^0\varphi(\xi_s^n)\, ds \to \int_0^t \mathscr{L}^0\varphi(\xi_s)\, ds \tag{5.11}$$

in $C([0,T])$. For every $s \in [0,t]$, rewrite

$$\begin{aligned}\mathscr{L}^0\varphi(\xi_s^n) - \mathscr{L}^0\varphi(\xi_s) &= \langle A\xi_s^n + b(\xi_s^n,\xi_s^n), h\rangle - \langle A\xi_s + b(\xi_s,\xi_s), h\rangle \\ &\quad + \int_H \psi_{\xi_s^n}(w)\, d\mu(w) - \int_H \psi_{\xi_s}(w)\, d\mu(w).\end{aligned}$$

Let us bound each term on the right hand side separately. Making use of the usual estimates on b, we have

$$\begin{aligned}|\langle A(\xi_s^n - \xi_s), h\rangle| &\le \|\xi_s^n - \xi_s\|_H \|h\|_{H^2}, \\ |\langle b(\xi_s^n,\xi_s^n) - b(\xi_s,\xi_s), h\rangle| &\le |\langle b(\xi_s^n, \xi_s^n - \xi_s), h\rangle| + |\langle b(\xi_s^n - \xi_s, \xi_s), h\rangle| \\ &\lesssim (\|\xi_s^n\|_H + \|\xi_s\|_H)\|\xi_s^n - \xi_s\|_H \|h\|_{H^{\theta_0}}, \\ \left|\int_H \psi_{\xi_s^n}(w)\, d\mu(w) - \int_H \psi_{\xi_s}(w)\, d\mu(w)\right| &\lesssim \|h\|_{H^{\theta_1}} \|\xi_s^n - \xi_s\|_H \int_H \|w\|^2_{H^{\theta_0}}\, d\mu(w) \\ &\lesssim \|h\|_{H^{\theta_1}} \|\xi_s^n - \xi_s\|_H.\end{aligned}$$

Putting all together, we obtain

$$|\mathscr{L}^0\varphi(\xi_s^n) - \mathscr{L}^0\varphi(\xi_s)| \lesssim (1 + \|\xi_s^n\|_H + \|\xi_s\|_H)\|\xi_s^n - \xi_s\|_H.$$

In particular, recalling that $\xi^n \to \xi$ in $L^2([0,T],H)$ we have

$$\begin{aligned}\sup_{t\in[0,T]} \left|\int_0^t \mathscr{L}^0\varphi(\xi_s^n)\, ds - \int_0^t \mathscr{L}^0\varphi(\xi_s)\, ds\right| &\le \int_0^T |\mathscr{L}^0\varphi(\xi_s^n) - \mathscr{L}^0\varphi(\xi_s)|\, ds \\ &\lesssim \left(\int_0^T (1 + \|\xi_s^n\|_H + \|\xi_s\|_H)^2\, ds\right)^{1/2} \left(\int_0^T \|\xi_s^n - \xi_s\|_H^2\, ds\right)^{1/2} \to 0.\end{aligned}$$

Step 3. By the weak convergence $\mathbb{Q}^n \to \mathbb{Q}$ and previous steps it is easy to show (cf. for instance [29, Theorem 3.1] or [21, Chapter 8.4]) that (ρ,ω) is a continuous square-integrable martingale on $(\mathcal{X}, \mathcal{A}, \{\mathcal{A}_t\}_{t\in[0,T]}, \mathbb{Q})$ with quadratic covariations ($\{e_k\}_{k\in\mathbb{N}}$ is a complete orthonormal system of H):

$$\begin{aligned}[][\rho,\rho]_t &= \int_0^t \|Q^{1/2} D_y\varphi_1(\xi_s)\|_H^2\, ds, \\ [\rho, \langle e_k,\omega\rangle]_t &= \int_0^t \langle Q^{1/2} D_y\varphi_1(\xi_s), Q^{1/2} e_k\rangle\, ds, \\ [\langle e_k,\omega\rangle, \langle e_k,\omega\rangle]_t &= \langle Q^{1/2} e_k, Q^{1/2} e_k\rangle t.\end{aligned}$$

This is basically due to the fact that ρ can be written as the sum of a martingale on $(\mathcal{X}, \mathcal{A}, \{\mathcal{A}_t^n\}_{t\in[0,T]}, \mathbb{Q}^n)$ plus remainder terms which are infinitesimal in mean square as

$n \to \infty$. By [21, Theorem 8.2], up to a possible enlargement of the underlying probability space, there exists a cylindrical Wiener process $\tilde{\omega}$ on $(\mathcal{X}, \mathcal{A}, \{\mathcal{A}_t\}_{t\in[0,T]}, \mathbb{Q})$ such that the following martingale representation formulae hold $\mathbb{Q}$-almost surely for every $t \in [0, T]$:

$$\omega_t = \int_0^t Q^{1/2} d\tilde{\omega}_s = Q^{1/2}\tilde{\omega}_t,$$

$$\rho_t = \int_0^t \langle D_y\varphi_1(\xi_s), Q^{1/2} d\tilde{\omega}_s\rangle = \int_0^t \langle D_y\varphi_1(\xi_s), d\omega_s\rangle.$$

In particular, since the auxiliary Wiener process $\tilde{\omega}$ satisfies $\omega_t = Q^{1/2}\tilde{\omega}_t$, the equation for ρ_t above holds true also in the original probability space, without necessarily taking an enlargement thereof. Thus, recalling (5.10) we have the following $\mathbb{Q}$-almost sure identity on the path space $\mathcal{X}$:

$$\varphi(\xi_t) = \varphi(\xi_0) + \int_0^t \mathscr{L}^0\varphi(\xi_s)\, ds + \int_0^t \langle D_y\varphi_1(\xi_s), d\omega_s\rangle, \quad \forall t \in [0, T],$$

which by $\mathbb{Q} = \mathbb{P} \circ (u, Q^{1/2}W)^{-1}$ and the explicit expression of φ_1 in (5.6) is equivalent to our conclusion. ■

5.3. *Itô–Stokes drift and Stratonovich corrector*

In this subsection, we provide an interpretation of the limiting equation in terms of different contributions to the dynamics. Recall that every weak accumulation point u of the family $\{u^\varepsilon_t\}_{\varepsilon\in(0,1)}$ satisfies, for every $\varphi \in F$ with $\varphi(u) = \langle u, h\rangle$ for some $h \in \mathscr{S}$, and all $t \in [0, T]$, the almost sure identity (5.9):

$$\varphi(u_t) = \varphi(u_0) + \int_0^t \mathscr{L}^0\varphi(u_s)\, ds + \int_0^t \langle b((-C)^{-1}Q^{1/2}dW_s, u_s), h\rangle,$$

where the limiting effective generator $\mathscr{L}^0$ is given by (5.7). For the reader's convenience, here we rewrite $\mathscr{L}^0\varphi$ more explicitly as

$$\begin{aligned}\mathscr{L}^0\varphi(u) &= \langle Au + b(u, u), h\rangle + \int_H \langle b(w, u), D_u\varphi_1(w)\rangle\, d\mu(w)\\ &\quad + \int_H \langle b(w, w), D_y\varphi_1(u)\rangle\, d\mu(w)\\ &= \langle Au + b(u, u), h\rangle + \int_H \langle b((-C)^{-1}w, b(w, u)), h\rangle\, d\mu(w)\\ &\quad + \int_H \langle b((-C)^{-1}b(w, w), u), h\rangle\, d\mu(w).\end{aligned}$$

Let us compare (5.9) with the dynamics of u^ε for $\varepsilon \in (0, 1)$. Of course, the term $\langle Au_s + b(u_s, u_s), h\rangle$ reflects the deterministic dynamics $\langle Au^\varepsilon_s + b(u^\varepsilon_s, u^\varepsilon_s), h\rangle$ in the evolution of u^ε.

On the other hand, the fast-oscillating term $\varepsilon^{-1/2}\langle b(y^\varepsilon_s, u^\varepsilon_s), h\rangle$ in the equation for u^ε is responsible for the additional terms in the limit. We can distinguish three different contributions:

- the Itô integral:

$$\int_0^t \langle b((-C)^{-1} Q^{1/2} dW_s, u_s), h\rangle; \tag{5.12}$$

- the Stratonovich corrector:

$$\int_0^t \int_H \langle b((-C)^{-1} w, b(w, u_s)), h\rangle \, d\mu(w)\, ds; \tag{5.13}$$

- the Itô–Stokes drift:

$$\int_0^t \int_H \langle b((-C)^{-1} b(w, w), u_s), h\rangle \, d\mu(w)\, ds. \tag{5.14}$$

The term named "Itô–Stokes drift" clearly equals $\int_0^t \langle b(r, u_s), h\rangle \, ds$ by the very definition of the Itô–Stokes drift velocity $r = \int_H (-C)^{-1} b(w, w)\, d\mu(w)$. Moreover, we shall see that the term called "Stratonovich corrector" above is indeed the Stratonovich corrector of the term called "Itô integral":

$$\int_0^t \langle D_y \varphi_1(u_s), Q^{1/2} \circ dW_s\rangle = \int_0^t \langle D_y \varphi_1(u_s), Q^{1/2} dW_s\rangle + \int_0^t \int_H \langle b(w, u_s), D_u \varphi_1(w)\rangle \, d\mu(w)\, ds. \tag{5.15}$$

Therefore, (5.9) is exactly the weak formulation of the equation in Theorem 5.1, which is thus proved.

We are left to check the validity of (5.15). Fix $h \in \mathscr{S}$, and let $W = \sum_k e_k W^k$, where $\{e_k\}_{k\in\mathbb{N}}$ is a complete orthonormal system in H and $\{W^k\}_{k\in\mathbb{N}}$ is a family of one-dimensional i.i.d. Wiener processes. One can rewrite the Itô integral (5.12) as

$$\int_0^t \langle b((-C)^{-1} Q^{1/2} dW_s, u_s), h\rangle = -\sum_{k\in\mathbb{N}} \int_0^t \langle b((-C)^{-1} Q^{1/2} e_k, h), u_s\rangle \, dW_s^k;$$

since for $h \in \mathscr{S}$ we have $b((-C)^{-1} Q^{1/2} e_k, h) \in \mathscr{S}$ as well, the quadratic variation between the processes $\langle b((-C)^{-1} Q^{1/2} e_k, h), u\rangle$ and W^k is given by

$$[\langle b((-C)^{-1} Q^{1/2} e_k, h), u\rangle, W^k]_t = -\int_0^t \langle b((-C)^{-1} Q^{1/2} e_k, u_s), b((-C)^{-1} Q^{1/2} e_k, h)\rangle \, ds. \tag{5.16}$$

On the other hand, (5.13) equals

$$\int_0^t \int_H \langle b((-C)^{-1} w, b(w, u_s)), h\rangle \, d\mu(w)\, ds = \sum_{k\in\mathbb{N}} \int_0^t \langle b((-C)^{-1} Q_\infty^{1/2} e_k, b(Q_\infty^{1/2} e_k, u_s)), h\rangle \, ds = -\sum_{k\in\mathbb{N}} \int_0^t \langle b((-C)^{-1} Q_\infty^{1/2} e_k, h), b(Q_\infty^{1/2} e_k, u_s)\rangle \, ds. \tag{5.17}$$

Recall that since C and Q commute, the covariance operator Q_∞ of the invariant measure $\mu = \mathcal{N}(0, Q_\infty)$ can be written as $Q_\infty = \frac{1}{2}(-C)^{-1}Q$. In particular, there exists a complete orthonormal system $\{e_k\}_{k\in\mathbb{N}}$ for H that diagonalizes C, Q and Q_∞ simultaneously

$$Ce_k = -\lambda_k e_k, \quad Qe_k = q_k e_k, \quad Q_\infty e_k = \frac{q_k}{2\lambda_k} e_k,$$

and therefore by (5.16) and (5.17) we finally get

$$\frac{1}{2}[\langle b((-C)^{-1}Q^{1/2}e_k, h), u\rangle, W^k]_t = \int_0^t \int_H \langle b((-C)^{-1}w, b(w, u_s)), h\rangle \, d\mu(w)\, ds.$$

Remark 5.9. (i) The attentive reader would have noticed that this is the first time we are actually using the fact that C and Q commute. In particular, the convergence result described in the previous sections still holds true when $CQ \neq QC$. In this case, we believe that

$$\sum_{k\in\mathbb{N}} \int_0^t \langle b((-C)^{-1}Q^{1/2}e_k, u_s), b((-C)^{-1}Q^{1/2}e_k, h)\rangle \, ds$$
$$\neq \sum_{k\in\mathbb{N}} \int_0^t \langle b((-C)^{-1}Q_\infty^{1/2}e_k, h), b(Q_\infty^{1/2}e_k, u_s)\rangle \, ds$$

and the limit equation cannot be interpreted as an equation with Stratonovich transport noise.

