PB-groupoids vs VB-groupoids

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Abstract. In this paper, we establish the principal bundle counterpart of the well-known and widely applied notion of vector bundle groupoid (VB-groupoid). In particular, we provide a general notion of principal bundle groupoid (PB-groupoid) as a diagram of Lie groupoids and principal bundles, together with the action of a (strict) Lie 2-groupoid. This recovers as particular cases various constructions involving structural Lie 2-groups, Lie groupoids and Lie groups.

Moreover, given any VB-groupoid, we detect which frames are compatible with the underlying structure and show that the sets of such frames has a natural PB-groupoid structure, with the action of the general linear 2-groupoid of the appropriate ranks. We use this construction, as well as that of the associated bundle induced by a 2-representation, to extend the classical correspondence between vector bundles and principal bundles to a new correspondence between VB-groupoids and PB-groupoids. We conclude with several examples and applications.

1. Introduction

In differential geometry, the notions of principal bundles and vector bundles over a smooth manifold M are intimately related by the standard correspondence

$$\left\{ \begin{array}{c} \text{Principal bundles over } M \\ \text{with structural Lie group } \mathrm{GL}(k, \mathbb{R}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Vector bundles over } M \\ \text{of rank } k \end{array} \right\}.$$

The overall goal of this paper is to (make sense and) generalise the correspondence above to "principal bundle objects" and "vector bundle objects" over a Lie groupoid $\mathcal{G} \rightrightarrows M$, namely

$$\left\{ \begin{array}{c} \text{PB-groupoids over } \mathcal{G} \rightrightarrows M \\ \text{with structural Lie 2-groupoid } \mathrm{GL}(l, k) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{VB-groupoids over } \mathcal{G} \rightrightarrows M \\ \text{of rank } (l, k) \end{array} \right\}.$$

VB-groupoids are already used in many geometric contexts. For instance, they appear when integrating fibrewise linear Poisson structures on vector bundles [15]; they are the

models of two-term representations up to homotopy [25], and in particular of the adjoint representation of a Lie groupoid [1]; their cohomology control the deformation theory of Lie groupoids [17]; they can be used for blow-up constructions in relation with index theory [18]. Accordingly, just as the frames of vector bundles encode them into principal bundles, and therefore constitute a fundamental tool in their study, our notion of PB-groupoids and the correspondence above will shed more light on VB-groupoids and their applications. See also the end of this introduction for further motivations.

Let us first quickly recall the main objects involved in the classical correspondence. Given a manifold M of dimension n, a frame at $x \in M$ is a linear isomorphism $\mathbb{R}^n \to T_x M$. The set of frames of M at all points, denoted by $\mathrm{Fr}(M)$ or $\mathrm{Fr}(TM)$, is a smooth manifold of dimension $n + n^2$, and is moreover a principal $\mathrm{GL}(n, \mathbb{R})$-bundle over M, with the right action given by the composition between frames and matrices. More generally, given any vector bundle $E \to M$ of rank k, its *frame bundle* is the set

$$\mathrm{Fr}(E) := \{\mathbb{R}^k \to E_x \text{ linear isomorphism}, x \in M\}.$$

It is a smooth manifold of dimension $n + k^2$ and a principal $\mathrm{GL}(k, \mathbb{R})$-bundle over M. The case $E = TM$ (where $k = n$) recovers $\mathrm{Fr}(M)$. In other words, one has the construction

$$\{\text{Vector bundles of rank } k\} \longrightarrow \{\text{Principal } \mathrm{GL}(k, \mathbb{R})\text{-bundles}\},$$
$$(E \to M) \mapsto (\mathrm{Fr}(E) \to M).$$

On the other hand, given a principal H-bundle $P \to M$ and a representation of H on a $\mathbb{R}$-vector space V, the *associated bundle* is

$$P[V] := (P \times V)/H.$$

If V has dimension k, then $P[V]$ is a vector bundle of rank k. Then one has the construction

$$\left\{ \begin{array}{c} \text{Principal } H\text{-bundles} \\ \text{and } H\text{-representations of dimension } k \end{array} \right\} \longrightarrow \{\text{Vector bundles of rank } k\},$$
$$(P \to M, V) \mapsto (P[V] \to M).$$

Taking $P = \mathrm{Fr}(E)$ and $V = \mathbb{R}^k$, for k the rank of $E \to M$, one has an isomorphism of vector bundles $P[V] \cong E$. Conversely, given a principal $\mathrm{GL}(k, \mathbb{R})$-bundle $P \to M$, one gets an isomorphism of principal bundles $\mathrm{Fr}(P[\mathbb{R}^k]) \cong P$ using the canonical $\mathrm{GL}(k, \mathbb{R})$-representation on $\mathbb{R}^k$.

In order to extend the PB-VB correspondence to the world of Lie groupoids, one needs first of all to understand the objects on both sides. A *VB-groupoid* is a vector bundle object in the category of Lie groupoid, i.e., a diagram

$$\begin{array}{ccc} E_{\mathcal{G}} & \longrightarrow & \mathcal{G} \\ \Downarrow & & \Downarrow \\ E_M & \longrightarrow & M \end{array}$$

consisting of two vector bundles and two Lie groupoids, such that the various morphisms are compatible with both structures (see Definition 2.1 later on). This notion was introduced in 1988 by Pradines [40] and it is now a standard concept in the theory of Lie groupoids; see, e.g., [9, 25, 35] for an overview.

On the other hand, the notion of *PB-groupoid* is not yet well established and the first goal of our paper is to fill this gap. If one tries to mimic the definition of VB-groupoid, i.e., a principal bundle object in the category of Lie groupoids, one encounters a diagram of the kind

$$
\begin{array}{ccc}
P_{\mathcal{G}} & \longrightarrow & \mathcal{G} \\
\downdownarrows & & \downdownarrows \\
P_M & \longrightarrow & M,
\end{array}
$$

with a principal action of a group object in the category of Lie groupoids. More precisely, one uses a Lie 2-group $H_2 \rightrightarrows H_1$, with H_2 acting on $P_{\mathcal{G}}$ and H_1 acting on P_M, as in the definition of principal bundle groupoids given in [23] by Garmendia and Paycha. However, as we will see, this notion is very restrictive when working with the frames of a VB-groupoid. Furthermore, besides standard principal bundles with a structure Lie group, one might be interested in considering principal bundles with structural Lie groupoids.

Indeed, in the literature there are various notions for "higher" principal bundles, including:

- *principal bundles over groupoids* as in Definition 2.33 of [32], *PBG-groupoids* as in Definition 2.1 of [34], and *G-groupoids* as in Definition 4.4 of [8], where a Lie group acts on a Lie groupoid;

- *generalized principal bundles* as in Theorem 2.1 of [12], where a Lie group bundle acts on a fibre bundle;

- *principal groupoid bundles* as in Section 5.7 of [38], Definition 2.1 in [41], Definition 3.1 in [28], and in Section 2.2 of [26], where a Lie groupoid acts on a smooth map;

- *principal 2-bundles* as in Definition 6.1.5 of [39], where a Lie 2-group acts on a fibred pair groupoid;

- *principal bundle groupoids* as in [23] and *principal 2-bundles over Lie groupoids* as in [14] and [27], where a Lie 2-group acts on a Lie groupoid.

In this article, we extend the definition of *PB-groupoid* to a diagram as above, with a (strict) Lie 2-*groupoid* acting free and properly on the Lie groupoid $P_{\mathcal{G}} \rightrightarrows P_M$. This notion not only generalises all the definitions above, but also fits as the natural structure carried by the "adapted frames" of a VB-groupoid.

Indeed, while the construction of the associated vector bundle generalises easily to the PB-groupoid case, that of the frame bundle does not: taking all the possible frames of the vector bundle $E_{\mathcal{G}} \to \mathcal{G}$ does not yield a groupoid structure over the frame bundle of $E_M \to M$. Accordingly, one has to restrict to a class of frames of $E_{\mathcal{G}}$ which are "adapted" to the VB-groupoid structure (which we will call *s-bisection frames*), obtaining a

diagram of the form

$$\begin{array}{ccc} \mathrm{Fr}^{\mathrm{sbis}}(E_{\mathscr{G}}) & \longrightarrow & \mathscr{G} \\ \Downarrow & & \Downarrow \\ \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M) & \longrightarrow & M. \end{array}$$

Moreover, $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathscr{G}})$ admits a natural (principal) action of the Lie 2-groupoid $\mathrm{GL}(l, k)$, where l is the rank of the core $C \to M$ of $E_{\mathscr{G}}$ and k the rank of the base $E_M \to M$, and defines therefore a PB-groupoid. Here $\mathrm{GL}(l, k)$ is a direct generalisation of the group of invertible matrices of a given dimension, which we describe explicitly but can also be seen as a particular case of the general linear 2-groupoid from [21] and of the 2-gauge groupoid from [27].

Main results and structure of the paper. Section 2 contains a solid background on VB-groupoids and related objects, including (strict) Lie 2-groupoids and (two versions of) 2-vector bundles, in order to make this article as self-contained as possible. Indeed, these last two topics are well known to the experts but we could find no detailed/complete sources in the literature. The most important object of this section, which will play a key role in the next ones, is the *general linear 2-groupoid* $\mathrm{GL}(l, k)$ (Definition 2.24).

Our contributions start in Section 3. First of all we introduce two notions of 2-*representation of Lie 2-groupoids* (Definition 3.8), one of which, to the best of our knowledge, has not appeared previously in the existing literature. Then we provide our first main definition: the *general notion of PB-groupoid* over a Lie groupoid (Definition 3.17). Finally, we describe the *associated bundle of a PB-groupoid* and a 2-representation, and we prove that it is a VB-groupoid (Theorem 3.22).

Section 4 is devoted to the opposite construction, namely the "*adapted frame bundle*" $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathscr{G}})$ *of a VB-groupoid* $E_{\mathscr{G}} \rightrightarrows E_M$ (Definition 4.3). The main result here, i.e., the fact that

$$\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathscr{G}}) \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$$

is a PB-groupoid (Theorem 4.16), is a consequence of the facts that it is a set-theoretical groupoid (Theorem 4.9), it is endowed with a natural $\mathrm{GL}(l, k)$-action (Theorem 4.13), and it is smooth (Theorem 4.15). We conclude with the proof that the constructions in Section 3 and 4 are inverse to each other, and therefore yield a 1-1 *correspondence between VB-groupoids and PB-groupoids* (Theorem 4.17).

