Rev. Mat. Iberoam. 42 (2026), no. 1, 123–186
DOI 10.4171/RMI/1564

Torsion-free modules over commutative domains of Krull dimension one

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Abstract. Let R be a domain of Krull dimension one. We study when the class $\mathcal{F}$ of modules over R that are arbitrary direct sums of finitely generated torsion-free modules is closed under direct summands. If R is local, we show that $\mathcal{F}$ is closed under direct summands if and only if any indecomposable, finitely generated, torsion-free module has local endomorphism ring. If, in addition, R is noetherian, this is equivalent to saying that the normalization of R is a local ring. If R is an h-local domain of Krull dimension 1 and $\mathcal{F}_R$ is closed under direct summands, then the property is inherited by the localizations of R at maximal ideals. Moreover, any localization of R at a maximal ideal, except maybe one, satisfies that any finitely generated ideal is 2-generated. The converse is true when the domain R is, in addition, integrally closed, or noetherian semilocal, or noetherian with module-finite normalization. Finally, over a commutative domain of finite character and with no restriction on the Krull dimension, we show that the isomorphism classes of countably generated modules in $\mathcal{F}$ are determined by their genus.

Contents

Mathematics Subject Classification 2020: 13C05 (primary); 13C60, 13G05, 16D40, 16D70 (secondary).
Keywords: torsion-free modules, h-local domain, infinite direct sum decomposition, 2-generated ideals, stable categories, relatively big projective modules.

Introduction

Let R be a commutative domain, let Λ be an R-algebra, and let $\mathcal{F}_\Lambda$ denote the class of Λ-modules that are direct sums of finitely generated Λ-modules which are torsion-free over R (we should write $\mathcal{F}$ if the ring is clear). The class $\mathcal{F}_\Lambda$ is closed under arbitrary direct sums, and we ask whether $\mathcal{F}_\Lambda$ is closed under direct summands. When Λ is noetherian, this is equivalent to the question whether $\mathcal{F}_\Lambda$ coincides with the class of pure projective Λ-modules which are torsion-free over R.

If R is a semihereditary domain, then the modules in $\mathcal{F}_R$ are projective modules. It is well known that, in this case, all projective modules are direct sums of finitely generated ideals of R. In particular, $\mathcal{F}_R$ is the class of all projective modules, and then it is closed under direct summands with no extra assumption on the Krull dimension of R.

The interest in these kinds of questions was awakened with the study of the so-called *generalized lattices*. Let R be a Dedekind domain with field of fractions Q, and let Λ be an R-order in a separable Q-algebra (hence, Λ is finitely generated and projective as R-module). Classically, a right Λ-module is a lattice if it is finitely generated and projective as an R-module. Dropping the finitely generated condition from the definition of lattice, we encounter the generalized Λ-lattices. In this context, the class $\mathcal{F}_\Lambda$ coincides with the (arbitrary) direct sums of Λ-lattices. If Λ is locally lattice-finite, the generalized lattices are precisely the direct summands of modules in $\mathcal{F}_\Lambda$. See Theorem 4 in [5] for the proof in the lattice-finite case, and Corollary on p. 112 of [30] for its extension to the locally lattice-finite case.

In Theorem 4 of [30], W. Rump proved that if generalized Λ-lattices are direct sums of lattices, then Λ is locally lattice-finite. Moreover, he answered the question of when $\mathcal{F}_\Lambda$ is closed under direct summands, finding that the answer depended on representation-theoretical data of Λ.

Příhoda, in the paper [28], translated the question to $\Lambda = R$ being a noetherian domain of Krull dimension 1 with module-finite normalization. In the local case, he proved that $\mathcal{F}_R$ is closed under direct summands if and only if its normalization (which is assumed to be finitely generated) is local. In the global case, and under the assumption that R is, in addition, lattice-finite (i.e., there are only finitely many indecomposable finitely generated torsion-free R-modules up to an isomorphism) he characterized when $\mathcal{F}_R$ is closed under direct summands by certain ring-theoretical properties of R that will also appear in Theorem B.

In the present paper, we consider the case $\Lambda = R$ being a commutative domain of Krull dimension 1, and we also follow the strategy of studying first the case of a local domain, and then extending to a suitable global situation. For the case of local domains, we get quite a satisfactory characterization, that is summarized in the following theorem.

Theorem A. (Corollary 3.3, Corollary 3.9) *Let R be a local domain of Krull dimension* 1. *The class $\mathcal{F}$ is closed under direct summands if and only if any finitely generated, indecomposable, torsion-free module has local endomorphism ring. If R, in addition, is noetherian, this is equivalent to having local integral closure.*

Therefore, in this situation, any module M in $\mathcal{F}$ can be written as $M = \bigoplus_{i\in I} N_i$, where each N_i is a finitely generated module with local endomorphism ring, and such *decomposition is unique. In addition, any direct summand of M is isomorphic to a direct sum of the modules $\{N_i\}_{i\in I}$.*

Our global setting will be that of h-local domains. Matlis introduced h-local domains in the 60s in [22] as commutative domains satisfying that:

(i) they have finite character; that is, every non-zero ideal is contained only in finitely many maximal ideals;

(ii) every non-zero prime ideal is contained only in one maximal ideal.

However, Jaffard appears to be the first author to give an equivalent definition of h-local domains in [15], but under the name of *domains of Dedekind type*. For the story and characterizations of h-local domains, the reader is referred to Fuchs and Salce's monograph (see Section 3 in Chapter IV of [8]) and to the paper by Olberding (see Theorem 2.1 in [25]).

A domain of Krull dimension 1 is h-local if and only if it has finite character, and noetherian domains of Krull dimension 1 are always h-local (equivalently, always have finite character).

If R is a domain such that $\mathcal{F}_R$ is closed under direct summands then, in particular, all projective modules are direct sums of finitely generated projective modules. Not so much is known about projective modules over arbitrary domains. However, Hinohara [14] proved that, over an h-local domain, an infinitely generated projective module, which is not finitely generated, is always free.

The following theorem summarizes our results for this class of domains,

Theorem B. (Theorem 6.6, Corollary 7.3, Corollary 7.4, Proposition 9.6, Theorem 9.8) *Let R be an h-local domain of Krull dimension* 1. *If the class $\mathcal{F}_R$ is closed under direct summands, then R satisfies that*

(1) *$\mathcal{F}_{R_\mathfrak{m}}$ is closed under direct summands for any maximal ideal $\mathfrak{m}$ of R*;

(2) *for any maximal ideal $\mathfrak{m}$ of R, except maybe one, all finitely generated ideals of $R_\mathfrak{m}$ are at most two-generated*;

(3) *the integral closure $\overline{R}$ of R in its field of fractions satisfies that $\mathcal{F}_{\overline{R}}$ is closed under direct summands.*

Conversely, if R is an h-local domain of Krull dimension 1 *satisfying* (1) *and* (2), *then $\mathcal{F}_R$ is closed under direct summands in the following situations*:

(a) *R is integrally closed, or*

(b) *R is semilocal noetherian, or*

(c) *R is noetherian and has module-finite normalization.*

That the closure of $\mathcal{F}$ under direct summands implies conditions (1) and (2) was already proven by Příhoda for noetherian domains of Krull dimension 1 with module-finite normalization [28]. The fact that a condition like (2) still holds in the generality we are working has been an interesting surprise, and its proof is one of the more involved in the paper.

A good description for torsion-free modules over a local ring with the 2-generated property for finitely generated ideals is only available in the noetherian case: if R has module-finite integral closure, then it is a Bass domain, and the theory goes back to Bass' fundamental paper [3]. The general case of noetherian domains with two-generated ideals remained open for a long time and was finally settled by Rush in [31]. In both cases, it is proven that finitely generated torsion-free modules are isomorphic to a direct sum of (finitely generated) ideals. As far as we know, it is an open question whether such a result could still hold outside the noetherian case.

We also study the isomorphism classes of modules in $\mathcal{F}$. For finitely generated modules, the isomorphism class of a module M is usually determined by its genus and some extra information on the *class group*. This is the case, for example, of Bass domains as it was shown in Theorem 4.2 of [20] and for general noetherian domains with 2-generated ideals [31]. For countable direct sums in $\mathcal{F}$ we show that the genus is enough to determine the isomorphism class. Now we state the precise result we prove, notice that it is for finitely generated R-algebras and that there is no assumption on the Krull dimension of the domain R.

Theorem C. (Theorem 8.4) *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be a module-finite R-algebra such that $\Lambda \otimes_R Q$ is a simple artinian ring. Let $M = \bigoplus_{i\in\mathbb{N}_0} A_i$ and $N = \bigoplus_{i\in\mathbb{N}_0} B_i$ be direct sums of non-zero finitely generated right Λ-modules which are torsion-free as R-modules. If M and N are in the same genus, then there are decompositions*

$$M = \bigoplus_{i\in\mathbb{N}_0} M_i \quad \text{and} \quad N = \bigoplus_{i\in\mathbb{N}_0} N_i$$

such that both M_i and N_i are finitely generated, and $M_i \cong N_i$ for every $i \in \mathbb{N}_0$. In particular, M and N are isomorphic.

Theorem C extends a result proven by Rump in [30] for Λ an order in a separable algebra over a Dedekind domain R.

Now we briefly describe the content of the different sections of the paper, taking the chance to highlight some of the main ideas.

To work in a problem like the closure of $\mathcal{F}$ under direct summands, we need some method to construct *interesting and possibly infinitely generated* direct summands. This is the main topic of Section 1, in which we exploit that projective modules can be lifted modulo the trace ideal of a projective module (cf. Theorem 1.9). This machinery for projective modules was developed in the noetherian setting in [26], extended to general rings in [11] and the connection with the study of the category Add(M), of direct summands of direct sums of copies of a finitely generated module M, was developed in [12] (cf. Theorem 1.11).

Section 2 is quite technical, but essential for the rest of the paper. In it, we survey the results needed on finitely generated (torsion-free) and/or finite rank modules over a domain R, as well as properties of their endomorphism rings. Given a domain R we also need to treat the case of finitely generated modules over an R-algebra Λ that is torsion-free as R-modules. We also introduce the concepts of domain of finite character and of h-local domain, developing the first basic properties of endomorphism rings of finitely generated modules over these classes of rings.

In Section 3, we characterize local domains of Krull dimension 1 satisfying that $\mathcal{F}$ is closed under direct summands. The results in this section prove Theorem A.

In Section 4, we develop tools to relate the property of being locally a direct summand with being a direct summand in the setting of finitely generated torsion-free modules. If R is a domain of finite character, and N and M are finitely generated torsion-free modules such that $N \otimes_R Q$ is a direct summand of $M \otimes_R Q$, then $N_{\mathfrak{m}}$ is a direct summand of $M_{\mathfrak{m}}$ for almost all maximal ideals $\mathfrak{m}$ of R. This means that we only have a finite number of "problematic maximal ideals". In the remainder of the section, we prove several results on how to proceed for these problematic maximal ideals. Analogous results were proven in the one-dimensional noetherian setting in [10], and were inspired by the representation theory of orders.

In Section 5, we study finitely generated modules over an algebra Λ over an h-local domain R. We prove the results that will serve as a bridge between the local and the global case. A key observation is the following: if M and N are finitely generated Λ-modules that are torsion-free as R-modules and satisfy $M \otimes_R Q \cong N \otimes_R Q$, where Q is the field of fractions of R, then the localizations of M and N at maximal ideals of R are isomorphic for almost all maximal ideals of R.

Then the problem is, given $\{X(\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{mSpec} R\}$ a family of right $\Lambda_{\mathfrak{m}}$-modules, when there is a right Λ-module N such that $N_{\mathfrak{m}} \cong X(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R. This question was already studied by Levy and Odenthal in [19] for R being a noetherian domain of Krull dimension 1, and the answer is that mild compatible conditions between the localizations are enough to warrant the existence of N. Here we give a generalized version of Levy–Odenthal's package deal theorem for localizations over h-local domains (cf. Theorem 5.7), and give another one for localizations of trace ideals of countably generated projective Λ-modules (cf. Theorem 5.13).

The case $\Lambda = R$ is particularly neat. The localization at a maximal ideal of a finitely generated torsion-free module is free for almost all maximal ideals (cf. Corollary 5.3). Conversely, a family $\{X(\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{mSpec} R\}$ of $R_{\mathfrak{m}}$-modules can be *glued together* in a finitely generated torsion-free R-module N provided all the modules $X_{\mathfrak{m}}$ have the same rank and they are free for almost all maximal ideals $\mathfrak{m}$ (cf. Corollary 5.8).

In Section 6, we exploit the theory developed in the previous sections, proving the first part of Theorem B. An important intermediate result is Theorem 6.2, which shows that if $\mathcal{F}_R$ is closed under direct summands, then finitely generated indecomposable torsion-free modules over different localizations at maximal ideals must have coprime rank. This type of result was first spotted by Příhoda in [28], and our proof follows the same strategy as the original result.

In Section 7, the remaining part of the characterization of integrally closed domains with $\mathcal{F}$ closed under direct summands, contained in Theorem B, is proven.

In Section 8, we give the proof of Theorem C. This section also includes examples showing that over a semilocal domain, being a direct summand of an infinite direct sum of finitely generated torsion-free modules could be satisfied locally but not globally. This contrasts with the case of finitely generated modules, cf. Corollary 4.5.

The noetherian part of Theorem B is proven in Section 9. We stress that Theorem 9.8 gives the converse of one of the main results in [28].

We close the paper in Section 10, with a family of examples of h-local domains of Krull dimension 1 satisfying Theorem B. These examples are of the form $R = K + xL[x]$, where $K \subseteq L$ is a field extension. If such an extension is finitely generated, then R is noetherian. If the extension is transcendental, then R is integrally closed in its field of fractions (and it is not a Prüfer domain!). Moreover, under mild restrictions over the dimension of the field extension, R has indecomposable finitely generated torsion-free modules of all ranks.

Throughout the paper, rings are associative with 1, and morphisms are unital. In each section we try to be very precise about the setting we are working with. Our main topic is (torsion-free) modules over commutative domains, so most of the time R is a commutative domain with field of fractions Q. We reserve Λ to denote an algebra over a domain R. If M is a Λ-module (usually finitely generated and torsion-free as an R-module), we denote by S its endomorphism ring.

1. Lifting direct summands modulo the stable category

Definition 1.1. Let R be a ring and let M be a right R-module. Then the *trace* of M in R is the two-sided ideal of R defined as

$$\mathrm{Tr}_R(M) = \sum_{f \in \mathrm{Hom}_R(M,R)} f(M).$$

To simplify the notation, we will just use $\mathrm{Tr}(M)$ to denote the trace in R of the module M when the ring R is clear. Our main interest will be in traces of projective modules. Recall that the trace of a projective module is always an idempotent ideal.

The following fact will be used throughout the paper.

Remark 1.2. Let R be a ring and M a right R-module. Let L be a two-sided ideal of R. Then $I = \{f \in \mathrm{End}_R(M) \mid f(M) \subseteq ML\}$ is a two-sided ideal of $\mathrm{End}_R(M)$.

In addition, the R-R bimodule structure of R gives a structure of left R-module to $\mathrm{Hom}_R(M, R)$ where $r \cdot \omega$, for $r \in R$ and $\omega \in \mathrm{Hom}_R(M, R)$, is defined as the module homomorphism $r \cdot \omega \colon M \to R$ such that $r \cdot \omega(x) = r\omega(x)$ for any $x \in M$. This gives a canonical morphism

$$\varphi : M \otimes_R \mathrm{Hom}_R(M, R) \to \mathrm{End}_R(M)$$

where, for $m \in M$ and $\omega \in \mathrm{Hom}_R(M, R)$, $\varphi(m \otimes \omega)$ is the endomorphism of M defined by $\varphi(m \otimes \omega)(x) = m\omega(x)$, for any $x \in M$. It is not difficult to show that

$$\varphi(M \otimes_R \mathrm{Hom}_R(M, R)) = \{f \in S \mid f \text{ factors through } R^n \text{ for some } n \geq 1\}.$$

Throughout the paper, we will use the notation

$$\varphi(M \otimes_R \mathrm{Hom}_R(M, R)) = M\,\mathrm{Hom}_R(M, R).$$

Lemma 1.3. *Let R be a ring. Let M be a right R-module with endomorphism ring $S = \mathrm{End}_R(M)$. Let $I = \mathrm{Tr}_R(M)$, and $J = \{f \in S \mid f(M) \subseteq MI\}$ (which is a two-sided ideal of S by Remark 1.2). Then $M\,\mathrm{Hom}_R(M, R) \subseteq J$.*

Proof. In the notation of Remark 1.2, and because of the definition of φ, it follows that $\varphi(m \otimes \omega)(M) \subseteq MI$. So that, also $\varphi(\sum_{i=1}^{n} m_i \otimes \omega_i)(M) \subseteq MI$ for $m_i \in M$ and $\omega_i \in \mathrm{Hom}_R(M, R)$. ∎

Lemma 1.4. *Let R be a ring and let P be a right R-module. Then, for any two-sided ideal I of R,*

(i) $(\mathrm{Tr}_R(P) + I)/I \subseteq \mathrm{Tr}_{R/I}(P/PI)$;

(ii) *if, in addition, P is projective, then* $\mathrm{Tr}_{R/I}(P/PI) = (\mathrm{Tr}_R(P) + I)/I$.

Proof. (i) Let $x \in \mathrm{Tr}_R(P)$. Then there exist $f_1, \dots, f_n \in \mathrm{Hom}_R(P, R)$ and $p_1, \dots, p_n \in P$ such that $x = \sum_{i=1}^n f_i(p_i)$. Then, for each $i = 1, \dots, n$, there is an induced homomorphism $\overline{f}_i \colon P/PI \to R/I$ given by $\overline{f}_i(p + PI) = f(p) + I$. Since $\overline{f}_i(P/PI) \subseteq \mathrm{Tr}_{R/I}(P/PI)$, in particular, $f_i(p_i) + I \in \mathrm{Tr}_{R/I}(P/PI)$. Since $x = \sum_{i=1}^n f_i(p_i)$, it follows that $x + I \in \mathrm{Tr}_{R/I}(P/PI)$.

Statement (ii) follows from the lifting property of projective modules, since every R/I-module homomorphism $\overline{f} \colon P/PI \to R/I$ lifts to an R-module homomorphism $f \colon P \to R$. ∎

Remark 1.5. In this section, we will make repeated use of the so-called *Eilenberg trick*, which implies, for example, that if P is a countably generated projective module and X is a direct summand of P, then $P^{(\omega)} \oplus X \cong P^{(\omega)}$.

We recall the definition of the class of modules generated by a projective module.

Definition 1.6. Let R be a ring. Let M be a right R-module. We define the *class* $\mathrm{Gen}(M)$ *of modules generated by M* as the class of right R-modules that are a homomorphic image of a direct sum of copies of M.

For further quoting, we recall the following characterization of the class $\mathrm{Gen}(P)$, when P is a projective module.

Lemma 1.7 (Lemma 2.10 in [12]). *Let R be a ring, and let P be a projective right R-module with trace ideal I. Then* $\mathrm{Gen}(P)$ *coincides with the class of right R-modules M such that $MI = M$.*

For further use, we note the following property of direct sums of finitely generated modules.

Lemma 1.8. *Let R be a ring, and assume there is a direct sum decomposition of right R-modules $M' \oplus M = X \oplus Y$, with M finitely generated and X, Y direct sums of finitely generated modules. Then there exist A and B, direct summands of X and Y, respectively, such that $M \oplus Z = A \oplus B$, with Z a finitely generated direct summand of M'.*

Proof. Since M is finitely generated, by the hypothesis, $M \subseteq A \oplus B$, with A and B being finitely generated direct summands of X and Y, respectively. Therefore, by the modular law, $A \oplus B = M \oplus Z$, where $Z = M' \cap (A \oplus B)$. ∎

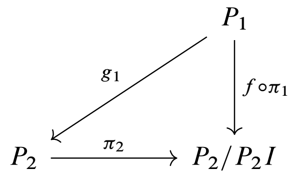
Theorem 1.9. *Let R be a ring, and let I be an ideal of R that is the trace of a countably generated projective right R-module Q.*

(i) (Theorem 3.1 in [11]) *Let P' be a countably generated projective right module over R/I. Then there exists a countably generated projective right R-module P such that $P/PI \cong P'$.*

(ii) *Let* P_1 *and* P_2 *be countably generated projective modules such that* $P_1/P_1I \cong P_2/P_2I$*. Then* $Q^{(\omega)} \oplus P_1 \cong Q^{(\omega)} \oplus P_2$*.*

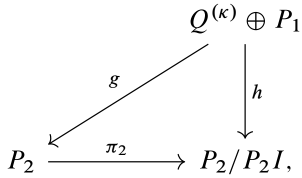
(iii) *Let* Q' *be a countably generated projective* R*-module such that* $\mathrm{Tr}_R(Q') = I$*. Then* $Q^{(\omega)} \cong (Q')^{(\omega)}$*.*

Proof. Statement (ii) follows from the proof of Lemma 2.5 in [26]. We give a self-contained proof for the reader's convenience. For each $i = 1, 2$, let $\pi_i \colon P_i \to P_i/P_iI$ denote the canonical projection. Let $f \colon P_1/P_1I \to P_2/P_2I$ be an isomorphism. Then there exists $g_1 \colon P_1 \to P_2$ such that the diagram



is commutative. Hence, $P_2 = g_1(P_1) + P_2I$.

Since I is the trace ideal of the countably generated projective module Q, it is idempotent and, by Lemma 1.7, there exists an onto module homomorphism $g_2 \colon Q^{(\kappa)} \to P_2I$, where κ is an infinite cardinal. Let $h \colon Q^{(\kappa)} \oplus P_1 \to P_2/P_2I$ be the module homomorphism defined by $h(q, p) = f \circ \pi_1(p)$ for any $(q, p) \in Q^{(\kappa)} \oplus P_1$. Then there is a commutative diagram



where g is the onto module homomorphism defined by $g(q, p) = g_2(q) + g_1(p)$ for any $(q, p) \in Q^{(\kappa)} \oplus P_1$. Since P_2 is countably generated, there exists $C \subseteq \kappa$ countably infinite such that $P_2 = g(Q^{(C)} \oplus P_1)$. Let $g' \colon Q^{(C)} \oplus P_1 \to P_2$ be the restriction of g to $Q^{(C)} \oplus P_1$. Since P_2 is projective, $Q^{(C)} \oplus P_1 = P_2' \oplus \operatorname{Ker} g'$ where P_2' is a submodule of $Q^{(C)} \oplus P_1$ isomorphic to P_2.

As $\operatorname{Ker} g' \subseteq \operatorname{Ker} h = Q^{(\kappa)} \oplus P_1I$, we deduce that

$$Q^{(C)} \oplus P_1I = \operatorname{Ker} g' \oplus (P_2' \cap (Q^{(C)} \oplus P_1I)).$$

By Lemma 1.7, $P_1I \in \mathrm{Gen}(Q)$. Hence, the countably generated module $\operatorname{Ker} g'$ is a homomorphic image of $Q^{(\omega)}$. Since it is also projective, it is a direct summand of $Q^{(\omega)}$. By the Eilenberg trick (cf. Remark 1.5), $Q^{(\omega)} \oplus \operatorname{Ker} g' \cong Q^{(\omega)}$. Therefore,

$$Q^{(\omega)} \oplus P_1 \cong Q^{(\omega)} \oplus P_2 \oplus \operatorname{Ker} g' \cong Q^{(\omega)} \oplus P_2,$$

as we wanted to prove.

Statement (iii) is a consequence of (ii). Taking $P_1 = Q^{(\omega)}$ and $P_2 = (Q')^{(\omega)}$, we deduce from (ii) that

$$Q^{(\omega)} \cong Q^{(\omega)} \oplus Q^{(\omega)} \cong Q^{(\omega)} \oplus (Q')^{(\omega)}.$$

Exchanging the roles of Q and Q', we deduce from (ii) that $(Q')^{(\omega)} \cong (Q')^{(\omega)} \oplus Q^{(\omega)}$, which yields the claimed isomorphism. ∎

As a corollary of Theorem 1.9, we show that direct sums of projective modules can be lifted modulo a trace ideal. This result is implicit in [12].

Corollary 1.10. *Let R be a ring, and let P be a countably generated projective right R-module. Let I be the trace of a countably generated projective right R-module Q. Assume that $P/PI = X' \oplus Y'$. Then there exist countably generated projective right R-modules X and Y such that $Q^{(\omega)} \oplus P \cong X \oplus Y$ and $X/XI \cong X'$, $Y/YI \cong Y'$ as R/I-modules.*

If, in addition, P is finitely generated, and X and Y are direct sums of finitely generated modules, then there exist A and B, isomorphic to direct summands of X and Y, respectively, such that $P \oplus Z = A \oplus B$, with Z a finitely generated direct summand of $Q^{(\omega)}$, and satisfying that $A/AI \cong X'$, $B/BI \cong Y'$.

Proof. By Theorem 1.9, there exist countably generated projective right R-modules X_1 and Y_1 such that $X_1/X_1I \cong X'$, $Y_1/Y_1I \cong Y'$. Hence, $P/PI \cong X_1/X_1I \oplus Y_1/Y_1I$.

Set $X = Q^{(\omega)} \oplus X_1$ and $Y = Q^{(\omega)} \oplus Y_1$. By Theorem 1.9, $Q^{(\omega)} \oplus P \cong X \oplus Y$.

To prove the final part of the statement, assume for simplicity that $Q^{(\omega)} \oplus P = X \oplus Y$. Notice that the existence of A, B and Z follows from Lemma 1.8. Therefore, there are direct sum decompositions $X = A \oplus A'$ and $Y = B \oplus B'$. Now

$$Q^{(\omega)} \oplus P = A \oplus A' \oplus B \oplus B' = P \oplus (Z \oplus A' \oplus B'),$$

so that the canonical projection $\pi\colon Q^{(\omega)} \oplus P \to Z \oplus A' \oplus B'$ induces an isomorphism $Q^{(\omega)} \to Z \oplus A' \oplus B'$. Therefore Z, A' and B' are modules in $\mathrm{Gen}(Q)$. By Lemma 1.7, we deduce that $X' \cong X/XI \cong A/AI$ and $Y' \cong Y/YI \cong B/BI$. This concludes the proof of the statement. ∎

Proposition 1.11 (Theorem 4.7 in [6]). *Let R be a ring. Let M be a right R-module. Let $S = \mathrm{End}_R(M)$. Then the functor $\mathrm{Hom}_R(M,-)$ induces a category equivalence between $\mathrm{add}(M)$ and the category of finitely generated projective right S-modules. The inverse of this equivalence is given by the functor $- \otimes_S M$.*

Assume, in addition, that M_R is finitely generated. Then the functor $\mathrm{Hom}_R(M,-)$ induces a category equivalence between $\mathrm{Add}(M)$ and the category of projective right S-modules.

Lemma 1.12 (Theorem 2.11 (i) in [12]). *Let R be a ring. Let M be a finitely generated right R-module. Let $S = \mathrm{End}_R(M)$.. Let X be an object of $\mathrm{Add}(M)$, and let $P_S = \mathrm{Hom}_R(M, X)$. Set $I = \mathrm{Tr}_S(P)$. Then,*

(i) *$I = \{f \in S \mid f$ factors through X^n for some $n \in \mathbb{N}\}$, that is,*

$$I = \mathrm{Hom}_R(X, M)\,\mathrm{Hom}_R(M, X).$$

(ii) *For any $Y \in \mathrm{Add}(M)$,*

$$\mathrm{Hom}_R(M, Y)I = \{f \in \mathrm{Hom}_R(M, Y) \mid f \text{ factors through } X^n \text{ for some } n \in \mathbb{N}\}.$$

Remark 1.13. Let M and M' be two right modules over a ring R, and with endomorphism rings S and S', respectively. Let

$$T := \operatorname{End}_R(M' \oplus M) = \begin{pmatrix} S' & \operatorname{Hom}_R(M, M') \\ \operatorname{Hom}_R(M', M) & S \end{pmatrix}.$$

Consider the finitely generated projective right T-module

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T \cong \operatorname{Hom}_R(M' \oplus M, M').$$

Its trace ideal is

$$I = T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T = \begin{pmatrix} S' & \operatorname{Hom}_R(M, M') \\ \operatorname{Hom}_R(M', M) & \operatorname{Hom}_R(M', M) \operatorname{Hom}_R(M, M') \end{pmatrix}.$$

Then $T/I \cong S/\operatorname{Hom}_R(M', M) \operatorname{Hom}_R(M, M')$.

We will use the case $M' = R$, so that $T/I \cong S/M \operatorname{Hom}_R(M, R)$. That is, it is the endomorphism ring of M_R in the stable category.

Corollary 1.14. *Let R be a ring. Let M be a finitely generated right R-module with endomorphism ring S. Let*

$$I = \{f \in S \mid f \text{ factors through } R^n \text{ for some } n \in \mathbb{N}\} = M \operatorname{Hom}_R(M, R).$$

Let e be an idempotent of S/I. Then there is a direct sum decomposition $R^{(\omega)} \oplus M = X \oplus Y$, such that

$$\operatorname{Hom}_R(M, X)/\operatorname{Hom}_R(M, X)I \cong e(S/I)$$

and

$$\operatorname{Hom}_R(M, Y)/\operatorname{Hom}_R(M, Y)I \cong (1-e)(S/I).$$

If, in addition, X and Y are direct sums of finitely generated modules, then there exist A and B, direct summands of X and Y, respectively, such that $M \oplus P = A \oplus B$ with P a finitely generated projective right R-module and satisfying that

$$\operatorname{Hom}_R(M, A)/\operatorname{Hom}_R(M, A)I \cong e(S/I)$$

and

$$\operatorname{Hom}_R(M, B)/\operatorname{Hom}_R(M, B)I \cong (1-e)(S/I).$$

Proof. Let $T = \operatorname{End}_R(R \oplus M)$. Considering Remark 1.13, $S/I \cong T/J$ with J the trace ideal of the projective right T-module $Q = \operatorname{Hom}_R(R \oplus M, R)$.

By Corollary 1.10, there is a direct sum decomposition $Q^{(\omega)} \oplus \operatorname{Hom}(R \oplus M, M) = X' \oplus Y'$ with $X'/X'J \cong e(S/I)$ and $Y'/Y'J \cong (1-e)(S/I)$.

Set $X = X' \otimes_T (R \oplus M)$ and $Y = Y' \otimes_T (R \oplus M)$. By Proposition 1.11, we have that $R^{(\omega)} \oplus M \cong X \oplus Y$, $R^{(\omega)} \oplus X \cong X$, $R^{(\omega)} \oplus Y \cong Y$, and X and Y are uniquely determined up to isomomorphism.