(ii) Assume that Q is isotropic, i.e. every basis of eigenvectors of A also diagonalizes Q. The Itô–Stokes drift then equals zero for the Navier–Stokes system,[4] since for the particular choice

$$e_{\mathbf{k},i} = a_{\mathbf{k},i}\cos(2\pi\mathbf{k}\cdot x) \quad \text{or} \quad e_{\mathbf{k},i} = a_{\mathbf{k},i}\sin(2\pi\mathbf{k}\cdot x),$$

where $\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, $i = 1, \dots, d-1$, and $a_{\mathbf{k},i} \in \mathbf{k}^\perp$ for all $\mathbf{k}, i$,

$$b(e_{\mathbf{k},i}, e_{\mathbf{k},i}) = -\Pi((e_{\mathbf{k},i}\cdot\nabla)e_{\mathbf{k},i}) = \pm\Pi\Big(\underbrace{(a_{\mathbf{k},i}\cdot 2\pi\mathbf{k})}_{=0}\cos(2\pi\mathbf{k}\cdot x)\sin(2\pi\mathbf{k}\cdot x)\Big) = 0.$$

6. Eddy viscosity and convergence to deterministic Navier–Stokes equations

In this section we restrict our attention to the Navier–Stokes system, and discuss the interesting problem of a varying covariance operator $Q = Q_N$, where $N \in \mathbb{N}$ is a parameter possibly depending on the scaling parameter $\varepsilon \in (0, 1)$. We shall identify conditions under which it is possible to prove convergence to a process u that solves the deterministic

[4]Similar conditions on Q can be found for other models, such as the primitive equations and the surface quasi-geostrophic equations.

Navier–Stokes equations with an additional dissipative term:

$$du_t = Au_t dt + b(u_t, u_t)dt + \kappa(u_t)dt, \tag{6.1}$$

for some (possibly unbounded) linear operator $\kappa : H \supset D(\kappa) \to H$, and the convergence is meant in the sense of Proposition 5.8.

It is well-known from experiments that, in fluids, turbulent regimes manifest a dissipation enhancement with respect to laminar flows. Following the literature we refer to the operator κ as *eddy viscosity*. To the best of our knowledge this is the first rigorous result showing that eddy viscosity in 3D Navier–Stokes can be deduced from additive noise, acting on idealized small scales of the fluid itself, in the limit of infinite separation of scales (cf. [59] for a similar result on the transport equation). In SPDE theory similar results have been proved for equations with Stratonovich transport noise, taking advantage of the explicit expression for the Stratonovich-to-Itô corrector and a suitable scaling limit to show that the Itô integral vanishes in the limit [27, 28, 31]. However, these results are mostly restricted to the two-dimensional case. The only result in dimension 3 we are aware of is [32], where Navier–Stokes equations in vorticity form are investigated; however, as admitted by the authors themselves, the noise used in [32] lacks physical meaning, since the vortex stretching term is neglected for technical limitations. Here, we apply their ideas of scaling parameters in such a way that, in the limit, the Itô integral does vanish whereas the Stratonovich corrector does not.

Accordingly, we consider a modification of (1.3) where the covariance operator depends on a parameter $N \in \mathbb{N}$:

$$\begin{cases} du_t^{\varepsilon,N} = Au_t^{\varepsilon,N} dt + b(u_t^{\varepsilon,N}, u_t^{\varepsilon,N})dt + \varepsilon^{-1/2} b(y_t^{\varepsilon,N}, u_t^{\varepsilon,N})dt, \\ dy_t^{\varepsilon,N} = \varepsilon^{-1} C y_t^{\varepsilon,N} dt + Ay_t^{\varepsilon,N} dt + b(u_t^{\varepsilon,N}, y_t^{\varepsilon,N})dt \\ \qquad\qquad + \varepsilon^{-1/2} b(y_t^{\varepsilon,N}, y_t^{\varepsilon,N})dt + \varepsilon^{-1/2} Q_N^{1/2} dW_t. \end{cases} \tag{6.2}$$

We assume that Q_N satisfies assumptions (Q1)–(Q2) for every fixed $N > 0$. It is easy to show that for every N there exists a bounded-energy family $\{(u^{\varepsilon,N}, y^{\varepsilon,N})\}_{\varepsilon\in(0,1)}$ of weak martingale solutions to (6.2). As a consequence, results of the previous sections hold true for the families $\{u^{\varepsilon,N}\}_{\varepsilon\in(0,1)}$ with N fixed, so that for every $N \in \mathbb{N}$ there exists a subsequence $\varepsilon_n \to 0$ such that $u^{\varepsilon_n,N} \to u^N$ in law as $n \to \infty$, where u^N satisfies for all $\varphi \in F$ and $t \in [0, T]$ the almost sure identity

$$\varphi(u_t^N) = \varphi(u_0) + \int_0^t \mathscr{L}^{0,N} \varphi(u_s^N)\, ds + \int_0^t \langle D_y \varphi_1(u_s^N), Q_N^{1/2} dW_s^N \rangle \tag{6.3}$$

with (μ^N below denotes the invariant measure of $dv = Cv dt + Q_N^{1/2} dW_t$)

$$\mathscr{L}^{0,N} \varphi(u) = \langle Au + b(u,u), h \rangle + \int_H \psi_u(w)\, d\mu^N(w).$$

In addition, the family $\{u^{\varepsilon,N}\}_{\varepsilon\in(0,1),\, N\in\mathbb{N}}$ can be chosen in such a way that the energy bound (S3) holds uniformly in $\varepsilon \in (0, 1)$ and $N \in \mathbb{N}$ (energy estimates on $u^{\varepsilon,N}$ do not involve the covariance operator Q_N).

When Q_N is isotropic, we have seen in Remark 5.9 (ii) that the Itô–Stokes drift vanishes. As for the Stratonovich corrector, recall that $Q_{N,\infty} = \frac{1}{2}(-C)^{-1}Q_N$ and there exists a simultaneous eigenbasis $\{e_k\}_{k\in\mathbb{N}}$ of C and Q_N such that $Ce_k = -\lambda_k e_k$ and $Q_N e_k = q_{N,k}e_k$, and therefore the candidate operator κ is formally given by

$$\kappa(u) := \lim_{N\to\infty} \kappa_N(u) := \lim_{N\to\infty} \sum_{k\in\mathbb{N}} \frac{q_{N,k}}{2\lambda_k^2} b(e_k, b(e_k, u)) \tag{6.4}$$

and it is easy to check by (B4) that κ is symmetric and

$$\langle \kappa(u), u\rangle = \lim_{N\to\infty} \sum_{k\in\mathbb{N}} \frac{q_{N,k}}{2\lambda_k^2} \langle b(e_k, b(e_k, u)), u\rangle = -\lim_{N\to\infty} \sum_{k\in\mathbb{N}} \frac{q_{N,k}}{2\lambda_k^2} \|b(e_k, u)\|_H^2.$$

Theorem 6.1. *Let Q_N satisfy* (Q1)–(Q2) *for every fixed $N \in \mathbb{N}$. Suppose Q_N is isotropic, and there exists some $\sigma_0 < \infty$ such that for every $N \in \mathbb{N}$ the operator κ_N given by* (6.4) *is well defined and continuous from H^{σ_0} to H, and for every fixed $u \in H^{\sigma_0}$ we have $\kappa_N(u) \to \kappa(u)$ in H as $N \to \infty$. Finally, assume*

$$\lim_{N\to\infty} \sum_{k\in\mathbb{N}} \|Q_N^{1/2} e_k\|_{H^{-1-2\gamma}}^2 = 0.$$

Then for every $\beta > 0$ the laws of the processes $\{u^{\varepsilon,N}\}_{\varepsilon\in(0,1),N\in\mathbb{N}}$ are tight as probability measures on $L^2([0,T],H) \cap C([0,T],H^{-\beta})$, and every weak accumulation point u of the family $\{u^{\varepsilon,N}\}_{\varepsilon\in(0,1),\,N\in\mathbb{N}}$ as $\varepsilon \to 0$ and then $N \to \infty$ is a weak solution of (6.1)*, meaning that for every $t \in [0,T]$ and $h \in \mathscr{S}$ we have almost surely*

$$\langle u_t, h\rangle = \langle u_0, h\rangle + \int_0^t \langle u_s, Ah\rangle\, ds + \int_0^t \langle b(u_s, u_s), h\rangle\, ds + \int_0^t \langle u_s, \kappa(h)\rangle\, ds.$$

Assume in addition that $u_0 \in H^1$ and there exists κ_0 sufficiently large such that $\langle \kappa(u), u\rangle \le -\kappa_0 \|u\|_{H^1}^2$ for every $u \in H^1$. Then $u \in L^\infty([0,T],H^1) \cap L^2([0,T],H^2)$ is the unique strong solution of (6.1) *on $[0,T]$, and the whole sequence $\{u^{\varepsilon,N}\}_{\varepsilon\in(0,1),\,N\in\mathbb{N}}$ converges in law to u as $\varepsilon \to 0$ and $N \to \infty$.*

Remark 6.2. A careful inspection of the convergence proof in Section 5.2 shows that we have the same result for the family $\{u^{\varepsilon,N_\varepsilon}\}_{\varepsilon\in(0,1)}$ when $\varepsilon \to 0$ and $N = N_\varepsilon$ satisfies the condition that there exists $\delta \in (0, \gamma - 1/4)$ such that

$$\varepsilon^{\delta/(2(1-\gamma))} \operatorname{Tr}((-A)^{\theta_0/2}(-C_\varepsilon)^{-1} Q_{N_\varepsilon}^{1/2}) \to 0.$$

Proof of Theorem 6.1. By Theorem 5.1 for every $N \in \mathbb{N}$ the family $\{u^{\varepsilon,N}\}_{\varepsilon\in(0,1)}$ converges in law as processes taking values in $L^2([0,T],H) \cap C([0,T],H^{-\beta})$ towards a weak martingale solution u^N of

$$\begin{aligned} du_t^N &= Au_t^N + b(u_t^N, u_t^N)dt + b((-C)^{-1} Q_N^{1/2} \circ dW_t^N, u_t^N) \\ &= Au_t^N + b(u_t^N, u_t^N)dt + \kappa_N(u_t^N)dt + b((-C)^{-1} Q_N^{1/2} dW_t^N, u_t^N). \end{aligned}$$

The Itô integral above vanishes when tested against smooth functions in the limit $N \to \infty$: indeed, for every given $h \in \mathscr{S}$ and $s, t \in [0, T]$ with $s < t$ we have

$$\mathbb{E}\left[\left(\int_s^t \langle b((-C)^{-1} Q_N^{1/2} dW_t^N, u_t^N), h\rangle\right)^2\right] \lesssim |t-s| \, \|h\|_{H^{\theta_1}}^2 \sum_{k\in\mathbb{N}} \|Q_N^{1/2} e_k\|_{H^{-1-2\gamma}}^2 \to 0.$$

In addition, by Simon's compactness criterion the family $\{u^N\}_{N\in\mathbb{N}}$ is tight in

$$L^2([0,T], H) \cap C([0,T], H^{-\beta})$$

since, on the one hand, uniform bounds in N for the expectation of the norm of u^N in $L^\infty([0,T], H) \cap L^2([0,T], H^1)$ are inherited from the approximating family $\{u^{\varepsilon,N}\}_{\varepsilon\in(0,1),\, N\in\mathbb{N}}$ (see Remark 5.2 (ii)), whereas by our assumption on κ and the Banach–Steinhaus theorem $\sup_{N\in\mathbb{N}} \|\kappa_N\|_{H^{\sigma_0}\to H} < \infty$ and thus

$$\sup_{N\in\mathbb{N}} \mathbb{E}[\|u^N\|_{W^{\alpha,p}([0,T],H^{-\sigma})}^p] \lesssim 1 + \sup_{N\in\mathbb{N}} \int_0^T \int_0^T \frac{\mathbb{E}[\|u_t^N - u_s^N\|_{H^{-\sigma}}^p]}{|t-s|^{1+\alpha p}} < \infty$$

for some σ sufficiently large, $p > 2$ and $\alpha > 1/p$.

To summarize the previous steps, by a diagonal argument there exists a subsequence $\{(\varepsilon_n, N_n)\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ and $N_n \to \infty$ such that u^{ε_n,N_n} converges weakly to some random variable u taking values in $L^2([0,T], H) \cap C([0,T], H^{-\beta})$. Moreover, arguing as in the proof of Proposition 5.8 it is easy to show that u is almost surely a weak solution of the deterministic equation (6.1) (notice that u could be random nonetheless, because of the lack of uniqueness in the limit equation).