Last, in Section 5 we illustrate our main results by *several interesting examples*, e.g., applying the frame bundle construction to the VB-groupoids with trivial core or with trivial base, as well as to various relevant cases of tangent Lie groupoids. Moreover, we discuss the *compatibility of our construction with duality*: the PB-groupoid associated to the dual $E_{\mathscr{G}}^*$ of a VB-groupoid $E_{\mathscr{G}}$ turns out to be canonically isomorphic to the PB-groupoid associated to $E_{\mathscr{G}}$ (Corollary 5.15).

Further motivations and future applications. On the one hand, this paper belongs to the recent programme of investigating classical objects of differential geometry compatible with a Lie groupoid. Such objects are often known as *multiplicative structures* and have naturally emerged in various contexts in Poisson geometry [10, 30]. In turn, this

programme is strictly related to (and motivated by) the description of such geometric objects on *manifolds with singularities*, e.g., leaf spaces of foliations or orbit spaces of group actions. Indeed, these spaces are not smooth manifolds but can be formalised in the framework of differentiable stacks, i.e., Morita equivalence classes of Lie groupoids [19]. In particular, the frame bundle of the tangent VB-groupoid $T\mathcal{G} \rightrightarrows TM$ (Example 5.5) represents the first step in order to obtain a notion of *frame bundle of a differentiable stack*. More generally, one could investigate principal bundles over a differentiable stack, complementing what done in [20] for VB-stacks, which led already to further insights and applications to representations up to homotopy.

On the other hand, an important application of the classical PB-VB correspondence is the possibility to encode geometric structures on the tangent bundle of a manifold (e.g., Riemannian structures, symplectic structures, complex structures, etc.) via reductions of the structure group of its frame bundle, giving rise to the well-known theory of G-structures [29, 43]. In several geometric problems where they arise, Lie groupoids are naturally equipped with compatible geometric structures. As a future application of the results in our paper, we aim at establishing a theory of *multiplicative G-structure-like objects on a Lie groupoid $\mathcal{G}$* in terms of reductions of the structural Lie 2-groupoid of its adapted frame bundle. Two crucial tasks in this problem will involve the generalisation of the classical tautological/soldering form $\theta \in \Omega^1(\mathrm{Fr}(M), \mathbb{R}^n)$ to the frame bundle of $T\mathcal{G}$, as well as the appropriate notion of principal connection for PB-groupoids. This last object would naturally pave the way to the development of a *Chern–Weil theory for Lie groupoids*, extending the results in [33], [4] and [27].

Last, one of the main principle of Lie theory is the strict relation between global and infinitesimal objects, which gives the possibility to transform non-linear problems into linear ones. In particular, it is known that multiplicative structures on Lie groupoids induce (interesting, and possibly easier) *infinitesimally multiplicative structures* on the corresponding Lie algebroids. Accordingly, we plan to adapt our construction of the adapted frame bundle to a VB-algebroid, obtaining what one should call *PB-algebroid*, which we expect to be the infinitesimal counterpart of our PB-groupoids.

Notations and conventions. All manifolds and maps are smooth, unless explicitly stated otherwise. Principal group(oid) actions are always considered from the right, while representations are considered from the left. The symbol $\mathrm{GL}(k)$ denotes the group of $k \times k$ invertible *real* matrices.

In many instances, we will consider several different groupoid structures, possibly on the same space. For all of them, we use

- the letters s and t to denote the source and target,
- the letter m and the sign $\circ$ or $\circ_m$ to denote the multiplication,
- the letter u for the unit and u_x to denote the unit at a point x in the base,
- and the letter τ or the symbol $(-)^\tau$ for the inverse and g^τ to denote the inverse of an arrow g.

We use tilde for the structure maps of VB-groupoids ($\tilde{s}$, $\tilde{t}$, $\tilde{m}$, etc.), and boldface for the structure maps of PB-groupoids ($\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$, etc.).

2. Background on 2-objects

Through this paper we will use, without recalling the basics, the theory of Lie groupoids, Lie algebroids, and principal bundles (with structural Lie groups or Lie groupoids). For an introduction on these topics we refer to [16, 35, 38].

2.1. VB-groupoids

In this section, we review some basics on VB-groupoids; for proofs and further details we refer to [9, 25, 35]. The reader familiar with this topic can safely skip this section but should take note of our emphasis on a non-standard terminology, which will be relevant in the later sections: the *rank* (l, k) of a VB-groupoid consists of the ranks of the core and of the base vector bundles.

Definition 2.1. A *VB-groupoid of rank* (l, k) is a commutative diagram

$$
\begin{array}{ccc}
E_{\mathcal{G}} & \xrightarrow{\pi_{\mathcal{G}}} & \mathcal{G} \\
\tilde{t} \downarrow\downarrow \tilde{s} & & t \downarrow\downarrow s \\
E_M & \xrightarrow{\pi_M} & M
\end{array}
$$

such that

- $\pi_{\mathcal{G}} \colon E_{\mathcal{G}} \to \mathcal{G}$ is a vector bundle of rank $l + k$ and $\pi_M \colon E_M \to M$ is a vector bundle of rank k;
- $\mathcal{G} \rightrightarrows M$ and $E_{\mathcal{G}} \rightrightarrows E_M$ are Lie groupoids;
- all the groupoid structure maps are morphisms of vector bundles. ◆

Equivalent definitions of a VB-groupoid can be found in Section 3.1 of [25]. For instance, the third bullet can be replaced by the following three conditions:

- $(\tilde{s}, s)$ and $(\tilde{t}, t)$ are morphisms of vector bundles;
- $(\pi_{\mathcal{G}}, \pi_M)$ is a morphism of Lie groupoids;
- the interchange law holds, i.e.,

$$
\tilde{m}(e_1 + e_3, e_2 + e_4) = \tilde{m}(e_1, e_2) + \tilde{m}(e_3, e_4)
$$

for every $(e_1, e_2), (e_3, e_4) \in E_{\mathcal{G}} {}_s\times_t E_{\mathcal{G}}$ such that $\pi_{\mathcal{G}}(e_1) = \pi_{\mathcal{G}}(e_3)$ and $\pi_{\mathcal{G}}(e_2) = \pi_{\mathcal{G}}(e_4)$.

Definition 2.2. The (right-)*core* of a VB-groupoid is

$$
C := \ker(\tilde{s})|_M = \{e \in E_{\mathcal{G}} : x \in M, \pi_{\mathcal{G}}(e) = u(x), \tilde{s}(e) = 0\} \subseteq E_{\mathcal{G}}.
$$

The *core-anchor* ρ is the restriction of $\tilde{t} \colon E_{\mathcal{G}} \to E_M$ to the core, i.e., $\rho := \tilde{t}|_C \colon C \to E_M$. ◆

The core has a natural structure of vector bundle of rank l over M, so that the core-anchor is a morphism of vector bundles over id_M. The *left-core* is defined analogously using $\tilde{t}$ instead of $\tilde{s}$, and it is canonically isomorphic to C.

Example 2.3. Given any Lie groupoid $\mathcal{G} \rightrightarrows M$, the tangent bundles $T\mathcal{G} \to \mathcal{G}$ and $TM \to M$ together with the differential of the structure maps of $\mathcal{G}$, define a VB-groupoid of rank

$$(\dim(\mathcal{G}) - \dim(M), \dim(M)),$$

called the *tangent VB-groupoid*:

$$
\begin{array}{ccc}
T\mathcal{G} & \xrightarrow{\;\pi_\mathcal{G}\;} & \mathcal{G} \\
dt \downdownarrows ds & & t \downdownarrows s \\
TM & \xrightarrow{\;\pi_M\;} & M.
\end{array}
$$

Its core coincides with the Lie algebroid $A \to M$ of $\mathcal{G}$ and its core-anchor with the anchor $\rho := dt|_A : A \to TM$ of A. $\diamond$

Example 2.4. Given any Lie groupoid $\mathcal{G} \rightrightarrows M$ with algebroid $A \to M$, the cotangent bundle $T^*\mathcal{G} \to \mathcal{G}$ and the dual bundle $A^* \to M$ define a VB-groupoid of rank

$$(\dim(M), \dim(\mathcal{G}) - \dim(M)),$$

called the *cotangent VB-groupoid*:

$$
\begin{array}{ccc}
T^*\mathcal{G} & \longrightarrow & \mathcal{G} \\
\tilde{t} \downdownarrows \tilde{s} & & t \downdownarrows s \\
A^* & \longrightarrow & M.
\end{array}
$$

The source and target maps of $T^*\mathcal{G} \rightrightarrows A^*$ are defined, for $g \in \mathcal{G}$, $\xi_g \in T_g^*\mathcal{G}$ and $v_{(-)} \in A_{(-)}$, as follows:

$$\tilde{s}(\xi_g)(v_{s(g)}) := -\xi_g(d_e L_g(d_{1_{s(g)}}\tau(v))) \quad \text{and} \quad \tilde{t}(\xi_g)(v_{t(g)}) := \xi_g(d_e R_g(v)).$$

The multiplication between two elements $\xi_1 \in T_{g_1}^*\mathcal{G}$ and $\xi_2 \in T_{g_2}^*\mathcal{G}$ such that $\tilde{s}(\xi_1) = \tilde{t}(\xi_2)$ is determined, for any composable $v_1 \in T_{g_1}\mathcal{G}$ and $v_2 \in T_{g_2}\mathcal{G}$, by

$$(\xi_1 \circ \xi_2)(dm(v_1, v_2)) := \xi_1(v_1) + \xi_2(v_2).$$

The core of $T^*\mathcal{G}$ coincides with the cotangent bundle $T^*M \to M$ and its core-anchor with the dual map $\rho^* : T^*M \to A^*$ of the anchor $\rho : A \to TM$. $\diamond$

Definition-Example 2.5. Given any VB-groupoid $E_\mathcal{G} \rightrightarrows E_M$ of rank (l, k) with core $C \to M$, the dual bundles $E_\mathcal{G}^* \rightrightarrows \mathcal{G}$ and $C^* \to M$ define a VB-groupoid of rank (k, l), called the *dual VB-groupoid*:

$$
\begin{array}{ccc}
E_\mathcal{G}^* & \longrightarrow & \mathcal{G} \\
\tilde{t} \downdownarrows \tilde{s} & & t \downdownarrows s \\
C^* & \longrightarrow & M.
\end{array}
$$

The source and target maps of $E_\mathcal{G}^* \rightrightarrows C^*$ are defined, for $g \in \mathcal{G}$, $\xi_g \in (E_\mathcal{G}^*)_g$ and $v_{(-)} \in C_{(-)}$, as follows:

$$\tilde{s}(\xi_g)(v_{s(g)}) := -\xi_g(\widetilde{m}(0_g, \tilde{\tau}(v))) \quad \text{and} \quad \tilde{t}(\xi_g)(v_{t(g)}) := \xi_g(\widetilde{m}(v, 0_g)).$$