The final part of the statement follows from Lemma 1.8 and Proposition 1.11. ∎

Lemma 1.15. *Let R be a ring. Let M be a right R-module with semiperfect endomorphism ring. If X is a right R-module such that $M \oplus X = Y_1 \oplus Y_2$ for suitable right R-modules Y_1 and Y_2, then for $i = 1, 2$, $Y_i \cong N_i \oplus Y_i'$, where $N_1 \oplus N_2 \cong M$ and $Y_1' \oplus Y_2' \cong X$.*

Proof. This is an easy consequence of the fact that modules with a semiperfect endomorphism ring are a finite direct sum of modules with local endomorphism ring, and that modules with local endomorphism ring satisfy the exchange property (cf. Lemma 2.7 and Theorem 2.8 in [6]). Namely, if $M \oplus X = Y_1 \oplus Y_2$, and M has local endomorphism ring, then we may assume that M is isomorphic to a direct summand of Y_1 so that $Y_1 \cong M \oplus Y_1'$ and, moreover, $X \cong Y_1' \oplus Y_2$. Now if $M = M_1 \oplus \cdots \oplus M_n$, with each M_i having local endomorphism ring, the statement follows by induction on n. ∎

Lemma 1.16. *Let R be a semiperfect ring. Let M be a right R-module. Then $\operatorname{Tr}_R(M) \subseteq J(R)$ if and only if M has no non-zero projective direct summands.*

Proof. Assume that $\operatorname{Tr}_R(M) \nsubseteq J(R)$. Hence, there exists $f \colon M \to R$ and $e^2 = e \in R$, with $eR/eJ(R)$ simple, such that $ef(M) \neq eJ(R)$. Therefore $eR = ef(M) + eJ(R)$. By Nakayama's lemma, $eR = ef(M)$, so eR is a projective direct summand of M.

The converse statement is clear because the trace of a non-zero projective module is never contained in $J(R)$. ∎

Lemma 1.17. *Let R be a semiperfect ring. Then:*

(i) *If M is a finitely generated module, $M = P \oplus M'$, with P a finitely generated projective module and M' with no non-zero projective direct summands. In addition, this decomposition is unique up to isomorphism.*

(ii) *If M_1 and M_2 are R-modules with no non-zero projective direct summand, then $M_1 \oplus M_2$ has the same property.*

Proof. Statement (i) is Theorem 3.15 in [6].

To prove statement (ii), notice that if P is an indecomposable projective R-module, then its endomorphism ring is local because R is semiperfect. Therefore, if such P is a direct summand of $M_1 \oplus M_2$, then it is a direct summand either of M_1 or of M_2, which is not possible by our assumptions. ∎

Proposition 1.18. *Let R be a semiperfect ring, and let M be a finitely generated right R-module with endomorphism ring S. Set $J = M \operatorname{Hom}_R(M, R)$. If every direct summand of $R^{(\omega)} \oplus M$ is a direct sum of finitely generated modules, then the following statements hold*:

(i) *Let $e \in S/J$ be an idempotent. Then there exists $f^2 = f \in S$ such that $(f + J)S/J \cong eS/J$ and $(1 - f + J)S/J \cong (1 - e)S/J$.*

(ii) *If M is indecomposable, then so is the right S-module S/J.*

(iii) *Every direct summand of $R^{(\omega)} \oplus M$ is of the form $P \oplus X$, where X is a direct summand of M and P is a direct summand of $R^{(\omega)}$ (that is, a countably generated projective module).*

Proof. (i) We consider the setting of Remark 1.13 with $M' = M$, and follow the notation on that remark.

By Corollary 1.14, there is a decomposition $P \oplus M = A \oplus B$ with P finitely generated projective and A and B finitely generated such that

$$\operatorname{Hom}_R(M, A)/\operatorname{Hom}_R(M, A)J \cong e(S/J)$$

and

$$\operatorname{Hom}_R(M, B)/\operatorname{Hom}_R(M, B)J \cong (1-e)(S/J).$$

By Lemma 1.15, $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, with $M \cong A_1 \oplus B_1$ and A_2 and B_2 projective R-modules. Therefore, $\operatorname{Hom}_R(M, A_1) \cong fS$ and $\operatorname{Hom}_R(M, B_1) \cong (1-f)S$ for $f^2 = f \in S$.

By Lemma 1.12,

$$\begin{aligned} e(S/J) &\cong \operatorname{Hom}_R(M, A)/\operatorname{Hom}_R(M, A)J \\ &\cong \operatorname{Hom}_R(M, A_1)/\operatorname{Hom}_R(M, A_1)J \cong (f+J)S/J. \end{aligned}$$

Similarly, $(1-e)S/J \cong (1-f+J)S/J$.

Statement (ii) is a consequence of (i).

(iii) Since R is semiperfect, $M = P \oplus M'$, where P is finitely generated projective and M' has no non-zero projective direct summands (cf. Lemma 1.17). Since $R^{(\omega)} \oplus P \cong R^{(\omega)}$, to prove our statement, we may assume that $M = M'$ has no non-zero projective direct summands.

Let N be a finitely generated direct summand of $R^{(\omega)} \oplus M$. Then, there exists $n \geq 0$ such that N is a direct summand of $R^n \oplus M$. By Lemma 1.15, we can deduce that N is of the required form.

Assume now that N is a direct summand of $R^{(\omega)} \oplus M$ that is not finitely generated. By hypothesis, $N = \bigoplus_{i \in A} N_i$, where N_i are finitely generated modules. By the previous step, for each $i \in A$, $N_i = P_i \oplus M_i$, with P_i finitely generated projective and M_i isomorphic to a direct summand of M. Notice that each M_i has no non-zero projective direct summands, by our assumptions.

Since the ring R is semiperfect, then $M/MJ(R)$ is a semisimple module of length ℓ. Then the maximal number of non-zero direct summands in a direct sum decomposition of M is ℓ. We claim that this implies that only ℓ of the direct summands M_i are different from zero.

Indeed, take $i_1, \dots, i_{\ell+1}$ to be $\ell + 1$ different elements of the index set A. Then $N' = \bigoplus_{j=1}^{\ell+1} M_{i_j}$ is a direct summand of $R^{(\omega)} \oplus M$. By Lemma 1.17, N' has no non-zero projective direct summands. By the first step, N' is a direct summand of M. Hence, there is some $M_{i_j} = \{0\}$.

Since only finitely many M_i's are different from zero, as before, we can deduce that its direct sum is a direct summand of M. Since $\bigoplus_{i \in A} P_i$ is a countably generated projective module, it is a direct summand of $R^{(\omega)}$. This finishes the proof of the claim. ∎

Now we want to interpret these results in terms of *lifting of idempotents* modulo an ideal. The following fact characterizing when two idempotents are conjugated by a unit will be useful.

Lemma 1.19 (Exercise 21.15 in [17]). *Let S be a ring, and let e and g be idempotents of S. Then there exists a unit $u \in S$ such that $ueu^{-1} = g$ if and only if $eS \cong gS$ and $(1-e)S \cong (1-g)S$.*

Lemma 1.20. *Let S be a ring, and let J be an ideal of S contained in $J(S)$. Let $e = e^2 \in S/J$. Assume that $S_S \cong X_1 \oplus X_2$ and $X_1/X_1J \cong e(S/J)$ and $X_2/X_2J \cong (1-e)(S/J)$.*

Then there exists $f^2 = f \in S$ such that $f + J = e$ and $fS \cong X_1$ and $(1-f)S \cong X_2$.

Proof. Let $g^2 = g \in S$ be such that there exist two isomorphisms $f_1\colon gS \to X_1$ and $f_2\colon (1-g)S \to X_2$. Then the restriction of f_1 and f_2 give isomorphisms $gJ \cong X_1J$ and $(1-g)J \cong X_2J$, so that

$$(g+J)S/J \cong gS/gJ \cong X_1/X_1J,$$
$$(1-g+J)S/J \cong (1-g)S/(1-g)J \cong X_2/X_2J.$$

By Lemma 1.19, there exists v a unit of S/J such that $v(g+J)v^{-1} = e$. Since J is contained in $J(S)$, any $u \in S$ such that $u + J = v$ is also a unit of S. Pick such an invertible element u, and set $f = ugu^{-1}$. Then f is an idempotent element of S with $f + J = v(g+J)v^{-1} = e$. Moreover, as f is conjugated of g, we can apply Lemma 1.19 to deduce that $fS \cong X_1$ and $(1-f)S \cong X_2$. ∎

Corollary 1.21. *Let R be a semiperfect ring. Let M be a finitely generated right R-module with endomorphism ring S. Assume that onto endomorphisms of M are bijective and that M has no non-zero projective direct summands. Then*:

(i) *S is semilocal and the ideal $J = M\,\mathrm{Hom}_R(M,R)$ is contained in $J(S)$.*

(ii) *If the direct summands of $R^{(\omega)} \oplus M$ are direct sum of finitely generated modules, then idempotents of S/J can be lifted to idempotents of S. If, in addition, $J(S)/J$ is a nil-ideal, then S is semiperfect.*

Proof. (i) By hypothesis and Lemma 1.16, $f \in J$ implies that $f(M) \subseteq MJ(R)$, so $f(M)$ is superfluous in M. Then $1 - f$ is onto, hence bijective. This implies that $J \subseteq J(S)$. The ring S is semilocal because $M/MJ(R)$ is a semisimple module of finite length, so we can apply the results in [13].

(ii) By Proposition 1.18, any direct summand of S/J can be lifted to a direct summand of S. By (i), $J \subseteq J(S)$, and we deduce that idempotents can be lifted modulo J from Lemma 1.20. If $J(S)/J$ is nil, then idempotents of $S/J(S)$ can be lifted to S/J. Hence, to S. Since, by (i), S is semilocal, we deduce that S is semiperfect. ∎

2. The endomorphism ring of a finitely generated module

As implicit in Corollary 1.21, the condition that onto endomorphisms of a finitely generated module are bijective is going to play an important role in our study. In this section, we recall some more or less well-known facts on when this happens, and some of the consequences it has.

We start with some well-known facts about the endomorphism ring of a finitely generated module over a commutative ring. We will use them throughout the paper, sometimes without previous acknowledgment.

Lemma 2.1. *Let R be a commutative ring. Let M be a finitely generated R-module with endomorphism ring S. Then*:

(i) (Vasconcelos, Proposition 1.2 in [32]) *Any onto endomorphism of M is bijective.*

(ii) *The central ring extension $R/\mathrm{ann}_R(M) \hookrightarrow S$ is integral* (*that is, any element of S satisfies a monic polynomial with coefficients in $R/\mathrm{ann}_R(M)$*).

(iii) *Let L be a two-sided ideal of R containing $\mathrm{ann}_R(M)$ and let $I = \{f \in S \mid f(M) \subseteq ML\}$. If M can be generated by ℓ elements, then for every $f \in I$, there exist*

$$b_1, \dots, b_\ell \in L$$

such that $f^\ell + f^{\ell-1}\bar{b}_\ell + \cdots + f\bar{b}_2 + \bar{b}_1 = 0$, where $\bar{b}_i$, $i = 1, \dots, \ell$, means the class of the element b_i in $R/\mathrm{ann}_R(M)$ viewed inside S.

Proof. Statements (ii) and (iii) are applications of the determinant trick. To prove (ii), let $M = m_1R + \cdots + m_\ell R$ and let $f \in S$. Then there exists a matrix $A \in M_\ell(R)$ such that

$$(f\,\mathrm{Id}_\ell - A)\begin{pmatrix} m_1 \\ \vdots \\ m_\ell \end{pmatrix} = 0.$$

Then we deduce that $\det(f\,\mathrm{Id}_\ell - A)m_i = 0$ for any $i = 1, \dots, \ell$. Hence, $\overline{\det(f\,\mathrm{Id}_\ell - A)} \in (R/\mathrm{ann}_R(M))[f]$ is zero. Hence, f is an integral element over $R/\mathrm{ann}_R(M)$.

To prove (iii), just observe that the coefficients of the polynomial $\overline{\det(f\,\mathrm{Id}_\ell - A)}$ satisfy the claimed properties if $f \in I$. ∎

Remark 2.2. Let R be a commutative ring, and let Λ be a (non-necessarily commutative) module-finite R-algebra. Let M be a finitely generated right Λ-module. Then the conclusions of Lemma 2.1 also hold for $S = \mathrm{End}_\Lambda(M)$. M_Λ being finitely generated and Λ being module-finite implies that M_R is also finitely generated as an R-module. Moreover, $\mathrm{End}_\Lambda(M)$ is a subring of $\mathrm{End}_R(M)$.

Goodearl in [9] characterized which rings satisfy that any onto endomorphism of a finitely generated module is bijective. The following result follows easily from such characterization, and it includes an extension of Lemma 2.1 and Remark 2.2.

Proposition 2.3. *Let R be a commutative ring, and let Λ be a* (*non-necessarily commutative*) *module-finite R-algebra. Let M be a finitely generated right Λ-module with endomorphism ring $S = \mathrm{End}_\Lambda(M)$. Then*:

(i) *Every finitely generated right S-module X satisfies that any onto endomorphism of X is bijective.*

(ii) *If S is semilocal, then so is the endomorphism ring of any finitely generated right* (*or left*) *S-module. In particular, if Λ is semilocal, then so is S.*

(iii) *$I = \{f \in \mathrm{End}_S(X) \mid f(X) \subseteq XJ(S)\}$ is a two-sided ideal of $\mathrm{End}_S(X)$ contained in $J(\mathrm{End}_S(X))$. In particular, $\{f \in S \mid f(M) \subseteq MJ(\Lambda)\}$ is a two-sided ideal of S contained in $J(S)$.*

Proof. (i) Since $M_n(S) \cong \mathrm{End}_\Lambda(M^n)$ for any $n \geq 1$, we deduce from Lemma 2.1 (ii) and Remark 2.2 that the extension $R/\mathrm{ann}_R(M) \to M_n(S)$ is integral for any $n \geq 1$. Now, the conclusion follows from Corollary 6 in [9].

(ii) The first part of the result is a particular case of Proposition 3.2 in [7]. Taking $M = \Lambda$ gives the second part of the statement.

(iii) By Remark 1.2, I is a two-sided ideal of S, so it is enough to show that $1 - f$ is bijective for every $f \in I$.

Let $f \in I$. Then $S = f(X) + (1 - f)(X) = XJ(S) + (1 - f)(X)$. By Nakayama's lemma, $X = (1 - f)(X)$. By (i), $1 - f$ is bijective. ■

Definition 2.4. A ring homomorphism $f\colon R \to S$ is said to be *local* if, for any $r \in R$, $f(r)$ is invertible in S if and only if r is invertible in R.

Lemma 2.5. *Let R be a commutative ring. Let $R \subseteq S$ be a central integral extension of rings, where S is not necessarily commutative. Then*:

(i) *For any $s \in S$, the inclusion $R[s] \hookrightarrow S$ is a local ring homomorphism. Therefore $J(S) \cap R[s] \subseteq J(R[s])$.*

(ii) *If R is a ring of Krull dimension 0, then $J(S)$ is a nil-ideal.*

Proof. (i) Let $s \in S$. To show that the inclusion $R[s] \subseteq S$ is a local homomorphism, we need to prove that if $x \in R[s]$ is invertible in S, then the inverse is in $R[s]$. This is a well-known fact for integral extensions of commutative rings. We recall the argument to see that it also works in our situation.

Since $R \subseteq S$ is an integral extension, the element x^{-1} satisfies $x^{-n} - r_1 x^{-n+1} - \cdots - r_n = 0$ for suitable $r_1, \dots, r_n \in R$. Since x^{-1} is invertible, we may assume $r_n \neq 0$. Therefore, multiplying by x^n we obtain the equality $1 = (r_1 + r_2 x + \cdots + r_n x^{n-1})x$. Therefore, $x^{-1} = r_1 + r_2 x + \cdots + r_n x^{n-1} \in R[s]$.

The second part of the statement follows immediately from the first part.

(ii) Assume now that R has Krull dimension 0. Let $s \in J(S)$. By (i), $s \in J(R[s])$. Since $R \subseteq R[s]$ is an integral extension of commutative rings, also $R[s]$ is a ring of Krull dimension 0. Hence, $J(R[s])$ coincides with the nilradical of $R[s]$. So we deduce that s is nilpotent. This shows that $J(S)$ is a nil-ideal. ■

Lemma 2.6. *Let R be a commutative ring, and let Λ be a module-finite R-algebra. If M is a finitely generated right Λ-module, then $\mathrm{End}_\Lambda(M)J(R) \subseteq J(\mathrm{End}_\Lambda(M))$.*

Proof. Let $f_1, \dots, f_k \in \mathrm{End}_\Lambda(M)$ and let $r_1, \dots, r_k \in J(R)$. Since M_R is finitely generated, by Lemma 2.1, to show that $1 - \sum_{i=1}^k f_i r_i$ is invertible it suffices to show that $1 - \sum_{i=1}^k f_i r_i$ is an onto Λ-endomorphism of M. To this aim, notice that

$$M = \Big(1 - \sum_{i=1}^k f_i r_i\Big)(M) + \Big(\sum_{i=1}^k f_i r_i\Big)(M) = \Big(1 - \sum_{i=1}^k f_i r_i\Big)(M) + MJ(R).$$

Since $MJ(R)$ is a small R-submodule of M, we deduce that

$$M = \Big(1 - \sum_{i=1}^k f_i r_i\Big)(M).$$ ■

Lemma 2.7. *Let R be a commutative ring, and let Λ be a module-finite R-algebra. Let M be a non-zero finitely generated right Λ-module with endomorphism ring $S = \mathrm{End}_\Lambda(M)$. Let $\varphi\colon R \to S$ denote the canonical homomorphism. Let $\mathfrak{n}$ be a two-sided maximal ideal of S. Then*:

(i) $\mathfrak{m} = \varphi^{-1}(\mathfrak{n})$ *is a maximal ideal of* R.

(ii) $I = \{f \in S \mid f(M) \subseteq M\mathfrak{m}\}$ *is a two-sided ideal of S contained in $\mathfrak{n}$ and satisfying* $\mathfrak{m} = \varphi^{-1}(I)$.

Proof. (i) By Lemma 2.1 and Remark 2.2, the morphism $\varphi\colon R \to S$ induces a central integral extension $R/\mathrm{ann}_R(M) \to S$. Then $S/\mathfrak{n}$ is a simple ring and $R/\varphi^{-1}(\mathfrak{n}) \to S/\mathfrak{n}$ is also a central integral extension. The center of the simple ring $S/\mathfrak{n}$ is always a field K (the two-sided ideal generated by a non-zero element x in the center must be the total and, being the element central, this ideal is just the set of multiples of x; this is to say that x is invertible in $S/\mathfrak{n}$ because x is central, so is its inverse). We deduce that $R/\varphi^{-1}(\mathfrak{n}) \to K$ is an integral extension of commutative rings. Hence, $R/\varphi^{-1}(\mathfrak{n})$ is a field (cf. Lemma 2.5) and $\mathfrak{m} = \varphi^{-1}(\mathfrak{n})$ is a maximal ideal of R.

(ii) By Remark 1.2, I is a two-sided ideal of S. Clearly, $\mathfrak{m} \subseteq \varphi^{-1}(I)$. Since $\mathrm{ann}_R(M)$ is contained in the maximal ideal $\mathfrak{m}$ of R and M_R is finitely generated, $M_\mathfrak{m} \neq \{0\}$. By Nakayama's lemma, $M \neq M\mathfrak{m}$, so we conclude that $1 \notin \varphi^{-1}(I)$. Since $\mathfrak{m}$ is a maximal ideal contained in $\varphi^{-1}(I)$, we deduce that $\mathfrak{m} = \varphi^{-1}(I)$.

By Lemma 2.1 (iii), $(I + \mathfrak{n})/\mathfrak{n}$ is a two-sided nil-ideal of $S/\mathfrak{n}$. Since $S/\mathfrak{n}$ is simple, we deduce that $(I + \mathfrak{n})/\mathfrak{n}$ is the zero ideal, so $I \subseteq \mathfrak{n}$, as claimed. ∎

2.1. Finitely generated torsion-free modules

Let R be a domain with field of fractions Q. An R-module M is *torsion-free* if the natural map $M \to M_Q$ is injective, where M_Q denotes the localization of an R-module M at $R \setminus \{0\}$. Sometimes we will also use the tensor product notation $M \otimes_R Q$.

It is well known that, over commutative rings, Hom and localization commute when the first variable of the Hom is a finitely presented module. The following lemma shows that this is true for finitely generated torsion-free modules. This result is crucial throughout the paper. For a similar result (and proof), the reader can check Lemma 3.5 in Warfield's paper [33].

Lemma 2.8. *Let R be a domain, and let Λ be an R-algebra. Let Σ be a multiplicative subset of R. If M is a finitely generated right Λ-module which is torsion-free as an R-module, then the canonical injective homomorphism*

$$\varphi : \mathrm{Hom}_\Lambda(M, N) \otimes_R R_\Sigma \to \mathrm{Hom}_{\Lambda \otimes_R R_\Sigma}(M \otimes_R R_\Sigma, N \otimes_R R_\Sigma)$$

is an isomorphism for every Λ-module N which is torsion-free as an R-module.

Proof. Note that the Λ-modules that are torsion-free as R-modules can be seen as submodules of its localization at Σ.

Let $f \in \mathrm{Hom}_{\Lambda \otimes_R R_\Sigma}(M \otimes_R R_\Sigma, N \otimes_R R_\Sigma)$. Let $m_1, \dots, m_n$ be a set of generators of M. Then, there exists $r \in \Sigma$, such that, for any $i \in \{1, \dots, n\}$, $f(m_i) = n_i \otimes \frac{1}{r}$

for $n_i \in N$. Therefore $g = fr|_M \in \mathrm{Hom}_\Lambda(M, N)$, so that $f = \varphi(g \otimes \frac{1}{r})$. This shows that φ is onto, as claimed in the statement. ∎

Let Λ be an algebra over a commutative domain R. We say that a short exact sequence

$$(*) \qquad 0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

of right Λ-modules is *locally split* if it is split when we apply the functor $R_{\mathfrak{m}} \otimes_R -$ for any maximal ideal $\mathfrak{m}$ of R. Therefore, if X is a left Λ-module we obtain an exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow L \otimes_\Lambda X \longrightarrow M \otimes_\Lambda X \longrightarrow N \otimes_\Lambda X \longrightarrow 0.$$

As $(*)$ is locally split, $R_{\mathfrak{m}} \otimes_R K = 0$ for any maximal ideal $\mathfrak{m}$ of R. Therefore $K = 0$. This proves that a locally split exact sequence of right Λ-modules is pure.

Now we show that the finitely generated Λ-modules that are torsion-free as R-modules are projective with respect to the class of locally split exact sequences of right Λ-modules that are torsion-free as R-modules.

Lemma 2.9. *Let R be a commutative domain, and let Λ be an R-algebra. Let L, M, N and X be right Λ-modules which are torsion-free as R-modules. Assume that there is a short exact sequence*

$$(2.1) \qquad 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0.$$

Then:

(i) *If X_Λ is finitely generated, and (2.1) is a locally split exact sequence, then the map $\mathrm{Hom}_\Lambda(X, g)\colon \mathrm{Hom}_\Lambda(X, M) \to \mathrm{Hom}_\Lambda(X, N)$ is onto. Equivalently, finitely generated torsion-free modules are projective with respect to the class of locally split exact sequences of torsion-free modules.*

(ii) *If N is a direct summand of a direct sum of finitely generated Λ-modules that are torsion-free as R-modules, then the exact sequence (2.1) splits if and only if it is locally split.*

Proof. (i) If $g_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a splitting epimorphism for every maximal ideal $\mathfrak{m}$ of R. Then, $\mathrm{Hom}_{\Lambda_{\mathfrak{m}}}(X_{\mathfrak{m}}, g_{\mathfrak{m}})\colon \mathrm{Hom}_{\Lambda_{\mathfrak{m}}}(X_{\mathfrak{m}}, M_{\mathfrak{m}}) \to \mathrm{Hom}_{\Lambda_{\mathfrak{m}}}(X_{\mathfrak{m}}, N_{\mathfrak{m}})$ is onto for every maximal ideal $\mathfrak{m}$ of R. By Lemma 2.8, we may identify $\mathrm{Hom}_{\Lambda_{\mathfrak{m}}}(X_{\mathfrak{m}}, g_{\mathfrak{m}})$ with $R_{\mathfrak{m}} \otimes_R \mathrm{Hom}_\Lambda(X, g)$. Therefore, $R_{\mathfrak{m}} \otimes_R \mathrm{Hom}_\Lambda(X, g)$ is onto for every maximal ideal $\mathfrak{m}$ of R, so $\mathrm{Hom}_\Lambda(X, g)$ is onto, as desired.

(ii) As the class of modules that are projective with respect to the class of locally split exact sequences is closed under direct sums and direct summands, we deduce the statement (ii) from (i). ∎

Corollary 2.10. *Let R be a commutative domain, and let Λ be an R-algebra. Let N be a direct summand of a direct sum of finitely generated right Λ-modules which are torsion-free as R-modules. Then N is a projective Λ-module if and only if $N_{\mathfrak{m}}$ is a projective $\Lambda_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Let $f: F \to N$ be an epimorphism of Λ-modules for some free Λ-module F. By Lemma 2.9(ii), f is a splitting epimorphism if and only if $f_{\mathfrak{m}}$ is a splitting epimorphism for every maximal ideal $\mathfrak{m}$ of R. This is equivalent to the statement. ∎

The following easy lemma will be useful throughout the paper.

Lemma 2.11. *Let R be a commutative domain, and let Λ be an R-algebra. Let Σ be a multiplicative subset of R. Let N be a right Λ-module which is torsion-free as an R-module. If M is a finitely generated Λ-submodule of N_Σ, then dM is isomorphic to a Λ-submodule of N for some $d \in \Sigma$.*

Proof. Note that, since N is torsion-free as an R-module, it can be seen as a Λ-submodule of N_Σ.

Let $m \in M$ and let $\{n_1/s_1, \dots, n_t/s_t\} \subseteq N_\Sigma$ be a finite set of Λ-generators of M. Then,

$$m = \frac{n_1}{1}\frac{a_1}{s_1} + \cdots + \frac{n_t}{1}\frac{a_t}{s_t} \in N_\Sigma$$

for some $a_1, \dots, a_t \in \Lambda$, and $s_1, \dots, s_t \in \Sigma$. Multiplying by $d = s_1 \cdots s_t \in \Sigma$,

$$dm = \frac{n_1 a_1' + \cdots + n_t a_t'}{1} \in \lambda(N),$$

where $\lambda: N \to N_\Sigma$ denotes the localization map. Therefore, $dM \subseteq \lambda(N) \cong N$. ∎

Remark 2.12. We note that Lemmas 2.8 and 2.11 could be stated for general commutative rings, provided that the multiplicative set Σ consists of non-zero divisors of R. In this general context, we mean that a module M is torsion-free if $md = 0$, for $m \in M$ and d a non-zero divisor of R, implies that $m = 0$.

Let R be a commutative domain, and let Q denote the field of fractions of R. The *rank* of an R-module M is the dimension of M_Q as a Q-vector space.

Remark 2.13. *Let R be a commutative domain with field of fractions Q. Let N be a torsion-free R-module of rank n. Then N contains a free R-module of rank n.* Note that, since N is torsion-free of rank n, we can see N as an essential submodule of its injective hull $E(M) \cong Q^n$. On the other hand, R^n has the same injective hull Q^n. Therefore, by Lemma 2.11, since R^n is a finitely generated R-submodule of $N_Q = Q^n$, $dR^n \subseteq N$ for some non-zero element $d \in R$.

Lemma 2.14. *Let R be a commutative domain with field of fractions Q. Let M and N be non-zero torsion-free R-modules of the same finite rank n. Let $f: M \to N$ be an R-module homomorphism. Then*:

(i) *f is onto if and only if it is bijective*;

(ii) *f is injective if and only if* $\operatorname{Im} f$ *is an essential submodule of N.*

Proof. Applying the functor $- \otimes_R Q$ to the exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow M \xrightarrow{f} N \longrightarrow 0,$$

we get the exact sequence

$$0 \longrightarrow \operatorname{Ker} f \otimes_R Q \longrightarrow M \otimes_R Q \xrightarrow{f \otimes_R \mathrm{Id}_Q} N \otimes_R Q \longrightarrow 0,$$

in which the Q-module homomorphism $f \otimes_R \mathrm{Id}_Q$ is an isomorphism because it is an onto morphism between Q-vector spaces of the same finite dimension. Therefore, $\operatorname{Ker} f \otimes_R Q = 0$ and $\operatorname{Ker} f$ being torsion-free, it must be zero. Hence, f is bijective.

The statement (ii) is easy to prove. ∎

Proposition 2.15. *Let R be a commutative domain with field of fractions Q, and let Λ be a torsion-free R-algebra of finite rank. Let M_Λ be a non-zero finitely generated right Λ-module which is torsion-free as an R-module. Then*:

(i) *any onto endomorphism of M is bijective*;

(ii) *if $J(R) \subseteq J(\Lambda)$, then* $\mathrm{End}_\Lambda(M)J(R) \subseteq J(\mathrm{End}_\Lambda(M))$;

(iii) *if Λ is semilocal, then so is* $\mathrm{End}_\Lambda(M)$.

Proof. (i) Since M is a homomorphic image of a finite number of copies of Λ, M is a torsion-free R-module of finite rank. Since a Λ-module homomorphism is also an R-module homomorphism, the statement follows from Lemma 2.14 (i).

(ii) Our hypothesis allows us to repeat the proof of Lemma 2.6 to conclude the statement.

(iii) If Λ is semilocal, then $MJ(\Lambda)$ is a small submodule of M and $M/MJ(\Lambda)$ is semisimple artinian of finite length. By (i), any onto endomorphism ring of M is bijective. As an application of [13], we deduce that $\mathrm{End}_\Lambda(M)$ is semilocal. ∎

2.2. Domains of finite character and h-local domains

Definition 2.16. A commutative domain R is said to have *finite character* if any non-zero element is contained only in a finite number of maximal ideals. A commutative domain of finite character is said to be *h-local* if, in addition, any non-zero prime ideal of R is contained in a unique maximal ideal.

The domain R has finite character if R/I is a semilocal ring for every non-zero ideal I of R. If R is h-local then it also satisfies that $R/\mathfrak{p}$ is a local domain for every non-zero prime ideal $\mathfrak{p}$ of R.

If R is a domain of Krull dimension 1, then the notion of h-locality and being of finite character coincide. They are equivalent to saying that, for every non-zero ideal I of R, R/I is a semiperfect ring whose Jacobson radical is a nil ideal. Because R/I is a ring of Krull dimension 0 with only a finite number of maximal ideals, $J(R/I)$ coincides with the nilradical of R/I.

Matlis introduced the notion of h-local domain in [22] as a generalization of local domains and noetherian domains of Krull dimension 1. In the next result, we recall the key property of h-local domains that we will use throughout the paper.

Lemma 2.17. *A commutative domain R is h-local if and only if R/I is semiperfect for every non-zero proper ideal I of R. In particular, if R is h-local, I is a non-zero proper ideal of R, and $\{\mathfrak{m}_1, \dots, \mathfrak{m}_\ell\}$ is the finite set of maximal ideals of R containing I, then*

(i) $I = I_1 \cdots I_\ell$, *where* $I_i = I_{\mathfrak{m}_i} \cap R$, *so* $I_1, \dots, I_\ell$ *are pairwise comaximal ideals of* R *such that each* I_i *is contained in exactly one maximal ideal of* R;

(ii) *the canonical map* $R/I \to (R/I)_{\mathfrak{m}_1} \times \cdots \times (R/I)_{\mathfrak{m}_\ell}$, *given by the localization at each component, is an isomorphism, and* $(R/I)_{\mathfrak{m}_i} = (R/I_i)_{\mathfrak{m}_i}$ *for* $i = 1, \dots, \ell$.

