Rev. Mat. Iberoam. 42 (2026), no. 1, 75–94
DOI 10.4171/RMI/1599

Obstructions to prescribed Q-curvature of complete conformal metrics on $\mathbb{R}^n$

Mingxiang Li

Abstract. We provide some obstructions to the prescribed Q-curvature problem for complete conformal metrics on $\mathbb{R}^n$ with finite total Q-curvature. One of them is a Bonnet–Mayer type theorem with respect to Q-curvature. Others are related to the decay rate of the prescribed functions.

1. Introduction

Given a smooth function $K(x)$ on the standard sphere $(\mathbb{S}^2, g_0)$, the well-known Nirenberg problem is to find a conformal metric $g = e^{2u} g_0$ such that its Gaussian curvature equals f. This is equivalent to solving the following conformally invariant equation on $\mathbb{S}^2$:

$$-\Delta_{\mathbb{S}^2} u(x) + 1 = K(x)\, e^{2u(x)}, \quad x \in \mathbb{S}^2, \tag{1.1}$$

where $\Delta_{\mathbb{S}^2}$ is the Laplace–Beltrami operator. The famous Chern–Gauss–Bonnet formula requires that $\sup K > 0$, which is an obvious obstruction. Surprisingly, there is another obstruction to (1.1), known as the Kazdan–Warner identity [23]. It can be stated as follows:

$$\int_{\mathbb{S}^2} \langle \nabla x_i, \nabla K \rangle\, e^{2u}\, \mathrm{d}\mu_{\mathbb{S}^2} = 0, \quad 1 \le i \le 3, \tag{1.2}$$

where x_i is the eigenfunction satisfying $-\Delta_{\mathbb{S}^2} x_i = 2x_i$. We refer the readers to [6, 7] for more information about the Nirenberg problem. A direct corollary of the identity (1.2) is that $f(x) = 1 + t x_i$, for any $t \neq 0$, cannot be the prescribed Gaussian curvature on $\mathbb{S}^2$.

For open surfaces, without restricting to the conformal class, some results have been established in [24] by Kazdan and Warner. In particular, Theorem 4.1 in [24] gives a necessary and sufficient condition for a smooth function on $\mathbb{R}^2$ to be the prescribed Gaussian curvature of a complete Riemannian metric. However, restricting to the conformal class, the situation becomes very subtle. In this paper, we focus on the conformal metrics of Euclidean space $\mathbb{R}^n$, where $n \geq 2$ is an even integer. It is better to start from the

Mathematics Subject Classification 2020: 58J90 (primary); 53C18 (secondary).
Keywords: prescribed Q-curvature, complete metric, conformally flat.

two dimensional case. Given a smooth function $f(x)$ on $\mathbb{R}^2$, we consider the following conformally invariant equation:

$$-\Delta u(x) = f(x)\, e^{2u(x)}, \quad x \in \mathbb{R}^2. \tag{1.3}$$

Indeed, via the stereographic projection, equation (1.1) can be transformed into (1.3). There are many works devoted to this equation (1.3), see, e.g., [9–12, 18, 25, 34, 36, 38, 39]. In particular, the case $f(x) \le 0$ is well understood, see [9, 12, 22, 36, 38].

In this paper, the completeness of the metrics will be taken into account. Under such geometric restriction, Cohn–Vossen [14] and Huber [17] gave a control of the Gaussian curvature integral. For the readers' convenience, a simplified version of their results can be stated as follows. Throughout this paper, φ^+ and φ^- denote the positive part and negative part of function φ, respectively.

Theorem 1.1 (Cohn–Vossen [14], Huber [17]). *Consider a complete metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^2$. If the negative part of its Gaussian curvature K_g is integrable in $(\mathbb{R}^2, e^{2u}|dx|^2)$, i.e., if*

$$\int_{\mathbb{R}^2} K_g^-\, e^{2u}\, \mathrm{d}x < +\infty,$$

then there holds

$$\int_{\mathbb{R}^2} K_g\, e^{2u}\, \mathrm{d}x \le 2\pi.$$

For higher dimensional cases $n \ge 4$ and a conformal metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^n$, the Q-curvature with respect to such metric satisfies the following conformally invariant equation:

$$(-\Delta)^{n/2} u(x) = Q_g(x)\, e^{nu(x)}, \quad x \in \mathbb{R}^n. \tag{1.4}$$

We say that the conformal metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^n$ has finite total Q-curvature if

$$\int_{\mathbb{R}^n} |Q_g|\, e^{nu}\, \mathrm{d}x < +\infty.$$

As in the two dimensional case, equation (1.4) also comes from the standard sphere through a stereographic projection. Concerning the prescribed Q-curvature on the standard sphere $\mathbb{S}^n$, one may refer to [2, 8, 21, 31, 43] for more details. From the analytic point of view, equation (1.4) is studied in [19, 25, 29, 33, 42]. From the geometric point of view, similar to Theorem 1.1, the Q-curvature integral is bounded from above under suitable geometric assumptions.

Theorem 1.2 (Chang–Qing–Yang [5], Fang [15], Ndiaye–Xiao [35]). *Consider a complete conformal metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^n$, where $n \ge 4$ is an even integer, with finite total Q-curvature. If the scalar curvature satisfies $R_g \ge 0$ near infinity, then*

$$\int_{\mathbb{R}^n} Q_g\, e^{nu}\, \mathrm{d}x \le \frac{(n-1)!\, |\mathbb{S}^n|}{2},$$

where $|\mathbb{S}^n|$ denotes the volume of standard sphere $\mathbb{S}^n$.

For more related results, we refer to [4, 26, 30, 41], and references therein.

Regarding equation (1.4), a natural question is for what kind of prescribed functions $f(x)$ on $\mathbb{R}^n$ we can find a complete conformal metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^n$ such that $Q_g = f$. Are there obstructions like the Kazdan–Warner identity (1.2)? To explore this question, it is better to start with the famous Bonnet–Mayer theorem, which shows that for a complete manifold (M^n, g_b), if the Ricci curvature is such that $\mathrm{Ric}_{g_b} \geq (n-1)g_b$, then (M, g_b) is compact. We refer to Chapter 6 of [37] for more details. Bonnet–Mayer's theorem poses an obvious obstruction for $n = 2$.

Theorem 1.3 (Bonnet–Mayer's theorem). *Given a smooth function $f(x) \geq 1$ on $\mathbb{R}^2$, there is no complete conformal metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^2$ such that its Gaussian curvature satisfies $K_g = f$.*

Firstly, we generalize this result to higher dimensions.

Theorem 1.4. *Given a smooth function $f(x)$ on $\mathbb{R}^n$, where $n \geq 2$ is an even integer, such that $f(x) \geq 1$ near infinity, there is no complete conformal metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^n$ with finite total Q-curvature such that its Q-curvature satisfies $Q_g = f$.*

One may ask whether $f \geq 1$ near infinity is a sharp barrier for the existence of complete conformal metrics and what kind of behaviors occur if $f(x)$ tends to zero near infinity. More precisely, consider a function f satisfying

$$|f(x)| \leq C(|x|+1)^{-s}, \quad s > 0, \tag{1.5}$$

where C is a positive constant which may be different from line to line throughout the paper. When $f(x)$ is positive somewhere and satisfies (1.5), the existence of solutions to the equation

$$(-\Delta)^{n/2} u(x) = f(x)\, e^{nu(x)}, \quad x \in \mathbb{R}^n, \tag{1.6}$$

has been established in Theorem 1 of [34] for $n = 2$ and in Theorem 2.1 of [3] for $n \geq 4$. Taking the completeness of metrics into account, Aviles [1] studied equation (1.6) for $n = 2$ and showed that, for f positive somewhere and $s \geq 2$ in (1.5), there exists a complete conformal metric. Besides, for $0 < s < 1$, if f satisfies

$$\lim_{|x|\to\infty} f(x)\,|x|^s = 1, \tag{1.7}$$

Aviles claimed also that there exists a complete metric (See Theorem A[1] in [1]). However, Cheng and Lin constructed a family of functions $f(x)$ satisfying (1.7) (see Theorem 1.1 in [13]) to show the non-existence of complete conformal metric, contradicting Aviles' claim. The example of Cheng and Lin tells us that the complete conformal metric has some obstructions, even if the given function f satisfies (1.5). Inspired by this, we shall provide another barrier.