The multiplication between two elements $\xi_1 \in (E_{\mathcal{G}}^*)_{g_1}$ and $\xi_2 \in (E_{\mathcal{G}}^*)_{g_2}$ such that $\tilde{s}(\xi_1) = \tilde{t}(\xi_2)$ is determined, for any composable $v_1 \in (E_{\mathcal{G}})_{g_1}$ and $v_2 \in (E_{\mathcal{G}})_{g_2}$, by

$$(\xi_1 \circ \xi_2)(\widetilde{m}(v_1, v_2)) := \xi_1(v_1) + \xi_2(v_2).$$

The core of $E_{\mathcal{G}}^*$ coincides with the dual vector bundle $E_M^* \to M$ and its core-anchor with the dual map $E_M^* \to C^*$ of the core-anchor $C \to E_M$ of $E_{\mathcal{G}}$. In particular, the cotangent VB-groupoid of a Lie groupoid (Example 2.4) is the dual VB-groupoid of its tangent VB-groupoid (Example 2.3).   ♦

Example 2.6. A *double vector bundle* (DVB) is a commutative diagram of vector bundles

$$\begin{array}{ccc} E & \xrightarrow{\pi_B^E} & B \\ \downarrow{\pi_A^E} & & \downarrow{\pi^B} \\ A & \xrightarrow{\pi^A} & M, \end{array}$$

where all the structure maps (i.e., projections, additions, multiplications by scalars and zero sections) are vector bundle morphisms; see, e.g., Chapter 9 of [35] for further details. Any DVB is a VB-groupoid with rank $(\mathrm{rank}(\pi_B^E) - \mathrm{rank}(\pi^A), \mathrm{rank}(\pi^A))$, since any vector bundle is a bundle of (abelian) Lie groups, i.e., it is a special Lie groupoid with $s = t$. Moreover, its core coincides with the DVB core, i.e., the intersection

$$C = (\pi^A)^{-1}(0_M) \cap (\pi^B)^{-1}(0_M). \qquad \diamond$$

Example 2.7 (Trivial core). Given a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a representation $E_M \to M$ of $\mathcal{G}$, the action groupoid $\mathcal{G} \times_M E_M \rightrightarrows E_M$ defines a VB-groupoid of rank $(0, k)$, together with the induced vector bundle $\mathcal{G} \times_M E_M \to \mathcal{G}$:

$$\begin{array}{ccc} \mathcal{G} \times_M E_M & \xrightarrow{\mathrm{pr}_1} & \mathcal{G} \\ \tilde{t}\downdownarrows\tilde{s} & & t\downdownarrows s \\ E_M & \longrightarrow & M. \end{array}$$

By construction, the core of $\mathcal{G} \times_M E_M$ is the zero-vector bundle. Conversely, any VB-groupoid $E_{\mathcal{G}} \rightrightarrows E_M$ with core $C = 0$ is isomorphic to the action groupoid $\mathcal{G} \times_M E_M \rightrightarrows E_M$, for the representation of $\mathcal{G}$ on E_M given by $g \cdot e := \tilde{t}_g(v)$, with $v \in (E_{\mathcal{G}})_g$ uniquely defined by $\tilde{s}_g(v) = e$.

Note that, if a tangent VB-groupoid $T\mathcal{G} \rightrightarrows TM$ (Example 2.3) has rank $(0, k)$, i.e., if its Lie algebroid $A \to M$ is the zero vector bundle, then $\mathcal{G} \rightrightarrows M$ must be an étale Lie groupoid. In this case, $T\mathcal{G} \cong \mathcal{G} \times_M TM$ for the canonical representation of any étale Lie groupoid $\mathcal{G} \rightrightarrows M$ on TM. $\diamond$

The dual of a VB-groupoid with trivial core does not have trivial core, but has trivial base. This means that such dual is not the VB-groupoid associated to the dual representation E_M^*. On the other hand, VB-groupoids with trivial base are encoded into representations of their core, as discussed in Example 2.8 below. Applying the general construction from Example 2.5 to Example 2.7, one obtains precisely the VB-groupoid with trivial base associated to the dual representation E_M^*.

Example 2.8 (Trivial base). Given a Lie groupoid $\mathscr{G} \rightrightarrows M$ and a rank l representation $C \to M$ of $\mathscr{G}$, one has the following Lie groupoid structure on $\mathscr{G} \times_M C \rightrightarrows M \times \{0\}$:

$$\tilde{s}(g,c) = s(g), \quad \tilde{t}(g,c) = t(g),$$
$$\tilde{m}((g_1,c_1),(g_2,c_2)) = (g_1 g_2, c_1 + g_2 \cdot c_2).$$

This defines a VB-groupoid of rank $(l,0)$, together with the induced vector bundle $\mathscr{G} \times_M C \to \mathscr{G}$:

$$
\begin{array}{ccc}
\mathscr{G} \times_M C & \xrightarrow{\ \mathrm{pr}_1\ } & \mathscr{G} \\
\tilde{t} \downdownarrows \tilde{s} & & t \downdownarrows s \\
M \times \{0\} & \longrightarrow & M.
\end{array}
$$

Conversely, any VB-groupoid $E_{\mathscr{G}} \rightrightarrows E_M = M \times \{0\}$ with core C is isomorphic to the Lie groupoid $\mathscr{G} \times_M C \rightrightarrows M \times 0$ discussed above, for the representation of $\mathscr{G}$ on C given by $g \cdot c = \tilde{m}(0_g, c)$.

   Note that, if a tangent VB-groupoid $T\mathscr{G} \rightrightarrows TM$ (Example 2.3) has rank $(l,0)$, i.e., if $TM \to M$ is the zero vector bundle, then M is 0-dimensional. If we assume that M is connected, then $G = \mathscr{G} \rightrightarrows \{*\}$ must be a Lie group and the core C its Lie algebra $\mathfrak{g}$. In this case, $TG \cong G \times \mathfrak{g}$ for the adjoint representation of any Lie group G on $\mathfrak{g}$. ◇

Example 2.9 (Pullback). Given a vector bundle $\pi \colon E_M \to M$ of rank k and a Lie groupoid $\mathscr{G} \rightrightarrows M$, the pullback groupoid of $\mathscr{G}$ with respect to π defines a VB-groupoid of rank (k,k):

$$
\begin{array}{ccc}
E_M \ {}_{\pi}\!\times_t \mathscr{G} \ {}_s\!\times_{\pi} E_M & \xrightarrow{\ \mathrm{pr}_2\ } & \mathscr{G} \\
\tilde{t} \downdownarrows \tilde{s} & & t \downdownarrows s \\
E_M & \xrightarrow{\quad \pi \quad} & M.
\end{array}
$$

The core of $E_M \ {}_{\pi}\!\times_t \mathscr{G} \ {}_s\!\times_{\pi} E_M$ coincides with E_M, and its core-anchor is the identity $E_M \to E_M$. ◇

The following simple example will be crucial for carrying out our theory; further insights will be provided later.

Definition-Example 2.10. Any pair (l,k) of natural numbers defines a VB-groupoid of rank (l,k), which we call the *canonical VB-groupoid* $\mathbb{R}^{(l,k)}$. We will use here the notation

$$\mathrm{GL}(l,k)_0 := \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k),$$

the reasons of which will be clear in Definition 2.24. The groupoid $\mathbb{R}^{(l,k)}$ is given by

$$
\begin{array}{ccc}
\mathbb{R}^{(l,k)}_2 := \mathbb{R}^l \times \mathbb{R}^k \times \mathrm{GL}(l,k)_0 & \xrightarrow{\ \mathrm{pr}\ } & \mathrm{GL}(l,k)_0 \\
\tilde{t} \downdownarrows \tilde{s} & & \downdownarrows \\
\mathbb{R}^{(l,k)}_1 := \mathbb{R}^k \times \mathrm{GL}(l,k)_0 & \xrightarrow{\ \mathrm{pr}\ } & \mathrm{GL}(l,k)_0.
\end{array}
$$

Here the rows are the trivial vector bundles of ranks $l + k$ and k over the manifold $GL(l, k)_0$. Moreover, the groupoid on the left column is the action groupoid of the following (abelian) action of $\mathbb{R}^l$ on $\mathbb{R}^k \times GL(l, k)_0$:

$$\mathbb{R}^l \times \mathbb{R}^k \times GL(l, k)_0 \to \mathbb{R}^k \times GL(l, k)_0, \quad (w, v, d) \mapsto (d(w) + v, d).$$

Finally, the groupoid $GL(l, k)_0 \rightrightarrows GL(l, k)_0$ on the right column is identified with the unit groupoid of $GL(l, k)_0$. A VB-groupoid with this last property is actually equivalent to an anchored 2-vector bundle (see Lemma 2.12 below).

The core of $\mathbb{R}^{(l,k)}$ is the trivial vector bundle of rank l over $GL(l, k)_0$, and the core-anchor is

$$\mathbb{R}^l \times GL(l, k)_0 \to \mathbb{R}^k \times GL(l, k)_0, \quad (w, d) \mapsto (d(w), d). \qquad \blacklozenge$$

2.2. 2-vector bundles

In the literature, there are various "higher" generalisations of vector bundles, often induced from abstract categorical points of view. With our specific applications in mind, here we consider two notions which we will use in the rest of the paper, and which might have appeared elsewhere under different names.

Definition 2.11. A 2-*graded vector bundle* is a pair (E_1, E_0) of vector bundles over the same manifold M. An *anchored* 2-*graded vector bundle* (or, shortly, an *anchored* 2-*vector bundle*) is a 2-graded vector bundle (E_1, E_0) endowed with a vector bundle map $\delta \colon E_1 \to E_0$ over the identity id_M, which we call *anchor*. This is represented by the diagram

$$
\begin{array}{ccc}
E_1 & \longrightarrow & M \\
\delta \downarrow & & \downarrow \mathrm{id}_M \\
E_0 & \longrightarrow & M
\end{array}
$$

and denoted, more compactly, by $E_1 \xrightarrow{\delta} E_0 \to M$. $\qquad \blacklozenge$

Anchored 2-vector bundles generalise the standard notion of *anchored vector bundle*, i.e., a vector bundle $E_1 \to M$ together with a vector bundle map $E_1 \to TM$; in our case, TM is replaced by an arbitrary vector bundle $E_0 \to M$. From a different point of view, an anchored 2-vector bundle is simply a chain complex (E_i, δ_i) of vector bundles over the same base, with $E_i = 0$ for every $i \neq 0, 1$.

