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On the cubic Shimura lift to PGL(3): The fundamental lemma

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Abstract. The classical Shimura correspondence lifts automorphic representations on the double cover of SL_2 to automorphic representations on PGL_2. Here we take key steps towards establishing a relative trace formula that would give a new global Shimura lift, from the triple cover of SL_3 to PGL_3, and also characterize the image of the lift. The characterization would be through the non-vanishing of a certain global period involving a function in the space of the automorphic minimal representation Θ_{SO_8} for split $SO_8(\mathbb{A})$, consistent with a conjecture of Bump, Friedberg and Ginzburg (2001). In this paper, we first analyze a global distribution on $PGL_3(\mathbb{A})$ involving this period and show that it is a sum of factorizable orbital integrals. The same is true for the Kuznetsov distribution attached to the triple cover of $SL_3(\mathbb{A})$. We then match the corresponding local orbital integrals for the unit elements of the spherical Hecke algebras; that is, we establish the fundamental lemma.

Keywords: Shimura correspondence, metaplectic group, cubic cover, relative trace formula, minimal representation, period.

Contents

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1. Introduction

The classical Shimura correspondence [24] lifts automorphic representations on the double cover of SL_2 to automorphic representations on PGL_2. Niwa [21] (working classically) and Waldspurger [25] (working adelically) showed that the map may be obtained as a theta lifting. Waldspurger also showed that the image of the lift is characterized by the non-vanishing of a certain period (the integral of an automorphic form over a cycle). Jacquet [14] then used his relative trace formula, which compares distributions on two different groups, one involving the relevant period, to give another proof of these facts.

Let $n \geq 2$ and let G be a split connected reductive algebraic group over a global field F with a full set of n-th roots of unity $\mu_n \subset F$. Let $\mathbb{A} = \mathbb{A}_F$ be the ring of adeles of F. Then one may define an n-fold cover $\widetilde{G}^{(n)}(\mathbb{A})$ of the adelic points of G, and it is natural to ask if there is an analogue of the Shimura map for $\widetilde{G}^{(n)}(\mathbb{A})$. The local Shimura correspondence was investigated by Savin [23], who proved (in generality) an isomorphism of Iwahori–Hecke algebras. Based on this work, for Cartan type A one expects a global functorial lift from genuine cuspidal automorphic representations on $\widetilde{\mathrm{SL}}_r^{(n)}(\mathbb{A})$ to automorphic representations on $\mathrm{PGL}_r(\mathbb{A})$ if n divides r, and a lift to automorphic representations on $\mathrm{SL}_r(\mathbb{A})$ if $(n,r)=1$. (See also Bump, Friedberg and Ginzburg [1, Section 1] for a discussion of this.) Such lifts have been studied since the 1980s, but progress in establishing a global Shimura correspondence has been obtained only in the cases $n=2$ or $r=2$. In this paper, we establish the fundamental lemma for a relative trace formula that will give the global Shimura lift in the case $n=r=3$; moreover, this project will characterize the image of the lift by means of a period involving a function in the space of the automorphic minimal representation for $\mathrm{SO}_8(\mathbb{A})$, confirming a conjecture of Bump, Friedberg and Ginzburg (2001).

To put this work in context, recall that Ginzburg, Rallis and Soudry [10] observed that SL_2 and its 3-fold cover $\widetilde{\mathrm{SL}}_2^{(3)}$ form a dual pair in the 3-fold cover of the exceptional group G_2, and used this to establish a lifting of genuine cuspidal automorphic representations on $\widetilde{\mathrm{SL}}_2^{(3)}(\mathbb{A})$ to automorphic representations on $\mathrm{SL}_2(\mathbb{A})$ and to determine its image. Let $\mathrm{Sym}^3\colon \mathrm{SL}_2 \to \mathrm{Sp}_4$ be the symmetric cube map, and let Θ_{Sp_4} be the theta representation on the metaplectic double cover of $\mathrm{Sp}_4(\mathbb{A})$. This cover splits on the image of Sym^3. Then they showed that an irreducible cuspidal automorphic representation τ of $\mathrm{SL}_2(\mathbb{A})$ is in the image of the rank-one cubic Shimura map if and only if the period integral

$$\int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \varphi(g)\theta(\mathrm{Sym}^3(g))\,dg \tag{1.1}$$

is non-zero for some φ in the space of τ and some θ in the space of Θ_{Sp_4}. Another proof of these results was given by Mao and Rallis [19] via a relative trace formula. However, for higher degree covers or higher rank special linear groups, there is no similar dual pair.

By comparing orbital integrals, Flicker [8] succeeded in establishing a Shimura-type correspondence for the n-fold cover of $\mathrm{GL}_2(\mathbb{A})$ (there is more than one cover, and he treated a specific one); in this generality, there is no known general characterization of the image by periods. Kazhdan and Patterson [16] and Flicker and Kazhdan [9] studied these orbital integrals for higher rank general linear groups, but were not able to obtain sufficient control to establish a correspondence. As for double covers, Mao [18] established the fundamental lemma for a relative trace formula for the double cover of GL_3, and the case of the double cover of GL_n has recently been treated by Do by first establishing the equal characteristic case [5] and then using model theoretic methods to move to characteristic zero [6].

Here we are concerned with the conjectured Shimura map from the cubic cover of SL_3 to PGL_3. The question of characterizing its image was considered by Bump, Friedberg and Ginzburg [1]. Let Θ_{SO_8} be the automorphic minimal representation on the split special orthogonal group $\mathrm{SO}_8(\mathbb{A})$. This representation was constructed by Ginzburg, Rallis and Soudry [11] as a multi-residue of a Borel Eisenstein series on SO_8. Let Ad denote the adjoint representation $\mathrm{Ad}\colon \mathrm{PGL}_3 \to \mathrm{SO}_8$ (see Section 3.1 below). Supposing that a Shimura lift exists in this case, they then conjectured the following.

Conjecture 1.1 (Bump, Friedberg, Ginzburg). *Let π be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_3(\mathbb{A})$. Then π is in the image of the cubic Shimura correspondence from $\widetilde{\mathrm{SL}}_3^{(3)}(\mathbb{A})$ if and only if the period*

$$\int_{\mathrm{PGL}_3(F)\backslash \mathrm{PGL}_3(\mathbb{A})} \varphi(g)\, \theta(\mathrm{Ad}(g))\, dg \tag{1.2}$$

is non-zero for some φ in the space of π and some θ in Θ_{SO_8}.

Those authors presented two pieces of evidence for this conjecture, the first from finite fields, and the second by supposing that π was not cuspidal but rather an Eisenstein series induced from cuspidal data τ on $\mathrm{GL}_2(\mathbb{A})$, formally unfolding period (1.2) in this case, and showing that the resulting integral is non-vanishing for some choice of data if and only if period (1.1) for τ is non-vanishing for some choice of data.

The study of periods in the context of Langlands type correspondences is a main theme of contemporary research; see, for example, Sakellaridis and Venkatesh [22]. However, the extension to metaplectic covers is not well developed. In particular, we do not know of an extension of Conjecture 1.1 to higher rank special linear groups or higher degree covers.

Here we take key steps towards establishing the existence of the global Shimura correspondence with $n = r = 3$ and towards characterizing its image as in Conjecture 1.1 above. Our approach, following Jacquet [14], Mao and Rallis [19], and Mao [18], is to establish a comparison of relative trace formulas. For an algebraic group H defined over F, we denote its automorphic quotient space by $[H] = H(F)\backslash H(\mathbb{A})$. For an affine variety X defined over F, denote by $\mathcal{S}(X(\mathbb{A}))$ the space of Schwartz–Bruhat functions on $X(\mathbb{A})$. The action of $f \in \mathcal{S}(H(\mathbb{A}))$ on $L^2([H])$, obtained by averaging the right regular representation

$$R(f)\phi(h) = \int_{H(\mathbb{A})} \phi(hy) f(y)\, dy,$$

is realized by the kernel function

$$\sum_{\gamma \in H(F)} f(h_1^{-1}\gamma h_2), \quad h_1, h_2 \in [H].$$

The relative trace formula compares two distributions obtained as integrals of these kernel functions for different groups, one integral involving the period and a Whittaker character and the other involving solely Whittaker characters.

Let $G = \mathrm{PGL}(3)$ be considered as an algebraic group defined over F, and let N be the standard maximal unipotent subgroup of G realized as the group of upper triangular 3×3 unipotent matrices. Fix a non-trivial character ψ of $F\backslash\mathbb{A}$. By abuse of notation, we further denote by ψ the generic character $\psi(n) = \psi(n_{1,2} + n_{2,3})$ of $[N]$. For $\Theta \in \Theta_{\mathrm{SO}_8}$, we consider the distribution $I(\Theta)$ on $G(\mathbb{A})$ defined by

$$I(f, \Theta) = \int_{[N]} \int_{[G]} \Big\{ \sum_{\gamma \in G(F)} f(g^{-1}\gamma n) \Big\} \Theta(\mathrm{Ad}(g))\psi(n)\, dg\, dn, \quad f \in \mathcal{S}(G(\mathbb{A})). \tag{1.3}$$

Let $G'(\mathbb{A})$ be the adelic 3-fold metaplectic cover of $\mathrm{SL}_3(\mathbb{A})$. It is a central extension of $\mathrm{SL}_3(\mathbb{A})$ satisfying the exact sequence

$$1 \to \mu_3 \to G'(\mathbb{A}) \to \mathrm{SL}_3(\mathbb{A}) \to 1.$$

(Since $G'(\mathbb{A})$ is not the adelic points of an algebraic group, this is an abuse of notation.) Denote by $\mathcal{S}(G'(\mathbb{A}))$ the space of genuine (that is, μ_3-equivariant) Schwartz–Bruhat functions on $G'(\mathbb{A})$. There is a splitting of $\mathrm{SL}_3(F)$ in $G'(\mathbb{A})$, and we denote by $G'(F)$ its image. The group $N(\mathbb{A})$ also splits in $G'(\mathbb{A})$, and we continue to denote by $N(\mathbb{A})$ the image of this splitting. The Kuznetsov trace formula is realized by the distribution on $G'(\mathbb{A})$ defined by

$$J(f') = \int_{[N]} \int_{[N]} \Big\{ \sum_{\gamma \in G'(F)} f'(n_1 \gamma n_2) \Big\} \psi(n_1 n_2)\, dn_1\, dn_2, \quad f' \in \mathcal{S}(G'(\mathbb{A})). \tag{1.4}$$

Our goal is to study a comparison between the two distributions $I(f, \Theta)$ and $J(f')$.

The first step is to *geometerize* the distribution $I(\Theta)$, that is, express it as a sum (over some geometric orbits) of distributions that are factorizable. In most relative trace formulas, this step is straightforward: one writes the sums using the Bruhat decomposition or some variant and unfolds to get a sum over double cosets of adelic orbital integrals, with only the relevant double cosets contributing. For factorizable test functions, each side is then factorizable.

The obstacle in our case is the automorphic minimal representation that appears in distribution (1.3). In contrast to the rank one case, this representation is not directly obtained from the Weil representation, so there is no easy way to express the functions in Θ_{SO_8} as sums that may then be unfolded. To address this, we make use of the dual pair $(\mathrm{SL}_2, \mathrm{SO}_8)$ inside Sp_{16}, and realize the automorphic minimal representation Θ_{SO_8} as the theta lift of the trivial representation from $\mathrm{SL}_2(\mathbb{A})$ to $\mathrm{SO}_8(\mathbb{A})$. This realization is due to Ginzburg, Rallis and Soudry [11, Theorem 6.9]; their proof relies on the work of Kudla and Rallis [17] (or in a classical language over $\mathbb{Q}$, the work of Deitmar and Krieg [4]). When applied to the groups at hand, these works establish that the suitably regularized theta lift of an Eisenstein series on SL_2 is an Eisenstein series on SO_8; then taking a residue, one obtains the desired realization of Θ_{SO_8}. Since we are aiming for a factorizable integral, it is perhaps surprising that it is helpful to introduce an additional integration over an automorphic quotient $[\mathrm{SL}_2]$. Nonetheless, it turns out that this realization allows us to do an unfolding and to express the distribution $I(f, \Theta)$ as a sum of factorizable orbital integrals.

An analogous geometric expansion for $J(f')$ follows directly from the Bruhat decomposition on G' (and is well known). This allows us to establish a one-to-one correspondence for the relevant orbits of the two distributions. To effect a global comparison, we must then achieve a comparison of the corresponding local orbital integrals for each family of relevant orbits up to a transfer factor whose product over all places is 1.

The main local result of this paper is a matching of these orbital integrals for the unit element of the spherical Hecke algebra, that is, the fundamental lemma for this relative trace formula. The calculations are rather elaborate, but as in prior work on relative trace formulas that give Shimura correspondences, there is an algebraic fact that is key. In Jacquet's work on the Shimura correspondence, this was the computation of a certain Salié sum; this fact was also key in Iwaniec's work [13] on the same topic with an eye towards analytic number theoretic applications. For the work of Mao and Rallis, key use was made of an identity relating a cubic exponential sum to a Kloosterman sum with a cubic character that is due to Duke and Iwaniec [7] (as these authors remark, the analogous fact at a real archimedean place is Nicholson's formula for the Airy integral!). The proof of that result in turn is based on the Davenport–Hasse relation. Their result applies when the additive character is of conductor 1, that is, to exponential sums of the form

$$\sum_{x \bmod p} \exp\Big(2\pi i \frac{ax^3 + bx}{p}\Big),$$

where p in our computation is the cardinality of the residue field. These sums are shown to be equal to certain Kloosterman sums modulo p with a cubic character. Remarkably, we observe that a similar relation is true for exponential sums that involve additive characters of higher conductor (Corollary 7.5). For higher conductor, the proof is based on the method of stationary phase.

In our work, the orbital integrals for the big cell frequently reduce to integrals of pairs of Kloosterman integrals with cubic characters. These integrals appear intractable. However, using this identity twice to make them into integrals of pairs of cubic exponential integrals, we are able to effect the desired comparison. It also sometimes happens that we encounter Kloosterman integrals without a cubic character. In this case for conductor 1, the result of Duke–Iwaniec is not applicable, but another identity, which expresses the characteristic function of the local condition $(ab^{-1})^3 \equiv -1 \bmod \mathfrak{p}$ in terms of a sum of three cubic exponential integrals, is used instead (Lemma 7.6).

We describe the contents of this paper. The first part of this paper gives the geometric expansions of the distributions we have described above. Our first task is to express the distribution $I(f, \Theta)$ as a sum of factorizable orbital integrals. In Section 2, we formulate a general unfolding principle and recall the definition of relevant orbits. We also specify our conventions on measures. Then, in Section 3, we use this principle and the theta lift to SO_8 to unfold the distribution $I(f, \Theta)$ and to express it as a sum of factorizable orbital integrals over relevant orbits (Proposition 3.2). In the following Section 4, we work with flag varieties to determine these relevant orbits explicitly (Theorem 4.2) and also write the orbital integrals for each relevant orbit in a form amenable for computation. We also need the geometric expansion for the Kuznetsov distribution $J(f')$, and this is presented in Section 5. This completes the reduction of the comparison of spectral sides of the global distributions to a comparison of a collection of explicitly given geometric local distributions.

The second part of this paper establishes the fundamental lemma. We begin with the big cell comparison, which is stated in Theorem 6.1, and whose proof occupies the following four sections. In Section 7, we develop the necessary local ingredients, including the comparison of cubic exponential and Kloosterman integrals, which is given (as a consequence of prior computations that are closely related to the method of stationary phase) in Section 7.5. The companion computation of cubic integrals when Duke–Iwaniec does not apply is in Section 7.6. The orbital integral $J(a, b)$ attached to the big cell is then evaluated in Section 8. This is the piece of the calculation that requires computations in the metaplectic group. Specifically, if $\mathcal{O}$ denotes the local ring of integers, then the embedding of $\mathrm{SL}_3(\mathcal{O})$ in the local covering group is given by means of a section that must be computed in order to work with the unit element in the corresponding Hecke algebra. In doing so, we make use of an algorithm for computing the local two-cocycle (defined by Matsumoto [20]) that is given in Bump and Hoffstein [3]. This leads to the complicated orbital integrals involving pairs of Kloosterman integrals that we then evaluate. The orbital integral $I(a, b)$ is evaluated in Section 9. The comparison is then completed in the brief Section 10. The final section, Section 11, matches the remaining orbital integrals with an explicit transfer factor. For the two one-parameter families of relevant orbits, the comparison once again requires replacing a Kloosterman integral by a cubic exponential integral, but this time only once.

Part I
Global theory: The geometric expansions

2. An unfolding principle; measures

Let F be a global field, $\mathbb{A} = \mathbb{A}_F$ the ring of adeles of F, X an affine F-variety and G an affine algebraic F-group acting on X on the right. We denote the action by $(x, g) \mapsto x^g$, $x \in X$, $g \in G$. Let δ_G be the modular quasicharacter of $G(\mathbb{A})$ and $[G] = G(F)\backslash G(\mathbb{A})$.

2.1. An unfolding principle

For $\xi \in X(F)$, we denote by G_ξ the stabilizer of ξ in G, an affine algebraic group defined over F. We say that a function $\mathrm{c}\colon X(\mathbb{A}) \times G(\mathbb{A}) \to \mathbb{C}$ is an *automorphic cocycle* if it satisfies the following two conditions:

- $\mathrm{c}(\xi, \gamma) = 1$, $\xi \in X(F)$, $\gamma \in G(F)$;
- $\mathrm{c}(\xi, gh) = \mathrm{c}(\xi, g)\mathrm{c}(\xi^g, h)$, $\xi \in X(F)$, $g, h \in G(\mathbb{A})$.

This implies that $g \mapsto \mathrm{c}(\xi, g)$ restricts to an automorphic character of $G_\xi(\mathbb{A})$ for every $\xi \in X(F)$.

Assume that c is an automorphic cocycle such that for every $\xi \in X(F)$

$$\int_{[G_\xi]} |\mathrm{c}(\xi, h)|\delta_G(h)\delta_{G_\xi}(h)^{-1}\, dh < \infty. \tag{2.1}$$

Consequently,

$$\int_{[G_\xi]} \mathrm{c}(\xi, h)\delta_G(h)\delta_{G_\xi}(h)^{-1}\, dh = \begin{cases} \mathrm{vol}([G_\xi]) & \text{if } (\mathrm{c}(\xi,\cdot)\delta_G)|_{G_\xi(\mathbb{A})} \equiv \delta_{G_\xi}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Remark 2.1. Note that if $[G_\xi]$ is compact for every $\xi \in X(F)$, then assumption (2.1) clearly holds. If $G(\mathbb{A})$ and $G_\xi(\mathbb{A})$ are unimodular and $\mathrm{c}(\xi,\cdot)|_{G_\xi(\mathbb{A})}$ is a unitary character, then for (2.1) to hold, it is enough to assume that $[G_\xi]$ is of finite volume.

Remark 2.2. Note that if $\mathrm{c}\colon X(\mathbb{A}) \times G(\mathbb{A}) \to \mathbb{C}$ is an automorphic cocycle and χ is an automorphic character of $G(\mathbb{A})$, then the function c_χ defined by $\mathrm{c}_\chi(x, g) = \chi(g)\mathrm{c}(x, g)$ is also an automorphic cocycle.

We say that $\xi \in X(F)$ is *relevant* if $\mathrm{c}(\xi,\cdot)\delta_G\delta_{G_\xi}^{-1}$ is trivial on $G_\xi(\mathbb{A})$. By our assumptions, for $\xi \in X(F)$, $\gamma \in G(F)$ and $h \in G_\xi(\mathbb{A})$ we have $\mathrm{c}(\xi^\gamma, \gamma^{-1}h\gamma) = \mathrm{c}(\xi, h)$. It is also clear that $\delta_G(\gamma^{-1}h\gamma) = \delta_G(h)$ and $\delta_{G_{\xi^\gamma}}(\gamma^{-1}h\gamma) = \delta_{G_\xi}(h)$. Therefore, ξ is relevant if and only if ξ^γ is relevant. We denote by (X/G) a set of representatives for the $G(F)$-orbits in $X(F)$ and by $(X/G)_{\mathrm{rel}}$ the subset of relevant elements in (X/G). We note the following formal unfolding principle. At this generality, we ignore convergence issues, hence the term principle. For $\phi \in \mathcal{S}(X(\mathbb{A}))$, formally, we have

$$\int_{[G]} \sum_{\xi \in X(F)} \phi(\xi^g)\mathrm{c}(\xi, g)\, dg = \sum_{\xi \in (X/G)_{\mathrm{rel}}} \mathrm{vol}([G_\xi]) \int_{G_\xi(\mathbb{A})\backslash G(\mathbb{A})} \phi(\xi^g)\mathrm{c}(\xi, g)\, dg. \tag{2.3}$$

Indeed, let (X/G) be a set of representatives for the $G(F)$-orbits in $X(F)$. Then

$$\sum_{\xi\in X(F)}\phi(\xi^g)\mathrm{c}(\xi,g)=\sum_{\xi\in(X/G)}\sum_{\gamma\in G_\xi(F)\backslash G(F)}\phi(\xi^{\gamma g})\mathrm{c}(\xi^\gamma,g)$$
$$=\sum_{\xi\in(X/G)}\sum_{\gamma\in G_\xi(F)\backslash G(F)}\phi(\xi^{\gamma g})\mathrm{c}(\xi,\gamma g).$$

On the left-hand side of (2.3), we exchange the order of integration over $[G]$ and summation over (X/G) and then unfold γ with g to obtain

$$\sum_{\xi\in(X/G)}\int_{G_\xi(F)\backslash G(\mathbb{A})}\phi(\xi^g)\mathrm{c}(\xi,g)\,dg.$$

Integrating in stages, this equals

$$\sum_{\xi\in(X/G)}\int_{G_\xi(\mathbb{A})\backslash G(\mathbb{A})}\phi(\xi^g)\int_{[G_\xi]}\mathrm{c}(\xi,hg)\delta_G(h)\delta_{G_\xi}(h)^{-1}\,dh\,dg$$
$$=\sum_{\xi\in(X/G)}\int_{G_\xi(\mathbb{A})\backslash G(\mathbb{A})}\phi(\xi^g)\mathrm{c}(\xi,g)\,dg\int_{[G_\xi]}\mathrm{c}(\xi,h)\delta_G(h)\delta_{G_\xi}(h)^{-1}\,dh.$$

It follows from (2.2) that this equals the right-hand side of (2.3).

2.2. *Measure conventions*

For a locally compact group H and a closed subgroup H' both unimodular and endowed with Haar measures dh, dh', respectively, we consider the H-invariant measure dx on $H'\backslash H$ such that

$$\int_H f(h)\,dh=\int_{H'\backslash H}\int_{H'}f(h'x)\,dh'\,dx$$

for every continuous function of compact support f on H.

Throughout the global part of this work, we apply the following conventions on invariant measures. Discrete subgroups are endowed with the counting measure. Let dx be the Haar measure on $\mathbb{A}$ normalized so that the volume of the compact quotient $F\backslash\mathbb{A}$ is one. Similarly, for a unipotent group U defined over F, the space $[U]$ is compact, and we endow $U(\mathbb{A})$ with the unique Haar measure so that the volume of $[U]$ is one.

Fix once and for all a Haar measure d^*x on $\mathbb{A}^*$. For a rank n split torus T defined over F, the Haar measure on $T(\mathbb{A})$ is determined by its isomorphism with $(\mathbb{A}^*)^n$.

Other measures used in the global part of this work are the Haar measures on $H(\mathbb{A})$, where H is either SL_2 or PGL(3). The Haar measure dh on $H(\mathbb{A})$ is normalized via the Iwasawa decomposition. That is, if $T(\mathbb{A})$ is the diagonal subgroup of $H(\mathbb{A})$, then

$$\int_{H(\mathbb{A})}f(h)\,dh=\int_{U(\mathbb{A})\times T(\mathbb{A})\times K}f(utk)\delta_B(t)^{-1}\,du\,dt\,dk$$

for any continuous function of compact support f on $H(\mathbb{A})$. Here K is the standard maximal compact subgroup of $H(\mathbb{A})$, dk is the Haar measure on K with total volume one, B is the standard Borel subgroup of H, U is the unipotent radical of B, and δ_B is the modular function of $B(\mathbb{A})$.

3. The geometrization of the distribution $I(f, \Theta)$

Throughout the rest of Part I, we suppose that the global field F contains a primitive cube root of unity ρ (that is, there exists $\rho \in F$ such that $\rho^2 + \rho + 1 = 0$), and we set $\mu_3 = \langle \rho \rangle$. Our goal is to geometrize the distribution $I(f, \Theta)$ given by (1.3).

3.1. The adjoint embedding

For $n \geq 2$, let w_n denote the $n \times n$ antidiagonal matrix $w_n = (\delta_{i,n+1-j}) \in \mathrm{GL}_n$. We realize the adjoint representation of PGL(3) as the map $\mathrm{PGL}(3) \to \mathrm{GL}(\mathfrak{sl}(3))$ defined by conjugation. Considering $\mathfrak{sl}(3)$ as a quadratic space with respect to the bilinear form $\langle x, y \rangle = \mathrm{Tr}(xy)$, the image lies in the special orthogonal group $\mathrm{SO}(\mathfrak{sl}(3))$. With respect to the basis

$$\{e_{1,3}, e_{2,3}, e_{1,2}, e_{1,1} - e_{2,2}, e_{2,2} - e_{3,3}, e_{2,1}, e_{3,2}, e_{3,1}\}$$

of $\mathfrak{sl}(3)$, where the $e_{i,j}$ are the standard elementary matrices, $\mathrm{SO}(\mathfrak{sl}(3))$ is isomorphic to $\mathrm{SO}(J)$, where

$$J = \begin{pmatrix} & & & w_3 \\ & 2 & -1 & \\ & -1 & 2 & \\ w_3 & & & \end{pmatrix}.$$

Since the field F contains the cube roots of unity, the group $\mathrm{SO}(J)$ is split over F and may be conjugated to $\mathrm{SO}_8 := \mathrm{SO}(w_8)$; indeed, if $g_0 = -\frac{1}{3}\left(\begin{smallmatrix} 1-\rho & 2+\rho \\ -(1+2\rho) & 1+2\rho \end{smallmatrix}\right)$, then ${}^t g_0 \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right) g_0 = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$. In this way, we realize the adjoint map as a homomorphism $\mathrm{Ad}\colon \mathrm{PGL}(3) \to \mathrm{SO}_8$. This is the map that is used in period (1.2). Restricted to $N(\mathbb{A})$, this leads to the expression

$$\mathrm{Ad}\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & -y & -(xy+\rho z) & -(xy+\rho^2 z) & x(xy-z) & -zy & z(xy-z) \\ & 1 & 0 & \rho^2 y & \rho y & xy-z & -y^2 & y(xy-z) \\ & & 1 & x & x & -x^2 & z & -zx \\ & & & 1 & 0 & -x & -\rho y & \rho xy + \rho^2 z \\ & & & & 1 & -x & -\rho^2 y & \rho^2 xy + \rho z \\ & & & & & 1 & 0 & y \\ & & & & & & 1 & -x \\ & & & & & & & 1 \end{pmatrix}.$$

It was also explicated in [1].

3.2. *First steps*

Since Θ is automorphic and Ad is defined over F, we may combine the integral over $[G]$ with the sum over $G(F)$ to obtain

$$I(f,\Theta) = \int_{[N]} \int_{G(\mathbb{A})} f(g^{-1}n)\Theta(\mathrm{Ad}(g))\psi(n)\,dg\,dn,$$

and after the change of variables $g \mapsto ng$, we obtain that

$$\begin{aligned} I(f,\Theta) &= \int_{[N]} \left\{ \int_{G(\mathbb{A})} f(g^{-1})\Theta(\mathrm{Ad}(n)\,\mathrm{Ad}(g))\,dg \right\} \psi(n)\,dn \\ &= \int_{[N]} \Theta_f(\mathrm{Ad}(n))\psi(n)\,dn, \end{aligned} \tag{3.1}$$

where $\Theta_f \in \Theta_{\mathrm{SO}_8}$ is defined by $\Theta_f(x) = \int_{G(\mathbb{A})} f(g^{-1})\Theta(x\,\mathrm{Ad}(g))\,dg$.

This means that in order to obtain a geometric expansion, we are reduced to showing that

$$\int_{[N]} \Theta(\mathrm{Ad}(n))\psi(n)\,dn$$

is a sum of factorizable integrals.

As noted in the introduction, by [11, Theorem 6.9], the minimal representation Θ_{SO_8} can be expressed as a theta lift using the Weil representation of the metaplectic double cover $\widetilde{\mathrm{Sp}}_{16}(\mathbb{A})$ of $\mathrm{Sp}_{16}(\mathbb{A})$. We now make this explicit and use this to show the desired expansion. To use the theta lift, we follow Deitmar and Krieg [4, Section 2] and Kudla and Rallis [17, Section 5] by imposing a condition on f (see Section 3.4 below) at one archimedean place. With this condition, the theta integral of the Eisenstein series is convergent, and so the computation applies.

3.3. *On the Weil representation*

We realize the group Sp_{16} as the group of automorphisms preserving the alternating matrix $\left(\begin{smallmatrix} 0 & w_8 \\ -w_8 & 0 \end{smallmatrix}\right)$. For $g \in \mathrm{GL}_n$, let $g^* = w_n\,{}^t g^{-1} w_n$.

For a matrix $Y \in M_n(\mathbb{A})$, let ${}_tY = w_n\,{}^tY w_n$. The equality $Y = {}_tY$ means that the matrix Y is symmetric with respect to the second diagonal, i.e., $Y = (y_{i,j})$, where $y_{i,j} = y_{n+1-j,n+1-i}$ (or equivalently, that w_nY is symmetric). It follows from the standard explicit formulas for the Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}_{16}(\mathbb{A})$ (see, e.g., [12]) that for $x \in \mathbb{A}^8$ and $\phi \in \mathcal{S}(\mathbb{A}^8)$, we have

$$\omega_\psi \begin{pmatrix} g & \\ & g^* \end{pmatrix} \phi(x) = \phi(xg), \quad g \in \mathrm{SL}_8(\mathbb{A})$$

and

$$\omega_\psi \begin{pmatrix} I_8 & Y \\ & I_8 \end{pmatrix} \phi(x) = \psi(xw_8\,{}^tY\,{}^tx)\phi(x) = \psi(xYw_8\,{}^tx)\phi(x), \quad Y = {}_tY \in M_8(\mathbb{A}).$$

Combining this gives

$$\omega_\psi\left(\begin{pmatrix} I_8 & Y \\ & I_8 \end{pmatrix}\begin{pmatrix} g & \\ & g^* \end{pmatrix}\right)\phi(x) = \psi(xYw_8\,{}^t x)\phi(xg). \tag{3.2}$$

Let $P = M \ltimes U$ be the Siegel parabolic subgroup of Sp_{16} with unipotent radical $U = \{\left(\begin{smallmatrix} I_8 & Y \\ & I_8 \end{smallmatrix}\right) : {}_tY = Y \in M_8\}$ isomorphic to the vector space of symmetric matrices in M_8 and $M = \{\mathrm{diag}(g, g^*) : g \in \mathrm{GL}_8\}$. Let $P^\circ = M^\circ \ltimes U$, where $M^\circ = \{\mathrm{diag}(g, g^*) : g \in \mathrm{SL}_8\}$ and let $\mathrm{p}_M : P \to M$ be the projection map to the Levi subgroup. We consider the right action of P° on affine 8-space $\mathbb{G}_a^8$ defined by

$$x^p = xg, \quad x \in \mathbb{G}_a^8, \quad p \in P^\circ, \quad \mathrm{p}_M(p) = \mathrm{diag}(g, g^*)$$

and the function $\mathrm{c} : \mathbb{A}^8 \times P^\circ(\mathbb{A}) \to \mathbb{C}$ defined by

$$\mathrm{c}(x, p) = \psi(xYw_8\,{}^t x), \quad x \in \mathbb{A}^8, \quad p = \begin{pmatrix} I_8 & Y \\ & I_8 \end{pmatrix}\mathrm{p}_M(p) \in P^\circ(\mathbb{A}).$$

We can rewrite (3.2) as

$$\omega_\psi(p)\phi(x) = \mathrm{c}(x, p)\phi(x^p), \quad x \in \mathbb{A}^8, \quad p \in P^\circ(\mathbb{A}). \tag{3.3}$$

We denote by P°_ξ the stabilizer in P° of $\xi \in F^8$ (considered as an algebraic group over F) and by $P^\circ(\mathbb{A})_x$ the stabilizer in $P^\circ(\mathbb{A})$ of $x \in \mathbb{A}^8$. Note that

$$\mathrm{c}(\xi, \gamma) = 1, \quad \xi \in F^8, \quad \gamma \in P^\circ(F).$$

Lemma 3.1. *The function* c *satisfies the cocycle condition*

$$\mathrm{c}(x, p_1p_2) = \mathrm{c}(x, p_1)\mathrm{c}(x^{p_1}, p_2), \quad x \in \mathbb{A}^8, \quad p_1, p_2 \in P^\circ(\mathbb{A}).$$

In particular, it is an automorphic cocycle.