The Kazdan–Warner identity (1.2) establishes an obstruction for the prescribed Q-curvature on $\mathbb{S}^n$. One could ask whether there are some other barriers from this perspective. The Kazdan–Warner identity on $\mathbb{R}^n$ is known as Pohozaev's identity, see, for instance, [11, 12, 25, 28, 44]. We provide an obstruction for the existence of complete conformal metric from this point of view.

Theorem 1.5. *Let $f(x)$ be a positive and smooth function on $\mathbb{R}^n$, where $n \geq 2$ is an even integer, such that*

$$\frac{x \cdot \nabla f(x)}{f(x)} \geq -\frac{n}{2}. \tag{1.8}$$

Then there is no complete conformal metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^n$ with finite total Q-curvature such that its Q-curvature satisfies $Q_g = f(x)$.

Remark 1.6. We will show that condition (1.8) is sharp to a certain extent in Section 4.

For non-positive functions, we also obtain a barrier.

Theorem 1.7. *Let $f(x)$ be a non-positive and smooth function on $\mathbb{R}^n$, where $n \geq 2$ is an even integer, such that*

$$f(x) \leq -C|x|^{-n}, \quad |x| \gg 1. \tag{1.9}$$

Then there is no complete conformal metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^n$ with finite total Q-curvature such that its Q-curvature satisfies $Q_g = f(x)$.

Remark 1.8. In fact, without completeness, the conclusion still holds for $n = 2$ by the result of Sattinger [38]. However, for $n \geq 4$ and $f(x) \equiv -1$, a result of Martinazzi [32] showed the existence of non-complete conformal metrics.

Furthermore, if the prescribed function may change sign, another barrier is established.

Theorem 1.9. *Let $f(x)$ be a smooth function on $\mathbb{R}^n$, where $n \geq 2$ is an even integer, such that*

$$f(x) \leq -C|x|^s, \quad \text{for } |x| \gg 1 \text{ and } s > 0. \tag{1.10}$$

Then there is no complete conformal metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^n$ with finite total Q-curvature such that its Q-curvature satisfies $Q_g = f(x)$.

Now, we briefly introduce the structure of this paper. In Section 2, some results established in [26] are reviewed for later use. Subsequently, we prove Theorems 1.4, 1.5, 1.7 and 1.9 in Section 3. Finally, the sharpness of condition (1.8) is discussed.

2. Integral estimates and Pohozaev's identity

Lemma 2.1. *Let $f(x)$ be a positive and smooth function on $\mathbb{R}^n$ such that, for $|x| \gg 1$ and $s \in \mathbb{R}$,*

$$\frac{x \cdot \nabla f(x)}{f(x)} \geq s.$$

There holds

$$f(x) \geq c_0|x|^s, \quad \text{for } |x| \gg 1,$$

where c_0 is a positive constant.

Proof. By assumption, there exists $t_1 > 0$ such that, for $|x| \geq t_1$,

$$\frac{x \cdot \nabla f(x)}{f(x)} \geq s.$$

Using this, one has, for $|x| > t_1$,

$$\log f(x) - \log f\Big(\frac{t_1}{|x|}x\Big) = \int_{t_1}^{|x|} \frac{x}{|x|} \cdot \nabla \log f\Big(t\,\frac{x}{|x|}\Big)\,\mathrm{d}t$$
$$= \int_{t_1}^{|x|} \Big(t\,\frac{x}{|x|} \cdot \nabla \log f\Big(t\,\frac{x}{|x|}\Big)\Big)\,\frac{1}{t}\,\mathrm{d}t \geq \int_{t_1}^{|x|} s\,\frac{1}{t}\,\mathrm{d}t = s\log|x| - s\log t_1,$$

which yields

$$f(x) \geq t_1^{-s}\Big(\min_{|y|=t_1} f(y)\Big)|x|^s, \quad |x| > t_1.$$

Thus, we finish the proof. ∎

Recall the conformally invariant equation

$$(-\Delta)^{n/2}u(x) = f(x)\,e^{nu(x)}, \quad x \in \mathbb{R}^n. \tag{2.1}$$

We say that a solution to (2.1) is normal if u satisfies the integral equation

$$u(x) = \frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{\mathbb{R}^n} \log\frac{|y|}{|x-y|}\,f(y)\,e^{nu(y)}\,\mathrm{d}y + C_0, \tag{2.2}$$

where C_0 is a constant. For more details about normal solutions, see Section 2 of [26].

Given a function $\varphi(x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, we can define the logarithmic potential

$$\mathcal{L}(\varphi)(x) := \frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{\mathbb{R}^n} \log\frac{|y|}{|x-y|}\,\varphi(y)\,\mathrm{d}y.$$

For brevity, we define

$$\alpha := \frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{\mathbb{R}^n} \varphi(y)\,\mathrm{d}y.$$

Further, $B_r(p)$ denotes the Euclidean ball with radius r centered at $p \in \mathbb{R}^n$ and $|B_r(p)|$ denotes its volume respect to the standard Euclidean metric.

The following lemmas register some properties of $\mathcal{L}(\varphi)$ established in [26]; we include the proofs for the readers' convenience.

Lemma 2.2. *For* $|x| \gg 1$,

$$\mathcal{L}(\varphi)(x) = (-\alpha + o(1))\log|x| + \frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{B_1(x)} \log\frac{1}{|x-y|}\,\varphi(y)\,\mathrm{d}y,$$

where $o(1) \to 0$ *as* $|x| \to \infty$.

Proof. Choose $|x|$ big enough so that $\log(2\log|x|) > 0$, and split $\mathbb{R}^n$ into three pieces,

$$A_1 = B_1(x), \quad A_2 = B_{\log|x|}(0), \quad \text{and} \quad A_3 = \mathbb{R}^n \backslash (A_1 \cup A_2).$$

For $y \in A_2$ and $|y| \geq 2$, we have

$$\Big| \log \frac{|x| \cdot |y|}{|x-y|} \Big| \leq \log(2\log|x|).$$

Respectively, for $|y| \leq 2$,

$$\Big| \log \frac{|x| \cdot |y|}{|x-y|} \Big| \leq |\log|y|| + C.$$

Thus

$$\Big| \int_{A_2} \log \frac{|y|}{|x-y|}\, \varphi(y)\, \mathrm{d}y + \log|x| \int_{A_2} \varphi(y)\, \mathrm{d}y \Big| \leq C \log\log|x| + C = o(1)\log|x|.$$