Proof. The first part is Theorem 4.9 in [4]. Statement (i) is proven in Lemma 5.1 of [24]. To prove (ii), notice that, by the Chinese remainder theorem, there is an isomorphism

$$\varphi : R/I \longrightarrow R/I_1 \times \cdots \times R/I_\ell.$$

Since each I_i is contained only in the maximal ideal $\mathfrak{m}_i$, we deduce that R/I_i is a local ring and that $R/I_i = (R/I_i)_{\mathfrak{m}_i}$. So that we have the isomorphisms claimed. ∎

Lemma 2.18. *Let* R *be a commutative domain with field of fractions* Q. *Let* M *be a non-zero finitely generated torsion-free module with endomorphism ring* S. *Then*:

(i) S *is a torsion-free* R*-module containing* R, *and* S *is integral over* R. *The restriction of the embedding* $R \subseteq S$ *gives an embedding* $J(R) \subseteq J(S)$.

(ii) *For any* $f \in S$, *there is some non-zero* $d \in R$ *such that* $df \in M \operatorname{Hom}_R(M, R)$. *In particular,* $M \operatorname{Hom}_R(M, R) \neq \{0\}$.

(iii) $S_Q \cong M_n(Q)$, *where* n *is the rank of* M. *So* S *is a subring of the simple artinian algebra* $M_n(Q)$.

(iv) *For any non-zero two-sided ideal* I *of* S, $I \cap R \neq \{0\}$.

(v) $S_{\mathfrak{m}} \cong \operatorname{End}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$ *is a semilocal ring for all maximal ideals* $\mathfrak{m}$ *of* R.

If, in addition, R *is h-local, then*:

(vi) *Let* I *be any non-zero two-sided ideal of* S. *Then* S/I *is a semilocal ring. Moreover,* $I_{\mathfrak{m}} = S_{\mathfrak{m}}$ *for almost all maximal ideals* $\mathfrak{m}$ *of* R. *If* R *has Krull dimension* 1, *then* $J(S/I)$ *is a nilideal, so* S/I *is semiperfect.*

(vii) *If* I *is the trace ideal of a non-zero projective right* S*-module, then* I *is contained only in a finite number of trace ideals of projective right* S*-modules.*

Proof. Recall that, since R is commutative, there is a ring morphism $\varphi: R \to S$ where, for any $r \in S$, $\varphi(r)$ is the morphism given by multiplication of r. Since M is a faithful R-module, φ is injective, and we may identify R as a subring of S. We will use this identification throughout the proof.

(i) Let $f \in S$ be such that there is some non-zero $r \in R$ with $rf = 0$. Equivalently, $rf(M) = \{0\}$ so that $f(M) = \{0\}$ because M is torsion-free.

The extension $R \subseteq S$ is integral because of Lemma 2.1. Repeating the arguments in Lemma 2.6, we deduce that $J(R) \subseteq J(S)$.

(ii) Since M is finitely generated and torsion-free, we can see M as an essential submodule of its injective hull $E(M) \cong Q^n$ for some $n \geq 1$. On the other hand, R^n has the same injective hull Q^n. Therefore, since M is finitely generated, by Lemma 2.11, $gM \subseteq R^n$ for some non-zero $g \in R \subseteq S$.

Now we proceed with the proof of the statement. The claim is clear for the zero endomorphism. Pick $f \in S \setminus \{0\}$. Since Q^n is an injective R-module, there exists

$$h \in \operatorname{Hom}_R(R^n, Q^n) \cong M_n(Q)$$

such that the diagram

$$\begin{array}{ccccc} M & \xrightarrow{f} & M & \hookrightarrow & Q^n \\ & {\scriptstyle g}\searrow & & \nearrow{\scriptstyle h} & \\ & & R^n & & \end{array}$$

commutes. Since R^n is finitely generated and M is essential in Q^n, by Lemma 2.11, $dh(R^n) \subseteq M$ for some non-zero $d \in R$, that is, $dh \in \operatorname{Hom}_R(R^n, M)$. Therefore, $df = (dh) \circ g \in M \operatorname{Hom}_R(M, R)$.

(iii) By Lemma 2.8, $S_Q \cong \operatorname{End}_Q(M_Q) \cong M_n(Q)$, where n is the rank of M. By (i), S is torsion-free as an R-module, so the localization map $S \to S_Q$ is injective.

(iv) By (iii), we can identify S with a subring of its central localization at the non-zero divisors of R which is the simple ring $M_n(Q)$. If I is a non-zero ideal of S, then $I_Q = IM_n(Q)$ is a non-zero ideal of $M_n(Q)$. Therefore $IM_n(Q) = M_n(Q)$. Therefore, there is some $f \in I$ and some non-zero $d \in R$ such that $f/d = \mathrm{Id}_n$. This implies that f is the endomorphism of M given by multiplication by d. So $f \in R \cap I$.

(v) By Lemma 2.8, $S_\mathfrak{m} \cong \operatorname{End}_{R_\mathfrak{m}}(M_\mathfrak{m})$. By Lemma 2.1, $S_\mathfrak{m}$ is semilocal.

From now on, we are assuming that R is h-local.

(vi) Let I be a non-zero two-sided ideal of S. The inclusion $\varphi \colon R \to S$ induces an injective ring homomorphism $R/(I \cap R) \to S/I$. By (iv), $I \cap R \neq \{0\}$ and, since R is h-local there are only finitely many maximal ideals of R, say $\mathfrak{m}_1, \dots, \mathfrak{m}_t$, containing $I \cap R$. By Lemma 2.17, the canonical homomorphisms

$$R/(I \cap R) \longrightarrow (R/(I \cap R))_{\mathfrak{m}_1} \times \cdots \times (R/(I \cap R))_{\mathfrak{m}_t}$$

and

$$S/I \longrightarrow (S/I)_{\mathfrak{m}_1} \times \cdots \times (S/I)_{\mathfrak{m}_t}$$

are isomorphisms. Therefore, we have a commutative diagram

$$\begin{array}{ccc} R/(I \cap R) & \hookrightarrow & S/I \\ \downarrow{\cong} & & \downarrow{\cong} \\ \big(R/(I \cap R)\big)_{\mathfrak{m}_1} \times \cdots \times \big(R/(I \cap R)\big)_{\mathfrak{m}_\ell} & \hookrightarrow & \big(S/I\big)_{\mathfrak{m}_1} \times \cdots \times \big(S/I\big)_{\mathfrak{m}_\ell}. \end{array}$$

By (v), S/I is a finite product of semilocal rings, so it is a semilocal ring.

Since $I \cap R$ is only contained in a finite number of maximal ideals of R, we have that $(I \cap R)_\mathfrak{m} = R_\mathfrak{m}$ for almost all maximal ideals $\mathfrak{m}$ of R. Therefore, $I_\mathfrak{m} = S_\mathfrak{m}$ for almost all maximal ideals $\mathfrak{m}$ of R.

Assume now that R has Krull dimension 1. As $R/(I \cap R) \subseteq S/I$ is an integral extension, and $R/(I \cap R)$ is a ring of Krull dimension 0, Lemma 2.5 implies that $J(S/I)$ is a nil-ideal. As idempotents can be lifted modulo a nilideal we deduce that S/I is a semiperfect ring.

The statement (vii) follows from (vi) because semilocal rings only have a finite number of trace ideals of projective modules. ∎

3. Local domains of Krull dimension one

Now we are ready to start our study of direct summands of direct sums of torsion-free modules. In this section, we will specialize to the case of local domains of Krull dimension one. The next lemma points out the key properties of endomorphism rings of torsion-free modules over such domains that we will need.

Lemma 3.1. *Let R be a commutative local domain of Krull dimension 1, and with field of fractions Q. Let M be a non-zero finitely generated torsion-free R-module with endomorphism ring S. If M does not contain a non-zero free direct summand, then*:

(i) $J = M\,\mathrm{Hom}_R(M,R) \subseteq J(S)$ *and* $J(S)/M\,\mathrm{Hom}_R(M,R)$ *is a nil-ideal of* S/J.

(ii) *If every direct summand of* $R^{(\omega)} \oplus M$ *is a direct sum of finitely generated modules, then S is semiperfect.*

Proof. By Lemma 2.1, M satisfies that any onto endomorphism ring of M is bijective.

(i) By Corollary 1.21,

$$J = M\,\mathrm{Hom}_R(M,R) \subseteq J(S).$$

The rest of the statement is included in Lemma 2.18.

(ii) Follows from (i) and Corollary 1.21. ■

Proposition 3.2. *Let R be a commutative local domain of Krull dimension 1, and with field of fractions Q. Let M be a finitely generated torsion-free R-module. The following statements are equivalent*:

(i) *every module in* $\mathrm{Add}(R \oplus M)$ *is a direct sum of finitely generated modules*;

(ii) *every direct summand of* $R^{(\omega)} \oplus M$ *is a direct sum of finitely generated modules*;

(iii) $\mathrm{End}_R(M)$ *is semiperfect*; *and*

(iv) *every finitely generated indecomposable module in* $\mathrm{Add}(R \oplus M)$ *has local endomorphism ring.*

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii) Write $M = F \oplus M'$, with F finitely generated free and M' has no projective direct summands. By Lemma 3.1 (ii), $\mathrm{End}_R(M')$ is semiperfect. As R is local, so is $\mathrm{End}_R(M)$.

Statement (iii) is equivalent to saying that $M = Y_1 \oplus \cdots \oplus Y_m$, where each Y_i has local endomorphism ring for $i \in \{1,\dots,n\}$ (cf. Theorem 3.14 in [6]).

In general, over a commutative ring, direct summands of direct sums of finitely generated modules with local endomorphism rings are also direct sums of finitely generated modules with local endomorphism rings, and such decomposition is unique (cf. Corollary 2.55 in [6]). Therefore, every element in $\mathrm{Add}(R \oplus M)$ is isomorphic to a direct sum of copies of $R, Y_1, \dots, Y_m$. This proves (i).

In particular, since every indecomposable in $\mathrm{Add}(R \oplus M)$ is isomorphic to one in $\{R, Y_1, \dots, Y_m\}$, we deduce (iv).

Note that (iv) also implies that M is a direct sum of indecomposable modules with local endomorphism rings, so this implies (iii). ■

Corollary 3.3. *Let R be a commutative local domain of Krull dimension* 1*, and with field of fractions Q. The following statements are equivalent*:

(i) *the class of direct sums of finitely generated torsion-free R-modules is closed under direct summands*;

(ii) *for any finitely generated torsion-free R-module M, every direct summand of $R^{(\omega)} \oplus M$ is a direct sum of finitely generated modules*;

(iii) *for any finitely generated torsion-free R-module M, every module in* $\mathrm{Add}(M)$ *is a direct sum of finitely generated modules*; *and*

(iv) *every finitely generated, indecomposable, torsion-free R-module has local endomorphism ring.*

In addition, the above equivalent conditions are fulfilled by any intermediate ring S between R and its integral closure.

Proof. The equivalence of the four statements is an immediate consequence of Proposition 3.2.

To finish the proof of the statement, let S be a ring such that $R \subseteq S \subseteq \overline{R}$. Notice that if X_S is a finitely generated torsion-free S-module then it has finite rank n. Identifying X with a submodule of Q^n,

$$\mathrm{End}_S(X) = \{A \in \mathrm{End}_Q(Q^n) \mid AX \subseteq X\} = \mathrm{End}_R(X).$$

In particular, X is indecomposable as an S-module if and only if it is indecomposable as an R-module.

If S is finitely generated over R, then also X is finitely generated over R, so by (iv), X_S is indecomposable if and only if $\mathrm{End}_S(X)$ is local.

Assume S is not finitely generated over R, and fix $x_1, \dots, x_r$ a family of generators of X_S. Then $\mathrm{End}_S(X)$ is the directed union of $\mathrm{End}_T(\sum_{i=1}^r x_i T)$, where T varies between all subrings of S that are finite extensions of S. If X_S is indecomposable, then so is the T-module $\sum_{i=1}^r x_i T$. By the previous step, $\mathrm{End}_T(\sum_{i=1}^r x_i T)$ is a local ring for any T. Therefore, $\mathrm{End}_S(X)$ is also a local ring. ∎

We do not know of a characterization of local domains (of Krull dimension 1) such that their indecomposable finitely generated torsion-free modules have local endomorphism ring. If we restrict the condition to finitely generated, torsion-free, rank one modules, then it is equivalent to having a local integral closure, as we show in the next result.

Proposition 3.4. *Let R be a commutative local domain of Krull dimension* 1*, and with field of fractions Q. The following statements are equivalent*:

(i) *the class of direct sums of finitely generated, rank-one, torsion-free R-modules is closed under direct summands*;

(ii) *for any finitely generated, rank-one, torsion-free R-module M, every direct summand of $R^{(\omega)} \oplus M$ is a direct sum of finitely generated modules*;

(iii) *every finitely generated, rank-one, torsion-free module has local endomorphism ring*; *and*

(iv) *the integral closure of R* (*into its field of fractions*) *is a local ring.*

Proof. The equivalence of (i), (ii) and (iii) follows from Proposition 3.2.

(iii) ⇒ (iv). By (iii), any finitely generated, integral extension of R inside Q is local. Let $S \subseteq Q$ be any integral extension of R inside Q. If S has two different maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, then for any $i = 1, 2$, choose $s_i \in \mathfrak{m}_i$ such that $s_1 + s_2 = 1$. Then $R \subseteq R[s_1, s_2]$ is a finitely generated integral extension of R. By the first part of the proof, $R[s_1, s_2]$ is a local ring, but its maximal ideal should contain s_1 and s_2, which contradicts our assumptions. Therefore, S is a local ring.

(iv) ⇒ (iii) If M is a finitely generated, rank-one, torsion-free module, then its endomorphism ring S is a subring of Q, integral over S. Hence, it is contained in the integral closure $\overline{R}$ of R. Since $S \subseteq \overline{R}$ is integral and $\overline{R}$ is local, so is S (cf. Theorem 9.3 in [23]). ■

In the case of noetherian domains of Krull dimension 1 with finitely generated integral closure, Příhoda in [28] proved that having local integral closure (so the integral closure is a discrete valuation ring) already implies that all indecomposable finitely generated torsion-free modules have local endomorphism ring.

The following criteria to ensure local endomorphism ring encodes Příhoda's ideas in Proposition 1 of [28]. It will allow us to prove that for any noetherian local domain of Krull dimension 1 having local integral closure is equivalent to the fact that indecomposable, finitely generated, torsion-free modules have local endomorphism ring.

Lemma 3.5. *Let R be a commutative local domain of Krull dimension* 1*, and with field of fractions Q. Let M_R be a finitely generated, indecomposable, torsion-free module. Assume that*:

(i) *there exists a local over ring T of R, of Krull dimension* 1*, contained in Q, such that MT_T is a direct sum of T-modules with local endomorphism ring*;

(ii) *the conductor ideal $\mathfrak{c} = \{r \in R \mid rT \subseteq R\}$ is non-zero.*

Then M_R has local endomorphism ring.

Proof. Since MT_T is a (finite) direct sum of modules with local endomorphism ring, then $\mathrm{End}_T(MT)$ is a semiperfect ring. Since $\mathfrak{c}$ is a non-zero proper ideal of both the rings R and T, it is contained in their corresponding maximal ideals. Since R and T have Krull dimension 1, $R/\mathfrak{c}$ and $T/\mathfrak{c}$ are rings of Krull dimension zero.

Let $S = \mathrm{End}_R(M)$ and $J = \{f \in S \mid f(M) \subseteq M\mathfrak{c}\}$. By Proposition 2.3, J is a two-sided ideal of S contained in $J(S)$. So it is enough to show that S/J is a local ring.

Since $R \subseteq T \subseteq Q = E(R)$, M can be identified with an essential submodule of Q^n for some n. Via this identification, $S = \{A \in M_n(Q) \mid AM \subseteq M\}$ and $\mathrm{End}_T(MT) = \{A \in M_n(Q) \mid AMT \subseteq MT\}$, so that S is a subring of the semiperfect ring $\mathrm{End}_T(MT_T)$. Moreover, $J = \{A \in M_n(Q) \mid AM \subseteq M\mathfrak{c}\} = \{A \in M_n(Q) \mid AMT \subseteq M\mathfrak{c}\}$, so it is also a two-sided ideal of $\mathrm{End}_T(MT_T)$ contained in its Jacobson radical.

By Lemma 2.1, $R/(J \cap R) \subseteq S/J$ is an integral extension. As $\mathfrak{c} \subseteq J \cap R$, $R/(J \cap R)$ has Krull dimension 0, and by Lemma 2.5, we deduce that $J(S)/J$ is a nil-ideal. So the idempotents of the semisimple artinian ring $S/J(S)$ can be lifted to S/J. Let $e \in S$ be such that $e + J$ is an idempotent of $S/J \subseteq \mathrm{End}_T(MT_T)/J$. Since $\mathrm{End}_T(MT_T)$ is a semiperfect ring, by Proposition 27.4 and Theorem 27.6 in [1], there exists an idempotent $e' \in \mathrm{End}_T(MT_T)$ such that $e - e' \in J$. But then $e'M \subseteq (e' - e)M + eM \subseteq M\mathfrak{c} + M = M$.

Hence $e' \in S$. Since M is indecomposable, e' is a trivial idempotent. Therefore, $e + J$ is also a trivial idempotent of S/J and we deduce that S/J is a local ring. ■

Lemma 3.6. *Let R be a commutative local domain of Krull dimension* 1*, and with local integral closure. Let M be a finitely generated indecomposable torsion-free R-module. Assume that there is a finitely generated integral extension T of R such that MT_T is a direct sum of T-modules with local endomorphism ring. Then M_R has local endomorphism ring.*

Proof. Since the integral closure of R is local, so is T. Moreover, T is also a ring of Krull dimension 1, and since T_R is finitely generated, the conductor ideal $\mathfrak{c} = \{r \in R \mid rT \subseteq R\}$ is different from zero. Therefore, the statement follows from Lemma 3.5. ■

Lemma 3.7. *Let R be a commutative local domain of Krull dimension* 1*, and with field of fractions Q. Assume that the integral closure $\overline{R}$ of R is a valuation domain. For any non-zero finitely generated torsion-free module M, there exists a finite integral extension T of R inside Q such that $MT \cong T \oplus M'$ for a suitable T-module M'.*

Proof. Let $n > 0$ be the rank of M. We may assume that M is an essential submodule of Q^n that contains R^n as an (essential) submodule. Since $\overline{R}$ is a valuation domain, the module $M\overline{R}_{\overline{R}}$ is free (see, for example, Corollary V.2.8 in [8]), so that $\mathrm{Tr}_{\overline{R}}(M\overline{R}) = \overline{R}$. Therefore, there exist $f_1, \dots, f_\ell \in \mathrm{Hom}_{\overline{R}}(M\overline{R}, \overline{R})$ and $m_1, \dots, m_\ell \in M$ such that $1 = f_1(m_1) + \dots + f_\ell(m_\ell)$.

Since $\overline{R}^n$ is a submodule of $M\overline{R}$,

$$\mathrm{Hom}_{\overline{R}}(M\overline{R}, \overline{R}) = \{(s_1, \dots, s_n) \in \overline{R}^n \mid (s_1, \dots, s_n)M \subseteq \overline{R}\}.$$

So that, for $i = 1, \dots, \ell$, $f_i = (s_1^i, \dots, s_n^i)$. Let T be the finite integral extension of R generated by the elements

$$\{s_j^i\}_{\substack{i=1,\dots,\ell \\ j=1,\dots,n}} \quad \text{and} \quad \bigcup_{i=1}^{\ell} f_i(G),$$

where G is a finite set of generators of M. By construction, $\mathrm{Tr}_T(MT) = T$ and then, by Lemma 1.16 we deduce that there exists M' such that $MT \cong T \oplus M'$. ■

Proposition 3.8. *Let R be a commutative local domain of Krull dimension* 1*, and with field of fractions Q. Assume that the integral closure $\overline{R}$ of R is a valuation domain. Let M_R be a finitely generated torsion-free module. Then M is a direct sum of modules with local endomorphism ring.*

Proof. We are going to prove the statement by induction on the rank n of M. If $n = 1$, Lemma 3.7 implies that $MT \cong T$. Then, since T must be also a local ring, we may conclude by Lemma 3.6. Now assume that $n > 1$ and that the statement is proven for modules of smaller rank. By the previous claim, there exists a finite integral extension of T such that $MT \cong T \oplus M'$ for a suitable T-module M' of smaller rank. By the inductive hypothesis, M' is a T-module that is a direct sum of modules with local endomorphism ring. By Lemma 3.6, we conclude that M is a direct sum of modules with local endomorphism ring. ■

Corollary 3.9. *Let R be a commutative local noetherian domain of Krull dimension* 1*, and with field of fractions Q. Then the following statements are equivalent*:

(i) *every direct summand of a direct sum of finitely generated, torsion-free modules of rank one is a direct sum of finitely generated modules*;

(ii) *for any finitely generated, torsion-free R-module M of rank one, every direct summand of $R^{(\omega)} \oplus M$ is a direct sum of finitely generated modules*;

(iii) *every finitely generated, torsion-free module of rank one has local endomorphism ring*;

(iv) *the integral closure of R is local*;

(v) *every direct summand of a direct sum of finitely generated, torsion-free modules is a direct sum of finitely generated modules*;

(vi) *for any finitely generated and torsion-free R-module M, every direct summand of $R^{(\omega)} \oplus M$ is a direct sum of finitely generated modules*;

(vii) *every finitely generated, indecomposable, and torsion-free module has local endomorphism ring.*

Proof. The first four statements are equivalent because of Proposition 3.4. Since R is a commutative local noetherian ring of Krull dimension 1, its integral closure is a discrete valuation ring. Hence, by Proposition 3.8, all finitely generated torsion-free modules are direct sums of modules with local endomorphism rings. Now, the equivalence of the remaining statements follows from Corollary 3.3. This finishes the proof of the equivalence of the statements. ∎

The reader is referred to Lemma 8.6 for examples that satisfy the conclusion of Corollary 3.9.

4. Local versus global direct summands

The following result, for finitely generated torsion-free modules over a domain R, gives a general relation between the property of being locally a direct summand and being a direct summand. Afterwards, we will see how the result can be refined when we assume, in addition, that the domain is of finite character.

Proposition 4.1. *Let R be a commutative domain, and Λ be an R-algebra. Let M and N be finitely generated right Λ-modules which are torsion-free as R-modules. If $M_{\mathfrak{m}}$ is a direct summand of $N_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R, then there exists $k > 0$ such that M is a direct summand of N^k.*

Proof. Let $\{\mathfrak{m}_\alpha\}_{\alpha\in\Omega}$ denote the set of all maximal ideals of R and fix $\alpha \in \Omega$. Since $M_{\mathfrak{m}_\alpha}$ is a direct summand of $N_{\mathfrak{m}_\alpha}$, there are $\Lambda_{\mathfrak{m}_\alpha}$-module homomorphisms $\tilde{f}_\alpha\colon N_{\mathfrak{m}_\alpha} \to M_{\mathfrak{m}_\alpha}$ and $\tilde{g}_\alpha\colon M_{\mathfrak{m}_\alpha} \to N_{\mathfrak{m}_\alpha}$ such that $\tilde{f}_\alpha \circ \tilde{g}_\alpha = \mathrm{Id}_{M_{\mathfrak{m}_\alpha}}$. By Lemma 2.8, there are Λ-homomorphisms $f_\alpha\colon N \to M$ and $g_\alpha\colon M \to N$ such that $\tilde{f}_\alpha = f_\alpha/s_\alpha$, $\tilde{g}_\alpha = g_\alpha/s_\alpha$ and $f_\alpha \circ g_\alpha = s_\alpha^2 \mathrm{Id}_M$ for some $s_\alpha \in R \setminus \mathfrak{m}_\alpha$.

Note that, by the definition of the elements s_α,

$$I = \sum_{\alpha \in \Omega} s_\alpha^2 R = R,$$

because no maximal ideal of R contains I. Therefore, there exist $k > 0$, $\alpha_1, \dots, \alpha_k \in \Omega$ and $r_1, \dots, r_k \in R$ such that $1 = \sum_{i=1}^k s_{\alpha_i}^2 r_i$. Hence

$$\sum_{i=1}^k r_i f_{\alpha_i} \circ g_{\alpha_i} = \Big(\sum_{i=1}^k s_{\alpha_i}^2 r_i \Big) \mathrm{Id}_M = \mathrm{Id}_M,$$

and we conclude that M is a direct summand of N^k. ∎

The following lemma is an extension of Lemma 2.1 in [10] to the case of finitely generated torsion-free modules over commutative domains of finite character.

Lemma 4.2. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be an R-algebra, and let M and N be right Λ-modules which are torsion-free as R-modules. Assume that there exists a homomorphism of Λ_Q-modules $F\colon M_Q \to N_Q$. If M is finitely generated, then*:

(i) *If F is a monomorphism* (*respectively, an epimorphism*)*, then there is a Λ-module homomorphism $f\colon M \to N$ such that the induced homomorphism $f_\mathfrak{m}\colon M_\mathfrak{m} \to N_\mathfrak{m}$ is a monomorphism* (*respectively, an epimorphism*) *for almost all maximal ideals $\mathfrak{m}$ of R.*

Moreover, if N is also finitely generated, then:

(ii) *If F is a splitting monomorphism* (*respectively, a splitting epimorphism*)*, then there is a Λ-module homomorphism $f\colon M \to N$ such that the induced homomorphism $f_\mathfrak{m}\colon M_\mathfrak{m} \to N_\mathfrak{m}$ is a splitting monomorphism* (*respectively, a splitting epimorphism*) *for almost all maximal ideals $\mathfrak{m}$ of R.*

(iii) *If F is an isomorphism, then there is a Λ-module homomorphism $f\colon M \to N$ such that the induced homomorphism $f_\mathfrak{m}\colon M_\mathfrak{m} \to N_\mathfrak{m}$ is an isomorphism for almost all maximal ideals $\mathfrak{m}$ of R.*

Proof. (i) By Lemma 2.8, there is a Λ-module homomorphism $f\colon M \to N$ such that $F = f/s$ for some non-zero $s \in R$. Since R is of finite character, s is contained only in finitely many maximal ideals of R. For any other maximal ideal $\mathfrak{m}$, $s/1$ is a unit in $R_\mathfrak{m}$. Therefore, $f_\mathfrak{m}$ is a monomorphism (respectively, an epimorphism) for every maximal ideal not containing s.

(ii) If F is a splitting monomorphism, then there is a Λ_Q-module homomorphism $G\colon N_Q \to M_Q$ such that $G \circ F = \mathrm{Id}_{M_Q}$. In particular, G is a splitting epimorphism. By Lemma 2.8, there are Λ-module homomorphisms $f\colon M \to N$ and $g\colon N \to M$ such that $F = f/s$, $G = g/s$ and $g \circ f = s^2 \mathrm{Id}_M$ for some non-zero $s \in R$. For any maximal ideal $\mathfrak{m}$ not containing s, $s/1$ is a unit in $R_\mathfrak{m}$. Therefore, $f_\mathfrak{m}$ is a splitting monomorphism and $g_\mathfrak{m}$ is a splitting epimorphism for every maximal ideal not containing s. The proof in the case where F is a splitting epimorphism follows by changing the roles of F and G.

The proof of (iii) follows from (i) and (ii) since F is an isomorphism if and only if it is a monomorphism and a splitting epimorphism. ∎

Corollary 4.3. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be an R-algebra, and let N be a finitely generated right Λ-module which is torsion-free as an R-module. If N_Q is projective, then $N_{\mathfrak{m}}$ is projective for almost all maximal ideals $\mathfrak{m}$ of R.*

Proof. Suppose N_Q is projective and let $\tilde{f}\colon F_Q \to N_Q$ be a splitting epimorphism for some finitely generated free Λ-module F. By Lemma 4.2, there is a Λ-module homomorphism $f\colon F \to N$ such that the induced homomorphism $f_{\mathfrak{m}}\colon F_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a splitting epimorphism for almost all maximal ideals $\mathfrak{m}$ of R. Thus, $N_{\mathfrak{m}}$ is projective for almost all maximal ideals $\mathfrak{m}$ of R. ∎

Lemma 4.4. *Let R be a commutative domain, and let Λ be a module-finite R-algebra. Let M and N be finitely generated right Λ-modules which are torsion-free as R-modules. If $N_{\mathfrak{m}}$ is a direct summand of $M_{\mathfrak{m}}$ for each $\mathfrak{m}$ in some finite subset $\mathcal{M} \subseteq \operatorname{mSpec} R$, then there is a Λ-module homomorphism $f\colon M \to N$ such that the induced homomorphism $f_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a splitting epimorphism for every $\mathfrak{m} \in \mathcal{M}$.*

Proof. Let $\mathcal{M} = \{\mathfrak{m}_1, \dots, \mathfrak{m}_k\}$ and fix $i \in \{1, \dots, k\}$. Since $N_{\mathfrak{m}_i}$ is a direct summand of $M_{\mathfrak{m}_i}$, there are $\Lambda_{\mathfrak{m}_i}$-module homomorphisms $\tilde{f}_i\colon M_{\mathfrak{m}_i} \to N_{\mathfrak{m}_i}$ and $\tilde{g}_i\colon N_{\mathfrak{m}_i} \to M_{\mathfrak{m}_i}$ such that $\tilde{f}_i \circ \tilde{g}_i = \mathrm{Id}_{N_{\mathfrak{m}_i}}$. By Lemma 2.8, there are Λ-module homomorphisms $f_i\colon M \to N$ and $g_i\colon N \to M$ such that $\tilde{f}_i = f_i/s_i$, $\tilde{g}_i = g_i/s_i$ and $f_i \circ g_i = s_i^2 \,\mathrm{Id}_N$ for some $s_i \notin \mathfrak{m}_i$. Note that, for any $r \notin \mathfrak{m}_i$, $(rf_i)_{\mathfrak{m}_i}$ is a splitting epimorphism.

For any $i \in \{1, \dots, k\}$, because $R = \mathfrak{m}_i + \bigcap_{j \neq i} \mathfrak{m}_j$, $1 = s_i + r_i$ with $s_i \in \mathfrak{m}_i$ and $r_i \in \bigcap_{j \neq i} \mathfrak{m}_j$. Let $f = \sum_{i=1}^{k} r_i f_i$. We claim that $f_{\mathfrak{m}_i}$ is a splitting epimorphism. Since $(r_i f_i)_{\mathfrak{m}_i}$ is a splitting epimorphism, there is a $\Lambda_{\mathfrak{m}_i}$-module homomorphism $h\colon N_{\mathfrak{m}_i} \to M_{\mathfrak{m}_i}$ such that $(r_i f_i)_{\mathfrak{m}_i} \circ h = \mathrm{Id}_{N_{\mathfrak{m}_i}}$. Hence

$$f_{\mathfrak{m}_i} \circ h = \mathrm{Id}_{N_{\mathfrak{m}_i}} + h', \quad \text{where } h' \in \operatorname{End}_{\Lambda_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i})\, \mathfrak{m}_i R_{\mathfrak{m}_i}.$$

By Lemma 2.6, $h' \in J(\operatorname{End}_{\Lambda_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i}))$. Therefore, $f_{\mathfrak{m}_i} \circ h$ is invertible and $f_{\mathfrak{m}_i}$ is a splitting epimorphism, as claimed. ∎

Corollary 4.5. *Let R be a commutative semilocal domain, and let Λ be a module-finite R-algebra. Let M and N be finitely generated right Λ-modules which are torsion-free as R-modules. Then:*

(i) *N is a direct summand of M if and only if $N_{\mathfrak{m}}$ is a direct summand of $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R.*

(ii) *N is isomorphic to M if and only if $N_{\mathfrak{m}}$ is isomorphic to $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R.*

Proof. Statement (i) follows immediately from Lemma 4.4. To prove (ii), observe that by (i), there exist K and K' finitely generated Λ-modules such that $M \cong N \oplus K$ and $N \cong M \oplus K'$. Since N and M are torsion-free R-modules of the same rank, we deduce that $K = K' = 0$. ∎

The following is an extension of Lemma 6.1 in [10] to the case of torsion-free modules over domains of finite character.