Proof. Let $p_i = \left(\begin{smallmatrix} I_8 & Y_i \\ & I_8 \end{smallmatrix}\right)\left(\begin{smallmatrix} g_i & \\ & g_i^* \end{smallmatrix}\right)$, $i = 1, 2$. Then $p_1p_2 = \left(\begin{smallmatrix} I_8 & Y_1 + g_1Y_2(g_1^*)^{-1} \\ & I_8 \end{smallmatrix}\right)\left(\begin{smallmatrix} g_1g_2 & \\ & (g_1g_2)^* \end{smallmatrix}\right)$. Since $g_1Y_2(g_1^*)^{-1}w_8 = g_1Y_2w_8\,{}^t g_1$, it follows that

$$\begin{aligned}\mathrm{c}(x, p_1p_2) &= \psi(xY_1w_8\,{}^t x + xg_1Y_2w_8\,{}^t(xg_1)) = \mathrm{c}(x, p_1)\mathrm{c}(xg_1, p_2) \\ &= \mathrm{c}(x, p_1)\mathrm{c}(x^{p_1}, p_2)\end{aligned}$$

as required. ∎

3.4. On the theta lift

We fix an embedding of $\mathrm{SL}_2 \times \mathrm{SO}_8$ in Sp_{16} such that $\mathrm{SL}_2 \times \mathrm{Ad}(N)$ embeds in P° (we later explicate such an embedding) and denote by $(g, h) \mapsto [g, h]$ the associated embedding of $\mathrm{SL}_2(\mathbb{A}) \times \mathrm{SO}_8(\mathbb{A})$ in $\widetilde{\mathrm{Sp}}_{16}(\mathbb{A})$. Note that the action of the Weil representation directly shows that $P^\circ(\mathbb{A})$ splits in $\widetilde{\mathrm{Sp}}_{16}(\mathbb{A})$. Let

$$\widetilde{\theta}_\phi(g) = \sum_{x \in F^8} \omega_\psi(g)\phi(x), \quad g \in \widetilde{\mathrm{Sp}}_{16}(\mathbb{A}).$$

There is a subspace $\mathcal{S}^\circ(\mathbb{A}^8)$ of $\mathcal{S}(\mathbb{A}^8)$ (the image of $\mathcal{S}(\mathbb{A}^8)$ under the Casimir operator in the center of the universal enveloping algebra of the Lie algebra of SL_2 at one fixed archimedean place) such that for $\phi \in \mathcal{S}^\circ(\mathbb{A}^8)$, we have that $g \mapsto \widetilde{\theta}_\phi([g,h])$ is a rapidly decreasing automorphic function on $\mathrm{SL}_2(\mathbb{A})$. For $\phi \in \mathcal{S}^\circ(\mathbb{A}^8)$, let

$$\Theta_\phi(g) = \int_{[\mathrm{SL}_2]} \widetilde{\theta}_\phi([h,g])\,dh, \quad g \in \mathrm{SO}_8(\mathbb{A}).$$

We then have

$$\Theta_{\mathrm{SO}_8} = \{\Theta_\phi : \phi \in \mathcal{S}^\circ(\mathbb{A}^8)\}.$$

We consider the distributions on $G(\mathbb{A})$ defined by

$$I(f,\phi) = \int_{[N]} \int_{G(\mathbb{A})} f(g^{-1}) \int_{[\mathrm{SL}_2]} \widetilde{\theta}_\phi([h, \mathrm{Ad}(ng)])\,dh\,dg\,\psi(n)\,dn,$$

where $f \in \mathcal{S}(G(\mathbb{A}))$, $\phi \in \mathcal{S}^\circ(\mathbb{A}^8)$.

Note that $I(f,\phi) = I(f,\Theta_\phi)$. Let

$$\phi[f](x) = \int_{G(\mathbb{A})} f(g^{-1})\omega_\psi([I_2, \mathrm{Ad}(g)])\phi(x)\,dg. \tag{3.4}$$

Then

$$\begin{aligned} I(f,\phi) &= \int_{[N]} \int_{[\mathrm{SL}_2]} \tilde{\theta}_{\phi[f]}([h, \mathrm{Ad}(n)])\,dh\,\psi(n)\,dn \\ &= \int_{[N]} \int_{[\mathrm{SL}_2]} \sum_{\xi \in F^8} \omega_\psi([h, \mathrm{Ad}(n)])\phi[f](\xi)\,dh\,\psi(n)\,dn. \end{aligned}$$

We therefore need to unfold integrals of the form

$$I(\phi) = \int_{[N]} \int_{[\mathrm{SL}_2]} \sum_{\xi \in F^8} \omega_\psi(h, \mathrm{Ad}(n))\phi(\xi)\,dh\,\psi(n)\,dn. \tag{3.5}$$

3.5. *The unfolding of I*

Let $H = \mathrm{SL}_2 \times N$. For $h = (g,n) \in H(\mathbb{A})$, by abuse of notation we write $[h] = [g, \mathrm{Ad}(n)]$, and we extend ψ to an automorphic character of $H(\mathbb{A})$ (still denoted by ψ) that is trivial on $\mathrm{SL}_2(\mathbb{A})$. With our choice of embedding of $H(\mathbb{A})$ and by (3.3), we have

$$I(\phi) = \int_{[H]} \sum_{\xi \in F^8} \phi(\xi^{[h]})\mathrm{c}_\psi(\xi, [h])\,dh,$$

where $\mathrm{c}_\psi(\xi, [g, \mathrm{Ad}(n)]) = \psi(n)\mathrm{c}(\xi, [g, \mathrm{Ad}(n)])$.

The $H(F)$-orbits in F^8, as well as the stabilizers H_ξ for orbit representatives ξ, are explicated in Section 4.2. In particular, we show in Theorem 4.2 that H_ξ is unimodular and $[H_\xi]$ has finite volume for any $\xi \in F^8$. Furthermore, it is a direct consequence of the explicit formula (4.2) below that for $\xi = 0$, we have $H_\xi = H$ so that $\xi = 0$ is not relevant and also that for any $\xi \neq 0$, the stabilizer H_ξ is unipotent so that $[H_\xi]$ is compact. By convention, in this case $[H_\xi]$ is of volume one.

Let Ξ_{rel} be a set of representatives for the $H(F)$-orbits in F^8 for which

$$\mathrm{c}_\psi(\xi,\cdot)|_{H_\xi(\mathbb{A})} \equiv 1.$$

As a consequence of Lemma 3.1, Remarks 2.1 and 2.2 and the orbit analysis summarized in Theorem 4.2, we can apply the unfolding principle of Section 2 to obtain the following.

Proposition 3.2. *The distribution I given in (3.5) is a sum of factorizable orbital integrals*

$$I(\phi) = \sum_{\xi \in \Xi_{\mathrm{rel}}} \mathcal{O}(\xi, \phi),$$

where for $\xi \in F^8$ relevant, the associated orbital integral is given by

$$\mathcal{O}(\xi,\phi) = \int_{H_\xi(\mathbb{A})\backslash H(\mathbb{A})} \phi(\xi^{[h]})\mathrm{c}_\psi(\xi,[h])\,dh.$$

In order to compare the distributions I and J, we must describe the quantities in this proposition precisely. This is accomplished in Theorem 4.2 below.

4. Determination of the relevant orbits and orbital integrals for *I*

4.1. Explication of the setup

Let $V = \mathrm{Span}(e, f)$ be the two-dimensional symplectic space with $\langle e, f\rangle = 1 = -\langle f, e\rangle$ and $W = \mathrm{Span}(e_1, \dots, e_8)$ with $\langle e_i, e_j\rangle = \delta_{i,9-j}$. Fixing the basis $\{e, f\}$ identifies SL_2 with $\mathrm{Sp}(V)$ and the basis $\{e_1, \dots, e_8\}$ identifies SO_8 with $\mathrm{SO}(W)$. We also consider $V \otimes W$ as a symplectic space with the product form. Fixing the basis

$$\{e\otimes e_1, f\otimes e_1, \dots, e\otimes e_4, f\otimes e_4, -e\otimes e_5, f\otimes e_5, \dots, -e\otimes e_8, f\otimes e_8\}$$

identifies Sp_{16} with $\mathrm{Sp}(V\otimes W)$. Via the tensor product embedding $\mathrm{Sp}(V)\times\mathrm{SO}(W) \hookrightarrow \mathrm{Sp}(V\otimes W)$, this defines an embedding of

$$\mathrm{SL}_2 \times \mathrm{SO}_8 \hookrightarrow \mathrm{Sp}_{16}.$$

We explicate this embedding.

Let $\lambda\colon \mathrm{GL}_2 \to \mathrm{GL}_8$ be the embedding $\lambda(h) = \mathrm{diag}(h,h,h,h)$. Note that $\lambda({}^t h) = {}^t\lambda(h)$ and $\lambda(h)^* = \lambda(h^*)$. The embedding of SL_2 in Sp_{16} is then $h \mapsto \mathrm{diag}(\lambda(h), \lambda(h^*))$.

The embedding of SO_8 in Sp_{16} can be described as follows. Let $\iota\colon M_4 \to M_8$ be the standard 'tensor with I_2 map', that is,

$$\iota(A) = (a_{i,j} I_2), \quad A = (a_{i,j}) \in M_4,$$

and let $\jmath\colon M_4 \to M_8$ be the twisted map $\jmath(A) = \lambda(\varepsilon_2)\imath(A) = \imath(A)\lambda(\varepsilon_2)$, where $\varepsilon_2 = \mathrm{diag}(1, -1)$. Then the embedding of SO_8 to Sp_{16} in terms of the 4×4 blocks is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \imath(a) & \jmath(b) \\ \jmath(c) & \imath(d) \end{pmatrix} = \mathrm{diag}(\lambda(\varepsilon_2), I_8) \begin{pmatrix} \imath(a) & \imath(b) \\ \imath(c) & \imath(d) \end{pmatrix} \mathrm{diag}(\lambda(\varepsilon_2), I_8).$$

Furthermore, we remark that the group homomorphism λ and the algebra homomorphism $\imath$ are such that $\lambda(\mathrm{GL}_2)$ and $\imath(M_4)$ commute with each other.

Recall that $\mathrm{Ad}(N)$ consists of upper-triangular unipotent matrices in SO_8. For

$$n = \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \in N(\mathbb{A}), \tag{4.1}$$

we write

$$\mathrm{Ad}(n) = \begin{pmatrix} I_4 & s \\ & I_4 \end{pmatrix} \begin{pmatrix} u & \\ & u^* \end{pmatrix}$$

with $u = u(n) \in \mathrm{GL}_4$ upper-triangular unipotent and $s = s(n) \in M_4$ such that sw_4 is alternating. We have

$$[h, \mathrm{Ad}(n)] = \begin{pmatrix} I_8 & \jmath(s) \\ & I_8 \end{pmatrix} \begin{pmatrix} \imath(u)\lambda(h) & \\ & \imath(u^*)\lambda(h^*) \end{pmatrix}, \quad h \in \mathrm{SL}_2(\mathbb{A}), \quad n \in N(\mathbb{A}).$$

Since $w_8 = \lambda(w_2)\imath(w_4)$, we see that $\jmath(s)w_8 = \imath(sw_4)\lambda(\varepsilon_2 w_2)$. We also note that

$$\xi^{[h,\mathrm{Ad}(n)]} = \xi\imath(u)\lambda(h).$$

Since $h\varepsilon_2 w_2\, {}^t h = \varepsilon_2 w_2$, it follows that $x\jmath(s)w_8\, {}^t x = x\lambda(h)\jmath(s)w_8\, {}^t\lambda(h)\, {}^t x$, $x \in \mathbb{A}^8$, $h \in \mathrm{SL}_2(\mathbb{A})$ and therefore

$$\mathrm{c}(\xi, [h, \mathrm{Ad}(n)]) = \mathrm{c}(\xi, [I_2, \mathrm{Ad}(n)]), \quad n \in N(\mathbb{A})$$

is independent of $h \in \mathrm{SL}_2(\mathbb{A})$.

We notice from the explicit formula for $\mathrm{Ad}(n)$ in Section 3.1 that

$$u = \begin{pmatrix} 1 & x & -y & -yx - z\rho \\ & 1 & & y\rho^2 \\ & & 1 & x \\ & & & 1 \end{pmatrix}$$

and

$$sw_4 = \begin{pmatrix} & (z - yx)y\rho^2 & zx\rho & -(yx + z\rho^2) \\ (yx - z)y\rho^2 & & -(z + xy\rho^2) & y\rho \\ -zx\rho & z + xy\rho^2 & & x \\ yx + z\rho^2 & -y\rho & -x & \end{pmatrix}.$$

Writing $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in F^8$ with $\xi_i \in F^2$, $i = 1, 2, 3, 4$, we therefore have

$$\xi^{[h,\mathrm{Ad}(n)]} = (\xi_1 h, (\xi_2 + x\xi_1)h, (\xi_3 - y\xi_1)h, (\xi_4 + x\xi_3 + y\rho^2\xi_2 - (yx + z\rho)\xi_1)h). \quad (4.2)$$

We also conclude that $\jmath(s)w_8$ equals

$$\begin{pmatrix} 0 & 0 & 0 & (z-xy)y\rho^2 & 0 & xz\rho & 0 & -(xy+z\rho^2) \\ 0 & 0 & (xy-z)y\rho^2 & 0 & -xz\rho & 0 & xy+z\rho^2 & 0 \\ 0 & (xy-z)y\rho^2 & 0 & 0 & 0 & -(xy\rho^2+z) & 0 & y\rho \\ (z-xy)y\rho^2 & 0 & 0 & 0 & xy\rho^2+z & 0 & -y\rho & 0 \\ 0 & -xz\rho & 0 & xy\rho^2+z & 0 & 0 & 0 & x \\ xz\rho & 0 & -(xy\rho^2+z) & 0 & 0 & 0 & -x & 0 \\ 0 & xy+z\rho^2 & 0 & -y\rho & 0 & -x & 0 & 0 \\ -(xy+z\rho^2) & 0 & y\rho & 0 & x & 0 & 0 & 0 \end{pmatrix}$$

and this gives us an explicit formula for $\mathfrak{c}(\xi, [h, \mathrm{Ad}(n)]) = \psi(\xi\,\jmath(s)w_8\,{}^t\xi)$.

4.2. Relevant orbits and their associated orbital integrals

We use the above notation and formulas in order to find an explicit set Ξ_{rel} of relevant orbit representatives, compute their stabilizers and explicate the associated factorizable orbital integrals. Let $e_1 = (1, 0)$, $e_2 = (0, 1)$ be the standard basis of F^2 and $N_2 = \{g \in \mathrm{SL}_2 : e_2 g = e_2\}$. Whenever we write $x = (x_1, x_2, x_3, x_4) \in \mathbb{A}^8$, we mean that each $x_i \in \mathbb{A}^2$. In particular, we write $x_i = 0$ if $x_i = (0, 0)$.

The following elementary observation will be used repeatedly in the orbit analysis.

Lemma 4.1. *Let $\{v_1, v_2\}$ be a basis of F^2. Then there exist $h \in \mathrm{SL}_2(F)$ and $a \in F^*$ such that $v_1 h = e_2$ and $v_2 h = a e_1$.*

Proof. Applying $h = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ if necessary, we may assume without loss of generality that $v_1 = (a, b)$ with $b \neq 0$. Then by applying $\begin{pmatrix} b & \\ -a & b^{-1} \end{pmatrix}$, we may assume without loss of generality that $v_1 = e_2$. Now write $v_2 = (a, b)$. Then $a \neq 0$, and applying $\begin{pmatrix} 1 & -a^{-1}b \\ & 1 \end{pmatrix}$ the lemma follows. ■

For $\phi \in \mathcal{S}(\mathbb{A}^8)$, we let $\phi_{K_2} \in \mathcal{S}(\mathbb{A}^8)$ be defined by

$$\phi_{K_2}(v) = \int_{K_2} \phi[v\lambda(k)]\, dk,$$

where K_2 is the standard maximal compact subgroup of $\mathrm{SL}_2(\mathbb{A})$.

Let $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in F^8$. It follows from (4.2) that the following are invariants of $H(F)$-orbits:

- $d_1(\xi) = \dim\langle \xi_1 \rangle$;
- $d_{1,2}(\xi) = \dim\langle \xi_1, \xi_2 \rangle$;
- $d_{1,3}(\xi) = \dim\langle \xi_1, \xi_3 \rangle$;
- $d_{1,2,3}(\xi) = \dim\langle \xi_1, \xi_2, \xi_3 \rangle$;
- $d_{1,2,3,4}(\xi) = \dim\langle \xi_1, \xi_2, \xi_3, \xi_4 \rangle$.

In the remainder of this section, we will analyze the orbits case by case using these invariants. The following result summarizes this analysis.

Theorem 4.2. (1) *For every $\xi \in F^8$, the stabilizer H_ξ is unimodular and $[H_\xi]$ has finite volume.*

(2) *A complete set of representatives for the $H(F)$-orbits that are relevant is given in Lemma 4.3 (orbits with $d_1 = 0$ and $d_{1,2} = d_{1,2,3,4} = 1$); Lemma 4.4 (orbits with $d_1 = 0$, $d_{1,2} = d_{1,2,3} = 1$ and $d_{1,2,3,4} = 2$); Lemma 4.6 (orbits with $d_1 = 1 = d_{1,2,3,4}$); Lemma 4.8 (orbits with $d_1 = 1 = d_{1,2}$ and $d_{1,2,3} = 2$); Lemma 4.9 (orbits with $d_1 = 1 = d_{1,3}$ and $d_{1,2} = 2$); and Lemma 4.10 (orbits with $d_1 = 1$ and $d_{1,2} = 2 = d_{1,3}$; these are the generic relevant orbits). For every relevant orbit representative $\xi \in \Xi_{\mathrm{rel}}$, the stabilizer H_ξ is unipotent and in particular $[H_\xi]$ is compact.*

When the orbit is relevant, in these lemmas we also write the associated orbital integral explicitly. There are other $H(F)$-orbits, and as part of the proof we will show that they are not relevant. Throughout this analysis, we implicitly use coordinates (4.1) for $n \in N(\mathbb{A})$.

4.2.1. Orbits with $d_{1,2} = 0$ are irrelevant. Suppose that $d_{1,2}(\xi) = 0$ and write $\xi = (0, 0, \xi_3, \xi_4)$. Note that

$$\mathrm{c}(\xi, [h, \mathrm{Ad}(n)]) = \psi(2x\xi_3\varepsilon_2 w_2\, {}^t\xi_4)$$

is independent of y (and z), and by (4.2) we have

$$\xi^{[h,\mathrm{Ad}(n)]} = (0, 0, \xi_3 h, (\xi_4 + x\xi_3)h).$$

In particular,

$$\left\{\left(I_2, \begin{pmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{pmatrix}\right) : y \in \mathbb{A}\right\} \subseteq H_\xi(\mathbb{A})$$

and therefore ξ is not relevant. We further compute the stabilizer for a chosen representative. We separate into four cases. If $d_{1,2,3,4}(\xi) = 0$ (that is, for $\xi = 0$), we have $H_\xi = H$. Otherwise, if $d_{1,2,3}(\xi) = 0$, then since SL_2 acts transitively on non-zero vectors in F^2, we may assume without loss of generality that $\xi_4 = e_2$ and then $H_\xi = N_2 \times N$. If $d_{1,2,3}(\xi) = 1 = d_{1,2,3,4}(\xi)$, then by the same argument we may assume that $(\xi_3, \xi_4) = (e_2, \alpha e_2)$ for some $\alpha \in F$ and in this case $H_\xi = N_2 \times \{n \in N : x = 0\}$. Finally, if $d_{1,2,3}(\xi) = 1$ and $d_{1,2,3,4}(\xi) = 2$, then by Lemma 4.1 we may assume without loss of generality that $\xi_3 = e_2$ and $\xi_4 = \alpha e_1$ for some $\alpha \in F^*$. In this case, $H_\xi = \{((\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}), n) \in N_2 \times N : \alpha b + x = 0\}$. In particular, in all cases H_ξ is unimodular and $[H_\xi]$ is of finite volume (and in fact unipotent unless $\xi = 0$).

4.2.2. The $H(F)$-orbits with invariants $d_1 = 0$ and $d_{1,2} = d_{1,2,3,4} = 1$.

Lemma 4.3. *A complete set of representatives for $H(F)$-orbits with invariants $d_1 = 0$ and $d_{1,2} = d_{1,2,3,4} = 1$ is given by $\{(0, e_2, \alpha e_2, 0) : \alpha \in F\}$. For $\xi = (0, e_2, \alpha e_2, 0)$, the* *stabilizer* $H_\xi = N_2 \times \{n \in N : x\alpha + y\rho^2 = 0\}$ *is unipotent. Furthermore,* ξ *is relevant if and only if* $\alpha = \rho^2$*, and in this case the associated orbital integral is*

$$I((0,0,0,1,0,\rho^2,0,0),\phi) = \int_{\mathbb{A}} \int_{\mathbb{A}^*} \phi_{K_2}[(0,0,0,t,0,\rho^2 t,0,xt)]|t|^2\, d^*t\, \psi(x)\, dx.$$

Proof. Let $\xi \in F^8$ be such that $d_1(\xi) = 0$ and $d_{1,2}(\xi) = d_{1,2,3,4}(\xi) = 1$. Then there exist non-zero $v \in F^2$ and $\alpha, \alpha' \in F$ such that $\xi = (0, v, \alpha v, \alpha' v)$. Since $\mathrm{SL}_2(F)$ acts transitively on non-zero vectors in F^2, there exists $h_0 \in \mathrm{SL}_2(F)$ such that $vh_0 = e_2$. Let

$$n_0 = \begin{pmatrix} 1 & & \\ & 1 & -\alpha'\rho \\ & & 1 \end{pmatrix};$$

then $\xi^{[h_0, \mathrm{Ad}(n_0)]} = (0, e_2, \alpha e_2, 0)$. Now for a general $g = (h, n) \in H(F)$, we have

$$(0, e_2, \alpha e_2, 0)^{[g]} = (0, e_2 h, \alpha e_2 h, (x\alpha + y\rho^2) e_2 h).$$

This is of the form $(0, e_2, \alpha' e_2, 0)$ for some $\alpha' \in F$ if and only if $h \in N_2(F)$ and $x\alpha + y\rho^2 = 0$. We conclude that $(0, e_2, \alpha e_2, 0)$ and $(0, e_2, \alpha' e_2, 0)$ are in the same $H(F)$-orbit if and only if $\alpha = \alpha'$ and furthermore

$$H_{(0,e_2,\alpha e_2,0)} = N_2 \times \{n \in N : x\alpha + y\rho^2 = 0\}.$$

We further observe that $\mathfrak{c}((0, e_2, \alpha e_2, 0), [g]) = 1$ for all $g \in H(\mathbb{A})$. It follows that the orbit containing $(0, e_2, \alpha e_2, 0)$ is relevant if and only if $\alpha = \rho^2$. Thus, for $\xi = (0, e_2, \rho^2 e_2, 0)$ we have

$$\mathcal{O}(\xi, \phi) = \int_{\mathbb{A}} \int_{N_2(\mathbb{A}) \backslash \mathrm{SL}_2(\mathbb{A})} \phi[(0,0,0,1,0,\rho^2,0,x)\lambda(h)]\, dh\, \psi(x)\, dx.$$

The lemma now follows using the Iwasawa decomposition. ∎

4.2.3. The $H(F)$-orbits with invariants $d_1 = 0$, $d_{1,2} = d_{1,2,3} = 1$ and $d_{1,2,3,4} = 2$.

Lemma 4.4. *A complete set of representatives for $H(F)$-orbits with invariants $d_1 = 0$, $d_{1,2} = d_{1,2,3} = 1$ and $d_{1,2,3,4} = 2$ is given by*

$$\{(0, e_2, \alpha e_2, \beta e_1) : \alpha \in F, \beta \in F^*\}.$$

For $\xi = (0, e_2, \alpha e_2, \beta e_1)$*, the stabilizer*

$$H_\xi = \left\{ \left(\begin{pmatrix} 1 & -\beta^{-1}(x\alpha + y\rho^2) \\ & 1 \end{pmatrix}, n \right) : n \in N \right\}$$

is unipotent. Furthermore, ξ *is relevant if and only if* $\alpha = \rho$ *and* $\beta = \frac{1}{2}\rho^2$*, and in this case the associated orbital integral is*

$$\mathcal{O}\Big(\Big(0,0,0,1,0,\rho,\frac{1}{2}\rho^2,0\Big),\phi\Big) = \int_{\mathbb{A}^*} \int_{\mathbb{A}} \phi_{K_2}\Big[\Big(0,0,0,t,0,\rho t,\frac{1}{2}\rho^2 t^{-1},a\Big)\Big]\, da\, |t|\, d^*t.$$

Proof. Let $\xi \in F^8$ be such that $d_1(\xi) = 0$, $d_{1,2}(\xi) = d_{1,2,3}(\xi) = 1$ and $d_{1,2,3,4}(\xi) = 2$. Then there exist a basis $\{v, w\}$ of F^2 and $\alpha \in F$ such that $\xi = (0, v, \alpha v, w)$. Applying Lemma 4.1, we have that there exist $h_0 \in \mathrm{SL}_2(F)$ and $\beta \in F^*$ such that

$$\xi^{[h_0, I_8]} = (0, e_2, \alpha e_2, \beta e_1).$$

Now for a general $g = (h, n) \in H(F)$, we have

$$(0, e_2, \alpha e_2, \beta e_1)^{[g]} = (0, e_2 h, \alpha e_2 h, [\beta e_1 + (x\alpha + y\rho^2) e_2] h).$$

This is of the form $(0, e_2, \alpha' e_2, \beta' e_1)$ for some $\alpha' \in F$ and $\beta' \in F^*$ if and only if $h = \left(\begin{smallmatrix} 1 & a \\ & 1 \end{smallmatrix}\right)$, where $\beta a + x\alpha + y\rho^2 = 0$, and in this case $(\alpha', \beta') = (\alpha, \beta)$. We conclude that $(0, e_2, \alpha e_2, \beta e_1)$ and $(0, e_2, \alpha' e_2, \beta' e_1)$ are in the same $H(F)$-orbit if and only if $(\alpha', \beta') = (\alpha, \beta)$ and furthermore $H_{(0, e_2, \alpha e_2, \beta e_1)}$ is as given in the statement of the lemma. We further observe that

$$\mathrm{c}_\psi((0, e_2, \alpha e_2, \beta e_1), [h, \mathrm{Ad}(n)]) = \psi[(1 - 2\beta\rho) y + (1 - 2\beta\alpha) x].$$

It follows that $(0, e_2, \alpha e_2, \beta e_1)$ is relevant if and only if $1 - 2\beta\rho = 1 - 2\beta\alpha = 0$, i.e., if and only if $\alpha = \rho$ and $\beta = \frac{1}{2}\rho^2$. Thus, for $\xi = (0, e_2, \rho e_2, \frac{1}{2}\rho^2 e_1)$ we have

$$\mathcal{O}(\xi, \phi) = \int_{\mathrm{SL}_2(\mathbb{A})} \phi\Big[\Big(0, 0, 0, 1, 0, \rho, \frac{1}{2}\rho^2, 0\Big)\lambda(h)\Big]\, dh.$$

Using the Iwasawa decomposition we can write this as

$$\int_{\mathbb{A}^*} \int_{\mathbb{A}} \phi_{K_2}\Big[\Big(0, 0, 0, t, 0, \rho t, \frac{1}{2}\rho^2 t^{-1}, \frac{1}{2}\rho^2 a t\Big)\Big]\, da\, |t|^2\, d^*t.$$

Making the change of variables $a \mapsto 2\rho t^{-1} a$, the lemma follows. ∎

4.2.4. The $H(F)$-orbits with invariants $d_1 = 0$, $d_{1,2} = 1$ and $d_{1,2,3} = 2$ are not relevant.

Lemma 4.5. *A complete set of representatives for $H(F)$-orbits with invariants $d_1 = 0$, $d_{1,2} = 1$ and $d_{1,2,3} = 2$ is*

$$\{(0, e_2, \beta e_1, 0) : \beta \in F^*\}.$$

None of them is relevant. Furthermore, for $\xi = (0, e_2, \beta e_1, 0)$ the stabilizer $H_\xi = \{I_2\} \times \{n \in N : x = y = 0\}$ is unipotent.

Proof. Let $\xi \in F^8$ be such that $d_1(\xi) = 0$, $d_{1,2}(\xi) = 1$ and $d_{1,2,3}(\xi) = 2$. Then there exist a basis $\{v, w\}$ of F^2 and $u \in F^2$ such that $\xi = (0, v, w, u)$. Applying Lemma 4.1, let $h_0 \in \mathrm{SL}_2(F)$ and $\beta \in F^*$ be such that $\{v h_0, w h_0\} = \{e_2, \beta e_1\}$. Let $x_0, y_0 \in F$ be such that $-u = x_0 \beta v + y_0 \rho^2 w$ and let

$$n_0 = \begin{pmatrix} 1 & x_0 & \\ & 1 & y_0 \\ & & 1 \end{pmatrix}.$$

Then

$$\xi^{[h_0,n_0]} = (0, e_2, \beta e_1, 0).$$

Now for a general $g = (h, n) \in H(F)$, we have

$$(0, e_2, \beta e_1, 0)^{[g]} = (0, e_2 h, \beta e_1 h, [x\beta e_1 + y\rho^2 e_2]h).$$

This is of the form $(0, e_2, \beta' e_1, 0)$ for some $\beta' \in F^*$ if and only if $h = I_2$ and $x = y = 0$, and in this case $\beta' = \beta$. We conclude that $(0, e_2, \beta e_1, 0)$ and $(0, e_2, \beta' e_1, 0)$ are in the same $H(F)$-orbit if and only if $\beta' = \beta$ and furthermore

$$H_{(0,e_2,\beta e_1,0)} = \{I_2\} \times \{n \in N : x = y = 0\}.$$

We further observe that

$$\mathrm{c}((0, e_2, \beta e_1, 0), [h, \mathrm{Ad}(n)]) = \psi[2\beta(xy\rho^2 + z)], \quad (h, n) \in H(\mathbb{A}).$$

The inclusion

$$\{I_2\} \times \left\{ \begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix} : z \in \mathbb{A} \right\} \subseteq H_{(0,e_2,\beta e_1,0)}(\mathbb{A})$$

therefore shows that $(0, e_2, \beta e_1, 0)$ is not relevant for any $\beta \neq 0$. ■

This completes the analysis of orbits with $d_1 = 0$.

