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A lower bound for the first eigenvalue of a minimal hypersurface in the sphere

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Abstract. Let Σ be a closed embedded minimal hypersurface in the unit sphere $\mathbb{S}^{m+1}$ and let $\Lambda = \max_\Sigma |A|$ be the norm of its second fundamental form. In this work, we prove that the first eigenvalue of the Laplacian of Σ satisfies

$$\lambda_1(\Sigma) > \frac{m}{2} + \frac{m(m+1)}{32\,(12\Lambda + m + 11)^2 + 8},$$

and $\lambda_1(\Sigma) = m$ when $\Lambda \le \sqrt{m}$. In particular, this estimate improves the one obtained recently in Duncan–Sire–Spruck (2024). The proof of our main result is based on a Rayleigh quotient estimate for a harmonic extension of an eigenfunction of the Laplacian of Σ in the spirit of Choi and Wang (1983).

1. Introduction

Let $\mathbb{S}^{m+1}$ denote the unit sphere in $\mathbb{R}^{m+2}$, and let Σ be a closed embedded hypersurface in $\mathbb{S}^{m+1}$. The eigenvalues of the Laplacian operator Δ on Σ form a discrete set of non-negative real numbers. We denote by $\lambda_1(\Sigma)$ the first nonzero eigenvalue. It is well known that Σ is minimal if and only if all coordinate functions in $\mathbb{R}^{m+2}$ restricted to Σ are eigenfunctions corresponding to eigenvalues m. This implies that $\lambda_1(\Sigma) \le m$.

On the other hand, it is interesting and important to find the sharp lower bound of λ_1 for minimal hypersurfaces in $\mathbb{S}^{m+1}$. Yau [1] conjectured that $\lambda_1(\Sigma) = m$. Choi and Wang [5] proved that $\lambda_1(\Sigma) \ge m/2$. This estimate was later refined in [2] by Barros-Bessa who gave the lower bound

$$\lambda_1(\Sigma) \ge \frac{m}{2} + \frac{m}{2}\,\rho(u), \tag{1.1}$$

where ρ is an explicit function and u is a solution to (2.3). Much progress has been made in proving Yau's conjecture after Choi–Wang's paper (see, for instance, [2,3,6,10,11,13]). Despite an extensive literature on the study of $\lambda_1(\Sigma)$ under additional assumptions on Σ, the estimate (1.1) has remained the strongest *explicit* lower bound that is known to hold

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for a general embedded minimal hypersurface in $\mathbb{S}^{n+1}$. The new estimate that we obtain in this work depends on the geometry of Σ, as we explain in detail next.

Given $x \in \Sigma$, let

$$|A|(x) := \Big(\sum_{i=1}^{m} k_i^2(x)\Big)^{1/2},$$

where $k_1(x), \dots, k_m(x)$ are the principal curvatures of Σ in x with respect to to the unit normal $\nu(x)$. We call $|A|$ *the norm of the second fundamental form with respect to* ν, A_ν, and define

$$\Lambda := \max_{\Sigma} |A|.$$

It is known (see [9]) that in the case $\Lambda \le \sqrt{m}$, then $\lambda_1(\Sigma) = m$. Therefore, our contribution concerns only the case $\Lambda > \sqrt{m}$. Precisely, we prove the following.

Theorem 1.1. *Let Σ be a closed embedded minimal hypersurface in the unit sphere $\mathbb{S}^{m+1}$ and let $\Lambda = \max_\Sigma |A|$ be the norm of its second fundamental form. Then, the first Laplacian eigenvalue of Σ satisfies*

$$\lambda_1(\Sigma) > \frac{m}{2} + \frac{m(m+1)}{32\,(12\Lambda + m + 11)^2 + 8}, \tag{1.2}$$

and $\lambda_1(\Sigma) = m$, when $\Lambda \le \sqrt{m}$.

Remark 1.2. A recent improvement to (1.1) is given by Duncan–Sire–Spruck in [6], where they proved that

$$\lambda_1 \ge \frac{m}{2} + \frac{a(m)}{\Lambda^6 + b(m)},$$

for specific functions $a(m)$ and $b(m)$ (see (3.23)). By a simple comparison of the order of growth, it is easy to see that our estimate is larger when Λ is big enough. In fact, a computation at the end of this paper shows that it is larger for every m and Λ.

Remark 1.3. Our estimate depends on the norm of second fundamental form and is not sharp. It would be interesting to find a lower bound which is bigger than $m/2$ and depends only on m.

We can combine (1.2) with the Yang–Yau inequality [12] to obtain an area bound for embedded minimal surfaces in $\mathbb{S}^3$ in terms of their genus. This plays a crucial role in the compactness theory of Choi–Schoen in [4]. They find a non-explicit constant $C(\chi)$ which is an upper bound for the norm of the second fundamental form of any compact minimal embedded minimal surface in $\mathbb{S}^3$ with Euler number χ.

On the other hand, Yau's conjecture is true for embedded minimal surfaces in S^3 which are invariant under a finite group of reflections (see [3]). And Zhao [13] proved that there is a lower bound depending on the genus. We also have the following corollary from Theorem 1.1.

Corollary 1.4. *Let $C(\chi)$ be the constant in Choi–Schoen's theorem. Then the first nonzero eigenvalue of the Laplacian of a compact embedded minimal surface Σ in $\mathbb{S}^3$ with Euler number χ satisfies*

$$\lambda_1(\Sigma) > 1 + \frac{3}{16(12C(\chi) + 13)^2 + 4}.$$

The remainder of the paper is divided into two sections. In Section 2, we recall Reilly's formula and reformulate a result in [2]. We also give a lower bound for $\lambda_1(\Sigma)$ in terms of the Rayleigh quotient of its harmonic extension. That is,

$$\lambda_1(\Sigma) \geq \frac{m}{1 + \sqrt{1 - (m+1)/Q}},$$

where

$$Q := \Big(\int_\Omega |\overline{\nabla} u|^2 \, dV \Big) \Big(\int_\Omega u^2 \, dV \Big)^{-1}$$

and u is a solution to

$$\begin{cases} \overline{\Delta} u = 0 & \text{in } \Omega, \\ u = \varphi & \text{in } \Sigma, \end{cases}$$

where the bars in the expressions above refer to operations in the ambient sphere. Here, φ is an eigenfunction of the Laplacian Δ associated to the eigenvalue $\lambda_1(\Sigma)$ and Ω is a component of $\mathbb{S}^{m+1} - \Sigma$ which is chosen appropriately. We call *u the harmonic extension of φ to Ω*. Then we prove Theorem 1.1 assuming the validity of an appropriate estimate for Q.

In Section 3, we make a quick review of the normal exponential map and we prove the estimate

$$\int_\Omega u^2 \, dV \geq C_2(m, \Lambda) \int_\Omega |\overline{\nabla} u|^2 \, dV, \tag{1.3}$$

which may be considered as an inverse Poincaré type inequality, i.e., C_2 is an upper estimate for Q. We finish the paper by comparing our estimate with the one in [6].