Proposition 4.6. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be a module-finite R-algebra, and let M, N and K be finitely generated right Λ-modules which are torsion-free as R-modules. Assume that*:

(i) *$M_{\mathfrak{m}}$ is a direct summand of $N_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R, and*

(ii) *M_Q is a direct summand of K_Q.*

Then M is a direct summand of $N \oplus K$.

Proof. It suffices to find Λ-module homomorphisms $f: N \to M$, $g: K \to M$ such that, for any $\mathfrak{m} \in \operatorname{mSpec} R$, either $f_{\mathfrak{m}}$ or $g_{\mathfrak{m}}$ is a splitting epimorphism. For then $(f, g): N \oplus K \to M$ is locally a splitting epimorphism and, by Lemma 2.9, is a splitting epimorphism.

By condition (ii), since M_Q is a direct summand of K_Q, there is a splitting epimorphism $\tilde{g}: K_Q \to M_Q$. By Lemma 4.2, there is a Λ-module homomorphism $g: K \to M$ such that the induced homomorphism $g_{\mathfrak{m}}: K_{\mathfrak{m}} \to M_{\mathfrak{m}}$ is a splitting epimorphism for almost all maximal ideals $\mathfrak{m}$ of R.

By condition (i), since $M_{\mathfrak{m}}$ is a direct summand of $N_{\mathfrak{m}}$, there is a splitting epimorphism $\tilde{f}_{\mathfrak{m}}: N_{\mathfrak{m}} \to M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R. Let $\mathcal{M} \subseteq \operatorname{mSpec} R$ be the finite set of maximal ideals such that $g_{\mathfrak{m}}$ is not a splitting epimorphism. By Lemma 4.4, there is a Λ-module homomorphism $f: N \to M$ such that the induced homomorphism $f_{\mathfrak{m}}: N_{\mathfrak{m}} \to M_{\mathfrak{m}}$ is a splitting epimorphism for every $\mathfrak{m} \in \mathcal{M}$. This finishes the proof of the proposition. ∎

5. Package deal theorems for localizations over h-local domains

In this section, we will develop tools to deal with modules over an algebra Λ over an h-local domain R. A particular instance of this situation is when Λ is the endomorphism ring of a finitely generated, torsion-free, R-module (recall Lemma 2.18). We are interested in knowing what modules over such endomorphism rings are.

In Subsection 5.1, we prove basic properties of the localization at maximal ideals of R of finitely generated modules over an algebra Λ over a domain of finite character.

In Subsection 5.2, we enter the study of constructing Λ-modules having prescribed localization at the maximal ideals of R, and proving the so-called *Package Deal Theorems*. To do that, we extend the methods of Levy and Odenthal from algebras over noetherian rings of Krull dimension 1 to algebras over h-local domains.

Finally, in Subsection 5.3, we prove a package deal theorem for traces of countably generated projective modules. Trace ideals are not, in general, finitely generated but, as we will see, satisfy enough finiteness conditions to be able to extend our methods to this case.

5.1. Finitely generated modules over domains of finite character

Lemma 5.1. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be an R-algebra, and let M be a finitely generated right Λ-module, which is torsion-free as an R-module. Let N be a Λ-submodule of M such that $M_Q = N_Q$. Then $M_{\mathfrak{m}} = N_{\mathfrak{m}}$ for almost all maximal ideals $\mathfrak{m}$ of R.*

In particular, $N_{\mathfrak{m}}$ is finitely generated as $\Lambda_{\mathfrak{m}}$-module for almost all maximal ideals $\mathfrak{m}$ of R.

Proof. Since M is torsion-free as an R-module, it can be seen as a Λ-submodule of $M_Q = N_Q$. Therefore, $M \subseteq N_Q$, and since M is finitely generated, by Lemma 2.11, $dM \subseteq N$ for some non-zero $d \in R$. Since R is of finite character and d is non-zero, d is contained only in finitely many maximal ideals of R. For any other maximal ideal $\mathfrak{m}$, $d/1$ is a unit in $R_\mathfrak{m}$, hence, $M_\mathfrak{m} = N_\mathfrak{m}$, and $N_\mathfrak{m}$ is finitely generated. ∎

Lemma 5.2. *Let R be a commutative domain of finite character with field of fractions Q. Let Y be a submodule of a finitely generated torsion-free R-module M. Then, there exist $y_1, \dots, y_r \in Y$ such that $Y_\mathfrak{m}$ is a finitely generated free $\Lambda_\mathfrak{m}$-module with basis $y_1/1, \dots, y_r/1$ for almost all maximal ideals $\mathfrak{m}$ of R.*

Proof. By Remark 2.13, we may assume that $M \leq R^s$ for $s = \operatorname{rank} M$, and then Y is a submodule of R^s.

Let $r = \operatorname{rank} Y$, and notice that $r \leq s$. Let $y_1, \dots, y_r \in Y$ be such that they are R-linearly independent (that is, they form a basis of Y_Q). Then, there is an embedding $f\colon R^r \to R^s$ given by $f(a_1, \dots, a_r) = \sum_{i=1}^r y_i a_i$. Since f_Q is a splitting monomorphism, by Lemma 4.2, $f_\mathfrak{m}$ is a splitting monomorphism for almost all maximal ideals of R, which implies that $Z_\mathfrak{m} = \operatorname{Im} f_\mathfrak{m} = \sum_{i=1}^r \frac{y_i}{1} R_\mathfrak{m}$ is a direct summand of $R^s_\mathfrak{m}$ for almost all maximal ideals $\mathfrak{m}$ of R. Since $Z_\mathfrak{m} \subseteq Y_\mathfrak{m}$ and they have the same rank, we deduce, from the modular law, that $Z_\mathfrak{m} = Y_\mathfrak{m}$ for almost all maximal ideals $\mathfrak{m}$. Therefore, $Y_\mathfrak{m}$ has $y_1/1, \dots, y_r/1$ as a basis for almost all maximal ideals $\mathfrak{m}$ of R. ∎

Corollary 5.3. *Let R be a commutative domain of finite character with field of fractions Q. Let M be a finitely generated R-module. Then:*

(i) *If M is torsion-free, then $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for almost all maximal ideals $\mathfrak{m}$ of R;*

(ii) *$M_\mathfrak{m}$ is a finitely presented $R_\mathfrak{m}$-module of projective dimension at most one for almost all maximal ideals $\mathfrak{m}$ of R.*

Proof. (i) Since M is torsion-free as an R-module, it can be seen as an essential submodule of its injective hull Q^n. By Remark 2.13, $dR^n \subseteq M$ for some non-zero $d \in R$. By Lemma 5.1 with $\Lambda = R$ and $N = dR^n$, $M_\mathfrak{m} = R^n_\mathfrak{m}$ for almost all maximal ideals $\mathfrak{m}$ of R, as claimed.

(ii) Assume now that M is a finitely generated R-module. Consider a presentation of M,

$$0 \longrightarrow N \longrightarrow R^n \longrightarrow M \longrightarrow 0.$$

By Lemma 5.2, $N_\mathfrak{m}$ is a finitely generated free module for almost all maximal ideals $\mathfrak{m}$ of R. This proves the claim. ∎

Corollary 5.4. *Let R be a commutative domain of finite character. Let M be a countable direct sum of finitely generated torsion-free R-modules. Then $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for all but countably many maximal ideals $\mathfrak{m}$ of R.*

Proof. Let $M = \bigoplus_{i \in \mathbb{N}} M_i$, where each M_i is a finitely generated torsion-free module. By Corollary 5.3, $(M_i)_\mathfrak{m}$ is free for almost all maximal ideals $\mathfrak{m}$ of R. Therefore, $M_\mathfrak{m}$ is free for all but countably many maximal ideals $\mathfrak{m}$ of R. ∎

5.2. Gluing localizations over h-local domains

It is very important to keep in mind the following structure result of torsion modules over h-local domains. It was already proved by Matlis, see Theorem 22 in [21], but since the result and its proof are quite important in our work, we also include a proof of it.

Lemma 5.5. *Let R be an h-local domain, and let Λ be an R-algebra. Let M be a Λ-module that is torsion as an R-module. Then $M \cong \bigoplus_{\mathfrak{m} \in \mathrm{mSpec}(R)} M_{\mathfrak{m}}$, each $M_{\mathfrak{m}}$ is a homomorphic image of M, and its structure as $\Lambda_{\mathfrak{m}}$-module is the same as the structure as Λ-module.*

Moreover, M is finitely generated as Λ-module if and only if there exists a finite set $\mathcal{S}$ of maximal ideals of R such that $M_{\mathfrak{m}} = \{0\}$ for any $\mathfrak{m} \notin \mathcal{S}$ and $M_{\mathfrak{m}}$ is a finitely generated $\Lambda_{\mathfrak{m}}$-module for any $\mathfrak{m} \in \mathcal{S}$

Proof. For any maximal ideal $\mathfrak{m}$ of R, let $\lambda_{\mathfrak{m}}\colon M \to M_{\mathfrak{m}}$ denote the localization map. Then there is an inclusion $\lambda\colon M \hookrightarrow \prod_{\mathfrak{m}\in\mathrm{mSpec}(R)} M_{\mathfrak{m}}$ defined by $m \mapsto (\lambda_{\mathfrak{m}}(m))_{\mathfrak{m}\in\mathrm{mSpec}(R)}$ for any $m \in M$.

For any $m \in M$, $m\Lambda$ is an $R/\mathrm{ann}_R(m)$-module. Being M torsion as an R-module, $\mathrm{ann}_R(m) \neq \{0\}$. Since R is h-local, $\mathrm{ann}_R(m)$ is contained only in finitely many maximal ideals of R. Therefore, $m\Lambda_{\mathfrak{m}} = \{0\}$ for almost all maximal ideals $\mathfrak{m}$ of R. Hence, λ has its image in $\bigoplus_{\mathfrak{m}\in\mathrm{mSpec}(R)} M_{\mathfrak{m}}$.

Let $\mathcal{S}$ denote the finite set of maximal ideals containing $\mathrm{ann}_R(m)$. By Lemma 2.17,

$$m\Lambda \cong \bigoplus_{\mathfrak{m}\in\mathcal{S}} (m\Lambda)_{\mathfrak{m}},$$

and also

$$\Lambda/\mathrm{ann}_R(m) \cong \prod_{\mathfrak{m}\in\mathcal{S}} (\Lambda/\mathrm{ann}_R(m))_{\mathfrak{m}}.$$

This allows us to conclude that, for any $m \in M$ and $\mathfrak{m} \in \mathrm{mSpec}(R)$, there exists $m' \in M$ such that $\lambda_{\mathfrak{n}}(m') = 0$ for any $\mathfrak{n} \neq \mathfrak{m}$ and such that $\lambda_{\mathfrak{m}}(m') = \lambda_{\mathfrak{m}}(m)$. This implies that λ induces an isomorphism $M \cong \bigoplus_{\mathfrak{m}\in\mathrm{mSpec}(R)} M_{\mathfrak{m}}$.

In particular, for any maximal ideal $\mathfrak{m}$ of R, $M_{\mathfrak{m}}$ is a homomorphic image of M and the structure as Λ-module is the same as the structure as $\Lambda_{\mathfrak{m}}$-module. So the statement about the finite generation of M easily follows from our previous discussion. ■

Next lemma will be used to determine when we get finitely generated modules in our package deal results.

Lemma 5.6. *Let R be an h-local domain with field of fractions Q. Let Λ be an R-algebra, and let M be a right Λ-module, which is torsion-free as an R-module. Let N be a Λ-submodule of M such that $M_Q = N_Q$. Then N is finitely generated as a Λ-module if and only if $N_{\mathfrak{m}}$ is finitely generated as a $\Lambda_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R.*

Proof. It is clear that if N is finitely generated as a Λ-module, then $N_{\mathfrak{m}}$ is finitely generated as a $\Lambda_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R.

To prove the converse implication, assume $N_{\mathfrak{m}}$ is finitely generated as a $\Lambda_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R. Since M is torsion-free as an R-module, it can be seen as a Λ-submodule of $M_Q = N_Q$. Therefore, $M \subseteq N_Q$, and since M is finitely generated as a Λ-module, by Lemma 2.11, $dM \subseteq N$ for some non-zero $d \in R$. Note that dM is also finitely generated. So we only need to prove that N/dM is a finitely generated $\Lambda/d\Lambda$-module.

Since R is h-local and d is non-zero, d is contained only in finitely many maximal ideals of R, say $\mathcal{M} = \{\mathfrak{n}_1, \dots, \mathfrak{n}_t\}$. By Lemma 2.17, there is an isomorphism

$$\varphi : R/dR \to (R/dR)_{\mathfrak{n}_1} \times \cdots \times (R/dR)_{\mathfrak{n}_t}.$$

Therefore, $N/dM \cong N_{\mathfrak{n}_1}/dM_{\mathfrak{n}_1} \oplus \cdots \oplus N_{\mathfrak{n}_t}/dM_{\mathfrak{n}_t}$. By Lemma 5.5, N/dM is finitely generated as a $\Lambda/d\Lambda$-module. Therefore, N is finitely generated as a Λ-module. ■

Package Deal Theorem 5.7 (Localization of submodules). *Let R be an h-local domain with field of fractions Q. Let Λ be an R-algebra, and let M be a finitely generated right Λ-module, which is torsion-free as an R-module. For each maximal ideal $\mathfrak{m}$ of R, let $X(\mathfrak{m})$ be a $\Lambda_{\mathfrak{m}}$-submodule of $M_{\mathfrak{m}}$, which is torsion-free as an $R_{\mathfrak{m}}$-module, and such that $X(\mathfrak{m})_Q = (M_{\mathfrak{m}})_Q$. Then the following statements are equivalent*:

(i) *There is a Λ-submodule $N \subseteq M$, which is torsion-free as an R-module, and such that $N_{\mathfrak{m}} = X(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R.*

(ii) *$X(\mathfrak{m}) = M_{\mathfrak{m}}$ for almost all maximal ideals $\mathfrak{m}$ of R.*

Moreover, if each of the $X(\mathfrak{m})$ is finitely generated as a $\Lambda_{\mathfrak{m}}$-module, then N is also finitely generated as a Λ-module.

Proof. (ii) ⇒ (i) Suppose that $X(\mathfrak{m}) = M_{\mathfrak{m}}$ for almost all maximal ideals $\mathfrak{m}$ of R. Let $\mathcal{M} = \{\mathfrak{n}_1, \dots, \mathfrak{n}_t\}$ denote the finite set of maximal ideals of R such that $X(\mathfrak{m}) \neq M_{\mathfrak{m}}$.

Fix $i \in \{1, \dots, t\}$. Since $M_{\mathfrak{n}_i}$ is torsion-free as an $R_{\mathfrak{n}_i}$-module, it can be seen as a $\Lambda_{\mathfrak{n}_i}$-submodule of $(M_{\mathfrak{n}_i})_Q = X(\mathfrak{n}_i)_Q$. Therefore, $M_{\mathfrak{n}_i} \subseteq X(\mathfrak{n}_i)_Q$, and since $M_{\mathfrak{n}_i}$ is finitely generated, by Lemma 2.11, $d_i M_{\mathfrak{n}_i} \subseteq X(\mathfrak{n}_i)$ for some non-zero $d_i \in R$. Since $X(\mathfrak{n}_i) \neq M_{\mathfrak{n}_i}$, $d_i \in \mathfrak{n}_i$ (otherwise $d_i/1$ would be a unit in $R_{\mathfrak{n}_i}$ and, hence $X(\mathfrak{n}_i) = M_{\mathfrak{n}_i}$, a contradiction).

Let $d = d_1 \cdots d_t \in \mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_t$. Since R is h-local and d is non-zero, d is contained only in finitely many maximal ideals of R, say $\{\mathfrak{m}_1 = \mathfrak{n}_1, \dots, \mathfrak{m}_t = \mathfrak{n}_t, \dots, \mathfrak{m}_k\}$. By Lemma 2.17, the canonical homomorphism

$$\varphi : R/dR \to (R/dR)_{\mathfrak{m}_1} \times \cdots \times (R/dR)_{\mathfrak{m}_k}$$

is an isomorphism. Therefore, there are non-zero $b_1, \dots, b_t, b \in R$ such that $\varphi(\bar{b}_i) = (\bar{0}, \dots, \bar{1}^{(i)}, \dots, \bar{0})$ and $\varphi(\bar{b}) = (\bar{0}, \overset{(t}{\dots}, \bar{0}, \bar{1}, \dots, \bar{1})$. Note that $b_i/1$ is a unit in $R_{\mathfrak{n}_i}$ for every $i = 1, \dots, t$, $b/1, b_j/1 \in dR_{\mathfrak{n}_i}$ for every $j \neq i$, and $b/1$ is a unit in $R_{\mathfrak{m}_i}$ for every $i = t+1, \dots, k$.

For each $i \in \{1, \dots, t\}$, let X_i' be the Λ-submodule of M generated by the numerators of some set of $\Lambda_{\mathfrak{n}_i}$-generators of $X(\mathfrak{n}_i)$. Let $N = b_1 X_1' + \cdots + b_t X_t' + bM + dM$, which is a Λ-submodule of M. Since $dM_{\mathfrak{n}_i} \subseteq X(\mathfrak{n}_i)$, $N_{\mathfrak{n}_i} = X(\mathfrak{n}_i) + dM_{\mathfrak{n}_i} = X(\mathfrak{n}_i)$ for every $i = 1, \dots, t$. On the other hand, $N_{\mathfrak{m}_i} = M_{\mathfrak{m}_i} + dM_{\mathfrak{m}_i} = M_{\mathfrak{m}_i}$ for every $i = t+1, \dots, k$. For any other maximal ideal $\mathfrak{m}$, $d/1$ is a unit in $R_{\mathfrak{m}}$, hence $N_{\mathfrak{m}} = M_{\mathfrak{m}}$, as claimed.

(i) ⇒ (ii) Conversely, let N be a Λ-submodule of $M \subseteq M_Q$, which is torsion-free as an R-module, and such that $N_{\mathfrak{m}} = X(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R. Since $M_Q = X(\mathfrak{m})_Q = (N_{\mathfrak{m}})_Q$, by Lemma 5.1, $M_{\mathfrak{m}} = N_{\mathfrak{m}} = X(\mathfrak{m})$ for almost all maximal ideals $\mathfrak{m}$ of R.

Finally, the last part of the statement follows from Lemma 5.6. This finishes the proof of the theorem. ∎

Corollary 5.8. *Let R be an h-local domain with field of fractions Q. For each maximal ideal $\mathfrak{m}$ of R, let $X(\mathfrak{m})$ be a finitely generated torsion-free $R_{\mathfrak{m}}$-module of rank n. Then the following statements are equivalent*:

(i) *There is a finitely generated torsion-free R-module N of rank n, such that $N_{\mathfrak{m}} \cong X(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R.*

(ii) *$X(\mathfrak{m})$ is a free $R_{\mathfrak{m}}$-module of rank n for almost all maximal ideals $\mathfrak{m}$ of R.*

Proof. (ii) ⇒ (i) Suppose that $X(\mathfrak{m})$ is a free $R_{\mathfrak{m}}$-module of rank n for almost all maximal ideals $\mathfrak{m}$ of R. Let $\mathcal{M} = \{\mathfrak{m}_1, \dots, \mathfrak{m}_t\}$ be the finite set of maximal ideals of R such that $X(\mathfrak{m})$ is not free.

Fix $i \in \{1, \dots, t\}$. Since $X(\mathfrak{m}_i)$ is finitely generated and torsion-free, we can see $X(\mathfrak{m}_i)$ as an essential submodule of its injective hull Q^n. On the other hand, $R^n_{\mathfrak{m}_i}$ has the same injective hull Q^n. Since $X(\mathfrak{m}_i)$ is finitely generated, by Lemma 2.11, we find that $d_i X(\mathfrak{m}_i) \subseteq R^n_{\mathfrak{m}_i}$ for some non-zero $d_i \in R$.

Let $d = d_1 \dots d_t \in R$ and let $M = d^{-1}R^n$. Note that $X(\mathfrak{m})$ is an $R_{\mathfrak{m}}$-submodule of $M_{\mathfrak{m}}$. Since R is h-local and d is non-zero, d is contained only in finitely many maximal ideals of R. For any other maximal ideal $\mathfrak{m}$, $d/1$ is a unit in $R_{\mathfrak{m}}$. Therefore, $M_{\mathfrak{m}} = R^n_{\mathfrak{m}} = X(\mathfrak{m})$ for almost all maximal ideals $\mathfrak{m}$ of R. By the package deal Theorem 5.7, there is a finitely generated torsion-free R-module N, such that $N_{\mathfrak{m}} = X(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R.

(i) ⇒ (ii) The converse is just Corollary 5.3. ∎

The results in Section 4 allow us to make the following considerations about the (non-)uniqueness of the modules constructed in Theorem 5.7 and Corollary 5.8.

Remark 5.9. It is well known that two finitely generated modules M and M' over a commutative domain R that are in the same genus (that is, with isomorphic localizations at maximal ideals of R) need not be isomorphic.

If M and M' are also torsion-free, by Proposition 4.1, $\mathrm{add}(M) = \mathrm{add}(M')$. If R has finite character, by Proposition 4.6, M is a direct summand of $M' \oplus M'$ and M' is a direct summand of $M \oplus M$. If R is semilocal, then $M \cong M'$ by Corollary 4.5.

5.3. Package deal for traces of projective modules

Now we want to focus on localizations of trace ideals of countably generated projective modules over suitable algebras over h-local domains. Trace ideals of countably generated projective right modules were characterized by Whitehead in [34] and with more detail by Herbera and Příhoda in [11]. We recall here this characterization.

Proposition 5.10 (Proposition 2.4 in [11]). *Let R be a ring, and let I be a two-sided ideal of R. Then I is the trace ideal of a countably generated projective right R-module if and only if there exists an ascending chain of finitely generated left ideals $(J_n)_{n\geq 1}$ such that $J_{n+1}J_n = J_n$ and $I = \bigcup_{n\geq 1} J_n R$.*

It is useful to keep in mind the following lemma, as it explains some modifications that can be made in the ascending chain in Proposition 5.10.

Lemma 5.11 (Lemma 2.2 in [11]). *Let R be a ring. Let $J_1 \subseteq J_2$ be finitely generated left ideals of R satisfying that $J_2 J_1 = J_1$. For $i = 1, 2$, fix A_i a finite set of generators of J_i. Let X be a finite subset of R such that $1 \in X$. For $i = 1, 2$, set*

$$J_i' = \sum_{\substack{x\in X\\ a\in A_i}} Rax.$$

Then $J_1' \subseteq J_2'$ and $J_2' J_1' = J_1'$. Moreover, for $i = 1, 2$, $J_i \subseteq J_i R = J_i' R$.

Lemma 5.12. *Let R be an h-local domain with field of fractions Q. Let Λ be a torsion-free R-algebra such that Λ_Q is a simple artinian ring. Let I be a non-zero two-sided ideal of Λ. Then $I_{\mathfrak{m}} = \Lambda_{\mathfrak{m}}$ for almost all maximal ideals $\mathfrak{m}$ of R.*

Proof. I_Q is a non-zero two-sided ideal of Λ_Q, so $I_Q = \Lambda_Q$. Therefore, there is a non-zero $q \in R$ such that $q = \sum a_i\lambda_i$ for $a_i \in I$ and $\lambda_i \in \Lambda$. Since R is h-local, q is invertible in almost all maximal ideals of R. The claim follows. ∎

Package Deal Theorem 5.13 (Localization of trace ideals). *Let R be an h-local domain with field of fractions Q. Let Λ be a torsion-free R-algebra such that Λ_Q is a simple artinian ring. For each maximal ideal $\mathfrak{m}$ of R, let $I(\mathfrak{m})$ be a non-zero two-sided ideal of $\Lambda_{\mathfrak{m}}$ which is the trace ideal of a countably generated projective right $\Lambda_{\mathfrak{m}}$-module. Then the following statements are equivalent*:

(i) *There is a two-sided ideal I of Λ, which is the trace ideal of a countably generated projective right Λ-module, and such that $I_{\mathfrak{m}} = I(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R.*

(ii) *$I(\mathfrak{m}) = \Lambda_{\mathfrak{m}}$ for almost all maximal ideals $\mathfrak{m}$ of R.*

Moreover, the ideal I that satisfies the equivalent conditions (i) *and* (ii) *is unique.*

Proof. By Lemmas 10.3 and 10.4 in [12], two trace ideals with isomorphic localizations at maximal ideals of R are equal. This implies that the ideal I in statement (i) is unique. Now we proceed to prove the equivalence of the two statements.

(ii) ⇒ (i) Let $\mathcal{M} = \{\mathfrak{n}_1, \dots, \mathfrak{n}_t\}$ and fix some $i \in \{1, \dots, t\}$. We may assume that $I(\mathfrak{n}_i) \neq \Lambda_{\mathfrak{n}_i}$. Since $I(\mathfrak{n}_i)$ is the trace ideal of a countably generated projective right $\Lambda_{\mathfrak{n}_i}$-module, by Proposition 5.10, there is an ascending chain of non-zero finitely generated left ideals $(J_{i,n})_{n\geq 1}$ of $\Lambda_{\mathfrak{n}_i}$ such that

$$J_{i,n+1}J_{i,n} = J_{i,n} \quad \text{and} \quad I(\mathfrak{n}_i) = \bigcup_{n\geq 1} J_{i,n}\Lambda_{\mathfrak{n}_i}.$$

Fix $n \geq 1$. Let $A_{i,n} \subseteq \Lambda$ be a finite set of $\Lambda_{\mathfrak{n}_i}$-generators of $J_{i,n}$. Since $J_{i,n}$ is non-zero and Λ_Q is simple artinian, $(J_{i,n})_Q\Lambda_Q = \Lambda_Q$. Since Λ is torsion-free, it can be seen as a Λ-submodule of $\Lambda_Q = (J_{i,n})_Q\Lambda_Q$. Therefore, $\Lambda \subseteq (J_{i,n})_Q\Lambda_Q$, and by Lemma 2.11, $d_{i,n}\Lambda_{\mathfrak{n}_i} \subseteq J_{i,n}\Lambda_{\mathfrak{n}_i}$ for some non-zero $d_{i,n} \in \mathfrak{n}_i$. Then,

$$\frac{1}{d_{i,n}}\sum_{j=1}^{m_{i,n}}\sum_{a_{i,n}\in A_{i,n}} \frac{r_{j,i,n}}{1}\frac{a_{i,n}}{1}\frac{s_{j,i,n}}{1} = 1_{\Lambda_Q} \in \Lambda_Q,$$

for some $r_{j,i,n}, s_{j,i,n} \in \Lambda$ for every $j = 1, \dots, m_{i,n}$.

Let

$$X_n = \{1_\Lambda\} \cup \bigcup_{i=1}^{t}\{s_{1,i,n}, s_{2,i,n}, \dots, s_{m_{i,n},i,n}\} \subseteq \Lambda$$

and define

$$J'_{i,n} = \sum_{\substack{x_n\in X_n\\ a_{i,n}\in A_{i,n}}} \Lambda_{\mathfrak{n}_i}a_{i,n}x_n.$$

By Lemma 5.11, $J'_{i,n} \subseteq J'_{i,n+1}$, $J'_{i,n+1}J'_{i,n} = J'_{i,n}$, and $J_{i,n} \subseteq J_{i,n}\Lambda_{\mathfrak{n}_i} = J'_{i,n}\Lambda_{\mathfrak{n}_i}$. Let $d_n = d_{1,n}\dots d_{t,n} \in \mathfrak{n}_1 \cap \dots \cap \mathfrak{n}_t$. Note that $d_n\Lambda_{\mathfrak{n}_i} \subseteq J'_{i,n}$ by construction. Define

$$L_{i,n} = \sum_{\substack{x_n\in X_n\\ a_{i,n}\in A_{i,n}}} \Lambda a_{i,n}x_n + \Lambda d_n,$$

which are left ideals of Λ. Note that $(L_{i,n})_{\mathfrak{n}_i} = J'_{i,n}$ because $d_n\Lambda_{\mathfrak{n}_i} \subseteq J'_{i,n}$. Since R is of finite character and d_n is non-zero, d_n is contained only in finitely many maximal ideals of R. For any other maximal ideal $\mathfrak{m}$, $d_n/1$ is a unit in $R_\mathfrak{m}$, hence $(L_{i,n})_\mathfrak{m} = \Lambda_\mathfrak{m}$. Let $\mathcal{M}_n = \{\mathfrak{m}_1 = \mathfrak{n}_1, \dots, \mathfrak{m}_t = \mathfrak{n}_t, \mathfrak{m}_{t+1}, \dots, \mathfrak{m}_{k_n}\}$ be the finite set of maximal ideals containing d_n. By Lemma 2.17, the canonical homomorphism

$$\varphi_n : R/d_nR \longrightarrow (R/d_nR)_{\mathfrak{m}_1} \times \dots \times (R/d_nR)_{\mathfrak{m}_{k_n}}$$

is an isomorphism. Therefore, there are non-zero elements $b_{1,n}, \dots, b_{t,n}, b_n \in R$ such that $\varphi_n(\bar{b}_{i,n}) = (\bar{0}, \dots, \bar{1}^{(i)}, \dots, \bar{0})$ and $\varphi_n(\bar{b}_n) = (\bar{0}, \overset{t)}{\dots}, \bar{0}, \bar{1}, \dots, \bar{1})$.

Let $J_n = L_{1,n}b_{1,n} + \dots + L_{t,n}b_{t,n} + \Lambda b_n + \Lambda d_n$. Note that $(J_n)_{\mathfrak{n}_i} = J'_{i,n}$, and $(J_n)_\mathfrak{m} = \Lambda_\mathfrak{m}$ for every $\mathfrak{m} \neq \mathfrak{n}_i$. Consider the short exact sequence of left Λ-modules

$$0 \longrightarrow J_{n+1} \longrightarrow J_{n+1} + J_n \longrightarrow (J_{n+1} + J_n)/J_{n+1} \longrightarrow 0.$$

Since localization is an exact functor,

$$0 \longrightarrow (J_{n+1})_\mathfrak{m} \longrightarrow (J_{n+1} + J_n)_\mathfrak{m} \longrightarrow (J_{n+1} + J_n)_\mathfrak{m}/(J_{n+1})_\mathfrak{m} \longrightarrow 0$$

is also a short exact sequence of left $\Lambda_\mathfrak{m}$-modules. Note that, since $J'_{i,n} \subseteq J'_{i,n+1}$, $(J_{n+1} + J_n)_{\mathfrak{n}_i}/(J_{n+1})_{\mathfrak{n}_i} = (J'_{i,n+1} + J'_{i,n})/J'_{i,n+1} = 0$ for every $i = 1, \dots, t$. On the other hand, $(J_{n+1} + J_n)_\mathfrak{m}/(J_{n+1})_\mathfrak{m} = (\Lambda_\mathfrak{m} + \Lambda_\mathfrak{m})/\Lambda_\mathfrak{m} = 0$ for every $\mathfrak{m} \neq \mathfrak{n}_i$. Therefore, $(J_{n+1} + J_n)/J_{n+1} = 0$ and we deduce that $(J_n)_{n\geq 1}$ is an increasing sequence of left ideals of Λ.