Another reason for the adjective "anchored" comes from the following lemma, which is part of the folklore knowledge in the field; we are adding it here for the sake of self-containment. Note that this lemma can also be seen as a special instance of the Dold–Kan correspondence between chain complexes of vector bundles and simplicial vector bundles; in this case, the chain complexes have only two non-zero terms and the corresponding simplicial vector bundles are the nerves of VB-groupoids over the units.

Lemma 2.12 (Anchored 2-vector bundles and VB-groupoids over the unit groupoid). *Any anchored* 2-*vector bundle* $E_1 \xrightarrow{\delta} E_0 \to M$ *induces a VB-groupoid over the unit group-*

oid $M \rightrightarrows M$:

$$
\begin{array}{ccc}
E_1 \times_M E_0 & \longrightarrow & M \\
\tilde{t} \downdownarrows \tilde{s} & & \Downarrow \\
E_0 & \longrightarrow & M,
\end{array}
$$

where $E_1 \times_M E_0$ is the action groupoid of the following (abelian) groupoid action of $E_1 \to M$ on $E_0 \to M$:

$$
E_1 \times_M E_0 \to E_0, \quad (v_1, v_0) \mapsto \delta(v_1) + v_0.
$$

Conversely, given a VB-groupoid $E_{\mathscr{G}} \rightrightarrows E_M$ over the unit groupoid $\mathscr{G} = M \rightrightarrows M$, then $E_1 := C$ and $E_0 := E_M$ define an anchored 2-vector bundle, with anchor map equal to the core-anchor $\delta := \tilde{t}|_{E_1} : E_1 \to E_0$ (Definition 2.2).

Remark 2.13 (2-graded vector bundles and VB-groupoids over the unit groupoid). Note that a 2-graded vector bundle would not be enough to define the VB-groupoid structure described in Lemma 2.12, since one needs the anchor map δ in order to define the groupoid structure on $E_1 \times_M E_0 \rightrightarrows E_0$. Conversely, any VB-groupoid $E_{\mathscr{G}} \rightrightarrows E_M$ over the unit groupoid $M \rightrightarrows M$ induces a 2-vector bundle ($E_1 := C$, $E_0 := E_M$), where the anchor $\delta : E_1 \to E_0$ is the core-anchor $C \to E_M$. ◇

Example 2.14 (2-graded vector spaces). When $M = \{*\}$, a 2-anchored vector space $V_1 \xrightarrow{\delta} V_0$ is simply a 2-term chain complex of vector spaces, hence it is equivalent to the notion of 2-vector spaces introduced in Definition 3.1 and Proposition 3.4 of [3], i.e., the structure underlying a strict Lie 2-algebra.

On the other hand, a 2-graded vector bundle boils down to a 2-graded vector space (V_1, V_0). In particular, any pair (l, k) of natural numbers defines the 2-graded vector bundle $(\mathbb{R}^l, \mathbb{R}^k)$ over a point.

Notice that there is no canonical anchor $\delta : \mathbb{R}^l \to \mathbb{R}^k$ making $(\mathbb{R}^l, \mathbb{R}^k)$ into an anchored 2-vector space. Each choice of such an anchor would yield, via Lemma 2.12, a VB-groupoid $\mathbb{R}^l \times \mathbb{R}^k \rightrightarrows \mathbb{R}^k$ over the trivial unit groupoid $\{*\} \rightrightarrows \{*\}$, with core-anchor equal to δ. ◇

Definition-Example 2.15. Any pair (l, k) of natural numbers defines an anchored 2-vector bundle $E^{(l,k)}$ over $M = \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k)$, which we call the *canonical 2-anchored vector bundle*, given by the trivial vector bundles

$$
E_1^{(l,k)} = \mathbb{R}^l \times \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \to \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k),
$$
$$
E_0^{(l,k)} = \mathbb{R}^k \times \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \to \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k)
$$

and the anchor map

$$
\delta : E_1^{(l,k)} \to E_0^{(l,k)}, \quad (v_1, d) \mapsto (d(v_1), d).
$$

Applying the correspondence from Lemma 2.12, the canonical 2-anchored vector bundle $E^{(l,k)}$ can be equivalently described precisely by the canonical VB-groupoid $\mathbb{R}^{(l,k)}$ from Example 2.10.

More generally, any 2-graded vector bundle (E_1, E_0) over M defines an anchored 2-vector bundle $(E_1 \times_M \text{Hom}(E_1, E_0), E_0 \times_M \text{Hom}(E_1, E_0))$ with anchor given by

$$\delta(v_1, d) = (d(v_1), d).$$

Notice that, for M a point and $(E_1, E_0) = (\mathbb{R}^l, \mathbb{R}^k)$, we recover precisely the canonical 2-anchored vector bundle $E^{(l,k)}$ described above. ◆

Example 2.16. Ordinary vector bundles can be viewed as anchored 2-vector bundles in three non-equivalent ways:

- when $E_0 = M$ (i.e., E_0 has rank 0) and δ coincides with the projection $E_1 \to M$;
- when $E_1 = M$ (i.e., E_1 has rank 0) and $\delta \colon M \to E_0$ is the zero section;
- when $E_0 = E_1$ and $\delta = \text{id}$. ◇

2.3. Lie 2-groupoids

As for 2-vector bundles, Lie 2-groupoids have several inequivalent definitions in the literature, as discussed in Remark 2.18 below. Here we follow, e.g., [37], [21] and [5], and pick the notion which will appear naturally when looking at the frame bundle of a VB-groupoid.

In general, since we will consider different groupoid structures on the same space, we will from now on adopt the notation s_{ij}, t_{ij}, m_{ij}, u_{ij} and τ_{ij} for the structure maps of a Lie groupoid $\mathcal{H}_i \rightrightarrows \mathcal{H}_j$; for $\mathcal{H}_i \rightrightarrows M$, we will use s_M, t_M, m_M, u_M and τ_M.

Definition 2.17. A *double Lie groupoid* (see, e.g., [6] or Definition 3.1 and 3.2 in [37]) is a commutative diagram of Lie groupoids

$$
\begin{array}{ccc}
\mathcal{H}_2 & \rightrightarrows & \mathcal{H}_1 \\
\downdownarrows & & \downdownarrows \\
\mathcal{H}_0 & \rightrightarrows & M,
\end{array}
$$

where the following three conditions are satisfied:

(1) all the source and targets maps are Lie groupoid morphisms;

(2) the interchange law

$$(g_1 \circ_{m_{20}} g_2) \circ_{m_{21}} (g_3 \circ_{m_{20}} g_4) = (g_1 \circ_{m_{21}} g_3) \circ_{m_{20}} (g_2 \circ_{m_{21}} g_4)$$

holds for all $g_i \in \mathcal{H}_2$ such that the compositions above make sense;

(3) the map $(s_{21}, s_{20}) \colon \mathcal{H}_2 \to \mathcal{H}_1 \, {}_{s_M}\!\times_{s_M} \mathcal{H}_0$ is a surjective submersion.

A *Lie 2-groupoid* $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ (see, e.g., Example 3.7 in [37]) is a double Lie groupoid where the base groupoid $\mathcal{H}_0 \rightrightarrows M = \mathcal{H}_0$ is the unit groupoid. In other words, it is a commutative diagram of Lie groupoids

$$
\begin{array}{ccc}
\mathcal{H}_2 & \rightrightarrows & \mathcal{H}_1 \\
& \searrow \quad \swarrow & \\
& \mathcal{H}_0 &
\end{array}
$$

such that conditions (1) and (2) above hold (condition (3) holds automatically, since $\mathcal{H}_1\ {}_{s_M}\times_{s_M}\ \mathcal{H}_0 \cong \mathcal{H}_1$ and the map (s_{21}, s_{20}) reduces to the surjective submersion s_{21}). See also Section 4 of [21] and Section 4.1 of [5] for an equivalent formulation in terms of 2-categories.   ◆

Remark 2.18 (Strict vs weak Lie 2-groupoids). The object we described above is also known as *strict* Lie 2-groupoid, in order to distinguish it from *weak* Lie 2-groupoids, defined as simplicial manifolds satisfying a horn-filling condition; see, e.g., [45] for the precise definition and further references, and Sections 2-4 of [21] for a comparison between the two notions. In this paper, we will never use the simplicial definition, and by "Lie 2-groupoid" we will always refer to the stricter notion from Definition 2.17. The same holds for Lie-2-groups, recalled in Definition 2.22 below.   ◇

Remark 2.19 (Anchored 2-vector bundles vs Lie 2-groupoids). While every vector bundle $\pi\colon E \to M$ is automatically a Lie groupoid with $s = t = \pi$, an anchored 2-vector bundle $E_1 \xrightarrow{\delta} E_0 \to M$ (Definition 2.11) is not *automatically* a Lie 2-groupoid (Definition 2.17). Indeed, the vector bundles $E_1 \to M$ and $E_0 \to M$ are Lie groupoids (being bundles of abelian groups), but the vector bundle map $\delta\colon E_1 \to E_0$ is simply a Lie groupoid morphism, i.e., it is not the same thing as a Lie groupoid structure, since we lack a splitting of δ to define a groupoid unit. On the other hand, by Lemma 2.12, any anchored 2-vector bundle $E_1 \xrightarrow{\delta} E_0 \to M$ *induces canonically* a VB-groupoid $E_1 \times_M E_0 \rightrightarrows E_0$ over the unit groupoid, which can be therefore seen, in view of Definition 2.17, as a special double Lie groupoid over the unit groupoid, i.e., a special Lie 2-groupoid.   ◇

Example 2.20 (Lie groupoids). Ordinary Lie (1-)groupoids can be viewed as Lie 2-groupoids when $\mathcal{H}_1 = \mathcal{H}_2$.   ◇

Example 2.21 (Lie groups). Ordinary Lie (1-)groups can be viewed as Lie 2-groupoids when $\mathcal{H}_1 = \mathcal{H}_2$ and $\mathcal{H}_0 = \{*\}$. This is of course a special instance (for $\mathcal{H}_0 = \{*\}$) of Example 2.20.   ◇

Notice that, given a Lie 2-groupoid $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$, the condition $\mathcal{H}_0 = \mathcal{H}_1$ is not sufficient to obtain a Lie (1-)groupoid: indeed, the interchange law would not be satisfied unless $\mathcal{H}_2$ is abelian. Furthermore, $\mathcal{H}_2$ would be forced to be a bundle of Lie groups: indeed, t_{21} is a groupoid morphism over $\mathrm{id}_{\mathcal{H}_0}$, hence $s_{20} = s_{10} \circ t_{21}$. But in the case $\mathcal{H}_1 = \mathcal{H}_0$ we have $s_{10} = t_{10}$, therefore $s_{20} = t_{10} \circ t_{21} = t_{20}$. For the same reason, the condition $\mathcal{H}_0 = \mathcal{H}_1 = \{*\}$ is not sufficient to obtain a Lie (1-)group.