4.2.5. The $H(F)$-orbits with invariants $d_1 = 1 = d_{1,2,3,4}$.

Lemma 4.6. *The set of all $\xi \in F^8$ with invariants $d_1 = 1 = d_{1,2,3,4}$ consists of a unique $H(F)$-orbit with representative $(e_2, 0, 0, 0)$ and it is relevant. The stabilizer $H_{(e_2,0,0,0)} = N_2 \times \{I_3\}$ is unipotent, and the associated orbital integral is*

$$\begin{aligned}
&\mathcal{O}((0, 1, 0, 0, 0, 0, 0, 0), \phi) \\
&\quad = \int_{\mathbb{A}} \int_{\mathbb{A}} \int_{\mathbb{A}} \int_{\mathbb{A}^*} \phi_{K_2}[(0, t, 0, xt, 0, -yt, 0, zt)] |t|^2 \, d^*t \, dz \, \psi(x + y) \, dx \, dy.
\end{aligned}$$

Proof. Assume that $d_1(\xi) = 1 = d_{1,2,3,4}(\xi)$. Then there exist non-zero $v \in F^2$ and $\alpha_i \in F$, $i = 1, 2, 3$, such that

$$\xi = (v, \alpha_1 v, \alpha_2 v, \alpha_3 v).$$

Let $h_0 \in \mathrm{SL}_2(F)$ be such that $v h_0 = e_2$ and let

$$n_0 = \begin{pmatrix} 1 & -\alpha_1 & \rho^2\alpha_3 + \rho\alpha_1\alpha_2 \\ & 1 & \alpha_2 \\ & & 1 \end{pmatrix}.$$

Then $\xi^{[h_0,n_0]} = (e_2, 0, 0, 0)$. For $g = (h, n) \in H(\mathbb{A})$, we have

$$(e_2, 0, 0, 0)^{[g]} = (e_2 h, x e_2 h, -y e_2 h, -(xy + z\rho) e_2 h),$$

and therefore $H_{(e_2,0,0,0)}(\mathbb{A}) = N_2(\mathbb{A}) \times \{I_3\}$. Furthermore, $\mathrm{c}((e_2, 0, 0, 0), [h]) = 1$, $h \in H(\mathbb{A})$. It follows that $\xi = (e_2, 0, 0, 0)$ is relevant and we have

$$\mathcal{O}(\xi, \phi) = \int_{\mathbb{A}} \int_{\mathbb{A}} \int_{\mathbb{A}} \int_{N_2(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \phi[(e_2, xe_2, -ye_2, -(xy + z\rho)e_2)\lambda(h)]\, dh \times \psi(x + y)\, dx\, dy\, dz.$$

The change of variables $z \mapsto -\rho^2(z + xy)$ gives

$$\mathcal{O}(\xi, \phi) = \int_{\mathbb{A}} \int_{\mathbb{A}} \int_{\mathbb{A}} \int_{N_2(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \phi[(e_2, xe_2, -ye_2, ze_2)\lambda(h)]\, dh\, dz\, \psi(x + y)\, dx\, dy.$$

The lemma follows by applying the Iwasawa decomposition. ■

4.2.6. The $H(F)$-orbits with invariants $d_1 = 1 = d_{1,2,3}$ and $d_{1,2,3,4} = 2$ are not relevant.

Lemma 4.7. *A complete set of representatives of $H(F)$-orbits with invariants $d_1 = 1 = d_{1,2,3}$ and $d_{1,2,3,4} = 2$ is*

$$\{(e_2, 0, 0, \beta e_1) : \beta \in F^*\}.$$

None of them is relevant. Furthermore, for $\xi = (e_2, 0, 0, \beta e_1)$ the stabilizer

$$H_\xi = \left\{ \left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix} \right) : \beta\rho^2 b = z \right\}$$

is unipotent.

Proof. Assume that $d_1(\xi) = 1 = d_{1,2,3}(\xi)$ and $d_{1,2,3,4}(\xi) = 2$. Then there exist a basis $\{v, w\}$ of F^2 and $\alpha_i \in F$, $i = 1, 2$, such that $\xi = (v, \alpha_1 v, \alpha_2 v, w)$. Applying Lemma 4.1, let $h_0 \in \mathrm{SL}_2(F)$ and $\beta \in F^*$ be such that $vh_0 = e_2$ and $wh_0 = \beta e_1$. Let

$$n_0 = \begin{pmatrix} 1 & -\alpha_1 & \rho\alpha_1\alpha_2 \\ & 1 & \alpha_2 \\ & & 1 \end{pmatrix}.$$

Then $\xi^{[h_0,n_0]} = (e_2, 0, 0, \beta e_1)$. For $g = (h, n) \in H(\mathbb{A})$, we have

$$(e_2, 0, 0, \beta e_1)^{[g]} = (e_2 h, xe_2 h, -ye_2 h, \beta e_1 h - (xy + z\rho)e_2 h).$$

This is of the form $(e_2, 0, 0, \beta' e_1)$ for some $\beta' \in F^*$ if and only if $x = y = 0$ and $h = \left(\begin{smallmatrix} 1 & a \\ & 1 \end{smallmatrix}\right)$ with $\beta a - z\rho = 0$. It follows that $(e_2, 0, 0, \beta e_1)$ and $(e_2, 0, 0, \beta' e_1)$ are in the same $H(F)$-orbit if and only if $\beta = \beta'$ and

$$H_{(e_2,0,0,\beta e_1)}(\mathbb{A}) = \left\{ \left(\begin{pmatrix} 1 & \beta^{-1} z\rho \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix} \right) : z \in \mathbb{A} \right\}.$$

We further note that

$$\mathrm{c}((e_2, 0, 0, \beta e_1), [g]) = \psi[2\beta(xy + z\rho^2)],$$

and therefore we can deduce that $(e_2, 0, 0, \beta e_1)$ is not relevant. ■

4.2.7. The $H(F)$-orbits with invariants $d_1 = 1 = d_{1,2}$ and $d_{1,2,3} = 2$.

Lemma 4.8. *A complete set of representatives of $H(F)$-orbits with invariants $d_1 = 1 = d_{1,2}$ and $d_{1,2,3} = 2$ is*

$$\{(e_2, \alpha e_2, \beta e_1, 0) : \alpha \in F,\ \beta \in F^*\}.$$

For $\xi = (e_2, \alpha e_2, \beta e_1, 0)$, the stabilizer $H_\xi = \{((\begin{smallmatrix}1 & b\\ & 1\end{smallmatrix}), n) \in N_2 \times N : x = 0,\ z = \rho\alpha y,\ \beta b = y\}$ is unipotent. Furthermore, ξ is relevant if and only if $1 + 2\beta\rho\alpha^2 = 0$. For $\alpha \in F^*$ and $\xi[\alpha] = (e_2, \alpha e_2, -\frac{1}{2}\rho^2\alpha^{-2}e_1, 0)$, the associated orbital integral is

$$\begin{aligned}\mathcal{O}(\xi[\alpha], \phi) = \int_{\mathbb{A}^*}\int_{\mathbb{A}}\int_{\mathbb{A}}\int_{\mathbb{A}} \phi_{K_2}\Big[\Big(0, t, 0, t(\alpha + x), -\frac{1}{2}\rho^2\alpha^{-2}t^{-1}, -ty,\\ -\frac{1}{2}\rho^2\alpha^{-2}t^{-1}x, t(y\rho^2\alpha - xy - z\rho)\Big)\Big]\\ \times \psi[-\rho\alpha^{-1}xy - \rho^2\alpha^{-1}z + \alpha^{-2}xz + x + y]\, dx\, dy\, dz |t|^2\, d^*t.\end{aligned}$$

Proof. Assume that $d_1(\xi) = 1 = d_{1,2}(\xi)$ and $d_{1,2,3}(\xi) = 2$. Then there exist a basis $\{v, w\}$ of F^2, $\alpha \in F$ and $u \in F^2$ such that $\xi = (v, \alpha v, w, u)$. Let $x_0, z_0 \in F$ be such that $u = z_0\rho v - x_0 w$ and let

$$n_0 = \begin{pmatrix}1 & x_0 & z_0\\ & 1 & \\ & & 1\end{pmatrix}.$$

Applying Lemma 4.1, let $h_0 \in \mathrm{SL}_2(F)$ and $\beta \in F^*$ be such that $vh_0 = e_2$ and $wh_0 = \beta e_1$. Then $\xi^{[h_0,n_0]} = (e_2, (\alpha + x_0)e_2, \beta e_1, 0)$. For $g = (h, n) \in H(\mathbb{A})$, we have

$$(e_2, \alpha e_2, \beta e_1, 0)^{[g]} = (e_2 h, (\alpha + x)e_2 h, (\beta e_1 - y e_2)h, (\beta x e_1 + (y\rho^2\alpha - xy - z\rho)e_2)h).$$

This is of the form $(e_2, \alpha' e_2, \beta' e_1, 0)$ for some $\alpha' \in F$ and $\beta' \in F^*$ if and only if $x = 0$, $z\rho = y\rho^2\alpha$ and $h = (\begin{smallmatrix}1 & a\\ & 1\end{smallmatrix})$ with $\beta a = y$. It follows that $(e_2, \alpha e_2, \beta e_1, 0)$ and $(e_2, \alpha' e_2, \beta' e_1, 0)$ are in the same $H(F)$-orbit if and only if $(\alpha, \beta) = (\alpha', \beta')$ and

$$H_{(e_2,\alpha e_2,\beta e_1,0)}(\mathbb{A}) = \left\{\left(\begin{pmatrix}1 & \beta^{-1}y\\ & 1\end{pmatrix}, \begin{pmatrix}1 & & y\rho\alpha\\ & 1 & y\\ & & 1\end{pmatrix}\right) : y \in \mathbb{A}\right\}.$$

We further note that

$$\mathrm{c}((e_2, \alpha e_2, \beta e_1, 0), [g]) = \psi[2\beta\alpha(xy\rho^2 + z) - 2\beta xz\rho].$$

We therefore have

$$\mathrm{c}_\psi((e_2, \alpha e_2, \beta e_1, 0), [g]) = \psi[(1 + 2\beta\alpha^2\rho)y],$$
$$g = \left(\begin{pmatrix}1 & \beta^{-1}y\\ & 1\end{pmatrix}, \begin{pmatrix}1 & & y\rho\alpha\\ & 1 & y\\ & & 1\end{pmatrix}\right) \in H_{(e_2,\alpha e_2,\beta e_1,0)}(\mathbb{A}).$$

We conclude that $(e_2, \alpha e_2, \beta e_1, 0)$ is relevant if and only if $1 + 2\beta\alpha^2\rho = 0$ and for $\beta = -\frac{1}{2}\rho^2\alpha^{-2}$, we have

$$\mathcal{O}((0,1,0,\alpha,\beta,0,0,0),\phi) = \int_{\mathbb{A}}\int_{\mathbb{A}}\int_{\mathrm{SL}_2(\mathbb{A})} \phi[(0,1,0,\alpha+x,\beta,0,\beta x,-z\rho)\lambda(h)]\,dh \times \psi(x-\rho^2 z-2\beta\rho xz)\,dx\,dz.$$

Alternatively, we can also express the orbital integral as the iterated integral

$$\int_{N_2(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})}\int_{\mathbb{A}}\int_{\mathbb{A}}\int_{\mathbb{A}} \phi[(0,1,0,\alpha+x,\beta,-y,\beta x, y\rho^2\alpha - xy - z\rho)\lambda(h)] \times \psi[2\beta\alpha(xy\rho^2+z) - 2\beta xz\rho + x + y]\,dx\,dy\,dz\,dh,$$

where the integral over $N_2(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})$ makes sense only after integration over y and z. Applying the Iwasawa decomposition, this equals

$$\int_{\mathbb{A}^*}\int_{\mathbb{A}}\int_{\mathbb{A}}\int_{\mathbb{A}} \phi_{K_2}[(0,t,0,t(\alpha+x),t^{-1}\beta,-ty,t^{-1}\beta x, t(y\rho^2\alpha - xy - z\rho))] \times \psi[2\beta\alpha(xy\rho^2+z) - 2\beta xz\rho + x + y]\,dx\,dy\,dz|t|^2\,d^*t.$$

The lemma follows. ■

4.2.8. The $H(F)$-orbits with invariants $d_1 = 1 = d_{1,3}$ and $d_{1,2} = 2$.

Lemma 4.9. *A complete set of representatives of $H(F)$-orbits with invariants $d_1 = 1 = d_{1,3}$ and $d_{1,2} = 2$ is*

$$\{(e_2, \beta e_1, \alpha e_2, 0) : \alpha \in F,\ \beta \in F^*\}.$$

For $\xi = (e_2, \beta e_1, \alpha e_2, 0)$, the stabilizer $H_\xi = \{((\begin{smallmatrix}1 & b\\ & 1\end{smallmatrix}), n) \in N_2 \times N : y = 0,\ z = \rho^2\alpha x,\ \beta b + x = 0\}$ is unipotent. Furthermore, ξ is relevant if and only if $\beta = \frac{1}{2}\rho\alpha^{-2}$. For $\alpha \in F^*$ and $\xi[\alpha] = (e_2, \frac{1}{2}\rho\alpha^{-2}e_1, \alpha e_2, 0)$, the associated orbital integral is

$$\mathcal{O}(\xi[\alpha],\phi) = \int_{\mathbb{A}^*}\int_{\mathbb{A}}\int_{\mathbb{A}}\int_{\mathbb{A}} \phi_{K_2}\Big[\Big(0,t,\frac{1}{2}\rho\alpha^{-2}t^{-1}, tx, 0, t(\alpha-y), \frac{1}{2}\alpha^{-2}t^{-1}y, \times t(x\alpha - xy - z\rho)\Big)\Big] \times \psi[\alpha^{-2}(xy-z)y - \rho(xy\rho^2+z)\alpha^{-1} + x + y]\,dx\,dy\,dz|t|^2\,d^*t.$$

Proof. Assume that $d_1(\xi) = 1 = d_{1,3}(\xi)$ and $d_{1,2}(\xi) = 2$. Then there exist a basis $\{v, w\}$ of F^2, $\alpha \in F$ and $u \in F^2$ such that $\xi = (v, w, \alpha v, u)$. Let $y_0, z_0 \in F$ be such that $u = z_0\rho v - y_0\rho^2 w$ and let

$$n_0 = \begin{pmatrix} 1 & & z_0 \\ & 1 & y_0 \\ & & 1 \end{pmatrix}.$$

Applying Lemma 4.1, let $h_0 \in \mathrm{SL}_2(F)$ and $\beta \in F^*$ be such that $vh_0 = e_2$ and $wh_0 = \beta e_1$. Then $\xi^{[h_0,n_0]} = (e_2, \beta e_1, (\alpha - y_0)e_2, 0)$. For $g = (h,n) \in H(\mathbb{A})$, we have

$$(e_2, \beta e_1, \alpha e_2, 0)^{[g]} = (e_2 h, (\beta e_1 + xe_2)h, (\alpha - y)e_2 h, (x\alpha e_2 + y\rho^2\beta e_1 - (xy + z\rho)e_2)h).$$

This is of the form $(e_2, \beta' e_1, \alpha' e_2, 0)$ for some $\alpha' \in F$ and $\beta' \in F^*$ if and only if $h = \left(\begin{smallmatrix} 1 & a \\ & 1 \end{smallmatrix}\right)$ with $\beta a + x = 0$, $y = 0$ and $z\rho = x\alpha$, and in this case $(\alpha, \beta) = (\alpha', \beta')$. It follows that $(e_2, \beta e_1, \alpha e_2, 0)$ and $(e_2, \beta' e_1, \alpha' e_2, 0)$ are in the same $H(F)$-orbit if and only if $(\alpha, \beta) = (\alpha', \beta')$ and

$$H_{(e_2, \beta e_1, \alpha e_2, 0)}(\mathbb{A}) = \left\{ \left(\begin{pmatrix} 1 & -\beta^{-1} x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & x & \rho^2 x \alpha \\ & 1 & \\ & & 1 \end{pmatrix} \right) : x \in \mathbb{A} \right\}.$$

We further note that

$$\mathrm{c}((e_2, \beta e_1, \alpha e_2, 0), [g]) = \psi[2\beta(xy - z)\rho^2 y - 2(xy\rho^2 + z)\alpha\beta].$$

We therefore have

$$\mathrm{c}_\psi((e_2, \beta e_1, \alpha e_2, 0), [g]) = \psi[(1 - 2\rho^2\alpha^2\beta)x],$$
$$g = \left(\begin{pmatrix} 1 & -\beta^{-1} x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & x & \rho^2 x \alpha \\ & 1 & \\ & & 1 \end{pmatrix} \right).$$

We conclude that $(e_2, \beta e_1, \alpha e_2, 0)$ is relevant if and only if $2\rho^2\alpha^2\beta = 1$, and in this case, the associated (iterated, as in the previous lemma) orbital integral is

$$\int_{N_2(\mathbb{A}) \backslash \mathrm{SL}_2(\mathbb{A})} \int_{\mathbb{A}} \int_{\mathbb{A}} \int_{\mathbb{A}} \phi[(0, 1, \beta, x, 0, \alpha - y, y\rho^2\beta, x\alpha - xy - z\rho)\lambda(h)]$$
$$\times \psi[2\rho^2\beta(xy - z)y - 2(xy\rho^2 + z)\alpha\beta + x + y]\, dx\, dy\, dz\, dh.$$

Again applying the Iwasawa decomposition, the lemma follows. ∎

4.2.9. The generic $H(F)$-orbits.

Lemma 4.10. *A complete set of representatives of $H(F)$-orbits with invariants $d_1 = 1$ and $d_{1,2} = 2 = d_{1,3}$ is*

$$\{(e_2, \beta e_1, \beta' e_1, 0) : \beta,\ \beta' \in F^*\}.$$

They all have trivial stabilizers and are, in particular, all relevant. For a, $b \in F^*$ and $\xi[a, b] = (e_2, ae_1, be_1, 0)$, the associated orbital integral is

$$\mathcal{O}(\xi[a, b], \phi) = \int_{\mathbb{A}} \int_{\mathbb{A}} \int_{\mathbb{A}} \int_{\mathbb{A}} \int_{\mathbb{A}^*} \phi_{K_2}\big[(0, t, t^{-1}a, t(x + as), t^{-1}b, t(bs - y),$$
$$t^{-1}(xb + y\rho^2 a), t[(xb + y\rho^2 a)s - (xy + z\rho)])\big]$$
$$\times |t|^2\, d^*t\, ds\, \psi[x + y + 2a(xy - z)y\rho^2 - 2bxz\rho]\, dx\, dy\, dz.$$

Proof. Let $\xi \in F^8$ be such that $d_1(\xi) = 1$ and $d_{1,2}(\xi) = 2 = d_{1,3}(\xi)$. Then there exist $v, w, w', u \in F^2$ such that $\{v, w\}$ and $\{v, w'\}$ are bases of F^2 and $\xi = (v, w, w', u)$. Let

$a, b \in F$ be such that $w' = av + bw$ and note that $b \neq 0$. Let $x_0, y_0, z_0 \in F$ be the unique common solutions to the equations

$$\begin{cases} w' - y_0 v = b(w + x_0 v), \\ u + x_0 w' + y_0 \rho^2 w - (x_0 y_0 + z_0 \rho) v = 0. \end{cases}$$

Applying Lemma 4.1, let $h_0 \in \mathrm{SL}_2(F)$ and $\beta \in F^*$ be such that

$$(vh_0, (w + x_0 v)h_0) = (e_2, \beta e_1),$$

and let

$$n_0 = \begin{pmatrix} 1 & x_0 & z_0 \\ & 1 & y_0 \\ & & 1 \end{pmatrix}.$$

Then

$$\xi^{[h_0, \mathrm{Ad}(n_0)]} = (e_2, \beta e_1, b\beta e_1, 0).$$

For $g = (h, n) \in H(F)$, we have

$$\begin{aligned} &(e_2, \beta e_1, \beta' e_1, 0)^{[g]} \\ &\quad = (e_2 h, (\beta e_1 + x e_2)h, (\beta' e_1 - y e_2)h, [(x\beta' + y\rho^2\beta)e_1 - (xy + z\rho)e_2]h), \end{aligned}$$

and we can deduce that this is of the form $(e_2, \beta_1 e_1, \beta_1' e_1, 0)$ for some $\beta_1, \beta_1' \in F^*$ if and only if g is the identity. Thus, $H_{(e_2, \beta e_1, \beta' e_1, 0)}$ is trivial and therefore $(e_2, \beta e_1, \beta' e_1, 0)$ is relevant. We also note that

$$\mathrm{c}((e_2, \beta e_1, \beta' e_1, 0), [g]) = \psi[2\beta(xy - z)y\rho^2 - 2\beta' xz\rho].$$

We therefore have

$$\begin{aligned} &\mathcal{O}((e_2, \beta e_1, \beta' e_1, 0), \phi) \\ &\quad = \int_{\mathbb{A}} \int_{\mathbb{A}} \int_{\mathbb{A}} \int_{\mathrm{SL}_2(\mathbb{A})} \phi[(0, 1, \beta, x, \beta', -y, x\beta' + y\rho^2\beta, -(xy + z\rho))\lambda(h)]\, dh \\ &\qquad \times \psi[x + y + 2\beta(xy - z)y\rho^2 - 2\beta' xz\rho]\, dx\, dy\, dz. \end{aligned}$$

The lemma follows by the Iwasawa decomposition. ∎

This completes the proof of Theorem 4.2.

5. Unfolding the distribution *J*

Recall that $G'(\mathbb{A})$ denotes the adelic 3-fold cover of $\mathrm{SL}_3(\mathbb{A})$ and $G'(F)$ denotes the embedded image of $\mathrm{SL}_3(F)$ in $G'(\mathbb{A})$. For notational convenience, we suppress this embedding and write elements of $G'(F)$ by matrices. We consider the action $g \cdot (n_1, n_2) = n_1^{-1} g n_2$ of $N(F) \times N(F)$ on $G'(F)$. The element g or its orbit is called relevant if

$(n_1, n_2) \mapsto \psi(n_1^{-1}n_2)$ is trivial on the stabilizer $(N \times N)_g(\mathbb{A})$ of g. For $a, b \in F^*$, let $t(a,b) = \mathrm{diag}(a, a^{-1}b, b^{-1})$, and let

$$w_1 = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}.$$

Then using the Bruhat decomposition, one sees that a complete set Ξ'_{rel} of representatives for the relevant orbits is given as follows:

(1) $t(a, a^2) = aI_3$, $a \in \mu_3 \subseteq F^*$;

(2) $t(a^{-2}, a^{-1})w_1w_2 = \begin{pmatrix} & & a^{-2} \\ a & & \\ & a & \end{pmatrix}$, $a \in F^*$;

(3) $t(-a, a^2)w_2w_1 = \begin{pmatrix} & a & \\ & & a \\ a^{-2} & & \end{pmatrix}$, $a \in F^*$;

(4) $t(b^{-1}, a^{-1})w_1w_2w_1 = \begin{pmatrix} & & b^{-1} \\ & -a^{-1}b & \\ a & & \end{pmatrix}$, $a, b \in F^*$.

(See, for example, [2], where the determination of relevant orbits is carried out in the language of Poincaré series; the adelic ingredients are the same.) Note that the notion of relevance does not depend on the cover, though the orbital integral that we consider does.

To give the needed orbital integrals, we fix once and for all an embedding of μ_3 in $\mathbb{C}^*$ and consider the genuine Hecke algebra $\mathcal{H}_{\mathrm{gen}}(G'(\mathbb{A})) = \{f' \in C_c^\infty(G'(\mathbb{A})) : f'(zg) = zf'(g)\}$. For $f' \in \mathcal{H}_{\mathrm{gen}}(G'(\mathbb{A}))$ and $g \in G'(F)$ relevant we consider the orbital integral

$$\mathcal{O}'(g, f') = \int_{N(\mathbb{A})\times N(\mathbb{A})/(N\times N)_g(\mathbb{A})} f'(n_1^{-1}gn_2)\psi(n_1^{-1}n_2)\, dn_1\, dn_2.$$

Here $N(\mathbb{A})$ is embedded in $G'(\mathbb{A})$ by the trivial section $n \mapsto (n, 1)$. Explicitly, according to the relevant orbits we have

(1) $\mathcal{O}'(t(a,a^2), f') = \int_{N(\mathbb{A})} f'(an)\psi(n)\, dn$, $a \in \mu_3$;

(2) $\mathcal{O}'(t(a^{-2}, a^{-1})w_1w_2, f') = \int_{N(\mathbb{A})\times U_{2,1}(\mathbb{A})} f'(n^{-1}t(a^{-2}, a^{-1})w_1w_2u) \times \psi(n^{-1}u)\, dn\, du$;

(3) $\mathcal{O}'(t(-a, a^2)w_2w_1, f') = \int_{U_{2,1}(\mathbb{A})\times N(\mathbb{A})} f'(u^{-1}t(-a, a^2)w_2w_1n)\psi(u^{-1}n)\, du\, dn$;

(4) $\mathcal{O}'(t(b^{-1}, a^{-1})w_1w_2w_1, f') = \int_{N(\mathbb{A})\times N(\mathbb{A})} f'(n_1^{-1}t(b^{-1}, a^{-1})w_1w_2w_1n_2)) \times \psi(n_1^{-1}n_2)\, dn_1\, dn_2$.

Here

$$U_{2,1}(\mathbb{A}) = \left\{ \begin{pmatrix} I_2 & v \\ & 1 \end{pmatrix} : v \in \mathbb{A}^2 \right\}.$$

Proposition 5.1. *The distribution J given in* (1.4) *is a sum of factorizable orbital integrals*

$$J(f') = \sum_{\xi' \in \Xi'_{\mathrm{rel}}} \mathcal{O}(\xi', f'),$$

where the relevant orbits and the associated orbital integrals are given above.

Part II
Local theory: The fundamental lemma

6. Notation and main result

Throughout Part II, we set the following notation. Let F denote a non-archimedean local field, $\mathcal{O}$ the ring of integers of F, $\mathfrak{p}$ its maximal ideal, ϖ a uniformizer in F (a generator of $\mathfrak{p}$), $k_F = \mathcal{O}/\mathfrak{p}$ the residual field of F, p the characteristic of k_F, q the number of elements in k_F, val: $F^* \to \mathbb{Z}$ the standard valuation so that $\mathrm{val}(u\varpi^n) = n$ for $u \in \mathcal{O}^*$, $n \in \mathbb{Z}$, and $S^n = \{s^n : s \in S\}$ for any subgroup S of F^* and $n \in \mathbb{N}$. Assume that F contains a primitive cube root of unity ρ and set $\mu_3 = \langle \rho \rangle$.

Fix a character ψ of F with conductor $\mathcal{O}$ (that is, ψ restricted to $\mathcal{O}$ is trivial and restricted to $\mathfrak{p}^{-1}$ is non-trivial). We denote by $\mathbf{1}_A$ the characteristic function of a set A. We normalize the Haar measure dx on F so that $\int_{\mathcal{O}} dx = 1$ and the Haar measure d^*x on F^* so that $\int_{\mathcal{O}^*} d^*x = 1$. Note that both d^*x and dx restrict to Haar measures on $\mathcal{O}^*$. However, they do not coincide on $\mathcal{O}^*$; we have $\int_{\mathcal{O}^*} dx = 1 - q^{-1}$.

The fundamental lemma for the unit Hecke element is an explicit identity between two families of local orbital integrals at almost all places, associated with unramified data and test functions that are the respective units of the spherical Hecke algebras. The unit element of the spherical Hecke algebra of $\mathrm{PGL}(3, F)$ is the function $f_0 = \mathbf{1}_{\mathrm{PGL}(3,\mathcal{O})}$. In light of the geometrization of the global distribution $I(f, \Theta_\phi) = I(f, \phi)$ in Section 3, local unramified data requires the choice of Schwartz–Bruhat function $\phi_0 = \mathbf{1}_{\mathcal{O}^8}$ in the local Weil representation. Note that for the local analogue of (3.4), we have $\phi_0[f_0] = \phi_0$. Note further that

$$\phi_0(x) = \int_{\mathrm{SL}_2(\mathcal{O})} \phi_0(x\,\mathrm{diag}(k,k,k,k))\,dk, \quad x \in F^8.$$

Here, the Haar measure dk on $\mathrm{SL}_2(\mathcal{O})$ has total volume one. The main work is to compare the two local orbital integrals corresponding to the big cells. These are families of integrals parameterized by $F^* \times F^*$. In light of Lemma 4.10 and the discussion of the prior paragraph, the first is defined by the formula

$$\begin{aligned} I(a,b) = \int_F\int_F\int_F\int_F\int_{F^*} &\mathbf{1}_{\mathcal{O}^8}\big[(0,t,t^{-1}a,t(x+as),t^{-1}b,t(bs-y),t^{-1}(xb+y\rho^2a),\\ &\qquad t[(xb+y\rho^2a)s-(xy+z\rho)])\big]|t|^2\,d^*t\,ds\\ &\times\psi[x+y+2a(xy-z)y\rho^2-2bxz\rho]\,dx\,dy\,dz. \end{aligned} \tag{6.1}$$

This is the local integral obtained from Lemma 4.10.

In order to define the second, we need to set some further notation. Consider the local metaplectic 3-fold cover $\widetilde{\mathrm{SL}}_3(F)$ of $\mathrm{SL}_3(F)$. It is a central extension

$$1 \to \mu_3 \to \widetilde{\mathrm{SL}}_3(F) \to \mathrm{SL}_3(F) \to 1.$$

As a set, we identify $\widetilde{\mathrm{SL}}_3(F)$ with $\mathrm{SL}_3(F) \times \mu_3$, and the group operation is given by

$$(g_1, z_1)(g_2, z_2) = (g_1g_2, z_1z_2\sigma(g_1, g_2)),$$

where σ is a certain 2-cocycle of $\mathrm{SL}_3(F)$. This cocycle is described, following Matsumoto [20], in Bump and Hoffstein [3, §2].

Let N be the group of upper triangular unipotent matrices in $\mathrm{SL}_3(F)$ and $K = \mathrm{SL}_3(\mathcal{O})$. The map $n \mapsto (n, 1)\colon N \to \widetilde{\mathrm{SL}}_3(F)$ is an imbedding of N in $\widetilde{\mathrm{SL}}_3(F)$ and this way we view N as a subgroup of $\widetilde{\mathrm{SL}}_3(F)$. We endow N with the Haar measure $du = \bigotimes_{i<j} du_{i,j}$. The group K also admits a splitting in $\widetilde{\mathrm{SL}}_3(F)$. There is a map $\kappa\colon K \to \mu_3$ such that the map $g \mapsto (g, \kappa(g))$ imbeds K into $\widetilde{\mathrm{SL}}_3(F)$. By abuse of notation, we also denote by ψ the character of N that satisfies $\psi(u) = \psi(u_{1,2} + u_{2,3})$, $u \in N$.