It is important to note that, as in [6], we also use Reilly's formula and an upper bound of the mean curvature of the parallel surfaces to Σ. However, we provide an upper bound of $|\overline{\nabla} u|^2_{L^2(\Omega)}$ that depends only on the geometry of Σ, via an elementary result on harmonic extensions.

We need to point out that our techniques can be improved and generalized in order to obtain an estimate for the first eigenvalue of a minimal surface embedded in an ambient of bounded sectional curvature. However, the sharp bound can only be achieved by proving $Q = m + 1$. Further work will be part of the PhD thesis of the second author. In fact, this work was in preparation before we had access to [6].

2. A first eigenvalue estimate via Rayleigh quotient

In this section. we review Reilly's formula and give a lower bound for the nonzero first eigenvalue $\lambda_1(\Sigma)$ in terms of the Rayleigh quotient of the harmonic extension of the corresponding eigenfunction. As a consequence, we will prove our main result, Theorem 1.1, assuming the inequality (1.3), which will be explicitly stated in Theorem 3.9.

From now on, Σ will denote a closed embedded hypersurface of $\mathbb{S}^{m+1}$. It follows that Σ divides the sphere into two components Ω_1 and Ω_2, where $\partial\Omega_1 = \partial\Omega_2 = \Sigma$ (see [5]). Set ν as the unit normal of Σ pointing outward to Ω_1 (and $-\nu$ as the unit normal of Σ and pointing outward to Ω_2) and let A_ν be the second fundamental form of Σ with respect to ν.

Let $\varphi \in C^\infty(\Sigma)$. We can assume, without loss of generality, that it satisfies the property

$$\int_\Sigma \langle A_\nu \nabla\varphi, \nabla\varphi\rangle \, dS \geq 0$$

and denote $\Omega := \Omega_1$. Otherwise, we can choose $\Omega = \Omega_2$. Let us denote all the functions of class C^2 that extend the function φ over $\overline{\Omega}$ as $C^2_\varphi(\overline{\Omega})$. The following equation is known as Reilly's formula (see [8]).

Lemma 2.1. *For all $u \in C^2_\varphi(\overline{\Omega})$, we have*

$$\begin{aligned}\int_\Omega \big[(\overline{\Delta} u)^2 - |\overline{\nabla}^2 u|^2 - \mathrm{Ric}_{\mathbb{S}^{m+1}(1)}(\overline{\nabla} u, \overline{\nabla} u)\big]\, dV \\ = \int_\Sigma \Big[\langle A_\nu \nabla\varphi, \nabla\varphi\rangle + 2\frac{\partial u}{\partial \nu}\Delta\varphi + H_\Sigma (u_\nu)^2\Big]\, dS.\end{aligned} \tag{2.1}$$

where $\overline{\Delta} u$, $\overline{\nabla} u$ and $\overline{\nabla}^2 u$ denote the Laplacian, gradient, and Hessian of u in Ω, respectively, while $\Delta\varphi$ and $\nabla\varphi$ denote the Laplacian and the gradient of φ in Σ with respect to the induced metric of Ω. On the other hand, $\partial u/\partial\nu := \langle\nabla u, \nu\rangle$ denotes the outward normal derivative of u in Σ, $\mathrm{Ric}_{\mathbb{S}^{m+1}}$ is the Ricci tensor of $\mathbb{S}^{m+1}$, and $H_\Sigma := \mathrm{trace}(A_\nu)$ is the mean curvature of Σ.

The following corollary follows from (2.1) using

$$|\overline{\nabla}^2 u|^2 \geq \frac{1}{m+1}(\overline{\Delta} u)^2.$$

Corollary 2.2. *Let Σ be a closed embedded minimal hypersurface in the unit sphere $\mathbb{S}^{m+1}$. For any $v \in C^2(\overline{\Omega})$, we assume that Ω is chosen so that*

$$\int_{\partial\Omega} \langle A_\nu(\nabla(v\,|_{\partial\Omega})), \nabla(v\,|_{\partial\Omega})\rangle \, dS \geq 0.$$

Then

$$\int_\Omega \Big[\frac{m}{m+1}(\overline{\Delta} v)^2 - m|\overline{\nabla} v|^2\Big]\, dV - 2\int_{\partial\Omega} \frac{\partial v}{\partial \nu}\Delta(v\,|_{\partial\Omega})\, dS \geq 0. \tag{2.2}$$

The following result is proven by Barros and Bessa in [2]. We include here a simpler proof.

Lemma 2.3. *Let Σ be a closed embedded minimal hypersurface in the unit sphere $\mathbb{S}^{m+1}$. Assume that φ is a eigenfunction of Δ on Σ corresponding to the eigenvalue $\lambda_1(\Sigma)$ and that u is its harmonic extension to Ω, i.e.,*

$$\begin{cases} \overline{\Delta} u = 0 & \text{in } \Omega, \\ u = \varphi & \text{in } \Sigma. \end{cases} \tag{2.3}$$

Then for all $t \in \mathbb{R}$,

$$(2\lambda_1(\Sigma) - m)Qt^2 + 2\lambda_1(\Sigma)t + \frac{m}{m+1} \geq 0, \tag{2.4}$$

where Q is defined by

$$Q := \frac{\int_\Omega |\overline{\nabla} u|^2\, dV}{\int_\Omega u^2\, dV}. \tag{2.5}$$

Proof. For $t \neq 0$, let g be the solution of the problem

$$\begin{cases} \overline{\Delta} g = u & \text{in } \Omega, \\ g = t\varphi & \text{in } \Sigma, \end{cases}$$

where u is a solution to (2.3). Then, from (2.2) applied to g, we have

$$\int_\Omega \Big(\frac{m}{m+1} u^2 - m|\overline{\nabla} g|^2\Big)\, dV + 2t\lambda_1(\Sigma) \int_{\partial\Omega} \varphi \frac{\partial g}{\partial \nu}\, dS \geq 0. \tag{2.6}$$

On the other hand, by (2.3) and Stokes' formula,

$$\int_\Omega \langle \overline{\nabla} g, \overline{\nabla} u\rangle\, dV = -\int_\Omega g \overline{\Delta} u\, dV + \int_\Sigma g \frac{\partial u}{\partial \nu}\, dS = t\int_\Sigma \varphi \frac{\partial u}{\partial \nu}\, dS = t\int_\Omega |\overline{\nabla} u|^2\, dV. \tag{2.7}$$

Hence, by (2.7),

$$\begin{aligned} 0 \leq \int_\Omega |\overline{\nabla} g - t\overline{\nabla} u|^2\, dV &= \int_\Omega \big(|\overline{\nabla} g|^2 - 2t\langle \overline{\nabla} g, \overline{\nabla} u\rangle + t^2 |\overline{\nabla} u|^2\big)\, dV \\ &= \int_\Omega |\overline{\nabla} g|^2\, dV - t^2 \int_\Omega |\overline{\nabla} u|^2\, dV. \end{aligned}$$