Note that $(J_{n+1})_{\mathfrak{n}_i}(J_n)_{\mathfrak{n}_i} = J'_{i,n+1}J'_{i,n} = J'_{i,n} = (J_n)_{\mathfrak{n}_i}$ for every $i = 1, \dots, t$; and also $(J_{n+1})_\mathfrak{m}(J_n)_\mathfrak{m} = \Lambda_\mathfrak{m}^2 = \Lambda_\mathfrak{m} = (J_n)_\mathfrak{m}$ for every $\mathfrak{m} \neq \mathfrak{n}_i$. Therefore, $J_{n+1}J_n = J_n$

for every $n \geq 1$, and by Proposition 5.10, $I = \bigcup_{n\geq 1} J_n\Lambda$ is a two-sided ideal of Λ such that it is the trace ideal of a countably generated projective right Λ-module, $I_{\mathfrak{n}_i} = I(\mathfrak{n}_i)$ and $I_{\mathfrak{m}} = \Lambda_{\mathfrak{m}}$ otherwise.

(i) $\Rightarrow$ (ii) The converse is just Lemma 5.12. ■

6. The global case

In this section, we study when the class $\mathcal{F}$, of direct sums of finitely generated torsion-free modules over an h-local domain R, is closed under direct summands. In the next proposition, we prove that this property *localizes* at maximal ideals of R, so the results from Section 3 hold for $R_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of R.

It is worth mentioning that we do not know how to prove directly that if $\mathcal{F}$ is closed under direct summands then the same holds for $R_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of R. The property that passes to the localization of R is that, for any finitely generated torsion-free R-module, every direct summand of $R^{(\omega)} \oplus X$ is a direct sum of finitely generated modules.

Proposition 6.1. *Let R be an h-local domain of Krull dimension* 1. *Let X be a finitely generated torsion-free R-module, and assume that every direct summand of $R^{(\omega)} \oplus X$ is a direct sum of finitely generated modules. Then, for any maximal ideal $\mathfrak{m}$ of R,* $\mathrm{End}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}})$ *is a semiperfect ring, so that $X_{\mathfrak{m}}$ satisfies the equivalent conditions of Proposition* 3.2.

Proof. Fix a maximal ideal $\mathfrak{m}$ of R, and set $M' = X_{\mathfrak{m}}$. By Lemma 2.18 (v), $\mathrm{End}_{R_{\mathfrak{m}}}(M')$ is a semilocal ring. We want to prove that it is semiperfect. We decompose $M' = F \oplus M$, where F is a finitely generated free $R_{\mathfrak{m}}$-module and M is a finitely generated module with no projective direct summands. Notice that $\mathrm{End}_{R_{\mathfrak{m}}}(M')$ is semiperfect if and only if so is $\mathrm{End}_{R_{\mathfrak{m}}}(M)$.

By Lemma 3.1 (i) (see also Corollary 1.21), to prove that $\mathrm{End}_{R_{\mathfrak{m}}}(M)$ is semiperfect, it suffices to show that the idempotents of $\mathrm{End}_{R_{\mathfrak{m}}}(M)/M\,\mathrm{Hom}_{R_{\mathfrak{m}}}(M, R_{\mathfrak{m}})$ can be lifted to idempotents of $\mathrm{End}_{R_{\mathfrak{m}}}(M)$. Therefore, we may assume that M is non-zero and that $J = M\,\mathrm{Hom}_{R_{\mathfrak{m}}}(M, R_{\mathfrak{m}})$ is a proper ideal of $\mathrm{End}_{R_{\mathfrak{m}}}(M)$ contained in $J(\mathrm{End}_{R_{\mathfrak{m}}}(M))$.

Let $e \in \mathrm{End}_{R_{\mathfrak{m}}}(M)/J$ be a non-trivial idempotent. By Lemma 1.20 and Proposition 1.11, to show that e can be lifted to an idempotent of $\mathrm{End}_{R_{\mathfrak{m}}}(M)$ we only need to find a decomposition $M \cong A_1 \oplus A_2$ such that, if we set $P_i = \mathrm{Hom}_{R_{\mathfrak{m}}}(M, A_i)$ for $i = 1, 2$, then $P_1/P_1J \cong e(\mathrm{End}_{R_{\mathfrak{m}}}(M)/J)$ and $P_2/P_2J \cong (1-e)(\mathrm{End}_{R_{\mathfrak{m}}}(M)/J)$.

As $M \cong A_1 \oplus A_2$, it is easy to see that, for $i = 1, 2$,

$$P_iJ = \{f \in \mathrm{Hom}_{R_{\mathfrak{m}}}(M, A_i) \mid f \text{ factors throught a free module}\} = A_i\mathrm{Hom}_{R_{\mathfrak{m}}}(M, R_{\mathfrak{m}}).$$

Therefore, we need to prove that $M \cong A_1 \oplus A_2$, with

$$\mathrm{Hom}_{R_{\mathfrak{m}}}(M, A_1)/A_1\mathrm{Hom}_{R_{\mathfrak{m}}}(M, R_{\mathfrak{m}}) \cong e(\mathrm{End}_{R_{\mathfrak{m}}}(M)/J)$$

and

$$\mathrm{Hom}_{R_{\mathfrak{m}}}(M, A_2)/A_2\mathrm{Hom}_{R_{\mathfrak{m}}}(M, R_{\mathfrak{m}}) \cong (1-e)(\mathrm{End}_{R_{\mathfrak{m}}}(M)/J).$$

By Lemma 2.18, there is an extension of rings $R/I \subseteq \mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X,R)$, where $0 \neq I = R \cap X\,\mathrm{Hom}_R(X,R)$. Since R is h-local, I is contained only in finitely many maximal ideals $\{\mathfrak{m} = \mathfrak{m}_1, \dots, \mathfrak{m}_\ell\}$ of R. So we have an isomorphism

$$\begin{aligned}\mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X,R) \to & \prod_{i=1}^{\ell} \big(\mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X,R)\big)_{\mathfrak{m}_i} \\ & \cong \prod_{i=1}^{\ell} \mathrm{End}_{R_{\mathfrak{m}_i}}(X_{\mathfrak{m}_i})/X_{\mathfrak{m}_i}\,\mathrm{Hom}_{R_{\mathfrak{m}_i}}(X_{\mathfrak{m}_i}, R_{\mathfrak{m}_i})\end{aligned}$$

given by taking localization at $\mathfrak{m}_i$ at each component (cf. Lemma 5.5).

Hence, there exists $\tilde{e}^2 = \tilde{e} \in \mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X,R)$ such that $\lambda_{\mathfrak{n}}(\tilde{e}) = 0$ for any maximal ideal $\mathfrak{n} \neq \mathfrak{m}$ and $\lambda_{\mathfrak{m}}(\tilde{e}) = e$, where

$$\begin{aligned}\lambda_{\mathfrak{n}}\colon \mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X,R) \to & \,(\mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X,R))_{\mathfrak{n}} \\ & \cong \mathrm{End}_{R_{\mathfrak{n}}}(X_{\mathfrak{n}})/X_{\mathfrak{n}}\,\mathrm{Hom}_{R_{\mathfrak{n}}}(X_{\mathfrak{n}}, R_{\mathfrak{n}})\end{aligned}$$

denotes the localization morphism.

By Corollary 1.14, $R^{(\omega)} \oplus X = X_1 \oplus X_2$, with

$$\begin{aligned}\mathrm{Hom}_R(R \oplus X, X_1)/X_1\,\mathrm{Hom}_R(R \oplus X, R) & \cong \tilde{e}(\mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X,R)) \\ & \cong e(\mathrm{End}_{R_{\mathfrak{m}}}(M)/J)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_R(R \oplus X, X_2)/X_2\,\mathrm{Hom}_R(R \oplus X, R) & \cong (1-\tilde{e})(\mathrm{End}_R(X)/X\,\mathrm{Hom}_R(X,R)) \\ \cong (1-e)(\mathrm{End}_{R_{\mathfrak{m}}}(M)/J) \times & \Big(\prod_{i=2}^{\ell} \mathrm{End}_{R_{\mathfrak{m}_i}}(X_{\mathfrak{m}_i})/X_{\mathfrak{m}_i}\,\mathrm{Hom}_{R_{\mathfrak{m}_i}}(X_{\mathfrak{m}_i}, R_{\mathfrak{m}_i})\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}& R_{\mathfrak{m}}^{(\omega)} \oplus M \cong R_{\mathfrak{m}}^{(\omega)} \oplus X_{\mathfrak{m}} = (X_1)_{\mathfrak{m}} \oplus (X_2)_{\mathfrak{m}}, \\ & \mathrm{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}} \oplus X_{\mathfrak{m}}, (X_1)_{\mathfrak{m}})/(X_1)_{\mathfrak{m}}\,\mathrm{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}} \oplus X_{\mathfrak{m}}, R_{\mathfrak{m}}) \cong e(\mathrm{End}_{R_{\mathfrak{m}}}(M)/J)\end{aligned}$$

and

$$\mathrm{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}} \oplus X_{\mathfrak{m}}, (X_2)_{\mathfrak{m}})/(X_2)_{\mathfrak{m}}\,\mathrm{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}} \oplus X_{\mathfrak{m}}, R_{\mathfrak{m}}) \cong (1-e)(\mathrm{End}_{R_{\mathfrak{m}}}(M)/J).$$

By hypothesis, for $i = 1, 2$, X_i is a direct sum of finitely generated R-modules; hence, $N_i = (X_i)_{\mathfrak{m}}$ is also a direct sum of finitely generated $R_{\mathfrak{m}}$-modules. By Lemma 1.15, we obtain a decomposition $M = A_1 \oplus A_2$ such that, for each $i \in \{1, 2\}$,

$$\begin{aligned}\mathrm{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}} \oplus M, N_i)/N_i\,\mathrm{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}} \oplus M, R_{\mathfrak{m}}) & \\ \cong \mathrm{Hom}_{R_{\mathfrak{m}}}(M, A_i)/A_i\,\mathrm{Hom}_{R_{\mathfrak{m}}}(M, R_{\mathfrak{m}}). &\end{aligned}$$

Therefore $\mathrm{Hom}_{R_{\mathfrak{m}}}(M, A_1)/A_1\,\mathrm{Hom}_{R_{\mathfrak{m}}}(M, R_{\mathfrak{m}}) \cong e(\mathrm{End}_{R_{\mathfrak{m}}}(M)/J)$ and

$$\mathrm{Hom}_{R_{\mathfrak{m}}}(M, A_2)/A_2\,\mathrm{Hom}_{R_{\mathfrak{m}}}(M, R_{\mathfrak{m}}) \cong (1-e)(\mathrm{End}_{R_{\mathfrak{m}}}(M)/J),$$

as we wanted. So we can conclude that e can be lifted to an idempotent of $\mathrm{End}_{R_{\mathfrak{m}}}(M)$ by an application of Lemma 1.20. ∎

Next result shows that, over an h-local domain R, to have the class $\mathcal{F}$ closed under direct summands we need that the ranks of indecomposable modules over different localizations at maximal ideals are coprime.

This somewhat surprising Theorem 6.2 is the extension of Lemma 4 in [28] to our setting. For its proof, we need our versions of the package deal Theorems 5.7 and 5.13, as well as the results from Section 1.

Theorem 6.2. *Let R be an h-local domain with at least two different maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$. For each $i = 1, 2$, let M_i be a finitely generated, indecomposable, torsion-free $R_{\mathfrak{m}_i}$-module with local endomorphism ring and with rank r_i. Then,*

(1) *for $i = 1, 2$, there exists a finitely generated, indecomposable, torsion-free R-module X_i such that $(X_i)_{\mathfrak{m}_i} \cong M_i$ and $X_{\mathfrak{m}} \cong R_{\mathfrak{m}}^{r_i}$ for every maximal ideal $\mathfrak{m}$ different from $\mathfrak{m}_i$;*

(2) *if r_1 and r_2 are not coprime, then $\operatorname{Add}(X_1 \oplus X_2)$ has elements that are not a direct sum of finitely generated modules.*

Proof. Corollary 5.8 ensures the existence of finitely generated, indecomposable, torsion-free R-modules X_1 and X_2 with the claimed properties in statement (1).

To prove (2), let $d = \gcd(r_1, r_2)$. We show that if $d > 1$, there exists a module in $\operatorname{Add}(X_1 \oplus X_2)$ which is not a direct sum of finitely generated torsion-free R-modules.

Let $\Lambda = \operatorname{End}_R(X_1 \oplus X_2)$. By Lemma 2.18 (i) and (iii), Λ is a torsion-free R-module and Λ_Q a simple artinian ring. Recall from Remark 1.13 that Λ can be identified with the matrix ring:

$$\Lambda = \begin{pmatrix} \operatorname{End}_R(X_1) & \operatorname{Hom}_R(X_2, X_1) \\ \operatorname{Hom}_R(X_1, X_2) & \operatorname{End}_R(X_2) \end{pmatrix}$$

By Lemma 2.8, $\Lambda_{\mathfrak{m}} \cong \operatorname{End}_{R_{\mathfrak{m}}}((X_1)_{\mathfrak{m}} \oplus (X_2)_{\mathfrak{m}})$ for every maximal ideal $\mathfrak{m}$ of R. Let

$$I_1 = \Lambda_{\mathfrak{m}_1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Lambda_{\mathfrak{m}_1} \quad \text{and} \quad I_2 = \Lambda_{\mathfrak{m}_2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Lambda_{\mathfrak{m}_2}.$$

By the package deal Theorem 5.13, there is a non-zero two-sided ideal I of Λ, which is the trace of a countably generated projective right Λ-module, such that $I_{\mathfrak{m}_i} = I_i$ $(i = 1, 2)$ and $I_{\mathfrak{m}} = \Lambda_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R different from $\mathfrak{m}_i$.

By Lemma 2.18 (iv), $I \cap R \neq \{0\}$, so Λ/I is a torsion R-module. Note that the only maximal ideals containing $I \cap R$ are $\mathfrak{m}_1$ and $\mathfrak{m}_2$ (otherwise, if there exists another maximal ideal $\mathfrak{n}$ containing $I \cap R$, $I_{\mathfrak{n}} \cap R_{\mathfrak{n}} \neq R_{\mathfrak{n}}$, and then $I_{\mathfrak{n}} \neq \Lambda_{\mathfrak{n}}$, a contradiction). By Lemma 2.18 (vi), the canonical homomorphism

$$\Lambda/I \to \bigoplus\nolimits_{\mathfrak{m} \in \operatorname{mSpec} R} (\Lambda/I)_{\mathfrak{m}} = (\Lambda/I)_{\mathfrak{m}_1} \times (\Lambda/I)_{\mathfrak{m}_2}$$

is an isomorphism.

From Remark 1.13,

$$I_1 = \begin{pmatrix} \operatorname{End}_{R_{\mathfrak{m}_1}}(M_1) & \operatorname{Hom}_{R_{\mathfrak{m}_1}}(M_1, R_{\mathfrak{m}_1}^{r_2}) \\ \operatorname{Hom}_{R_{\mathfrak{m}_1}}(R_{\mathfrak{m}_1}^{r_2}, M_1) & \operatorname{Hom}_{R_{\mathfrak{m}_1}}(M_1, R_{\mathfrak{m}_1}^{r_2}) \operatorname{Hom}_{R_{\mathfrak{m}_1}}(R_{\mathfrak{m}_1}^{r_2}, M_1) \end{pmatrix}.$$

Therefore, $(\Lambda/I)_{\mathfrak{m}_1} \cong \Lambda_{\mathfrak{m}_1}/I_1 \cong M_{r_2}(R_{\mathfrak{m}_1})/J$, where

$$J = \operatorname{Hom}_{R_{\mathfrak{m}_1}}(M_1, R_{\mathfrak{m}_1}^{r_2}) \operatorname{Hom}_{R_{\mathfrak{m}_1}}(R_{\mathfrak{m}_1}^{r_2}, M_1).$$

Notice that $J \neq \mathrm{End}_{R_{\mathfrak{m}_1}}(R_{\mathfrak{m}_1}^{r_2})$, because this would imply that the identity map of $R_{\mathfrak{m}_1}^{r_2}$ is in J, so that $R_{\mathfrak{m}_1}^{r_2}$ is a direct summand of M_1^n for some $n \in \mathbb{N}$. Since M_1 has local endomorphism ring, the Krull–Schmidt theorem implies that $M_1 \cong R_{\mathfrak{m}_1}$, but we are assuming that $r_1 > 1$, a contradiction.

Therefore, there is an isomorphism $\varphi\colon \Lambda/I \to M_{r_2}(R_{\mathfrak{m}_1})/J \times (\Lambda/I)_{\mathfrak{m}_2}$. Then there is an idempotent element $e \in \Lambda/I$ such that $\varphi(e) = (E_{11} + J, 0)$, where E_{11} is the idempotent matrix with 1 in the 1,1-entry and zeros elsewhere. Since $M_{r_2}(R_{\mathfrak{m}_1})E_{11}M_{r_2}(R_{\mathfrak{m}_1}) = M_{r_2}(R_{\mathfrak{m}_1})$, we deduce that $E_{11} \notin J$.

By Theorem 1.9 (i), there is a countably generated projective right Λ-module Q such that $Q/QI \cong e(\Lambda/I)$. We claim that such Q is neither finitely generated nor a direct sum of finitely generated modules.

Recall that $\Lambda_{\mathfrak{m}_1} \cong \mathrm{End}_{R_{\mathfrak{m}_1}}(M_1 \oplus R_{\mathfrak{m}_1}^{r_2})$. Since M_1 has local endomorphism ring, there are only two finitely generated indecomposable projective $\Lambda_{\mathfrak{m}_1}$-modules up to isomorphism, namely

$$P_{1a} = \mathrm{Hom}_{R_{\mathfrak{m}_1}}(M_1 \oplus R_{\mathfrak{m}_1}^{r_2}, M_1) \quad \text{and} \quad P_{1b} = \mathrm{Hom}_{R_{\mathfrak{m}_1}}(M_1 \oplus R_{\mathfrak{m}_1}^{r_2}, R_{\mathfrak{m}_1}).$$

Notice that P_{1a} and P_{1b} are not isomorphic because M_1 has rank $r_1 > 1$. In addition, by the Krull–Schmidt theorem, all projective $\Lambda_{\mathfrak{m}_1}$-modules can be written in a unique way as a direct sum of copies of P_{1a} and P_{1b}.

By the definition of P_{1a}, $\mathrm{Tr}(P_{1a}) = I_1$. On the other hand,

$$(Q/QI)_{\mathfrak{m}_1} \cong (e(\Lambda/I))_{\mathfrak{m}_1} \cong (E_{11} + J)M_{r_2}(R_{\mathfrak{m}_1}),$$

so $Q_{\mathfrak{m}_1} \cong P_{1a}^{(\kappa_1)} \oplus P_{1b}$, where κ_1 is, at most, a countable cardinal.

Similarly, there are two finitely generated indecomposable projective $\Lambda_{\mathfrak{m}_2}$-modules up to isomorphism

$$P_{2a} = \mathrm{Hom}_{R_{\mathfrak{m}_2}}(R_{\mathfrak{m}_2}^{r_1} \oplus M_2, M_2) \quad \text{and} \quad P_{2b} = \mathrm{Hom}_{R_{\mathfrak{m}_2}}(R_{\mathfrak{m}_2}^{r_1} \oplus M_2, R_{\mathfrak{m}_2}).$$

By the definition of P_{2a}, $\mathrm{Tr}(P_{2a}) = I_2$. Thus, since $(Q/QI)_{\mathfrak{m}_2} \cong (e(\Lambda/I))_{\mathfrak{m}_2} = \{0\}$, we have $Q_{\mathfrak{m}_2} \cong P_{2a}^{(\kappa_2)}$, where κ_2 is, at most, a countable cardinal.

Let $L = Q \otimes_\Lambda (X_1 \oplus X_2)$, which is in $\mathrm{Add}(X_1 \oplus X_2)$. By Proposition 1.11, L is a direct summand of $(X_1 \oplus X_2)^{(\omega)}$ and $\mathrm{Hom}_R(X_1 \oplus X_2, L) \cong Q$. Moreover, by Proposition 1.11,

$$(*) \quad L_{\mathfrak{m}_1} \cong M_1^{(\kappa_1)} \oplus R_{\mathfrak{m}_1} \quad \text{and} \quad (**) \quad L_{\mathfrak{m}_2} \cong M_2^{(\kappa_2)}.$$

If Q is finitely generated, κ_1 and κ_2 are finite. But then, by the isomorphism $(*)$, the rank of L is congruent to 1 modulo d and, by the isomorphism $(**)$, the rank of L is divisible by d, a contradiction.

To prove the second statement in the claim, assume that $Q = \bigoplus_{i\in\mathbb{N}} Q_i$, where the Q_i's are finitely generated. Since $Q/QI \cong e(\Lambda/I)$ is finitely generated, $Q_i = Q_i I$ for almost all $i \in \mathbb{N}$. If $I_0 := \{i \in \mathbb{N} \mid Q_i \neq Q_i I\}$, then $Q_f = \bigoplus_{i\in I_0} Q_i$ is a finitely generated projective right Λ-module such that $Q_f/Q_f I \cong e(\Lambda/I)$. By the previous part of the proof, such Q_f does not exist, hence Q is not a direct sum of finitely generated modules. By Proposition 1.11, $L \in \mathrm{Add}(X_1 \oplus X_2)$ is not a direct sum of finitely generated modules. This finishes the proof of (2). ∎

Next proposition reviews a construction, that goes back to an idea of Bass [3], of an indecomposable finitely generated torsion-free module of rank two over any local domain that has a finitely generated ideal that cannot be generated by two elements.

This construction is particularly relevant to us in view of Theorem 6.2. It is going to imply that if $\mathcal{F}$ is closed under direct summands, then finitely generated ideals of $R_\mathfrak{m}$ are two-generated for all maximal ideals of R except maybe one, see Theorem 6.6.

Proposition 6.3. *Let R be a commutative local domain, with field of fractions Q, and maximal ideal $\mathfrak{m}$. Let a,b,c be elements in R such that the ideal $K = aR + bR + cR$ cannot be generated by two elements. Set $\alpha = (a,b,c) \in R^3$ and let $H = R^3 \cap Q\alpha$. Then:*

(i) *$M = R^3/H$ is an indecomposable, finitely generated torsion-free module of rank 2.*

(ii) *If $\overline{R}$ is local, then so is $\operatorname{End}_R(M)$.*

Proof. (i) Let $k = R/\mathfrak{m}$ be the residue field of R. Let $M = A_1/H \oplus A_2/H$ be a non-trivial decomposition with A_i R-submodules of R^3 containing H. Consider $M \otimes_R k \cong (A_1/H \otimes_R k) \oplus (A_2/H \otimes_R k)$. Then, since $M \otimes_R k$ is at most a three-dimensional vector space, one of the direct summands is one-dimensional. We may assume that it is $A_2/H \otimes_R k$, which in turn implies that $A_2/H \cong R^3/A_1$ is isomorphic to R. Therefore, the exact sequence

$$0 \longrightarrow A_1 \longrightarrow R^3 \longrightarrow R^3/A_1 \longrightarrow 0$$

splits, and we deduce that A_1 is a 2-generated free module. So that if v_1 and v_2 is a basis of A_1, there exists $r_1, r_2 \in R$ such that $(a,b,c) = v_1r_1 + v_2r_2$. This implies that $K \subseteq r_1R + r_2R$. Since v_1 and v_2 can be completed to a basis of R^3, it follows that $K = r_1R + r_2R$, which contradicts the assumption that K cannot be generated by two elements. This finishes the proof that M is indecomposable.

(ii) The module M fits into the exact sequence

$$0 \longrightarrow H \longrightarrow R^3 \longrightarrow M \longrightarrow 0.$$

Let $T = \{f \in \operatorname{End}_R(R^3) \mid f(H) \subseteq H\}$, which is a subring of $\operatorname{End}_R(R^3)$. Then there is an onto ring endomorphism $\varphi\colon T \to \operatorname{End}_R(M)$. We will prove that T is a local ring, and then so is $\operatorname{End}_R(M)$.

Restriction to H induces a ring morphism $\psi\colon T \to \operatorname{End}_R(H)$ with kernel $I = \{f \in T \mid f(H) = 0\}$.

We claim that the embedding $T \hookrightarrow \operatorname{End}_R(R^3)$ is local. Indeed, let $f \in T$ be such that f is invertible in $\operatorname{End}_R(R^3)$. Then there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \longrightarrow & R^3 & \longrightarrow & M & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \psi(f)} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi(f)} & & \\
0 & \longrightarrow & H & \longrightarrow & R^3 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

Since f is invertible, it is onto and then so is $\varphi(f)$. Since M is a finitely generated module over a commutative ring, we deduce that $\varphi(f)$ is bijective. Now the snake lemma implies that $\psi(f)$ is also bijective, and then we can deduce that $f^{-1} \in T$.

Now we prove that $I \subseteq J(T)$. Notice that if $f \in I$, then it induces a module homomorphism $\tilde{f}: M \to R^3$ such that $\operatorname{Im} f = \operatorname{Im} \tilde{f}$. Since M is indecomposable, and R is local $\operatorname{Im} \tilde{f} \subseteq R^3\mathfrak{m}$. Hence, $1 - f$ is invertible in $\operatorname{End}_R(R^3)$, and since the embedding $T \hookrightarrow \operatorname{End}_R(R^3)$ is local, we deduce that $1 - f$ is invertible in T. This proves that the two-sided ideal $I \subseteq J(T)$.

The image of the ring morphism ψ is the ring

$$S = \{g \in \operatorname{End}_R(H) \mid g \text{ can be extended to an endomorphism of } R^3\}.$$

Hence $T/I \cong S$, and T is local provided S is local. Notice that for any $g \in S$ its extension to R^3 satisfies a monic polynomial of degree 3 with coefficients in R, so g also satisfies that polynomial. This is to say that the extension $R \hookrightarrow S$ is integral.

Since H is a torsion-free module of rank 1, $\operatorname{End}_R(H)$ can be identified with a subring of Q. Then S is a subring of $\overline{R}$. Since $\overline{R}$ is local, so is S. This finishes the proof of the result. ∎

The next corollary shows that the conclusions of Proposition 6.3 can be extended to any ring S between R and its integral closure.

Corollary 6.4. *Let R be a commutative local domain, with field of fractions Q and maximal ideal $\mathfrak{m}$ such that $\overline{R}$ is local. Let S be an intermediate ring $R \subseteq S \subseteq \overline{R}$. Let a, b, c be elements in S such that the ideal $K = aS + bS + cS$ cannot be generated by two elements. Set $\alpha = (a, b, c) \in S^3$ and let $H = S^3 \cap Q\alpha$. Then*:

(i) *$M = S^3/H$ is a torsion-free R-module of rank 2 and $\operatorname{End}_R(M) = \operatorname{End}_S(M)$ is a local ring. In particular, M is indecomposable.*

(ii) *Consider the ring $T = R[a, b, c]$, which is a finite integral extension of R. Let $H' = H \cap T^3$. Then $M' = T^3/H'$ is an indecomposable, finitely generated R-module of rank 2 with local endomorphism ring.*

Proof. Since $\overline{R}$ is local, so is any intermediate ring $R \subseteq S \subseteq \overline{R}$.

Notice also that an S-module M has finite rank n as S-module if and only if it has finite rank n as R-module because $M \otimes_S Q = M \otimes_S S \otimes_R Q = M \otimes_R Q$. This implies, in addition, that $\operatorname{End}_S(M) \cong \{A \in M_n(Q) \mid AM \subseteq M\}$ coincides with $\operatorname{End}_R(M)$.

Therefore, the result is a consequence of Proposition 6.3 applied to S in (i) and to T in (ii). ∎

The following lemma is a variation for domains of finite character of a well-known fact about bounds on the number of generators of finitely generated ideals.

Lemma 6.5 (Proposition 1.4 in [2]). *Let R be a commutative domain of finite character. Let $k \geq 2$. If I is a non-zero finitely generated ideal of R such that $IR_{\mathfrak{m}}$ is k-generated for every maximal ideal $\mathfrak{m}$ of R, then I is also k-generated.*

Proof. Let I be a non-zero ideal of R. Since R is of finite character, I is contained only in finitely many maximal ideals $\{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ of R. First, observe that there exists $\alpha \in I$ such that $\frac{\alpha}{1} \notin \mathfrak{m}_i I_{\mathfrak{m}_i}$ for every $i = 1, \dots, n$. Indeed, let $\alpha_i \in I$ be such that $\frac{\alpha_i}{1} \notin \mathfrak{m}_i I_{\mathfrak{m}_i}$ and let $e_1, \dots, e_n$ be such that $e_i - 1 \in \mathfrak{m}_i$ for every $i = 1, \dots, n$ and $e_i \in \mathfrak{m}_j$ whenever $1 \leq i \neq j \leq n$. Then let $\alpha = \sum_{i=1}^n e_i \alpha_i$.

Since R is of finite character, α is contained only in finitely many maximal ideals of R, say $\mathcal{M} = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n, \mathfrak{m}_{n+1}, \dots, \mathfrak{m}_{n+\ell}\}$.

For each $i = 1, \dots, n$, there are $\alpha_{i,1}, \dots, \alpha_{i,k-1} \in I$ such that

$$I_{\mathfrak{m}_i} = \frac{\alpha}{1} R_{\mathfrak{m}_i} + \sum_{j=1}^{k-1} \frac{\alpha_{i,j}}{1} R_{\mathfrak{m}_i}.$$

For each $i = n+1, \dots, n+\ell$, $j = 1, \dots, k-1$ let $\alpha_{i,1} \in I \setminus \mathfrak{m}_i$ and $\alpha_{i,j} = 0$ if $j > 1$. Notice that $I_{\mathfrak{m}_i} = \frac{\alpha}{1} R_{\mathfrak{m}_i} + \sum_{j=1}^{k-1} \frac{\alpha_{i,j}}{1} R_{\mathfrak{m}_i}$ for every $i = 1, \dots, n+\ell$. As before, consider $f_1, \dots, f_{n+\ell} \in R$ such that $f_i - 1 \in \mathfrak{m}_i$ and $f_i \in \mathfrak{m}_j$ if $1 \le j \ne i \le n+\ell$. Set $\beta_{i,j} = f_i \alpha_{i,j}$ for every $i = 1, \dots, n+\ell$ and $j = 1, \dots, k-1$. For $j = 1, \dots, k-1$ set $\alpha_j := \sum_{i=1}^{n+\ell} \beta_{i,j}$. We claim that $I = \alpha R + \sum_{j=1}^{k-1} \alpha_j R$. It is sufficient to verify this equality locally. If $\mathfrak{m} \notin \mathcal{M}$, then both sides localize to $R_{\mathfrak{m}}$. Also,

$$\alpha R_{\mathfrak{m}_i} + \sum_{j=1}^{k-1} \alpha_j R_{\mathfrak{m}_i} + \mathfrak{m}_i I_{\mathfrak{m}_i} = \alpha R_{\mathfrak{m}_i} + \sum_{j=1}^{k-1} \alpha_{i,j} R_{\mathfrak{m}_i} + \mathfrak{m}_i I_{\mathfrak{m}_i}.$$

Since I is finitely generated, we can conclude by Nakayama's lemma. ∎

Now we are ready to prove the main result of the section.