Fix an embedding of μ_3 in $\mathbb{C}^*$ and let $f_0'\colon \widetilde{\mathrm{SL}}_3(F) \to \mathbb{C}$ be defined by

$$f_0'(g, z) = \begin{cases} z\kappa(g)^{-1} & \text{if } g \in K, \\ 0 & \text{otherwise,} \end{cases} \qquad g \in \mathrm{SL}_3(F), \quad z \in \mu_3.$$

The function f_0' is the unit element of the genuine spherical Hecke algebra of $\widetilde{\mathrm{SL}}_3(F)$. The second family of integrals is defined by

$$J(a, b) = \int_N \int_N f_0'((u_1 g_{a,b} u_2, 1))\psi(u_1 u_2)\, du_1\, du_2,$$

where

$$g_{a,b} = \begin{pmatrix} & & b^{-1} \\ & -a^{-1}b & \\ a & & \end{pmatrix} \in \mathrm{SL}_3(F).$$

Let $(\cdot, \cdot)_3\colon F^* \times F^* \to \mu_3$ be the cubic Hilbert symbol. Our main result is the following comparison.

Theorem 6.1 (The fundamental lemma for the big cell orbital integrals-unit element). *Assume $p > 3$. For any $a, b \in F^*$, we have*

$$I(a, b) = (c, d)_3\, J(c, d), \quad \text{where } c = -54a,\ d = 54b.$$

In the final section, we also compare the orbital integrals for the other relevant orbits (Theorem 11.7).

7. Ingredients needed for the comparison

We develop the ingredients that are needed to establish Theorem 6.1. From now on, we assume that $p > 3$.

7.1. The cubic Hilbert symbol

We begin by recalling the basic properties of the cubic Hilbert symbol. They will be used throughout our computation without further mention. For $x, y, z \in F^*$, we have

- $(y, x)_3 = \overline{(x, y)_3}$;
- $(xy, z)_3 = (x, z)_3 (y, z)_3$;

- $(x, 1-x)_3 = 1, x \neq 1$;
- $(x, y)_3 = 1$ for all $y \in F^*$ if and only if $x \in F^{*3}$;
- $(x, u)_3 = 1$ for all $u \in \mathcal{O}^*$ if and only if $3|\mathrm{val}(x)$.

The last property is a consequence of our assumption that the residual characteristic $p > 3$. We further observe that by this assumption and Hensel's lemma, $1 + \mathfrak{p} \subseteq F^{*3}$ and consequently

$$(x + y, z)_3 = (x, z)_3 \quad \text{whenever } |y| < |x|.$$

7.2. Cubic Gauss sums

For future reference, we recall here a formula that is essentially the calculation of the absolute value squared of a cubic Gauss sum. This result is standard and the proof is omitted.

Lemma 7.1. *Let $a \in F$ be such that $|a| = q$. Then*

$$\int_{\mathcal{O}^*} \int_{\mathcal{O}^*} (a, uv^{-1})_3 \, \psi(a(u+v)) \, du \, dv = q^{-1}.$$

7.3. Kloosterman integrals

For $y \in F^*$ and $a, b \in F$, consider the integral

$$\mathcal{K}(y; a, b) = \int_{\mathcal{O}^*} (y, u)_3 \, \psi(au + bu^{-1}) \, du.$$

We provide a formula for $\mathcal{K}(y; a, b)$ in terms of Kloosterman sums over the residual field, the function

$$\delta_{3|\mathrm{val}(x)} = \begin{cases} 1 & \text{if } 3 \mid \mathrm{val}(x), \\ 0 & \text{otherwise}, \end{cases} \qquad x \in F^*,$$

and the function $\mathfrak{s}$ on F^* defined by

$$\mathfrak{s}(x) = \sum_{z \in \mathfrak{p}^{\ell - e}/\mathfrak{p}^{\ell}} \psi(xz^2),$$

where $\ell \in \mathbb{Z}$ and $e \in \{0, 1\}$ are such that $|x| = q^{2\ell - e}$. Note that $\ell = \lfloor \frac{1-\mathrm{val}(x)}{2} \rfloor$ and the summand $\psi(xz^2)$ is well defined (independent of the class of z mod $\mathfrak{p}^{\ell}$). In particular, $\mathfrak{s}(x) = 1$ if $\mathrm{val}(x)$ is even. It is easy to verify the basic property of $\mathfrak{s}$:

$$\mathfrak{s}(xy^2) = \mathfrak{s}(x), \quad x, y \in F^*. \tag{7.1}$$

In the evaluation below and subsequent formulas, if $x \in F^*$ is a square, then we write $\sqrt{x}$ for a square root of x in F; all formulas will be independent of the choice of square root.

Lemma 7.2. *We have*

$$\mathcal{K}(y;a,b)=\begin{cases}(1-q^{-1})\delta_{3|\mathrm{val}(y)}, & \max(|a|,|b|)\le 1,\\ q^{-1}\sum_{u\in k_F^*}(y,u)_3\,\psi(au+bu^{-1}), & \max(|a|,|b|)=q,\\ q^{-\lfloor\frac{1-\mathrm{val}(a)}{2}\rfloor}(y,ab^{-1})_3\sum_{\varepsilon=\pm1}\psi(2\varepsilon\sqrt{ab})\mathfrak{s}(\varepsilon\sqrt{ab}), & \max(|a|,|b|)>q,\\ & a^{-1}b\in\mathcal{O}^{*2},\\ 0, & \max(|a|,|b|)>q,\\ & a^{-1}b\notin\mathcal{O}^{*2}.\end{cases}$$

Proof. If $\max(|a|,|b|)\le q$, then the integrand $(y,u)_3\psi(au+bu^{-1})$ depends only on $u+\mathfrak{p}$ and therefore

$$\mathcal{K}(y;a,b)=q^{-1}\sum_{u\in k_F^*}(y,u)_3\psi(au+bu^{-1}).$$

If in addition $a,b\in\mathcal{O}$, then $\psi(au+bu^{-1})=1$ for all $u\in k_F^*$. This explains the formula when $\max(|a|,|b|)\le q$.

Assume now that $\max(|a|,|b|)\ge q^2$ and let $\ell\ge 1$ and $e\in\{0,1\}$ be such that $\max(|a|,|b|)=q^{2\ell-e}$. We decompose the integral by writing $u=v(1+z)$ with $v\in\mathcal{O}^*/(1+\mathfrak{p}^\ell)$ and $z\in\mathfrak{p}^\ell$. Since $(1+z)^{-1}=\sum_{i=0}^\infty(-z)^i\in 1-z+\mathfrak{p}^{2\ell}$ and $bv^{-1}\mathfrak{p}^{2\ell}\subseteq\mathcal{O}$, we conclude that

$$\mathcal{K}(y;a,b)=\sum_{v\in\mathcal{O}^*/(1+\mathfrak{p}^\ell)}(y,v)_3\psi(av+bv^{-1})\int_{\mathfrak{p}^\ell}\psi((av-bv^{-1})z)\,dz.$$

We have

$$\int_{\mathfrak{p}^\ell}\psi((av-bv^{-1})z)\,dz=\begin{cases}q^{-\ell} & \text{if } av-bv^{-1}\in\mathfrak{p}^{-\ell},\\ 0 & \text{otherwise.}\end{cases}$$

Note that $av-bv^{-1}\in\mathfrak{p}^{-\ell}$ if and only if $\varpi^{2\ell-e}(av^2-b)\in\mathfrak{p}^{\ell-e}$ and by assumption $\ell-e\ge 1$. By Hensel's lemma, there exists $v\in\mathcal{O}^*$ such that $av-bv^{-1}\in\mathfrak{p}^{-\ell}$ if and only if $a^{-1}b\in\mathcal{O}^{*2}$, and in this case v is of this form if and only if $v\in\pm v_0+\mathfrak{p}^{\ell-e}$ where $v_0^2=a^{-1}b$. Note that for $v=\pm v_0(1+z)$ with $z\in\mathfrak{p}^{\ell-e}$, we have

$$(y,v)_3=(y,\pm v_0)_3=(y,v_0^{-2})_3=(y,ab^{-1})_3,$$
$$av+bv^{-1}\in\pm(av_0+bv_0^{-1}+(av_0-bv_0^{-1})z+bv_0^{-1}z^2)+b\mathfrak{p}^{3(\ell-e)},$$

and by assumption $av_0-bv_0^{-1}=0$ and $b\mathfrak{p}^{3(\ell-e)}\subseteq\mathcal{O}$, so that

$$\psi(av+bv^{-1})=\psi(\pm(av_0+bv_0^{-1})+bv_0^{-1}z^2).$$

Note further that $av_0+bv_0^{-1}=2av_0=2bv_0^{-1}$ and that bv_0^{-1} is a square root of ab. The lemma readily follows. ■

We remark that the analysis above of the case $\max(|a|,|b|)\ge q^2$ is tantamount to an evaluation of the sum using the method of stationary phase.

7.4. Cubic exponential integrals

For $a, b \in F$, consider the integrals

$$\mathcal{C}(a,b) = \int_{\mathcal{O}} \psi(ax + bx^3)\,dx \quad \text{and} \quad \mathcal{C}_0(a,b) = \int_{\mathcal{O}^*} \psi(au + bu^3)\,du.$$

Lemma 7.3. *We have*

$$\mathcal{C}(a,b) = \begin{cases} 1, & \max(|a|,|b|) \le 1, \\ 0, & |b| < |a| = q, \\ q^{-1} \sum_{x \in k_F} \psi(ax + bx^3), & |a| \le |b| = q, \\ 0, & \max(|b|,q) < |a|, \\ q^{-\lfloor \frac{1-\mathrm{val}(a)}{2} \rfloor} \sum_{\varepsilon = \pm 1} \psi\Big(\varepsilon \frac{2}{9} a \sqrt{-3ab^{-1}}\Big) \mathfrak{s}\Big(\varepsilon \frac{1}{9} a \sqrt{-3ab^{-1}}\Big), & |a| = |b| > q, \\ & ab^{-1} \in \mathcal{O}^{*2}, \\ 0, & |a| = |b| > q, \\ & ab^{-1} \notin \mathcal{O}^{*2} \end{cases}$$

and

$$\mathcal{C}_0(a,b) = \begin{cases} 1 - q^{-1}, & \max(|a|,|b|) \le 1, \\ -q^{-1}, & |b| < |a| = q, \\ q^{-1} \sum_{u \in k_F^*} \psi(au + bu^3), & |a| \le |b| = q, \\ q^{-\lfloor \frac{1-\mathrm{val}(a)}{2} \rfloor} \sum_{\varepsilon = \pm 1} \psi\Big(\varepsilon \frac{2}{9} a \sqrt{-3ab^{-1}}\Big) \mathfrak{s}\Big(\varepsilon \frac{1}{9} a \sqrt{-3ab^{-1}}\Big), & \max(|a|,|b|) > q, \\ & ab^{-1} \in \mathcal{O}^{*2}, \\ 0 & \textit{otherwise.} \end{cases}$$

In particular,

$$\mathcal{C}(a,b) = \begin{cases} q^{-1} + \mathcal{C}_0(a,b), & \max(|a|,|b|) \le q, \\ \mathcal{C}_0(a,b), & \max(|b|,q^2) \le |a|. \end{cases}$$

Remark 7.4. The lemma excludes the computation of $\mathcal{C}(a,b)$ when $\max(|a|,q) < |b|$. In fact, in this case we have

$$\mathcal{C}(a,b) = \begin{cases} q^{-\ell} \sum_{x \in \mathfrak{p}^{\lfloor \frac{\ell+1-e}{2} \rfloor}/\mathfrak{p}^{\ell}} \psi(ax + bx^3) & \text{if } |a| \le q^{\ell}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\ell \ge 1$ and $e \in \{0,1\}$ are such that $|b| = q^{2\ell - e}$. However, we omit the proof for this case since it is never used in our work.

Proof. The two formulas are straightforward when $\max(|a|,|b|) \le q$. Assume now that $\max(|b|,q^2) \le |a|$, and let $\ell \ge 1$ and $e \in \{0,1\}$ be such that $|a| = q^{2\ell-e}$. Since $b\mathfrak{p}^{2\ell} \subseteq \mathcal{O}$, for $y \in \mathcal{O}$ we have

$$\int_{y+\mathfrak{p}^\ell} \psi(ax+bx^3)\,dx = \psi(ay+by^3)\int_{\mathfrak{p}^\ell} \psi((a+3by^2)z)\,dz$$
$$= \begin{cases} q^{-\ell}\psi(ay+by^3) & \text{if } a+3by^2 \in \mathfrak{p}^{-\ell}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\mathcal{C}(a,b) = q^{-\ell} \sum_{\substack{x\in\mathcal{O}/\mathfrak{p}^\ell \\ a+3bx^2\in\mathfrak{p}^{-\ell}}} \psi(ax+bx^3) \quad \text{and} \quad \mathcal{C}_0(a,b) = q^{-\ell} \sum_{\substack{u\in\mathcal{O}^*/(1+\mathfrak{p}^\ell) \\ a+3bu^2\in\mathfrak{p}^{-\ell}}} \psi(au+bu^3).$$

Note that for $y \in \mathcal{O}$, we have $a+3by^2 \in \mathfrak{p}^{-\ell}$ if and only if $\varpi^{2\ell-e}(a+3by^2) \in \mathfrak{p}^{\ell-e}$ and that by assumption $\mathfrak{p}^{\ell-e} \subseteq \mathfrak{p}$ while $\varpi^{2\ell-e}a \in \mathcal{O}^*$. Consequently, if $y \in \mathcal{O}$ is such that $a+3by^2 \in \mathfrak{p}^{-\ell}$, then $|a| = |b|$ and $y \in \mathcal{O}^*$. It follows that indeed in this case $\mathcal{C}(a,b) = \mathcal{C}_0(a,b)$. Note that $-3 = (1+2\rho)^2 \in \mathcal{O}^*$. It further follows from Hensel's lemma that there exists $y \in \mathcal{O}^*$ such that $a+3by^2 \in \mathfrak{p}^{-\ell}$ if and only if $ab^{-1} \in \mathcal{O}^{*2}$. The vanishing of $\mathcal{C}(a,b)$ when $\max(|b|,q) < |a|$ and when $|a| = |b| > q$ and $ab^{-1} \notin \mathcal{O}^{*2}$ follows.

Assume now that $|a| = |b| > q$ and $ab^{-1} \in \mathcal{O}^{*2}$, and let $v_0 \in \mathcal{O}^*$ be such that $v_0^2 = -3ab^{-1}$. Then for $y \in \mathcal{O}$, we have $a+3by^2 \in \mathfrak{p}^{-\ell}$ if and only if $y \in \pm\frac{1}{3}v_0 + \mathfrak{p}^{\ell-e}$. Since $b\mathfrak{p}^{3(\ell-e)} \subseteq \mathcal{O}$, for $z \in \mathfrak{p}^{\ell-e}$ and $x = \pm(\frac{1}{3}v_0 + z)$ we have

$$ax+bx^3 \in \pm\Big(\frac{2}{9}av_0 + bv_0z^2\Big) + \mathcal{O}.$$

Consequently,

$$\mathcal{C}(a,b) = q^{-\ell}\sum_{\varepsilon=\pm1} \psi\Big(\varepsilon\frac{2}{9}av_0\Big)\mathfrak{s}(\varepsilon bv_0).$$

Since by assumption $\frac{1}{9}ab^{-1} \in F^{*2}$, we conclude from (7.1) that $\mathfrak{s}(\varepsilon bv_0) = \mathfrak{s}(\varepsilon\frac{1}{9}av_0)$ and the lemma follows. ■

Once again, the proof above may be regarded as a use of the method of stationary phase.

For future reference, for $\ell \in \mathbb{Z}$ let

$$\mathcal{C}_\ell(a,b) = \int_{\mathrm{val}(x)=\ell} \psi(ax+bx^3)\,dx.$$

The change of variables $x \mapsto tx$ shows that

$$\mathcal{C}_\ell(a,b) = |t|\mathcal{C}_{\ell-\mathrm{val}(t)}(ta,t^3b), \quad t \in F^*. \tag{7.2}$$

As an immediate consequence of Lemma 7.3, we record here that

$$\mathcal{C}_\ell(a,b) = 0 \quad \text{whenever } \max(q^{-\ell}|a|, q^{-3\ell}|b|) > q \text{ and } |a^{-1}b| \ne q^{2\ell}. \tag{7.3}$$

7.5. The comparison of cubic exponential integrals and Kloosterman integrals

Duke and Iwaniec [7] established the following identity between cubic exponential and Kloosterman sums. Let χ be an order three character of k_F^* (χ and χ^{-1} are the only such characters) and ξ a non-trivial character of k_F. Then

$$\sum_{x \in k_F} \xi(x + ax^3) = \chi(a)^{-1} \sum_{u \in k_F^*} \chi(u)\xi(u - 3^{-3}a^{-1}u^{-1}). \tag{7.4}$$

This identity, together with the work above which may be regarded as an application of the method of stationary phase, gives the following key comparison of cubic exponential and Kloosterman sums.

Corollary 7.5. *We have*

$$\mathcal{C}(a, -3^{-3}c^{-1}d^{-1}a^3) = (t, c^{-1}d)_3\mathcal{K}(t; c, d)$$

whenever either

- $|a| = |c| = |d| = q$ *and* $3 \nmid \mathrm{val}(t)$, *or*
- $|a| = |c| = |d| > q$.

Proof. If $|a| = |c| = |d| = q$, then

$$\mathcal{C}(a, -3^{-3}c^{-1}d^{-1}a^3) = q^{-1} \sum_{x \in k_F} \psi(ax - 3^{-3}c^{-1}d^{-1}a^3x^3)$$

while

$$\mathcal{K}(t; c, d) = q^{-1} \sum_{u \in k_F^*} (t, u)_3\psi(cu + du^{-1}).$$

If in addition $3 \nmid \mathrm{val}(t)$, then the character $(t, \cdot)_3$ is not trivial on $\mathcal{O}^*$ and we apply (7.4) with $\xi(x + \mathfrak{p}) = \psi(ax)$, $x \in \mathcal{O}$, and $\chi(u + \mathfrak{p}) = (t, u)_3$, $u \in \mathcal{O}^*$, to obtain

$$\mathcal{C}(a, -3^{-3}c^{-1}d^{-1}a^3) = q^{-1}(t, -3^3cda^{-2})_3 \sum_{u \in k_F^*} (t, u)_3\xi(u + cda^{-2}u^{-1}).$$

Note that $(t, -3^3cda^{-2})_3(t, a^{-1}c)_3 = (t, c^{-1}d)_3$ and therefore the change of variables $u \mapsto a^{-1}cu$ proves the desired identity.

If $|a| = |c| = |d| > q$, the lemma follows by comparing the formulas in Lemmas 7.2 and 7.3. ∎

7.6. An additional computation involving cubic exponential integrals

We present one additional computation involving a cubic exponential integral. We will use this evaluation when an orbital integral involves Kloosterman integrals without a cubic character – that is, integrals $\mathcal{K}(t; c, d)$ with $|c| = |d| = q$ but with $3 \mid \mathrm{val}(t)$ – and so the result of Duke–Iwaniec is not applicable.

Lemma 7.6. *For $|a| = |b| \leq q^{-3}$, we have*

$$3 + |a|^{-1} \sum_{k=0}^{2} \mathcal{C}_{\mathrm{val}(a)-1}(b^{-1} + a^{-1}\rho^k, -3^{-3}a^{-1}b^{-1}) = \begin{cases} q, & -(ab^{-1})^3 \in 1 + \mathfrak{p}, \\ 0, & -(ab^{-1})^3 \notin 1 + \mathfrak{p}. \end{cases}$$

Proof. Note that by (7.2) with $t = \varpi^{-1}a$, we have

$$\begin{aligned} &|a|^{-1}\mathcal{C}_{\mathrm{val}(a)-1}(b^{-1} + a^{-1}\rho^k, -3^{-3}a^{-1}b^{-1}) \\ &\qquad = q\mathcal{C}_0(\varpi^{-1}(ab^{-1} + \rho^k), -3^{-3}\varpi^{-3}a^2b^{-1}) \end{aligned}$$

and by assumption

$$-3^{-3}\varpi^{-3}a^2b^{-1} \in \mathcal{O},$$

so that by Lemma 7.3 we have

$$q\mathcal{C}_0(\varpi^{-1}(ab^{-1} + \rho^k), -3^{-3}\varpi^{-3}a^2b^{-1}) = \begin{cases} q-1, & -ab^{-1} \in \rho^k + \mathfrak{p}, \\ -1, & -ab^{-1} \notin \rho^k + \mathfrak{p}. \end{cases}$$

Since $-ab^{-1} \in \rho^k + \mathfrak{p}$ for at most one $k \in \{0, 1, 2\}$ and this is the case if and only if $(-ab^{-1})^3 \in 1 + \mathfrak{p}$, the lemma follows. ∎

8. The integral $J(a, b)$

In this section, we establish the following formula for $J(a, b)$.

Proposition 8.1. *For $|b| \leq |a|$, we have*

$$(a,b)_3 J(a,b)$$
$$= \begin{cases} 1, & |a|=|b|=1, \\ 2q, & |a|=|b|=q^{-1}, \\ 3|a|^{-1}+|ab|^{-1} \displaystyle\sum_{\ell=\lfloor \frac{\mathrm{val}(a)+1}{2} \rfloor}^{\mathrm{val}(a)-1} \sum_{k=0}^{2} \mathcal{C}_\ell(b^{-1}+\rho^k a^{-1}, -3^{-3}a^{-1}b^{-1}), & |a|=|b|\leq q^{-2}, \\ |b|^{-1}\mathcal{C}(b^{-1}, -3^{-3}a^{-1}b^{-1}), & |b| < |a| = 1, \\ |ab|^{-1} \displaystyle\sum_{k=0}^{2} \mathcal{C}_{\frac{\mathrm{val}(a)}{2}}(b^{-1} + \rho^k a^{-1}, -3^{-3}a^{-1}b^{-1}), & |b| < |a| < 1,\ 2 \mid \mathrm{val}(a), \\ 0, & |b| < |a| < 1,\ 2 \nmid \mathrm{val}(a) \\ 0, & |a| > 1. \end{cases}$$

Furthermore,

$$J(b,a) = \overline{J(a,b)}, \quad a, b \in F^*. \tag{8.1}$$

8.1. *Fixing coordinates*

By definition,

$$J(a,b)=\int_{A[a,b]}\kappa(u_1g_{a,b}u_2)^{-1}\psi(u_1u_2)\,du_1\,du_2,$$

where

$$A[a,b]=\{(u_1,u_2)\in N\times N : u_1g_{a,b}u_2\in K\}.$$

We fix coordinates on $N\times N$ as follows. In our computation, we set

$$u_i=\begin{pmatrix}1 & x_i & z_i\\ & 1 & y_i\\ & & 1\end{pmatrix}\in N,\quad i=1,2,$$

so that

$$u_1g_{a,b}u_2=\begin{pmatrix}az_1 & az_1x_2-a^{-1}bx_1 & az_1z_2-a^{-1}bx_1y_2+b^{-1}\\ ay_1 & ay_1x_2-a^{-1}b & ay_1z_2-a^{-1}by_2\\ a & ax_2 & az_2\end{pmatrix}. \tag{8.2}$$

We also recall that an element $g=(g_{i,j})\in \mathrm{SL}_3(F)$ satisfies $g\in K$ if and only if $g_{i,j}\in\mathcal{O}$ for all $1\le i,j\le 3$.

8.2. *A functional equation*

We observe the following properties on the big Bruhat cell.

Lemma 8.2. *Let $u_1,u_2\in N$, $a,b\in F^*$ and $g=u_1g_{a,b}u_2$. Then*

- $\sigma(g,g^{-1})=1.$
- *Consequently, if in addition $g\in K$, then $\kappa(g^{-1})=\kappa(g)^{-1}=\overline{\kappa(g)}$.*

Proof. The second part is immediate from the first since

$$1=\kappa(I_3)=\kappa(gg^{-1})=\kappa(g)\kappa(g^{-1})\sigma(g,g^{-1}).$$

For the first part, note that $g_{a,b}^{-1}=g_{b,a}$ and

$$\sigma(g,g^{-1})=\sigma(g_{a,b},g_{b,a})=1$$

by the explicit formulas for σ on monomial matrices in [3, §2]. ∎

The following consequence of the lemma reduces the computation of J to the case $|b|\le|a|$.

Corollary 8.3. *We have*

$$J(b,a)=\overline{J(a,b)},\quad a,b\in F^*.$$

Proof. As $g_{a,b}^{-1}=g_{b,a}$, the map $(u_1,u_2)\mapsto(u_2^{-1},u_1^{-1})$ maps $A[a,b]$ bijectively to $A[b,a]$. Since $\psi(u_2^{-1}u_1^{-1})=\overline{\psi(u_1u_2)}$, the corollary is now immediate from Lemma 8.2. ∎

8.3. Computation of κ

We provide a formula for $\kappa(g)$ for almost all g in the intersection of K with the big Bruhat cell (outside a measure zero set). Throughout the computation, we freely use coordinates of g as in (8.2).

The fact that $g \mapsto (g, \kappa(g))$ is a splitting of K in $\widetilde{\mathrm{SL}}_3(F)$ is expressed by the equality

$$\kappa(g_1 g_2) = \kappa(g_1)\kappa(g_2)\sigma(g_1, g_2), \quad g_1,\, g_2 \in K.$$

Our computation uses properties of κ, which we recall below, and an algorithm of Bump and Hoffstein for computing $\sigma(g_1, g_2)$. The algorithm is based on an explicit Bruhat decomposition of g_1, and it will be more convenient to write such a decomposition with factors in $\mathrm{GL}_3(F)$ rather than in $\mathrm{SL}_3(F)$. For this reason, we consider $\widetilde{\mathrm{SL}}_3(F)$ as a subgroup of the cubic cover $\widetilde{\mathrm{GL}}_3(F)$ of $\mathrm{GL}_3(F)$ constructed in Bump and Hoffstein [3] (following Matsumoto). We then freely use the algorithm for the computation of the two-cocycle σ in [3, §2]. In particular, in terms of the Bruhat decomposition, for $h \in \mathrm{GL}_3(F)$ we denote by $R(h)$ the unique monomial (i.e., scaled permutation) matrix such that $h \in NR(h)N$. Furthermore, we continue to denote by κ an extension to a splitting of $\mathrm{GL}_3(\mathcal{O})$ in $\widetilde{\mathrm{GL}}_3(F)$ and freely use the following properties of κ (see [15, §1]):

- the splitting κ is trivial on signed permutation matrices and on upper-triangular matrices in $\mathrm{GL}_3(\mathcal{O})$,
- $\kappa(\mathrm{diag}(g, 1)) = \kappa(\mathrm{diag}(1, g)) = \kappa_2(g)$, $g \in \mathrm{GL}_2(\mathcal{O})$, where

$$\kappa_2(g) = \begin{cases} (c, d \det g^{-1})_3 & \text{if } 0 < |c| < 1, \\ 1 & \text{otherwise,} \end{cases}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}).$$

Lemma 8.4. *Let*

$$g = \begin{pmatrix} 0 & a & b \\ u & 0 & 0 \\ x & c & d \end{pmatrix} \in K$$

with $x, c \neq 0$. Then

$$\kappa(g) = (u, x)_3 \kappa_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} (u, x)_3 & \text{if } |c| = 1, \\ (c, d)_3 (u, xc^{-1})_3 & \text{otherwise.} \end{cases}$$

Proof. Note that $g = ksh$ with

$$k = \begin{pmatrix} 1 & & \\ & 1 & \\ & u^{-1}x & 1 \end{pmatrix}, \quad s = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \quad h = \begin{pmatrix} u & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \in K,$$

and therefore $\kappa(g) = \kappa(ks)\kappa(h)\sigma(ks, h)$. Clearly, $\kappa(h) = \kappa_2\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\kappa(ks) = \sigma(k, s)$. We compute that $\sigma(k, s) = 1$ and $\sigma(ks, h) = (u, x)_3$ using the algorithm in [3, §2]. ∎

Lemma 8.5. *Let*

$$g = \begin{pmatrix} a & 0 & b \\ c & 0 & d \\ x & u & 0 \end{pmatrix} \in K$$

with $x, c \neq 0$. *Then*

$$\kappa(g) = \kappa_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} (c, u^{-1}x)_3 = \begin{cases} (c, dx)_3, & |c| < 1, \\ (c, x)_3, & |c| = 1. \end{cases}$$

Proof. Note that $g = hsk$ with

$$h = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & u \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 1 & 0 & 0 \\ u^{-1}x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K,$$

and therefore $\kappa(g) = \kappa(h)\kappa(sk)\sigma(h, sk)$. Clearly, $\kappa(h) = \kappa_2\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\kappa(sk) = \sigma(s, k)$. We compute that $\sigma(s, k) = 1$ and $\sigma(h, sk) = (c, u^{-1}x)_3$ using the algorithm from [3, §2]. ∎

Lemma 8.6. *Let*

$$g = \begin{pmatrix} a & 0 & b \\ x & u & 0 \\ c & y & d \end{pmatrix} \in K$$

with $u \in \mathcal{O}^*$ *and* $x, c, xy - cu \neq 0$. *Then*

$$\kappa(g) = (x, c)_3 (c - u^{-1}xy, u^{-1}x)_3 \kappa_2 \begin{pmatrix} a & b \\ c - u^{-1}xy & d \end{pmatrix}$$

$$= \begin{cases} (x, c)_3 (c - u^{-1}xy, xd)_3, & |cu - xy| < 1, \\ (x, c)_3 (c - u^{-1}xy, x)_3, & |cu - xy| = 1. \end{cases}$$

Proof. Note first that $g = sk$, where

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} a & 0 & b \\ c & y & d \\ x & u & 0 \end{pmatrix} \in K,$$

so that $\kappa(g) = \kappa(s)\kappa(k)\sigma(s, k)$. Note that $\kappa(s) = 1$, and we compute that

$$\sigma(s, k) = (x, c)_3$$

using the algorithm in [3, §2]. In order to compute $\kappa(k)$, we note that $k = nhk'$, where

$$n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & u^{-1}y \\ 0 & 0 & 1 \end{pmatrix} \in N \cap K, \quad h = \begin{pmatrix} a & b & 0 \\ c - u^{-1}xy & d & 0 \\ 0 & 0 & u \end{pmatrix}, \quad k' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ u^{-1}x & 1 & 0 \end{pmatrix} \in K,$$

and therefore $\kappa(k) = \kappa(hk') = \kappa(h)\kappa(k')\sigma(h,k')$. Clearly, $\kappa(h) = \kappa_2\left(\begin{smallmatrix} a & b \\ c-u^{-1}xy & d\end{smallmatrix}\right)$ (note that $\det h = \det g = 1$). Furthermore, it is easy to see that $\kappa(sk') = 1$ and therefore also that $\kappa(k') = \sigma(s, sk') = 1$. We further compute that

$$\sigma(h,k') = (c - u^{-1}xy, u^{-1}x)_3$$

using the algorithm in [3, §2]. The lemma follows. ■

Lemma 8.7. *Let $g = u_1 g_{a,b} u_2 \in K$ with $a, b \in F^*$ and $u_1, u_2 \in N$.*

(1) *If $|a| = 1 = |b|$, then $\kappa(g) = 1$.*

(2) *If $|a| = 1 > |b|$, then $\kappa(g) = (b,a)_3(a^{-1}b, y_2)_3$.*

(3) *If $|a|, |b| < 1$, then at most one of ay_1 and ax_2 is in $\mathcal{O}^*$ and*