Therefore,

$$\int_\Omega |\overline{\nabla} g|^2\, dV \geq t^2 \int_\Omega |\overline{\nabla} u|^2\, dV. \tag{2.8}$$

Similarly,

$$\int_\Sigma \varphi \frac{\partial g}{\partial \nu}\, dS = \int_\Omega \langle \overline{\nabla} u, \overline{\nabla} g\rangle\, dV + \int_\Omega u \overline{\Delta} g\, dV = t\int_\Omega |\overline{\nabla} u|^2\, dV + \int_\Omega u^2\, dV. \tag{2.9}$$

From (2.6) and (2.9), we have

$$\int_\Omega \Big(\frac{m}{m+1} u^2 - m|\overline{\nabla} g|^2\Big)\, dV + 2t^2\lambda_1(\Sigma) \int_\Omega |\overline{\nabla} u|^2\, dV + 2t\lambda_1(\Sigma) \int_\Omega u^2\, dV \geq 0,$$

and so from (2.8),

$$(2\lambda_1(\Sigma) - m)t^2 \int_\Omega |\overline{\nabla} u|^2\, dV + 2\lambda_1(\Sigma) t \int_\Omega u^2\, dV + \frac{m}{m+1} \int_\Omega u^2\, dV \geq 0.$$

Then (2.4) follows by dividing this last inequality by $\int_\Omega u^2\, dV$ and using the definition of Q in (2.5). ∎

Next we obtain a first estimate for $\lambda_1(\Sigma)$ which is a corollary of Barros–Bessa's theorem.

Theorem 2.4. *Let Σ be a closed embedded minimal hypersurface in the unit sphere $\mathbb{S}^{m+1}$. Then*

$$\lambda_1(\Sigma) \geq \frac{m}{1+\sqrt{1-(m+1)/Q}}. \tag{2.10}$$

Proof. First note that, choosing any t in (2.4) such that $2\lambda_1(\Sigma)t + m/(m+1) < 0$, then we trivially have $\lambda_1(\Sigma) > m/2$. On the other hand, by choosing

$$t = \frac{-\lambda_1(\Sigma)}{(m+1)(2\lambda_1(\Sigma)-m)}$$

in (2.4), we have that

$$\begin{aligned}
\frac{Q\lambda_1^2(\Sigma)}{(m+1)^2(2\lambda_1(\Sigma)-m)} &\geq \frac{2\lambda_1^2(\Sigma)}{(m+1)(2\lambda_1(\Sigma)-m)} - \frac{m}{m+1} \\
&= \frac{1}{(m+1)(2\lambda_1(\Sigma)-m)}\left(2\lambda_1^2(\Sigma) - m(2\lambda_1(\Sigma)-m)\right) \\
&= \frac{1}{(m+1)(2\lambda_1(\Sigma)-m)}\left(\lambda_1^2(\Sigma) + (\lambda_1(\Sigma)-m)^2\right) \\
&\geq \frac{\lambda_1^2(\Sigma)}{(m+1)(2\lambda_1(\Sigma)-m)}.
\end{aligned}$$

Hence $Q \geq m+1$ (see also [2]) and so (2.10) is well defined. Note that if $Q = m+1$ in the inequality above, then $\lambda_1(\Sigma) = m$, so we can assume $Q > m+1$.

Again from (2.4), we have

$$2\lambda_1(\Sigma)(Qt^2+t) \geq mQt^2 - \frac{m}{m+1}.$$

Then

$$\lambda_1(\Sigma) \geq \frac{m}{2}\max_{t(Qt+1)>0}\beta(t), \tag{2.11}$$

where

$$\beta(t) := 1 - \frac{1}{Qt+1} - \frac{1}{(m+1)(Qt^2+t)}.$$

Note that

$$\beta'(t) = \frac{Q(m+1)t^2 + 2Qt + 1}{(m+1)t^2(Qt+1)^2}.$$

At the points where $t(Qt+1) > 0$, we have that $\beta'(t) = 0$ if and only if

$$t = \frac{-2Q - \sqrt{4Q^2 - 4Q(m+1)}}{2Q(m+1)} = \frac{-1-\sqrt{1-(m+1)/Q}}{m+1} =: t_0.$$

Note that, in particular, β has no critical points in the interval $(0, +\infty)$. It follows that

$$\begin{aligned}\max_{t(Qt+1)>0} \beta(t) = \beta(t_0) &= \frac{Q(m+1)t_0^2 - 1}{(m+1)t_0(Qt_0+1)} = \frac{-2}{t_0(m+1)} \\ &= \frac{2}{1+\sqrt{1-(m+1)/Q}}.\end{aligned} \tag{2.12}$$

The above equality is valid from the fact that $\beta'(t_0) = 0$, i.e., $Q(m+1)t_0^2 + 2Qt_0 + 1 = 0$.

Therefore, from (2.11) and (2.12) we have (2.10). ∎

Next we prove Theorem 1.1 assuming Theorem 3.9.

Proof of Theorem 1.1. It is well known that when $\Lambda \le \sqrt{m}$, Σ is either a great sphere $\mathbb{S}^n$ or a Clifford torus and so $\lambda_1(\Sigma) = m$. Therefore, we can now assume that $\Lambda \ge \sqrt{m}$. Let $\varphi \in C^\infty(\Sigma)$ be the eigenfunction corresponding to the first nonzero eigenvalue $\lambda_1(\Sigma)$ and let $u \in C^2_\varphi(\overline{\Omega})$ be its harmonic extension to Ω as before. It follows from Theorem 3.9 that $Q < 4(12\Lambda + m + 11)^2 + 1$. Therefore we have from (2.10) that

$$\lambda_1(\Sigma) \ge \frac{m}{1+\sqrt{1-(m+1)Q^{-1}}} = \frac{m}{2} + m\Big(\frac{1}{1+\sqrt{1-(m+1)Q^{-1}}} - \frac{1}{2}\Big). \tag{2.13}$$

On the other hand, since for all $0 \le x < 1$,

$$\frac{x}{8} < \frac{1}{1+\sqrt{1-x}} - \frac{1}{2},$$

we can consider $x = (m+1)Q^{-1}$ in (2.13) and deduce from Theorem 3.9 that

$$\lambda_1(\Sigma) > \frac{m}{2} + \frac{m(m+1)}{8} Q^{-1} > \frac{m}{2} + \frac{m(m+1)}{32(12\Lambda + m + 11)^2 + 8}.$$

The proof is then complete. ∎

3. Gradient estimate via an inverse Poincaré-type inequality

The aim of this section is to prove an inverse Poincaré-type inequality in Theorem 3.9. To do that, we recall first some preliminary results concerning the normal exponential map.