Finally, under the stronger assumptions on u_0 and κ it is well-known [63] that Navier–Stokes equations have a unique strong solution on $[0,T]$, where $T = T(u_0, \nu, \kappa_0)$ can be made arbitrarily large by taking κ_0 sufficiently large. By weak-strong uniqueness, u is the strong solution, and therefore the whole sequence $\{u^{\varepsilon,N}\}_{\varepsilon\in(0,1),\, N\in\mathbb{N}}$ converges since its limit does not depend on the subsequence. ∎

Remark 6.3. As an example we recall the scaling limit construction in [32]. Let $d = 3$, $\mathbb{Z}_+^d \cup \mathbb{Z}_-^d$ be a partition of $\mathbb{Z}_0^d := \mathbb{Z}^d \setminus \{\mathbf{0}\}$ such that $\mathbb{Z}_+^d = -\mathbb{Z}_-^d$, $\mathbf{k} \in \mathbb{Z}_0^d$, and $\{a_{\mathbf{k},i}\}_{i=1,2}$ be an orthonormal basis of $\mathbf{k}^\perp$ such that $\{a_{\mathbf{k},1}, a_{\mathbf{k},2}, \mathbf{k}^\perp/|\mathbf{k}|\}$ is right-handed and $a_{-\mathbf{k},i} = -a_{\mathbf{k},i}$ for every $\mathbf{k} \in \mathbb{Z}_+^d$. Introducing the Fourier basis $e_{\mathbf{k},i}(x) := \sqrt{2} a_{\mathbf{k},i} \cos(2\pi \mathbf{k}\cdot x)$ for $\mathbf{k} \in \mathbb{Z}_+^d$, and $e_{\mathbf{k},i}(x) := \sqrt{2} a_{\mathbf{k},i} \sin(2\pi \mathbf{k}\cdot x)$ for $\mathbf{k} \in \mathbb{Z}_-^d$, and the covariance operators Q_N given by

$$Q_N^{1/2} e_{\mathbf{k},i} := C_{\kappa_0} \left(\sum_{N\le|\mathbf{k}|\le 2N} \frac{1}{|\mathbf{k}|^{2\delta}}\right)^{-1/2} \frac{e_{\mathbf{k},i}}{|\mathbf{k}|^\delta} \mathbf{1}_{\{N\le|\mathbf{k}|\le 2N\}}, \quad \delta > 0,$$

we are in the setting of Theorem 6.1 with limit eddy viscosity κ given by a multiple of the Laplace operator: $\kappa(u) = C'_{\kappa_0} \Delta u$ for every $u \in H^2$ [32, Theorem 5.1]. Moreover, for every $\kappa_0 > 0$ the constant C_{κ_0} can be chosen in such a way that $\langle \kappa(u), u\rangle \le -\kappa_0 \|u\|_{H^1}^2$ for every $u \in H^1$, so that the second part of our theorem holds, too.

7. Other models

In the main body of this paper our principal focus was on the system of 3D Navier–Stokes equations. In this section we present other models of interest, namely the 2D Navier–Stokes equations with strong friction and unitary viscosity at small scales, the 2D surface quasi-geostrophic equations and the primitive equations. We shall see how the proof of Theorem 5.1 (as well as the preliminary work needed in the proof) can be adapted to those models with little changes in the arguments. Notice that the results in Section 6 will not be extended to other models, but remain a peculiarity of the Navier–Stokes equations.

7.1. 2D Navier–Stokes with pure friction

In the 2D case, we can weaken the assumptions on the operator C. Unless some additional dissipation mechanism is present at small scales, assumption (C2) is not completely satisfactory and we would therefore consider pure friction, i.e. $C = -\chi \operatorname{Id}$ for some $\chi > 0$.

For instance, comparing our model with [37], where the authors study L^∞ solutions of 2D Navier–Stokes equations in vorticity form, it could be sensible and physically meaningful to consider instead the system

$$\begin{cases} du^\varepsilon_t = \nu\Delta u^\varepsilon_t dt - \Pi(u^\varepsilon_t \cdot \nabla)u^\varepsilon_t dt - \varepsilon^{-1/2}\Pi(y^\varepsilon_t \cdot \nabla)u^\varepsilon_t dt, \\ dy^\varepsilon_t = \nu\Delta y^\varepsilon_t dt - \chi\varepsilon^{-1} y^\varepsilon_t dt - \Pi(u^\varepsilon_t \cdot \nabla)y^\varepsilon_t dt \\ \qquad\quad - \varepsilon^{-1/2}\Pi(y^\varepsilon_t \cdot \nabla)y^\varepsilon_t dt + \varepsilon^{-1/2} d\Pi\mathcal{W}_t, \end{cases} \tag{7.1}$$

where the large dissipation has the form of a friction term ($C = -\chi Id$). The previous equations are posed in the two-dimensional torus $\mathbb{T}^2$, and take the abstract form

$$\begin{cases} du^\varepsilon_t = Au^\varepsilon_t dt + b(u^\varepsilon_t, u^\varepsilon_t)dt + \varepsilon^{-1/2} b(y^\varepsilon_t, u^\varepsilon_t)dt, \\ dy^\varepsilon_t = Ay^\varepsilon_t dt - \chi\varepsilon^{-1} y^\varepsilon_t dt + b(u^\varepsilon_t, y^\varepsilon_t)dt + \varepsilon^{-1/2} b(y^\varepsilon_t, y^\varepsilon_t)dt + \varepsilon^{-1/2} Q^{1/2} dW_t. \end{cases}$$

Therefore, in this section, we restrict to the 2D case and prove that our result indeed extends to pure friction provided smoother initial data and noise are taken.

In this setting, our analysis revolves around the ε-dependent Ornstein–Uhlenbeck semigroup associated to the small-scale equation

$$dY^y_t = (\varepsilon A - \chi)Y^y_t dt + Q^{1/2} dW_t, \quad Y^y_0 = y.$$

Assume (Q1), and replace (Q2) with the assumption $\int_H \|w\|^2_{H^4} \, d\mathcal{N}(0, Q)(w) < \infty$. Moreover, suppose the initial datum y_0 is in H^1 (whereas $u_0 \in H$ as usual). Then we can prove an analogue of Theorem 5.1:

Theorem 7.1. *Under the previous assumptions, let $\{(u^\varepsilon, y^\varepsilon)\}_{\varepsilon\in(0,1)}$ be a bounded-energy family of probabilistically strong solutions to (7.1) on a common stochastic basis*

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W),$$

which exists by Proposition 2.3 and pathwise uniqueness. Then, for every $\beta > 0$, u^ε *converges in* $L^2([0,T],H) \cap C([0,T],H^{-\beta})$ *in probability as* $\varepsilon \to 0$ *towards the unique solution* u *of the limiting equation (here the Itô–Stokes drift velocity* r *is defined as* $r = \chi^{-1}\int b(w,w)\,d\mu(w)$, *and* $\mu = \mathcal{N}(0,Q_\infty)$ *with* $Q_\infty = (2\chi)^{-1}Q$)

$$\begin{aligned} du_t &= Au_t dt + b(u_t,u_t)dt + \chi^{-1} b(Q^{1/2}\circ dW_t, u_t) + b(r,u_t)dt \\ &= \nu\Delta u_t dt - \Pi(u_t\cdot\nabla)u_t dt - \chi^{-1}\Pi(Q^{1/2}\circ dW_t\cdot\nabla)u_t - \Pi(r\cdot\nabla)u_t dt. \end{aligned}$$

Here we limit ourselves to point out the major modifications needed to prove the result in the present setting (for simplicity we take $\chi = 1$ in the following):

• The conclusion of Lemma 2.5 changes into

$$\sup_{\substack{\varepsilon\in(0,1)\\ n\in\mathbb{N}}} \sup_{t\in[0,T]} \int_0^t \mathbb{E}[\|y_s^{\varepsilon,n}\|_H^{2p}]\,ds \lesssim 1,$$

with minor modifications in the proof; notice that this is still sufficient to prove the analogue of Lemma 2.6 in the present setting.

• Making use of the key identity $\langle b(u,u), Au\rangle = 0$ [63, Lemma 3.1], one can improve Proposition 2.7 in the following way: applying the Itô formula to

$$\|\zeta_t^{\varepsilon,n}\|_{H^1}^2 = \langle \zeta_t^{\varepsilon,n}, (-A)\zeta_t^{\varepsilon,n}\rangle$$

and the Young inequality, one gets

$$\begin{aligned} \frac{1}{2}\|\zeta_t^{\varepsilon,n}\|_{H^1}^2 - \frac{1}{2}\|y_0\|_{H^1}^2 &+ \int_0^t \|\zeta_s^{\varepsilon,n}\|_{H^2}^2\,ds + \varepsilon^{-1}\int_0^t \|\zeta_s^{\varepsilon,n}\|_{H^1}^2\,ds \\ &= -\int_0^t \langle b(u_s^{\varepsilon,n}, \zeta_s^{\varepsilon,n} + Y_s^{\varepsilon,n}), A\zeta_s^{\varepsilon,n}\rangle\,ds \\ &\quad + \varepsilon^{-1/2}\int_0^t \langle b(\zeta_s^{\varepsilon,n} + Y_s^{\varepsilon,n}, \zeta_s^{\varepsilon,n} + Y_s^{\varepsilon,n}), AY_s^{\varepsilon,n}\rangle\,ds \\ &\le \int_0^t \|u_s^{\varepsilon,n}\|_{H^1}\|\zeta_s^{\varepsilon,n}\|_{H^1}\|\zeta_s^{\varepsilon,n}\|_{H^2}\,ds \\ &\quad + \int_0^t \|u_s^{\varepsilon,n}\|_{L^2}\|Y_s^{\varepsilon,n}\|_{H^4}\|\zeta_s^{\varepsilon,n}\|_{H^2}\,ds \\ &\quad + \varepsilon^{-1/2}\int_0^t \|\zeta_s^{\varepsilon,n} + Y_s^{\varepsilon,n}\|_H\|\zeta_s^{\varepsilon,n} + Y_s^{\varepsilon,n}\|_{H^1}\|Y_s^{\varepsilon,n}\|_{H^4}\,ds \\ &\le \frac{1}{2}\int_0^t \|\zeta_s^{\varepsilon,n}\|_{H^2}^2\,ds + M\int_0^t \|u_s^{\varepsilon,n}\|_{H^1}^2\|\zeta_s^{\varepsilon,n}\|_{H^1}^2\,ds \\ &\quad + \frac{1}{2}\varepsilon^{-1}\int_0^t \|\zeta_s^{\varepsilon,n}\|_{H^1}^2\,ds + M\int_0^t \|\zeta_s^{\varepsilon,n}\|_H^4\,ds \\ &\quad + M\int_0^t \|Y_s^{\varepsilon,n}\|_{H^4}^4\,ds \end{aligned}$$

for some unimportant finite constant M. For the first inequality above we have used the identity for $u, \zeta \in \mathscr{S}$ (with repeated indices summed over)

$$\begin{aligned}\frac{1}{\nu}\langle b(u,\zeta), A\zeta\rangle &= -\int_{\mathbb{T}^2} u^i \partial_i \zeta^j \partial_{kk}\zeta^j \, dx \\ &= \int_{\mathbb{T}^2} \partial_k (u^i \partial_i \zeta^j)\partial_k \zeta^j \, dx \\ &= \int_{\mathbb{T}^2} \partial_k u^i \partial_i \zeta^j \partial_k \zeta^j \, dx + \int_{\mathbb{T}^2} u^i \partial_i \left(\frac{1}{2}\partial_k \zeta^j \partial_k \zeta^j\right) dx \\ &= \int_{\mathbb{T}^2} \partial_k u^i \partial_i \zeta^j \partial_k \zeta^j \, dx - \int_{\mathbb{T}^2} \underbrace{\mathrm{div}(u)}_{=0}\left(\frac{1}{2}\partial_k \zeta^j \partial_k \zeta^j\right) dx,\end{aligned}$$

and the bound (recall the Sobolev embedding $H^{1/2} \subset L^4$)

$$\begin{aligned}&\left|\int_{\mathbb{T}^2} \partial_k u^i \partial_i \zeta^j \partial_k \zeta^j \, dx\right| \\ &\quad\lesssim \left(\int_{\mathbb{T}^2} |\partial_k u^i|^2 \, dx\right)^{1/2}\left(\int_{\mathbb{T}^2} |\partial_i \zeta^j|^4 \, dx\right)^{1/4}\left(\int_{\mathbb{T}^2} |\partial_k \zeta^j|^4 \, dx\right)^{1/4} \\ &\quad\lesssim \|u\|_{H^1}\|\nabla\zeta\|_{L^4}^2 \lesssim \|u\|_{H^1}\|\nabla\zeta\|_{H^{1/2}}^2 \lesssim \|u\|_{H^1}\|\zeta\|_{H^1}\|\zeta\|_{H^2}.\end{aligned}$$

Therefore, because $y_0 \in H^1$, for ε sufficiently small we have

$$\sup_{t\in[0,T]}\left(\mathbb{E}[\|\zeta_t^{\varepsilon,n}\|_{H^1}^2] + \int_0^t \mathbb{E}[\|\zeta_s^{\varepsilon,n}\|_{H^2}^2]\, ds + \varepsilon^{-1}\int_0^t \mathbb{E}[\|\zeta_s^{\varepsilon,n}\|_{H^1}^2]\, ds\right) \lesssim 1. \quad (7.2)$$

Similarly, for every $p \geq 2$ one can prove

$$\sup_{t\in[0,T]}\left(\mathbb{E}[\|\zeta_t^{\varepsilon,n}\|_{H^1}^p] + \int_0^t \mathbb{E}[\|\zeta_s^{\varepsilon,n}\|_{H^1}^{p-2}\|\zeta_s^{\varepsilon,n}\|_{H^2}^2]\, ds + \varepsilon^{-1}\int_0^t \mathbb{E}[\|\zeta_s^{\varepsilon,n}\|_{H^1}^p]\, ds\right) \lesssim 1.$$

• Thanks to the bounds (7.2), the following estimates for the time increments of $u^\varepsilon, y^\varepsilon$ hold true (Lemma 5.5).

$$\begin{aligned}\mathbb{E}[\|u_t^\varepsilon - u_s^\varepsilon\|_{H^{-1}}^p] &\lesssim \varepsilon^{-1/2}|t-s|^{p/2}, \\ \mathbb{E}[\|Y_t^\varepsilon - Y_s^\varepsilon\|_H^p] &\lesssim \varepsilon^{-1/2}|t-s|^{p/2}, \\ \mathbb{E}[\|y_t^\varepsilon - y_s^\varepsilon - (Y_t^\varepsilon - Y_s^\varepsilon)\|_H^p] &\lesssim \varepsilon^{-1/2}|t-s|^{p/2}.\end{aligned}$$

• The results of Section 3 can be obtained by replacing C_ε with $\varepsilon A - \mathrm{Id}$. As a consequence, the Poisson equation improves regularity of the datum (cf. Corollary 3.5 and Proposition 3.6) only at the price of factors ε^{-1}, whereas preserves regularity uniformly in ε.