For $y \in A_3$, it is not hard to check that

$$\frac{1}{|x|+1} \leq \frac{|y|}{|x-y|} \leq |x| + 1.$$

With this estimate, we bound the integral over A_3 as follows:

$$\Big| \int_{A_3} \log \frac{|y|}{|x-y|}\, \varphi(y)\, \mathrm{d}y \Big| \leq \log(|x|+1) \int_{A_3} |\varphi|\, \mathrm{d}y.$$

For $y \in B_1(x)$, one has $1 \leq |y| \leq |x| + 1$, and then

$$\Big| \int_{A_1} \log|y|\, \varphi\, \mathrm{d}y \Big| \leq \log(|x|+1) \int_{A_1} |\varphi|\, \mathrm{d}y.$$

Since $f \in L^1(\mathbb{R}^n)$, we have $\int_{A_3 \cup A_1} |\varphi|\, \mathrm{d}y \to 0$ as $|x| \to \infty$, and thus,

$$\frac{2}{(n-1)!\, |\mathbb{S}^n|} \int_{A_2} \varphi(y)\, \mathrm{d}y = \alpha + o(1).$$

This gives, finally,

$$\mathcal{L}(\varphi)(x) = (-\alpha + o(1))\log|x| + \frac{2}{(n-1)!\, |\mathbb{S}^n|} \int_{B_1(x)} \log \frac{1}{|x-y|}\, \varphi(y)\, \mathrm{d}y. \qquad \blacksquare$$

Lemma 2.3. *For* $0 < r_1 < 1$ *fixed and* $|x| \gg 1$,

$$\frac{1}{|B_{r_1|x|}(x)|} \int_{B_{r_1|x|}(x)} \mathcal{L}(\varphi)(y)\, \mathrm{d}y = (-\alpha + o(1))\log|x|.$$

Proof. By direct computations and Fubini's theorem, one has

$$\begin{aligned}
&\int_{B_{r_1|x|}(x)} \Big| \int_{B_1(z)} \log \frac{1}{|z-y|} \varphi(y)\, \mathrm{d}y \Big| \mathrm{d}z \le \int_{B_{r_1|x|}(x)} \int_{B_1(z)} \frac{1}{|z-y|} |\varphi(y)|\, \mathrm{d}y\, \mathrm{d}z \\
&\le \int_{B_{r_1|x|}(x)} \int_{B_{r_1|x|+1}(x)} \frac{1}{|z-y|} |\varphi(y)|\, \mathrm{d}y\, \mathrm{d}z \le \int_{B_{r_1|x|+1}(x)} |\varphi(y)| \int_{B_{r_1|x|}(x)} \frac{1}{|z-y|}\, \mathrm{d}z\, \mathrm{d}y \\
&\le \int_{B_{r_1|x|+1}(x)} |\varphi(y)| \int_{B_{2r_1|x|+1}(0)} \frac{1}{|z|}\, \mathrm{d}z\, \mathrm{d}y \le C|x|^{n-1}.
\end{aligned}$$

Thus

$$\frac{1}{|B_{r_1|x|}(x)|} \int_{B_{r_1|x|}(x)} \int_{B_1(z)} \log \frac{1}{|z-y|} \varphi(y)\, \mathrm{d}y\, \mathrm{d}z = O(|x|^{-1}). \tag{2.3}$$

For $y \in B_{r_1|x|}(x)$, there holds

$$\Big| \log \frac{|y|}{|x|} \Big| \le \log \frac{1}{1-r_1} + \log(1+r_1) \le C. \tag{2.4}$$

Using (2.3), (2.4) and Lemma 2.2, we conclude that

$$\frac{1}{|B_{r_1|x|}(x)|} \int_{B_{r_1|x|}(x)} \mathcal{L}(\varphi)(y)\, \mathrm{d}y = (-\alpha + o(1)) \log|x|. \qquad \blacksquare$$

Lemma 2.4. *If $\varphi \ge 0$ near infinity, we have, for $|x| \gg 1$,*

$$\mathcal{L}(\varphi)(x) \ge -\alpha \log|x| - C.$$

Proof. By a direct computation, we have

$$\begin{aligned}
&\frac{(n-1)!\,|\mathbb{S}^n|}{2} (\mathcal{L}(\varphi)(x) + \alpha \log|x|) \\
&\quad = \int_{\mathbb{R}^n} \log \frac{|x| \cdot (|y|+1)}{|x-y|} \varphi(y)\, \mathrm{d}y + \int_{\mathbb{R}^n} \log \frac{|y|}{|y|+1} \varphi(y)\, \mathrm{d}y \\
&\quad = \int_{\mathbb{R}^n} \log \frac{|x| \cdot (|y|+1)}{|x-y|} \varphi^+(y)\, \mathrm{d}y - \int_{\mathbb{R}^n} \log \frac{|x| \cdot (|y|+1)}{|x-y|} \varphi^-(y)\, \mathrm{d}y \\
&\qquad + \int_{\mathbb{R}^n} \log \frac{|y|}{|y|+1} \varphi(y)\, \mathrm{d}y.
\end{aligned}$$

For $|x| \ge 1$, it is easy to check that

$$\frac{|x| \cdot (|y|+1)}{|x-y|} \ge 1,$$

and thus,

$$\log \frac{|x| \cdot (|y|+1)}{|x-y|} \ge 0.$$

Immediately, one has

$$\int_{\mathbb{R}^n} \log \frac{|x| \cdot (|y|+1)}{|x-y|} \, \varphi^+(y) \, dy \geq 0.$$

By assumption, φ^- has compact support, and thus there exists $R_1 > 0$ such that $\operatorname{supp}(\varphi^-) \subset B_{R_1}(0)$. And for $|x| \geq 2R_1$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \log \frac{|x| \cdot (|y|+1)}{|x-y|} \, \varphi^-(y) \, dy &= \int_{B_{R_1}(0)} \log \frac{|x| \cdot (|y|+1)}{|x-y|} \, \varphi^-(y) \, dy \\ &\leq \log(2|R_1|+2) \int_{B_{R_1}(0)} \varphi^-(y) \, dy \leq C,\end{aligned}$$

where we use the fact that $|x|/|x-y| \leq 2$ for $|x| \geq 2R_1$ and $y \in B_{R_1}(0)$.

Since $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, one has

$$\begin{aligned}&\Big| \int_{\mathbb{R}^n} \log \frac{|y|}{|y|+1} \, \varphi(y) \, dy \Big| \\ &\leq \Big| \int_{B_2(0)} \log \frac{|y|}{|y|+1} \, \varphi(y) \, dy \Big| + \Big| \int_{\mathbb{R}^n \setminus B_2(0)} \log \frac{|y|}{|y|+1} \, \varphi(y) \, dy \Big| \\ &\leq C \int_{B_2(0)} |\log|y|| \, dy + C \int_{B_2(0)} |\log(|y|+1)| \, dy + \log \frac{3}{2} \int_{\mathbb{R}^n \setminus B_2(0)} |\varphi(y)| \, dy \leq C.\end{aligned}$$