  (a) *if $|ay_1| = 1$, then $\kappa(g) = (b,a)_3(y_1y_2, ab^{-1})_3(y_2, y_1)_3$,*

  (b) *if $|ax_2| = 1$, then $\kappa(g) = (b, ax_2)_3(z_2x_2^{-1} - y_2, b^{-1}ax_2)_3$,*

  (c) *if $|ay_1|, |ax_2| < 1$, then $ay_1x_2 - a^{-1}b \in \mathcal{O}^*$, and if in addition $y_1 \neq 0$, then*

$$\kappa(g) = (b,a)_3(y_1, ab^{-1})_3(ay_1(x_2y_2 - z_2), ay_1x_2 - a^{-1}b)_3(b, x_2y_2 - z_2)_3.$$

Proof. If $|a| = 1 = |b|$, then $u_1, u_2 \in K$, and therefore $\kappa(g) = \kappa(g_{a,b}) = 1$. If $|a| = 1 > |b|$, then $x_2, y_1, z_1, z_2 \in \mathcal{O}$, and hence $\kappa(g) = \kappa\left(\begin{smallmatrix} & g_0 \\ a & \end{smallmatrix}\right)$, where $g_0 = \left(\begin{smallmatrix} -a^{-1}bx_1 & b^{-1}-x_1y_2a^{-1}b \\ -a^{-1}b & -a^{-1}by_2\end{smallmatrix}\right)$. Note that $\left(\begin{smallmatrix} & g_0 \\ a & \end{smallmatrix}\right) = \left(\begin{smallmatrix} g_0 & \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} & I_2 \\ a & \end{smallmatrix}\right)$ and $\sigma(\left(\begin{smallmatrix} g_0 & \\ & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} & I_2 \\ a & \end{smallmatrix}\right)) = 1$. Therefore,

$$\kappa\begin{pmatrix} & g_0 \\ a & \end{pmatrix} = \kappa_2(g_0) = (-a^{-1}b, -by_2)_3 = (b,a)_3(a^{-1}b, y_2)_3.$$

For the rest of the proof, assume that $|a|, |b| < 1$. Since the (2,2)-entry of g is $ay_1x_2 - a^{-1}b \in \mathcal{O}$, we cannot have both ay_1 and ax_2 in $\mathcal{O}^*$. If $|ay_1| = 1$, then

$$k = \begin{pmatrix} 1 & \frac{a^{-1}b - y_1ax_2}{y_1a} & \frac{a^{-1}by_2 - y_1z_2a}{y_1a} \\ & 1 & \\ & & 1\end{pmatrix}, \quad k' = \begin{pmatrix} 1 & -\frac{z_1a}{y_1a} & \\ & 1 & \\ & & 1\end{pmatrix} \in N \cap K,$$

so that $\kappa(g) = \kappa(k'gk)$. Note that

$$k'gk = \begin{pmatrix} 0 & \frac{(z_1 - x_1y_1)b}{y_1a} & \frac{(z_1-x_1y_1)y_2b}{y_1a} + b^{-1} \\ y_1a & 0 & 0 \\ a & \frac{b}{y_1a} & \frac{y_2b}{y_1a}\end{pmatrix},$$

and from Lemma 8.4 it follows that

$$\kappa(g) = (b(ay_1)^{-1}, b(ay_1)^{-1}y_2)_3(ay_1, ab^{-1}ay_1)_3 = (b,a)_3(y_1y_2, ab^{-1})_3(y_2, y_1)_3.$$

If $|ax_2| = 1$, then

$$k = \begin{pmatrix} 1 & & \\ & 1 & \frac{-z_2a}{x_2a} \\ & & 1\end{pmatrix}, \quad k' = \begin{pmatrix} 1 & & \frac{x_1a^{-1}b - z_1x_2a}{x_2a} \\ & 1 & \frac{a^{-1}b - y_1x_2a}{x_2a} \\ & & 1\end{pmatrix} \in N \cap K,$$

so that $\kappa(g) = \kappa(k'gk)$. Note that

$$k'gk = \begin{pmatrix} x_1x_2^{-1}a^{-1}b & 0 & (z_2x_2^{-1} - y_2)x_1a^{-1}b + b^{-1} \\ x_2^{-1}a^{-1}b & 0 & (z_2x_2^{-1} - y_2)a^{-1}b \\ a & x_2a & 0 \end{pmatrix},$$

and from Lemma 8.5 it follows that

$$\kappa(g) = (x_2^{-1}a^{-1}b, (z_2x_2^{-1} - y_2)b)_3 = (b, ax_2)_3(z_2x_2^{-1} - y_2, b^{-1}ax_2)_3.$$

Finally, assume that $|ax_2|, |ay_1| < 1$. Any 2×2 minor of an element of K must have an entry in $\mathcal{O}^*$. Since the (1,3)-minor of g is $\left(\begin{smallmatrix} ay_1 & ay_1x_2 - a^{-1}b \\ a & ax_2 \end{smallmatrix}\right)$, the condition $g \in K$ implies that $ay_1x_2 - a^{-1}b \in \mathcal{O}^*$. Therefore,

$$k = \begin{pmatrix} 1 & & \\ & 1 & \frac{a^{-1}by_2 - y_1z_2a}{y_1ax_2 - a^{-1}b} \\ & & 1 \end{pmatrix}, \quad k' = \begin{pmatrix} 1 & \frac{x_1a^{-1}b - z_1x_2a}{y_1ax_2 - a^{-1}b} & \\ & 1 & \\ & & 1 \end{pmatrix} \in N \cap K$$

and $\kappa(g) = \kappa(k'gk)$. Note that $k'gk$ has the form

$$k'gk = \begin{pmatrix} * & 0 & * \\ y_1a & y_1ax_2 - a^{-1}b & 0 \\ a & x_2a & \frac{(x_2y_2 - z_2)b}{y_1ax_2 - a^{-1}b} \end{pmatrix}.$$

Furthermore, under the assumption that $y_1 \neq 0$, the rest of the computation follows from Lemma 8.6. ■

8.4. *Evaluation of $J(a, b)$: Some simple cases*

8.4.1. *The case* $\max(|a|, |b|) > 1$.

Lemma 8.8. *If* $\max(|a|, |b|) > 1$, *then*

$$J(a, b) = 0.$$

Proof. For $g = (g_{i,j}) \in \mathrm{SL}_3(F)$, let $\Delta_1(g) = g_{3,1}$ and $\Delta_2(g) = \det\left(\begin{smallmatrix} g_{2,1} & g_{2,2} \\ g_{3,1} & g_{3,2} \end{smallmatrix}\right)$. Note that $\Delta_1(g) = a$ and $\Delta_2(g) = b$ for every $g \in Ng_{a,b}N$. Consequently, $A[a, b]$ is empty and therefore $J(a, b) = 0$ unless $a, b \in \mathcal{O}$. ■

8.4.2. *The case* $|a| = |b| = 1$.

Lemma 8.9. *If $a, b \in \mathcal{O}^*$, then*

$$(a, b)_3\, J(a, b) = 1.$$

Proof. If $a, b \in \mathcal{O}^*$, then $(a, b)_3 = 1$, and it easily follows from (8.2) that $A[a, b] = (N \cap K) \times (N \cap K)$. Since κ is bi-$(N \cap K)$-invariant, $\kappa(g_{a,b}) = 1$ and ψ is trivial on $N \cap K$, the lemma follows. ■

8.4.3. The case $1 = |a| > |b|$.

Lemma 8.10. *If* $1 = |a| > |b|$, *then*

$$(a,b)_3\, J(a,b) = |b|^{-1}\mathcal{C}(b^{-1}, -3^{-3}a^{-1}b^{-1}).$$

Proof. It follows from (8.2) and Lemma 8.7 (2) that

$$J(a,b) = (a,b)_3 \int (ab^{-1}, y_2)_3 \psi(x_1 + y_2)\, dx_1\, dy_2,$$

where the integral is over x_1 and y_2 such that $x_1 b, y_2 b, b^{-1} - x_1 y_2 a^{-1} b \in \mathcal{O}$ or equivalently such that

$$y_2 b \in \mathcal{O}^*, \quad x_1 \in (y_2 b)^{-1} ab^{-1} + \mathcal{O}.$$

Integrating over x_1, we obtain

$$J(a,b) = (a,b)_3 \int_{|y_2|=|b|^{-1}} (ab^{-1}, y_2)_3 \psi(y_2 + ab^{-2} y_2^{-1})\, dy_2,$$

and after the change of variables $y_2 \mapsto b^{-1} y_2$ this becomes

$$|b|^{-1} \int_{\mathcal{O}^*} (ab^{-1}, y_2)_3 \psi(b^{-1} y_2 + ab^{-1} y_2^{-1})\, dy_2 = |b|^{-1} \mathcal{K}(ab^{-1}; b^{-1}, ab^{-1}).$$

The lemma now follows from Corollary 7.5. ∎

8.5. Calculation of $J(a,b)$ *when* $1 > |a| \geq |b|$*: Subdivision and evaluations of* J_1, J_2

Assume now that $1 > |a| \geq |b|$ and consider the conditions

(1) $|ay_1| = 1$;

(2) $|ax_2| = 1$;

(3) $|ay_1|, |ax_2| < 1$.

Let

$$J_i(a,b) = \int_{A_i[a,b]} \kappa(u_1 g_{a,b} u_2)^{-1} \psi(u_1 u_2)\, du_1\, du_2,$$

where $A_i[a,b]$ is the intersection of $A[a,b]$ with the set defined by the condition (i) for $i = 1, 2, 3$. By Lemma 8.7, we have

$$J(a,b) = J_1(a,b) + J_2(a,b) + J_3(a,b). \tag{8.3}$$

8.5.1. Computation of J_1.

Lemma 8.11. *Let* $1 > |a| \geq |b|$, *then*

$$(a,b)_3\, J_1(a,b) = \begin{cases} q & \text{if } |a| = |b| = q^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that according to Lemma 8.7, we have

$$J_1(a,b) = (a,b)_3 \int (y_1y_2, a^{-1}b)_3 (y_1, y_2)_3 \times \psi(x_1 + x_2 + y_1 + y_2)\, dx_1\, dx_2\, dy_1\, dy_2\, dz_1\, dz_2,$$

where the integral is over

$$ay_1 \in \mathcal{O}^*, \quad az_1, az_2,\ ax_2,\ y_1x_2a - a^{-1}b,\ y_1z_2a - y_2a^{-1}b,\ z_1x_2a - x_1a^{-1}b,\ z_1z_2a - x_1y_2a^{-1}b + b^{-1} \in \mathcal{O}$$

or equivalently

$$ay_1 \in \mathcal{O}^*, \quad az_1,\ az_2,\ x_2,\ y_1z_2a - y_2a^{-1}b,\ x_1a^{-1}b,\ z_1z_2a - x_1y_2a^{-1}b + b^{-1} \in \mathcal{O}. \tag{8.4}$$

We first show that $J_1(a,b) = 0$ unless $|a| = |b| = q^{-1}$. Indeed, for $u, v \in \mathcal{O}^*$ the change of variables

$$(x_1, y_1, z_1, y_2, z_2) \mapsto (ux_1, vy_1, uvz_1, u^{-1}y_2, u^{-1}v^{-1}z_2)$$

preserves the domain of integration. Averaging over u and v, we see that the integral $J_1(a,b)$ factors through

$$\int_{\mathcal{O}^*} (v, a^{-1}by_2)_3 \psi(vy_1)\, dv \int_{\mathcal{O}^*} (u, ab^{-1}y_1)_3 \psi(ux_1 + u^{-1}y_2)\, du.$$

Note that in the domain of integration $|y_1| = |a|^{-1} > 1$ and for such y_1, it follows from Lemma 7.2 that $\int_{\mathcal{O}^*} (v, a^{-1}by_2)_3 \psi(vy_1)\, dv = 0$ unless $|a| = q^{-1}$. Next, assume that $|b| < |a| = q^{-1}$. Note that in domain (8.4), we have

$$|y_2| = |b|^{-1} > q \quad \text{and} \quad |x_1| = |ab^{-1}| < |y_2|. \tag{8.5}$$

Indeed, in the domain of integration

$$|ay_1z_2|,\ |az_1z_2| \le |a|^{-1} < |b|^{-1}$$

and consequently,

$$|a^{-1}by_2| \le |a|^{-1} \quad \text{and} \quad |a^{-1}bx_1y_2| = |b|^{-1}.$$

Combined with the condition $a^{-1}bx_1 \in \mathcal{O}$, we obtain (8.5). Consequently, it follows from Lemma 7.2 that $\int_{\mathcal{O}^*} (u, ab^{-1}y_1)_3 \psi(ux_1 + u^{-1}y_2)\, du = 0$.

Assume now that $|a| = |b| = q^{-1}$. The domain of integration (8.4) is equivalently characterized by

$$|y_1| = |y_2| = q, \quad x_2,\ az_1 \in \mathcal{O}, \quad z_2 \in a^{-2}by_1^{-1}y_2 + \mathcal{O},$$
$$x_1 \in ab^{-1}y_2^{-1}(az_1z_2 + b^{-1}) + \mathfrak{p},$$

and in particular, the integrand is independent of x_1, x_2, z_1 and z_2 in this domain. After integrating over x_1 and x_2, we further integrate over z_1 and z_2. After the change of variables

$$(y_1, y_2) \mapsto (a^{-1}y_1, a^{-1}y_2),$$

we have

$$J_1(a,b) = (b,a)_3 q^2 \int_{\mathcal{O}^*} \int_{\mathcal{O}^*} (a^{-1}, y_2 y_1^{-1})_3 \psi(a^{-1}(y_1 + y_2))\, dy_1\, dy_2.$$

The lemma now follows from Lemma 7.1. ∎

8.5.2. Computation of J_2.

Lemma 8.12. *Let $1 > |a| \geq |b|$, then*

$$(a,b)_3\, J_2(a,b) = \begin{cases} q & \text{if } |a| = |b| = q^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that according to Lemma 8.7, we have

$$J_2(a,b) = (a,b)_3 \int (x_2, b)_3 (b^{-1}ax_2, z_2x_2^{-1} - y_2)_3 \times \psi(x_1 + x_2 + y_1 + y_2)\, dx_1\, dx_2\, dy_1\, dy_2\, dz_1\, dz_2,$$

where the integral is over

$$ax_2 \in \mathcal{O}^*, \quad az_1, az_2, ay_1, y_1x_2a - a^{-1}b, y_1z_2a - y_2a^{-1}b, z_1x_2a - x_1a^{-1}b, z_1z_2a - x_1y_2a^{-1}b + b^{-1} \in \mathcal{O}.$$

Note further that

$$z_1z_2a - x_1y_2a^{-1}b + b^{-1} = x_2^{-1}z_2(z_1x_2a - x_1a^{-1}b) + a^{-1}bx_1(x_2^{-1}z_2 - y_2) + b^{-1},$$

and the domain of integration is equivalently characterized by

$$ax_2 \in \mathcal{O}^*, \quad az_1, az_2, y_1, y_2a^{-1}b, z_1x_2a - x_1a^{-1}b, a^{-1}bx_1(x_2^{-1}z_2 - y_2) + b^{-1} \in \mathcal{O}.$$

Since the change of variables

$$(x_1, x_2, y_2, z_1, z_2) \mapsto (u^{-1}x_1, vx_2, uy_2, u^{-1}v^{-1}z_1, uvz_2)$$

preserves the domain of integration for every $u, v \in \mathcal{O}^*$, an argument analogous to the proof of Lemma 8.11 shows that $J_2(a,b) = 0$ unless $|a| = |b| = q^{-1}$.

Assume now that $|a| = |b| = q^{-1}$. After the change of variables $y_2 \mapsto y_2 + x_2^{-1}z_2$, we have

$$J_2(a,b) = (a,b)_3 \int (x_2, b)_3 (b^{-1}ax_2, y_2)_3 \psi(x_1 + x_2)\, dx_1\, dx_2\, dy_1\, dy_2\, dz_1\, dz_2,$$

where the integral is over

$$|x_2| = q, \quad az_1,\, az_2,\, y_1,\, y_2,\, z_1x_2a - x_1a^{-1}b,\, a^{-1}bx_1y_2 - b^{-1} \in \mathcal{O}$$

or equivalently

$$|x_1| = |x_2| = q, \quad az_2,\, y_1 \in \mathcal{O}, \quad z_1 \in a^{-2}bx_2^{-1}x_1 + \mathcal{O}, \quad y_2 \in ab^{-2}x_1^{-1} + \mathfrak{p}.$$

Note that in this domain $ax_2, y_2 \in \mathcal{O}^*$, and therefore

$$(x_2, b)_3(b^{-1}ax_2, y_2)_3 = (x_2y_2, b)_3 = (ax_1^{-1}x_2, b)_3.$$

That is, the integrand is independent of y_1, z_1, y_2, z_2 in the domain of integration. Integrating over these variables and applying the change of variables

$$(x_1, x_2) \mapsto (a^{-1}x_1, a^{-1}x_2),$$

we obtain that

$$J_2(a,b) = (b,a)_3q^2 \int_{\mathcal{O}^*}\int_{\mathcal{O}^*} (b, x_1x_2^{-1})_3\psi(a^{-1}(x_1 + x_2))\, dx_1\, dx_2.$$

The lemma now follows from Lemma 7.1. ■

8.6. The calculation of $J(a,b)$ when $1 > |a| \geq |b|$: Evaluation of J_3

We turn to the evaluation of the third summand that contributes to $J(a,b)$. Recall that by definition,

$$J_3(a,b) = \int_{A_3[a,b]} \kappa(u_1g_{a,b}u_2)^{-1}\psi(u_1u_2)\, du_1\, du_2,$$

where $A_3[a,b]$ is defined by the following conditions (coordinates are as in (8.2)):

(a) $ay_1 \in \mathfrak{p}$;

(b) $ax_2 \in \mathfrak{p}$;

(c) $az_1 \in \mathcal{O}$;

(d) $az_2 \in \mathcal{O}$;

(e) $y_1x_2a - a^{-1}b \in \mathcal{O}$;

(f) $y_1z_2a - y_2a^{-1}b \in \mathcal{O}$;

(g) $z_1x_2a - x_1a^{-1}b \in \mathcal{O}$;

(h) $z_1z_2a - x_1y_2a^{-1}b + b^{-1} \in \mathcal{O}$.

In the proof of the following simple lemma, we refer to this list of defining conditions.

Lemma 8.13. *Let $a, b \in F^*$ be such that $|a| < 1$.*

(1) *Every element of $A_3[a,b]$ satisfies $bx_1 \in \mathfrak{p}$.*

(2) *Every element of $A_3[a,b]$ satisfies $ay_1x_2 - a^{-1}b, az_1, az_2 \in \mathcal{O}^*$.*

Proof. It follows from (b) and (c) that $|az_1x_2| < |z_1| \le |a|^{-1}$. Part (1) of the lemma therefore follows from (g). For an element of $A_3[a,b]$, matrix (8.2) reduces mod $\mathfrak{p}$ to upper triangular form. Part (2) follows. ■

Note that the subset of $A_3[a,b]$ with $y_1 = 0$ is of measure zero (in fact, it is empty if $|b| < |a|$). It therefore follows from Lemma 8.7 that

$$J_3(a,b) = (a,b)_3 \int_{A_3'[a,b]} (y_1, a^{-1}b)_3 (ay_1x_2 - a^{-1}b, ay_1(x_2y_2 - z_2))_3 (x_2y_2 - z_2, b)_3 \\ \times \psi(x_1 + x_2 + y_1 + y_2)\, dx_1\, dx_2\, dy_1\, dy_2\, dz_1\, dz_2, \qquad (8.6)$$

where $A_3'[a,b]$ is the subset of elements of $A_3[a,b]$ satisfying $y_1 \ne 0$.

8.6.1. Integration over z_1. We begin the computation of J_3 by integrating over the single variable z_1. We use the coordinates $(x_1, x_2, y_1, y_2, z_2)$ in F^5. For a set D in F^5, let D' be the set of elements in D such that $y_1 \ne 0$. Let $D[a,b]$ be the domain defined by the following conditions:

(a) $ay_1 \in \mathfrak{p}$;

(b) $ax_2 \in \mathfrak{p}$;

(c) $az_2 \in \mathcal{O}^*$;

(d) $y_1 - a^{-1}by_2 \in \mathcal{O}$;

(e) $x_2 - a^{-1}bx_1 \in \mathcal{O}$;

(f) $|1 - bx_1y_2| = |ab^{-1}|$.

In the rest of the section, we refer to this list of conditions.

Lemma 8.14. *Let $|b| \le |a| < 1$. We have*

$$J_3(a,b) = (a,b)_3 |a^{-1}b| \int_{D'[a,b]} (y_1, ab^{-1}(ay_1x_2 - a^{-1}b))_3 \\ \times (ax_2y_2 - 1, ay_1x_2 - a^{-1}b)_3 (b, az_2(ax_2y_2 - 1))_3 \\ \times \psi(-(bz_2)^{-1}(1 - bx_1y_2)^{-1}x_1 + y_1 + x_2 + az_2y_2) \\ \times dx_1\, dx_2\, dy_1\, dy_2\, dz_2.$$

Proof. Applying the change of variables

$$x_1 \mapsto az_1x_1, \quad y_2 \mapsto az_2y_2$$

to (8.6) and using Lemma 8.13, we have

$$J_3(a,b) = (a,b)_3 \int (y_1, a^{-1}b)_3 (ay_1x_2 - a^{-1}b, y_1(ax_2y_2 - 1))_3 (z_2(ax_2y_2 - 1), b)_3 \\ \times \psi(az_1x_1 + y_1 + x_2 + az_2y_2)\, dx_1\, dx_2\, dy_1\, dy_2\, dz_1\, dz_2,$$

where integration is now over the domain defined by $y_1 \ne 0$ and

$$ay_1,\ ax_2 \in \mathfrak{p}, \quad az_1,\ az_2 \in \mathcal{O}^*, \\ ay_1x_2,\ y_1 - a^{-1}by_2,\ x_2 - a^{-1}bx_1, az_1z_2(1 - bx_1y_2) + b^{-1} \in \mathcal{O}.$$

Note that the conditions

$$az_1,\ az_2 \in \mathcal{O}^*, \quad az_1z_2(1-bx_1y_2)+b^{-1} \in \mathcal{O}$$

are equivalent to the conditions

$$az_2 \in \mathcal{O}^*, \quad |1-bx_1y_2| = |ab^{-1}|, \quad z_1 \in -(az_2)^{-1}(1-bx_1y_2)^{-1}b^{-1} + a^{-1}b\mathcal{O}.$$

Furthermore, it follows from Lemma 8.13 that $bx_1 \in \mathfrak{p} \subseteq \mathcal{O}$, and therefore in the domain of integration

$$\psi(az_1x_1) = \psi(-(bz_2)^{-1}(1-bx_1y_2)^{-1}x_1),$$

so that the integrand is independent of z_1 in its $a^{-1}b\mathcal{O}$ coset. Integrating over z_1, we obtain

$$\begin{aligned} J_3(a,b) = (a,b)_3|a^{-1}b| \int & (y_1, a^{-1}b)_3(ay_1x_2 - a^{-1}b, y_1(ax_2y_2-1))_3 \\ & \times (z_2(ax_2y_2-1), b)_3 \\ & \times \psi(-(bz_2)^{-1}(1-bx_1y_2)^{-1}x_1 + y_1 + x_2 + az_2y_2) \\ & \times dx_1\, dx_2\, dy_1\, dy_2\, dz_2, \end{aligned}$$

where the integral is over the subdomain of elements of $D'[a,b]$ that satisfy $ay_1x_2 \in \mathcal{O}$. However, this condition is satisfied by any element of $D[a,b]$. Indeed, if for an element of $D'[a,b]$ either x_2 or y_1 is in $\mathcal{O}$, this follows from conditions (a) and (b), otherwise it follows from (d) and (e) that $|x_2| = |a^{-1}bx_1|$ and $|y_1| = |a^{-1}by_2|$ and therefore $|ay_1x_2| = |a^{-1}b^2x_1y_2|$. Equality (f) implies that $|bx_1y_2| \le |ab^{-1}|$. The statement and the lemma follow. ∎

8.6.2. On the domain $D[a,b]$. Let $D_0[a,b]$ be the subset of elements in $D[a,b]$ that satisfy either $a^{-1}bx_1 \in \mathcal{O}$ or $a^{-1}by_2 \in \mathcal{O}$, or equivalently (by (d) and (e)) either $x_2 \in \mathcal{O}$ or $y_1 \in \mathcal{O}$, and let $D_{>0}[a,b] = D[a,b] \setminus D_0[a,b]$ be its complement. Further, let $D_{0x}[a,b]$ (resp. $D_{0y}[a,b]$) be the subset of elements in $D[a,b]$ that satisfy $x_2 \in \mathcal{O}$ (resp. $y_1 \in \mathcal{O}$).

Lemma 8.15. *Let* $|b| \le |a| < 1$.

(1) *The set* $D[a,b]$ *is preserved by the interchange of variables*

$$(x_1, y_1) \leftrightarrow (y_2, x_2). \tag{8.7}$$

(2) *Every element of* $D[a,b]$ *satisfies* $by_2, bx_1 \in \mathfrak{p}$.

(3) *The set* $D_0[a,b]$ *is empty unless* $|a| = |b|$.

(4) *If* $|a| = |b|$, *then the set* $D_{0x}[a,b]$ *is defined by the conditions*

$$x_1,\ x_2 \in \mathcal{O}, \quad ay_1 \in \mathfrak{p}, \quad y_2 \in ab^{-1}y_1 + \mathcal{O}, \quad az_2 \in \mathcal{O}^*. \tag{8.8}$$

(5) *If* $|a| = |b|$, *then the set* $D_{0y}[a,b]$ *is defined by the conditions*

$$y_1,\ y_2 \in \mathcal{O}, \quad ax_2 \in \mathfrak{p}, \quad x_1 \in ab^{-1}x_2 + \mathcal{O}, \quad az_2 \in \mathcal{O}^*.$$

(6) *If* $|a| = |b|$, *then every element of* $D'_0[a,b]$ *satisfies*

$$(y_1, a^{-1}b)_3(ay_1x_2 - a^{-1}b, y_1(ax_2y_2 - 1))_3(z_2(ax_2y_2 - 1), b)_3 = (z_2, b)_3.$$

(7) *For every element of* $D'_{>0}[a,b]$, *we have*

$$\begin{aligned}(y_1, a^{-1}b)_3(ay_1x_2 - a^{-1}b, y_1(ax_2y_2 - 1))_3(z_2(ax_2y_2 - 1), b)_3 \\ = (x_1, bx_1y_2 - 1)_3(z_2, b)_3.\end{aligned}$$

Proof. Part (1) is straightforward from the defining properties (a)–(f) of $D[a,b]$. It follows from (a) and (d) that an element of $D[a,b]$ satisfies $a^{-1}by_2 \in a^{-1}\mathfrak{p}$ or equivalently $by_2 \in \mathfrak{p}$. Part (2) now follows further applying symmetry (8.7).

By part (2), an element of $D_{0x}[a,b]$ satisfies $bx_1y_2 \in ay_2\mathcal{O} \subseteq ab^{-1}\mathfrak{p}$. If $|b| < |a|$, this contradicts (f). By symmetry (8.7), if $|b| < |a|$, then $D_{0y}[a,b]$ is also empty. Part (3) follows.

Suppose that $|a| = |b|$. It is easy to see that an element of $D_{0x}[a,b]$ satisfies conditions (8.8). For the reverse inclusion, note that if an element of F^5 satisfies (8.8), then $by_2 \in ay_1 + b\mathcal{O} \subseteq \mathfrak{p}$, and therefore also $bx_1y_2 \in \mathfrak{p}$. Consequently, $|1 - bx_1y_2| = 1 = |ab^{-1}|$. Part (4) now easily follows. Part (5) follows from part (4) and symmetry (8.7). Every element of $D_0[a,b]$ satisfies $ay_1x_2, ax_2y_2 \in \mathfrak{p}$ while, by assumption, $a^{-1}b \in \mathcal{O}^*$, and therefore

$$(t, ay_1x_2 - a^{-1}b)_3 = (t, a^{-1}b)_3 \quad \text{and} \quad (t, ax_2y_2 - 1)_3 = 1, \quad t \in F^*.$$

Part (6) now follows by properties of the Hilbert symbol.