• The first corrector φ_1^ε is given by $\varphi_1^\varepsilon(u,y) = \phi_u^\varepsilon(y) = \langle b((-\varepsilon A + \mathrm{Id})^{-1}y, u), h\rangle$. In particular, by interpolation we can prove the following analogue of Proposition 4.1: for every $\theta \in [0,1]$ and $s \in \mathbb{R}$ there exists $\theta_1 = \theta_1(s)$ such that

$$\|D_y\varphi_1^\varepsilon(u)\|_{H^{2\theta+s}} \lesssim \varepsilon^{-\theta}\|h\|_{H^{\theta_1}}\|u\|_{H^s}, \quad \|D_u\varphi_1^\varepsilon(y)\|_{H^{2\theta+s}} \lesssim \varepsilon^{-\theta}\|h\|_{H^{\theta_1}}\|y\|_{H^s}.$$

• The linearization trick relies on Proposition 4.2. As usual, denote $\zeta = y - Y$. Then for every $\theta \in (1,2)$ the following bounds hold true:

$$\begin{aligned}
|\langle b(\zeta,u), D_u\varphi_1(Y)\rangle| &\lesssim \|\zeta\|_{H^1}\|u\|_H\|Y\|_{H^\theta}\|h\|_{\theta_1},\\
|\langle b(y,u), D_u\varphi_1(\zeta)\rangle| &\lesssim \|b(y,u)\|_{H^{-1}}\|\zeta\|_{H^1}\|h\|_{\theta_1}\\
&\lesssim \|y\|_{H^1}\|u\|_{H^{\theta-1}}\|\zeta\|_{H^1}\|h\|_{\theta_1},\\
|\langle b(\zeta,Y), D_y\varphi_1(u)\rangle &\lesssim \|\zeta\|_{H^1}\|Y\|_{H^\theta}\|u\|_H\|h\|_{\theta_1},\\
|\langle b(y,\zeta), D_y\varphi_1(u)| &\lesssim \|y\|_{H^1}\|\zeta\|_{H^1}\|u\|_{H^{\theta-1}}\|h\|_{\theta_1}.
\end{aligned}$$

• To find the second corrector φ_2^ε, we apply the analogue of Proposition 3.6 to $\Psi_u^\varepsilon = \psi_u^\varepsilon - \int_H \psi_u^\varepsilon(w)\,d\mu(w)$, where

$$\psi_u^\varepsilon = \langle b(\cdot,u), D_u\varphi_1^\varepsilon\rangle + \langle b(\cdot,\cdot), D_y\varphi_1^\varepsilon\rangle \in \mathcal{E}_\theta, \quad \|\psi_u^\varepsilon\|_{\mathcal{E}_\theta} \lesssim \|h\|_{H^{\theta_1}}\|u\|_H$$

for every $\theta > 1$. As a consequence, we get $\Phi_u^\varepsilon \in \mathcal{E}_1 \cap D(\mathscr{L}_y^\varepsilon)$ satisfying $\mathscr{L}_y^\varepsilon\Phi_u^\varepsilon = -\Psi_u^\varepsilon$, and defining $\varphi_2^\varepsilon(u,Y) = \Phi_u^\varepsilon(Y)$ we have $\|\varphi_2^\varepsilon(u,\cdot)\|_{\mathcal{E}_1} \lesssim \|h\|_{H^{\theta_1}}\|u\|_H$. Concerning regularity of $D_u\varphi_2^\varepsilon(Y)$ (Proposition 4.5), we notice that $\langle D_u^\varepsilon\Psi_u, v\rangle \in \mathcal{E}$ for every $v \in H$, with $\|\langle D_u\Psi_u^\varepsilon, v\rangle\|_{\mathcal{E}} \lesssim \|h\|_{H^{\theta_1}}\|v\|_H$, and with arguments similar to those in the proof of Proposition 4.5 the same holds true for $\langle D_u\varphi_2^\varepsilon, v\rangle$.

• To prove Lemma 5.6, we follow an approach similar to that of the proof in Section 5. The main difference in the proof consists in recovering a suitable bound for $|\varphi_1^k(u_s^\varepsilon, y_s^\varepsilon) - \varphi_1^k(u_t^\varepsilon, y_t^\varepsilon)|$ for $s,t \in [0,T]$. For given $s,t \in [0,T]$ with $s < t$ we control

$$\begin{aligned}
|\varphi_1^k(u_s^\varepsilon, y_s^\varepsilon) - \varphi_1^k(u_t^\varepsilon, y_t^\varepsilon)| &\lesssim \varepsilon^{-1/4}\|u_s^\varepsilon - u_t^\varepsilon\|_{H^{-1}}^{1/4}\|u_s^\varepsilon - u_t^\varepsilon\|_H^{3/4}\|y_s^\varepsilon\|_{H^{1/2}}\|e_k\|_{H^{\theta_0}}\\
&\quad + \|y_s^\varepsilon - y_t^\varepsilon\|_{H^{-1}}^{1/2}\|y_s^\varepsilon - y_t^\varepsilon\|_{H^1}^{1/2}\|u_t^\varepsilon\|_H\|e_k\|_{H^{\theta_0}},
\end{aligned}$$

where we have used $\|(-\varepsilon A + \mathrm{Id})^{-1}y\|_{H^1} \lesssim \varepsilon^{-1/4}\|y\|_{H^{1/2}}$ in the first estimate. By the analogue of Lemma 5.5 showed above, we have, for some $\alpha > 0$,

$$\varepsilon^{p/2}\mathbb{E}[|\varphi_1^k(u_s^\varepsilon, y_s^\varepsilon) - \varphi_1^k(u_t^\varepsilon, y_t^\varepsilon)|^p] \lesssim \|e_k\|_{\theta_0}^p|t-s|^{\alpha p},$$

and using H^2 bounds on y^ε we arrive at

$$\mathbb{E}[\|u_t^\varepsilon - u_s^\varepsilon\|_{H^{-\sigma}}^p] \lesssim |t-s|^{\alpha p}.$$

• To identify the equation solved by the limiting process u, we argue as in the proof of Proposition 5.8.

7.2. 2D surface quasi-geostrophic equations

On the space $H = L^2(\mathbb{T}^2)$ of zero-mean square integrable vorticity fields of the two-dimensional torus $\mathbb{T}^2$ we consider the surface quasi-geostrophic system

$$\begin{cases} d\xi^\varepsilon_t = -(-\nu\Delta)^{1/2}\xi^\varepsilon_t dt - u^\varepsilon_t\cdot\nabla\xi^\varepsilon_t dt - \varepsilon^{-1/2}y^\varepsilon_t\cdot\nabla\xi^\varepsilon_t dt, \\ d\eta^\varepsilon_t = \varepsilon^{-1}C\eta^\varepsilon_t + (-\nu\Delta)^{1/2}\eta^\varepsilon_t dt - (u^\varepsilon_t\cdot\nabla)\eta^\varepsilon_t dt - \varepsilon^{-1/2}(y^\varepsilon_t\cdot\nabla)\eta^\varepsilon_t dt \\ \qquad + \varepsilon^{-1/2}d\mathcal{W}_t, \\ u^\varepsilon_t = -\nabla^\perp(-\Delta)^{-1/2}\xi^\varepsilon_t, \\ y^\varepsilon_t = -\nabla^\perp(-\Delta)^{-1/2}\eta^\varepsilon_t, \end{cases} \tag{7.3}$$

where $\nu > 0$, $\nabla^\perp = (-\partial_y, \partial_x)$, the velocity fields u^ε, y^ε are reconstructed from the vorticity fields ξ^ε, η^ε via the Riesz transform $\mathcal{R} = -\nabla^\perp(-\Delta)^{-1/2}$, and $\mathcal{W} = Q^{1/2}W$ is a Wiener process on H.

We remark that in this setting elements of H are scalar functions, whereas in the context of Navier–Stokes equations they were vector-valued. The Sobolev spaces H^s for $s \in \mathbb{R}$ are then defined accordingly. Notice that the Riesz transform preserves Sobolev regularity componentwise.

We can set the previous system as an abstract equation in H defining $\Lambda\xi = -(-\nu\Delta)^{1/2}\xi$ and the nonlinear operator b as

$$b(\xi,\eta) = \nabla^\perp(-\Delta)^{-1/2}\xi\cdot\nabla\eta = -\mathcal{R}\xi\cdot\nabla\eta.$$

Sometimes in the literature concerning surface quasi-geostrophic equations the operator Λ is defined to be equal to $(-\nu\Delta)^{1/2}$; in particular notice that with our convention on the sign, Λ is a negative definite operator, and system (7.3) reads

$$\begin{cases} d\xi^\varepsilon_t = \Lambda\xi^\varepsilon_t dt + b(\xi^\varepsilon_t,\xi^\varepsilon_t)dt + \varepsilon^{-1/2}b(\eta^\varepsilon_t,\xi^\varepsilon_t)dt, \\ d\eta^\varepsilon_t = \varepsilon^{-1}C\eta^\varepsilon_t dt + \Lambda\eta^\varepsilon_t dt + b(\xi^\varepsilon_t,\eta^\varepsilon_t)dt + \varepsilon^{-1/2}b(\eta^\varepsilon_t,\eta^\varepsilon_t)dt + \varepsilon^{-1/2}Q^{1/2}dW_t. \end{cases} \tag{7.4}$$

Properties (B1)–(B4) hold true for the nonlinear operator b (and $d = 2$), so most of the arguments shown in previous sections for the Navier–Stokes equations in velocity form can be adapted to the 2D surface quasi-geostrophic equations; the main difference between the two models lies in the dissipative operator Λ, which is less regularizing in the latter case. However, there are two main factors that help us deal with loss of regularity: first, borrowing ideas from [61], we can prove additional bounds for the η component of solutions to (7.4) in $\mathcal{B}([0,T], L^p(\Omega, L^p(\mathbb{T}^2)))$, assuming the initial condition η_0 is in $H \cap L^p(\mathbb{T}^2)$ for $p > 2$; second, having restricted the surface quasi-geostrophic equations to a two-dimensional space, one can take advantage of better Sobolev embeddings and better estimates for the product of Sobolev functions to circumvent the difficulties coming from a less regularizing operator Λ. As a consequence, we can allow $\Gamma \geq \gamma > 1/4$ in (C2), but to exploit $L^p = L^p(\mathbb{T}^2)$ integrability of η we need in addition that C generates a semigroup e^{Ct} satisfying $\|e^{Ct}\eta\|_{L^q} \lesssim \|\eta\|_{L^q}$ uniformly in $t > 0$ for some $q > p$, so that by the arguments of [61, Lemma 5.5] there exists c_0 sufficiently small such that

$$\int_{\mathbb{T}^2} |\eta(x)|^p\eta(x)(C - c_0\,\mathrm{Id})\eta(x)\,dx \geq 0. \tag{7.5}$$

Notice that, in particular, the choice $C = \Lambda$ is allowed in the present setting. Finally, assumptions (Q1)–(Q2) must be modified by replacing A with Λ.

Here we state the analogue of Theorem 5.1. In the case of the surface quasi-geostrophic equations, the limiting equation below has been studied in [60].