Combining these estimates, for $|x| \gg 1$, we obtain that

$$\mathcal{L}(\varphi)(x) \geq -\alpha \log|x| - C. \qquad \blacksquare$$

Lemma 2.5. *For $R \gg 1$,*

$$\int_{B_R(0)} |\mathcal{L}(\varphi)(x)| \, dx = O((\log R) \cdot R^n).$$

Proof. A direct computation and Fubini's theorem yield that

$$\begin{aligned}\int_{B_R(0)} |\mathcal{L}(\varphi)| \, dx &\leq C \int_{B_R(0)} \int_{\mathbb{R}^n \setminus B_{2R}(0)} \Big| \log \frac{|y|}{|x-y|} \Big| \cdot |\varphi(y)| \, dy \, dx \\ &\quad + C \int_{B_R(0)} \int_{B_{2R}(0)} \Big| \log \frac{|y|}{|x-y|} \Big| \cdot |\varphi(y)| \, dy \, dx \\ &\leq C \int_{B_R(0)} \int_{\mathbb{R}^n \setminus B_{2R}(0)} \Big| \log \frac{|y|}{|x-y|} \Big| \cdot |\varphi(y)| \, dy \, dx \\ &\quad + C \int_{B_R(0)} \int_{B_{2R}(0)} |\log|y|| \cdot |\varphi(y)| \, dy \, dx \\ &\quad + C \int_{B_R(0)} \int_{B_{2R}(0)} |\log|x-y|| \cdot |\varphi(y)| \, dy \, dx.\end{aligned}$$

We deal with these three terms one by one. For $|x| \le R$ and $y \in \mathbb{R}^n \backslash B_{2R}(0)$, it is easy to verify that

$$\frac{1}{2} \le \frac{|y|}{|x-y|} \le 2.$$

Thus, the first term can be controlled as follows:

$$\begin{aligned}\int_{B_R(0)} \int_{\mathbb{R}^n \backslash B_{2R}(0)} \left| \log \frac{|y|}{|x-y|} \right| \cdot |\varphi(y)| \, \mathrm{d}y \, \mathrm{d}x &\le \log 2 \int_{B_R(0)} \int_{\mathbb{R}^n \backslash B_{2R}(0)} |\varphi(y)| \, \mathrm{d}y \, \mathrm{d}x \\ &\le CR^n.\end{aligned}$$

As for the second term, one has

$$\begin{aligned}&\int_{B_R(0)} \int_{B_{2R}(0)} |\log|y|| \cdot |\varphi(y)| \, \mathrm{d}y \, \mathrm{d}x \\ &\le \int_{B_R(0)} \int_{B_1(0)} |\log|y|| \cdot |\varphi(y)| \, \mathrm{d}y \, \mathrm{d}x + \int_{B_R(0)} \int_{B_{2R}(0) \backslash B_1(0)} |\log|y|| \cdot |\varphi(y)| \, \mathrm{d}y \, \mathrm{d}x \\ &\le CR^n \int_{B_1(0)} |\log|y|| \cdot |\varphi(y)| \, \mathrm{d}y + CR^n \int_{B_{2R}(0) \backslash B_1(0)} |\log|y|| \cdot |\varphi(y)| \, \mathrm{d}y \\ &\le CR^n + CR^n \log(2R) \int_{B_{2R}(0) \backslash B_1(0)} |\varphi(y)| \, \mathrm{d}y \le CR^n \log R.\end{aligned}$$

Finally, the last term can be dealt with using Fubini's theorem:

$$\begin{aligned}\int_{B_R(0)} \int_{B_{2R}(0)} |\log|x-y|| \cdot |\varphi(y)| \, \mathrm{d}y \, \mathrm{d}x &\le \int_{B_{2R}(0)} |\varphi(y)| \mathrm{d}y \int_{B_{3R}(0)} |\log|z|| \, \mathrm{d}z \\ &\le CR^n \log R.\end{aligned}$$

Combining these estimates, one has

$$\int_{B_R(0)} |\mathcal{L}(\varphi)(x)| \, \mathrm{d}x = O((\log R) \cdot R^n).$$ ■

We say the conformal metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^n$ with finite total Q-curvature is a normal metric if u is a normal solution to (1.4). To characterize the normal metric, a volume entropy $\tau(g)$ was introduced in [26]; it is defined as

$$\tau(g) := \limsup_{R \to \infty} \frac{\log \int_{B_R(0)} e^{nu} \, \mathrm{d}x}{\log |B_R(0)|}.$$

Theorem 2.6 (Theorem 1.1 in [26]). *Consider a complete metric $g = e^{2u}|dx|^2$ with finite total Q-curvature on $\mathbb{R}^n$, where $n \ge 2$ is an even integer. The metric g is normal if and only if $\tau(g)$ is finite. Moreover, if $\tau(g)$ is finite, then*

$$\tau(g) = 1 - \frac{2}{(n-1)! \, |\mathbb{S}^n|} \int_{\mathbb{R}^n} Q_g e^{nu} \, \mathrm{d}x.$$

In [26], the following geodesic distance $d_g(\cdot,\cdot)$ comparison identity was established; it will be used in Section 4 to show some metrics are complete.

Theorem 2.7 (Theorem 1.4 in [26]). *Consider a conformal metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^n$ with finite total Q-curvature, where $n \geq 2$ is an even integer. Assume that the metric g is normal. Then, for each fixed point p,*

$$\lim_{|x|\to\infty} \frac{\log d_g(x,p)}{\log|x-p|} = \Big(1 - \frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{\mathbb{R}^n} Q_g\, e^{nu}\,\mathrm{d}x\Big)^+,$$

where, for a constant c, c^+ is equal to c if $c \geq 0$, and is equal to 0 otherwise.

The following Pohozaev-type inequality is inspired by the work of Xu (see Theorem 2.1 in [44]). One may also refer to [25] and Lemma 3.1 in [28].

Lemma 2.8. *Suppose that $u(x)$ is a smooth solution to the integral equation*

$$u(x) = \frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{\mathbb{R}^n} \log\frac{|y|}{|x-y|}\, Q(y)\, e^{nu(y)}\,\mathrm{d}y + C_0,$$

where C_0 is a constant, $Q e^{nu} \in L^1(\mathbb{R}^n)$ and the smooth function $Q(x)$ does not change sign near infinity. Then there exists a sequence $R_i \to \infty$ such that

$$\limsup_{i\to\infty} \frac{4}{n!\,|\mathbb{S}^n|}\int_{B_{R_i}(0)} x\cdot\nabla Q\, e^{nu}\,\mathrm{d}x \leq \alpha_0(\alpha_0 - 2),$$

where α_0 denotes the normalized Q-curvature integral:

$$\alpha_0 := \frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{\mathbb{R}^n} Q\, e^{nu}\,\mathrm{d}x.$$

Proof. A direct computation gives

$$\langle x, \nabla u\rangle = -\frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{\mathbb{R}^n} \frac{\langle x, x-y\rangle}{|x-y|^2}\, Q(y)\, e^{nu(y)}\,\mathrm{d}y$$

Multiplying by $Q e^{nu(x)}$ and integrating over the ball $B_R(0)$ for any $R > 0$, we obtain

$$\begin{aligned}\int_{B_R(0)} Q e^{nu(x)}\Big[-\frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{\mathbb{R}^n}\frac{\langle x, x-y\rangle}{|x-y|^2}\, Q(y) e^{nu(y)}\,\mathrm{d}y\Big]\,\mathrm{d}x \\ = \int_{B_R(0)} Q e^{nu(x)}\langle x, \nabla u(x)\rangle\,\mathrm{d}x.\end{aligned} \tag{2.5}$$

Using $x = \frac{1}{2}((x+y) + (x-y))$, for the left-hand side of (2.5), one has the following identity:

$$\begin{aligned}\text{l.h.s of (2.5)} &= \frac{1}{2}\int_{B_R(0)} Q e^{nu(x)}\Big[-\frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{\mathbb{R}^n} Q e^{nu(y)}\,\mathrm{d}y\Big]\,\mathrm{d}x \\ &+ \frac{1}{2}\int_{B_R(0)} Q e^{nu(x)}\Big[-\frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{\mathbb{R}^n}\frac{\langle x+y, x-y\rangle}{|x-y|^2}\, Q e^{nu(y)}\,\mathrm{d}y\Big]\,\mathrm{d}x.\end{aligned}$$