For part (7), we return to the more general setting $|b| \le |a| < 1$. For an element of $D_{>0}[a,b]$, conditions (d) and (e) imply that

$$|x_2| = |a^{-1}bx_1| > 1 \quad \text{and} \quad |y_1| = |a^{-1}by_2| > 1.$$

In particular, from conditions (d) and (e) it follows that

$$(ay_1, t)_3 = (by_2, t)_3, \quad t \in F^*$$

and it easily follows that

$$ay_1x_2 - a^{-1}b \in a^{-1}b(bx_1y_2 - 1) + \mathfrak{p} \quad \text{and} \quad ax_2y_2 - 1 \in bx_1y_2 - 1 + ab^{-1}\mathfrak{p}.$$

Combined with (f), it follows that

$$(t, ay_1x_2 - a^{-1}b)_3 = (t, a^{-1}b(bx_1y_2 - 1))_3 \quad \text{and} \quad (t, ax_2y_2 - 1)_3 = (t, bx_1y_2 - 1)_3$$

for all $t \in F^*$. Part (7) now follows from the properties of Hilbert symbols. ■

Based on Lemma 8.15, if $|b| \le |a| < 1$, we have

$$J_3(a,b) = \mathcal{J}_0(a,b) + \mathcal{J}_{>0}(a,b), \tag{8.9}$$

where

$$\mathcal{J}_{>0}(a,b) = |a^{-1}b| \int_{D'_{>0}[a,b]} (x_1, bx_1y_2 - 1)_3 (az_2, b)_3 \times \psi(-(bz_2)^{-1}(1 - bx_1y_2)^{-1}x_1 + y_1 + x_2 + az_2y_2) \times dx_1\, dx_2\, dy_1\, dy_2\, dz_2 \tag{8.10}$$

and $\mathcal{J}_0(a,b) = 0$ unless $|a| = |b|$ in which case

$$\mathcal{J}_0(a,b) = \int_{D'_0[a,b]} (az_2, b)_3 \psi(-(bz_2)^{-1}(1 - bx_1y_2)^{-1}x_1 + y_1 + x_2 + az_2y_2) \times dx_1\, dx_2\, dy_1\, dy_2\, dz_2.$$

8.6.3. The computation of $\mathcal{J}_0$.

Lemma 8.16. *Let $|a| = |b| < 1$. We have*

$$(a,b)_3 \mathcal{J}_0(a,b) = \begin{cases} 2|a|^{-1}, & 3 \nmid \operatorname{val}(a) > 1, \\ (1 + q^{-1})|a|^{-1}, & 3 \mid \operatorname{val}(a), \\ 0, & \operatorname{val}(a) = 1. \end{cases}$$

Proof. Suppose that $|a| = |b| < 1$. Write J_1 (resp. J_2, resp. J_3) for the integral of

$$(az_2, b)_3 \psi(-(bz_2)^{-1}(1 - bx_1y_2)^{-1}x_1 + y_1 + x_2 + az_2y_2)$$

over the domain $D'_{0x}[a,b]$ (resp. $D'_{0y}[a,b]$, resp. $D'_{0x}[a,b] \cap D'_{0y}[a,b]$) so that

$$\mathcal{J}_0(a,b) = \mathrm{J}_1 + \mathrm{J}_2 - \mathrm{J}_3.$$

By Lemma 8.15 (4), an element of $D_{0x}[a,b]$ satisfies $-(bz_2)^{-1}(1 - bx_1y_2)^{-1}x_1, x_2 \in \mathcal{O}$ and $az_2y_2 \in a^2b^{-1}z_2y_1 + \mathcal{O}$. Therefore,

$$\mathrm{J}_1 = \int_{a^{-1}\mathcal{O}^*} \int_{a^{-1}\mathfrak{p}} \int_{ab^{-1}y_1 + \mathcal{O}} \int_{\mathcal{O}} \int_{\mathcal{O}} (az_2, b)_3 \psi((a^2b^{-1}z_2 + 1)y_1)\, dx_1\, dx_2\, dy_2\, dy_1\, dz_2.$$

Integrating over x_1, x_2 and y_2 and applying the change of variables $z_2 \mapsto a^{-2}bz_2$, we have

$$(a,b)_3\, \mathrm{J}_1 = |a|^{-1} \int_{\mathcal{O}^*} (z_2, b)_3 \int_{a^{-1}\mathfrak{p}} \psi((z_2 + 1)y_1)\, dy_1\, dz_2.$$

If $|a| = q^{-1}$, then $a^{-1}\mathfrak{p} = \mathcal{O}$ and

$$\int_{\mathcal{O}} \psi((z_2 + 1)y_1)\, dy_1 = 1$$

is independent of $z_2 \in \mathcal{O}^*$. Since $\operatorname{val}(b) = 1$, we conclude in this case that

$$(a,b)_3\, \mathrm{J}_1 = |a|^{-1} \int_{\mathcal{O}^*} (z_2, b)_3\, dz_2 = 0.$$

If $|a| < q^{-1}$, then $a\mathfrak{p}^{-1} \subseteq \mathfrak{p}$ and

$$\int_{a^{-1}\mathfrak{p}} \psi((z_2+1)y_1)\,dy_1 = \begin{cases} q^{-1}|a|^{-1}, & z_2 \in -1 + a\mathfrak{p}^{-1}, \\ 0, & z_2 \in \mathcal{O}^* \setminus (-1 + a\mathfrak{p}^{-1}). \end{cases}$$

Consequently,

$$(a,b)_3\,\mathrm{J}_1 = |a|^{-2}q^{-1}\int_{-1+a\mathfrak{p}^{-1}} (z_2,b)\,dz_2 = |a|^{-1}.$$

By part (5) of Lemma 8.15, an element of $D_{0y}[a,b]$ satisfies $y_1, az_2y_2 \in \mathcal{O}$ and

$$-(bz_2)^{-1}(1-bx_1y_2)^{-1}x_1 \in -ab^{-1}(bz_2)^{-1}(1-bx_1y_2)^{-1}x_2 + \mathcal{O}.$$

Therefore,

$$\mathrm{J}_2 = \int_{a^{-1}\mathcal{O}^*}\int_{a^{-1}\mathfrak{p}}\int_{ab^{-1}x_2+\mathcal{O}}\int_{\mathcal{O}}\int_{\mathcal{O}} (az_2,b)_3\psi((1-ab^{-1}(bz_2)^{-1}(1-bx_1y_2)^{-1})x_2) \times dy_1\,dy_2\,dx_1\,dx_2\,dz_2.$$

Note that in the domain of integration, we have $bx_1y_2 \in \mathfrak{p}$. After applying the change of variables

$$z_2 \mapsto ab^{-2}(1-bx_1y_2)^{-1}z_2,$$

the integrand becomes independent of $x_1 \in ab^{-1}x_2 + \mathcal{O}$ and $y_1, y_2 \in \mathcal{O}$, and after integrating over these three variables, we obtain

$$(a,b)_3\,\mathrm{J}_2 = |a|^{-1}\int_{\mathcal{O}^*}\int_{a^{-1}\mathfrak{p}} (z_2,b)_3\psi((1-z_2^{-1})x_2)\,dx_2\,dz_2.$$

If $|a| = q^{-1}$, we again see that the integral factors through $\int_{\mathcal{O}^*}(z_2,b)\,dz_2$ and therefore $\mathrm{J}_2 = 0$. If $|a| < q^{-1}$, then the computation is similar to that of J_1. We have

$$(a,b)_3\mathrm{J}_2 = |a|^{-2}q^{-1}\int_{1+a\mathfrak{p}^{-1}} (z_2,b)\,dz_2 = |a|^{-1}.$$

Finally, it easily follows from parts (4) and (5) of Lemma 8.15 that $D_{0x}[a,b] \cap D_{0y}[a,b]$ is characterized by the conditions

$$x_1,\ x_2,\ y_1,\ y_2 \in \mathcal{O}, \quad az_2 \in \mathcal{O}^*.$$

Consequently, after integrating over x_1, x_2, y_1 and y_2 and applying the change of variables

$$z_2 \mapsto a^{-2}bz_2,$$

we have

$$(a,b)_3\mathrm{J}_3 = |a|^{-1}\int_{\mathcal{O}^*} (z_2,b)\,dz_2 = |a|^{-1}(1-q^{-1})\delta_{3|\mathrm{val}(b)}.$$

The lemma follows. ∎

8.6.4. A simplification of $\mathcal{J}_{>0}(a,b)$.

Lemma 8.17. *Let $|b| \le |a| < 1$. Then*

$$\mathcal{J}_{>0}(a,b) = \sum_{\ell=\lfloor \frac{\mathrm{val}(a)+1}{2} \rfloor}^{\mathrm{val}(a)-1} \mathcal{J}_\ell(a,b),$$

where

$$\mathcal{J}_\ell(a,b) = |a|^{-2} \int_L (b^{-1}\varpi^\ell, y_2 - 1)_3 \mathcal{K}(b; -ab^{-2}\varpi^\ell(1-y_2)^{-1}, \varpi^{-\ell} y_2) \times \mathcal{K}(b^{-1}(y_2-1)^{-1}; a^{-1}\varpi^\ell, a^{-1}by_2\varpi^{-\ell})\, dy_2,$$

where $L = \{|y_2| = q^{-2\ell}|b|^{-1}, |1-y_2| = |ab^{-1}|\}$. In particular,

$$\mathcal{J}_{>0}(a,b) = 0 \quad \text{if } |a| = q^{-1}.$$

Proof. Note that the domain $D_{>0}[a,b]$ is characterized by the conditions

$$|ab^{-1}| < |x_1|, |y_2| < |b|^{-1}, \quad az_2 \in \mathcal{O}^*,$$
$$y_1 - a^{-1}by_2,\ x_2 - a^{-1}bx_1 \in \mathcal{O}, \quad |1 - bx_1y_2| = |ab^{-1}|.$$

In particular, if $|a| = q^{-1}$, then $D_{>0}[a,b]$ is empty and $\mathcal{J}_{>0}(a,b) = 0$. For the rest of this section, assume that $|a| < q^{-1}$. Applying the change of variables $y_1 \mapsto y_1 + a^{-1}by_2$ and $x_2 \mapsto x_2 + a^{-1}bx_1$ to (8.10), the integrand becomes independent of $y_1, x_2 \in \mathcal{O}$. After integrating over these two variables, we obtain the expression

$$\mathcal{J}_{>0}(a,b) = |a^{-1}b| \int (x_1, bx_1y_2 - 1)_3 (az_2, b)_3 \times \psi((a^{-1}b - (bz_2)^{-1}(1 - bx_1y_2)^{-1})x_1 + (a^{-1}b + az_2)y_2) dx_1\, dy_2\, dz_2,$$

where integration is over the domain defined by the conditions

$$az_2 \in \mathcal{O}^*, \quad |ab^{-1}| < |x_1|, |y_2| < |b|^{-1} \quad \text{and} \quad |1 - bx_1y_2| = |ab^{-1}|.$$

Next, applying the change of variables

$$x_1 \mapsto az_2x_1, \quad y_2 \mapsto (az_2)^{-1}y_2,$$

we have

$$\mathcal{J}_{>0}(a,b) = |a^{-1}b| \int (x_1, bx_1y_2 - 1)_3 (az_2, b(bx_1y_2 - 1))_3 \times \psi(bz_2x_1 - ab^{-1}(1 - bx_1y_2)^{-1}x_1 + a^{-2}by_2z_2^{-1} + y_2)\, dx_1\, dy_2\, dz_2,$$

where integration is over the same domain. Now applying the change of variables

$$z_2 \mapsto z_2x_1^{-1}, \quad y_2 \mapsto y_2(bx_1)^{-1},$$

we have

$$\mathcal{J}_{>0}(a,b) = |a|^{-1} \int |x_1|^{-2}(b,x_1)_3(az_2,b(y_2-1))_3 \\ \times \psi(bz_2 - ab^{-1}(1-y_2)^{-1}x_1 + a^{-2}y_2z_2^{-1} + b^{-1}y_2x_1^{-1})\, dx_1\, dy_2\, dz_2,$$

where integration is over the domain defined by

$$|ab^{-1}| < |x_1| = |az_2| < |b|^{-1}, \quad |ax_1| < |y_2| < |x_1| \quad \text{and} \quad |1-y_2| = |ab^{-1}|.$$

Note further that the condition $|1-y_2| = |ab^{-1}|$ implies that $|y_2| \le |ab^{-1}|$ and combined with the condition $|ab^{-1}| < |x_1|$ it implies that $|y_2| < |x_1|$. Thus the condition $|y_2| < |x_1|$ may be omitted.

We express the integral $\mathcal{J}_{>0}(a,b)$ as a sum over $\ell = 1, \dots, \mathrm{val}(a) - 1$ of integrals over the subdomain, where $|x_1| = |b|^{-1}q^{-\ell}$, that is, over the domain defined by the conditions

$$|x_1| = |b|^{-1}q^{-\ell}, \quad |z_2| = |ab|^{-1}q^{-\ell}, \quad q^{-\ell}|b^{-1}a| < |y_2| \quad \text{and} \quad |1-y_2| = |ab^{-1}|.$$

Applying the change of variables

$$x_1 \mapsto b^{-1}\varpi^{\ell}x_1, \quad z_2 \mapsto a^{-1}b^{-1}\varpi^{\ell}z_2$$

to the ℓ-th summand, we have

$$\begin{aligned}\mathcal{J}_{>0}(a,b) &= |a|^{-2} \sum_{\ell=1}^{\mathrm{val}(a)-1} \int (b^{-1}\varpi^{\ell}, y_2-1)_3 \int_{\mathcal{O}^*}\int_{\mathcal{O}^*} (b,x_1)_3(z_2,b(y_2-1))_3 \\ &\qquad \times \psi\big(a^{-1}\varpi^{\ell}z_2 - ab^{-2}\varpi^{\ell}(1-y_2)^{-1}x_1 + a^{-1}b\varpi^{-\ell}y_2z_2^{-1} \\ &\qquad\quad + \varpi^{-\ell}y_2x_1^{-1}\big)\, dx_1\, dz_2\, dy_2 \\ &= |a|^{-2} \sum_{\ell=1}^{\mathrm{val}(a)-1} \int (b^{-1}\varpi^{\ell}, y_2-1)_3 \mathcal{K}(b; -ab^{-2}\varpi^{\ell}(1-y_2)^{-1}, \varpi^{-\ell}y_2) \\ &\qquad \times \mathcal{K}(b^{-1}(y_2-1)^{-1}; a^{-1}\varpi^{\ell}, a^{-1}by_2\varpi^{-\ell})\, dy_2,\end{aligned}$$

where in the ℓ-th integral y_2 is integrated over the domain defined by the conditions

$$q^{-\ell}|b^{-1}a| < |y_2| \quad \text{and} \quad |1-y_2| = |ab^{-1}|.$$

It follows from Lemma 7.2 that in the domain of integration above, we have

$$\mathcal{K}(b^{-1}(y_2-1)^{-1}; a^{-1}\varpi^{\ell}, a^{-1}by_2\varpi^{-\ell}) = 0$$

unless

$$|y_2| = q^{-2\ell}|b|^{-1}.$$

Indeed, if $\ell < \mathrm{val}(a) - 1$, then $|a^{-1}\varpi^{\ell}| > q$ and for $\ell = \mathrm{val}(a) - 1$, we have $|a^{-1}\varpi^{\ell}| = q$ while in the domain of integration $|a^{-1}by_2\varpi^{-\ell}| \ge q$. Since in the domain of the ℓ-th integral we have $|y_2| \le |ab^{-1}|$ and since $q^{-2\ell} \le |a|$ if and only if $\ell \ge \lfloor \frac{\mathrm{val}(a)+1}{2} \rfloor$, the lemma follows. ∎

In order to proceed with the computation of $\mathcal{J}_\ell(a,b)$, we separate it into two cases:

- Case 1: $\ell < \mathrm{val}(b) - 1$ or $\ell = \mathrm{val}(b) - 1$ and $3 \nmid \mathrm{val}(b)$.
- Case 2: $\ell = \mathrm{val}(b) - 1$ and $3 \mid \mathrm{val}(b)$.

The computation in Case 1 requires some preparation that we carry out first.

8.6.5. Let $a, b \in F^*$, and let ℓ be an integer such that either

- $|a| = |b| \le q^{-2}$ and $\frac{\mathrm{val}(a)}{2} \le \ell \le \mathrm{val}(a) - 1$, or
- $|b| < |a| < 1$ and $\ell = \frac{\mathrm{val}(a)}{2}$.

For $j \ge 1$, let

$$\Gamma_j = \{(x,y) \in \mathcal{O} \times \mathcal{O} : x^3 - y^3 \in a\mathfrak{p}^{-j}\},$$

and let

$$G_j^\ell(a,b) = \int_{\Gamma_j} \psi(\varpi^\ell b^{-1} x - 3^{-3}\varpi^{3\ell} a^{-1} b^{-1} x^3 + \varpi^\ell a^{-1} y)\, dx\, dy.$$

Note that the dependence of Γ_j on a is only via $\mathrm{val}(a)$. Our goal in this subsection is to compute $G_\ell^\ell(a,b)$ and $G_{\ell+1}^\ell(a,b)$. Let $j \in \{\ell, \ell+1\}$ and set

$$m = \mathrm{val}(a) - j.$$

Note that with the above assumptions $m \ge 0$. We begin with the explication of Γ_j.

Lemma 8.18. *We have the disjoint union*

$$\Gamma_j = (\mathfrak{p}^{-\lfloor -\frac{m}{3} \rfloor} \times \mathfrak{p}^{-\lfloor -\frac{m}{3} \rfloor}) \sqcup \bigsqcup_{n=0}^{\lfloor \frac{m-1}{3} \rfloor} \bigsqcup_{k=0}^{2} \{(x,y) : |x| = q^{-n},\ y \in \rho^k x + \mathfrak{p}^{m-2n}\}.$$

(In particular, if $m = 0$, then $\Gamma_j = \mathcal{O} \times \mathcal{O}$.)

Proof. The case $m = 0$ is straightforward. Assume that $m > 0$. By definition, we have

$$\Gamma_j = \{(x,y) \in \mathcal{O} \times \mathcal{O} : x^3 - y^3 \in \mathfrak{p}^m\}.$$

Note that for $(x,y) \in \Gamma_j$, we have $x^3 \in \mathfrak{p}^m$ if and only if $y^3 \in \mathfrak{p}^m$ and that for $x \in F$, we have $x^3 \in \mathfrak{p}^m$ if and only if $x \in \mathfrak{p}^{-\lfloor -\frac{m}{3} \rfloor}$. Suppose that $(x,y) \in \Gamma_j$ with $x^3 \notin \mathfrak{p}^m$ and set $|x| = q^{-n}$. By assumption, $0 \le n \le \lfloor \frac{m-1}{3} \rfloor$ and thus $|x| = |y|$. Note that in the decomposition

$$y^3 - x^3 = (y - x)(y - \rho x)(y - \rho^2 x) \in \mathfrak{p}^m,$$

two of the three factors must also be of absolute value q^{-n} and the third must therefore be in $\mathfrak{p}^{m-2n}$. This gives the inclusion of Γ_j in the disjoint union. The above decomposition also explains the other inclusion. ■

Lemma 8.19. *With the above assumptions, we have*

$$G_{\ell+1}^{\ell}(a,b)=0$$

and

$$G_{\ell}^{\ell}(a,b)=q^{-2}\delta_{\ell,\mathrm{val}(a)-1}\delta_{\mathrm{val}(a),\mathrm{val}(b)}+q^{2\ell}|a|\sum_{k=0}^{2}\mathcal{C}_{\ell}(b^{-1}+a^{-1}\rho^{k},-3^{-3}a^{-1}b^{-1}).$$

In particular, if $|a|=|b|$, *then*

$$G_{\ell}^{\ell}(a,b)=G_{\ell}^{\ell}(b,a).$$

Proof. It follows from Lemma 8.18 that

$$G_{j}^{i}(a,b)=\mathcal{G}_{0}(a,b)+\sum_{n=0}^{\lfloor\frac{m-1}{3}\rfloor}\sum_{k=0}^{2}\mathcal{G}_{n,k}(a,b),$$

where

$$\mathcal{G}_{0}(a,b)=\int_{\mathfrak{p}^{-\lfloor-\frac{m}{3}\rfloor}}\psi(\varpi^{\ell}a^{-1}y)\,dy\int_{\mathfrak{p}^{-\lfloor-\frac{m}{3}\rfloor}}\psi(\varpi^{\ell}b^{-1}x-3^{-3}\varpi^{3\ell}a^{-1}b^{-1}x^{3})\,dx$$

and

$$\begin{aligned}\mathcal{G}_{n,k}(a,b)=&\int_{\mathfrak{p}^{m-2n}}\psi(\varpi^{\ell}a^{-1}z)\,dz\\&\times\int_{|x|=q^{-n}}\psi(\varpi^{\ell}(b^{-1}+\rho^{k}a^{-1})x-3^{-3}\varpi^{3\ell}a^{-1}b^{-1}x^{3})\,dx.\end{aligned}$$

Note that writing $j=\ell+e$ with $e\in\{0,1\}$, since $|\varpi^{\ell}a^{-1}|=q^{m+e}$, we have

$$\int_{\mathfrak{p}^{-\lfloor-\frac{m}{3}\rfloor}}\psi(\varpi^{\ell}a^{-1}y)\,dy=\begin{cases}q^{\lfloor-\frac{m}{3}\rfloor}, & e=0\text{ and }m\le 1,\\ 0, & e=1\text{ or }m>1.\end{cases}$$

In our setup, the conditions $e=0$ and $m\le 1$ are met if and only if one of these conditions holds:

(1) $\mathrm{val}(b)=\mathrm{val}(a)$, $\ell=\mathrm{val}(a)-1$ and $e=0$, so that $m=1$;

(2) $\mathrm{val}(b)>\mathrm{val}(a)=2$ and $e=0$, so that $\ell=1$ and $m=1$.

Note that in both these cases $-\lfloor-\frac{m}{3}\rfloor=1$ and

$$\begin{aligned}\int_{\mathfrak{p}}\psi(\varpi^{\ell}b^{-1}x-3^{-3}\varpi^{3\ell}a^{-1}b^{-1}x^{3})\,dx&=q^{-1}\mathcal{C}(\varpi^{\ell+1}b^{-1},-3^{-3}\varpi^{3(\ell+1)}a^{-1}b^{-1})\\&=\begin{cases}q^{-1} & \text{in case (1)},\\ 0 & \text{in case (2)}.\end{cases}\end{aligned}$$

The last equality is a consequence of Lemma 7.3. Indeed, in case (1) both $\varpi^{\ell+1}b^{-1}$ and $-3^{-3}\varpi^{3(\ell+1)}a^{-1}b^{-1}$ are in $\mathcal{O}$, and in case (2) we have $1 < |\varpi^{\ell+1}b^{-1}| = q^{-2}|b|^{-1}$ while $|-3^{-3}\varpi^{3(\ell+1)}a^{-1}b^{-1}| = q^{-4}|b|^{-1}$. We conclude that

$$\mathcal{G}_0(a,b) = \begin{cases} q^{-2} & \text{if } \mathrm{val}(a) = \mathrm{val}(b),\ \ell = j = \mathrm{val}(a) - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$\int_{\mathfrak{p}^{m-2n}} \psi(\varpi^{\ell}a^{-1}z)\,dz = \begin{cases} q^{\ell}|a| & \text{if } n = 0 \text{ and } \ell = j, \\ 0 & \text{otherwise.} \end{cases}$$

Also, applying the change of variables $x \mapsto \varpi^{-\ell}x$, we have

$$\int_{\mathcal{O}^*} \psi(\varpi^{\ell}(b^{-1} + \rho^k a^{-1})x - 3^{-3}\varpi^{3\ell}a^{-1}b^{-1}x^3)\,dx$$
$$= q^{\ell}\mathcal{C}_{\ell}(b^{-1} + a^{-1}\rho^k, -3^{-3}a^{-1}b^{-1}).$$

Therefore,

$$\mathcal{G}_{n,k}(a,b) = \begin{cases} q^{2\ell}|a|\mathcal{C}_{\ell}(b^{-1} + a^{-1}\rho^k, -3^{-3}a^{-1}b^{-1}) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The formulas for $G^{\ell}_{\ell+1}(a,b)$ and $G^{\ell}_{\ell}(a,b)$ follow. Applying (7.2) with $t = \rho^{-k}$, we observe that if $|a| = |b|$, then the sum $\sum_{k=0}^{2} \mathcal{C}_{\ell}(b^{-1} + a^{-1}\rho^k, -3^{-3}a^{-1}b^{-1})$ is symmetric with respect to a and b. The last equality and the lemma therefore follow from the formula for $G^{\ell}_{\ell}(a,b)$. ■

8.6.6. Computation of $\mathcal{J}_{\ell}(a,b)$*: Case* 1. Here we apply Corollary 7.5 in order to provide a formula for $\mathcal{J}_{\ell}(a,b)$ whenever either $\ell < \mathrm{val}(b) - 1$ or $\ell = \mathrm{val}(b) - 1$ and $3 \nmid \mathrm{val}(b)$.

Lemma 8.20. *If either*

- $|a| = |b| \le q^{-2}$ *and* $\frac{\mathrm{val}(a)}{2} \le \ell \le \mathrm{val}(a) - 2$*, or*
- $|a| = |b| \le q^{-2}$, $\ell = \mathrm{val}(a) - 1$ *and* $3 \nmid \mathrm{val}(b)$*, or*
- $|b| < |a| < 1$ *and* $\ell = \frac{\mathrm{val}(a)}{2}$ *(here necessarily* $2 \mid \mathrm{val}(a)$*),*

then

$$(a,b)_3\,\mathcal{J}_{\ell}(a,b) = \delta_{\ell,\mathrm{val}(a)-1}\delta_{\mathrm{val}(a),\mathrm{val}(b)}|a|^{-1}$$
$$+ |ab|^{-1}\sum_{k=0}^{2}\mathcal{C}_{\ell}(b^{-1} + a^{-1}\rho^k, -3^{-3}a^{-1}b^{-1}).$$

If $|b| < |a| < 1$ *and* $\frac{\mathrm{val}(a)}{2} < \ell \le \mathrm{val}(a) - 1$*, then* $\mathcal{J}_{\ell}(a,b) = 0$.

Proof. With the assumption that either $\ell < \mathrm{val}(b) - 1$ or $\ell = \mathrm{val}(b) - 1$ and $3 \nmid \mathrm{val}(b)$, by Corollary 7.5, in the domain of integration we have

$$\begin{aligned}&\mathcal{K}(b; -ab^{-2}\varpi^{\ell}(1-y_2)^{-1}, \varpi^{-\ell}y_2)\\&\quad= (b, \varpi^{2\ell}ay_2^{-1}(1-y_2)^{-1})_3\mathcal{C}(\varpi^{\ell}b^{-1}, 3^{-3}\varpi^{3\ell}a^{-1}b^{-1}(y_2^{-1}-1))\end{aligned}$$

and

$$\begin{aligned}&\mathcal{K}(b^{-1}(y_2-1)^{-1}; a^{-1}\varpi^{\ell}, a^{-1}by_2\varpi^{-\ell})\\&\quad= (b^{-1}(y_2-1)^{-1}, \varpi^{2\ell}b^{-1}y_2^{-1})_3\mathcal{C}(a^{-1}\varpi^{\ell}, -3^{-3}\varpi^{3\ell}a^{-1}b^{-1}y_2^{-1}).\end{aligned}$$

Note further that

$$\begin{aligned}&(b, \varpi^{2\ell}ay_2^{-1}(1-y_2)^{-1})_3(b^{-1}(y_2-1)^{-1}, \varpi^{2\ell}b^{-1}y_2^{-1})_3\\&\quad= (b,a)_3(y_2-1, b^{-1}\varpi^{\ell})_3.\end{aligned}$$

Plugging this into the formula in Lemma 8.17 defining $\mathcal{J}_\ell(a,b)$, we obtain that

$$\begin{aligned}(a,b)_3\mathcal{J}_\ell(a,b) = |a|^{-2}\int_L &\mathcal{C}(\varpi^{\ell}b^{-1}, 3^{-3}\varpi^{3\ell}a^{-1}b^{-1}(y_2^{-1}-1))\\&\times\mathcal{C}(a^{-1}\varpi^{\ell}, -3^{-3}\varpi^{3\ell}a^{-1}b^{-1}y_2^{-1})\,dy_2,\end{aligned}$$

where $L = \{|y_2| = q^{-2\ell}|b|^{-1}, |1-y_2| = |ab^{-1}|\}$.

Note that if $|b| < |a| < 1$, then the domain of integration is empty (so that $\mathcal{J}_\ell(a,b) = 0$) unless $2\ell = \mathrm{val}(a)$. For the rest of the proof, we assume that either $|a| = |b|$ or $2\ell = \mathrm{val}(a)$.

The domain of integration over y_2 is explicated further as follows. If either

- $|b| < |a|$, $\mathrm{val}(a)$ is even and $\ell = \frac{\mathrm{val}(a)}{2}$, or
- $|a| = |b|$ and $\ell > \frac{\mathrm{val}(a)}{2}$,

then the domain is characterized by $|y_2| = q^{-2\ell}|b|^{-1}$. If $|b| = |a|$, $\mathrm{val}(a)$ is even and $\ell = \frac{\mathrm{val}(a)}{2}$, then the domain is $\mathcal{O}^* \setminus (1+\mathfrak{p})$. In order to unify notation, let

$$\delta = \begin{cases} 1 & \text{if } |b| = |a|, \mathrm{val}(a) \text{ is even and } \ell = \dfrac{\mathrm{val}(a)}{2},\\ 0 & \text{otherwise.}\end{cases}$$

Writing the cubic exponential integrals attached to the functions $\mathcal{C}$, we obtain

$$\begin{aligned}(a,b)_3\mathcal{J}_\ell(a,b) = |a|^{-2}\int_{\mathcal{O}}\int_{\mathcal{O}}&\Big[\int_L\psi(3^{-3}\varpi^{3\ell}a^{-1}b^{-1}(x^3-y^3)y_2^{-1})\,dy_2\Big]\\&\times\psi(\varpi^{\ell}b^{-1}x - 3^{-3}\varpi^{3\ell}a^{-1}b^{-1}x^3 + a^{-1}\varpi^{\ell}y)\,dx\,dy,\end{aligned}$$

where $L = \{|y_2| = q^{-2\ell}|b|^{-1}, |1-y_2| = |ab^{-1}|\}$.

The inner integral over y_2 may be evaluated using the straightforward formulas

$$\int_{|y_2|=q^{-2\ell}|b|^{-1}} \psi(ty_2^{-1})\,dy_2 = \begin{cases} q^{-2\ell}|b|^{-1}(1-q^{-1}), & t \in b^{-1}\mathfrak{p}^{2\ell}, \\ -q^{-2\ell-1}|b|^{-1}, & |t| = q^{2\ell+1}|b|, \\ 0, & |t| > q^{2\ell+1}|b| \end{cases} \tag{8.11}$$

and

$$\int_{\mathcal{O}^*\setminus(1+\mathfrak{p})} \psi(ty_2^{-1})\,dy_2 = \begin{cases} 1-2q^{-1}, & t \in \mathcal{O}, \\ -q^{-1}[1+\psi(t)], & |t| = q, \\ 0, & |t| > q. \end{cases}$$

In the notation of Section 8.6.5, we conclude that

$$\begin{aligned}(a,b)_3|a^2b|q^{2\ell}\mathcal{J}_\ell(a,b) = {}&(1-\delta)[(1-q^{-1})G_\ell^\ell(a,b) - q^{-1}(G_{\ell+1}^\ell(a,b) - G_\ell^\ell(a,b))] \\ &+ \delta\big[(1-2q^{-1})G_\ell^\ell(a,b) - q^{-1}(G_{\ell+1}^\ell(a,b) - G_\ell^\ell(a,b)) \\ &- q^{-1}(G_{\ell+1}^\ell(b,a) - G_\ell^\ell(b,a))\big] = G_\ell^\ell(a,b).\end{aligned}$$

The last equality and the lemma follow from Lemma 8.19. ∎

8.6.7. Computation of $\mathcal{J}_{\mathrm{val}(a)-1}(a,b)$: Case 2.

Lemma 8.21. *Assume that $|a| = |b| < 1$ and $3 \mid \mathrm{val}(b)$. Then*

$$|a|(a,b)_3\mathcal{J}_{\mathrm{val}(a)-1}(a,b) = \begin{cases} q-(1+q^{-1}), & -(ab^{-1})^3 \in 1+\mathfrak{p}, \\ -(1+q^{-1}), & -(ab^{-1})^3 \in \mathcal{O}^* \setminus (1+\mathfrak{p}). \end{cases}$$

Proof. Since $3 \mid \mathrm{val}(b) = \mathrm{val}(a) > 0$, for $\ell = \mathrm{val}(a) - 1$, we have

$$q^{-2\ell}|b|^{-1} = q^2|a| < 1.$$

It follows from Lemma 8.17 that

$$\begin{aligned}\mathcal{J}_{\mathrm{val}(a)-1}(a,b) = |a|^{-2}\int_{\mathcal{O}^*}\int_{\mathcal{O}^*} &\psi(-ab^{-2}\varpi^{\mathrm{val}(a)-1}u + a^{-1}\varpi^{\mathrm{val}(a)-1}v) \\ \times &\int_{|y_2|=q^2|a|} \psi(\varpi^{1-\mathrm{val}(a)}(u^{-1}+a^{-1}bv^{-1})y_2)\,dy_2\,du\,dv.\end{aligned}$$

Note that for $u, v \in \mathcal{O}^*$, we have

$$u^{-1}+a^{-1}bv^{-1} \in \begin{cases} \mathfrak{p} & \text{if } v \in -a^{-1}bu+\mathfrak{p}, \\ \mathcal{O}^* & \text{otherwise.} \end{cases}$$

Applying (8.11) to integrate over y_2, we deduce that

$$\int_{|y_2|=q^2|a|} \psi(\varpi^{1-\mathrm{val}(a)}(u^{-1}+a^{-1}bv^{-1})y_2)\,dy_2 = \begin{cases} (q^2-q)|a| & \text{if } v \in -a^{-1}bu+\mathfrak{p}, \\ -q|a| & \text{otherwise,} \end{cases}$$

and therefore

$$\mathcal{J}_{\mathrm{val}(a)-1}(a,b) = |a|^{-1}\Big[(q-1)\int_{\mathcal{O}^*}\psi(-\varpi^{\mathrm{val}(a)-1}(ab^{-2}+a^{-2}b)u)\,du \\ -q\int_{\mathcal{O}^*}\psi(-ab^{-2}\varpi^{\mathrm{val}(a)-1}u) \\ \times\int_{\mathcal{O}^*\setminus(-a^{-1}bu+\mathfrak{p})}\psi(a^{-1}\varpi^{\mathrm{val}(a)-1}v)\,dv\,du\Big].$$

We have

$$-q\int_{\mathcal{O}^*\setminus(-a^{-1}bu+\mathfrak{p})}\psi(a^{-1}\varpi^{\mathrm{val}(a)-1}v)\,dv = 1+\psi(-a^{-2}b\varpi^{\mathrm{val}(a)-1}u).$$

Applying (8.11) again, we obtain

$$\mathcal{J}_{\mathrm{val}(a)-1}(a,b) = |a|^{-1}\Big[q\int_{\mathcal{O}^*}\psi(-\varpi^{\mathrm{val}(a)-1}(ab^{-2}+a^{-2}b)u)\,du - q^{-1}\Big]$$

and

$$\int_{\mathcal{O}^*}\psi(-\varpi^{\mathrm{val}(a)-1}(ab^{-2}+a^{-2}b)u)\,du = \begin{cases}1-q^{-1}, & -(ab^{-1})^3\in 1+\mathfrak{p},\\ -q^{-1}, & -(ab^{-1})^3\in\mathcal{O}^*\setminus(1+\mathfrak{p}).\end{cases}$$

Since $(a,b)_3 = 1$ in this case, the lemma readily follows. ∎

8.6.8. Completion of the proof of Proposition 8.1. The functional equation (8.1) is proved in Corollary 8.3. Assume that $|b|\le|a|$. The computation of $J(a,b)$ for the cases $|a|>1$, $1=|a|=|b|$ and $1=|a|>|b|$ is taken care of in Lemmas 8.8, 8.9 and 8.10, respectively. In the case $|a|=q^{-1}$, it follows from (8.9), Lemmas 8.16 and 8.17 that $J_3(a,b)=0$ and the formula is now a consequence of (8.3) and Lemmas 8.11 and 8.12.