Definition 2.22. A *Lie 2-group* is a Lie 2-groupoid where $\mathcal{H}_0$ is a singleton; see, e.g., Section 3.1 of [24] for a more explicit definition.

Given any Lie 2-groupoid $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ and an element $d \in \mathcal{H}_0$, then the pair $\big((\mathcal{H}_2)_d, (\mathcal{H}_1)_d\big)$ of isotropy groups forms a Lie 2-group, called the *isotropy 2-group* at d.
                                                                                    ◆

Notice that the isotropy 2-group at d can be defined replacing $(\mathcal{H}_2)_d$ by the set of elements of $\mathcal{H}_2$ starting and ending at points in $(\mathcal{H}_1)_d$. This is a consequence of the definition of Lie 2-groupoid, more precisely, the facts that

$$t_{20} = t_{10} \circ t_{21} = t_{10} \circ s_{21} \quad \text{and} \quad s_{20} = s_{10} \circ s_{21} = s_{10} \circ t_{21}.$$

Example 2.23 (Crossed modules). A *crossed module of Lie groups* is a pair of Lie groups (H, G) together with a map $d \colon H \to G$ and an action $C \colon G \to \mathrm{Aut}(H)$, satisfying, for any $g \in G$ and $h, h' \in H$,

$$d(C_g h) = g d(h) g^{-1} \quad \text{and} \quad C_{d(h)} h' = h h' h^{-1}.$$

It is well known in the literature that Lie 2-groups and crossed modules of Lie groups are in 1-1 correspondence; in the set-theoretical setting, the first published proof seems to be that of Theorem 1 in [7]. Let us remind the reader of this correspondence, which makes crossed modules a special case of Lie 2-groupoids.

Given a crossed module (H, G, d, C), one defines a Lie 2-groupoid $H \times G \rightrightarrows G \rightrightarrows \{*\}$, where the groupoid structure on $H \times G \rightrightarrows G$ is the action groupoid of the canonical action of H in G given by $(h, g) \mapsto d(h)g$, and the groupoid structure of $H \times G \rightrightarrows \{*\}$ is given by the semidirect product $H \rtimes G$ induced by C.

On the other hand, given a Lie 2-group $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \{*\}$, the pair $(H = \ker(s_{21})$, $G = \mathcal{H}_1)$ is a crossed module, together with $d := t_{21}|_H \colon H \to G$ and the conjugate action C of the Lie subgroup $u(G) \subset (\mathcal{H}_2 \rightrightarrows \{*\})$ on $H \subset \mathcal{H}_2$, i.e., for all $g \in G$ and $h \in H$,

$$C_g h := (u_{21}(g)) \circ_{20} h \circ_{20} (u_{21}(g))^{\tau_{20}}. \qquad \diamond$$

The following example of Lie 2-groupoid will be fundamental in the rest of the paper.

Definition-Example 2.24. Any pair (l, k) of natural numbers defines a Lie 2-groupoid $\mathrm{GL}(l, k)$, which we call the *general linear 2-groupoid of rank* (l, k), given by

$$\mathrm{GL}(l,k)_2 := \left\{ \left(d, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix} \right) \in \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \times \mathrm{GL}(l+k) : \begin{array}{l} (I_l + Jd) \in \mathrm{GL}(l) \\ (I_k + dJ) \in \mathrm{GL}(k) \end{array} \right\}$$

$$\downarrow\downarrow$$

$$\mathrm{GL}(l,k)_1 := \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \times \mathrm{GL}(l) \times \mathrm{GL}(k)$$

$$\downarrow\downarrow$$

$$\mathrm{GL}(l,k)_0 := \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k).$$

- The groupoid structure on $\mathrm{GL}(l, k)_2 \rightrightarrows \mathrm{GL}(l, k)_0$ is the unique one with source and target maps $t_{20}, s_{20} \colon \mathrm{GL}(l, k)_2 \to \mathrm{GL}(l, k)_0$ given by

$$d \xleftarrow{\;t_{20}\;} \left(d, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix} \right) \xrightarrow{\;s_{20}\;} ((I + dJ)B)^{-1} dA,$$

and multiplication map $m_{20} \colon \mathrm{GL}(l, k)_2 {}_{s_{20}}\times_{t_{20}} \mathrm{GL}(l, k)_2 \to \mathrm{GL}(l, k)_2$ given by the matrix multiplication:

$$\left(\left(d, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix} \right), \left(t_{20}\left(d, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix} \right), \begin{pmatrix} A' & J'B' \\ 0 & B' \end{pmatrix} \right) \right)$$

$$\xmapsto{\;m_{20}\;} \left(d, \begin{pmatrix} AA' & (AJ'B^{-1} + J)BB' \\ 0 & BB' \end{pmatrix} \right).$$

- The groupoid structure on $\mathrm{GL}(l,k)_2 \rightrightarrows \mathrm{GL}(l,k)_1$ is the unique one with source and target maps $t_{21}, s_{21}\colon \mathrm{GL}(l,k)_2 \to \mathrm{GL}(l,k)_1$ given by

$$(d,\,A,\,(I+dJ)B) \xleftarrow{\ t_{21}\ } \left(d,\,\begin{pmatrix} A & JB \\ 0 & B \end{pmatrix}\right) \xrightarrow{\ s_{21}\ } (d,\,(I+Jd)^{-1}A,\,B),$$

and multiplication map $m_{21}\colon \mathrm{GL}(l,k)_2 \ {}_{s_{21}}\!\times_{t_{21}} \mathrm{GL}(l,k)_2 \to \mathrm{GL}(l,k)_2$ given by:

$$\left(\left(d,\begin{pmatrix} A & J(I+dJ')B' \\ 0 & (I+dJ')B' \end{pmatrix}\right),\left(d,\begin{pmatrix} (I+Jd)^{-1}A & J'B' \\ 0 & B' \end{pmatrix}\right)\right)$$

$$\xmapsto{\ m_{21}\ } \left(d,\begin{pmatrix} A & (JdJ'+J+J')B' \\ 0 & B' \end{pmatrix}\right).$$

- The groupoid $\mathrm{GL}(l,k)_1 \rightrightarrows \mathrm{GL}(l,k)_0$ is given by the action groupoid of the canonical right action of $\mathrm{GL}(l) \times \mathrm{GL}(k)$ on $\mathrm{GL}(l,k)_0$, given by $d \cdot (A,B) := B^{-1} \circ d \circ A$, i.e.,

$$d \xleftarrow{\ t_{10}\ } (d,A,B) \xmapsto{\ s_{10}\ } B^{-1} \circ d \circ A.$$

Moreover, note that the isotropy Lie 2-group $((\mathrm{GL}(l,k)_2)_d,(\mathrm{GL}(l,k)_1)_d)$ of $\mathrm{GL}(l,k)$ at any element $d \in \mathrm{GL}(l,k)_0$, as in Definition 2.22, recovers precisely the data of the crossed module (K_1, K_0) associated to d defined in Section 3.1 of [42], which was introduced to integrate Lie 2-algebras. ◆

Definition-Example 2.25. Given any 2-graded vector bundle $E = (E_1, E_0)$ over X, its *general linear* 2-*groupoid* $\mathrm{GL}(E)$ is the Lie 2-groupoid defined as follows:

- $\mathrm{GL}(E)_0 = \mathrm{Hom}(E_1, E_0)$;
- an element of $\mathrm{GL}(E)_1$ with source $d_x\colon (E_1)_x \to (E_0)_x$ and target $d_y\colon (E_1)_y \to (E_0)_y$ is a couple of linear isomorphisms $A\colon (E_1)_x \to (E_1)_y$ and $B\colon (E_0)_x \to (E_0)_y$ such that $d_y \circ A = B \circ d_x$;
- an element of $\mathrm{GL}(E)_2$ with source (d_x, d_y, A, B) and target (d_x, d_y, A', B') is given by a map $J\colon (E_0)_x \to (E_1)_y$ such that $J \circ d_x = A - A'$ and $d_y \circ J = B - B'$.

The structure maps of $\mathrm{GL}(E)$ are analogous to those of $\mathrm{GL}(l,k)$ from Definition 2.24; for more details, see Section 5 of [21] and Section 4.3 of [5], where this concept was first introduced (see also Remark 2.26 below). More precisely, in Remark 5.6 of [21] the Lie 2-groupoid $\mathrm{GL}(E)$ was denoted by $\mathrm{GL}'(E)$, in order to distinguish it from a similar notion which allows more general arrows, while in [5] it was called 2-gauge groupoid and denoted by 2-Gau(E). ◆

Remark 2.26 (General linear 2-groupoid and representations up to homotopy). The Lie 2-groupoid $\mathrm{GL}(E)$ was introduced in [21] in order to interpret 2-term representations up to homotopy (RUTHs) of a Lie groupoid $\mathcal{G}$ on a 2-graded vector bundle E as morphisms $\mathcal{G} \to \mathrm{GL}(E)$, in analogy to what happens for ordinary representations of Lie groupoids on a vector bundle. Moreover, $\mathrm{GL}(E)$ is actually equivalent to the gauge 2-groupoid 2-Gau(E) from Section 4.3 of [5], which was used in order to integrate a 2-term RUTH of a Lie algebroid to a 2-term RUTH of a Lie groupoid. The general idea of "representing" RUTHs was first announced in [36]. ◇

Example 2.27. Applying the construction of Example 2.25 to the canonical 2-graded vector space $(\mathbb{R}^l, \mathbb{R}^k)$ over $X = \{*\}$ yields precisely the general Lie 2-groupoid $\mathrm{GL}(l, k)$ from Example 2.24.

On the other hand, the general linear 2-groupoid of the 2-graded vector bundle underlying the canonical anchored 2-vector bundle $E^{(l,k)}$ from Example 2.15 is intuitively $\mathrm{GL}(l, k)$ "enlarged" by multiplying by a suitable amount of copies of $\mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k)$; more precisely,

- $\mathrm{GL}(E^{(l,k)})_0 = \mathrm{GL}(l, k)_0 \times \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k)$;
- $\mathrm{GL}(E^{(l,k)})_i \cong \mathrm{GL}(l, k)_i \times \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \times \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k)$ for $i \in \{1, 2\}$.