In what follows, $N\Sigma := \{(x, v) : x \in \Sigma, v \in T_x^\perp \Sigma\}$ will denote the normal bundle of Σ, $U\Sigma := \{(x, v) \in N\Sigma : |v| = 1\}$ will denote the normal unit bundle of Σ, and $\exp^\perp : N\Sigma \to \mathbb{S}^{m+1}$ defined by $\exp^\perp(x, v) := \exp_x(v)$ will denote the normal exponential map on Σ. This map is well defined in Σ since Σ is embedded with compact closure on the sphere $\mathbb{S}^{m+1}$ (see, for instance, [7]). Let $\theta_\Sigma : N\Sigma \to \mathbb{R}$ denote the Jacobian determinant of the normal exponential map $\exp^\perp$. On the other hand, let $\Phi_t : \Sigma \to \mathbb{S}^{m+1}$ be defined by $\Phi_t(x) := \exp^\perp(x, t\nu(x))$ and let

$$\Sigma_t := \Phi_t(\Sigma) = \{\exp^\perp(x, t\nu(x)) : x \in \Sigma\}.$$

By Proposition 2.1 in [6], if we define

$$k_{\max} := \max_{\Sigma, i} |k_i|$$

and

$$T_\Sigma := \arctan(k_{\max}^{-1}), \tag{3.1}$$

we have

$$T_\Sigma \leq \sup\{t > 0 : \Phi_t \colon \Sigma \to \Sigma_t \text{ is a diffeomorphism}\} =: t_*.$$

Similarly, we have

$$T_\Sigma \leq -\inf\{t < 0 : \Phi_t \colon \Sigma \to \Sigma_t \text{ is a diffeomorphism}\} := t^*.$$

Defining $\operatorname{minfoc}(\Sigma) := \min\{t_*, t^*\}$, we have that

$$T_\Sigma \leq \operatorname{minfoc}(\Sigma).$$

The following lemma corresponds to Lemma 10.9 in [7].

Lemma 3.1. *For all* $0 \leq t < \operatorname{minfoc}(\Sigma)$,

$$\frac{d}{dt} \ln \theta_\Sigma(x, t\nu(x)) = -H_{\Sigma_t},$$

where H_{Σ_t} is the mean curvature of Σ_t in $\Phi_t(x)$.

On the other hand, if Σ is minimal and $0 \leq t < T_\Sigma$, we have (see [6])

$$H_{\Sigma_t} = \sum_{i=1}^{m} \frac{\tan t (k_i^2 + 1)}{1 - k_i \tan t} \geq 0,$$

where k_i are the principal curvatures of Σ. Then, from Lemma 3.1 we have the following:

$$\theta(t, x) := \theta_\Sigma(x, t\nu(x)) \leq \theta_\Sigma(x, 0) = 1. \tag{3.2}$$

Lemma 3.2. *For each* $0 \leq t < T_\Sigma$ *and* $x \in \Sigma$, *we have*

$$\cos t - k_i(x) \sin t = \frac{\sin(\theta_i(x) - t)}{\sin \theta_i(x)} \geq \frac{\sin(T_\Sigma - t)}{\sin T_\Sigma} > 0,$$

where $\cot \theta_i(x) = k_i(x)$.

Proof. By the definition of T_Σ in (3.1), we have that $k_i(x) \leq\mid k_i(x) \mid\leq k_{\max} = \cot T_\Sigma$. Therefore,

$$\cos t - k_i(x) \sin t \geq \cos t - \cot T_\Sigma \sin t = \frac{\sin(T_\Sigma - t)}{\sin T_\Sigma} > 0,$$

where we have used that $0 < T_\Sigma - t \leq T_\Sigma < \pi/2$. ∎

Given $\varphi \in C^\infty(\Sigma)$, we define

$$\widetilde{\varphi}(\Phi_t(x)) := \varphi(x).$$

The function $\widetilde{\varphi}$ is called *the normal extension of* φ, and it is well defined in the set

$$\{y \in \mathbb{S}^{m+1} : y = \Phi_t(x), x \in \Sigma, |t| < T_\Sigma\}.$$

Lemma 3.3. *For each* $0 \le t < T_\Sigma$ *and* $x \in \Sigma$, *we have*

$$|\nabla^T \widetilde{\varphi}|^2(\Phi_t(x)) \le \frac{\sin^2 T_\Sigma}{\sin^2(T_\Sigma - t)} |\nabla\varphi|^2(x),$$

where $\widetilde{\varphi}$ *is the normal extension of* φ *into* Ω, $\nabla^T\widetilde{\varphi}(\Phi_t(x))$ *denotes the gradient of* $\widetilde{\varphi}|_{\Sigma_t}$ *in* $y = \Phi_t(x)$, *and* $\Phi_t(x) = \Phi(t,x)$.

Proof. For $x \in \Sigma$, let $\{E_i(t) := P_t e_i\}$ be an orthonormal basis of $T_{\Phi_t(x)}\Sigma_t$, where $\{e_i\}$ is an orthonormal basis of $T_x\Sigma$ such that $A_{\nu(x)}e_i = k_i(x)e_i$ and such that $P_t e_i$ corresponds to the parallel transport of e_i along the geodesic $\gamma_x(t) := \Phi_t(x)$ from $\gamma_x(0) = x$ to $\gamma_x(t) = \Phi_t(x)$. It follows that

$$\begin{aligned} d(\Phi_t)_x e_i &= P_t((\cos t)e_i + (\sin t)\nu'(x)e_i) \\ &= P_t((\cos t)e_i - (\sin t)A_{\nu(x)}e_i) = (\cos t - k_i(x)\sin t)E_i(t). \end{aligned} \tag{3.3}$$

We have from (3.3) that

$$\begin{aligned} |\nabla^T\widetilde{\varphi}|^2(\Phi_t(x)) &= \sum_i \langle \nabla^T\widetilde{\varphi}(\Phi_t(x)), E_i(t)\rangle^2 = \sum_i \Big\langle \nabla^T\widetilde{\varphi}(\Phi_t(x)), \frac{d(\Phi_t)_x e_i}{\cos t - k_i(x)\sin t}\Big\rangle^2 \\ &= \sum_i \frac{\langle \nabla^T\widetilde{\varphi}(\Phi(t,x)), d\Phi_{(t,x)}(0,e_i)\rangle^2}{(\cos t - k_i(x)\sin t)^2} = \sum_i \frac{(d(\widetilde{\varphi}|_{\Sigma_t}\circ\Phi)_{(t,x)}(0,e_i))^2}{(\cos t - k_i(x)\sin t)^2} \\ &= \sum_i \frac{\langle (0,\nabla\varphi(x)), (0,e_i)\rangle^2}{(\cos t - k_i(x)\sin t)^2}. \end{aligned}$$

And so,

$$|\nabla^T\widetilde{\varphi}|^2(\Phi_t(x)) = \sum_i \frac{\langle \nabla\varphi(x), e_i\rangle^2}{(\cos t - k_i(x)\sin t)^2}. \tag{3.4}$$

Using Lemma 3.2 in formula (3.4), the proof of the lemma is concluded. ∎

Next we are going to construct a transition function wich will be a key technical tool in the proof of our main result. For any $a < b$, we define

$$\psi_{a,b}(t) := 1 - g\Big(\frac{t^2 - a^2}{b^2 - a^2}\Big), \tag{3.5}$$

where the function $g\colon \mathbb{R} \to [0,1]$ is defined by

$$g(t) := \frac{f(t)}{f(t) + f(1-t)}, \quad \text{with} \quad f(t) := \begin{cases} e^{-1/t} & t > 0, \\ 0 & t \le 0. \end{cases}$$

It follows that

(i) $g(t) \geq 0$ for all $t \in \mathbb{R}$.