Theorem 6.6. *Let R be a commutative domain of finite character, and of Krull dimension* 1. *Assume that for any finitely generated torsion-free R-module X, every element of* Add(X) *is a direct sum of finitely generated modules. Then R satisfies the following properties*:

(1) *For any maximal ideal $\mathfrak{m}$ of R and any ring S such that $R_{\mathfrak{m}} \subseteq S \subseteq \overline{R_{\mathfrak{m}}}$, we have that finitely generated indecomposable torsion-free S-modules have local endomorphism ring.*

(2) *For any maximal ideal $\mathfrak{m}$ of $R_{\mathfrak{m}}$, except maybe one maximal ideal $\mathfrak{m}_0$, all finitely generated ideals of $R_{\mathfrak{m}}$ are at most two-generated. Then $\overline{R_{\mathfrak{m}}}$ is a valuation ring for any maximal ideal $\mathfrak{m} \neq \mathfrak{m}_0$.*

(3) *For each maximal ideal $\mathfrak{m}$ of R, there is a unique maximal ideal of $\overline{R}$ lying over $\mathfrak{m}$. In particular, $\overline{R}$ has also finite character.*

If $\overline{R_{\mathfrak{m}}}$ satisfies the two-generated property for any maximal ideal $\mathfrak{m}$, then $\overline{R}$ is a Prüfer domain of Krull dimension 1, *so any finitely generated ideal of R is, at most, two-generated.*

Proof. By Corollary 3.3 and Proposition 6.1, we deduce that for any maximal ideal $\mathfrak{m}$ of R, any ring S such that $R_{\mathfrak{m}} \subseteq S \subseteq \overline{R_{\mathfrak{m}}}$ satisfies that every finitely generated indecomposable torsion-free S-module has local endomorphism ring. This shows (1).

Notice that (1) implies that $\overline{R}$ is also a ring of finite character because there is just one maximal ideal of $\overline{R}$ lying over each maximal ideal of R. This shows (3).

By Theorem 6.2, the ranks of two finitely generated, indecomposable, torsion-free modules over different localizations of R at maximal ideals must be coprime. In view of Corollary 6.4, we deduce that finitely generated ideals of $R_{\mathfrak{m}}$ are 2-generated for any maximal ideal $\mathfrak{m}$ of R except maybe for one that we will denote by $\mathfrak{m}_0$. By Proposition III.1.11 in [8], $\overline{R_{\mathfrak{m}}}$ is a valuation ring for any maximal ideal $\mathfrak{m} \neq \mathfrak{m}_0$. This proves statement (2).

If the maximal ideal $\mathfrak{m}_0$ does not exist, then $\overline{R}$ is a Prüfer domain of Krull dimension 1. Moreover, by Lemma 6.5, any finitely generated ideal of R is, at most, 2-generated. ■

7. The integrally closed case

In this section, we will use the results developed until now to show that the converse of Theorem 6.6 is true for integrally closed h-local domains of Krull dimension 1. This will be done in Corollary 7.3.

We recall the following definition, which will be needed for this section.

Definition 7.1. Let R be a ring. Let M be a right R-module. A submodule N of M is called *relatively divisible* or an RD-*submodule* if $Nr = N \cap Mr$ for each $r \in R$.

In particular, this definition tells us that if for some $r \in R$, $a \in N$, the equation $xr = a$ has a solution in M, then it has a solution in N as well. In this sense, pure submodules are a generalization of RD-submodules to the case where instead of an equation, we have a system of linear equations which has a solution in N^n whenever it has a solution in M^n, for some positive integer $n \geq 1$.

For a discussion of the basic properties of RD-submodules, the reader is referred to Section 7 of Chapter I in [8].

Proposition 7.2. *Let R be an h-local domain of Krull dimension* 1, *and with field of fractions Q. Assume that $R_{\mathfrak{m}}$ is a valuation domain for all maximal ideals $\mathfrak{m}$ of R except maybe one maximal ideal $\mathfrak{m}_0$.*

Then the class of R-modules that are direct sums of finitely generated torsion-free R-modules is closed under direct summands if and only if the class of direct sums of finitely generated torsion-free $R_{\mathfrak{m}_0}$-modules is closed under direct summands.

Proof. By Proposition 6.1, if the class of R-modules that are direct sums of finitely generated torsion-free R-modules is closed under direct summands, then so is the corresponding class for $R_{\mathfrak{m}_0}$. We need to prove the converse result.

Let $\{N_i \mid i \in \mathbb{N}\}$ be a countable family of non-zero finitely generated torsion-free R-modules, and set $N = \bigoplus_{i\in\mathbb{N}} N_i$. Let A be a direct summand of N. Then $A_{\mathfrak{m}_0}$ is a direct summand of $N_{\mathfrak{m}_0}$. By hypothesis, $A_{\mathfrak{m}_0}$ is a direct sum of finitely generated $R_{\mathfrak{m}_0}$-modules, say $A_{\mathfrak{m}_0} = \bigoplus_{i\in\mathbb{N}} X_i$. For $i \in \mathbb{N}$, let $A_i := \{a \in A \mid a/1 \in \bigoplus_{1\leq j\leq i} X_j\}$. We claim that A_i is an RD-submodule of A. Let $a \in A$ and $r \in R \setminus \{0\}$ be such that $(a/1)(r/1) \in \bigoplus_{1\leq j\leq i} X_j$, since $r/1$ is not a zero divisor of X_ℓ for any ℓ, we deduce that $a/1 \in A_i$. This finishes the proof of the claim.

Note also that there is an $\ell \geq 1$ such that $\bigoplus_{1\leq j\leq i} X_j$ is a submodule of $\bigoplus_{i=1}^{\ell} (N_i)_{\mathfrak{m}_0}$. Hence, since all modules involved are torsion-free, $A_i \leq \bigoplus_{i=1}^{\ell} N_i$. We claim that this is also an RD-embedding. Assume that $n \in \bigoplus_{i=1}^{\ell} N_i$ is such that there exists $r \in R$ such that $nr \in A_i \leq A$. As A is an RD-submodule of N, we deduce that $n \in A$, and since A_i is an RD-submodule of A, we deduce that $n \in A_i$.

Now we shall prove that A_i is finitely generated. Since A_i is a submodule of a free module R^r, by Lemma 5.2 there exist $a_1, \dots, a_s \in A_i$ and a finite set of maximal ideals of the form $\mathcal{S} = \{\mathfrak{m}_0, \mathfrak{m}_1, \dots, \mathfrak{m}_t\}$ such that $(A_i)_\mathfrak{m} = \sum_{j=1}^s \frac{a_j}{1} R_\mathfrak{m}$ for any maximal ideal $\mathfrak{m} \notin \mathcal{S}$. In particular, $(A_i)_Q = \sum_{j=1}^s \frac{a_j}{1} Q$, and in the exact sequence

$$0 \longrightarrow \sum_{j=1}^s a_j R \longrightarrow A_i \longrightarrow A_i/\Big(\sum_{j=1}^s a_j R\Big) \longrightarrow 0,$$

the module $A_i/(\sum_{j=1}^s a_j R)$ is torsion. Since R is an h-local domain of Krull dimension 1,

$$A_i/\Big(\sum_{j=1}^s a_j R\Big) = \bigoplus_{\mathfrak{m}\in \mathrm{mSpec}(R)} \Big(A_i/\Big(\sum_{j=1}^s a_j R\Big)\Big)_\mathfrak{m} = \bigoplus_{\mathfrak{m}\in\mathcal{S}} \Big(A_i/\Big(\sum_{j=1}^s a_j R\Big)\Big)_\mathfrak{m}.$$

Hence, to prove the claim, it is enough to show that $(A_i)_{\mathfrak{m}_j}$ is finitely generated as a $\Lambda_{\mathfrak{m}_j}$-module for any $j = 0, \dots, t$, cf. Lemma 5.5.

Note that $(A_i)_{\mathfrak{m}_0} = \bigoplus_{1\le j\le i} X_j$ is finitely generated. Assume now that $j \ge 1$. Since A_i is an RD-submodule of $\bigoplus_{k=1}^\ell N_k$, we deduce that $(A_i)_{\mathfrak{m}_j}$ is an RD-submodule of $\bigoplus_{k=1}^\ell (N_k)_{\mathfrak{m}_j}$. Since $R_{\mathfrak{m}_j}$ is a valuation domain, and the module $\bigoplus_{k=1}^\ell (N_k)_{\mathfrak{m}_j}/(A_i)_{\mathfrak{m}_j}$ is finitely generated and torsion-free, it is projective. Therefore, $(A_i)_{\mathfrak{m}_j}$ is a direct summand of $\bigoplus_{k=1}^\ell (N_k)_{\mathfrak{m}_j}$, so it is finitely generated, as claimed.

Now, A is a union of a chain of finitely generated modules $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. For any $i \in \mathbb{N}$, the exact sequence $0 \to A_i \to A_{i+1} \to A_{i+1}/A_i \to 0$ splits upon localization at $\mathfrak{m}_0$ by construction. It also splits when localized at other maximal ideals since it is RD-exact, $R_\mathfrak{m}$ is a valuation domain, and all the involved modules are finitely generated and torsion-free, and therefore projective.

By Lemma 2.9, the sequence $0 \to A_i \to A_{i+1} \to A_{i+1}/A_i \to 0$ splits in R for any $i \in \mathbb{N}$. Therefore, $A \cong A_1 \oplus (\bigoplus_{i\in\mathbb{N}} A_{i+1}/A_i)$ is a direct sum of finitely generated modules. ∎

Corollary 7.3. *Let R be an h-local domain of Krull dimension 1 that is integrally closed in its field of fractions Q. Then the following statements are equivalent*:

- (i) *The class of R-modules that are direct sums of finitely generated torsion-free R-modules is closed under direct summands.*
- (ii) *For any finitely generated torsion-free R-module X, every element in* $\mathrm{Add}(X)$ *is a direct sum of finitely generated modules.*
- (iii) *The ring R satisfies one of the following conditions*:
  - (1) *R is a Prüfer domain*; *or*
  - (2) *there is a maximal ideal $\mathfrak{m}_0$ such that for every maximal ideal $\mathfrak{m} \ne \mathfrak{m}_0$, $R_\mathfrak{m}$ is a valuation domain, $R_{\mathfrak{m}_0}$ is an integrally closed domain that is not a valuation domain, and every indecomposable finitely generated torsion-free $R_{\mathfrak{m}_0}$-module has local endomorphism ring.*

Proof. It is clear that (i) implies (ii). Theorem 6.6 shows that (ii) implies (iii).

Assume (iii) holds. If R is a Prüfer domain, then it is semihereditary, so finitely generated torsion-free modules are projective, and all projective modules are the direct sum of finitely generated ideals. So (i) is trivially satisfied.

Now assume that R satisfies the condition (2). By Corollary 3.3, $R_{\mathfrak{m}_0}$ satisfies that the class of finitely generated torsion-free modules is closed under direct summands. Then (i) follows from Proposition 7.2. ■

Corollary 7.4. *Let R be an h-local domain of Krull dimension 1. Assume that the class of R-modules that are direct sums of finitely generated torsion-free modules is closed under direct summands. Then its integral closure also satisfies this property.*

Proof. By Theorem 6.6, $\overline{R}$ satisfies conditions (1) and (2) of part (iii) of Corollary 7.3, so we can deduce that the class of finitely generated torsion-free $\overline{R}$-modules is closed by direct summands. ■

8. Infinite direct sums are determined by the genus

Definition 8.1. Let R be a commutative ring, and let Λ be an R-algebra. Let M and N be right Λ-modules. We say that M and N are *in the same genus* if $M_\mathfrak{m}$ is isomorphic to $N_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R.

Lemma 8.2. *Let R be a commutative ring, and let Λ be a module-finite R-algebra. Let M, N and X be right Λ-modules. If X is finitely generated over R and $M \oplus X$ and $N \oplus X$ are in the same genus, then M and N are in the same genus.*

Proof. Recall that if a module Y has semilocal endomorphism ring, then it has the cancellation property, that is, $Y \oplus A \cong Y \oplus B$ implies $A \cong B$ (see, for example, Theorem 4.5 in [6]).

By Proposition 2.3, $X_\mathfrak{m}$ has semilocal endomorphism ring. Therefore, if $M_\mathfrak{m} \oplus X_\mathfrak{m} \cong N_\mathfrak{m} \oplus X_\mathfrak{m}$, then $M_\mathfrak{m} \cong N_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R. ■

Lemma 8.3. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be a module-finite R-algebra such that Λ_Q is a simple artinian ring. Let $M = \bigoplus_{i \in I} M_i$ be an infinite direct sum of non-zero finitely generated right Λ-modules which are torsion-free as R-modules, and let N be a finitely generated right Λ-module which is torsion-free as an R-module. If $N_\mathfrak{m}$ is a direct summand of $M_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R, then N is a direct summand of M.*

Proof. Suppose that $N_\mathfrak{m}$ is a direct summand of $M_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R. Since M_Q is a free Λ_Q-module and N is finitely generated, there is some finite subset $I_1 \subseteq I$ such that N_Q is a direct summand of $\bigoplus_{i \in I_1} (M_i)_Q$. By Lemma 4.2, there is a Λ-module homomorphism $f \colon \bigoplus_{i \in I_1} M_i \to N$ such that the induced homomorphism $f_\mathfrak{m} \colon \bigoplus_{i \in I_1} (M_i)_\mathfrak{m} \to N_\mathfrak{m}$ is a splitting epimorphism for almost all maximal ideals $\mathfrak{m}$ of R.

Let $\mathcal{M}$ be the finite set of maximal ideals $\mathfrak{m}$ such that $f_\mathfrak{m}$ is not a splitting epimorphism. Since N is finitely generated, there is some finite subset $I_2 \subseteq I$ containing I_1 such that $N_\mathfrak{m}$ is a direct summand of $\bigoplus_{i \in I_2} (M_i)_\mathfrak{m}$ for every $\mathfrak{m} \in \mathcal{M}$. Then $N_\mathfrak{m}$ is a direct summand of $\bigoplus_{i \in I_2} (M_i)_\mathfrak{m}$ for all $\mathfrak{m} \in \operatorname{mSpec} R$.

Since Λ_Q is simple artinian and N is finitely generated, there exists $I_3 \subseteq I \setminus I_2$ finite such that N_Q is a direct summand of $\bigoplus_{i\in I_3}(M_i)_Q$. Then, by Proposition 4.6, N is a direct summand of $\bigoplus_{i\in I_0} M_i$, where $I_0 = I_2 \cup I_3$. Hence, N is a direct summand of M. ■

Theorem 8.4. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be a module-finite R-algebra such that Λ_Q is a simple artinian ring. Let $M = \bigoplus_{i\in\mathbb{N}_0} A_i$ and $N = \bigoplus_{i\in\mathbb{N}_0} B_i$ be direct sums of non-zero finitely generated right Λ-modules which are torsion-free as R-modules. If M and N are in the same genus, then there are decompositions*

$$M = \bigoplus_{i\in\mathbb{N}_0} M_i \quad \text{and} \quad N = \bigoplus_{i\in\mathbb{N}_0} N_i$$

such that both M_i and N_i are finitely generated, and $M_i \cong N_i$ for every $i \in \mathbb{N}_0$. In particular, M and N are isomorphic.

Proof. Let $\{\mathfrak{m}_\alpha\}_{\alpha\in\Omega}$ denote the set of maximal ideals of R. Let $\{a_i\}_{i\in\mathbb{N}_0}$ and $\{b_j\}_{j\in\mathbb{N}_0}$ be countable sets of generators for M and N, respectively. By Lemma 2.3, $(A_i)_{\mathfrak{m}_\alpha}$ and $(B_i)_{\mathfrak{m}_\alpha}$ have semilocal endomorphism rings for every $\alpha \in \Omega$, and $i \in \mathbb{N}$. Hence, both A_i and B_i satisfy the cancellation property locally for every $i \in \mathbb{N}$.

We claim that there exist $m_0, n_0 \in \mathbb{N}$ and decompositions

$$M = M_0 \oplus Z_0 \oplus \bigoplus_{i=m_0+1}^{\infty} A_i \quad \text{and} \quad N = N_0 \oplus \bigoplus_{j=n_0+1}^{\infty} B_j$$

for some finitely generated modules M_0, N_0 and Z_0, such that M_0 is isomorphic to N_0, $a_0 \in M_0$, $b_0 \in N_0$, and $Z_0 \oplus \bigoplus_{i=m_0+1}^{\infty} A_i$ and $\bigoplus_{j=n_0+1}^{\infty} B_j$ are in the same genus.

Let $m_0' = \min\{m \in \mathbb{N}_0 \mid a_0 \in \bigoplus_{i=0}^{m} A_i\}$, and consider $X_0 = \bigoplus_{i=0}^{m_0'} A_i$. By Lemma 8.3 applied to X_0, there is $n_0' \in \mathbb{N}_0$ such that X_0 is isomorphic to a direct summand of $\bigoplus_{j=0}^{n_0'} B_j$. Therefore, we can write

$$M = X_0 \oplus \bigoplus_{i=m_0'+1}^{\infty} A_i \quad \text{and} \quad N = Y_0 \oplus Z_0' \oplus \bigoplus_{j=n_0'+1}^{\infty} B_j$$

with $X_0 \cong Y_0$ and $Y_0 \oplus Z_0' = \bigoplus_{j=0}^{n_0'} B_j$. Thanks to Lemma 8.2, $\bigoplus_{i=m_0+1}^{\infty} A_i$ and $Z_0' \oplus \bigoplus_{j=n_0'+1}^{\infty} B_j$ are in the same genus.

Let $n_0 = \max\{n_0' + 1, \min\{n \in \mathbb{N}_0 \mid b_0 \in \bigoplus_{j=0}^{n} B_j\}\}$, and consider $Y_1 = Z_0' \oplus \bigoplus_{j=n_0'+1}^{n_0} B_j$. By Lemma 8.3 applied to Y_1, there is $m_0 > m_0'$ such that Y_1 is isomorphic to a direct summand of $\bigoplus_{i=m_0'+1}^{m_0} A_i$. Then

$$M = X_0 \oplus X_1 \oplus Z_0 \oplus \bigoplus_{i=m_0+1}^{\infty} A_i \quad \text{and} \quad N = Y_0 \oplus Y_1 \oplus \bigoplus_{j=n_0+1}^{\infty} B_j,$$

with $X_0 \cong Y_0$ and $X_1 \cong Y_1$. By Lemma 8.2, $Z_0 \oplus \bigoplus_{i=m_0+1}^{\infty} A_i$ and $\bigoplus_{j=n_0+1}^{\infty} B_j$ are in the same genus. Take $M_0 = X_0 \oplus X_1$ and $N_0 = Y_0 \oplus Y_1$. We can repeat this argument ω times to obtain ascending chains of positive integers $m_0 < \cdots < m_\ell < \cdots$ and $n_0 < \cdots < n_\ell < \cdots$, and decompositions

$$M = \left(\bigoplus_{i=0}^{\ell} M_i\right) \oplus Z_\ell \oplus \left(\bigoplus_{i=m_\ell+1}^{\infty} A_i\right) \quad \text{and} \quad N = \left(\bigoplus_{i=0}^{\ell} N_i\right) \oplus \left(\bigoplus_{j=n_\ell+1}^{\infty} B_j\right)$$

such that M_i is finitely generated and isomorphic to N_i for every $i = 0, \dots, \ell, a_0, \dots, a_\ell \in \bigoplus_{i=0}^{\ell} M_i$, $b_0, \dots, b_\ell \in \bigoplus_{i=0}^{\ell} N_i$, and $Z_\ell \oplus (\bigoplus_{i=m_\ell+1}^{\infty} A_i)$ and $\bigoplus_{j=n_\ell+1}^{\infty} B_j$ are in the same genus. Therefore, we obtain two families of finitely generated Λ-modules which are torsion-free as R-modules $\{M_i\}_{i\in\mathbb{N}_0} \subseteq M$ and $\{N_i\}_{i\in\mathbb{N}_0} \subseteq N$ such that M_i is finitely generated and isomorphic to N_i, $\{a_i\}_{i\in\mathbb{N}_0} \subseteq \bigoplus_{i\in\mathbb{N}_0} M_i$, and $\{b_i\}_{i\in\mathbb{N}_0} \subseteq \bigoplus_{i\in\mathbb{N}_0} N_i$. We deduce that $M = \bigoplus_{i\in\mathbb{N}_0} M_i$ and $N = \bigoplus_{i\in\mathbb{N}_0} N_i$. Therefore, M is isomorphic to N. This finishes the proof of the proposition. ■

The following proposition generalizes Lemma 8.3.

Proposition 8.5. *Let R be a commutative domain of finite character with field of fractions Q. Let Λ be a module-finite R-algebra such that Λ_Q is a simple artinian ring. Let $M = \bigoplus_{i\in I} M_i$ be an infinite direct sum of non-zero finitely generated right Λ-modules which are torsion-free as R-modules, and let A be a direct summand of M of infinite rank. Let N be a finitely generated right Λ-module which is torsion-free as an R-module. If $N_\mathfrak{m}$ is isomorphic to a direct summand of $A_\mathfrak{m}$ for each maximal ideal $\mathfrak{m}$ of R, then N is isomorphic to a direct summand of A.*

Proof. Since A is a direct summand of M, there are Λ-homomorphisms $\iota\colon A \hookrightarrow M$ and $\pi\colon M \to A$ such that $\pi\iota = 1_A$. For every subset $J \subseteq I$, let $\pi_J\colon M \to \bigoplus_{j\in J} M_j$ denote the canonical projection, and let $\iota_J\colon \bigoplus_{j\in J} M_j \hookrightarrow M$ denote the canonical embedding.

Since N_Q is a direct summand of A_Q, there are Λ_Q-homomorphisms $f\colon N_Q \to A_Q$ and $g\colon A_Q \to N_Q$ such that $gf = 1_{N_Q}$. By Lemma 2.8, there is a Λ-module homomorphism $f_0\colon N \to A$ such that $f = f_0/s$ for some non-zero $s \in R$. Let J_0 be a finite subset of I such that $\operatorname{Im} f_0 \subseteq \bigoplus_{j\in J_0} M_j$. Hence $g(\iota_{J_0}\pi_{J_0}\iota)_Q f = 1_{N_Q}$.

Again, by Lemma 2.8, there is a Λ-homomorphism $g'\colon \bigoplus_{j\in J_0} M_j \to N$ such that $g(\iota_{J_0})_Q = g'/t$ for some non-zero $t \in R$. Let $g_0 = g'\pi_{J_0}\iota$, and note that $g_0/t\colon A_Q \to N_Q$ is a Λ_Q-homomorphism satisfying that $(g_0/t)f = 1_{N_Q}$. Hence, we may assume that there exists a Λ-homomorphism $g_0\colon A \to N$ that factors through $\pi_{J_0}\iota$, and such that g is of the form g_0/t for some non-zero $t \in R$. In particular, $g_0 f_0 = st1_N$. Also note that $g_0(A \cap \bigoplus_{i\in I\setminus J_0} M_i) = 0$ and $\operatorname{Im} f_0 \subseteq \bigoplus_{j\in J_0} M_j$.

Let $r_0 = st \in R$. Since R is of finite character, r_0 is contained only in finitely many maximal ideals of R, say $\mathcal{M} = \{\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_k\}$. A similar argument as above shows that there are Λ-homomorphisms $f_1, \dots, f_k\colon N \to A$, $g_1, \dots, g_k\colon A \to N$, and non-zero elements $r_1, \dots, r_k \in R$ such that $g_i f_i = r_i 1_N$ and $r_i \notin \mathfrak{m}_i$, for each $i = 1, \dots, k$. These morphisms can be chosen such that there are finite subsets $J_1, J_2, \dots, J_k \subseteq I$ such that $g_j(A \cap \bigoplus_{i\in I\setminus J_j} M_i) = 0$ and $\operatorname{Im} f_j \subseteq \bigoplus_{i\in J_j} M_i$.

Let $J = \bigcup_{i=1}^{k} J_i$, and let $J' := I \setminus J$. Consider an exact sequence

$$0 \longrightarrow X \xrightarrow{\ \nu\ } A \xrightarrow{\ \pi_J|_A\ } \bigoplus_{j\in J} M_j,$$

where $X = A \cap \bigoplus_{j\in J'} M_j$. Since A is of infinite rank, X has infinite rank as well, and hence there is a Λ_Q-monomorphism $f'\colon N_Q \to X_Q$. Then $\nu_Q f'$ splits because Λ_Q is simple artinian. Let $g'\colon A_Q \to N_Q$ be a Λ_Q-homomorphism such that $g'\nu_Q f' = 1_{N_Q}$. As in the first part of the proof, we may assume that g' factors through $(\pi_K\iota)_Q$ for some finite subset $K \subseteq J'$. Therefore, there are Λ-homomorphisms $f_\infty\colon N \to A$ and $g_\infty\colon A \to N$

such that $g_\infty f_\infty = r_\infty 1_N$ for some non-zero $r_\infty \in R$. Moreover, $\operatorname{Im} f_\infty \subseteq X$ and $g_\infty(A \cap \bigoplus_{j\in J} M_j) = 0$.

Since R is of finite character, r_∞ is contained only in finitely many maximal ideals of R, say $\mathcal{N} = \{\mathfrak{n}_1, \mathfrak{n}_2, \dots, \mathfrak{n}_\ell\}$. By the Chinese remainder theorem, we can find $e_1, \dots, e_\ell \in R$ such that $e_j \equiv 0 \mod \mathfrak{n}_i$ whenever $j \neq i$ and $e_j \equiv 1 \mod \mathfrak{n}_j$, for every $j = 1, \dots, \ell$. Note that any maximal ideal of R either does not contain r_0 or is in $\mathcal{M}$. Hence, by the definition of the $r_1, \dots, r_k$, no maximal ideal of R contains the whole set $\{r_0, \dots, r_k\}$.

For every $j = 1, \dots, \ell$, let $i(j) \in \{0, \dots, k\}$ be such that $r_{i(j)} \notin \mathfrak{n}_j$. Let $g\colon A \to N$ be the Λ-homomorphism given by $g = g_\infty + \sum_{j=1}^{\ell} e_j g_{i(j)}$.

We claim that g is a locally split epimorphism, so by Lemma 2.9 it is a split epimorphism. Note that $g f_\infty = g_\infty f_\infty = r_\infty 1_N$, so $g_\mathfrak{n}$ splits if $\mathfrak{n}$ does not contain r_∞. On the other hand,

$$g f_{i(t)} = \sum_{j=1}^{\ell} e_j\, g_{i(j)}\, f_{i(t)} = e_t\, r_{i(t)}\, 1_N + \sum_{\substack{1\le j\le \ell\\ j\neq t}} e_j\, g_{i(j)}\, f_{i(t)}.$$

When we localize at $\mathfrak{n}_t$, the first summand is an invertible element, while the second summand is in $J(\operatorname{End}_{\Lambda_{\mathfrak{n}_t}}(N_{\mathfrak{n}_t}))$ by Lemma 2.6. Hence, $g_\mathfrak{n}$ splits for any maximal ideal of R, and we conclude with the proof of the statement. ■

Now we are going to give an example showing that Lemma 8.3 is not true for direct summands that are not finitely generated. We start with the following well-known lemma, that will provide us a source of noetherian local domains of Krull dimension 1 with local integral closure. So that indecomposable finitely generated torsion-free modules have local endomorphism ring (see, for example, Corollary 3.9).

Lemma 8.6. *Let $\alpha_1, \dots, \alpha_n$ be (non-zero) coprime elements of $\mathbb{N}$. Let K be a field, let $R = K[t^{\alpha_1}, \dots, t^{\alpha_n}]$ and let $\mathfrak{m} = t^{\alpha_1}R + \cdots + t^{\alpha_n}R$. Set $\Sigma = R \setminus \mathfrak{m}$. Then:*

- (i) *R is a noetherian domain of Krull dimension 1 and field of quotients $K(t)$. The integral closure of R into its field of fractions is $K[t]$.*
- (ii) *The integral closure of $R_\mathfrak{m}$ into its field of fractions is $K[t]_\Sigma = K[t]_{(t)}$. In particular, it is a local ring.*
- (iii) *Every finitely generated, indecomposable, torsion-free $R_\mathfrak{m}$-module has local endomorphism ring.*

Proof. (i) The field of fractions of R is a subfield of $K(t)$ that contains R and also t (because $\alpha_1, \dots, \alpha_n$ are coprime), so it coincides with $K(t)$. Since $K[t]$ is a PID, it is integrally closed, and being an integral extension of R, it is the integral closure.

(ii) By (i), the extension $R_\mathfrak{m} \subseteq K[t]_\Sigma$ is integral. If $\mathfrak{n}$ is a maximal ideal of $K[t]_\Sigma$, then $\mathfrak{n} \cap R = \mathfrak{m}$. Therefore $t \in \mathfrak{n}$, so that $\mathfrak{n} = tK[t]_\Sigma$, and $K[t]_\Sigma$ is a local ring.

The statement (iii) follows from (ii) and Corollary 3.9. ■

Example 8.7. *There are a semilocal noetherian domain R of Krull dimension 1 and R-modules M and N that can be written as infinite direct sums of non-zero finitely generated torsion-free R-modules, such that $N_\mathfrak{m}$ is a direct summand of $M_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R, but N is not a direct summand of M.*

Proof. Let K be an infinite field. Let $R_1 = K[t^2, t^3]_{(t^2,t^3)}$, and let $R_2 = K[t^3, t^7]_{(t^3,t^7)}$. So Lemma 8.6 applies to R_1 and R_2.

The local domain R_1 is a Bass domain with just two indecomposable finitely generated torsion-free modules (or rank one) up to isomorphism. Namely, $X = R_1$ and $Y = \mathfrak{m}_1$ where $\mathfrak{m}_1$ denotes the maximal ideal of R_1. Note that, by Lemma 8.6, X and Y have local endomorphism ring.

The domain R_2 has infinitely many indecomposable finitely generated torsion-free modules of all ranks ≥ 2, because it fails to satisfy the Drozd–Roiter conditions (cf. Theorem 4.2 in [18]). More precisely, if we denote by $\mathfrak{m}_2$ the maximal ideal of R_2, the R_2-module $(\mathfrak{m}_2 K[t]_{(t)} + R_2)/R_2$ is not cyclic because a minimal set of generators is t^4 and t^5.

We single out an infinite family $Z_1, Z_2, \dots$ of indecomposable finitely generated torsion-free R_2-modules of rank 2. Note that, by Lemma 8.6, such modules have local endomorphism ring.

Let $\varphi\colon K(t) \to K(t)$ be the automorphism that fixes K and such that $\varphi(t) = t + 1$. Let R be the ring that fits in the pull-back diagram

$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \big\downarrow & & \big\downarrow \\ R_2 & \xrightarrow{\ \varphi'\ } & K(t), \end{array}$$

where $\varphi'\colon R_2 \hookrightarrow K(t) \xrightarrow{\varphi} K(t)$.

By Theorem 4.4 in [35], R is a noetherian domain of Krull dimension 1 with exactly two maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ and satisfying that $R_{\mathfrak{m}} \cong R_1$ and $R_{\mathfrak{n}} \cong R_2$.