It is also interesting to consider the case of a 2-graded vector bundle E where either $E_0 \to X$ or $E_1 \to X$ has rank 0 (Example 2.16); then E is an ordinary vector bundle and its general linear 2-groupoid recovers the general linear (1-)groupoid $\mathrm{GL}(E) \rightrightarrows X$, called sometimes the gauge or Atiyah groupoid of E. ◇

3. From PB-groupoids to VB-groupoids

3.1. 2-Actions

In order to define a PB-groupoid, we discuss first actions of Lie 2-groupoids on Lie groupoids along pairs of smooth maps, which generalise ordinary actions of Lie groupoids on manifolds along smooth maps (see, e.g., Section 5.3 of [38]). For other particular cases of 2-actions present in the literature, we refer to the works cited in Section 3.4.

Definition 3.1. A 2-*action* of a Lie 2-groupoid $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ (Definition 2.17) on a Lie groupoid $P_2 \rightrightarrows P_1$ consists of

- two smooth maps $\mu_2 \colon P_2 \to \mathcal{H}_0$ and $\mu_1 \colon P_1 \to \mathcal{H}_0$;
- a (right) action of $\mathcal{H}_2 \rightrightarrows \mathcal{H}_0$ on P_2 along μ_2;
- a (right) action of $\mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ on P_1 along μ_1;

such that

(1) for any $p \in P_2$, one has $\mu_2(p) = \mu_1(\mathbf{s}(p)) = \mu_1(\mathbf{t}(p))$;

(2) the action maps define a Lie groupoid morphism

$$
\begin{array}{ccc}
P_2 \; {}_{\mu_2}\!\times_{t_{20}} \mathcal{H}_2 & \xrightarrow{\;\Phi\;} & P_2 \\
{\scriptstyle (\mathbf{t},t_{21})} \Big\downarrow\Big\downarrow {\scriptstyle (\mathbf{s},s_{21})} & & {\scriptstyle \mathbf{t}}\Big\downarrow\Big\downarrow{\scriptstyle \mathbf{s}} \\
P_1 \; {}_{\mu_1}\!\times_{t_{10}} \mathcal{H}_1 & \xrightarrow{\;\varphi\;} & P_1,
\end{array}
$$

where on the left column the Lie groupoid structure is given by restricting the product groupoid $P_2 \times \mathcal{H}_2 \rightrightarrows P_1 \times \mathcal{H}_1$. ♦

Note that the submanifold $P_2 \; {}_{\mu_2}\!\times_{t_{20}} \mathcal{H}_2 \subseteq P_2 \times \mathcal{H}_2$ defines a subgroupoid of the product groupoid $P_2 \times \mathcal{H}_2 \rightrightarrows P_1 \times \mathcal{H}_1$ precisely because μ_2 and μ_1 satisfy condition (1). Moreover, the action groupoid $P_2 \; {}_{\mu_2}\!\times_{t_{20}} \mathcal{H}_2 \rightrightarrows P_2$ defines a double Lie groupoid over

the action groupoid $P_1 \; {}_{\mu_1}\times_{t_{10}} \mathcal{H}_1 \rightrightarrows P_1$ precisely because the action maps (Φ, ϕ) form a Lie groupoid morphism.

Example 3.2 (Actions of Lie 2-groupoids on themselves). Any Lie 2-groupoid $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ acts on $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1$, say from the right, with $\mu_2 = s_{20}$ and $\mu_1 = s_{10}$. Indeed, the action is given by the standard Lie groupoid actions of $\mathcal{H}_2 \rightrightarrows \mathcal{H}_0$ on $\mathcal{H}_2$ along μ_2 and of $\mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ on $\mathcal{H}_1$ along μ_1. Condition (1) of Definition 3.1 is automatically fulfilled by the fact that a Lie 2-groupoid is a commutative diagram, while condition (2) follows from the interchange law. $\diamond$

Example 3.3 (Actions of Lie groupoids). If $\mathcal{H}_1 = \mathcal{H}_2$ (Example 2.20), then condition (1) in Definition 3.1 becomes trivial, while condition (2) implies the following special property: every global bisection b of $\mathcal{H}_2 \rightrightarrows \mathcal{H}_0$ defines a Lie groupoid automorphism of $P_2 \rightrightarrows P_1$, given by

$$P_2 \to P_2, \quad p_2 \mapsto p_2 \cdot b(\mu_2(p_2)),$$

$$P_1 \to P_1, \quad p_1 \mapsto p_1 \cdot b(\mu_1(p_1)).$$

Here with bisection we mean a t-section b such that $s \circ b$ is a diffeomorphism.

In particular, if $P_2 = P_1$ (i.e., P is the unit groupoid), Definition 3.1 boils down to an action of a Lie groupoid on a manifold along $\mu_2 = \mu_1$, and the automorphism above becomes the standard diffeomorphism $P \cong P$ associated to any $\mathcal{H}_2$-action on P and any bisection $b \in \mathrm{Bis}(\mathcal{G})$. $\diamond$

Example 3.4 (Actions of Lie groups). If $\mathcal{H}_2 = \mathcal{H}_1 = G$ and $\mathcal{H}_0 = \{*\}$ (Example 2.21), condition (1) in Definition 3.1 becomes trivial, while condition (2) implies the following special property: every element $g \in G$ determines a Lie groupoid isomorphism of $P_2 \rightrightarrows P_1$, simply given by $(\Phi(\cdot, g), \varphi(\cdot, g))$.

In particular, if $P_2 = P_1$ (i.e., if P is the unit groupoid), Definition 3.1 boils down to an action of a Lie group on a manifold, and the isomorphism above becomes the standard diffeomorphism $P \cong P$ associated to any G-action on P and any element $g \in G$.

These are, of course, special instances (for $\mathcal{H}_0 = \{*\}$) of the cases examined in the previous Example 3.3. $\diamond$

Example 3.5 (Actions of Lie 2-groups). If $\mathcal{H}_0 = \{*\}$, i.e., if $(\mathcal{H}_2 = G_2, \mathcal{H}_1 = G_1)$ is a Lie 2-group (Definition 2.22), then Definition 3.1 boils down to an action of a Lie 2-group $G_2 \rightrightarrows G_1$ on a Lie groupoid, i.e., a G_2-action on P_2 and a G_1-action on P_1 such that (Φ, ϕ) is a Lie groupoid morphism (since condition (1) becomes trivial). This is also called a *strict action* in Section 14 of [13]. $\diamond$

Remark 3.6 (Actions of bundles of abelian Lie groups). If $\mathcal{H}_0 = \mathcal{H}_1$ (as briefly discussed after Example 2.20 and 2.21), then Definition 3.1 boils down to an action Φ of the bundle of abelian Lie groups $\mathcal{H}_2 \rightrightarrows \mathcal{H}_0$ on P_2 which is completely encoded by the product $\cdot$ in P_2 and by a map $\phi \colon P_1 \times \mathcal{H}_2 \to P_2$ taking values in the isotropy group bundle of $P_2 \rightrightarrows P_1$.

Indeed, the $\mathcal{H}_1$-action φ is forced to be trivial by the commutativity of the diagram and the fact that $u_{21} = u_{20}$:

$$\varphi(p_1, g_1) = \varphi((\mathbf{s}, s_{21})(\mathbf{u}(p_1), u_{21}(g_1))) = \mathbf{s}(\Phi(\mathbf{u}(p_1), u_{20}(g_1))) = \mathbf{s}(\mathbf{u}(p_1)) = p_1.$$

Moreover,

$$\Phi(p, g) = \Phi\big((p, u_{20}(\mu_2(p))) \circ (\mathbf{u}(\mathbf{s}(p)), g)\big)$$
$$= \Phi(p, u_{20}(\mu_2(p))) \cdot \Phi(\mathbf{u}(\mathbf{s}(p)), g) = p \cdot \phi(\mathbf{s}(p), g),$$

where we set $\phi(p_1, g) := \Phi(\mathbf{u}(p_1), g)$. Last, ϕ takes automatically values in the isotropy group bundle of $P_2 \rightrightarrows P_1$. ◇

Given an action of a Lie group G on a manifold M, each element $g \in G$ gives rise to a diffeomorphism $M \to M$. This phenomenon holds also in more generality, with actions of Lie 2-groupoids on Lie groupoids, provided we replace g with a section and the group $\mathrm{Diff}(M)$ with a suitable group. This fact is discussed in the proposition below, which will not be used elsewhere in the paper.

Proposition 3.7. *Given a Lie 2-groupoid $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ acting on a Lie groupoid $P_2 \rightrightarrows P_1$ with moment map $\mu \colon P_2 \to \mathcal{H}_0$, any bisection $b \colon \mathcal{H}_0 \to \mathcal{H}_2$ gives rise to a diffeomorphism*

$$\phi_b \colon P_2 \to P_2, \quad \mathfrak{p} \mapsto \mathfrak{p} \cdot (b(\mu(\mathfrak{p}))).$$

Moreover,

- *ϕ is an affine morphism between Lie groupoids (see below);*
- *if b takes values in the units of $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1$, then ϕ is an actual Lie groupoid morphism.*

Here with bisection we mean a t_{20}-section b such that $s_{20} \circ b$ is a diffeomorphism. Moreover, we call a smooth map $\phi \colon P_2 \to Q_2$ between Lie groupoids $P_2 \rightrightarrows P_1$ and $Q_2 \rightrightarrows Q_1$ an *affine morphism* if there is a Lie groupoid morphism $\varphi_0 \colon P_2 \to Q_2$ and a t-section $\sigma \colon Q_1 \to Q_2$ such that, for any $\mathfrak{p} \in P_2$, the following equation holds:

$$\phi(\mathfrak{p}) = \varphi_0(\mathfrak{p}) \circ_{Q_2} \sigma(t(\varphi_0(\mathfrak{p}))).$$

Affine-morphisms between Lie groupoids should not be confused with the algebraic notion of affine morphism between schemes; the adjective "affine" here makes reference to affine transformations on a vector space. Indeed, if two vector spaces V and W are seen as Lie groupoids over the point $\{*\}$, an affine morphism is an affine transformation of the kind

$$v \mapsto A(v) + w_0,$$

where $A \colon V \to W$ is a linear map and hence a Lie groupoid morphism, and $* \mapsto w_0 \in W$ is a t-section.