(ii) $g(t) = 0$ for all $t \in (-\infty, 0]$.

(iii) $\lim_{t\to 0^+} g(t) = 0$ and $\lim_{t\to 1^-} g(t) = 1$.

(iv) The function

$$g'(t) = \frac{e^{1/(t(1-t))}(1-2t+2t^2)}{(e^{1/(1-t)} + e^{1/t})^2 (t-1)^2 t^2}$$

is such that $0 < g'(t) \leq 2$, for all $t \in (0,1)$ and $\lim_{t\to 0^+} g'(t) = \lim_{t\to 1^-} g'(t) = 0$.

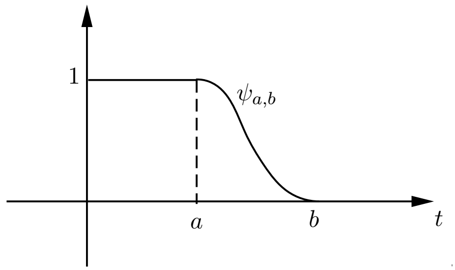

Figure 1. Graph of $\psi_{a,b}$.

For $t \in [a,b]$, it follows from (3.5) that

$$|\psi'_{a,b}(t)| = g'\Big(\frac{t^2-a^2}{b^2-a^2}\Big)\frac{2t}{b^2-a^2} \leq 2\Big(\frac{2t}{b^2-a^2}\Big) = \frac{4t}{b^2-a^2}.$$

Therefore,

$$\int_a^b (\psi'_{a,b}(t))^2\,dt \leq \frac{16}{(b^2-a^2)^2}\int_a^b t^2\,dt = \frac{16(b^3-a^3)}{3(b^2-a^2)^2}.$$

In particular, if we denote $\overline{\psi}_{\rho,c} := \psi_{a,b}$ for the special choices $a = b/c$ for some $c > 1$, and $b = \rho T_\Sigma$ for some $0 < \rho < 1$, we have that

$$\int_{\rho T_\Sigma/c}^{\rho T_\Sigma} (\overline{\psi}'_{\rho,c}(t))^2\,dt \leq \frac{16}{3\rho T_\Sigma}\,\frac{c(c^3-1)}{(c^2-1)^2}. \tag{3.6}$$

Lemma 3.4. *For any $0 < \rho < 1$ and $c > 1$, the function $v_{\rho,c}\colon \overline{\Omega} \to \mathbb{R}$ defined by*

$$v_{\rho,c}(\Phi_t(x)) = \overline{\psi}_{\rho,c}(t)\varphi(x) \tag{3.7}$$

satisfies

$$\int_\Omega |\overline{\nabla} v_{\rho,c}|^2\,dV = \int_0^{\rho T_\Sigma}\int_\Sigma (\overline{\psi}'_{\rho,c}(t))^2\varphi^2(x)\,\theta(t,x)\,dS\,dt + \int_0^{\rho T_\Sigma}\int_\Sigma (\overline{\psi}_{\rho,c}(t))^2\,|\nabla^T\widetilde{\varphi}|^2(\Phi_t(x))\,\theta(t,x)\,dS\,dt.$$

Proof. Let $\bar{v}_{\rho,c}\colon [0,\rho T_\Sigma]\times\Sigma\to\mathbb{R}$ be the function defined by

$$\bar{v}_{\rho,c}(t,x) := \overline{\psi}_{\rho,c}(t)\varphi(x).$$

It follows that $\bar{v}_{\rho,c} = v_{\rho,c}\circ\Phi$ and

$$\overline{\nabla}\bar{v}_{\rho,c}(t,x) = (\overline{\psi}'_{\rho,c}(t)\varphi(x), \overline{\psi}_{\rho,c}(t)\nabla\varphi(x)). \tag{3.8}$$

For $t\in[0,\rho T_\Sigma]$ and $x\in\Sigma$, consider $\{E_i(t)\}$ be as in the proof of Lemma 3.3. Then

$$\begin{aligned}|\overline{\nabla}v_{\rho,c}|^2(\Phi_t(x)) &= \sum_i \langle\overline{\nabla}v_{\rho,c}(\Phi_t(x)), E_i(t)\rangle^2 + \langle\overline{\nabla}v_{\rho,c}(\Phi_t(x)), \gamma_x'(t)\rangle^2\\ &= \sum_i \Big\langle\overline{\nabla}v_{\rho,c}(\Phi(t,x)), \frac{d\Phi_{(t,x)}(0,e_i)}{\cos t - k_i(x)\sin t}\Big\rangle^2 + \langle\overline{\nabla}v_{\rho,c}(\Phi(t,x)), d\Phi_{(t,x)}(1,0)\rangle^2,\end{aligned}$$

where the last equality is a consequence of (3.3). It follows that

$$\begin{aligned}|\overline{\nabla}v_{\rho,c}|^2(\Phi_t(x)) &= \sum_i \frac{\langle\overline{\nabla}v_{\rho,c}(\Phi(t,x)), d\Phi_{(t,x)}(0,e_i)\rangle^2}{(\cos t - k_i(x)\sin t)^2}\\ &\quad + \langle\overline{\nabla}v_{\rho,c}(\Phi(t,x)), d\Phi_{(t,x)}(1,0)\rangle^2\\ &= \sum_i \frac{(d(v_{\rho,c}\circ\Phi)_{(t,x)}(0,e_i))^2}{(\cos t - k_i(x)\sin t)^2} + \big(d(v_{\rho,c}\circ\Phi)_{(t,x)}(1,0)\big)^2\\ &= \sum_i \frac{\langle\overline{\nabla}\bar{v}_{\rho,c}(t,x),(0,e_i)\rangle^2}{(\cos t - k_i(x)\sin t)^2} + \langle\overline{\nabla}\bar{v}_{\rho,c}(t,x),(1,0)\rangle^2.\end{aligned}$$