Assume now that either $|b|\le|a|\le q^{-2}$. It follows from Lemmas 8.11 and 8.12 that $J_1(a,b)=J_2(a,b)=0$, and therefore from (8.3) and (8.9) we have that

$$J(a,b)=J_3(a,b)=\mathcal{J}_0(a,b)+\mathcal{J}_{>0}(a,b).$$

Combining Lemmas 7.6, 8.16 and 8.20 for $\ell=\mathrm{val}(a)-1$ and Lemma 8.21, we conclude that

$$(a,b)_3[\mathcal{J}_0(a,b)+\mathcal{J}_{\mathrm{val}(a)-1}(a,b)] \\ = \begin{cases}3|a|^{-1}+|ab|^{-1}\sum_{k=0}^{2}\mathcal{C}_{\mathrm{val}(a)-1}(b^{-1}+a^{-1}\rho^k,-3^{-3}a^{-1}b^{-1}), & |a|=|b|,\\ \delta_{\mathrm{val}(a),2}|ab|^{-1}\sum_{k=0}^{2}\mathcal{C}_1(b^{-1}+a^{-1}\rho^k,-3^{-3}a^{-1}b^{-1}), & |b|<|a|.\end{cases}$$

Applying Lemmas 8.17 and 8.20 for $\ell<\mathrm{val}(a)-1$, the formula follows. This completes the proof of the proposition.

9. The integral $I(a,b)$

In this section, we provide the following formula for $I(a,b)$.

Proposition 9.1. *For $|b| \le |a|$, we have*

$$I(a,b) = \begin{cases} 1, & |a| = |b| = 1, \\ 2q, & |a| = |b| = q^{-1}, \\ 3|a|^{-1} + |ab|^{-1} \sum_{\ell=\lfloor \frac{\mathrm{val}(a)+1}{2} \rfloor}^{\mathrm{val}(a)-1} \sum_{k=0}^{2} \mathcal{C}_\ell(b^{-1} - \rho^k a^{-1}, 2a^{-1}b^{-1}), & |a| = |b| \le q^{-2}, \\ |b|^{-1}\mathcal{C}(b^{-1}, 2a^{-1}b^{-1}), & |b| < |a| = 1, \\ |ab|^{-1} \sum_{k=0}^{2} \mathcal{C}_{\frac{\mathrm{val}(a)}{2}}(b^{-1} - \rho^k a^{-1}, 2a^{-1}b^{-1}), & |b| < |a| < 1, \\ & 2 \mid \mathrm{val}(a), \\ 0, & |b| < |a| < 1, \\ & 2 \nmid \mathrm{val}(a), \\ 0, & |a| > 1. \end{cases}$$

Furthermore,

$$I(b,a) = \overline{I(-a,-b)}, \quad a, b \in F^*. \tag{9.1}$$

9.1. *A first simplification and functional equation*

In this section, we show that $I(a,b)$ satisfies a functional equation that allows us to reduce the computation to the case $|b| \le |a|$.

9.1.1. We begin by observing some basic properties of the integral $I(a,b)$.

Lemma 9.2. *We have the following properties of $I(a,b)$.*

(1) *The integral $I(a,b)$ takes real values, that is, $\overline{I(a,b)} = I(a,b)$.*

(2) *$I(a,b) = 0$ unless $a, b \in \mathcal{O}$.*

(3) *For $a, b \in \mathcal{O}$, we have*

$$I(a,b) = \sum_{j=0}^{\min(\mathrm{val}(a),\mathrm{val}(b))} I(j;a,b), \tag{9.2}$$

where

$$I(j;a,b) = q^{-2j} \int \psi[x + y + 2a\rho^2 xy^2 - (2\rho b x + 2\rho^2 a y)z]\, ds\, dx\, dy\, dz$$

and the integral is over $x, y, z, s \in F$ such that

$$x + as,\ y - bs,\ (xb + y\rho^2 a)s - (xy + z\rho) \in \mathfrak{p}^{-j}, \quad xb + y\rho^2 a \in \mathfrak{p}^j.$$

Proof. Changing order of integration, it follows from definition (6.1) that

$$I(a,b)=\int_{\max(|a|,|b|)\le|t|\le1}|t|^2\Big[\int\psi[x+y+2a(xy-z)y\rho^2-2bxz\rho]\,ds\,dx\,dy\,dz\Big]\,d^*t,$$

where the inner integral is over $x,y,z,s\in F$ such that

$$t(x+as),\ t(bs-y),\ t^{-1}(xb+y\rho^2a),\ t[(xb+y\rho^2a)s-(xy+z\rho)]\in\mathcal{O}.$$

The second part immediately follows. Furthermore, the change of variables

$$(x,y,s)\mapsto-(x,y,s)$$

preserves the domain of integration and transforms the integrand into its conjugate. The first part follows.

Assume that $a,b\in\mathcal{O}$. The inner integral over x, y, z, s depends only on $|t|$ and the third part of the lemma also follows, observing that $I(j;a,b)$ is the integral over the part of the domain where $|t|=q^{-j}$. ∎

9.1.2. In the next lemma, a change of variables in z simplifies the domain of integration and allows us to integrate over z and simplify the expression for $I(j;a,b)$. We then observe that the resulting expression satisfies a symmetry between a and b.

Lemma 9.3. *For $a,b\in\mathcal{O}$ and $0\le j\le\min(\mathrm{val}(a),\mathrm{val}(b))$, we have*

$$I(j;a,b)=q^{-j}\int\psi\big[x+y+2(bx^2y-axy^2+(abxy-b^2x^2-a^2y^2)s)\big]\,ds\,dx\,dy,\tag{9.3}$$

where the integral is over $x,y,s\in F$ such that

$$x+as,\ y-bs\in\mathfrak{p}^{-j},\quad bx,\ ay\in\mathfrak{p}^{j}.$$

Consequently,

$$I(a,b)=I(-b,-a),\quad a,b\in F^*.\tag{9.4}$$

Proof. After the change of variables

$$z\mapsto z+b\rho^2xs+a\rho ys-\rho^2xy,$$

we have

$$I(j;a,b)=q^{-2j}\int\psi\big[x+y+2a\rho^2xy^2-(2\rho bx+2\rho^2ay)(z+b\rho^2xs+a\rho ys-\rho^2xy)\big]\,ds\,dx\,dy\,dz,$$

where the integral is over $x,y,z,s\in F$ such that

$$x+as,\ y-bs,\ z\in\mathfrak{p}^{-j},\quad xb+y\rho^2a\in\mathfrak{p}^{j}.$$

After integrating over z this becomes, we have

$$q^{-j}\int \psi[x+y+2a\rho^2xy^2-(2\rho bx+2\rho^2ay)(b\rho^2xs+a\rho ys-\rho^2xy)]\,ds\,dx\,dy$$
$$=q^{-j}\int \psi[x+y+2(bx^2y-axy^2+(abxy-b^2x^2-a^2y^2)s)]\,ds\,dx\,dy,$$

where the integral is now over $x, y, s \in F$ such that

$$x+as,\ y-bs\in\mathfrak{p}^{-j},\quad xb+y\rho^2a,\ xb+y\rho a\in\mathfrak{p}^j$$

or equivalently

$$x+as,\ y-bs\in\mathfrak{p}^{-j},\quad xb,\ ya\in\mathfrak{p}^j.$$

This completes the proof of (9.3).

Applying (9.3), an interchange between x and y shows that $I(j;a,b)=I(j;-b,-a)$. The functional equation (9.4) immediately follows from (9.2) and Lemma 9.2 (2). ∎

This reduces the computation of $I(a,b)$ to the case where $|b|\le|a|\le 1$.

9.2. *The case* $|b|\le|a|\le 1$

Our goal in this section is to compute $I(j;a,b)$ for $|b|\le|a|\le 1$ and $0\le j\le \mathrm{val}(a)$.

9.2.1. First, we further simplify the expression for $I(j;a,b)$ for the case $|b|\le|a|\le 1$. In the following lemma, we first apply to (9.3) a change of variables in s that under the assumption $|b|\le|a|$ simplifies the domain of integration and allows us to integrate over s. Motivated by the polynomial decomposition

$$X^2+Y^2-XY=(X+\rho Y)(X+\rho^2Y),$$

we follow up with a change of variables in y that further simplifies the domain of integration.

Lemma 9.4. *For* $|b|\le|a|\le 1$ *and* $0\le j\le\min(\mathrm{val}(a),\mathrm{val}(b))$, *we have*

$$I(j;a,b)=|a|^{-1}\int \psi[(1-\rho a^{-1}b)x-(1+2\rho)y+2a^{-1}b^2x^3]\,dx\,dy, \tag{9.5}$$

where the integral is over $x, y\in F$ *such that*

$$bx-\rho ay\in a\mathfrak{p}^{-j},\quad bx,\ ay,\ y(bx+ay)\in\mathfrak{p}^j. \tag{9.6}$$

Proof. Apply to (9.3) the change of variables $s\mapsto s-a^{-1}x$ to obtain

$$I(j;a,b)=q^{-j}\int \psi\big[x+y+2(bx^2y-axy^2+(abxy-b^2x^2-a^2y^2)(s-a^{-1}x))\big]\,ds\,dx\,dy$$
$$=q^{-j}\int \psi[x+y+2(a^{-1}b^2x^3+(abxy-b^2x^2-a^2y^2)s)]\,ds\,dx\,dy,$$

where the integral is now over $x, y, s \in F$ such that

$$as,\ bs - (a^{-1}bx + y) \in \mathfrak{p}^{-j}, \quad xb,\ ya \in \mathfrak{p}^{j}$$

or, under our assumption that $|b| \le |a|$, equivalently such that

$$as,\ a^{-1}bx + y \in \mathfrak{p}^{-j}, \quad xb,\ ya \in \mathfrak{p}^{j}.$$

Now integrating over s, we have

$$I(j;a,b) = |a|^{-1} \int \psi[x + y + 2a^{-1}b^2x^3]\,dx\,dy,$$

where the integral is now over $x, y \in F$ such that

$$a^{-1}bx + y \in \mathfrak{p}^{-j}, \quad xb,\ ya \in \mathfrak{p}^{j}, \quad abxy - b^2x^2 - a^2y^2 \in a\mathfrak{p}^{j}.$$

After the change of variables

$$y \mapsto -(\rho a^{-1}bx + (1+2\rho)y),$$

we have

$$I(j;a,b) = |a|^{-1} \int \psi[(1-\rho a^{-1}b)x - (1+2\rho)y + 2a^{-1}b^2x^3]\,dx\,dy,$$

where the integral is now over $x, y \in F$ such that

$$(1-\rho)bx - (1+2\rho)ay \in a\mathfrak{p}^{-j}, \quad bx,\ ay,\ y(bx+ay) \in \mathfrak{p}^{j}.$$

Note that $1 + 2\rho = (1-\rho)\rho$ and $1 - \rho \in \mathcal{O}^*$, so that the first condition is equivalent to $bx - \rho ay \in a\mathfrak{p}^{-j}$ and the lemma follows. ∎

9.2.2. By splitting the domain of integration into three parts, we express $I(j;a,b)$ as a sum of three integrals that we subsequently evaluate separately.

Let $|b| \le |a| \le 1$, fix an integer j such that $0 \le j \le \operatorname{val}(a)$ and set

$$m = \min\Big(j, \operatorname{val}(a) - j, \Big\lfloor \frac{\operatorname{val}(a) - j}{2} \Big\rfloor\Big).$$

By definition,

$$\mathfrak{p}^{-m} = \mathfrak{p}^{-j} \cap \mathfrak{p}^{j-\operatorname{val}(a)} \cap \mathfrak{p}^{-\lfloor \frac{\operatorname{val}(a)-j}{2} \rfloor}.$$

Note further that $y \in \mathfrak{p}^{-\lfloor \frac{\operatorname{val}(a)-j}{2} \rfloor}$ if and only if $ay^2 \in \mathfrak{p}^{j}$. Consequently, we have

$$y \in \mathfrak{p}^{-m} \quad \text{if and only if} \quad ay \in a\mathfrak{p}^{-j} \cap \mathfrak{p}^{j} \text{ and } ay^2 \in \mathfrak{p}^{j}. \tag{9.7}$$

It follows that the conditions

$$y \notin \mathfrak{p}^{-m} \quad \text{and} \quad ay \in \mathfrak{p}^{j}$$

are equivalent to the conditions

$$(y \notin \mathfrak{p}^{-j} \text{ or } ay^2 \notin \mathfrak{p}^j) \quad \text{and} \quad ay \in \mathfrak{p}^j.$$

Consequently, the set of $x, y \in F$ satisfying (9.6) partitions into three parts with the disjoint extra conditions:

(1) $y \in \mathfrak{p}^{-m}$,

(2) $y \notin \mathfrak{p}^{-j}$,

(3) $y \in \mathfrak{p}^{-j}$ and $ay^2 \notin \mathfrak{p}^j$.

Write

$$I(j; a, b) = I_1(a,b) + I_2(a,b) + I_3(a,b),$$

where $I_\ell(a,b)$ is defined by integral (9.5) over $x, y \in F$ satisfying (9.6) as well as the condition (ℓ) above, $\ell = 1, 2, 3$.

For future reference, we further point out that it is immediate from the definitions that

$$m \geq 0 \text{ and } m = 0 \quad \text{if and only if} \quad j \in \{0, \mathrm{val}(a) - 1, \mathrm{val}(a)\}. \tag{9.8}$$

9.2.3. In the notation of Section 9.2.2, we compute $I_1(a,b)$.

Lemma 9.5. *We have*

$$I_1(a,b) = \begin{cases} |b|^{-1}\mathcal{C}(ab^{-1}, 2a^2b^{-1}), & j \in \{0, \mathrm{val}(a)\}, \\ q|b|^{-1}\mathcal{C}(\varpi^{-1}(\rho^2ab^{-1} - 1), 2\varpi^{-3}a^2b^{-1}), & 1 \leq j = \mathrm{val}(a) - 1, \\ |ab|^{-1}\mathcal{C}_j(b^{-1} - \rho a^{-1}, 2a^{-1}b^{-1}), & \left\lfloor \frac{\mathrm{val}(a)+1}{2} \right\rfloor \leq j \leq \mathrm{val}(a) - 2, \\ 0, & 1 \leq j \leq \left\lfloor \frac{\mathrm{val}(a)-1}{2} \right\rfloor. \end{cases}$$

Proof. It follows from (9.7) that the conditions

$$x \neq 0, \quad y \in \mathfrak{p}^{-m}, \quad bx - \rho ay \in a\mathfrak{p}^{-j}, \quad bx, ay, y(bx + ay) \in \mathfrak{p}^j$$

and

$$x \neq 0, \quad x \in b^{-1}(\mathfrak{p}^j \cap a\mathfrak{p}^{-j}), \quad y \in \mathfrak{p}^{-m} \cap (bx)^{-1}\mathfrak{p}^j$$

are equivalent. Excluding the measure zero set where $x = 0$, we conclude that

$$I_1(a,b) = |a|^{-1} \int_{b^{-1}(\mathfrak{p}^j \cap a\mathfrak{p}^{-j})} \psi[(1 - \rho a^{-1}b)x + 2a^{-1}b^2x^3] \times \left\{ \int_{\mathfrak{p}^{-m} \cap (bx)^{-1}\mathfrak{p}^j} \psi[-(1+2\rho)y]\, dy \right\} dx.$$

We have

$$\int_{\mathfrak{p}^{-m} \cap (bx)^{-1}\mathfrak{p}^j} \psi[-(1+2\rho)y]\, dy = \begin{cases} 1, & m = 0 \text{ or } |bx| = q^{-j}, \\ 0, & m > 0 \text{ and } |bx| < q^{-j}. \end{cases}$$

Note that the set of $x \in b^{-1}(\mathfrak{p}^j \cap a\mathfrak{p}^{-j})$ such that $|bx| = q^{-j}$ is empty unless $\frac{\mathrm{val}(a)}{2} \le j$ and in this case, it is precisely the set of x such that $|bx| = q^{-j}$. Taking (9.8) into consideration, we have

$$I_1(a,b) = \begin{cases} |a|^{-1} \displaystyle\int_{b^{-1}(\mathfrak{p}^j \cap a\mathfrak{p}^{-j})} \psi[(1-\rho a^{-1}b)x + 2a^{-1}b^2x^3]\,dx, \\ \hfill j \in \{0, \mathrm{val}(a)-1, \mathrm{val}(a)\}, \\ |a|^{-1} \displaystyle\int_{|bx|=q^{-j}} \psi[(1-\rho a^{-1}b)x + 2a^{-1}b^2x^3]\,dx, \\ \hfill \max\left(1, \dfrac{\mathrm{val}(a)}{2}\right) \le j \le \mathrm{val}(a)-2, \\ 0, \hfill 1 \le j < \dfrac{\mathrm{val}(a)}{2}. \end{cases}$$

We further observe that

$$\mathfrak{p}^j \cap a\mathfrak{p}^{-j} = \begin{cases} a\mathcal{O}, & j = 0 \text{ or } j = \mathrm{val}(a), \\ a\mathfrak{p}^{-1}, & j = \mathrm{val}(a) - 1 \ge 1, \end{cases}$$

and after the change of variables $x \mapsto b^{-1}ax$, we obtain

$$|a|^{-1} \int_{b^{-1}a\mathcal{O}} \psi[(1-\rho a^{-1}b)x + 2a^{-1}b^2x^3]\,dx = |b|^{-1}\mathcal{C}(ab^{-1}, 2a^2b^{-1}).$$

Then after the change of variables $x \mapsto b^{-1}a\varpi^{-1}x$, we have

$$\begin{aligned} |a|^{-1} &\int_{b^{-1}a\mathfrak{p}^{-1}} \psi[(1-\rho a^{-1}b)x + 2a^{-1}b^2x^3]\,dx \\ &= q|b|^{-1}\,\mathcal{C}(\varpi^{-1}(ab^{-1}-\rho), 2\varpi^{-3}a^2b^{-1}). \end{aligned}$$

And last, after the change of variables $x \mapsto b^{-1}x$, we get

$$|a|^{-1} \int_{|bx|=q^{-j}} \psi[(1-\rho a^{-1}b)x + 2a^{-1}b^2x^3]\,dx = |ab|^{-1}\mathcal{C}_j(b^{-1}-\rho a^{-1}, 2a^{-1}b^{-1}).$$

The lemma readily follows. ■

9.2.4. In the notation of Section 9.2.2, we compute $I_2(a,b)$.

Lemma 9.6. *Let*

$$D = \{(x,y) \in F^2 : y \notin \mathfrak{p}^{-j},\ bx - \rho ay \in a\mathfrak{p}^{-j},\ bx, ay, y(bx+ay) \in \mathfrak{p}^j\}.$$

and

$$D' = \{(x,y) \in F^2 : |ab^{-1}|q^j < |x| \le |a|^{\frac{1}{2}}|b|^{-1}q^{-\frac{j}{2}},\ y \in \rho^2a^{-1}bx + \mathfrak{p}^{-j}\}.$$

Then $D = D'$.

Proof. It is easy to see that

$$y \notin \mathfrak{p}^{-j}, \quad bx - \rho a y \in a\mathfrak{p}^{-j}$$

if and only if

$$|ab^{-1}|q^j < |x|, \quad y \in \rho^2 a^{-1} bx + \mathfrak{p}^{-j}$$

and that these conditions imply

$$|bx| = |ay| > |a|q^j.$$

Assume now that $(x, y) \in D$. From the first two conditions defining D, we have that $|bx - \rho a y| < |bx| = |ay|$, and therefore we must have $|bx + ay| = |bx|$. Consequently, the condition $y(bx + ay) \in \mathfrak{p}^j$ implies that $|a^{-1}b^2x^2| = |y(bx + ay)| \leq q^{-j}$ and it follows that $(x, y) \in D'$.

Conversely, if $(x, y) \in D'$, then the first two conditions defining D are satisfied and $|ay| = |bx|$. The weak inequality in the definition of D' now implies that $bx, ay, bxy, ay^2 \in \mathfrak{p}^j$ and therefore also $y(bx + ay) \in \mathfrak{p}^j$ so that $(x, y) \in D$. Then lemma follows. ∎

Lemma 9.7. *We have*

$$I_2(a, b) = \begin{cases} |ab|^{-1} \displaystyle\sum_{\ell = \lfloor \frac{\mathrm{val}(a)+1}{2} \rfloor}^{\mathrm{val}(a)-1} \mathcal{C}_\ell(b^{-1} - a^{-1}, 2a^{-1}b^{-1}), & j = 0, \\ 0, & j \geq 1. \end{cases}$$

Proof. As a consequence of Lemma 9.6, we have that $|a|I_2(a, b)$ equals

$$\sum_{\ell = \lfloor \frac{\mathrm{val}(a)+j+1}{2} \rfloor - \mathrm{val}(b)}^{\mathrm{val}(a)-\mathrm{val}(b)-j-1} \int_{\mathrm{val}(x)=\ell} \psi[(1 - \rho a^{-1} b)x + 2a^{-1}b^2x^3] \times \left\{ \int_{\rho^2 a^{-1} bx + \mathfrak{p}^{-j}} \psi(-(1 + 2\rho)y)\, dy \right\} dx.$$

Since $-(1 + 2\rho)\rho^2 = \rho - 1$, we have that

$$\int_{\rho^2 a^{-1} bx + \mathfrak{p}^{-j}} \psi(-(1 + 2\rho)y)\, dy = \begin{cases} \psi((\rho - 1)a^{-1}bx), & j = 0, \\ 0, & j \geq 1. \end{cases}$$

We conclude that if $j \geq 1$, then $I_2(a, b) = 0$, and if $j = 0$, then

$$I_2(a, b) = |a|^{-1} \sum_{\ell = \lfloor \frac{\mathrm{val}(a)+1}{2} \rfloor - \mathrm{val}(b)}^{\mathrm{val}(a)-\mathrm{val}(b)-1} \int_{\mathrm{val}(x)=\ell} \psi[(1 - a^{-1}b)x + 2a^{-1}b^2x^3]\, dx.$$

The change of variables $x \mapsto b^{-1}x$ shows that

$$\int_{\mathrm{val}(x)=\ell} \psi[(1 - a^{-1}b)x + 2a^{-1}b^2x^3]\, dx = |b|^{-1}\mathcal{C}_{\ell + \mathrm{val}(b)}(b^{-1} - a^{-1}, 2a^{-1}b^{-1}).$$

After the change $\ell \mapsto \ell - \mathrm{val}(b)$ in the index of summation, the lemma follows. ∎

9.2.5. In the notation of Section 9.2.2, we compute $I_3(a, b)$.

Lemma 9.8. *Let*

$$D = \{(x, y) \in F^2 : y \in \mathfrak{p}^{-j},\ ay^2 \notin \mathfrak{p}^j,\ bx - \rho ay \in a\mathfrak{p}^{-j},\ bx, ay, y(bx + ay) \in \mathfrak{p}^j\}$$

and

$$D' = \{(x, y) \in F^2 : |b|^{-1}|a|^{\frac{1}{2}}q^{-\frac{j}{2}} < |x| \le |b|^{-1} \min(|a|q^j, q^{-j}),\\ y \in -a^{-1}bx + (bx)^{-1}\mathfrak{p}^j\}.$$

Then $D = D'$ and D is empty unless $j < \mathrm{val}(a) < 3j$.

Proof. Note that if $ay^2 \notin \mathfrak{p}^j$ and $y(bx + ay) \in \mathfrak{p}^j$, then $|bx| = |ay|$. It follows that $bx \notin y^{-1}\mathfrak{p}^j = a(bx)^{-1}\mathfrak{p}^j$, and therefore x satisfies the strong inequality in the definition of D'. Furthermore, $y \in \mathfrak{p}^{-j}$ and $bx - \rho ay \in a\mathfrak{p}^{-j}$ imply that $bx \in a\mathfrak{p}^{-j}$, and together with $bx \in \mathfrak{p}^j$ this gives that x also satisfies the weak inequality. Consequently, for $(x, y) \in D$ we have $y^{-1}\mathfrak{p}^j = ab^{-1}x^{-1}\mathfrak{p}^j$, and therefore it is easy to see that $D \subseteq D'$.

For $(x, y) \in D'$, the strong inequality in the definition of D' implies that $(bx)^2 \notin a\mathfrak{p}^j$, while the condition on y implies that $bx(ay + bx) \in a\mathfrak{p}^j$. Therefore, $|ay| = |bx|$, and from the weak inequality it follows that $ay, bx \in a\mathfrak{p}^{-j} \cap \mathfrak{p}^j$, and from the strong inequality we conclude that $|ay^2| = |a^{-1}(bx)^2| > q^{-j}$, and therefore $D' \subseteq D$.

The condition on j follows from the fact that if D is not empty, then $|b|^{-1}|a|^{\frac{1}{2}}q^{-\frac{j}{2}} < |b|^{-1} \min(|a|q^j, q^{-j})$. ∎

Lemma 9.9. *We have*

$$I_3(a, b) = \begin{cases} |ab|^{-1}\mathcal{C}_j(b^{-1} - \rho^2 a^{-1}, 2a^{-1}b^{-1}), & \left\lfloor \frac{\mathrm{val}(a) + 1}{2} \right\rfloor \le j \le \mathrm{val}(a) - 1, \\ 0, & j = \mathrm{val}(a) \text{ or } j \le \left\lfloor \frac{\mathrm{val}(a) - 1}{2} \right\rfloor. \end{cases}$$

Proof. It follows from Lemma 9.8 that $I_3(a, b) = 0$ unless

$$\left\lfloor \frac{\mathrm{val}(a)}{3} \right\rfloor + 1 \le j \le \mathrm{val}(a) - 1.$$

For the rest of the proof, assume that these inequalities hold. Note that

$$\int_{-a^{-1}bx + (bx)^{-1}\mathfrak{p}^j} \psi(-(1 + 2\rho)y)\, dy = \begin{cases} \psi((1 + 2\rho)a^{-1}bx), & |bx| = q^{-j}, \\ 0, & |bx| < q^{-j}. \end{cases}$$

It therefore further follows from Lemma 9.8 that

$$I_3(a, b) = |a|^{-1} \int \psi[(1 - \rho^2 a^{-1}b)x + 2a^{-1}b^2x^3]\, dx,$$

where the integral is over all $x \in F$ such that

$$|bx| = q^{-j} \quad \text{and} \quad |b|^{-1}|a|^{\frac{1}{2}}q^{-\frac{j}{2}} < |x| \le |b|^{-1} \min(|a|q^j, q^{-j}).$$

Note that this domain is empty and therefore $I_3(a,b) = 0$ unless $j \geq \frac{\mathrm{val}(a)}{2}$ and when this is the case the domain is defined by $|bx| = q^{-j}$. Applying the change of variables $x \mapsto b^{-1}x$, we obtain in this case that

$$I_3(a,b) = |ab|^{-1}\mathcal{C}_j(b^{-1} - \rho^2 a^{-1}, 2a^{-1}b^{-1}).$$

The lemma follows. ∎

9.3. *Completion of proof of Proposition 9.1*

The functional equation (9.1) is a consequence of part (1) of Lemma 9.2 and equation (9.4). If $a \notin \mathcal{O}$, then $I(a,b) = 0$ by part (2) of Lemma 9.2.

Assume for the rest of this section that $|b| \leq |a| \leq 1$. Combining Lemmas 9.5, 9.7, 9.9 and (9.2), we have that

$$\begin{aligned} I(a,b) = (1 + \delta_{\mathrm{val}(a)\geq 1})|b|^{-1}\mathcal{C}(ab^{-1}, 2a^2b^{-1}) \\ + \delta_{\mathrm{val}(a)\geq 2}\{q|b|^{-1}\mathcal{C}(\varpi^{-1}(\rho^2 ab^{-1} - 1), 2\varpi^{-3}a^2b^{-1}) \\ + |ab|^{-1}\mathcal{C}_{\mathrm{val}(a)-1}(b^{-1} - a^{-1}, 2a^{-1}b^{-1}) \\ + |ab|^{-1}\mathcal{C}_{\mathrm{val}(a)-1}(b^{-1} - \rho^2 a^{-1}, 2a^{-1}b^{-1})\} \\ + |ab|^{-1}\sum_{\ell=\lfloor\frac{\mathrm{val}(a)+1}{2}\rfloor}^{\mathrm{val}(a)-2}\sum_{k=0}^{2}\mathcal{C}_\ell(b^{-1} - \rho^k a^{-1}, 2a^{-1}b^{-1}), \end{aligned}$$

where for $i = 1, 2$,

$$\delta_{\mathrm{val}(a)\geq i} = \begin{cases} 1 & \text{if } \mathrm{val}(a) \geq i, \\ 0 & \text{otherwise.} \end{cases}$$

The evaluation of $I(a,b)$ may now be obtained from this expression by applying Lemma 7.3, equations (7.2) and (7.3), and the identity

$$\mathcal{C}(x,y) = \mathcal{C}(ux, u^3y), \quad x,\ y \in F,\ u \in \mathcal{O}^*. \tag{9.9}$$

In detail, Proposition 9.1 is a straightforward consequence of the following seven identities:

(1) $$\mathcal{C}(ab^{-1}, 2a^2b^{-1}) = \begin{cases} 1, & |a| = |b| \leq 1, \\ \mathcal{C}(b^{-1}, 2a^{-1}b^{-1}), & 1 = |a| > |b|, \\ 0, & |b| < |a| \leq q^{-1}. \end{cases}$$

Indeed, for the case $1 = |a| > |b|$ apply (9.9) with $u = a^{-1}$, and in the other cases apply Lemma 7.3.