Theorem 7.2. *Under the previous assumptions, let $\{(\xi^\varepsilon, \eta^\varepsilon)\}_{\varepsilon\in(0,1)}$ be a bounded-energy family of martingale solutions to (7.3). Then for every $\beta > 0$, the laws of the processes $\{\xi^\varepsilon\}_{\varepsilon\in(0,1)}$ are tight as probability measures on $L^2([0,T],H) \cap C([0,T],H^{-\beta})$, and every weak accumulation point $(\xi, Q^{1/2}W)$ of $(\xi^\varepsilon, Q^{1/2}W^\varepsilon)$ as $\varepsilon \to 0$ is an analytically weak solution of the equation with transport noise and Itô–Stokes drift velocity $r = \int_H (-C)^{-1} b(w,w)\, d\mu(w)$:*

$$
\begin{aligned}
d\xi_t &= \Lambda \xi_t dt + b(\xi_t, \xi_t)dt + b((-C)^{-1}Q^{1/2} \circ dW_t, \xi_t) + b(r, \xi_t)dt \\
&= -\nu^{1/2}(-\Delta)^{1/2}\xi_t dt + \nabla^\perp(-\Delta)^{-1/2}\xi_t \cdot \nabla \xi_t dt \\
&\quad + \nabla^\perp(-\Delta)^{-1/2}(-C)^{-1}Q^{1/2} \circ dW_t \cdot \nabla\xi_t + \nabla^\perp(-\Delta)^{-1/2} r \cdot \nabla \xi_t dt.
\end{aligned}
$$

If in addition pathwise uniqueness holds for the limit equation then the whole sequence converges in law; moreover, convergence in $\mathbb{P}$-probability holds true if solutions to (7.3) are probabilistically strong.

Remark 7.3. In [61] the authors prove existence of probabilistically strong solutions to (7.4) only in the subcritical case $\Lambda = -(-\nu\Delta)^\alpha$, $\alpha > 1/2$. Concerning the limiting equation, [5, Theorem 3.3] estiblishes local well-posedness of probabilistically strong solutions under suitable regularity assumptions on the initial datum ξ_0.

Let us therefore illustrate the main differences in the proof:

• The existence of weak solutions $(\xi^\varepsilon, \eta^\varepsilon)$ to (7.4) with bounded η component in the space $\mathcal{B}([0,T], L^p(\Omega, L^p(\mathbb{T}^2)))$, $p > 2$, can be proved along the lines of [61, Theorem 3.3], under the hypothesis $\eta_0 \in H \cap L^p(\mathbb{T}^2)$. We point out that no L^p bound on the large-scale process at time zero is required, making this assumption not very demanding from the modelling point of view (the small-scale process η^ε has a typical decorrelation time of order ε, so that we care very little about the initial datum η_0). Let us only discuss why the bound in $\mathcal{B}([0,T], L^p(\Omega, L^p(\mathbb{T}^2)))$ for η^ε is indeed uniform in ε.

For the sake of a clear presentation, we present only a formal argument which relies on an L^p Itô formula for η^ε developed in [48], and a positivity lemma from [61]. To make the argument rigorous, one should perform the computations below at the level of particular smooth approximations of η^ε and check that the bounds obtained pass to the limit, as done in full detail in the proof of [61, Theorem 3.3]. It is worth noticing that the aforementioned approximation is different from Galerkin's, since it is necessary that (7.7) below holds true for the approximation process, too. We refer to [61] for details.

Let us therefore turn to the actual computations. Applying formally the L^p Itô formula [48, Lemma 5.1] to η^ε one has, for every $t \in [0,T]$,

$$\begin{aligned}\|\eta_t^\varepsilon\|_{L^p}^p &= \|\eta_0\|_{L^p}^p + p\varepsilon^{-1}\int_0^t\int_{\mathbb{T}^2}|\eta_s^\varepsilon(x)|^{p-2}\eta_s^\varepsilon(x)C\eta_s^\varepsilon(x)\,dx\,ds\\ &\quad + p\int_0^t\int_{\mathbb{T}^2}|\eta_s^\varepsilon(x)|^{p-2}\eta_s^\varepsilon(x)\Lambda\eta_s^\varepsilon(x)\,dx\,ds\\ &\quad + p\int_0^t\int_{\mathbb{T}^2}|\eta_s^\varepsilon(x)|^{p-2}\eta_s^\varepsilon(x)b(\xi_s^\varepsilon,\eta_s^\varepsilon)(x)\,dx\,ds\\ &\quad + p\varepsilon^{-1/2}\int_0^t\int_{\mathbb{T}^2}|\eta_s^\varepsilon(x)|^{p-2}\eta_s^\varepsilon(x)b(\eta_s^\varepsilon,\eta_s^\varepsilon)(x)\,dx\,ds\\ &\quad + p\varepsilon^{-1/2}\int_0^t\int_{\mathbb{T}^2}|\eta_s^\varepsilon(x)|^{p-2}\eta_s^\varepsilon(x)Q^{1/2}dW_s(x)\,dx\\ &\quad + \frac{p(p-1)}{2}\varepsilon^{-1}\int_0^t\int_{\mathbb{T}^2}|\eta_s^\varepsilon(x)|^{p-2}\,\mathrm{Tr}(Q)\,dx\,ds. \end{aligned}\tag{7.6}$$

Moreover, by integration by parts, it is easy to check that for every η,ξ sufficiently regular and $p>2$,

$$\int_{\mathbb{T}^2}|\eta|^{p-2}\eta b(\xi,\eta)\,dx = -(p-1)\int_{\mathbb{T}^2}|\eta|^{p-2}\eta b(\xi,\eta)\,dx. \tag{7.7}$$

Notice that the two quantities above differ only by the factor $p-1\neq 1$, therefore

$$\int_{\mathbb{T}^2}|\eta|^{p-2}\eta b(\xi,\eta)\,dx = 0. \tag{7.8}$$

Hence, taking into account (7.8), (7.5) and [61, Lemma 5.5], arguing as in the proof of Lemma 2.5 we deduce

$$\sup_{\varepsilon\in(0,1)}\sup_{t\in[0,T]}\mathbb{E}[\|\eta_t^\varepsilon\|_{L^p}^p]\lesssim 1. \tag{7.9}$$

• Lemma 2.6 becomes: for every $p\geq 2$,

$$\sup_{\substack{\varepsilon\in(0,1)\\ n\in\mathbb{N}}}\sup_{t\in[0,T]}\left(\mathbb{E}[\|\eta_t^\varepsilon\|_H^p]+\int_0^t\mathbb{E}[\|\eta_s^\varepsilon\|_H^{p-2}\|\eta_s^\varepsilon\|_{H^{1/2}}^2]\,ds\right)\lesssim 1,$$

with identical proof. Proposition 2.7 remains the same.

• The results of Section 3 have to be suitably modified taking into account that the semigroup generated by $C_\varepsilon = C+\varepsilon\Lambda$ is less regularizing than the semigroup generated by $C+\varepsilon A$, as for the Navier–Stokes system. The precise meaning of this statement is that additional spatial regularity comes at higher price (in terms of factors ε^{-1}). In particular, we need to interpolate between the Sobolev spaces H^γ and $H^{1/2}$ (instead of H^γ and H^1).

• To find the corrector φ_1^ε, one simply applies Proposition 3.6 as in the main body of this paper. The resulting corrector is

$$\varphi_1^\varepsilon(\xi,\eta) = \langle b((-C-\varepsilon\Lambda)^{-1}\eta,\xi),h\rangle.$$

• In order to fully exploit L^p integrability of the small-scale process η^ε, we introduce the (homogeneous) Bessel potential spaces on the torus,

$$H^{s,p} := \{\xi \in \mathscr{S}' : (-\Lambda)^s \xi \in L^p\}, \quad s \in \mathbb{R},\ p \in (1,\infty),$$

which are Banach spaces when equipped with the norm

$$\|\xi\|_{H^{s,p}} := \|(-\Lambda)^s \xi\|_{L^p}.$$

Bessel potential spaces are always reflexive, and $(H^{s,p})^* = H^{-s,q}$ where $1/p + 1/q = 1$. Of course, $H^{0,p} = L^p$ for every $p \in (1,\infty)$, and $H^{s,2} = H^s$ for every $s \in \mathbb{R}$, with equivalence of norms.

In Proposition 4.1, the same argument yields in fact $D_\eta \varphi_1^\varepsilon(\xi), D_\xi \varphi_1^\varepsilon(\eta) \in H^{2\theta+s,p}$ for all $\xi, \eta \in H^{s,p}$, $s \in \mathbb{R}$, $p \in (1,\infty)$, $\theta \in [\gamma, 1/2]$, with (here $\theta_1 = \theta_1(s,p)$ and $\gamma \le 1/2$ without loss of generality)

$$\|D_\eta \varphi_1^\varepsilon(\xi)\|_{H^{2\theta+s,p}} \lesssim \varepsilon^{-\frac{\theta-\gamma}{1/2-\gamma}} \|h\|_{H^{\theta_1}} \|\xi\|_{H^{s,p}},$$
$$\|D_\xi \varphi_1^\varepsilon(\eta)\|_{H^{2\theta+s,p}} \lesssim \varepsilon^{-\frac{\theta-\gamma}{1/2-\gamma}} \|h\|_{H^{\theta_1}} \|\eta\|_{H^{s,p}}.$$

• The linearization trick relies on Proposition 4.2. As usual, denote $\zeta = y - Y$. In the present setting, we can prove the following bounds for any p sufficiently large and δ sufficiently small (we use $b : L^p \times H^{1/2-\delta} \to H^{-1}$ continuous by Sobolev embedding, and assume $\gamma \le 1/3$ without loss of generality):

$$|\langle b(\zeta,\xi), D_\xi \varphi_1(Y)\rangle| \lesssim \|\zeta\|_H \|h\|_{H^{\theta_1}} \|Y\|_{H^{\theta_0-\gamma}} \|\xi\|_H,$$
$$|\langle b(\eta,\xi), D_\xi \varphi_1(\zeta)\rangle| \lesssim \varepsilon^{-\frac{1-3\gamma}{1-2\gamma}} \|\zeta\|_{H^\gamma} \|h\|_{H^{\theta_1}} \|\xi\|_{H^{1/2-\delta}} \|\eta\|_{L^p},$$
$$|\langle b(\zeta,Y), D_\eta \varphi_1(\xi)\rangle| \lesssim \|\zeta\|_H \|h\|_{H^{\theta_1}} \|Y\|_{H^{\theta_0-\gamma}} \|\xi\|_H,$$
$$|\langle b(\eta,\zeta), D_\eta \varphi_1(\xi)\rangle| \lesssim \varepsilon^{-\frac{1-3\gamma}{1-2\gamma}} \|\zeta\|_{H^\gamma} \|h\|_{H^{\theta_1}} \|\xi\|_{H^{1/2-\delta}} \|\eta\|_{L^p}.$$

• To find φ_2^ε, we apply Proposition 3.6 to $\Psi_\xi^\varepsilon = \psi_\xi^\varepsilon - \int_H \psi_\xi^\varepsilon(w)\, d\mu(w)$, where for every $\theta > 1-\gamma$,

$$\psi_\xi^\varepsilon = \langle b(\cdot,\xi), D_\xi \varphi_1^\varepsilon\rangle + \langle b(\cdot,\cdot), D_\eta \varphi_1^\varepsilon\rangle \in \mathcal{E}_\theta, \quad \|\psi_\xi^\varepsilon\|_{\mathcal{E}_\theta} \lesssim \|h\|_{H^{\theta_1}} \|\xi\|_H.$$

As a result, we get $\Phi_\xi^\varepsilon \in \mathcal{E}_{\theta-\gamma} \cap D(\mathscr{L}_\eta^\varepsilon)$ satisfying $\mathscr{L}_\eta^\varepsilon \Phi_\xi^\varepsilon = -\Psi_\xi^\varepsilon$, and defining $\varphi_2^\varepsilon(\xi, Y) = \Phi_\xi^\varepsilon(Y)$ we have $\|\varphi_2^\varepsilon(\xi,\cdot)\|_{\mathcal{E}_{\theta-\gamma}} \lesssim \|h\|_{\theta_1} \|\xi\|_H$.

• For every $0 < \theta < \theta_0 + 2\gamma - 1$ and $v \in H^{-\theta}$ we have $\langle v, D_\xi \Psi_\xi^\varepsilon(\cdot)\rangle \in \mathcal{E}_{\theta_0}$ with

$$\|\langle v, D_\xi \Psi_\xi^\varepsilon(\cdot)\rangle\|_{\mathcal{E}_{\theta_0}} \lesssim \|h\|_{H^{\theta_1}} \|v\|_{H^{-\theta}}.$$

Therefore, Proposition 4.5 becomes

$$\|\langle v, D_\xi \varphi_2^\varepsilon(\xi,\cdot)\rangle\|_{\mathcal{E}_{\theta_0}} \lesssim \|h\|_{H^{\theta_1}} \|v\|_{H^{-\theta}}.$$

• The Itô formula (Lemma 5.3) can be proved in a similar fashion, exploiting the L^p integrability of η^ε and Proposition 4.1.