Now, we deal with the last term of above equation by changing variables x and y:

$$\begin{aligned}
&\int_{B_R(0)} Q(x)\, e^{nu(x)} \Big[\int_{\mathbb{R}^n} \frac{\langle x+y, x-y\rangle}{|x-y|^2}\, Q(y) e^{nu(y)}\, \mathrm{d}y \Big] \mathrm{d}x \\
&\quad = \int_{B_R(0)} Q(x)\, e^{nu(x)} \Big[\int_{\mathbb{R}^n \setminus B_R(0)} \frac{\langle x+y, x-y\rangle}{|x-y|^2}\, Q(y) e^{nu(y)}\, \mathrm{d}y \Big] \mathrm{d}x \\
&\quad = \int_{B_{R/2}(0)} Q(x)\, e^{nu(x)} \Big[\int_{\mathbb{R}^n \setminus B_R(0)} \frac{\langle x+y, x-y\rangle}{|x-y|^2}\, Q(y) e^{nu(y)}\, \mathrm{d}y \Big] \mathrm{d}x \\
&\qquad + \int_{B_R(0)\setminus B_{R/2}(0)} Q(x)\, e^{nu(x)} \Big[\int_{\mathbb{R}^n \setminus B_{2R}(0)} \frac{\langle x+y, x-y\rangle}{|x-y|^2}\, Q(y) e^{nu(y)}\, \mathrm{d}y \Big] \mathrm{d}x \\
&\qquad + \int_{B_R(0)\setminus B_{R/2}(0)} Q(x)\, e^{nu(x)} \Big[\int_{B_{2R(0)} \setminus B_R(0)} \frac{\langle x+y, x-y\rangle}{|x-y|^2}\, Q(y) e^{nu(y)}\, \mathrm{d}y \Big] \mathrm{d}x \\
&\quad =: I_1(R) + I_2(R) + I_3(R).
\end{aligned}$$

For $x \in B_{R/2}(0)$ and $y \in \mathbb{R}^n \setminus B_R(0)$, one has

$$\Big| \frac{\langle x+y, x-y\rangle}{|x-y|^2} \Big| \le \frac{|x+y|}{|x-y|} \le 3,$$

and thus,

$$|I_1(R)| \le 3 \int_{B_{R/2}(0)} |Q(x)|\, e^{nu(x)}\, \mathrm{d}x \int_{\mathbb{R}^n \setminus B_R(0)} |Q(y)|\, e^{nu(y)}\, \mathrm{d}y.$$

Similarly,

$$|I_2(R)| \le 3 \int_{B_R(0)\setminus B_{R/2}(0)} |Q(x)|\, e^{nu(x)}\, \mathrm{d}x \int_{\mathbb{R}^n \setminus B_{2R}(0)} |Q(y)|\, e^{nu(y)}\, \mathrm{d}y.$$

Then both $|I_1(R)|$ and $|I_2(R)|$ tend to zero as $R \to \infty$, because $Q e^{nu} \in L^1(\mathbb{R}^n)$.

For the term $I_3(R)$, and since Q does not change sign near infinity, for $R \gg 1$, one has

$$I_3(R) = \int_{B_R(0)\setminus B_{R/2}(0)} Q(x)\, e^{nu(x)} \Big[\int_{B_{2R(0)} \setminus B_R(0)} \frac{x^2 - y^2}{|x-y|^2}\, Q(y)\, e^{nu(y)}\, \mathrm{d}y \Big] \mathrm{d}x \le 0.$$

As for the right-hand side of (2.5), by using the divergence theorem, we have

$$\begin{aligned}
\text{r.h.s of (2.5)} &= \frac{1}{n} \int_{B_R(0)} Q(x) \langle x, \nabla e^{nu(x)} \rangle\, \mathrm{d}x \\
&= - \int_{B_R(0)} \Big(Q(x) + \frac{1}{n} \langle x, \nabla Q(x) \rangle \Big) e^{nu(x)}\, \mathrm{d}x + \frac{1}{n} \int_{\partial B_R(0)} Q(x)\, e^{nu(x)} R\, \mathrm{d}\sigma.
\end{aligned}$$

Since $Q(x)\, e^{nu(x)} \in L^1(\mathbb{R}^n)$, there exist a sequence $R_i \to \infty$ such that

$$\lim_{i \to \infty} R_i \int_{\partial B_{R_i}(0)} |Q| e^{nu}\, \mathrm{d}\sigma = 0.$$

Therefore,

$$\frac{1}{n}\int_{B_{R_i}(0)}\langle x\cdot\nabla Q\rangle e^{nu}\,\mathrm{d}x$$
$$= -\int_{B_{R_i}(0)} Q e^{nu}\,\mathrm{d}x + \frac{1}{n}R_i\int_{\partial B_{R_i}(0)} Q e^{nu}\,\mathrm{d}\sigma + \frac{\alpha_0}{2}\int_{B_{R_i}(0)} Q e^{nu(x)}\,\mathrm{d}x + I_1(R_i) + I_2(R_i) + I_3(R_i)$$
$$\leq -\int_{B_{R_i}(0)} Q e^{nu}\,\mathrm{d}x + \frac{1}{n}R_i\int_{\partial B_{R_i}(0)} Q e^{nu}\,\mathrm{d}\sigma + \frac{\alpha_0}{2}\int_{B_{R_i}(0)} Q e^{nu(x)}\,\mathrm{d}x + I_1(R_i) + I_2(R_i),$$

which yields

$$\limsup_{i\to\infty}\frac{4}{n!\,|\mathbb{S}^n|}\int_{B_{R_i}(0)} x\cdot\nabla Q e^{nu}\,\mathrm{d}x \leq \alpha_0(\alpha_0-2). \qquad \blacksquare$$

3. Proofs

Throughout our proofs, we will argue by contradiction and suppose that there exists a complete conformal metric

$$g = e^{2u}|dx|^2$$

with finite total Q-curvature such that $Q_g = f$ which satisfies

$$(-\Delta)^{n/2}u = fe^{nu}$$

with $fe^{nu}\in L^1(\mathbb{R}^n)$. For brevity, set

$$\beta = \frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{\mathbb{R}^n} fe^{nu}\,\mathrm{d}x.$$

Proof of Theorem 1.4. Since $f\geq 1$ near infinity and $fe^{nu}\in L^1(\mathbb{R}^n)$, one has $e^{nu}\in L^1(\mathbb{R}^n)$, from which we deduce that $\tau(g)=0$. Thanks to Theorem 2.6, we have that the metric g is normal and

$$\frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{\mathbb{R}^n} fe^{nu}\,\mathrm{d}x = 1. \tag{3.1}$$