Proof. First notice that ϕ_b is well defined because b is a t_{20}-section. It is smooth since it the composition of smooth maps, and its inverse is the smooth map

$$\mathfrak{p} \mapsto \mathfrak{p} \cdot \tau(b((s_{20} \circ b)^{-1}(\mu(\mathfrak{p})))).$$

Moreover, ϕ_b induces another t_{20}-section

$$b_1 \colon \mathcal{H}_0 \to \mathcal{H}_2, \quad g \mapsto u(t(b(g))),$$

which takes values in the units of $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1$, so that

$$\phi_b(\mathfrak{p}) = \mathfrak{p} \cdot b(\mu(\mathfrak{p})) = (\mathfrak{p} \circ_{P_2} u(s(\mathfrak{p}))) \cdot (b_1(\mu(\mathfrak{p})) \circ_{m_{21}} b(\mu(\mathfrak{p})))$$
$$= (\mathfrak{p} \cdot b_1(\mu(\mathfrak{p}))) \circ_{P_2} (u(s(\mathfrak{p})) \cdot b(\mu(\mathfrak{p}))) = (\phi_{b_1}(\mathfrak{p})) \circ_{P_2} (u(s(\mathfrak{p})) \cdot b(\mu(s(\mathfrak{p})))).$$

Then

$$\sigma : P_1 \to P_2, \quad p \mapsto \phi_b(u(p)) = u(p) \cdot b(\mu(p)),$$

is an s-section, which can be interpreted as a "translation" between ϕ_{b_1} and the map ϕ_b. We conclude that ϕ_b is an affine morphism.

Last, if b takes values in $\mathcal{H}_1$, viewed as a submanifold of $\mathcal{H}_2$, then it is a Lie groupoid morphism:

$$\phi_b(\mathfrak{p} \circ_{P_2} \mathfrak{q}) = (\mathfrak{p} \circ_{P_2} \mathfrak{q}) \cdot b(\mu(\mathfrak{p} \circ_{P_2} \mathfrak{q})) = (\mathfrak{p} \circ_{P_2} \mathfrak{q}) \cdot (b(\mu(\mathfrak{p})) \circ_{m_{21}} b(\mu(\mathfrak{q})))$$
$$= \phi_b(\mathfrak{p}) \circ_{P_2} \phi_b(\mathfrak{q}). \qquad \blacksquare$$

Proposition 3.7 actually combines the morphisms associated to any t_{20}-section discussed in Examples 3.6 and 3.3.

3.2. 2-Representations

We now introduce two (related) notions of 2-representation of Lie 2-groupoids: one for 2-graded vector bundles $(E_1, E_0) \to X$, and another one for anchored 2-vector bundles $E_1 \xrightarrow{\delta} E_0 \to X$. The first one coincides with Definition 4.18 in [5], while, as far we are aware, the second one is not present anywhere in the literature.

In order to compare them, it will be useful to notice that a section

$$\sigma : X \to \mathrm{GL}(E_1, E_0)_0 = \mathrm{Hom}(E_1, E_0)$$

is canonically equivalent to an anchor map $\delta_\sigma \colon E_1 \to E_0$, via the formula

$$(\delta_\sigma)_x : (E_1)_x \to (E_0)_x, \quad e \mapsto \sigma(x)(e).$$

Definition 3.8. A 2-*graded representation of a Lie* 2-*groupoid* $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ (Definition 2.17) on a 2-graded vector bundle (E_1, E_0) over X (Definition 2.11) is a morphism ϕ from the Lie 2-groupoid $\mathcal{H}_i$ to the general Lie 2-groupoid $\mathrm{GL}(E_1, E_0)_i$ (Definition 2.25):

$$
\begin{array}{ccc}
\mathcal{H}_2 & \xrightarrow{\phi_2} & \mathrm{GL}(E_1, E_0)_2 \\
\Downarrow\Downarrow & & \Downarrow\Downarrow \\
\mathcal{H}_1 & \xrightarrow{\phi_1} & \mathrm{GL}(E_1, E_0)_1 \\
\Downarrow\Downarrow & & \Downarrow\Downarrow \\
\mathcal{H}_0 & \xrightarrow{\phi_0} & \mathrm{GL}(E_1, E_0)_0.
\end{array}
$$

A 2-*anchored representation of a Lie* 2-*groupoid* $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows X$ on an anchored 2-vector bundle $E_1 \xrightarrow{\delta} E_0 \to X$ is a morphism ϕ from the Lie 2-groupoid $\mathcal{H}_i$ to the general Lie 2-groupoid $\mathrm{GL}(E_1, E_0)_i$ which takes values in its restriction to the image of the

section σ corresponding to the anchor δ:

$$
\begin{array}{ccccc}
\mathcal{H}_2 & \xrightarrow{\phi_2} & \mathrm{GL}(E_1, E_0)_2|_{\sigma(X)} & \hookrightarrow & \mathrm{GL}(E_1, E_0)_2 \\
\big\Downarrow & & \big\Downarrow & & \big\Downarrow \\
\mathcal{H}_1 & \xrightarrow{\phi_1} & \mathrm{GL}(E_1, E_0)_1|_{\sigma(X)} & \hookrightarrow & \mathrm{GL}(E_1, E_0)_1 \\
\big\Downarrow & & \big\Downarrow & & \big\Downarrow \\
X & \xrightarrow{\ \ \sigma\ \ } & \sigma(X) & \hookrightarrow & \mathrm{GL}(E_1, E_0)_0.
\end{array}
\qquad \blacklozenge
$$

These two notions can be related by the following example.

Example 3.9. Let (E_1, E_0) be a 2-graded vector bundle over a manifold X. There are two canonical 2-representations of the Lie 2-groupoid $\mathrm{GL}(E_1, E_0)$:

- a 2-graded representation on (E_1, E_0), given by the identity morphisms;
- a 2-anchored representation on its associated anchored 2-vector bundle

$$
E_1 \times_X \mathrm{Hom}(E_1, E_0) \to E_0 \times_X \mathrm{Hom}(E_1, E_0) \to \mathrm{Hom}(E_1, E_0)
$$

(see Definition 2.15), given by the identity morphisms.

Notice that, while the first claim is trivial, the second one relies on the following fact: there is a canonical Lie 2-groupoid isomorphism between $\mathrm{GL}(E_1, E_0)$ and the restriction of $\mathrm{GL}(E_1 \times_X \mathrm{Hom}(E_1, E_0), E_0 \times_X \mathrm{Hom}(E_1, E_0))$ to $\sigma(\mathrm{Hom}(E_1, E_0))$, where

$$
\sigma : \mathrm{Hom}(E_1, E_0) \to \mathrm{GL}(E_1 \times \mathrm{Hom}(E_1, E_0), E_0 \times \mathrm{Hom}(E_1, E_0))_0
$$
$$
= \mathrm{Hom}(E_1, E_0) \times_X \mathrm{Hom}(E_1, E_0)
$$

is the section induced by the anchor map $\delta \colon E_1 \times_X \mathrm{Hom}(E_1, E_0) \to E_0 \times_X \mathrm{Hom}(E_1, E_0)$, i.e., the diagonal of $\mathrm{Hom}(E_1, E_0)$.

This example is a direct generalisation of the fact that any vector space V carries a canonical representation of the Lie group $\mathrm{GL}(V)$, and any vector bundle $E \to X$ carries a canonical representation of the Lie groupoid $\mathrm{GL}(E) \rightrightarrows X$. $\diamond$

The following proposition generalises the standard characterisation of representations of Lie group(oid)s as linear actions. Notice however that it holds only for representations on anchored 2-vector bundles; for 2-graded vector bundles the only approach is the one described above as morphism in $\mathrm{GL}(E_1, E_0)$.

Proposition 3.10. *Let $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows X$ be a Lie 2-groupoid and $E \rightrightarrows E_0$ a VB-groupoid over the unit groupoid $X \rightrightarrows X$. Recall that, by Lemma 2.12, there is a 2-anchored vector bundle (E_1, E_0) such that $E = E_1 \times_X E_0$ and the anchor δ of (E_1, E_0) is the core-anchor of E. Then there is a 1-1 correspondence between*

- *2-anchored representations (ϕ_2, ϕ_1, ϕ_0) of $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows X$ on $E_1 \xrightarrow{\delta} E_0 \to X$, in the sense of Definition 3.8;*
- *2-actions (in the sense of Definition 3.1) of $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows X$ on the Lie groupoid $E \rightrightarrows E_0$ which are linear.*

Proof. Given $\phi = (\phi_2, \phi_1, \phi_0)$, let $(\hat{\phi}_2, \hat{\phi}_1)$ be the compositions of ϕ_2 with the projection to $\mathrm{GL}(E_1 \times_X E_0)$, and that of ϕ_1 with the projection to $\mathrm{GL}(E_0)$, respectively. Then ϕ induces a Lie 2-groupoid action on $E \rightrightarrows E_0$, given by a linear action of $\mathcal{H}_2$ on $E = E_1 \times_X E_0$, namely

$$\mathcal{H}_2 \,_{t_{20}}\times_X (E_1 \times_X E_0) \to (E_1 \times_X E_0), \quad (g, v) \mapsto \hat{\phi}_2(g)v,$$

and a linear action of $\mathcal{H}_1$ on E_0, namely

$$\mathcal{H}_1 \,_{t_{10}}\times_X E_0 \to E_0, \quad (g, v) \mapsto \hat{\phi}_1(g)v.$$

The opposite direction is analogous, but in the backward direction. ∎

Example 3.11 (Representations of Lie groupoids). When $\mathcal{H}_2 = \mathcal{H}_1 = \mathcal{G}$ is a Lie groupoid over X (Example 2.20), then a 2-anchored representation boils down to a pair of ordinary representations of the Lie groupoid $\mathcal{G} \rightrightarrows X$ on the vector bundles $E_1 \times_X E_0 \to X$ and $E_0 \to X$.

As a special case, when $X = \{*\}$ and $\mathcal{H}_2 = \mathcal{H}_1 = G$ is a Lie group (Example 2.21), a 2-anchored representation boils down to a pair of ordinary representations of the Lie group G, one on the vector space E_0 and one on the cartesian product $E_1 \times E_0$. In particular, if $E_0 = 0$ we have a single G-representation on E_1, and if $E_1 = 0$ we have two identical G-representations on E_0. ◊

Example 3.12 (Representations of Lie 2-groups). When $X = \{*\}$, i.e., when $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1$ is a Lie 2-group, then a 2-anchored representation boils down to an ordinary representation of a Lie 2-group on a pair of vector spaces; see, e.g., Theorem 3.5 in [42], which fits in the general categorical setting described in Section 2.2.2 of [2]. ◊

3.3. Principal 2-actions

Definition 3.13. A *principal* 2-*action* of a Lie 2-groupoid $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ on a Lie groupoid $P_2 \rightrightarrows P_1$ is a 2-action (Definition 3.1) such that both the $\mathcal{H}_2$-action on P_2 and the $\mathcal{H}_1$ on P_1 are free and proper. ♦

From the standard theory of principal groupoid bundles, since the $\mathcal{H}_2$-action on P_2 is free and proper, its quotient $M_2 := P_2/\mathcal{H}_2$ is a smooth manifold and the projection $\pi_2 \colon P_2 \to M_2$ is a surjective submersion; similarly with $M_1 := P_1/\mathcal{H}_1$ and $\pi_1 \colon P_1 \to M_1$. Moreover, one has the following consequence.