- In order to prove tightness (Lemma 5.6) we estimate

$$|\varphi_1^{k,\varepsilon}(\xi_s^\varepsilon,\eta_s^\varepsilon)-\varphi_1^{k,\varepsilon}(\xi_t^\varepsilon,\eta_t^\varepsilon)| \lesssim \|\xi_s^\varepsilon-\xi_t^\varepsilon\|_{H^{-2\gamma}}\|\eta_s^\varepsilon\|_H\|e_k\|_{H^{\theta_1}} + \|\eta_s^\varepsilon-\eta_t^\varepsilon\|_{H^{-2\gamma}}\|\xi_t^\varepsilon\|_H\|e_k\|_{H^{\theta_0}},$$

and by interpolation it is sufficient to prove an easy analogue of Lemma 5.5 for the norm $H^{-\theta_0}$. To control the term involving $\Phi^{k,\varepsilon}$, we use Proposition 4.1 with $\theta=\gamma\geq 1/4$ and the continuity of $b: L^p\times H^{1/2-\delta}\to H^{1/2+2\gamma}$ for p sufficiently large and δ sufficiently small. As for the bound on $\Phi_1^{k,\varepsilon}$, and in particular on the term $\langle b(\xi_s^\varepsilon,\xi_s^\varepsilon), D_\xi\varphi_1^\varepsilon(\eta_s^\varepsilon)\rangle$, we use the Sobolev embedding $H^{1/4}\subset L^{8/3}$ and interpolation between $H^{\gamma+1/2,p}$ and $H^{\gamma+1-\delta,2}$ to control $\|D_\xi\varphi_1^\varepsilon(\eta_s^\varepsilon)\|_{H^{1,4}}$. Indeed, $1=(1-\theta)(\gamma+1-\delta)+\theta(\gamma+1/2)$ for $\theta=\frac{\gamma-\delta}{1/2-\delta}$, and thus $1/4=(1-\theta)/2+\theta/p$ for p sufficiently large since $1-\theta<1/2$ for $\gamma>1/4$ and δ sufficiently small. The identification of the Itô–Stokes drift and the Stratonovich corrector goes as for the Navier–Stokes equations.

7.3. *Primitive equations*

Let $\mathbb{T}^d=(\mathbb{R}/\mathbb{Z})^d$ denote the d-dimensional torus, identified with points $(\mathbf{x},z)\in[0,1]^{d-1}\times[0,1]$ with periodic boundary conditions, and consider the system of primitive equations

$$\begin{cases} du_t^\varepsilon = \nu\Delta u_t^\varepsilon dt-(u_t^\varepsilon\cdot\nabla_{\mathbf{x}})u_t^\varepsilon dt - v_t^\varepsilon\partial_z u_t^\varepsilon dt \\ \qquad -\varepsilon^{-1/2}(y_t^\varepsilon\cdot\nabla_{\mathbf{x}})u_t^\varepsilon dt-\varepsilon^{-1/2}w_t^\varepsilon\partial_z u_t^\varepsilon dt+\nabla_{\mathbf{x}}p_t^\varepsilon dt, \\ dy_t^\varepsilon = \varepsilon^{-1}Cy_t^\varepsilon dt+\nu\Delta y_t^\varepsilon dt-(u_t^\varepsilon\cdot\nabla_{\mathbf{x}})y_t^\varepsilon dt-v_t^\varepsilon\partial_z y_t^\varepsilon dt \\ \qquad -\varepsilon^{-1/2}(y_t^\varepsilon\cdot\nabla_{\mathbf{x}})y_t^\varepsilon dt-\varepsilon^{-1/2}w_t^\varepsilon\partial_z y_t^\varepsilon dt+\varepsilon^{-1/2}d\mathcal{W}_t+\nabla_{\mathbf{x}}q_t^\varepsilon dt, \\ \partial_z p_t^\varepsilon=0, \quad \partial_z q_t^\varepsilon=0, \\ \operatorname{div}_{\mathbf{x}} u_t^\varepsilon+\partial_z v_t^\varepsilon=0, \quad \operatorname{div}_{\mathbf{x}} y_t^\varepsilon+\partial_z w_t^\varepsilon=0, \end{cases} \tag{7.10}$$

where $\nu>0$ and $p^\varepsilon,q^\varepsilon$ are pressure fields. The conditions $\partial_z p_t^\varepsilon=0$, $\partial_z q_t^\varepsilon=0$ correspond to the so called hydrostatic approximation, whereas $\operatorname{div}_{\mathbf{x}} u_t^\varepsilon+\partial_z v_t^\varepsilon=0$ and $\operatorname{div}_{\mathbf{x}} y_t^\varepsilon+\partial_z w_t^\varepsilon=0$ prescribe incompressibility of the composite velocity fields $(u^\varepsilon,v^\varepsilon)$ and $(y^\varepsilon,w^\varepsilon)$.

Define H as the space of horizontal velocity fields on the torus:

$$H=\left\{u\in[L^2(\mathbb{T}^d)]^{d-1}: \int_{\mathbb{T}^d}u(\mathbf{x},z)\,d\mathbf{x}\,dz=0, \int_0^1\operatorname{div}_{\mathbf{x}} u(\mathbf{x},z)\,dz=0\right\}.$$

For any $u\in H$, the incompressibility condition $\operatorname{div}_{\mathbf{x}} u_t+\partial_z v_t=0$ uniquely determines the vertical velocity v via the relation

$$v(\mathbf{x},z)=-\int_0^z \operatorname{div}_{\mathbf{x}} u_t(\mathbf{x},z')\,dz', \tag{7.11}$$

so that (7.10) reduces to a system for the unknown horizontal velocities u^ε, y^ε only.

Projecting (7.10) via the L^2 projector Π, $(\Pi u)(\mathbf{x},z)=u(\mathbf{x},z)-\int_0^1 u(\mathbf{x},z')\,dz'$, we can eliminate pressures and obtain an equivalent system on H:

$$\begin{cases} du_t^\varepsilon = \nu\Delta u_t^\varepsilon dt - \Pi(u_t^\varepsilon\cdot\nabla_{\mathbf{x}})u_t^\varepsilon dt - \Pi v_t^\varepsilon\partial_z u_t^\varepsilon dt \\ \qquad\quad - \varepsilon^{-1/2}\Pi(y_t^\varepsilon\cdot\nabla_{\mathbf{x}})u_t^\varepsilon dt - \varepsilon^{-1/2}\Pi w_t\partial_z u_t^\varepsilon dt, \\ dy_t^\varepsilon = \varepsilon^{-1}Cy_t^\varepsilon dt + \nu\Delta y_t^\varepsilon dt - \Pi(u_t^\varepsilon\cdot\nabla_{\mathbf{x}})y_t^\varepsilon dt - \Pi v_t^\varepsilon\partial_z y_t^\varepsilon dt \\ \qquad\quad - \varepsilon^{-1/2}\Pi(y_t^\varepsilon\cdot\nabla_{\mathbf{x}})y_t^\varepsilon dt - \varepsilon^{-1/2}\Pi w_t^\varepsilon\partial_z y_t^\varepsilon dt + \varepsilon^{-1/2}\Pi d\mathcal{W}_t, \\ \operatorname{div}_{\mathbf{x}} u_t^\varepsilon + \partial_z v_t^\varepsilon = 0, \quad \operatorname{div}_{\mathbf{x}} y_t^\varepsilon + \partial_z w_t^\varepsilon = 0, \end{cases}$$

and thus we can recast the primitive equations in an abstract setting with

$$Au = \nu\Delta u, \quad b(u,u') = -\Pi(u\cdot\nabla_{\mathbf{x}})u' - \Pi v\partial_z u', \quad Q^{1/2}W = \Pi\mathcal{W},$$

with v recovered from u by (7.11). It is immediate to verify the antisymmetric property (B4) of the nonlinear term b above. However, b is less regular than the nonlinear term of the Navier–Stokes equations since the expression of $b(u,u')$ involves the horizontal gradient of u, which is not compensated by the integral in the vertical direction: indeed,

(B1) $b : H^s \times H^{\theta_0} \to H^{s-1}$ is bilinear and continuous for all $s \in \mathbb{R}$, $s < d/2$, and $\theta_0 > 1 + d/2$.

(B2) $b : H^s \times H^{\theta_1} \to H^{s-1}$ is bilinear and continuous for all $s \in \mathbb{R}$, $s \geq d/2$ and $\theta_1 > 1 + s$.

(B3) $b : H^s \times H^r \to H^{s+r-2-d/2}$ is bilinear and continuous for $s, r \in (1 - d/2, 1 + d/2)$ with $s + r > 1$.

Due to the very mild regularity of b, to prove our convergence result one could either (i) restrict to dimension $d = 2$, and include bounds on $\partial_z u^\varepsilon, \partial_z v^\varepsilon$ in the space

$$L^p(\Omega, L^\infty([0,T], H) \cap L^2([0,T], H^1))$$

in the notion of solution to (7.10), relying on a priori estimates as those carried out in [42]; or (ii) require a sufficiently strong dissipation C at small scales, satisfying (C1) and (C2) for some $\Gamma \geq \gamma > 9/8$. The introduction of the operator C at small scales allows us to bound solutions y^ε uniformly in $L^p(\Omega, L^2([0,T], H^\gamma))$, with minor modifications in the proof of Proposition 2.3. We will also retain (Q1)–(Q2) unchanged.

Under the previous assumptions, Theorem 5.1 applied to primitive equations yields the following.

Theorem 7.4. *Let $\{(u^\varepsilon, y^\varepsilon)\}_{\varepsilon\in(0,1)}$ be a bounded-energy family of martingale solutions to (7.10). Then for every $\beta > 0$, the laws of the processes $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ are tight as probability measures on $L^2([0,T], H) \cap C([0,T], H^{-\beta})$, and every weak accumulation point $(u, Q^{1/2}W)$ of $(u^\varepsilon, Q^{1/2}W^\varepsilon)$ as $\varepsilon \to 0$ is an analytically weak solution of the equation with transport noise and Itô–Stokes drift velocity $r = \int_H (-C)^{-1} b(w,w)\, d\mu(w)$:*

$$\begin{aligned} du_t &= Au_t dt + b(u_t,u_t)dt + b((-C)^{-1}Q^{1/2}\circ dW_t, u_t) + b(r,u_t)dt \\ &= \nu\Delta u_t dt - \Pi(u_t\cdot\nabla_{\mathbf{x}})u_t dt - \Pi v_t\partial_z u_t dt \\ &\quad - \Pi((-C)^{-1}Q^{1/2}\circ dW_t\cdot\nabla_{\mathbf{x}})u_t - \Pi dw_t\partial_z u_t - \Pi(r\cdot\nabla_{\mathbf{x}})u_t dt - \Pi q\partial_z u_t dt, \end{aligned}$$

where v, w, q are defined implicitly by the incompressibility conditions

$$\operatorname{div}_{\mathbf{x}} u_t + \partial_z v_t = 0, \quad \operatorname{div}_{\mathbf{x}} (-C)^{-1} Q^{1/2} W_t + \partial_z w_t = 0, \quad \operatorname{div}_{\mathbf{x}} r + \partial_z q = 0.$$

If in addition pathwise uniqueness holds for the limit equation then the whole sequence converges in law; moreover, convergence in $\mathbb{P}$-probability holds true if solutions to (7.10) are probabilistically strong.