From (3.8), we have

$$\begin{aligned}|\overline{\nabla}v_{\rho,c}|^2(\Phi_t(x)) &= \sum_i \frac{(\overline{\psi}_{\rho,c}(t)\langle\nabla\varphi(x),e_i\rangle)^2}{(\cos t - k_i(x)\sin t)^2} + (\overline{\psi}'_{\rho,c}(t)\varphi(x))^2\\ &= (\overline{\psi}_{\rho,c}(t))^2\sum_i \frac{\langle\nabla\varphi(x),e_i\rangle^2}{(\cos t - k_i(x)\sin t)^2} + (\overline{\psi}'_{\rho,c}(t))^2\varphi^2(x)\\ &= (\overline{\psi}_{\rho,c}(t))^2|\nabla^T\tilde{\varphi}|^2(\Phi_t(x)) + (\overline{\psi}'_{\rho,c}(t))^2\varphi^2(x),\end{aligned} \tag{3.9}$$

where the last equality is a consequence of (3.4). On the other hand, from (3.3) we have

$$\int_\Omega |\overline{\nabla}v_{\rho,c}|^2\,dV = \int_0^{\rho T_\Sigma}\int_\Sigma |\overline{\nabla}v_{\rho,c}|^2(\Phi_t(x))\theta(t,x)\;dS\,dt. \tag{3.10}$$

We conclude the proof of the lemma by substituting (3.9) into (3.10). ∎

In order to obtain an upper estimate for $|\nabla u|^2_{L^2(\Omega)}$, we will use the fact that the harmonic extension u minimizes the Dirichlet energy in $C^\infty_\varphi(\Sigma)$.

Lemma 3.5. *Let u be the harmonic extension of $\varphi\in C^\infty(\Sigma)$. For all $v\in C^\infty_\varphi(\overline{\Omega})$, we have*

$$\int_\Omega |\overline{\nabla}u|^2\,dV \le \int_\Omega |\overline{\nabla}v|^2\,dV.$$

Proof. Since u is harmonic, by Stokes' theorem,

$$\int_\Omega \langle \overline{\nabla} v, \overline{\nabla} u \rangle dV = \int_\Sigma v \frac{\partial u}{\partial \nu} \, dS = \int_\Sigma u \frac{\partial u}{\partial \nu} \, dS = \int_\Omega |\overline{\nabla} u|^2 \, dV.$$

And so,

$$\begin{aligned} 0 &\leq \int_\Omega |\overline{\nabla}(u - v)|^2 \, dV \\ &= \int_\Omega |\overline{\nabla} u|^2 \, dV + \int_\Omega |\overline{\nabla} v|^2 \, dV - 2 \int_\Omega \langle \overline{\nabla} u, \overline{\nabla} v \rangle dV \\ &= \int_\Omega |\overline{\nabla} u|^2 \, dV + \int_\Omega |\overline{\nabla} v|^2 \, dV - 2 \int_\Omega |\overline{\nabla} u|^2 \, dV, \end{aligned}$$

and we are done. ∎

Proposition 3.6. *Let u be the harmonic extension of $\varphi \in C^\infty(\Sigma)$. Then, if φ is a first eigenfunction of the Laplacian satisfying $\int_\Sigma \varphi^2 \, dS = 1$, we have*

$$\int_\Omega |\overline{\nabla} u|^2 \leq C_1,$$

where

$$C_1 = C_1(k_{\max}) := \frac{32}{3 \arctan(1/k_{\max})} + \frac{\lambda_1(\Sigma)}{\sqrt{1 + k_{\max}^2}}.$$

Proof. Let $0 < \rho < 1$, $c > 1$ and let $v_{\rho,c} \colon \overline{\Omega} \to \mathbb{R}$ be the function defined by (3.7).

From Lemma 3.4 and Lemma 3.5, we have

$$\begin{aligned} \int_\Omega |\overline{\nabla} u|^2 \, dV &\leq \int_\Omega |\overline{\nabla} v_{\rho,c}|^2 \, dV \\ &= \int_0^{\rho T_\Sigma} \int_\Sigma (\overline{\psi}'_{\rho,c}(t))^2 \, \widetilde{\varphi}^2(\Phi_t(x)) \theta(t,x) \, dS \, dt \\ &\quad + \int_0^{\rho T_\Sigma} \int_\Sigma (\overline{\psi}_{\rho,c}(t))^2 \, |\nabla^T \widetilde{\varphi}|^2(\Phi_t(x)) \theta(t,x) \, dS \, dt \\ &\leq \int_{\rho T_\Sigma / c}^{\rho T_\Sigma} (\overline{\psi}'_{\rho,c}(t))^2 \, dt + \int_0^{\rho T_\Sigma} \int_\Sigma (\overline{\psi}_{\rho,c}(t))^2 \, |\nabla^T \widetilde{\varphi}|^2(\Phi_t(x)) \, dS \, dt, \end{aligned} \tag{3.11}$$

where the last inequality is a consequence of the condition $\int_\Sigma \varphi^2 \, d\Sigma = 1$ and (3.2).

Then, by (3.6) and Lemma 3.3, we can rewrite (3.11) as

$$\begin{aligned} \int_\Omega |\overline{\nabla} u|^2 \, dV &\leq \frac{16}{3\rho T_\Sigma} \frac{c(c^3 - 1)}{(c^2 - 1)^2} \\ &\quad + \sin^2(T_\Sigma) \int_0^{\rho T_\Sigma} \csc^2(T_\Sigma - t) dt \int_\Sigma |\nabla \varphi|^2 \, dS. \end{aligned} \tag{3.12}$$

On the other hand, note that

$$\lambda_1(\Sigma) = \int_\Sigma |\nabla \varphi|^2 \, dS.$$

Then it follows that

$$\sin^2(T_\Sigma)\int_0^{\rho T_\Sigma}\csc^2(T_\Sigma - t)\,dt\int_\Sigma |\nabla\varphi|^2\,dS$$
$$= \lambda_1(\Sigma)\sin^2(T_\Sigma)\big(\cot((1-\rho)T_\Sigma)-\cot(T_\Sigma)\big) = \lambda_1(\Sigma)\,\frac{\sin(T_\Sigma)\sin(\rho T_\Sigma)}{\sin((1-\rho)T_\Sigma)},$$

where in the last equality we have used that

$$\cot(A-B) = \frac{\cot A\cot B + 1}{\cot B - \cot A}\cdot$$

Then (3.12) stands as

$$\int_\Omega |\overline{\nabla} u|^2\,dV \le \frac{16}{3\rho T_\Sigma}\,\frac{c(c^3-1)}{(c^2-1)^2} + \lambda_1(\Sigma)\,\frac{\sin(T_\Sigma)\sin(\rho T_\Sigma)}{\sin((1-\rho)T_\Sigma)}\cdot$$

Making $c\to+\infty$, choosing $\rho = 1/2$ and from the definition of T_Σ in (3.1), it follows that

$$\int_\Omega |\overline{\nabla} u|^2\,dV \le \frac{32}{3T_\Sigma} + \lambda_1(\Sigma)\sin(T_\Sigma) = \frac{32}{3\arctan(1/k_{\max})} + \lambda_1(\Sigma)\sin\Big(\arctan\Big(\frac{1}{k_{\max}}\Big)\Big)$$
$$= \frac{32}{3\arctan(1/k_{\max})} + \frac{\lambda_1(\Sigma)}{\sqrt{1+k_{\max}^2}}\cdot \qquad \blacksquare$$

Lemma 3.7. *For all $0\le t < T_\Sigma$ and $f\in C^1(\overline{\Omega})$, we have*

$$\frac{d}{dt}\Big(\int_{\Sigma_t} f(y)\,dS_t\Big) = \int_{\Sigma_t}\langle\nabla f(y),\nabla d(y)\rangle\,dS_t - \int_{\Sigma_t} f(y)H_{\Sigma_t}\,dS_t,$$

where d is the signed distance to Σ in $\mathbb{S}^{m+1}$, i.e.,

$$d(y) = \begin{cases} \operatorname{dist}(y,\Sigma) & \text{if } x\in\overline{\Omega},\\ -\operatorname{dist}(y,\Sigma) & \text{if } x\in\overline{\Omega}^c,\end{cases}$$

and H_{Σ_t} is the mean curvature of the hypersurface Σ_t.