(2) If $|b| \leq |a| = q^{-2}$, then

$$\mathcal{C}(\varpi^{-1}(\rho^2 ab^{-1} - 1), 2\varpi^{-3}a^2b^{-1}) = \mathcal{C}(\varpi(b^{-1} - \rho a^{-1}), 2\varpi^3 a^{-1}b^{-1}). \tag{9.10}$$

Indeed, in this case apply (9.9) with $u = \rho\varpi^2 a^{-1} \in \mathcal{O}^*$.

(3) If $|b| < |a| = q^{-2}$, then

$$\mathcal{C}(\varpi^{-1}(\rho^2 ab^{-1} - 1), 2\varpi^{-3}a^2b^{-1}) = q\,\mathcal{C}_1(b^{-1} - \rho a^{-1}, 2a^{-1}b^{-1}).$$

Indeed, apply identity (9.10) and note that

$$|\varpi(b^{-1} - \rho a^{-1})| = |2\varpi^3 a^{-1}b^{-1}| = q^{-1}|b|^{-1} \ge q^2,$$

so that by Lemma 7.3 we have

$$\mathcal{C}(\varpi^{-1}(\rho^2 ab^{-1} - 1), 2\varpi^{-3}a^2b^{-1}) = \mathcal{C}_0(\varpi(b^{-1} - \rho a^{-1}), 2\varpi^3 a^{-1}b^{-1}).$$

The identity now follows from (7.2) applied with $t = \varpi$.

(4) If $|b| = |a| = q^{-2}$, then

$$\mathcal{C}(\varpi^{-1}(\rho^2 ab^{-1} - 1), 2\varpi^{-3}a^2b^{-1}) = q^{-1} + q\,\mathcal{C}_1(b^{-1} - \rho a^{-1}, 2a^{-1}b^{-1}).$$

Indeed, apply identity (9.10) and note that

$$|\varpi(b^{-1} - \rho a^{-1})| \le |2\varpi^3 a^{-1}b^{-1}| = q,$$

so that by Lemma 7.3 we have

$$\mathcal{C}(\varpi^{-1}(\rho^2 ab^{-1} - 1), 2\varpi^{-3}a^2b^{-1}) = q^{-1} + \mathcal{C}_0(\varpi(b^{-1} - \rho a^{-1}), 2\varpi^3 a^{-1}b^{-1}).$$

The identity now follows from (7.2) applied with $t = \varpi$.

(5) If $|b| < |a| \le q^{-3}$, then $\mathcal{C}(\varpi^{-1}(\rho^2 ab^{-1} - 1), 2\varpi^{-3}a^2b^{-1}) = 0$. Indeed, since $\max(|2\varpi^{-3}a^2b^{-1}|, q) < |\varpi^{-1}(\rho^2 ab^{-1} - 1)|$, this is immediate from Lemma 7.3.

(6) If $|b| < |a| \le q^{-3}$, then $\mathcal{C}_\ell(b^{-1} - \rho^k a^{-1}, 2a^{-1}b^{-1}) = 0$ whenever $\ell \le \mathrm{val}(a) - 1$, $\mathrm{val}(a) \ne 2\ell$ and $k \in \{0, 1, 2\}$. Indeed, since

$$|b^{-1} - \rho^k a^{-1}| = |b|^{-1} > |a|^{-1} \ge q^{\ell+1},$$

this is immediate from (7.3).

(7) If $|b| = |a| \le q^{-3}$, then

$$q\,\mathcal{C}(\varpi^{-1}(\rho^2 ab^{-1} - 1), 2\varpi^{-3}a^2b^{-1}) = 1 + |a|^{-1}\mathcal{C}_{\mathrm{val}(a)-1}(b^{-1} - \rho a^{-1}, 2a^{-1}b^{-1}).$$

Indeed, in this case

$$|\varpi^{-1}(\rho^2 ab^{-1} - 1)| \le q \quad \text{while } 2\varpi^{-3}a^2b^{-1} \in \mathcal{O},$$

so that by Lemma 7.3 we have

$$q\,\mathcal{C}(\varpi^{-1}(\rho^2 ab^{-1} - 1), 2\varpi^{-3}a^2b^{-1}) = 1 + q\,\mathcal{C}_0(\varpi^{-1}(\rho^2 ab^{-1} - 1), 2\varpi^{-3}a^2b^{-1}).$$

The identity now follows from (7.2) applied with $t = \rho^2\varpi^{-1}a$.

This completes the proof of Proposition 9.1.

10. The comparison of $I(a, b)$ and $J(a, b)$: Proof of Theorem 6.1

Let $a, b \in F^*$, $c = -54a$ and $d = 54b$. It follows from the functional equations (8.1) and (9.1) that without loss of generality, we may assume that $|b| \le |a|$. Applying (9.9) with $u = 54$, we have

$$\mathcal{C}(d^{-1}, -3^{-3}c^{-1}d^{-1}) = \mathcal{C}(b^{-1}, 2a^{-1}b^{-1}),$$

and similarly applying (7.2) with $t = 54$, we have

$$\mathcal{C}_\ell(d^{-1} + \rho^k c^{-1}, -3^{-3}c^{-1}d^{-1}) = \mathcal{C}_\ell(b^{-1} - \rho^k a^{-1}, 2a^{-1}b^{-1}).$$

The proof of the theorem now follows from Propositions 8.1 and 9.1.

11. The comparison of orbital integrals for the other relevant orbits

In this section, we compute and match the local contributions to the non-big cell relevant orbits for the unit elements in the appropriate Hecke algebras.

11.1. The I integrals

We begin with the I integrals. The relevant orbits are given in Theorem 4.2.

There are three relevant orbits indexed by a single power of ρ (see Lemmas 4.3, 4.4, and 4.6). It is straightforward to check that for each such orbit, the local integral evaluates to 1. For example, the integral arising from Lemma 4.3 is

$$\int_F \int_{F^*} \int_{K_2} \mathbf{1}_{\mathcal{O}^8}[(0, 0, 0, t, 0, \rho^2 t, 0, xt)\lambda(k)]|t|^2 \, dk \, d^*t \; \psi(x) \, dx$$
$$= \int_{K_2} dk \int_{|t|,|xt|\le 1} |t|^2 \psi(x) \, d^*t \, dx = \int_{0<|t|\le 1} \Big(\int_{|x|\le |t|^{-1}} \psi(x) \, dx \Big) |t|^2 \, d^*t.$$

The inner integral in x is zero unless $|t| = 1$, and so the evaluation follows.

We turn to the local contribution to Lemma 4.8 when the local test function is the characteristic function of $\mathrm{PGL}_3(\mathcal{O})$. This is given, for $a \in F^*$ (a replaces α in the notation of the lemma), by

$$I_1(a) = \int \psi(-\rho a^{-1}xy - \rho^2 a^{-1}z + a^{-2}xz + x + y) \, dx \, dy \, dz |t|^2 \, d^*t,$$

where the integral is over the domain

$$t, t(a + x), a^{-2}t^{-1}, ty, a^{-2}t^{-1}x, t(\rho a y - \rho^2 xy - z) \in \mathcal{O}.$$

Note that the domain of integration is empty unless $|a| \ge 1$. When $|a| = 1$, the domain is

$$t \in \mathcal{O}^*, \quad x, y, z \in \mathcal{O}, \tag{11.1}$$

and we deduce that $I_1(a) = 1$.

Assume now that $|a| = q^n > 1$. We can write

$$I_1(a) = \sum_{j=0}^{2n} I_{1,j}(a),$$

where

$$I_{1,j}(a) = q^{-2j} \int \psi(-\rho a^{-1}xy - \rho^2 a^{-1}z + a^{-2}xz + x + y)\,dx\,dy\,dz,$$

where the domain of integration is

$$|y|,\ |a+x|,\ |\rho a y - \rho^2 xy - z| \le q^j, \quad |x| \le q^{2n-j}.$$

After the change of variables $z \mapsto z + \rho a y - \rho^2 xy$, we have

$$I_{1,j}(a) = q^{-2j} \int_{|x| \le q^{2n-j},\, |a+x| \le q^j} \psi(x) \int_{\mathfrak{p}^{-j}} \psi[\rho a^{-1}x(1-\rho a^{-1}x)y]\,dy \\ \times \int_{\mathfrak{p}^{-j}} \psi[-\rho^2 a^{-1}(1-\rho a^{-1}x)z]\,dz\,dx = \int \psi(x)\,dx,$$

where the last integral is over the domain

$$|x| \le q^{2n-j}, \quad |a+x| \le q^j, \quad |a^{-1}x(1-\rho a^{-1}x)|, \quad |a^{-1}(1-\rho a^{-1}x)| \le q^{-j}$$

or equivalently

$$|x|,\ |x(a-\rho x)|,\ |a-\rho x| \le q^{2n-j}, \quad |a+x| \le q^j.$$

If $j < n$, then the condition $|a+x| \le q^j < q^n = |a|$ implies that $|x| = |a-\rho x| = q^n$ and therefore $|x(a-\rho x)| = q^{2n}$. The domain of integration is therefore empty unless $j = 0$ in which case the domain is $x \in -a + \mathcal{O}$. We conclude that

$$I_{1,j}(a) = \begin{cases} \psi(-a), & j = 0, \\ 0, & 0 < j < n. \end{cases}$$

If $j > n$, then the condition $|a-\rho x| \le q^{2n-j} < q^n = |a|$ implies that $|x| = q^n > q^{2n-j}$, and therefore again the domain of integration is empty, so that $I_{1,j}(a) = 0$.

For $j = n$, note that the domain of integration becomes $\mathcal{O} \sqcup \rho^2 a + \mathcal{O}$, and therefore

$$I_{1,n}(a) = 1 + \psi(\rho^2 a).$$

We conclude that the following holds.

Lemma 11.1. *If $a \in F^*$, then*

$$I_1(a) = \begin{cases} \psi(-a) + 1 + \psi(\rho^2 a), & |a| > 1, \\ 1, & |a| = 1, \\ 0, & |a| < 1. \end{cases}$$

■

Next we turn to the local integral coming from the relevant orbits of Lemma 4.9 in the case that the local component of ϕ is the characteristic function of $\mathrm{PGL}_3(\mathcal{O})$. For $a \in F^*$, this is

$$I_2(a) = \int \psi[a^{-2}(xy - z)y - \rho(xy\rho^2 + z)a^{-1} + x + y]\, dx\, dy\, dz |t|^2\, d^*t,$$

where the integral is over the domain

$$t,\ a^{-2}t^{-1},\ tx,\ t(a-y),\ a^{-2}t^{-1}y,\ t(xa - xy - z\rho) \in \mathcal{O}.$$

The first two conditions imply that the domain is empty unless $|a| \geq 1$. When $|a| = 1$, the domain is once again (11.1) and it is easy to see that $I_2(a) = 1$. If $|a| = q^n > 1$, then since $|t| \leq 1$ and $|t|^{-1} \leq |a|^2 = q^{2n}$, we may write

$$I_2(a) = \sum_{j=0}^{2n} I_{2,j}(a),$$

where in $I_{2,j}(a)$ the domain of integration is limited to $|t| = q^{-j}$. After the change of variables $z \mapsto z + \rho^2 ax - \rho^2 xy$, we see that

$$I_{2,j}(a) = q^{-2j} \int \psi(-a^{-2}y(\rho xy + z + \rho^2 ax) - \rho z a^{-1} + y)\, dx\, dy\, dz,$$

where the domain of integration is

$$|x|, |a-y|, |z| \leq q^j, \quad |y| \leq q^{2n-j}.$$

The integration in z gives zero unless $|1 + \rho^2 a^{-1} y| \leq q^{-j}|a|$, and the integral in x gives zero unless $|a^{-1}y(a^{-1}y + \rho)| \leq q^{-j}$. Thus we arrive at

$$I_{2,j}(a) = \int \psi(y)\, dy,$$

where the domain of integration is

$$|y| \leq q^{2n-j}, \quad |a-y| \leq q^j, \quad |1 + \rho^2 a^{-1} y| \leq q^{n-j}, \quad |y(1 + \rho^2 a^{-1} y)| \leq q^{n-j}.$$

If $j < n$, then since $|a| = q^n$, the second inequality implies that $|y| = q^n$ and $|a^{-1}y - 1| < 1$. But this condition implies that $|1 + \rho^2 a^{-1} y| = 1$, so the last inequality is possible only when $j = 0$. In that case, the domain reduces to $y \in a + \mathcal{O}$. Thus

$$I_{2,j}(a) = \begin{cases} \psi(a), & j = 0, \\ 0, & 0 < j < n. \end{cases}$$

If $j > n$, then in the domain of integration we have $|1 + \rho^2 a^{-1} y| < 1$, and this implies that $|y| = |a| = q^n$ which contradicts $|y| \leq q^{2n-j}$. In this case, the domain is thus empty. Finally, if $j = n$, then the inequality $|y| \leq q^n$ implies the second and third inequalities above, and in view of the inequality $|y(1 + \rho^2 a^{-1} y)| \leq 1$, we see that the domain of integration is the union of the two $\mathcal{O}$-cosets $y \in \mathcal{O}$ and $y \in -\rho a + \mathcal{O}$. It follows that $I_{2,n}(a) = 1 + \psi(-\rho a)$. We conclude that the following holds.

Lemma 11.2. *If $a \in F^*$, then*

$$I_2(a) = \begin{cases} 1 + \psi(a) + \psi(-\rho a), & |a| > 1, \\ 1, & |a| = 1, \\ 0, & |a| < 1. \end{cases} \qquad \blacksquare$$

11.2. The J integrals

The non-big cell relevant orbits are given by

(1) $\zeta I_3, \zeta \in \mu_3$;

(2) $\begin{pmatrix} & a^{-2} \\ aI_2 & \end{pmatrix}, a \in F^*$;

(3) $\begin{pmatrix} & aI_2 \\ a^{-2} & \end{pmatrix}, a \in F^*$.

For the first family of three orbits, the local integral for the unit element is given by

$$\int_{N(F)} f_0'(\zeta n)\psi(n)\, dn, \quad \zeta \in \mu_3.$$

Since

$$f_0'(\zeta n) = \begin{cases} 1 & \text{if } n \in N \cap K, \\ 0 & \text{otherwise,} \end{cases}$$

each integral is 1.

We turn to the second family of orbits. For $i = 1, 2$, write the n_i that appears in the integration as

$$n_i = \begin{pmatrix} 1 & x_i & z_i \\ & 1 & y_i \\ & & 1 \end{pmatrix}.$$

Since we are modding out by the stabilizer, we may take $y_1 = 0$. Multiplying, we see that the local integral is given by

$$J_1(a) := \int_{F^5} f_0'\left(\begin{pmatrix} -ax_1 & -ax_1x_2 - az_1 & a^{-2} - a(z_1y_2 + x_1z_2) \\ a & ax_2 & az_2 \\ 0 & a & ay_2 \end{pmatrix}\right) \times \psi(-x_1 + x_2 + y_2)\, d(*).$$

Since f_0' vanishes away from $K \times \mu_3$, we see that $J_1(a) = 0$ unless $|a| \le 1$. In addition, in the support of the integrand, the variables of integration must satisfy the inequalities

$$|x_1|,\ |x_2|,\ |y_2|,\ |z_2|,\ |x_1x_2 + z_1| \le |a|^{-1}, \quad |a^{-2} - a(z_1y_2 + x_1z_2)| \le 1.$$

If $|a| = 1$, the domain of integration is $\mathcal{O}^5$, and it follows directly from the factorization of the argument of f_0' that $\kappa = 1$ in this domain. Thus the integral is 1.

Suppose from now on that $|a| < 1$. Then in the support of the integrand, the diagonal entries must be units. Thus the domain of integration is given by

$$\begin{aligned}&|x_1| = |x_2| = |y_2| = |a|^{-1},\\ &z_1 \in -x_1x_2 + a^{-1}\mathcal{O}, \quad z_2 \in x_1^{-1}a^{-3} - x_1^{-1}z_1y_2 + \mathcal{O}, \quad |a^{-3} + x_1x_2y_2| \le |a|^{-2}.\end{aligned} \tag{11.2}$$

Note that the last condition is required since $|z_2| \le |a|^{-1}$. We have the following evaluation.

Lemma 11.3. *Suppose that* $|a| < 1$ *and that conditions* (11.2) *hold. Then*

$$\kappa\left(\begin{pmatrix} -ax_1 & -ax_1x_2 - az_1 & a^{-2} - a(z_1y_2 + x_1z_2) \\ a & ax_2 & az_2 \\ 0 & a & ay_2 \end{pmatrix}\right) = (a, x_2y_2^2)_3.$$

Proof. We describe the proof in brief. Since $ay_2 \in \mathcal{O}^*$, left multiplication by an element of $N \cap K$ reduces the computation to the evaluation of $\kappa(g_1)$, where

$$g_1 = \begin{pmatrix} -ax_1 & -ax_1x_2 - y_2^{-1}a^2 + y_2^{-1}ax_1z_2 & 0 \\ a & ax_2 - ay_2^{-1}z_2 & 0 \\ 0 & a & ay_2 \end{pmatrix}.$$

We have $y_2^{-1} \in \mathcal{O}$. If

$$g_2 = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -y_2^{-1} & 1 \end{pmatrix},$$

then

$$g_1g_2 = \begin{pmatrix} -ax_1 & -ax_1x_2 - y_2^{-1}a^2 + y_2^{-1}ax_1z_2 & 0 \\ a & ax_2 - ay_2^{-1}z_2 & 0 \\ 0 & 0 & ay_2 \end{pmatrix}.$$

Since g_1g_2 is block diagonal, it is easy to evaluate

$$\kappa(g_1g_2) = (a, a(x_2 - y_2^{-1}z_2)ay_2)_3 = (a, x_2y_2 - z_2)_3 = (a, x_2y_2)_3.$$

In the last equality above, we have used that $|z_2| < |x_2y_2|$ in domain (11.2). Furthermore, $\kappa(g_2) = 1$. We conclude that the value we require is given by $(a, x_2y_2)_3\sigma(g_1, g_2)^{-1}$. Applying the algorithm of Bump–Hoffstein to compute σ, one finds that $\sigma(g_1, g_2) = (y_2, a)_3$. The result follows. ∎

We arrive at the expression

$$J_1(a) = \int (a, x_2^2y_2)_3\psi(-x_1 + x_2 + y_2)\, dz_2\, dz_1\, dx_1\, dx_2\, dy_2,$$

where the integration is over domain (11.2). We carry out the inner integrations in z_2 and then z_1 to obtain

$$|a|^{-1} \int (a, x_2^2y_2)_3\psi(-x_1 + x_2 + y_2)\, dx_1\, dx_2\, dy_2,$$

where the integration is over

$$|x_1| = |x_2| = |y_2| = |a|^{-1}, \quad |a^{-1} + a^2 x_1 x_2 y_2| \le 1.$$

Now we change $x_1 \to a^{-1}x_1$, $x_2 \to a^{-1}x_2$, and $y_2 \to a^{-1}y_2$ to obtain

$$|a|^{-4} \int (a, x_2^2 y_2)_3 \psi(a^{-1}(-x_1 + x_2 + y_2))\, dx_1\, dx_2\, dy_2,$$

where the integration is over

$$|x_1| = |x_2| = |y_2| = 1, \quad |1 + x_1 x_2 y_2| \le |a|.$$

After changing $x_1 \mapsto \frac{x_1}{x_2 y_2}$, the last inequality becomes $|1 + x_1| \le |a|$. This allows us to do the x_1 integral. We arrive at

$$J_1(a) = |a|^{-3} \int_{|x_2|=|y_2|=1} (a, x_2^2 y_2)_3 \psi\Big(a^{-1}\Big(x_2 + y_2 + \frac{1}{x_2 y_2}\Big)\Big)\, dx_2\, dy_2.$$

We now make use of Corollary 7.5 to replace the integral in x_2 by the integral of a cubic (so the parameters c, d, t of the corollary are given by $c = a^{-1}$, $d = (a y_2)^{-1}$, $t = a^2$; note that $(t, c^{-1}d)_3 = (a^2, y_2)_3^{-1}$). We see that

$$J_1(a) = |a|^{-3} \int_{|y_2|=1} \mathcal{C}(a^{-1}, -3^{-3} y_2 a^{-1}) \psi(a^{-1} y_2)\, dy_2.$$

Note that in the integrand above, the symbol $(a^2, y_2)_3$ obtained from the corollary cancels the symbol $(a, y_2)_3$ obtained from κ.

We evaluate the last integral by using the definition of $\mathcal{C}$. Substituting this definition and changing $x \to 3x$, we have

$$\begin{aligned} J_1(a) &= |a|^{-3} \int_{\mathcal{O}^*} \int_{\mathcal{O}} \psi(a^{-1}x - 3^{-3} y_2 a^{-1} x^3 + y_2 a^{-1})\, dx\, dy_2 \\ &= |a|^{-3} \int_{\mathcal{O}} \psi(3a^{-1}x) \int_{\mathcal{O}^*} \psi(y_2 a^{-1}(1 - x^3))\, dy_2\, dx. \end{aligned}$$

Note that

- $|a^{-1}(1 - x^3)| \le 1$ if and only if $x \in \rho^k + a\mathcal{O}$, $k = 0, 1, 2$;
- $|a^{-1}(1 - x^3)| = q$ if and only if $x \in \rho^k + \varpi^{-1} a \mathcal{O}^*$, $k = 0, 1, 2$,

and the inner integral is zero if $|a^{-1}(1 - x^3)| > q$. If $x \in \rho^k + a\mathcal{O}$, $k = 0, 1, 2$, then the inner integral is $1 - q^{-1}$, and so this piece contributes

$$(1 - q^{-1})|a|^{-2} \sum_{k=0}^{2} \psi(3\rho^k a^{-1}).$$

If $x \in \rho^k + \varpi^{-1} a \mathcal{O}^*$ for some $k = 0, 1, 2$, then the inner integral is $-q^{-1}$. If in addition $|a| < q^{-1}$, then the sets $\rho^k + \varpi^{-1} a \mathcal{O}^*$ are disjoint for $k = 0, 1, 2$, so this piece contributes

$$-q^{-1}|a|^{-3} \sum_{k=0}^{2} \psi(3\rho^k a^{-1}) \int_{\varpi^{-1} a \mathcal{O}^*} \psi(3a^{-1}x)\, dx = q^{-1}|a|^{-2} \sum_{k=0}^{2} \psi(3\rho^k a^{-1}).$$

If $|a| = q^{-1}$, then the sets $\rho^k + \varpi^{-1} a\mathcal{O}^* = \rho^k + \mathcal{O}^*$, $k = 0, 1, 2$, are no longer disjoint, however, their union equals $\mathcal{O} \setminus (\bigsqcup_{k=0}^2 (\rho^k + \mathfrak{p}))$. Consequently, in this case this piece contributes

$$-q^2 \int_{\mathcal{O}\setminus(\bigsqcup_{k=0}^2(\rho^k+\mathfrak{p}))} \psi(3a^{-1}x)\,dx = q^2 \sum_{k=0}^{2} \int_{\rho^k+\mathfrak{p}} \psi(3a^{-1}x)\,dx = q \sum_{k=0}^{2} \psi(3\rho^k a^{-1}).$$

We arrive at the following evaluation.

Lemma 11.4. *The orbital integral for the orbit* $\left(\begin{smallmatrix} & a^{-2} \\ aI_2 & \end{smallmatrix}\right)$, $a \in F^*$, *is given by*

$$J_1(a) = \begin{cases} |a|^{-2} \sum_{k=0}^{2} \psi(3\rho^k a^{-1}), & |a| < 1, \\ 1, & |a| = 1, \\ 0, & |a| > 1. \end{cases}$$

We turn to the evaluation of the orbital integral for the orbit $\left(\begin{smallmatrix} & aI_2 \\ a^{-2} & \end{smallmatrix}\right)$ with $a \in F^*$. In this case, modding out by the stabilizer we may take $x_1 = 0$. Then the integral of concern is

$$J_2(a) := \int_{F^5} f_0'\left(\begin{pmatrix} -a^{-2}z_1 & a - a^{-2}x_2z_1 & ay_2 - a^{-2}z_1z_2 \\ -a^{-2}y_1 & -a^{-2}x_2y_1 & a - a^{-2}y_1z_2 \\ a^{-2} & a^{-2}x_2 & a^{-2}z_2 \end{pmatrix}\right) \times \psi(-y_1 + x_2 + y_2)\,d(*).$$

Since f_0' is supported in K, we see that $J_2(a) = 0$ unless $|a| \geq 1$, and in this case the domain of integration is determined by the conditions

$$|x_2|,\ |y_1|,\ |z_2|,\ |z_1|,\ |x_2y_1| \leq |a|^2,$$
$$|a - a^{-2}y_1z_2|,\ |a - a^{-2}x_2z_1|,\ |ay_2 - a^{-2}z_1z_2| \leq 1.$$

If $|a| = 1$, then the domain of integration is $\mathcal{O}^5$, $n_1, n_2 \in N \cap K$, and the integrand is identically 1, so the integral is 1.

Suppose from now on that $|a| > 1$. Then since $|a - a^{-2}y_1z_2| \leq 1$, we must have $|y_1z_2| = |a|^3$, and since $|a - a^{-2}x_2z_1| \leq 1$, we must have $|x_2z_1| = |a|^3$. But then

$$|x_2y_1z_1z_2| = |a|^6.$$

Since $|x_2y_1| \leq |a|^2$, this gives $|z_1z_2| \geq |a|^4$. Since $|z_1|, |z_2| \leq |a|^2$, we conclude that $|z_1| = |z_2| = |a|^2$, $|x_2| = |y_1| = |a|$. The last condition now also implies that $|y_2| = |a|$.

We conclude that the domain may be rewritten as

$$\begin{aligned} &|z_1| = |z_2| = |a|^2, \\ &y_1 \in a^3 z_2^{-1} + \mathcal{O}, \quad x_2 \in a^3 z_1^{-1} + \mathcal{O}, \quad y_2 \in a^{-3} z_1 z_2 + a^{-1}\mathcal{O}, \end{aligned} \tag{11.3}$$

and on this domain, we have

$$\psi(-y_1+x_2+y_2)=\psi(a^3z_1^{-1}-a^3z_2^{-1}+a^{-3}z_1z_2). \tag{11.4}$$

The next step is the evaluation of f_0' in this domain.

Lemma 11.5. *In domain* (11.3), *we have*

$$\kappa\left(\begin{pmatrix} -a^{-2}z_1 & a-a^{-2}x_2z_1 & ay_2-a^{-2}z_1z_2 \\ -a^{-2}y_1 & -a^{-2}x_2y_1 & a-a^{-2}y_1z_2 \\ a^{-2} & a^{-2}x_2 & a^{-2}z_2 \end{pmatrix}\right)=(a,z_1^{-1}z_2)_3(z_1^{-1},z_2)_3.$$

Proof. The proof is similar to the proof of Lemma 11.3, but in place of g_1, g_2 there we use

$$g_1=\begin{pmatrix} -ay_2z_2^{-1} & a-ax_2y_2z_2^{-1} & 0 \\ -az_2^{-1} & -x_2z_2^{-1}a & 0 \\ a^{-2} & a^{-2}x_2 & a^{-2}z_2 \end{pmatrix}, \quad g_2=\begin{pmatrix} 1 & & \\ & 1 & \\ -z_2^{-1} & -x_2z_2^{-1} & 1 \end{pmatrix}.$$

Since g_1g_2 is block diagonal, one sees that $\kappa(g_1g_2)=(a,x_2z_2^{-1})_3(x_2,z_2)_3$ while again $\kappa(g_2)=1$ and so the value we require is $(a,x_2z_2^{-1})_3(x_2,z_2)_3\sigma(g_1,g_2)^{-1}$. For this case, a computation using the algorithm of Bump–Hoffstein shows that $\sigma(g_1,g_2)=(a,z_2)_3$. Using conditions (11.3), the lemma follows. ■

Using this expression and performing the integrations over x_2, y_1 and y_2 (after using (11.4), the integrand is independent of these variables), we obtain

$$J_2(a)=|a|^{-1}\int_{|z_1|=|z_2|=|a|^2}(a,z_1z_2^{-1})_3(z_1,z_2)_3\,\psi(a^3z_1^{-1}-a^3z_2^{-1}+a^{-3}z_1z_2)\,dz_1\,dz_2.$$

Changing z_i to a^2z_i for $i=1,2$, we get

$$J_2(a)=|a|^3\int_{\mathcal{O}^*\times\mathcal{O}^*}(z_1,az_2)_3(a,z_2)_3\psi(a(z_1^{-1}-z_2^{-1}+z_1z_2))\,dz_1\,dz_2.$$

We again make use of Corollary 7.5. Now we replace the integral in z_1 by the integral of a cubic (the parameters c, d, t of the corollary are given by $c=az_2$, $d=a$, $t=a^2z_2^2$; note that $(t,c^{-1}d)_3=(a,z_2)_3$). Just as in the prior case, the remaining cubic residue symbols cancel. We see that

$$\begin{aligned} J_2(a)&=|a|^3\int_{\mathcal{O}^*}\mathcal{C}(a,-3^{-3}az_2^{-1})\,\psi(-az_2^{-1})dz_2 \\ &=|a|^3\int_{\mathcal{O}^*}\int_{\mathcal{O}}\psi(ax-3^{-3}az_2^{-1}x^3-az_2^{-1})\,dx\,dz_2. \end{aligned}$$

Sending $x\to 3x$, $z_2\to -z_2^{-1}$ and reordering the integral gives

$$J_2(a)=|a|^3\int_{\mathcal{O}}\psi(3ax)\int_{\mathcal{O}^*}\psi(az_2(1+x^3))\,dz_2\,dx.$$

The evaluation proceeds as in the prior case of computing J_1, with contributions whenever either $x\in-\rho^k+a^{-1}\mathcal{O}$ or $x\in-\rho^k+\varpi^{-1}a^{-1}\mathcal{O}^*$, $k=0,1,2$. We obtain the following evaluation.

Lemma 11.6. *The orbital integral for the orbit* $\begin{pmatrix} & aI_2 \\ a^{-2} & \end{pmatrix}$, $a \in F^*$, *is given by*

$$J_2(a) = \begin{cases} |a|^2 \sum_{k=0}^{2} \psi(-3\rho^k a), & |a| > 1, \\ 1, & |a| = 1, \\ 0, & |a| < 1. \end{cases}$$

11.3. Comparison

The comparison is given by the following theorem.

Theorem 11.7. *The relevant I and J orbital integrals match up to transfer factors:*

(1) *The orbital integrals attached to the three singleton relevant orbits for I and for J each have value* 1.

(2) *Let* $a = 3(\rho - \rho^2)c^{-1}$. *Then* $I_1(a) = |c|^2 \psi(-3\rho c^{-1}) J_1(c)$.

(3) *Let* $a = 3(\rho - 1)c$. *Then* $I_2(a) = |c|^{-2} \psi(3\rho c) J_2(c)$.

The theorem follows at once from the evaluations above. We remark that there are other possible matchings available. First, for all $c \in F^*$ we have $J_i(\rho^k c) = J_i(c)$ for $i = 1, 2$ and $k = 1, 2$. In addition, we have the following matchings:

(1) If $a = 3(\rho - 1)c$, then $I_1(a) = |c|^{-2} \psi(3c) J_2(c)$.

(2) If $a = 3(\rho^2 - 1)c^{-1}$, then $I_2(a) = |c|^2 \psi(-3c^{-1}) J_1(c)$.

This completes the proof of the fundamental lemma.

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