We refer to [15, Theorem 2.6] for conditions ensuring pathwise uniqueness of the limit equations and the approximating system (7.10). See also [2, 3] for additional reference. Let us finally take a look at the main modifications needed to prove the analogue of Theorem 5.1 for the solution of the primitive equations. More specifically, we can prove:

- The statement of Lemma 2.6 holds, although the bound (2.3) has to be modified to

$$\begin{aligned}
\|\zeta_t^{\varepsilon,n}\|_H^p &+ \varepsilon^{-1} p M \int_0^t \|\zeta_s^{\varepsilon,n}\|_H^{p-2} \|\zeta_s^{\varepsilon,n}\|_{H^\gamma}^2 \, ds \\
&\le \|y_0\|_H^p + p \int_0^t \|\zeta_s^{\varepsilon,n}\|_H^{p-2} \langle b(u_s^{\varepsilon,n}, Y_s^{\varepsilon,n}), \zeta_s^{\varepsilon,n} \rangle \, ds \\
&\quad + \varepsilon^{-1/2} p \int_0^t \|\zeta_s^{\varepsilon,n}\|_H^{p-2} \langle b(y_s^{\varepsilon,n}, Y_s^{\varepsilon,n}), \zeta_s^{\varepsilon,n} \rangle \, ds \\
&\le \|y_0\|^p + M_1 \int_0^t \|\zeta_s^{\varepsilon,n}\|_H^{p-2} \|\zeta_s^{\varepsilon,n}\|_{H^\gamma} \|u_s^{\varepsilon,n}\|_{H^{1-\gamma}} \|Y_s^{\varepsilon,n}\|_{H^{\theta_0}} \, ds \\
&\quad + \varepsilon^{-1/2} M_1 \int_0^t \|\zeta_s^{\varepsilon,n}\|_H^{p-2} \|\zeta_s^{\varepsilon,n}\|_{H^\gamma} \|y_s^{\varepsilon,n}\|_{H^{1-\gamma}} \|Y_s^{\varepsilon,n}\|_{H^{\theta_0}} \, ds,
\end{aligned}$$

but still can be controlled with the same techniques;

- The following variant of Proposition 4.1 holds: for any $u, y \in H^s$ and $s \in \mathbb{R}$, we have

$$D_y \varphi_1^\varepsilon(u), D_u \varphi_1^\varepsilon(y) \in H^{2\gamma+s-1},$$

with

$$\|D_y \varphi_1^\varepsilon(u)\|_{H^{2\gamma+s-1}} \lesssim \|h\|_{H^{\theta_1}} \|u\|_{H^s}, \quad \|D_u \varphi_1^\varepsilon(y)\|_{H^{2\gamma+s-1}} \lesssim \|h\|_{H^{\theta_1}} \|y\|_{H^s}.$$

- For every $\delta > 0$ sufficiently small, and all $u \in H^1$, $y \in H^\gamma$ and $Y \in H^{\theta_0 - \gamma}$,

$$\begin{aligned}
|\langle b(\zeta, u), D_u \varphi_1^\varepsilon(Y) \rangle| &\lesssim \|\zeta\|_H \|h\|_{H^{\theta_1}} \|Y\|_{H^{\theta_0-\gamma}} \|u\|_{H^1}, \\
|\langle b(y, u), D_u \varphi_1^\varepsilon(\zeta) \rangle| &\lesssim \|\zeta\|_{H^\gamma} \|h\|_{H^{\theta_1}} \|y\|_{H^{\gamma-\delta}} \|u\|_H, \\
|\langle b(\zeta, Y), D_y \varphi_1^\varepsilon(u) \rangle| &\lesssim \|\zeta\|_H \|h\|_{H^{\theta_1}} \|Y\|_{H^{\theta_0-\gamma}} \|u\|_{H^1}, \\
|\langle b(y, \zeta), D_y \varphi_1^\varepsilon(u) \rangle| &\lesssim \|\zeta\|_{H^\gamma} \|h\|_{H^{\theta_1}} \|y\|_{H^{\gamma-\delta}} \|u\|_H,
\end{aligned}$$

where we have used $b : H^{\gamma-\delta} \times H^{3\gamma-1} \to H$ and $b : H^{\gamma-\delta} \times H^{2\gamma-1} \to H^{-\gamma}$ continuous by (B3), for $\gamma > 9/8$ and our choice of δ.

- The map $\psi_u^\varepsilon = \langle b(\cdot, u), D_u\varphi_1^\varepsilon\rangle + \langle b(\cdot,\cdot), D_y\varphi_1^\varepsilon\rangle$ is in $\mathcal{E}_{7/4-\gamma}$, with

$$\|\psi_u^\varepsilon\|_{\mathcal{E}_{7/4-\gamma}} \lesssim \|h\|_{H^{\theta_1}}\|u\|_H.$$

For all $v \in H^{1-\theta}$ and $\theta < \theta_0 - 1$, we have $\langle v, D_u\Psi_u^\varepsilon(\cdot)\rangle \in \mathcal{E}_{\theta_0}$ with $\|\langle v, D_u\Psi_u^\varepsilon(\cdot)\rangle\|_{\mathcal{E}_{\theta_0}} \lesssim \|h\|_{H^{\theta_1}}\|v\|_{H^{1-\theta}}$. As usual, we can construct φ_2^ε applying Proposition 3.6.

- The proof of the Itô formula (Lemma 5.3) needs the $H^{2\gamma-1}$ estimate of derivatives of Φ: $\|D_u\Phi(u,y)\|_{H^{2\gamma-1}}, \|D_y\Phi(u,y)\|_{H^{2\gamma-1}} \lesssim 1 + \|u\|_H + \|y\|_H$, coming from Proposition 4.1.

- In Lemma 5.6, we rely on suitable bounds on suitable negative Sobolev norms of the time increments $u_s^\varepsilon - u_t^\varepsilon$, $y_s^\varepsilon - y_t^\varepsilon$, analogous to those of Lemma 5.5. Here we need to use $b : H^1 \times H^{\theta_0} \to H$ continuous. To control the terms involving $\Phi^{\varepsilon,k}$ and $\Phi_1^{\varepsilon,k}$ we use the usual arguments and the continuity of $b : H^\gamma \times H^{3\gamma-1-\delta} \to H$ for $\gamma > 9/8$ and δ sufficiently small.

- Finally, the proof of Proposition 5.8 is similar, with the use of the estimate

$$|\langle b(u_s^{\varepsilon_n}, u_s^{\varepsilon_n}) - b(u,u), h\rangle| \lesssim (\|u_s^{\varepsilon_n}\|_{H^1} + \|u_s\|_{H^1})\|u_s^{\varepsilon_n} - u_s\|_H\|h\|_{H^{\theta_0}},$$

and introducing bounds on the $L^2([0,T], H^1)$ norm in the definition of the path space $\mathcal{X}$. The identification of the Itô–Stokes drift and the Stratonovich corrector goes as for the Navier–Stokes equations.

Appendix A. Proof of Proposition 2.3

Existence of weak martingale solutions to (1.3) is by now classical, at least when $\varepsilon \in (0,1)$ is fixed [29]. For completeness we provide a brief proof of our existence result for bounded-energy families of weak martingale solutions (Proposition 2.3).

We recall the following result, which is an immediate corollary of the Ascoli–Arzelà theorem.

Lemma A.1. *Let E be a separable Banach space and let $F \subset E$ be a dense subset. Let $\{f^n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions $f^n : [0,T] \to E^*$. Assume that for every $t \in [0,T]$ the sequence $\{f_t^n\}_{n\in\mathbb{N}}$ is equibounded in E^*, and for any fixed $h \in F$ the sequence of real-valued functions $\{t \mapsto \langle f_t^n, h\rangle\}_{n\in\mathbb{N}}$ is equicontinuous. Then $f^n \in C([0,T], (E^*)_w)$ for every $n \in \mathbb{N}$, and there exists $f \in C([0,T], (E^*)_w)$ such that, up to a subsequence,*

$$f^n \to f \quad \textit{strongly in } C([0,T], (E^*)_w).$$

We are ready to prove our existence result.

Proof of Proposition 2.3. Fix a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$. Since the Galerkin system (2.2) is finite-dimensional for all $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$, it is classical that for every $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$ a strong solution to (2.2) exists on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$.

Hereafter, we fix $\varepsilon \in (0,1)$ and we focus on the sequences $\{u^{\varepsilon,n}\}_{n\in\mathbb{N}}$ and $\{y^{\varepsilon,n}\}_{n\in\mathbb{N}}$.

In [29] it is shown (minor modifications of Theorem 3.1) that for every ε the families of the laws of the processes $\{u^{\varepsilon,n}\}_{n\in\mathbb{N}}$ and $\{y^{\varepsilon,n}\}_{n\in\mathbb{N}}$ are tight in $L^2([0,T],H)\cap C([0,T],H^{-\theta_0})$, and (up to a change in the underlying probability space) they converge almost surely to progressively measurable processes

$$u^\varepsilon, y^\varepsilon \in C([0,T],H_w)\cap L^2([0,T],H^1)$$

that satisfy (S1)–(S2).

We need to show that (S3)–(S4) hold true. Applying the Itô formula to the function $\|u^{\varepsilon,n}_t\|^2_H$, $t\in[0,T]$ we get, for all $t\in[0,T]$ and $\varepsilon\in(0,1)$,

$$\|u^{\varepsilon,n}_t\|^2_H + 2\int_0^t \|u^{\varepsilon,n}_s\|^2_{H^1}\,ds = \|\Pi_n u_0\|^2_H.$$

This implies (S3) since $u^{\varepsilon,n}\to u^\varepsilon$ in $L^2([0,T],H)\cap C([0,T],H^{-\theta_0})$. Moreover, recalling (2.4) and the fact that $\mathbb{E}[\|Y^{\varepsilon,n}_t\|^p_{H^1}]\lesssim 1$ uniformly in ε, n and $t\in[0,T]$, we also have, for every p,

$$\sup_{\substack{\varepsilon\in(0,1)\\ n\in\mathbb{N}}} \mathbb{E}\left[\left(\int_0^T \|y^{\varepsilon,n}_s\|^2_{H^1}\,ds\right)^{p/2}\right]\lesssim 1,$$

and the uniform bound on y^ε in $L^p(\Omega,L^2([0,T],H^1))$ descends readily. We are left to check the uniform bound on y^ε in $\mathcal{B}([0,T],L^p(\Omega,H))$. By Lemma 2.6,

$$\sup_{\substack{\varepsilon\in(0,1)\\ n\in\mathbb{N}}}\sup_{t\in[0,T]} \mathbb{E}[\|y^{\varepsilon,n}_t\|^p_H]\lesssim 1, \tag{A.1}$$

thus for every $p<\infty$ the functions $\{y^{\varepsilon,n}\}_{n\in\mathbb{N}}$ are measurable maps from $[0,T]$ with values in $L^p(\Omega,H)=(L^q(\Omega,H))^*$, $1/p+1/q=1$, that are equibounded in $L^p(\Omega,H)$ for every fixed $t\in[0,T]$ (actually uniformly in $t\in[0,T]$). Moreover, for every fixed $h\in L^\infty(\Omega,\mathscr{S})$ and $s,t\in[0,T]$ with $s<t$, we have

$$\begin{aligned}
|\mathbb{E}[\langle y^{\varepsilon,n}_t - y^{\varepsilon,n}_s, h\rangle]| &\le \varepsilon^{-1}\int_s^t \mathbb{E}[|\langle y^{\varepsilon,n}_r, C_\varepsilon h\rangle|]\,dr + \int_s^t \mathbb{E}[|\langle b(u^{\varepsilon,n}_r, y^{\varepsilon,n}_r), h\rangle|]\,dr\\
&\quad + \varepsilon^{-1/2}\int_s^t \mathbb{E}[|\langle b(y^{\varepsilon,n}_r, y^{\varepsilon,n}_r), h\rangle|]\,dr\\
&\quad + \varepsilon^{-1/2}\mathbb{E}[\langle \Pi_n Q^{1/2}(W_t - W_s), h\rangle]\\
&\lesssim |t-s|(1+\varepsilon^{-1})\|h\|_{L^\infty(\Omega,H^{\theta_0})} + |t-s|^{1/2}\varepsilon^{-1/2}\|h\|_{L^\infty(\Omega,H)},
\end{aligned}$$

meaning that the sequence $\{t\mapsto\langle y^{\varepsilon,n}_t, h\rangle\}_{n\in\mathbb{N}}$ of real-valued functions is equicontinuous for every fixed $h\in L^\infty(\Omega,\mathscr{S})$. Since $L^\infty(\Omega,\mathscr{S})$ is dense in $L^q(\Omega,H)$, by Lemma A.1 we have, up to a subsequence (not relabelled),

$$y^{\varepsilon,n}\to y^\varepsilon \quad \text{strongly in } C([0,T],(L^p(\Omega,H))_w).$$

By the Banach–Steinhaus theorem, $C([0,T],(L^p(\Omega,H))_w)\subset\mathcal{B}([0,T],L^p(\Omega,H))$ and therefore y^ε inherits the bound (A.1), which is uniform in ε. The proof is complete. ■

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References

[1] Agresti, A.: Delayed blow-up and enhanced diffusion by transport noise for systems of reaction-diffusion equations. Stoch. Partial Differential Equations Anal. Comput. (online, 2023)

[2] Agresti, A., Hieber, M., Hussein, A., Saal, M.: The stochastic primitive equations with transport noise and turbulent pressure. Stoch. Partial Differential Equations Anal. Comput. **12**, 53–133 (2024) Zbl 07858450 MR 4709539

[3] Agresti, A., Hieber, M., Hussein, A., Saal, M.: The stochastic primitive equations with non-isothermal turbulent pressure. arXiv:2210.05973 (2022)

[4] Agresti, A., Veraar, M.: Reaction-diffusion equations with transport noise and critical superlinear diffusion: local well-posedness and positivity. J. Differential Equations **368**, 247–300 (2023) Zbl 1518.35442 MR 4598499

[5] Alonso-Orán, D., Rohde, C., Tang, H.: A local-in-time theory for singular SDEs with applications to fluid models with transport noise. J. Nonlinear Sci. **31**, article no. 98, 55 pp. (2021) Zbl 1477.60105 MR 4328430

[6] Barrué, G., de Bouard, A., Debussche, A.: From the stochastic Zakharov system to the stochastic nonlinear Schrödinger equation. To appear

[7] Bauer, W., Chandramouli, P., Chapron, B., Li, L., Mémin, E.: Deciphering the role of small-scale inhomogeneity on geophysical flow structuration: a stochastic approach. J. Physical Oceanography **50**, 983–1003 (2020)