Proof. Making the change of variable $y = \Phi_t(x)$,

$$\frac{d}{dt}\Big(\int_{\Sigma_t} f(y)\,dS_t\Big) = \frac{d}{dt}\Big(\int_\Sigma f(\Phi_t(x))\,\theta(t,x)\,dS\Big)$$
$$= \int_\Sigma\langle\overline{\nabla} f(\Phi_t(x)),\overline{\nabla} d(\Phi_t(x))\rangle\,\theta(t,x)\,dS + \int_\Sigma f(\Phi_t(x))\,\theta'(t,x)\,dS$$
$$= \int_\Sigma\langle\overline{\nabla} f(\Phi_t(x)),\overline{\nabla} d(\Phi_t(x))\rangle\theta(t,x)\,dS - \int_\Sigma f(\Phi_t(x))H_{\Sigma_t}(\Phi_t(x))\theta(t,x)\,dS$$
$$= \int_{\Sigma_t}\langle\overline{\nabla} f(y),\overline{\nabla} d(y)\rangle\,dS_t - \int_{\Sigma_t} f(y)H_{\Sigma_t}(y)\,dS_t.$$

Here we have used that

$$H_{\Sigma_t}(\Phi_t(x)) = -\frac{\theta'(t,x)}{\theta(t,x)}$$

(see Lemma 10.9 in [7]), where $\theta'(t,x)$ denotes the derivative of $\theta(t,x)$ with respect to the first variable. ∎

The following result is a consequence of Lemma 3.5 in [6].

Lemma 3.8. *Let $0 < \varepsilon \le \Lambda/2$. Then for $t \in [0, \arctan(\varepsilon/\Lambda^2)]$,*

$$H_{\Sigma_t} \le 2\Lambda.$$

Proof. Let $0 < \varepsilon \le \Lambda/2$ and $t \in [0, \arctan(\varepsilon/\Lambda^2)]$. From Lemma 3.5 in [6], it follows that

$$H_{\Sigma_t} \le \frac{\Lambda\varepsilon}{\Lambda-\varepsilon}\Big(\frac{m}{\Lambda^2}+1\Big).$$

On the other hand, since $\varepsilon \le \Lambda/2$ and $m \le \Lambda^2$, we have that

$$\frac{\Lambda\varepsilon}{\Lambda-\varepsilon}\Big(\frac{m}{\Lambda^2}+1\Big) \le \frac{2\Lambda\varepsilon}{\Lambda-\varepsilon} \le \frac{\Lambda^2}{\Lambda-\varepsilon} \le \frac{2\Lambda^2}{\Lambda} = 2\Lambda.$$

We conclude that $H_{\Sigma_t} \le 2\Lambda$. ∎

Theorem 3.9. *Let Σ be a closed embedded minimal hypersurface in the unit sphere $\mathbb{S}^{m+1}$ and let $\Lambda = \max_\Sigma |A|$ be the norm of its second fundamental form. Assume that $\Lambda > \sqrt{m}$ and that φ is a first eigenfunction of the Laplacian satisfying $\int_\Sigma \varphi^2\, dS = 1$. Then the harmonic extension u of φ satisfies*

$$\int_\Omega u^2\, dV > \frac{1}{4(12\Lambda+m+11)^2+1}\int_\Omega |\overline{\nabla} u|^2\, dV.$$

Proof. For $t \ge 0$, let

$$\Omega(t) := \{y \in \Omega : d(y) > t\} \quad \text{and} \quad \eta(t) := \int_{\Omega(t)} u^2.$$

It follows that $\eta(0) = \int_\Omega u^2\, dV$. Moreover, from the coarea formula,

$$\begin{aligned}
\eta'_+(t) &= \lim_{\varepsilon\to 0^+}\frac{\eta(t+\varepsilon)-\eta(t)}{\varepsilon} = \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\Big(\int_{\Omega_{t+\varepsilon}} u^2\, dV - \int_{\Omega_t} u^2\, dV\Big)\\
&= -\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_{\{t\le d(y)\le t+\varepsilon\}} u^2\, dV = -\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{\Sigma_s} u^2\, dS_s ds\\
&= -\int_{\Sigma_t} u^2\, dS_t,
\end{aligned}$$

(analogously $\eta'_-(t) = -\int_{\Sigma_t} u^2\, dS_t$), and so $\eta'(0) = -\int_\Sigma \varphi^2 = -1$.

From Lemma 3.7, we have

$$\eta''(t) = -\frac{d}{dt}\Big(\int_{\Sigma_t} u^2\, dS_t\Big) = -\int_{\Sigma_t} \langle \overline{\nabla}(u^2), \overline{\nabla} d\rangle\, dS_t + \int_{\Sigma_t} u^2 H_{\Sigma_t}\, dS_t$$
$$\leq -\int_{\Sigma_t} \langle \overline{\nabla}(u^2), \overline{\nabla} d\rangle\, dS_t + 2\Lambda \int_{\Sigma_t} u^2\, dS_t = \int_{\Omega(t)} \overline{\Delta}(u^2)\, dV - 2\Lambda\eta'(t),$$

where in the last two steps we have used Lemma 3.8 (so here $t \in [0, \arctan(\varepsilon/\Lambda^2)]$) and Stokes' formula, respectively. Therefore, by Proposition 3.6,

$$\eta''(t) + 2\Lambda\eta'(t) \leq 2\int_{\Omega(t)} |\overline{\nabla} u|^2 \leq 2\int_{\Omega} |\overline{\nabla} u|^2 \leq 2C_1, \tag{3.13}$$

for all $t \in [0, \arctan(\varepsilon/\Lambda^2)]$. Multiplying by $e^{2\Lambda t}$ both sides of (3.13) and integrating from 0 to t, we have

$$\eta'(t) \leq 2C_1\Big(\frac{1 - e^{-2\Lambda t}}{2\Lambda}\Big) - e^{-2\Lambda t}. \tag{3.14}$$