Since u is normal and $f\geq 1$ near infinity, Lemma 2.4 and (3.1) yield that, for $|y|\gg 1$,

$$u(y) = \mathcal{L}(fe^{nu})(y) + C \geq -\log|y| - C.$$

In particular, for $|x|\gg 1$ and any $y\in B_{|x|/2}(x)$, there holds

$$u(y)\geq -\log\left(\frac{3}{2}|x|\right) - C = -\log|x| - C. \tag{3.2}$$

For $|x| \gg 1$, Jensen's inequality, the estimate (3.2) and the fact that $f \geq 1$ near infinity show that

$$\begin{aligned}\int_{B_{|x|/2}(x)} fe^{nu}\,\mathrm{d}y &\geq \int_{B_{|x|/2}(x)} e^{nu}\,\mathrm{d}y \\ &\geq |B_{|x|/2}(x)| \exp\Big(\frac{1}{|B_{|x|/2}(x)|}\int_{B_{|x|/2}(x)} nu(y)\,\mathrm{d}y\Big) \geq C|x|^n \cdot |x|^{-n} \geq C,\end{aligned}$$

which contradicts $fe^{nu} \in L^1(\mathbb{R}^n)$. This finishes the proof. ■

Proof of Theorem 1.5. Due to the condition (1.8), Lemma 2.1 yields

$$f(x) \geq C|x|^{-n/2}, \quad |x| > 1. \tag{3.3}$$

Since $fe^{nu} \in L^1(\mathbb{R}^n)$, the estimate (3.3) shows that for $R > 1$,

$$\begin{aligned}\int_{B_R(0)} e^{nu}\,\mathrm{d}x &\leq \int_{B_1(0)} e^{nu}\,\mathrm{d}x + \int_{B_R(0)\backslash B_1(0)} e^{nu}\,\mathrm{d}x \\ &\leq C + C\int_{B_R(0)\backslash B_1(0)} f(x)|x|^{n/2} e^{nu(x)}\,\mathrm{d}x \\ &\leq C + CR^{n/2}\int_{B_R(0)\backslash B_1(0)} f(x) e^{nu(x)}\,\mathrm{d}x \leq CR^{n/2},\end{aligned}$$

which gives that

$$\tau(g) \leq \frac{1}{2}.$$

Theorem 2.6 shows that u is normal and

$$\beta = 1 - \tau(g) \leq 1, \tag{3.4}$$

where we have used the fact $\tau(g) \geq 0$. By the definition of $\tau(g)$, one has

$$\tau(g) \geq \lim_{R\to\infty} \inf \frac{\log \int_{B_1(0)} e^{nu}\,\mathrm{d}x}{\log |B_R(0)|} = 0. \tag{3.5}$$

By condition (1.8), one has

$$\begin{aligned}\frac{4}{n!\,|\mathbb{S}^n|}\int_{B_R(0)} x\cdot\nabla fe^{nu}\,\mathrm{d}x &\geq \frac{4}{n!\,|\mathbb{S}^n|}\int_{B_R(0)} -\frac{n}{2} fe^{nu}\,\mathrm{d}x \\ &= -\frac{2}{(n-1)!\,|\mathbb{S}^n|}\int_{B_R(0)} fe^{nu}\,\mathrm{d}x.\end{aligned}$$

Thanks to Lemma 2.8 and choosing a suitable sequence, we get

$$\beta(\beta - 2) \geq -\beta,$$

which yields

$$\beta \geq 1,$$

where we have used the fact $\beta > 0$ since $f > 0$.

If the equality holds, one must have

$$x \cdot \nabla f(x) = -\frac{n}{2} f(x), \quad \text{a.e.},$$

which is impossible by choosing sufficiently small $\delta > 0$ such that

$$x \cdot \nabla f(x) + \frac{n}{2} f(x) > 0$$

for $x \in B_\delta(0)$. Hence we obtain that

$$\beta > 1,$$

which contradicts (3.4). The proof is complete. ∎

Proof of Theorem 1.7. By assumption (1.9), there exists $R_1 > 0$ such that for $|x| \geq R_1$,

$$|f(x)| \geq C|x|^{-n}.$$

For $R > R_1 + 1$, there holds

$$\begin{aligned}
\int_{B_R(0)} e^{nu} \, dx &= \int_{B_{R_1}(0)} e^{nu} \, dx + \int_{B_R(0)\backslash B_{R_1}(0)} e^{nu} \, dx \\
&\leq C - C \int_{B_R(0)\backslash B_{R_1}(0)} f(x)|x|^n e^{nu} \, dx \\
&\leq C - CR^n \int_{B_R(0)\backslash B_{R_1}(0)} f(x) e^{nu} \, dx \leq CR^n,
\end{aligned}$$

which yields that $\tau(g)$ is finite. Using Theorem 2.6, we show that $u(x)$ is normal satisfying

$$u(x) = \mathcal{L}(fe^{nu}) + C.$$

Now, by Jensen's inequality and Lemma 2.3, for $|x| \gg 1$, one has

$$\begin{aligned}
\int_{B_{|x|/2}(x)} |f(y)|e^{nu(y)} \, dy &\geq C \int_{B_{|x|/2}(x)} |y|^{-n} e^{nu(y)} \, dy \\
&\geq C|x|^{-n} \int_{B_{|x|/2}(x)} e^{n\mathcal{L}(fe^{nu})(y)} \, dy \\
&\geq C|x|^{-n}|B_{|x|/2}(x)| \exp\left(\frac{1}{|B_{|x|/2}(x)|} \int_{B_{|x|/2}(x)} n\mathcal{L}(fe^{nu})(y) \, dy\right) \\
&\geq C|x|^{-n} \cdot |x|^n \cdot |x|^{-n\beta + o(1)}.
\end{aligned}$$

From $fe^{nu} \in L^1(\mathbb{R}^n)$, we deduce that

$$\beta \geq 0. \tag{3.6}$$

However, since $f \leq 0$ satisfies (1.9), we must have $\beta < 0$, which contradicts (3.6). ∎

Proof of Theorem 1.9. By the assumption (1.10) and the fact that $fe^{nu} \in L^1(\mathbb{R}^n)$, and as in the proof of Theorem 1.7, one has $\tau(g)$ is finite and then $u(x)$ is normal.

Jensen's inequality and Lemma 2.3 yield, for $|x| \gg 1$,

$$\begin{aligned}\int_{B_{|x|/2}(x)} |f(y)|e^{nu(y)}\,\mathrm{d}y &\geq C\int_{B_{|x|/2}(x)} |y|^s e^{nu(y)}\,\mathrm{d}y \geq C|x|^s \int_{B_{|x|/2}(x)} e^{n\mathcal{L}(fe^{nu})(y)}\,\mathrm{d}y \\ &\geq C|x|^s|B_{|x|/2}(x)|\exp\Big(\frac{1}{|B_{|x|/2}(x)|}\int_{B_{|x|/2}(x)} n\mathcal{L}(fe^{nu})(y)\,\mathrm{d}y\Big) \\ &\geq C|x|^s\cdot|x|^n\cdot|x|^{-n\beta+o(1)},\end{aligned}$$

which yields

$$\beta \geq 1+\frac{s}{n} > 1.$$

However, from Theorem 2.6, one deduces that $\beta \leq 1$ due to (3.5), which is the desired contradiction. ∎

4. Sharp decay rate for f

With help of Lemma 2.1, the condition (1.8) deduces that

$$f(x) \geq C|x|^{-n/2},\ |x| > 1.$$

For $n = 2$, the results in [13, 18, 34, 40] ensure that for a positive function $f(x)$ satisfying

$$f(x) \leq C|x|^{-l}, |x| \gg 1,\ l > 0,$$

the solutions to equation (1.3) exist. We state now a simplified version of McOwen's result [34].