[8] Bedrossian, J., Blumenthal, A., Punshon-Smith, S.: Almost-sure enhanced dissipation and uniform-in-diffusivity exponential mixing for advection-diffusion by stochastic Navier–Stokes. Probab. Theory Related Fields **179**, 777–834 (2021) Zbl 1465.37099 MR 4242626

[9] Bedrossian, J., Blumenthal, A., Punshon-Smith, S.: Almost-sure exponential mixing of passive scalars by the stochastic Navier–Stokes equations. Ann. Probab. **50**, 241–303 (2022) Zbl 1513.37037 MR 4385127

[10] Bedrossian, J., Blumenthal, A., Punshon-Smith, S.: The Batchelor spectrum of passive scalar turbulence in stochastic fluid mechanics at fixed Reynolds number. Comm. Pure Appl. Math. **75**, 1237–1291 (2022) Zbl 1506.76048 MR 4415776

[11] Bedrossian, J., Blumenthal, A., Punshon-Smith, S.: Lagrangian chaos and scalar advection in stochastic fluid mechanics. J. Eur. Math. Soc. **24**, 1893–1990 (2022) Zbl 1496.76040 MR 4404792

[12] Bessaih, H., Ferrario, B.: Invariant measures for stochastic damped 2D Euler equations. Comm. Math. Phys. **377**, 531–549 (2020) Zbl 1440.35252 MR 4107937

[13] Brzeźniak, Z., Capiński, M., Flandoli, F.: Stochastic Navier–Stokes equations with multiplicative noise. Stochastic Anal. Appl. **10**, 523–532 (1992) Zbl 0762.35083 MR 1185046

[14] Brzeźniak, Z., Flandoli, F., Maurelli, M.: Existence and uniqueness for stochastic 2D Euler flows with bounded vorticity. Arch. Ration. Mech. Anal. **221**, 107–142 (2016) Zbl 1342.35231 MR 3483892

[15] Brzeźniak, Z., Slavík, J.: Well-posedness of the 3D stochastic primitive equations with multiplicative and transport noise. J. Differential Equations **296**, 617–676 (2021) Zbl 1473.35576 MR 4275634

[16] Buckmaster, T., Vicol, V.: Nonuniqueness of weak solutions to the Navier–Stokes equation. Ann. of Math. (2) **189**, 101–144 (2019) Zbl 1412.35215 MR 3898708

[17] Constantin, P., Iyer, G.: A stochastic Lagrangian representation of the three-dimensional incompressible Navier–Stokes equations. Comm. Pure Appl. Math. **61**, 330–345 (2008) Zbl 1156.60048 MR 2376844

[18] Cotter, C. J., Gottwald, G. A., Holm, D. D.: Stochastic partial differential fluid equations as a diffusive limit of deterministic Lagrangian multi-time dynamics. Proc. Roy. Soc. London A **473**, article no. 20170388, 10 pp. (2017) Zbl 1402.76101 MR 3710332

[19] Da Prato, G., Debussche, A.: Ergodicity for the 3D stochastic Navier–Stokes equations. J. Math. Pures Appl. (9) **82**, 877–947 (2003) Zbl 1109.60047 MR 2005200

[20] Da Prato, G., Zabczyk, J.: Second order partial differential equations in Hilbert spaces. London Math. Soc. Lecture Note Ser. 293, Cambridge University Press, Cambridge (2002) Zbl 1012.35001 MR 1985790

[21] Da Prato, G., Zabczyk, J.: Stochastic equations in infinite dimensions. 2nd ed., Encyclopedia Math. Appl. 152, Cambridge University Press, Cambridge (2014) Zbl 1317.60077 MR 3236753

[22] de Bouard, A., Gazeau, M.: A diffusion approximation theorem for a nonlinear PDE with application to random birefringent optical fibers. Ann. Appl. Probab. **22**, 2460–2504 (2012) Zbl 1261.35156 MR 3024974

[23] Debussche, A., De Moor, S., Vovelle, J.: Diffusion limit for the radiative transfer equation perturbed by a Markovian process. Asymptot. Anal. **98**, 31–58 (2016) Zbl 1344.35148 MR 3502371

[24] Debussche, A., Vovelle, J.: Diffusion limit for a stochastic kinetic problem. Comm. Pure Appl. Anal. **11**, 2305–2326 (2012) Zbl 1267.35022 MR 2912749

[25] Debussche, A., Vovelle, J.: Diffusion-approximation in stochastically forced kinetic equations. Tunis. J. Math. **3**, 1–53 (2021) Zbl 1442.35286 MR 4103765

[26] Flandoli, F., Galeati, L., Luo, D.: Delayed blow-up by transport noise. Comm. Partial Differential Equations **46**, 1757–1788 (2021) Zbl 1477.60096 MR 4304694

[27] Flandoli, F., Galeati, L., Luo, D.: Eddy heat exchange at the boundary under white noise turbulence. Philos. Trans. Roy. Soc. A **380**, article no. 20210096, 13 pp. (2022) MR 4402221

[28] Flandoli, F., Galeati, L., Luo, D.: Quantitative convergence rates for scaling limit of SPDEs with transport noise. J. Differential Equations **394**, 237–277 (2024) Zbl 07832967 MR 4714774

[29] Flandoli, F., Gątarek, D.: Martingale and stationary solutions for stochastic Navier–Stokes equations. Probab. Theory Related Fields **102**, 367–391 (1995) Zbl 0831.60072 MR 1339739

[30] Flandoli, F., Gubinelli, M., Priola, E.: Well-posedness of the transport equation by stochastic perturbation. Invent. Math. **180**, 1–53 (2010) Zbl 1200.35226 MR 2593276

[31] Flandoli, F., Luo, D.: Convergence of transport noise to Ornstein–Uhlenbeck for 2D Euler equations under the enstrophy measure. Ann. Probab. **48**, 264–295 (2020) Zbl 1440.35234 MR 4079436

[32] Flandoli, F., Luo, D.: High mode transport noise improves vorticity blow-up control in 3D Navier–Stokes equations. Probab. Theory Related Fields **180**, 309–363 (2021) Zbl 1469.60205 MR 4265023

[33] Flandoli, F., Luo, D., Luongo, E.: 2D Smagorinsky-type large eddy models as limits of stochastic PDEs. J. Nonlinear Sci. **34**, article no. 54, 25 pp. (2024) Zbl 07839652 MR 4732394

[34] Flandoli, F., Maurelli, M., Neklyudov, M.: Noise prevents infinite stretching of the passive field in a stochastic vector advection equation. J. Math. Fluid Mech. **16**, 805–822 (2014) Zbl 1322.60109 MR 3267550

[35] Flandoli, F., Olivera, C.: Well-posedness of the vector advection equations by stochastic perturbation. J. Evol. Equations **18**, 277–301 (2018) Zbl 1394.60068 MR 3820404

[36] Flandoli, F., Pappalettera, U.: 2D Euler equations with Stratonovich transport noise as a large-scale stochastic model reduction. J. Nonlinear Sci. **31**, article no. 24, 39 pp. (2021) Zbl 1466.76010 MR 4215938

[37] Flandoli, F., Pappalettera, U.: From additive to transport noise in 2D fluid dynamics. Stoch. Partial Differential Equations Anal. Comput. **10**, 964–1004 (2022) Zbl 1499.60217 MR 4491507

[38] Flandoli, F., Romito, M.: Markov selections for the 3D stochastic Navier–Stokes equations. Probab. Theory Related Fields **140**, 407–458 (2008) Zbl 1133.76016 MR 2365480

[39] Fouque J.-P., Garnier, J., Papanicolaou, G., Sølna, K.: Wave propagation and time reversal in randomly layered media. Springer (2007) Zbl 1386.74001 MR 2327824

[40] Garnier, J., Mertz, L.: A control variate method driven by diffusion approximation. Comm. Pure Appl. Math. **75**, 455–492 (2022) Zbl 1489.93118 MR 4373175

[41] Gess, B., Yaroslavtsev, I.: Stabilization by transport noise and enhanced dissipation in the Kraichnan model. arXiv:2104.03949 (2021)

[42] Glatt-Holtz, N., Ziane, M.: The stochastic primitive equations in two space dimensions with multiplicative noise. Discrete Contin. Dynam. Systems Ser. B **10**, 801–822 (2008) Zbl 1147.76013 MR 2434911

[43] Gyöngy, I., Krylov, N.: Existence of strong solutions for Itô's stochastic equations via approximations. Probab. Theory Related Fields **105**, 143–158 (1996) Zbl 0847.60038 MR 1392450

[44] Hairer, M., Mattingly, J. C.: Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing. Ann. of Math. (2) **164**, 993–1032 (2006) Zbl 1130.37038 MR 2259251

[45] Hofmanová, M., Zhu, R., Zhu, X.: Nonuniqueness in law of stochastic 3D Navier–Stokes equations. J. Eur. Math. Soc. **26**, 163–260 (2024) Zbl 07815180 MR 4705650

[46] Holm, D. D.: Variational principles for stochastic fluid dynamics. Proc. Roy. Soc. London A **471**, article no. 20140963, 19 pp. (2015) Zbl 1371.35219 MR 3325187

[47] Kreĭn, S. G., Petunin, Yu. I.: Scales of Banach spaces. Uspekhi Mat. Nauk **21**, no. 2, 89–168 (1966) (in Russian) Zbl 0173.15702 MR 0193499

[48] Krylov, N. V.: Itô's formula for the L_p-norm of stochastic W_p^1-valued processes. Probab. Theory Related Fields **147**, 583–605 (2010) Zbl 1232.60050 MR 2639716

[49] Kurtz, T. G.: A limit theorem for perturbed operator semigroups with applications to random evolutions. J. Funct. Anal. **12**, 55–67 (1973) Zbl 0246.47053 MR 0365224

[50] Luo, D.: Enhanced dissipation for stochastic Navier–Stokes equations with transport noise. J. Dynam. Differential Equations (online, 2023)

[51] Luo, D., Tang, B.: Stochastic inviscid Leray–α model with transport noise: convergence rates and CLT. Nonlinear Anal. **234**, article no. 113301, 34 pp. (2023) Zbl 1518.60058 MR 4592348

[52] Majda, A. J., Kramer, P. R.: Simplified models for turbulent diffusion: theory, numerical modelling, and physical phenomena. Phys. Rep. **314**, 237–574 (1999) MR 1699757

[53] Majda, A. J., Timofeyev, I., Vanden Eijnden, E.: A mathematical framework for stochastic climate models. Comm. Pure Appl. Math. **54**, 891–974 (2001) Zbl 1017.86001 MR 1829529

[54] Mémin, E.: Fluid flow dynamics under location uncertainty. Geophys. Astrophys. Fluid Dynam. **108**, 119–146 (2014) Zbl 07657767 MR 3223587

[55] Mikulevicius, R., Rozovskii, B. L.: Stochastic Navier–Stokes equations for turbulent flows. SIAM J. Math. Anal. **35**, 1250–1310 (2004) Zbl 1062.60061 MR 2050201

[56] Mikulevicius, R., Rozovskii, B. L.: Global L_2-solutions of stochastic Navier–Stokes equations. Ann. Probab. **33**, 137–176 (2005) Zbl 1098.60062 MR 2118862

[57] Panton, R. L.: Turbulent flows. John Wiley & Sons (2013)

[58] Papanicolaou, G. C., Stroock, D., Varadhan, S. R. S.: Martingale approach to some limit theorems. In: Papers from the Duke Turbulence Conference (Durham, NC, 1976), Duke University, Durham, NC, article no. 6, 120 pp. (1977) Zbl 0387.60067 MR 0461684

[59] Pappalettera, U.: Quantitative mixing and dissipation enhancement property of Ornstein–Uhlenbeck flow. Comm. Partial Differential Equations **47**, 2309–2340 (2022) Zbl 1505.35376 MR 4526894

[60] Resseguier, V., Mémin, E., Chapron, B.: Geophysical flows under location uncertainty, Part II Quasi-geostrophy and efficient ensemble spreading. Geophys. Astrophys. Fluid Dynam. **111**, 177–208 (2017) Zbl 1506.76026 MR 3648563

[61] Röckner, M., Zhu, R., Zhu, X.: Sub and supercritical stochastic quasi-geostrophic equation. Ann. Probab. **43**, 1202–1273 (2015) Zbl 1322.60121 MR 3342662

[62] Simon, J.: Compact sets in the space $L^p(0,T;B)$. Ann. Mat. Pura Appl. (4) **146**, 65–96 (1987) Zbl 0629.46031 MR 0916688

[63] Temam, R.: Navier–Stokes equations and nonlinear functional analysis. 2nd ed., CBMS-NSF Regional Conf. Ser. Appl. Math. 66, Society for Industrial and Applied Mathematics, Philadelphia, PA (1995) Zbl 0833.35110 MR 1318914

[64] van den Bremer, T. S., Breivik, Ø.: Stokes drift. Philos. Trans. Roy. Soc. A **376**, article no. 20170104, 23 pp. (2018) Zbl 1404.76056 MR 3744205