Now we can integrate from 0 to $T_\varepsilon := \arctan(\varepsilon/\Lambda^2)$ in (3.14) and deduce that

$$\eta(T_\varepsilon) - \eta(0) \leq \frac{1}{\Lambda} C_1\Big(T_\varepsilon + \frac{e^{-2\Lambda T_\varepsilon} - 1}{2\Lambda}\Big) + \frac{e^{-2\Lambda T_\varepsilon} - 1}{2\Lambda}. \tag{3.15}$$

Considering that $\eta(T_\varepsilon) > 0$ in (3.15), it follows that

$$\begin{aligned} \eta(0) &> \frac{1 - e^{-2\Lambda T_\varepsilon}}{2\Lambda} - \frac{1}{\Lambda} C_1\Big(T_\varepsilon + \frac{e^{-2\Lambda T_\varepsilon} - 1}{2\Lambda}\Big) \\ &= \frac{1 - e^{-2\Lambda T_\varepsilon}}{2\Lambda T_\varepsilon} T_\varepsilon - \frac{C_1}{2}\Big(\frac{2\Lambda T_\varepsilon + e^{-2\Lambda T_\varepsilon} - 1}{\Lambda^2 T_\varepsilon^2}\Big) T_\varepsilon^2. \end{aligned} \tag{3.16}$$

On the other hand, for all $x \geq 0$,

$$\frac{x + e^{-x} - 1}{x^2} \leq \frac{1}{2}. \tag{3.17}$$

Using the inequality (3.17) and the fact that $T_\varepsilon < \varepsilon/\Lambda^2$ in (3.16), we have

$$\begin{aligned} \int_\Omega u^2\, dV &= \eta(0) > T_\varepsilon(1 - \Lambda T_\varepsilon) - C_1 T_\varepsilon^2 = T_\varepsilon\big(1 - (\Lambda + C_1)T_\varepsilon\big) \\ &> T_\varepsilon\Big(1 - \frac{\varepsilon}{\Lambda^2}(\Lambda + C_1)\Big) = T_\varepsilon\Big(1 - \varepsilon\Big(\frac{1}{\Lambda} + \frac{C_1}{\Lambda^2}\Big)\Big). \end{aligned} \tag{3.18}$$

From (3.18), it follows that for any $0 < \varepsilon \leq \frac{1}{2}(1/\Lambda + C_1/\Lambda^2)^{-1}$,

$$\int_\Omega u^2\, dV > \frac{T_\varepsilon}{2} = \frac{1}{2}\arctan\Big(\frac{\varepsilon}{\Lambda^2}\Big).$$

In particular, for $\varepsilon = \frac{1}{2}(1/\Lambda + C_1/\Lambda^2)^{-1}$, we have

$$\int_\Omega u^2\, dV > \frac{1}{2}\arctan\Big(\frac{1}{2(\Lambda + C_1)}\Big).$$

Finally, this last inequality joint with Proposition 3.6 lead us to

$$\frac{\int_\Omega u^2\,dV}{\int_\Omega |\overline{\nabla} u|^2\,dV} > \frac{\arctan\left(\frac{1}{2(\Lambda+C_1)}\right)}{2C_1}. \tag{3.19}$$

On the other hand, note that

$$1 \le k_{\max} m^2 \le \frac{m-1}{m}\Lambda^2.$$

Then, using the fact that $\lambda_1(\Sigma) \le m$ and since $\arctan x \ge x/(1+x^2)$ for $x \in [0,1]$, we have

$$\begin{aligned} C_1 &= \frac{32}{3\arctan(1/k_{\max})} + \frac{\lambda_1(\Sigma)}{\sqrt{1+k_{\max}^2}} \le \frac{32(1+k_{\max}^2)}{3k_{\max}} + \frac{\lambda_1(\Sigma)}{\sqrt{1+k_{\max}^2}} \\ &< \frac{32k_{\max}}{3} + \frac{m+11}{k_{\max}} < \frac{32\Lambda}{3} + \frac{m+11}{k_{\max}} < 11\Lambda + m + 11. \end{aligned} \tag{3.20}$$

We now define

$$C_2 = C_2(m,\Lambda) := \frac{\arctan\left(\frac{1}{2(\Lambda+C_1)}\right)}{2C_1}. \tag{3.21}$$

From (3.20) and (3.21), we have

$$\begin{aligned} C_2 &= \frac{\arctan\left(\frac{1}{2\Lambda+2C_1}\right)}{2C_1} \ge \frac{2\Lambda+2C_1}{2C_1[(2\Lambda+2C_1)^2+1]} \\ &\ge \frac{1}{(2\Lambda+2C_1)^2+1} > \frac{1}{4(12\Lambda+m+11)^2+1}. \end{aligned} \tag{3.22}$$

Combining (3.19) and (3.22), we have

$$\frac{\int_\Omega u^2\,dV}{\int_\Omega |\overline{\nabla} u|^2\,dV} > \frac{1}{4(12\Lambda+m+11)^2+1}.$$

This completes the proof the theorem. ∎

We conclude the paper by comparing our estimate and the lower bound for $\lambda_1(\Sigma)$ obtained by Duncan, Sire, and Spruck in [6]. In their work, it is established that given a closed and embedded minimal hypersurface Σ in $\mathbb{S}^{m+1}$ with $\Lambda = \max_\Sigma |A| \ge \sqrt{m}$, then

$$\lambda_1(\Sigma) \ge \frac{m}{2} + \frac{a(m)}{\Lambda^6 + b(m)},$$

where

$$\begin{aligned} a(m) &:= \frac{3\sqrt{m}(m-1)}{3200}\left(m\arctan\left(\frac{1}{3\sqrt{m}}\right)\right)^3, \\ b(m) &:= \frac{5(m-1)}{8\sqrt{m}}\left(m\arctan\left(\frac{1}{3\sqrt{m}}\right)\right)^3. \end{aligned} \tag{3.23}$$

Since $x/2 \le \arctan x \le x$ when $x \in [0,1]$, we have $\sqrt{m}/6 \le m \arctan\left(\frac{1}{3\sqrt{m}}\right) \le \sqrt{m}/3$, then from (3.23) we deduce

$$a(m) \le \frac{(m-1)m^2}{28800} \quad \text{and} \quad b(m) \ge \frac{5(m-1)m}{1728}. \tag{3.24}$$

Then, since $m \ge 2$ and $\Lambda \ge \sqrt{m}$, we trivially obtain from (3.24) that

$$\begin{aligned} \frac{a(m)}{\Lambda^6 + b(m)} &< \frac{(m-1)m^2}{28800\,\Lambda^6 + \frac{28800}{175}} < \frac{(m+1)m\Lambda^2}{28800\,\Lambda^6 + 164} \\ &< \frac{(m+1)m}{32\,(12\Lambda + \Lambda^2 + 11)^2 + 8} < \frac{(m+1)m}{32\,(12\Lambda + m + 11)^2 + 8}. \end{aligned}$$

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