Theorem 4.1 (McOwen, [34]). *Let f be a smooth function which is positive somewhere and satisfies*

$$f(x) = O(|x|^{-l}),$$

where $l > 0$, $|x| \gg 1$ and C is a positive constant. Then for any $\alpha \in (\max\{0, 2-l\}, 2)$, there exist a solution $u(x)$ to the equation

$$-\Delta u = fe^{2u}, \quad \text{on } \mathbb{R}^2$$

satisfying

$$\alpha = \frac{1}{2\pi}\int_{\mathbb{R}^2} fe^{2u}\,\mathrm{d}x.$$

Theorem 4.2. *Consider a positive and smooth function f on $\mathbb{R}^2$ satisfying*

$$C^{-1}|x|^{-l} \leq f(x) \leq C|x|^{-l}, \tag{4.1}$$

where $l > 1$, $|x| \gg 1$ and where C is a positive constant. Then there exists a complete metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^2$ with finite total curvature such that its Gaussian curvature satisfies $K_g = f$.

Proof. Thanks to Theorem 4.1 and the fact that $l > 1$, we can find a solution $u(x)$ to the equation

$$-\Delta u = f e^{2u} \tag{4.2}$$

satisfying

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} f e^{2u}\,\mathrm{d}x < 1. \tag{4.3}$$

By (4.1) and (4.3), one has, for $R \gg 1$,

$$\int_{B_R(0)} e^{2u}\,\mathrm{d}x \le CR^l.$$

Now, using that

$$\frac{2}{l+4}u^+ \le e^{\frac{2}{l+4}u}$$

and Hölder's inequality, one has

$$\begin{aligned}\int_{B_R(0)} u^+(x)\,\mathrm{d}x &\le \frac{l+4}{2}\int_{B_R(0)} e^{\frac{2}{l+4}u}\,\mathrm{d}x\\ &\le \frac{l+4}{2}\Big(\int_{B_R(0)} e^{2u}\,\mathrm{d}x\Big)^{1/(l+4)}|B_R(0)|^{\frac{l+3}{l+4}} \le CR^{\frac{l}{l+4}}R^{\frac{2(l+3)}{l+4}} = CR^{3-\frac{6}{l+4}},\end{aligned}$$

which yields

$$\frac{1}{|B_R(0)|}\int_{B_R(0)} u^+\,\mathrm{d}x = o(R). \tag{4.4}$$

Set

$$v(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\frac{|y|}{|x-y|}f(y)e^{2u(y)}\,\mathrm{d}y \quad \text{and} \quad P(x) := u - v$$

satisfying

$$\Delta P = 0. \tag{4.5}$$

Using Lemma 2.5, there holds, for $R \gg 1$,

$$\frac{1}{|B_R(0)|}\int_{B_R(0)} |v(x)|\,\mathrm{d}x = O(\log R) = o(R). \tag{4.6}$$

Combing the estimate (4.4) with (4.6), one has

$$\frac{1}{|B_R(0)|}\int_{B_R(0)} P^+(x)\,\mathrm{d}x \le \frac{1}{|B_R(0)|}\int_{B_R(0)}\big(u^+(x) + |v(x)|\big)\,\mathrm{d}x = o(R). \tag{4.7}$$

By Liouville's theorem, (4.5) and (4.7), we deduce that

$$P(x) \equiv C.$$

Thus $u(x)$ is a normal solution to (4.2). By Theorem 2.7,

$$\lim_{|x|\to\infty} \frac{\log d_g(x,p)}{\log|x-p|} = 1 - \frac{1}{2\pi}\int_{\mathbb{R}^2} fe^{2u}\,\mathrm{d}x > 0,$$

which yields that the conformal metric g is complete (see Theorem 5.7.1 in [37]). ∎

For $n \geq 4$, a result analogous to Theorem 4.1 might still hold by using methods from [34] or [3]. A crucial ingredient would be Proposition 1 in [34], which is a singular type Moser–Trudinger inequality, also established in Theorem 6 of [40]. For higher dimensional cases, such a singular type Adams–Moser–Trudinger inequality has been established in Theorem 4.6 of [16] and Theorem 2.4 of [20]. With the help of such inequality, one may consider a suitable functional and find its minimizer, which would be a normal solution with finite total Q-curvature less than $(n-1)!\,|\mathbb{S}^n|/2$. Finally, using Theorem 2.7, one may show that such normal metric is complete.

However, this is only the general idea; implementing it and refining it is no easy task. We leave it as an open question.

Question 1. *Let $f(x)$ be a smooth function positive somewhere on $\mathbb{R}^n$, where $n \geq 2$ is an even integer, such that*

$$f(x) = O(|x|^{-s}), \quad s > \frac{n}{2}.$$

Is it true that there is a complete conformal metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^n$ with finite total Q-curvature such that its Q-curvature satisfies $Q_g = f(x)$?

Remark 4.3. After submitting this work, this question has been answered by the author in a joint work with Biao Ma, see [27].

Acknowledgements. The author would like to thank Professor Xingwang Xu, Dong Ye, Xia Huang, and Biao Ma for helpful discussions. The author is also deeply grateful to the referees for their valuable suggestions and comments.

References

[1] Aviles, P.: Conformal complete metrics with prescribed nonnegative Gaussian curvature in $\mathbb{R}^2$. *Invent. Math.* **83** (1986), no. 3, 519–544. Zbl 0592.53034 MR 0827365

[2] Brendle, S.: Convergence of the Q-curvature flow on S^4. *Adv. Math.* **205** (2006), no. 1, 1–32. Zbl 1104.53064 MR 2254307

[3] Chang, S.-Y. A. and Chen, W.: A note on a class of higher order conformally covariant equations. *Discrete Contin. Dynam. Systems* **7** (2001), no. 2, 275–281. Zbl 1014.35025 MR 1808400

[4] Chang, S.-Y. A., Qing, J. and Yang, P. C.: Compactification of a class of conformally flat 4-manifold. *Invent. Math.* **142** (2000), no. 1, 65–93. Zbl 0990.53026 MR 1784799

[5] Chang, S.-Y. A., Qing, J. and Yang, P. C.: On the Chern–Gauss–Bonnet integral for conformal metrics on $\mathbb{R}^4$. *Duke Math. J.* **103** (2000), no. 3, 523–544. Zbl 0971.53028 MR 1763657

[6] Chang, S.-Y. A. and Yang, P. C.: Prescribing Gaussian curvature on S^2. *Acta Math.* **159** (1987), no. 3-4, 215–259. Zbl 0636.53053 MR 0908146

[7] Chang, S.-Y. A. and Yang, P. C.: Conformal deformation of metrics on S^2. *J. Differential Geom.* **27** (1988), no. 2, 259–296. Zbl 0649.53022 MR 0925123

[8] Chang, S.-Y. A. and Yang, P. C.: Extremal metrics of zeta function determinants on 4-manifolds. *Ann. of Math. (2)* **142** (1995), no. 1, 171–212. Zbl 0842.58011 MR 1338677

[9] Chen, H., Ye, D. and Zhou, F.: On Gaussian curvature equation in $\mathbb{R}^2$ with prescribed nonpositive curvature. *Discrete Contin. Dyn. Syst.* **40** (2020), no. 6, 3201–3214. Zbl 1439.35508 MR 4097497

[10] Chen, W. X. and Li, C.: Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.* **63** (1991), no. 3, 615–622. Zbl 0768.35025 MR 1121147

[11] Chen, W. X. and Li, C.: Qualitative properties of solutions to some nonlinear elliptic equations in $\mathbb{R}^2$. *Duke Math. J.* **71** (1993), no. 2, 427–439. Zbl 0923.35055 MR 1233443