Apply the results on [35] (or just Corollary 5.8) to construct two sequences of finitely generated torsion-free R-modules $M_1, M_2, \dots, N_1, N_2, \dots$ such that

$$(M_i)_{\mathfrak{n}} = (N_i)_{\mathfrak{n}} = Z_i, \quad (M_i)_{\mathfrak{m}} = X \oplus Y \quad \text{and} \quad (N_i)_{\mathfrak{m}} = X \oplus X.$$

Call $M = \bigoplus_{i \in \mathbb{N}} M_i$ and $N = \bigoplus_{i \in \mathbb{N}} N_i$. Then N is locally a direct summand of M, but N is not isomorphic to a direct summand of M. Indeed, if $N \oplus N' \cong M$, then

$$\big(\textstyle\bigoplus_{i \in \mathbb{N}} Z_i\big) \oplus N'_{\mathfrak{n}} \cong \textstyle\bigoplus_{i \in \mathbb{N}} Z_i.$$

Since $\{Z_i\}_{i \in \mathbb{N}}$ are non-isomorphic and have local endomorphism rings, $N'_{\mathfrak{n}} = \{0\}$. Hence, as N' is torsion-free, $N' = \{0\}$. But $N_{\mathfrak{m}} \not\cong M_{\mathfrak{m}}$, a contradiction. ∎

Remark 8.8. By Proposition 9.6, the ring R in Example 8.7 satisfies that the class of direct sums of finitely generated torsion-free modules is closed under direct summands.

Proposition 8.9. *Let R be a commutative domain of finite character. Let M be a torsion-free R-module of countable rank. Let $\mathfrak{m}_0$ be a maximal ideal of R. Assume that*

(a) *$M_{\mathfrak{m}}$ is a direct sum of finitely generated torsion-free modules of rank one with local endomorphism ring for every maximal ideal $\mathfrak{m} \neq \mathfrak{m}_0$, and*

(b) *$M_{\mathfrak{m}_0}$ is a direct sum of finitely generated torsion-free modules.*

Then M is in the same genus as a direct sum of finitely generated torsion-free modules if and only if

(i) *$M_{\mathfrak{m}}$ is a free module for all but countably many maximal ideals $\mathfrak{m}$ of R.*

(ii) *If $\mathcal{M} = \{\mathfrak{m} \in \operatorname{mSpec} R \mid M_{\mathfrak{m}}$ is not free$\}$, and if $r_{\mathfrak{m}}$ denotes the number of free direct summands in the decomposition of $M_{\mathfrak{m}}$, then, for every $b \in \mathbb{N}$, the set $\{\mathfrak{m} \in \mathcal{M} \mid r_{\mathfrak{m}} \leq b\}$ is finite.*

Proof. Let $N = \bigoplus_{i\in\mathbb{N}} N_i$, where each N_i is a finitely generated torsion-free module, and assume that M and N are in the same genus. Statement (i) follows from Corollary 5.4. If $\mathcal{M}$ is finite, (ii) is clear. Assume $\mathcal{M}$ is infinite and let $b \in \mathbb{N}$. Then there are only finitely many maximal ideals $\mathfrak{m} \in \mathcal{M}$ such that at least one of $(N_1)_{\mathfrak{m}}, \dots, (N_b)_{\mathfrak{m}}$ is not free, so $r_{\mathfrak{m}} \geq b$ for almost all $\mathfrak{m} \in \mathcal{M}$. This proves (ii).

Conversely, assume that (i) and (ii) are satisfied and let $M_{\mathfrak{m}} = \bigoplus_{i\in\mathbb{N}} M_{\mathfrak{m},i}$, where each $M_{\mathfrak{m},i}$ is a finitely generated torsion-free module and such that $M_{\mathfrak{m},i}$ has rank one for every maximal ideal $\mathfrak{m} \neq \mathfrak{m}_0$. For each $i \in \mathbb{N}$, let d_i denote the rank of the module $M_{\mathfrak{m}_0,i}$.

First, assume that $\mathcal{M}$ is finite. For each $i \in \mathbb{N}$, apply the package deal Theorem 5.7 to

$$X_i(\mathfrak{m}_0) = M_{\mathfrak{m}_0,i} \quad \text{and} \quad X_i(\mathfrak{m}) = \bigoplus_{i=d_0+\cdots+d_{i-1}+1}^{d_0+\cdots+d_i} M_{\mathfrak{m},i} \quad \text{for each } \mathfrak{m} \neq \mathfrak{m}_0$$

(note that, since $\mathcal{M}$ is finite, $X_i(\mathfrak{m})$ is free for almost all maximal ideals $\mathfrak{m}$ of R). Then, for each $j \in \mathbb{N}$, there is an R-module N_j such that $(N_j)_{\mathfrak{m}} \cong X_j(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R. Therefore, $\bigoplus_{j\in\mathbb{N}} N_j$ is in the same genus as M.

Now, assume that $\mathcal{M}$ is infinite and consider the case where $r_{\mathfrak{m}}$ is finite for every maximal ideal $\mathfrak{m}$ of R. First, write the elements of $\mathcal{M}$ in a sequence $\mathfrak{m}_1, \mathfrak{m}_2, \dots$ in such a way that $r_{\mathfrak{m}_1} \leq r_{\mathfrak{m}_2} \leq \cdots$ is non-decreasing, and assume that the direct summands in the decompositions of $M_{\mathfrak{m}}$ are indexed such that $M_{\mathfrak{m},1}, \dots, M_{\mathfrak{m},r_{\mathfrak{m}}}$ are the free direct summands. For each $i \in \mathbb{N}$, apply the package deal Theorem 5.7 to

$$X_i(\mathfrak{m}_0) = M_{\mathfrak{m}_0,i} \quad \text{and} \quad X_i(\mathfrak{m}) = \bigoplus_{j=d_0+\cdots+d_{i-1}+1}^{d_0+\cdots+d_i} M_{\mathfrak{m},j} \quad \text{for each } \mathfrak{m} \neq \mathfrak{m}_0$$

(note that, with this reordering and considering (i) and (ii) with $b = d_0 + \cdots + d_i$, $M_{\mathfrak{m},i}$ is free for almost all maximal ideals). Then, for each $j \in \mathbb{N}$, there is an R-module N_j such that $(N_j)_{\mathfrak{m}} \cong X_j(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R. Therefore, $\bigoplus_{j\in\mathbb{N}} N_j$ is in the same genus as M.

Finally, assume that $\mathcal{M}$ is infinite and there is at least one maximal ideal $\mathfrak{m}$ with $r_{\mathfrak{m}}$ infinite. Let $\mathfrak{m}_1, \mathfrak{m}_2, \dots$ be a (finite or infinite) list of elements in $\{\mathfrak{m} \in \mathcal{M} \mid r_{\mathfrak{m}}$ is finite$\}$ and let $\mathfrak{n}_1, \mathfrak{n}_2, \dots$ be a (finite or infinite) list of elements in $\{\mathfrak{n} \in \mathcal{M} \mid r_{\mathfrak{n}}$ is infinite$\}$. The first sequence is chosen such that $r_{\mathfrak{m}_1} \leq r_{\mathfrak{m}_2} \leq \cdots$ is non-decreasing, and again we assume the direct summands in the decompositions of $M_{\mathfrak{m}_i}$ are indexed such that $M_{\mathfrak{m},1}, \dots, M_{\mathfrak{m},r_{\mathfrak{m}_i}}$ are the free direct summands. Moreover, we assume that the direct summands in the decompositions of $M_{\mathfrak{n}_i}$ are indexed such that $M_{\mathfrak{n}_i,1}, \dots, M_{\mathfrak{n}_i,i}$ are free.

Let us check that each $M_{\mathfrak{m},j}$ is free for almost all $\mathfrak{m} \in \mathcal{M}$. If $M_{\mathfrak{n}_i,j}$ is not free, then $i < j$. If $M_{\mathfrak{m},j}$ is not free for infinitely many $\mathfrak{m}$'s with finite $r_{\mathfrak{m}}$, then $b = j - 1$ would contradict (ii). Hence, as before, we find finitely generated torsion-free R-modules N_j such that $(N_j)_{\mathfrak{m}} \cong X_j(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R. Therefore, $\bigoplus_{j\in\mathbb{N}} N_j$ is in the same genus as M. ∎

9. The noetherian case

The results of Section 8 allow us to show in this section that the converse of Theorem 6.6 is true for semilocal noetherian domains of Krull dimension 1 (cf. Proposition 9.6), and this result will allow us also to prove the converse of Theorem 6.6 for noetherian domains of Krull dimension 1 and with finitely generated integral closure in Theorem 9.8.

Let $\mathcal{C}$ be a pre-additive category, let M be an object of $\mathcal{C}$, and let I be a two-sided ideal of the ring $\mathrm{End}_{\mathcal{C}}(M)$. Recall that the *ideal of $\mathcal{C}$ associated to I* is the ideal $\mathcal{A}_I$ of the category $\mathcal{C}$ defined as follows. A morphism $f\colon X \to Y$ belongs to $\mathcal{A}_I(X,Y)$ if and only if $\beta f \alpha \in I$ for every pair of morphisms $\alpha\colon M \to X$ and $\beta\colon Y \to M$ in the category $\mathcal{C}$.

Notice that if $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$. Then the restriction of $\mathcal{A}_I$ to $\mathcal{C}'$ gives an ideal of the category $\mathcal{C}'$.

Remark 9.1. Let R be a commutative ring, and let Λ be a module-finite R-algebra. Let M be a non-zero finitely generated right Λ-module with endomorphism ring $S = \mathrm{End}_\Lambda(M)$. Let $\varphi\colon R \to S$ denote the canonical homomorphism. Let $\mathfrak{n}$ be a two-sided maximal ideal of S, and let $\mathfrak{m} = \varphi^{-1}(\mathfrak{n})$. By Lemma 2.7 (i), $\mathfrak{m}$ is a maximal ideal of R.

Let X and Y be two objects in Mod-Λ. If $f \in \mathrm{Hom}_\Lambda(X,Y)$ is such that $f(X) \subseteq Y\mathfrak{m}$, then $f \in \mathcal{A}_\mathfrak{n}$. Indeed, $\beta f \alpha(M) \subseteq M\mathfrak{m}$ for every pair of morphisms $\alpha\colon M \to X$ and $\beta\colon Y \to M$. Hence, by Lemma 2.7 (ii), $\beta f \alpha \in \mathfrak{n}$.

If $\mathcal{C}$ is a full subcategory of Mod-Λ, we will still denote by $\mathcal{A}_\mathfrak{n}$ the restriction to $\mathcal{C}$ of the ideal defined in the whole module category.

Lemma 9.2. *Let R be a commutative ring, and let $\mathfrak{m}$ be a maximal ideal of the ring R. Let $f\colon X \to Y$ be a homomorphism of R-modules such that $f_\mathfrak{m}(X_\mathfrak{m}) \subseteq Y_\mathfrak{m}\mathfrak{m}R_\mathfrak{m}$. Then $f(X) \subseteq Y\mathfrak{m}$.*

Proof. Let $x \in X$. Then there exist $r_i \in \mathfrak{m}$, $y_i \in Y$ and $s \in R \setminus \mathfrak{m}$ such that

$$\frac{f(x)}{1} = \sum_{i=1}^{n} \frac{y_i r_i}{1}\,\frac{1}{s} = \frac{y}{s}, \qquad y \in Y\mathfrak{m}.$$

Therefore, there exists $t \notin \mathfrak{m}$ such that $f(x)st = yt$. Then there exists $u \notin \mathfrak{m}$ such that

$$1 - stu \in \mathfrak{m} \quad \text{and} \quad f(x) = f(x)(1-stu) + f(x)stu = f(x)(1-stu) + ytu \in Y\mathfrak{m}. \qquad \blacksquare$$

Proposition 9.3. *Under the assumptions of Remark 9.1, let $N = \bigoplus_{i\in\mathbb{N}} M_i$ be a direct sum of finitely generated right Λ-modules which are torsion-free as R-modules, and consider two direct summands A and A' of N. Assume that $\mathcal{C}$ contains M, A, A', and M_i, for each $i \in \mathbb{N}$. If $A_\mathfrak{m} \cong A'_\mathfrak{m}$, then A is isomorphic to A' in the factor category $\mathcal{C}/\mathcal{A}_\mathfrak{n}$.*

Proof. We have to check that there are homomorphisms $f\colon A \to A'$ and $g\colon A' \to A$ such that $\mathrm{Id}_A - gf, \mathrm{Id}_{A'} - fg \in \mathcal{A}_\mathfrak{n}$. By Remark 9.1 and Lemma 9.2, it is sufficient to find $f\colon A \to A'$ and $g\colon A' \to A$ such that

$$(\mathrm{Id}_A - gf)_\mathfrak{m}(A_\mathfrak{m}) \subseteq A_\mathfrak{m}\mathfrak{m}R_\mathfrak{m} \quad \text{and} \quad (\mathrm{Id}_{A'} - fg)_\mathfrak{m}(A'_\mathfrak{m}) \subseteq A'_\mathfrak{m}\mathfrak{m}R_\mathfrak{m}.$$

Assume now that $A_\mathfrak{m} \cong A'_\mathfrak{m}$. Let $\alpha\colon A_\mathfrak{m} \to A'_\mathfrak{m}$ and $\beta\colon A'_\mathfrak{m} \to A_\mathfrak{m}$ denote mutually inverse isomorphisms. Then, $\gamma = \iota_{A'_\mathfrak{m}} \alpha \pi_{A_\mathfrak{m}}$ and $\delta = \iota_{A_\mathfrak{m}} \beta \pi_{A'_\mathfrak{m}}$ are elements that belong to $\mathrm{End}_{R_\mathfrak{m}}(\bigoplus_{i \in \mathbb{N}} (M_i)_\mathfrak{m})$, and such that the following diagram commutes:

$$\begin{array}{ccc}
\bigoplus_{i\in\mathbb{N}}(M_i)_\mathfrak{m} & \underset{\delta}{\overset{\gamma}{\rightleftarrows}} & \bigoplus_{i\in\mathbb{N}}(M_i)_\mathfrak{m} \\
\iota_{A_\mathfrak{m}} \uparrow \downarrow \pi_{A_\mathfrak{m}} & & \iota_{A'_\mathfrak{m}} \uparrow \downarrow \pi_{A'_\mathfrak{m}} \\
A_\mathfrak{m} & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & A'_\mathfrak{m}.
\end{array}$$

By Lemma 2.8, we can consider γ as a column-finite matrix such that the i-th column represents

$$\mathrm{Hom}_{\Lambda_\mathfrak{m}}((M_i)_\mathfrak{m}, \textstyle\bigoplus_{j \in I} (M_j)_\mathfrak{m}) \cong \mathrm{Hom}_\Lambda(M_i, \bigoplus_{j\in I} M_j) \otimes_R R_\mathfrak{m}.$$

For each $i \in \mathbb{N}$, let $s_i \notin \mathfrak{m}$ be the product of the denominators in the i-th column, and define $\tau = \bigoplus_{i\in\mathbb{N}} s_i \mathrm{Id}_{(M_i)_\mathfrak{m}} \in \mathrm{Aut}_{\Lambda_\mathfrak{m}}(N_\mathfrak{m})$. By Lemma 2.8, there exists $h \in \mathrm{End}_\Lambda(N)$ such that $\gamma \circ \tau = h_\mathfrak{m}$. For each $i \in \mathbb{N}$, take $t_i \notin \mathfrak{m}$ such that $1 - t_i s_i \in \mathfrak{m}$ and define $\theta = \bigoplus_{i \in \mathbb{N}} t_i \mathrm{Id}_{M_i} \in \mathrm{End}_\Lambda(N)$. Similarly, define τ' and θ' starting with the endomorphism δ, i.e., $\delta \circ \tau' = h'_\mathfrak{m}$ for some $h' \in \mathrm{End}_\Lambda(N)$. Note that $\tau\theta_\mathfrak{m} = \mathrm{Id}_{N_\mathfrak{m}} - r$ for some $r \in \mathrm{End}_{\Lambda_\mathfrak{m}}(N_\mathfrak{m})$ with $\mathrm{Im}\, r \subseteq (N_\mathfrak{m})\mathfrak{m} R_\mathfrak{m}$. Similarly, there exists $r' \in \mathrm{End}_{\Lambda_\mathfrak{m}}(N_\mathfrak{m})$ with $\mathrm{Im}\, r' \subseteq (N_\mathfrak{m})\mathfrak{m}R_\mathfrak{m}$ such that $\tau'\theta'_\mathfrak{m} = \mathrm{Id}_{N_\mathfrak{m}} - r'$.

Set

$$f = \pi_{A'} h \theta \iota_A \quad \text{and} \quad g = \pi_A h' \theta' \iota_{A'}.$$

Then

$$\begin{aligned}
(\mathrm{Id}_A - gf)_\mathfrak{m} &= (\mathrm{Id}_A - \pi_A h' \theta' \iota_{A'} \pi_{A'} h \theta \iota_A)_\mathfrak{m} \\
&= \mathrm{Id}_{A_\mathfrak{m}} - \pi_{A_\mathfrak{m}} \delta \tau' \theta'_\mathfrak{m} \iota_{A'_\mathfrak{m}} \pi_{A'_\mathfrak{m}} \gamma \tau \theta_\mathfrak{m} \iota_{A_\mathfrak{m}} \\
&= \mathrm{Id}_{A_\mathfrak{m}} - \pi_{A_\mathfrak{m}} \iota_{A_\mathfrak{m}} \beta \pi_{A'_\mathfrak{m}} \tau' \theta'_\mathfrak{m} \iota_{A'_\mathfrak{m}} \pi_{A'_\mathfrak{m}} \iota_{A'_\mathfrak{m}} \alpha \pi_{A_\mathfrak{m}} \tau \theta_\mathfrak{m} \iota_{A_\mathfrak{m}} \\
&= \mathrm{Id}_{A_\mathfrak{m}} - \pi_{A_\mathfrak{m}} \iota_{A_\mathfrak{m}} \beta \pi_{A'_\mathfrak{m}} (\mathrm{Id}_{N_\mathfrak{m}} - r') \iota_{A'_\mathfrak{m}} \pi_{A'_\mathfrak{m}} \iota_{A'_\mathfrak{m}} \alpha \pi_{A_\mathfrak{m}} (\mathrm{Id}_{N_\mathfrak{m}} - r) \iota_{A_\mathfrak{m}} \\
&= \mathrm{Id}_{A_\mathfrak{m}} - \pi_{A_\mathfrak{m}} \iota_{A_\mathfrak{m}} \beta \pi_{A'_\mathfrak{m}} 1 \iota_{A'_\mathfrak{m}} \pi_{A'_\mathfrak{m}} \iota_{A'_\mathfrak{m}} \alpha \pi_{A_\mathfrak{m}} 1 \iota_{A_\mathfrak{m}} + s,
\end{aligned}$$

where $s \in \mathrm{End}_{\Lambda_\mathfrak{m}}(A_\mathfrak{m})$ satisfies that $\mathrm{Im}\, s \subseteq A_\mathfrak{m} \mathfrak{m} R_\mathfrak{m}$. A symmetric computation shows that $\mathrm{Im}(\mathrm{Id}_{A'} - fg)_\mathfrak{m} \subseteq A'_\mathfrak{m} \mathfrak{m} R_\mathfrak{m}$. ∎

Remark 9.4. The statement in Proposition 9.3 is also true if we assume that N is a direct sum of finitely presented right Λ-modules instead of a direct sum of finitely generated right Λ-modules which are torsion-free as R-modules using Theorem 3.18 in [29] instead of Lemma 2.8.

Corollary 9.5. *Let R be a commutative ring, and let Λ be a module-finite R-algebra. Let $M = \bigoplus_{i \in \mathbb{N}} M_i$ be a direct sum of non-zero finitely presented right Λ-modules with semilocal endomorphism rings. If A and A' are direct summands of M, then $A \cong A'$ if and only if $A_\mathfrak{m} \cong A'_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Consider $\mathcal{C} = \mathrm{Add}(\bigoplus_{i\in\mathbb{N}} M_i)$. We aim to apply Proposition 3.6 in [27]. We have to check if $\mathfrak{n}$ is a maximal ideal of $\mathrm{End}_\Lambda(M_i)$ for some $i \in \mathbb{N}$, then A and A' are isomorphic objects of the factor category $\mathcal{C}/\mathcal{A}_\mathfrak{n}$. Note that the preimage of $\mathfrak{n}$ in the canonical homomorphism $R \to \mathrm{End}_\Lambda(N_i)$ is a maximal ideal of R by Lemma 2.7(i), so we can apply Proposition 9.3 and Remark 9.4. ∎

From now on, we only consider the case when R is a noetherian domain of Krull dimension 1. We can try to find appropriate generalizations reflecting the results in the previous parts of the paper. For the semilocal case below, we need Krull dimension 1 because of Corollary 3.9.

Proposition 9.6. *Let R be a semilocal noetherian domain of Krull dimension 1. Let $\mathfrak{m}_0, \mathfrak{m}_1, \dots, \mathfrak{m}_k$ be the list of maximal ideals of R. Assume that*

(i) *$R_{\mathfrak{m}_i}$ has the two-generator property for $i = 1, \dots, k$,*

(ii) *the normalization of $R_{\mathfrak{m}_i}$ is a discrete valuation domain for every $i = 0, 1, \dots, k$.*

Then every pure projective torsion-free R-module is a direct sum of finitely presented modules.

Proof. By Kaplansky's theorem, see Theorem 1 in [16], every pure projective module is a direct sum of countably generated modules. Therefore, we may consider A to be a direct summand of $\bigoplus_{i\in\mathbb{N}} M_i$, where each M_i is a non-zero finitely presented torsion-free R-module, say $A \oplus A' = \bigoplus_{i\in\mathbb{N}} M_i$. Also, we may assume that A is not finitely generated.

If $\mathfrak{m}$ is a maximal ideal of R, then $A_\mathfrak{m} \oplus A'_\mathfrak{m} = \bigoplus_{i\in\mathbb{N}} (M_i)_\mathfrak{m}$, and by (ii) and Corollary 3.9, $A_\mathfrak{m}$ is a direct sum of finitely generated $R_\mathfrak{m}$-modules with local endomorphism ring.

For each $i = 0, \dots, k$, let $A_{\mathfrak{m}_i} = \bigoplus_{j\in\mathbb{N}} N_{i,j}$, where $N_{i,j}$ is a finitely generated indecomposable $R_{\mathfrak{m}_i}$-module. In Theorem 4.3 of [31], Rush proved that over a noetherian ring with the two-generator property any finitely generated indecomposable torsion-free module has rank 1, thus (i) implies that $N_{i,j}$ is a module of rank 1 if $i \geq 1$.

Therefore, the module A fulfills the hypothesis of Proposition 8.9, and since the number of maximal ideals of R is finite, we can deduce that A is in the same genus as a module B that is a direct sum of finitely generated torsion-free modules. Since A and B are direct summands of $B \oplus (\bigoplus_{i\in\mathbb{N}} M_i)$, we deduce from Corollary 9.5 that $A \cong B$. ∎

The following lemma is a variation of Proposition 7.2 adapted to the noetherian setting.

Lemma 9.7. *Let R be a noetherian domain of Krull dimension 1, and with module-finite normalization $\overline{R}$. Let $\mathcal{M} = \{\mathfrak{m}_1, \dots, \mathfrak{m}_k\}$ be the list of maximal ideals of R such that $R_\mathfrak{m}$ is a principal ideal domain for any maximal ideal $\mathfrak{m} \notin \mathcal{M}$, and let $\Sigma = R \setminus \bigcup_{i=1}^k \mathfrak{m}_i$. Further, let $M \subseteq \overline{R}^{(\omega)}$ be such that M_Σ is a direct sum of finitely generated R_Σ-modules. Then M is a direct sum of finitely generated modules.*

Proof. We may assume that the rank of M is infinite. Let $M_\Sigma = \bigoplus_{i=1}^\infty M_i$, where each M_i is a non-zero finitely generated R_Σ-module. For each $i = 1, 2, \dots$, let $N_i := \{m \in M \mid m/1 \in \bigoplus_{j=1}^i M_j\}$. Note that $N_i \subseteq \overline{R}^\ell$ for some ℓ, hence N_i is a finitely generated submodule of M. Further, we claim that N_i is an RD-submodule of M. Consider $m \in M, n \in N_i$

and $0 \neq r \in R$ such that $mr = n$. Then $(m/1)(r/1) = n/1$ in M_Σ. Since $r/1$ is not a zero-divisor on each M_t, $m/1$ is an element of $\bigoplus_{j=1}^{i} M_j$.

We claim that for every $i \in \mathbb{N}$, the inclusion $\iota_i \colon N_i \to N_{i+1}$ splits. By Lemma 2.9, it is enough to check that $\iota_i \otimes_R R_\mathfrak{m}$ splits for every maximal ideal $\mathfrak{m}$ of R. Since $\iota_i \otimes_R R_\mathfrak{m} \cong \iota_i \otimes_R R_\Sigma \otimes_{R_\Sigma} R_\mathfrak{m}$ for any $\mathfrak{m} \in \mathcal{M}$ and $\iota_i \otimes_R R_\Sigma$ splits, $\iota_i \otimes_R R_\mathfrak{m}$ splits for any $\mathfrak{m} \in \mathcal{M}$.

Now suppose that $\mathfrak{m} \notin \mathcal{M}$. Then it is easy to check that $(N_i)_\mathfrak{m}$ is an RD-submodule of $(N_{i+1})_\mathfrak{m}$ and since $R_\mathfrak{m}$ is a valuation domain, also a pure submodule of $(N_i)_\mathfrak{m}$. The pure exact sequence

$$0 \longrightarrow (N_i)_\mathfrak{m} \xrightarrow{\iota \otimes_R R_\mathfrak{m}} (N_{i+1})_\mathfrak{m} \longrightarrow (N_{i+1}/N_i)_\mathfrak{m} \longrightarrow 0$$

splits, since the right-hand term is finitely presented. From this, we conclude that ι_i splits.

Overall, M is a union of the chain $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ where each N_i splits in N_{i+1}, i.e., there exists a submodule $D_{i+1} \subseteq N_{i+1}$ such that $N_{i+1} = N_i \oplus D_{i+1}$. Further let $D_1 := N_1$. Then $M = \bigoplus_{i=1}^{\infty} D_i$ is a direct sum of finitely generated R-modules. ∎

Theorem 9.8. *Let R be a noetherian domain of Krull dimension* 1*, and with module-finite normalization $\overline{R}$. Then the following statements are equivalent*:

(1) *Every pure projective torsion-free R-module is a direct sum of finitely presented modules.*

(2) *For any finitely generated, torsion-free R-module X, every element in* $\mathrm{Add}(X)$ *is a direct sum of finitely generated modules.*

(3) *R satisfies the following two conditions*:

   (i) *there exists at most one maximal ideal $\mathfrak{m}_0$ of R such that $R_{\mathfrak{m}_0}$ is not a Bass domain, and*

   (ii) *the normalization of $R_\mathfrak{m}$ is a discrete valuation domain for every maximal ideal $\mathfrak{m}$ of R.*

Proof. It is clear that (1) implies (2). Statement (2) implies (3) by Theorem 6.6. We only need to prove that (3) implies (1).

Note that, if R is integrally closed, then R is a Dedekind domain, and pure projective torsion-free modules are projective. Assume that $R \neq \overline{R}$, and let $\mathcal{M} = \{\mathfrak{m}_1, \dots, \mathfrak{m}_k\}$ be a finite set of maximal ideals of R such that $R_\mathfrak{m}$ is integrally closed whenever $\mathfrak{m} \notin \mathcal{M}$. Let $\Sigma := R \setminus \bigcup_{i=1}^{k} \mathfrak{m}_i$. Therefore, R_Σ is a semilocal ring satisfying the assumptions of Proposition 9.6. Hence, any pure projective torsion-free R_Σ-module is a direct sum of finitely presented modules.

Let A be a countably generated pure projective torsion-free R-module, i.e., a direct summand of $\bigoplus_{i \in \mathbb{N}} M_i$, where each M_i is a finitely generated torsion-free module. Note that each M_i can be considered as a submodule of $M_i \overline{R}$ (the $\overline{R}$-submodule of $(M_i)_0$ generated by M_i) which is a projective $\overline{R}$-module. Hence, we may consider A as a submodule of $\overline{R}^{(\omega)}$.

Since A_Σ is a pure projective torsion-free R_Σ-module, it has to be a direct sum of finitely generated modules. By Lemma 9.7, A is a direct sum of finitely generated modules. ∎

10. A family of examples

In this section, we provide a family of examples of h-local domains of Krull dimension 1, not necessarily noetherian, that exemplifies well the situations described in Theorem 6.6 and also in Corollary 7.3, as we show that they satisfy that direct summands of finitely generated torsion-free modules are direct sums of finitely generated modules. This family of examples was suggested to us by Carmelo Antonio Finocchiaro and Paolo Zanardo.

Let $K \subseteq L$ be a field extension, and consider the ring $R = K + xL[x]$ and its localization $R_{\mathfrak{m}}$ at the maximal ideal $\mathfrak{m} = xL[x]$. The field of fractions of R is $L(x)$, and R fits in the pull-back diagram of rings

$$\text{(C1)} \qquad \begin{array}{ccc} R = K + xL[x] & \xhookrightarrow{\text{incl.}} & L[x] \\ \Big\downarrow \text{ev}_0 & & \Big\downarrow \text{ev}_0 \\ K & \xhookrightarrow{\text{incl.}} & L, \end{array}$$

where ev_0 denotes the evaluation at 0. In fact, this diagram is a conductor square, with conductor ideal $\mathfrak{c} = \mathfrak{m}$ since $R/\mathfrak{m} \cong K$ and $L[x]/\mathfrak{m} \cong L$.

Note that $R_{\mathfrak{m}} = K + xL[x]_{\mathfrak{m}}$, since every element in $R_{\mathfrak{m}}$ can be written as

$$\frac{a + xp(x)}{1 + xq(x)} = a + \frac{x(p(x) - aq(x))}{1 + xq(x)},$$

with $a \in K$, $p(x), q(x) \in L[x]$.

The localization of (C1) at $\mathfrak{m}$ gives also a conductor square for $R_{\mathfrak{m}}$,

$$\text{(C2)} \qquad \begin{array}{ccc} R_{\mathfrak{m}} = K + xL[x]_{\mathfrak{m}} & \xhookrightarrow{\text{incl.}} & L[x]_{\mathfrak{m}} \\ \Big\downarrow \text{ev}_0 & & \Big\downarrow \text{ev}_0 \\ K & \xhookrightarrow{\text{incl.}} & L, \end{array}$$

with conductor $\mathfrak{c}_{\mathfrak{m}} = xL[x]_{\mathfrak{m}}$.

Lemma 10.1. *Let $K \subseteq L$ be a field extension, let R denote the ring $K + xL[x]$, and let $\mathfrak{m} = xL[x]$. Then*:

(i) *If I is an ideal of R, then $IL = L[x]$ if and only if $I = R$.*

(ii) *Every maximal ideal $\mathfrak{n} \neq \mathfrak{m}$ of R is generated by a polynomial $q(x) \in R$ that is irreducible in $L[x]$ with $q(0) = 1$.*

(iii) *$R_{\mathfrak{n}} = L[x]_{\mathfrak{n}}$, that is, $R_{\mathfrak{n}}$ is a discrete valuation ring for every maximal ideal $\mathfrak{n} \neq \mathfrak{m}$ in R.*

(iv) *R is an h-local domain of Krull dimension* 1.