[12] Cheng, K.-S. and Lin, C.-S.: On the asymptotic behavior of solutions of the conformal Gaussian curvature equations in $\mathbb{R}^2$. *Math. Ann.* **308** (1997), no. 1, 119–139. Zbl 0871.35014 MR 1446203

[13] Cheng, K.-S. and Lin, C.-S.: Compactness of conformal metrics with positive Gaussian curvature in $\mathbb{R}^2$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **26** (1998), no. 1, 31–45. Zbl 0933.35056 MR 1633051

[14] Cohn-Vossen, S.: Kürzeste Wege und Totalkrümmung auf Flächen. *Compositio Math.* **2** (1935), 69–133. Zbl 0011.22501 MR 1556908

[15] Fang, H.: On a conformal Gauss–Bonnet–Chern inequality for LCF manifolds and related topics. *Calc. Var. Partial Differential Equations* **23** (2005), no. 4, 469–496. Zbl 1083.53035 MR 2153034

[16] Fang, H. and Ma, B.: Constant Q-curvature metrics on conic 4-manifolds. *Adv. Calc. Var.* **15** (2022), no. 2, 235–264. Zbl 1501.53042 MR 4399823

[17] Huber, A.: On subharmonic functions and differential geometry in the large. *Comment. Math. Helv.* **32** (1957), 13–72. Zbl 0080.15001 MR 0094452

[18] Hulin, D. and Troyanov, M.: Prescribing curvature on open surfaces. *Math. Ann.* **293** (1992), no. 2, 277–315. Zbl 0799.53047 MR 1166122

[19] Hyder, A., Mancini, G. and Martinazzi, L.: Local and nonlocal singular Liouville equations in Euclidean spaces. *Int. Math. Res. Not. IMRN* (2021), no. 15, 11393–11425. Zbl 1489.35139 MR 4294121

[20] Jevnikar, A., Sire, Y. and Yang, W.: Prescribing Q-curvature on even-dimensional manifolds with conical singularities. *Rev. Mat. Iberoam.* **41** (2025), no. 1, 1–28. Zbl 1565.53040 MR 4864995

[21] Jin, T., Li, Y. and Xiong, J.: On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions. *J. Eur. Math. Soc. (JEMS)* **16** (2014), no. 6, 1111–1171. Zbl 1300.53041 MR 3226738

[22] Kalka, M. and Yang, D.: On conformal deformation of nonpositive curvature on noncompact surfaces. *Duke Math. J.* **72** (1993), no. 2, 405–430. Zbl 0809.53039 MR 1248678

[23] Kazdan, J. L. and Warner, F. W.: Curvature functions for compact 2-manifolds. *Ann. of Math. (2)* **99** (1974), no. 1, 14–47. Zbl 0273.53034 MR 0343205

[24] Kazdan, J. L. and Warner, F. W.: Curvature functions for open 2-manifolds. *Ann. of Math. (2)* **99** (1974), no. 2, 203–219. Zbl 0278.53031 MR 0343206

[25] Li, M.: A Liouville-type theorem in conformally invariant equations. *Math. Ann.* **389** (2024), no. 3, 2499–2517. Zbl 1541.35171 MR 4753069

[26] Li, M.: The total Q-curvature, volume entropy and polynomial growth polyharmonic functions. *Adv. Math.* **450** (2024), article no. 109768, 43 pp. Zbl 1551.53032 MR 4754948

[27] Li, M. and Ma, B.: Existence of complete conformal metrics on $\mathbb{R}^n$ with prescribed Q-curvature. Preprint 2025, arXiv:2503.23689v1.

[28] Li, M. and Wei, J.: Higher order Bol's inequality and its applications. Preprint 2024, arXiv:2308.11388v2.

[29] Lin, C.-S.: A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$. *Comment. Math. Helv.* **73** (1998), no. 2, 206–231. Zbl 0933.35057 MR 1611691

[30] Lu, Z. and Wang, Y.: On locally conformally flat manifolds with finite total Q-curvature. *Calc. Var. Partial Differential Equations* **56** (2017), no. 4, article no. 98, 24 pp. Zbl 1373.53051 MR 3663637

[31] Malchiodi, A. and Struwe, M.: Q-curvature flow on S^4. *J. Differential Geom.* **73** (2006), no. 1, 1–44. Zbl 1099.53034 MR 2217518

[32] Martinazzi, L.: Conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **19** (2008), no. 4, 279–292. Zbl 1194.35154 MR 2465680

[33] Martinazzi, L.: Classification of solutions to the higher order Liouville's equation on $\mathbb{R}^{2m}$. *Math. Z.* **263** (2009), no. 2, 307–329. Zbl 1172.53026 MR 2534120

[34] McOwen, R. C.: Conformal metrics in $\mathbb{R}^2$ with prescribed Gaussian curvature and positive total curvature. *Indiana Univ. Math. J.* **34** (1985), no. 1, 97–104. Zbl 0546.53032 MR 0773395

[35] Ndiaye, C. B. and Xiao, J.: An upper bound of the total Q-curvature and its isoperimetric deficit for higher-dimensional conformal Euclidean metrics. *Calc. Var. Partial Differential Equations* **38** (2010), no. 1-2, 1–27. Zbl 1198.53009 MR 2610523

[36] Ni, W. M.: On the elliptic equation $\Delta u + K(x)e^{2u} = 0$ and conformal metrics with prescribed Gaussian curvatures. *Invent. Math.* **66** (1982), no. 2, 343–352. Zbl 0487.35042 MR 0656628

[37] Petersen, P.: *Riemannian geometry*. Third edition. Grad. Texts in Math. 171, Springer, Cham, 2016. Zbl 1417.53001 MR 3469435

[38] Sattinger, D. H.: Conformal metrics in $\mathbb{R}^2$ with prescribed curvature. *Indiana Univ. Math. J.* **22** (1972/73), 1–4. Zbl 0236.53009 MR 0305307

[39] Struwe, M.: "Bubbling" of the prescribed curvature flow on the torus. *J. Eur. Math. Soc. (JEMS)* **22** (2020), no. 10, 3223–3262. Zbl 1458.35151 MR 4153107

[40] Troyanov, M.: Prescribing curvature on compact surfaces with conical singularities. *Trans. Amer. Math. Soc.* **324** (1991), no. 2, 793–821. Zbl 0724.53023 MR 1005085

[41] Wang, Y.: The isoperimetric inequality and Q-curvature. *Adv. Math.* **281** (2015), 823–844. Zbl 1323.52008 MR 3366853

[42] Wei, J. and Xu, X.: Classification of solutions of higher order conformally invariant equations. *Math. Ann.* **313** (1999), no. 2, 207–228. Zbl 0940.35082 MR 1679783

[43] Wei, J. and Xu, X.: Prescribing Q-curvature problem on $\mathbb{S}^n$. *J. Funct. Anal.* **257** (2009), no. 7, 1995–2023. Zbl 1172.58005 MR 2548028

[44] Xu, X.: Uniqueness and non-existence theorems for conformally invariant equations. *J. Funct. Anal.* **222** (2005), no. 1, 1–28. Zbl 1069.53013 MR 2129763

Received February 26, 2024; revised August 3, 2025.

Mingxiang Li
Department of Mathematics & Institute of Mathematical Sciences, The Chinese University of Hong Kong
Shatin, NT 999077 Hong Kong, P. R. China;
mingxiangli@cuhk.edu.hk