Proof. (i) Suppose that $IL = L[x]$, so $y_0 + y_1 a_1 + \cdots + y_n a_n = 1$ for some elements $y_i \in I$ and $a_i \in L \setminus K$. Then, for every polynomial $q(x) \in L[x]$, $y_0 xq(x) + y_1 a_1 xq(x) + \cdots + y_n a_n xq(x) = xq(x) \in I$. Therefore $xL[x] \subseteq I$. Since $xL[x]$ is a maximal ideal of R and $I \not\subseteq xL[x]$, $I = K + xL[x]$.

(ii) Let $\mathfrak{n} \neq \mathfrak{m}$ be a maximal ideal of R. By (i) and because $\mathfrak{n}$ is maximal, $\mathfrak{n}L \neq L[x]$ and $\mathfrak{n}L \cap R = \mathfrak{n}$. There exists $p(x) \in L[x]$ with $p(0) = 1$ such that $p(x)L[x] = \mathfrak{n}L$. Since $p(x) \in \mathfrak{n}L \cap R$, $p(x) \in \mathfrak{n}$. Notice that if $q(x) \in L[x]$ and $p(x)q(x) \in R$, then also $q(x) \in R$. Therefore $p(x)R = \mathfrak{n}$. If $p(x) = q_1(x)q_2(x)$ in $L[x]$, and since $p(0) = 1$, we can assume that $q_1(0) = q_2(0) = 1$, so that it is also a decomposition in R. Since $\mathfrak{n}$ is maximal, either $q_1(x)$ or $q_2(x) \in \mathfrak{n}$, which implies that either $q_1(x)$ or $q_2(x)$ is equal 1. This shows that $p(x)$ is irreducible in $L[x]$.

(iii) Since the element $x \in R \setminus \mathfrak{n}$ is invertible in $R_\mathfrak{n}$, we have $a = ax/x \in R_\mathfrak{n}$ for every element $a \in L$, so we deduce that $L \subseteq R_\mathfrak{n}$, that is, $L[x]_\mathfrak{n} \subseteq R_\mathfrak{n}$.

(iv) Let $p(x)$ be a non-zero polynomial in R. If $p(0) \neq 0$, $p(x)$ can be written as $p(x) = kq_1(x) \cdots q_n(x)$, where $k = p(0) \in K^*$ and $q_i(x)$ are irreducible polynomials in $L[x]$ with $q_i(0) = 1$ for every $i = 1, \dots, n$ so, in particular, $q_i(x) \in R$. This decomposition is unique up to a unit (an element in K) because $L[x]$ is a UFD, and in this case, $p(x)$ is only contained in the maximal ideals generated by the irreducible polynomials $q_i(x)$ for $i = 1, \dots, n$.

If $p(0) = 0$, then $p(x)$ can be written as $p(x) = x^m q(x)$, where $m \geq 1$ and $q(x) \in L[x]$ with $q(0) \neq 0$. Then $q(x)$ can be written as $q(x) = lq_1(x) \cdots q_n(x)$, where $l = q(0) \in L^*$ and $q_i(x)$ are irreducible polynomials in $L[x]$ with $q_i(0) = 1$ for every $i = 1, \dots, n$, so again, $q_i(x) \in R$. Therefore, $p(x)$ is only contained in the maximal ideals generated by the irreducible polynomials $q_i(x)$ for $i = 1, \dots, n$ and it is also contained in $\mathfrak{m}$. Therefore, R has finite character.

Now we prove that $R_\mathfrak{m}$ has Krull dimension 1. Let $\mathfrak{p}$ be a non-zero prime ideal of $R_\mathfrak{m}$ and let $0 \neq g \in \mathfrak{p}$. We can assume, up to a unit of $R_\mathfrak{m}$, that g is of the form $x^n a$, with $a \in L \setminus \{0\}$ and $n \geq 1$. If $n > 1$, then $x^n a = x \cdot (x^{n-1}a)$ which implies that either x or $x^{n-1}a$ are in $\mathfrak{p}$. By induction on n, we can deduce that $\mathfrak{p}$ contains an element of the form xa with $a \in L \setminus \{0\}$, so it also contains $x^2 b = xa \cdot xa^{-1}b$ for any $b \in L$. As $(xb)^2 \in \mathfrak{p}$ for any $b \in L$, we deduce that $xb \in \mathfrak{p}$. Hence $\mathfrak{p} = \mathfrak{m}$.

Since the Krull dimension of R localized at any maximal ideal is 1, we deduce that R has Krull dimension 1 and, being of finite character, it is h-local. ∎

From (iii), it follows that finitely generated torsion-free $R_\mathfrak{n}$-modules are free for every maximal ideal $\mathfrak{n} \neq \mathfrak{m}$ in R.

Remark 10.2. Assume $K \subsetneq L$.

(1) *R is integrally closed if and only if the extension $K \subseteq L$ is purely transcendental.* If $\alpha \in L \setminus K$ is algebraic over K, then α is integral over R. So if R is integrally closed, $K \subseteq L$ is a purely transcendental extension.

To prove the converse, let $f \in L(x)$ satisfy a polynomial equation $f^n + a_{n-1} f^{n-1} + \cdots + a_0 = 0$ with $a_i \in R$ for $i = 0, \dots, n-1$. Since $L[x]$ is already integrally closed, we can assume that $f \in L[x]$. Let f_0 denote the evaluation at 0 of f. Then f_0 satisfies a polynomial equation in K, so f_0 is algebraic over K. Since L is purely transcendental, we deduce that $f_0 \in K$, so $f \in R$.

(2) *$R_\mathfrak{m}$ is noetherian if and only if the extension $K \subseteq L$ has finite degree.* Indeed, let $(a_i)_{i \in A}$ be a basis of L as K-vector space. The maximal ideal $\mathfrak{m} = xL[x]$ is generated by $\mathcal{A} = \{xa_i\}_{i \in A}$ and no proper subset of $\mathcal{A}$ generates $\mathfrak{m}$. Then $R_\mathfrak{m}$ is noetherian if and only if A is finite.

(3) $R_{\mathfrak{m}}$ *is not a valuation ring.* Note that every element $a \in L \setminus K$ is in its field of fractions $L(x)$, but neither a nor a^{-1} belong to $R_{\mathfrak{m}}$.

Now we will study torsion-free modules over the ring R and over the ring $R_{\mathfrak{m}}$. The inclusion of R into the principal ideal domain $L[x]$ allows us to use the techniques of the *conductor square and artinian pairs* (see [18]). First, we fix some notation.

We shall describe a class of torsion-free modules over the ring T that can be either R of $R_{\mathfrak{m}}$. The ring T is included in a principal ideal domain S, where $S = L[x]$ if $T = R$ and $S = L[x]_{\mathfrak{m}}$ if $T = R_{\mathfrak{m}}$. The field of fractions of T coincides with the one of S and it is $Q = L(x)$. We will denote by $\mathfrak{c}$ the conductor of both conductor squares (C1) and (C2).

We denote by $\lambda\colon T \to Q$ and $\lambda'\colon S \to Q$ the corresponding localization maps, and by $\varepsilon\colon T \to S$ the ring inclusion. Then if M_T is a torsion-free T-module, there is a commutative diagram

$$\begin{array}{ccc} M & \xhookrightarrow{\ M\otimes\lambda\ } & M \otimes_T Q \\ & \searrow^{M\otimes\varepsilon} & \uparrow{\scriptstyle M\otimes\lambda'} \\ & & M \otimes_T S. \end{array}$$

Therefore, we can identify M_T with an essential submodule of $Q^{(A)}$ for a suitable set A, and then, we set $(M \otimes \lambda')(M \otimes_T S) = MS$.

In general, $M \otimes \lambda'$ is not an injective map; it is when M_T is projective or, more generally, when M_T is flat.

Lemma 10.3. *Let M_T be a torsion-free module over T. Then*:

(i) $MS = ML$.

(ii) $ML \cap M \supseteq M\mathfrak{c}$.

(iii) *As a K-vector space, $M = V \oplus M\mathfrak{c}$, where V is any complement of $M\mathfrak{c}$ in M. Moreover, as L-vector space, $ML = VL \oplus M\mathfrak{c}$. Setting $W = VL$, it follows that* $\dim_K(V) \geq \dim_L(W)$.

(iv) *As an L-vector space, $ML = W \oplus M\mathfrak{c}$, where W is any complement of $M\mathfrak{c}$ in M. Moreover, $V = W \cap M$ is a K-vector space such that $M = V \oplus M\mathfrak{c}$ and $W = VL$.*

(v) *For any pair V, W chosen as in* (iii) *or* (iv)*, $V \cong M/M\mathfrak{c}$ and $W \cong ML/M\mathfrak{c}$. In addition, there is a pull-back diagram*

$$(*) \qquad \begin{array}{ccc} M & \xhookrightarrow{\ \text{incl.}\ } & ML \\ \big\downarrow\!\!\!\downarrow & & \big\downarrow\!\!\!\downarrow \\ V & \xhookrightarrow{\ \text{incl.}\ } & W. \end{array}$$

Proof. Statement (i) follows because $TS = TL$, cf. Lemma 10.1(i). Statement (ii) is clear.

The ring T fits in an exact sequence of T-modules

$$0 \longrightarrow \mathfrak{c} \longrightarrow T \longrightarrow K \longrightarrow 0$$

shows that $(*)$ is a pull-back diagram. ∎

Definition 10.4. The category $\mathcal{B}$ of modules over the artinian pair $K \hookrightarrow L$ has as objects the inclusions $V \hookrightarrow W$ where V is a K-subspace of the L-vector space W satisfying that $VL = W$.

If $V \hookrightarrow W$ and $V' \hookrightarrow W'$ are two objects of $\mathcal{B}$ a morphism between them consists of one K-linear map $f : V \to V'$ and one L-linear map $g : W \to W'$ making the diagram

$$\begin{array}{ccc} V & \xhookrightarrow{\text{incl.}} & W \\ \big\downarrow f & & \big\downarrow g \\ V' & \xhookrightarrow{\text{incl.}} & W' \end{array}$$

commutative.

Corollary 10.5. *As usual, T denotes the ring $R = K + xL[x]$ or $R_{\mathfrak{m}}$ where $\mathfrak{m} = xL[x]$. Let $\mathcal{A}_T$ be the category of torsion-free modules over T. Let $\mathcal{B}$ be the category of modules over the artinian pair $K \hookrightarrow L$. Then there is a functor F_T that assigns to each object M_T in $\mathcal{A}_T$ the object of $\mathcal{B}$, $M/M\mathfrak{c} \hookrightarrow ML/M\mathfrak{c}$.*

Moreover, $F_R(M) = F_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$ for any object M of $\mathcal{A}_R$.

Proof. Lemma 10.3 (v), and the fact that a morphism between torsion-free modules over T, induce a morphism between the modules over the artinian pair, imply the existence of such a functor.

If M is an object in $\mathcal{A}_R$, then $M/M\mathfrak{c}$ is an R-module and also an $R_{\mathfrak{m}}$-module. Also, $ML/M\mathfrak{c}$ is an $L[X]$ module as well as an $L[x]_{\mathfrak{m}}$-module. So, the second part of the statement is clear. ■

For a general torsion-free module M the pull-back diagram of Lemma 10.3 (v) can be trivial. Take, for example, $M = Q$. Then $M = ML = M\mathfrak{c}$ and $V = W = 0$. So, in the notation of Corollary 10.5, $F_T(Q) = 0$.

To get a better correspondence, we need to restrict the class of torsion-free modules we are interested in. We shall consider the class

$$\mathcal{C}_T = \{M \mid M \text{ is a torsion-free } T\text{-module such that } ML \text{ is a free } S\text{-module}\}$$

Notice that $\mathcal{C}_T$ contains all finitely generated torsion-free modules, and it is closed by arbitrary direct sums and direct summands. In addition, $M = S \in \mathcal{C}_T$.

Assume that ML is a free S-module, then we can fix an S-basis $\mathcal{B} = \{v_i\}_{i \in A}$. Then if W is the L-vector space generated by $\mathcal{B}$, $M\mathfrak{c} = W\mathfrak{c}$ and $ML = W \oplus W\mathfrak{c}$. Therefore, by the modular law and by Lemma 10.3, $M = V \oplus W\mathfrak{c}$ where $V = W \cap M$.

If M_1 and M_2 are two modules in $\mathcal{C}_T$, then, for $i = 1, 2$,

$$M_i L = W_i \oplus W_i\mathfrak{c} \quad \text{and} \quad M_i = V_i \oplus W_i\mathfrak{c}, \quad \text{where } V_i = W_i \cap M_i.$$

Therefore, if (f, g) is a morphism between the artinian pairs $V_1 \hookrightarrow W_1$ and $V_2 \hookrightarrow W_2$ or, equivalently, f is a K-linear map and g is an L-linear map, and there is a commutative diagram

$$(*)\qquad \begin{array}{ccc} V_1 & \xhookrightarrow{\text{incl.}} & W_1 \\ \Big\downarrow f & & \Big\downarrow g \\ V_2 & \xhookrightarrow{\text{incl.}} & W_2, \end{array}$$

then f can be extended to an S-linear map

$$\tilde{g} : M_1 L = W_1 \oplus W_1 \mathfrak{c} \to M_2 L = W_2 \oplus W_1 \mathfrak{c}$$

by setting $\tilde{g}(wx^n) = g(w)x^n$ for any $w \in W_1$ and any $n \geq 0$. Notice that g is an isomorphism if and only if $\tilde{g}$ is an isomorphism because $\widetilde{g^{-1}} = (\tilde{g})^{-1}$.

As a consequence, we have the following.

Corollary 10.6. *The functor F_T described in Corollary 10.5 is full when restricted to the category $\mathcal{C}_T$ and it reflects isomorphisms. Therefore,*

(i) *two objects M_1 and M_2 of $\mathcal{C}_R$ are isomorphic if and only if $(M_1)_{\mathfrak{m}}$ and $(M_2)_{\mathfrak{m}}$ are isomorphic;*

(ii) *M_T is projective if and only if $M \in \mathcal{C}_T$ and the inclusion $M/M\mathfrak{c} \hookrightarrow ML/M\mathfrak{c}$ sends K-basis of $M/M\mathfrak{c}$ to L-basis of $ML/M\mathfrak{c}$ if and only if M_T is a free T-module.*

Proof. The remarks before the statement prove that the functor F_T restricted to the category $\mathcal{C}_T$ is full and reflects isomorphisms.

The statement (i) follows because, by Corollary 10.5, $F_R(M) = F_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$ for any object M of $\mathcal{C}_R$ and, by the first part of the statement, two modules in $\mathcal{C}_R$ are isomorphic if and only if their corresponding artinian pairs are isomorphic if and only if their localizations at $\mathfrak{m}$ are isomorphic.

To prove (ii), notice that since over $R_{\mathfrak{m}}$ all projective modules are free, it follows by (i) that all projective modules over R, since they are modules in $\mathcal{C}_R$, are isomorphic to a free module. It is easy to check that this happens if and only if a K-basis of $V \cong M/M\mathfrak{c}$ is also an L-basis of $W = ML/M\mathfrak{c}$. ∎

Now we summarize some results on finitely generated torsion-free modules over $R = K + xL[x]$. As observed before, these are always in $\mathcal{C}_T$ as well as infinite direct sums of them.

Lemma 10.7. *Let $K \subseteq L$ be a field extension, let R denote the ring $K + xL[x]$, and $\mathfrak{m} = xL[x]$. Let M, N and $\{M_i\}_{i \in I}$ be finitely generated, torsion-free R-modules. Then:*

(i) *Every finitely generated indecomposable torsion-free $R_{\mathfrak{m}}$-module has local endomorphism ring.*

(ii) *$M \cong N$ if and only if $M_{\mathfrak{m}} \cong N_{\mathfrak{m}}$.*

(iii) *M is indecomposable if and only if $M_{\mathfrak{m}}$ is indecomposable.*

(iv) *If M and $\{M_i\}_{i \in I}$ are indecomposable and M is a direct summand of $\bigoplus_{i \in I} M_i$, then $M \cong M_i$ for some $i \in I$.*

Proof. (i) This follows from Lemma 3.5, taking $T = L[x]_{\mathfrak{m}}$. Note that $\mathfrak{m}$ is the conductor, so it is different from zero as required in the hypothesis of the Lemma.

Statement (ii) is included in Corollary 10.6.

(iii) Suppose that $M_{\mathfrak{m}} = A \oplus B$, and that M is indecomposable. Let A' and B' be R-submodules of A and B generated by some finite set of $R_{\mathfrak{m}}$-generators of A and B, respectively. Then $(A' \oplus B')_{\mathfrak{m}} = A \oplus B = M_{\mathfrak{m}}$ and by (ii), we deduce that $A' \oplus B' \cong M$. Therefore, $A' = 0$ or $B' = 0$, that is, $A = 0$ or $B = 0$, and $M_{\mathfrak{m}}$ is indecomposable. The other implication is clear.

(iv) Suppose that M is a direct summand of $\bigoplus_{i \in I} M_i$. Then $M_{\mathfrak{m}}$ is also a direct summand of $(\bigoplus_{i \in I} M_i)_{\mathfrak{m}} \cong \bigoplus_{i \in I} (M_i)_{\mathfrak{m}}$. Since $(M_i)_{\mathfrak{m}}$ has local endomorphism ring, it satisfies the Krull–Schmidt property, that is, $M_{\mathfrak{m}} \cong (M_i)_{\mathfrak{m}}$ for some $i \in I$. By (ii), we deduce that $M \cong M_i$ for some $i \in I$. ∎

Corollary 10.8. *Let $K \subseteq L$ be a field extension, let R denote the ring $K + xL[x]$. Then the class of modules that are direct sums of finitely generated torsion-free R-modules is closed under direct summands.*

Proof. By Lemmas 10.1 and 10.7, the result follows from Proposition 7.2. ∎

Now we want to explicitly construct finitely generated, indecomposable, torsion-free T-modules where T denotes either $R = K + xL[x]$ or $R_{\mathfrak{m}}$. First, we will specify better how these modules and their endomorphism rings can look like.

Recall that if $T = R = K + xL[x]$, then $S = L[x]$, and if $T = R_{\mathfrak{m}}$, then $S = L[x]_{\mathfrak{m}}$.

Remark 10.9. Let M_T be a finitely generated torsion-free T-module of rank n. As ML is a finitely generated free S-module, we may assume that M is a T-submodule of S^n such that $ML = S^n$, in particular, $(xS)^n \subseteq M$. Therefore, $M_T = V + (xS)^n$, where V is a K-subspace of L^n satisfying $VL = L^n$. Notice that, this gives us a very explicit construction of a T-module $M \in \mathcal{C}$ such that $F(M)$ (cf. Corollary 10.5) has as an image the module over the artinian pair $V \hookrightarrow W$.

Now we can also identify the endomorphism ring of M_T with a subring of $M_n(Q)$, in fact $\mathrm{End}_T(M_T) = \{A \in M_n(Q) \mid AM \subseteq M\}$. Since $(xS)^n \subseteq M$, if $A \in M_n(Q)$ represents an endomorphism of M_T, then $A \in M_n(S)$, i.e., $\mathrm{End}_T(M_T) = \{A \in M_n(S) \mid AM \subseteq M\}$. Every matrix $A \in M_n(S)$ can be uniquely decomposed as the sum of a matrix $B \in M_n(L)$ and a matrix $C \in M_n(xS)$. Since $M_n(xS) \subseteq \mathrm{End}_T(M)$, $A \in \mathrm{End}_T(M)$ if and only if $BV \subseteq V$. This is to say that if $A \in \mathrm{End}_T(M_T)$, then $F(A) = B$.

Lemma 10.10. *Let $K \subsetneq L$ be a field extension, let R denote the ring $K + xL[x]$, and let $\mathfrak{m} = xL[x]$. Let M be a finitely generated torsion-free right $R_{\mathfrak{m}}$-module of rank n. Then $M_n(xL[x]_{\mathfrak{m}}) \subseteq J(\mathrm{End}_{R_{\mathfrak{m}}}(M))$.*

Proof. Note that $J(R_{\mathfrak{m}}) = \mathfrak{m}R_{\mathfrak{m}}$. Then, for every $A \in M_n(xL[x]_{\mathfrak{m}})$, we have

$$M = AM + (I - A)M \subseteq J(R_{\mathfrak{m}})M + (I - A)M.$$

By Nakayama's lemma, $M = (I - A)M$ and, by Lemma 2.1 (i), $I - A$ is bijective. Hence $A \in J(\mathrm{End}_{R_{\mathfrak{m}}}(M))$. ∎

The following is a modification of Construction 3.13 in [18] to build indecomposable modules over artinian pairs. Using Remark 10.9, this immediately yields the existence of indecomposable finitely generated T-modules of arbitrary finite rank $n \geq 2$.

Construction 10.11. Let $n \geq 2$ be a fixed positive integer, and suppose we have chosen $\alpha, \beta \in L$ with $\{1, \alpha, \beta, \alpha^2, \alpha\beta, \beta^2\}$ linearly independent over K. Let I be the identity $n \times n$ matrix and H be the nilpotent $n \times n$ matrix with 1 below the diagonal and 0 elsewhere. For $t \in K$, we consider the $n \times 2n$ matrix,

$$\Psi_t := [I \mid \alpha I + \beta(tI + H)] = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & \alpha + t\beta & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \beta & \alpha + t\beta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & \alpha + t\beta \end{array}\right].$$

Let V_t be the K-subspace of L^n spanned by the columns of Ψ_t. Let $A \in \operatorname{Hom}_K(V_t, V_u)$, where

$$\operatorname{Hom}_K(V_t, V_u) = \{A \in M_n(L) \mid AV_t \subseteq V_u\}.$$

The condition $AV_t \subseteq V_u$ implies that there is a $2n \times 2n$ matrix $\theta \in M_{2n}(K)$ such that $A\Psi_t = \Psi_u \theta$. Write $\theta = \left[\begin{smallmatrix} C & D \\ P & Q \end{smallmatrix}\right]$, where $C, D, P, Q \in M_n(K)$. Then, using the condition $A\Psi_t = \Psi_u \theta$, we have the following two equations:

$$\begin{cases} A = C + \alpha P + \beta(uI + H)P, \\ \alpha A + \beta A(tI + H) = D + \alpha Q + \beta(uI + H)Q. \end{cases}$$

Substituting the first equation into the second and combining terms, we get the following:

$$\begin{aligned} -D + \alpha(C - Q) + \beta(tC - uQ + CH - HQ) + \alpha^2 P \\ + \alpha\beta(tP + uP + HP + PH) + \beta^2(tuP + HPH + tHP + uPH) = 0. \end{aligned}$$

From the linear independence of $\{1, \alpha, \beta, \alpha^2, \alpha\beta, \beta^2\}$, we have

$$D = P = 0, \quad A = C = Q \quad \text{and} \quad (t - u)A + AH = HA.$$

In particular, we deduce that $A \in M_n(K)$, and if A is an isomorphism and $t \neq u$, then the third equation above gives a contradiction since the left side is invertible and the right side is not. Thus, if $V_t \cong V_u$, then $t = u$. To see that V_t is indecomposable, we take $u = t$ and suppose that A is idempotent. But $AH = HA$, and it follows that A is in $K[H]$, which is a local ring. Therefore $A = 0$ or I, as desired.

By Remark 10.9, $M_t = V_t + (xS)^n$, where V_t is the K-subspace of L^n constructed above, is an indecomposable T-module of rank n (as usual, $T = R = K + xL[x]$ or $T = R_{\mathfrak{m}}$). Notice that, in view of Corollary 10.5, there are, at least, $|K|$ different isomorphism classes of such modules.

Acknowledgements. We are very grateful to the anonymous referees for their careful reading of the paper and for a number of interesting observations that helped to improve its readability. We would also like to thank Carmelo Antonio Finocchiaro and Paolo Zanardo for suggesting the family of examples in Section 10.

Funding. The research of the first author was supported by the pre-doctoral grant 2021FI-B00913 of the Generalitat de Catalunya.

The second author was supported by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M). The first and second authors were partially supported by the projects MIMECO PID2020-113047GB-I00 and PID2023-147110NB-I00 financed by the Spanish Government, and by "Laboratori d'Interaccions entre Geometria, Àlgebra i Topologia" (LIGAT) with reference number 2021 SGR 01015 financed by the Generalitat de Catalunya.

The third author was supported by Czech Science Foundation grant GAČR 23-05148S

References

[1] Anderson, F. W. and Fuller, K. R.: *Rings and categories of modules*. Second edition. Grad. Texts in Math. 13, Springer, New York, 1992. Zbl 0765.16001 MR 1245487

[2] Bass, H.: Torsion free and projective modules. *Trans. Amer. Math. Soc.* **102** (1962), no. 2, 319–327. Zbl 0103.02304 MR 0140542

[3] Bass, H.: Big projective modules are free. *Illinois J. Math.* **7** (1963), no. 1, 24–31. Zbl 0115.26003 MR 0143789

[4] Bazzoni, S. and Salce, L.: Strongly flat covers. *J. London Math. Soc. (2)* **66** (2002), no. 2, 276–294. Zbl 1009.13003 MR 1920402

[5] Butler, M. C. R., Campbell, J. M. and Kovács, L. G.: On infinite rank integral representations of groups and orders of finite lattice type. *Arch. Math. (Basel)* **83** (2004), no. 4, 297–308. Zbl 1065.16013 MR 2096802

[6] Facchini, A.: *Module theory. Endomorphism rings and direct sum decompositions in some classes of modules*. Progr. Math. 167, Birkhäuser, Basel, 1998. Zbl 0930.16001 MR 1634015

[7] Facchini, A. and Herbera, D.: Local morphisms and modules with a semilocal endomorphism ring. *Algebr. Represent. Theory* **9** (2006), no. 4, 403–422. Zbl 1130.16014 MR 2250654

[8] Fuchs, L. and Salce, L.: *Modules over non-Noetherian domains*. Math. Surveys Monogr. 84, American Mathematical Society, Providence, RI, 2001. Zbl 0973.13001 MR 1794715

[9] Goodearl, K. R.: Surjective endomorphisms of finitely generated modules. *Comm. Algebra* **15** (1987), no. 3, 589–609. Zbl 0612.16020 MR 0882800

[10] Guralnick, R. M. and Levy, L. S.: Cancellation and direct summands in dimension 1. *J. Algebra* **142** (1991), no. 2, 310–347. Zbl 0745.16004 MR 1127066

[11] Herbera, D. and Příhoda, P.: Reconstructing projective modules from its trace ideal. *J. Algebra* **416** (2014), 25–57. Zbl 1333.16005 MR 3232793

[12] Herbera, D., Příhoda, P. and Wiegand, R.: Big pure projective modules over commutative noetherian rings: Comparison with the completion. *Forum Math.* **37** (2025), no. 4, 1103–1146. Zbl 08059674 MR 4915561

[13] Herbera, D. and Shamsuddin, A.: Modules with semi-local endomorphism ring. *Proc. Amer. Math. Soc.* **123** (1995), no. 12, 3593–3600. Zbl 0843.16017 MR 1277114

[14] Hinohara, Y.: Supplement to "Projective modules over weakly noetherian rings". *J. Math. Soc. Japan* **15** (1963), 474–475. Zbl 0127.25903 MR 0184976

[15] Jaffard, P.: Théorie arithmétique des anneaux du type de Dedekind. *Bull. Soc. Math. France* **80** (1952), 61–100. Zbl 0049.02202 MR 0052396

[16] Kaplansky, I.: Projective modules. *Ann. of Math. (2)* **68** (1958), no. 2, 372–377. Zbl 0083.25802 MR 0100017

[17] Lam, T. Y.: *A first course in noncommutative rings*. Second edition. Grad. Texts in Math. 131, Springer, New York, 2001. Zbl 0980.16001 MR 1838439

[18] Leuschke, G. J. and Wiegand, R.: *Cohen–Macaulay representations*. Math. Surveys Monogr. 181, American Mathematical Society, Providence, RI, 2012. Zbl 1252.13001 MR 2919145

[19] Levy, L. S. and Odenthal, C. J.: Package deal theorems and splitting orders in dimension 1. *Trans. Amer. Math. Soc.* **348** (1996), no. 9, 3457–3503. Zbl 0858.16017 MR 1351493

[20] Levy, L. S. and Wiegand, R.: Dedekind-like behavior of rings with 2-generated ideals. *J. Pure Appl. Algebra* **37** (1985), no. 1, 41–58. Zbl 0616.13008 MR 0794792

[21] Matlis, E.: Cotorsion modules. *Mem. Amer. Math. Soc.* **49** (1964), 66 pp. Zbl 0135.07801 MR 0178025

[22] Matlis, E.: *Torsion-free modules*. Chicago Lect. Math., University of Chicago Press, Chicago-London, 1972. Zbl 0298.13001 MR 0344237

[23] Matsumura, H.: *Commutative ring theory*. Cambridge Stud. Adv. Math. 8, Cambridge University Press, Cambridge, 1986. Zbl 0603.13001 MR 0879273

[24] Olberding, B.: Stability, duality, 2-generated ideals and a canonical decomposition of modules. *Rend. Sem. Mat. Univ. Padova* **106** (2001), 261–290. Zbl 1072.13506 MR 1876223

[25] Olberding, B.: Characterizations and constructions of h-local domains. In *Models, modules and abelian groups*, pp. 385–406. Walter de Gruyter, Berlin, 2008. Zbl 1182.13014 MR 2513254

[26] Příhoda, P.: Fair-sized projective modules. *Rend. Semin. Mat. Univ. Padova* **123** (2010), 141–167. Zbl 1214.16003 MR 2683295

[27] Příhoda, P.: Classifying modules in Add of a class of modules with semilocal endomorphism rings. In *Advances in rings, modules and factorizations*, pp. 269–282. Springer Proc. Math. Stat. 321, Springer, Cham, 2020. Zbl 1440.16007 MR 4113969

[28] Příhoda, P.: Generalized lattices over one-dimensional noetherian domains. *J. Commut. Algebra* **14** (2022), no. 3, 443–453. Zbl 1502.13031 MR 4493000

[29] Reiner, I.: *Maximal orders*. London Mathematical Society Monographs 5, Academic Press Harcourt Brace Jovanovich, London-New York, 1975. Zbl 0305.16001 MR 0393100

[30] Rump, W.: Large lattices over orders. *Proc. London Math. Soc. (3)* **91** (2005), no. 1, 105–128. Zbl 1087.16011 MR 2149531

[31] Rush, D. E.: Rings with two-generated ideals. *J. Pure Appl. Algebra* **73** (1991), no. 3, 257–275. Zbl 0734.13010 MR 1124788

[32] Vasconcelos, W. V.: On finitely generated flat modules. *Trans. Amer. Math. Soc.* **138** (1969), 505–512. Zbl 0175.03603 MR 0238839

[33] Warfield, R. B., Jr.: Cancellation of modules and groups and stable range of endomorphism rings. *Pacific J. Math.* **91** (1980), no. 2, 457–485. Zbl 0484.16017 MR 0615693

[34] Whitehead, J. M.: Projective modules and their trace ideals. *Comm. Algebra* **8** (1980), no. 19, 1873–1901. Zbl 0447.16018 MR 0588450

[35] Wiegand, R. and Wiegand, S.: Bounds for one-dimensional rings of finite Cohen–Macaulay type. *J. Pure Appl. Algebra* **93** (1994), no. 3, 311–342. Zbl 0813.13013 MR 1275969

Received July 4, 2024; revised April 9, 2025.

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