Proposition 3.14. *Any 2-action of $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ on $P_2 \rightrightarrows P_1$ is principal if and only if the induced Lie groupoid morphism (Φ, ϕ), as in the diagram below, is an isomorphism:*

$$
\begin{array}{ccc}
P_2 \,_{\mu_2}\times_t \mathcal{H}_2 & \xrightarrow{\ \Phi\ } & P_2 \times_{\pi_2} P_2 \\
\tilde{t} \downarrow\downarrow \tilde{s} & & \downarrow\downarrow \\
P_1 \,_{\mu_1}\times_t \mathcal{H}_1 & \xrightarrow{\ \phi\ } & P_1 \times_{\pi_1} P_1.
\end{array}
$$

Proof. The second condition of Definition 3.1 implies that (Φ, ϕ) is a Lie groupoid morphism, while the principality of each action is equivalent to Φ and ϕ being diffeomorphisms. ∎

Proposition 3.15. *Any principal 2-groupoid action of $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ on $P_2 \rightrightarrows P_1$ induces a unique Lie groupoid structure on $P_2/\mathcal{H}_2 \rightrightarrows P_1/\mathcal{H}_1$ such that the quotient map from $P_2 \rightrightarrows P_1$ to $P_2/\mathcal{H}_2 \rightrightarrows P_1/\mathcal{H}_1$ is a fibration of Lie groupoids.*

Proof. We will use the following result from Theorem 2.4.6 in [35]. Let $\mathcal{R} \subset P_2 \times P_2$ be the graph of an equivalence relation $\sim_2$ in P_2 and let $R \subset P_1 \times P_1$ be the graph of an equivalence relation $\sim_1$ in P_1. If

(1) $\mathcal{R} \rightrightarrows R$ is a Lie subgroupoid of the cartesian product $P_2 \times P_2 \rightrightarrows P_1 \times P_1$;

(2) the map $\mathcal{R} \to P_2 \,_{\mathbf{s}}\times_{\mathrm{pr}_1} R, (p,q) \mapsto (p, \mathbf{s}(p), \mathbf{s}(q))$ is a surjective submersion;

then there is a unique Lie groupoid structure in $(P_2/\sim_2) \rightrightarrows (P_1/\sim_1)$, with the quotient map being a fibration. If $\sim_2$ and $\sim_1$ are given by the quotients by the $\mathcal{H}_2$ and $\mathcal{H}_1$-actions, then condition (1) is satisfied by Proposition 3.14, while condition (2) is equivalent to the map

$$P_2 \,_{\mu_2}\times_{\mathbf{t}} \to P_2 \,_{\mu_1 \circ \mathbf{s}}\times_{\mathbf{t}} \mathcal{H}_1, \quad (p,g) \mapsto (p, \mathbf{s}(g)),$$

being a surjective submersion, which is clearly true.   ■

Example 3.16. Let $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ be a Lie 2-groupoid. Its 2-action on itself from Example 3.2 is principal, since the (1-)actions of $\mathcal{H}_2 \rightrightarrows \mathcal{H}_0$ and of $\mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ on themselves are principal. The quotient of both groupoid actions is isomorphic to $\mathcal{H}_0$, so that the quotient of the principal 2-groupoid action is the unit groupoid $\mathcal{H}_0 \rightrightarrows \mathcal{H}_0$.   ◇

3.4. PB-groupoids

Definition 3.17. A *PB-groupoid* with structural 2-groupoid $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ is a commutative diagram

$$
\begin{array}{ccc}
P_{\mathcal{G}} & \xrightarrow{\ \Pi_{\mathcal{G}}\ } & \mathcal{G} \\
{\scriptstyle \mathbf{t}}\big\downarrow\big\downarrow{\scriptstyle \mathbf{s}} & & {\scriptstyle t}\big\downarrow\big\downarrow{\scriptstyle s} \\
P_M & \xrightarrow{\ \Pi_M\ } & M
\end{array}
$$

such that

- $P_{\mathcal{G}} \rightrightarrows P_M$ and $\mathcal{G} \rightrightarrows M$ are Lie groupoids;
- $\Pi_{\mathcal{G}}$ and Π_M are surjective submersions;

together with a principal 2-action (Definition 3.13) of $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ on $P_{\mathcal{G}} \rightrightarrows P_M$ and a Lie groupoid isomorphism φ between $(P_{\mathcal{G}}/\mathcal{H}_2) \rightrightarrows (P_M/\mathcal{H}_1)$ (see Proposition 3.15) and $\mathcal{G} \rightrightarrows M$ such that $\Pi_{\mathcal{G}} = \varphi \circ \mathrm{pr}$, where $\mathrm{pr}: P_{\mathcal{G}} \to P_{\mathcal{G}}/\mathcal{H}_2$ denotes the quotient map.   ♦

Notice that $\Pi_{\mathcal{G}}$ is a Lie groupoid fibration by Proposition 3.15, hence, in particular, it is a Lie groupoid morphism.

Example 3.18 (PB-groupoids with structural Lie 2-groups). If the structural Lie 2-groupoid is a Lie 2-group (Definition 2.22), then a PB-groupoid in the sense of Definition 3.17 boils down to a *principal bundle groupoid* in the sense of Definition 1.15 in [23] and to a *principal 2-bundle over a Lie groupoid* in the sense of Definition 3.4 in [14] and of Definition 5.9 in [27]; in this last work, the definition is formulated via the equivalent

condition from Proposition 3.14. A version of these objects involving differentiable stacks was considered in Definition 3.22 of [11].

Principal 2-*bundles* as in Definition 6.1.5 of [39] consist of a Lie 2-group $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1$ acting on a Lie groupoid $P_\mathcal{G} \rightrightarrows P_M$, together with a map $P_\mathcal{G} \to Y$ such that the Lie groupoids $\mathcal{H}_2 \times P_\mathcal{G} \rightrightarrows P_\mathcal{G}$ and $P_\mathcal{G} \times_Y P_\mathcal{G} \rightrightarrows P_\mathcal{G}$ are Morita equivalent. As proved in Proposition 1.32 of [23], these objects are the same things as PB-groupoids with structural Lie 2-group $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1$ over fibred pair groupoid $\mathcal{G} = M \times_Y M \rightrightarrows M$, for a submersion $M \to Y$.

Last, note that *semi-strict principal* 2-*bundles* from Definition 2.9 in [44] are defined in terms of local trivialisations and cannot be directly comparable with (particular cases of) our Definition 2.22 since they involve actions of Lie 2-groups on *smooth* 2-*spaces* (objects more general than a Lie groupoid). $\Diamond$

Example 3.19 (PB-groupoids with structural Lie groupoids). If the structural Lie 2-groupoid is a Lie groupoid, with $\mathcal{H}_2 = \mathcal{H}_1$ (Example 2.20), then a PB-groupoid in the sense of Definition 3.17 yields a pair $(P_\mathcal{G} \to \mathcal{G}, P_M \to M)$ of standard $(\mathcal{H}_2 \rightrightarrows \mathcal{H}_0)$-principal bundles (in the sense of, e.g., Section 5.7 in [38], Definition 2.1 in [41], Definition 3.1 in [28] or Section 2.2 in [26]). In particular, when $\mathcal{H}_2 \rightrightarrows \mathcal{H}_0$ is a bundle of Lie groups and $\mathcal{G} = M$, we recover *generalized principal bundles* as in Theorem 2.1 of [12]. $\Diamond$

Example 3.20 (PB-groupoids with structural Lie groups). If the structural Lie 2-groupoid is of the form $H \rightrightarrows H \rightrightarrows \{*\}$, then a PB-groupoid boils down to a *principal bundle over a groupoid* in the sense of Definition 2.33 in [32], to a *PBG-groupoid* in the sense of Definition 2.1 in [34] and to a *G-groupoid* in the sense of Definition 4.4 in [8]. This is a particular case, for $\mathcal{H}_0 = \{*\}$, of Example 3.19.

Last, note that *G-principal groupoids* from Definition 2.2 in [22] cannot be directly comparable with (particular cases of) our Definition 3.17, since they involve principal actions of Lie groups on the total space of Lie groupoids, which induce non-principal actions on the base space. $\Diamond$

Example 3.21 (Lie 2-groupoids as PB-groupoids). Let $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows \mathcal{H}_0$ be a Lie 2-groupoid. Then $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1$ defines a PB-groupoid over the unit groupoid $\mathcal{H}_0 \rightrightarrows \mathcal{H}_0$, with $\Pi_\mathcal{G}$ and Π_M given, respectively, by t_{20} and t_{10}, and with the principal action on itself (see Example 3.16). $\Diamond$

3.5. First half of the correspondence

Let $\mathcal{H} \rightrightarrows X$ be a Lie groupoid. Recall that, given a principal $\mathcal{H}$-bundle $P \to M$, with moment map $\mu: P \to X$, and a $\mathcal{H}$-representation $E \to X$, then its *associated bundle* is the quotient

$$P[E] := (P \times_X E)/\mathcal{H},$$

which is a vector bundle over M with rank equal to the rank of E. This construction generalises the standard associated bundles for principal groups bundles, and can in turn be generalised to PB-groupoids.

Theorem 3.22. *Let $P_\mathcal{G} \rightrightarrows P_M$ be a PB-groupoid with structural Lie 2-groupoid $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows X$ (Definition 3.17), and $E_1 \to E_0 \to X$ a 2-anchored representation of $\mathcal{H}$ (Defin-*

ition 3.8), *which we view as a linear action of* $\mathcal{H}$ *on* $E_1 \times_X E_0 \rightrightarrows E_0$, *as in Proposition* 3.10. *Then the quotients*

$$P[E]_2 := (P_{\mathcal{G}} \times_X (E_1 \times_X E_0))/\mathcal{H}_2 \quad and \quad P[E]_1 := (P_M \times_X E_0)/\mathcal{H}_1$$

define a VB-groupoid

$$\begin{array}{ccc} P[E]_2 & \xrightarrow{\pi_{\mathcal{G}}} & \mathcal{G} \\ \tilde{t} \downarrow\downarrow \tilde{s} & & t \downarrow\downarrow s \\ P[E]_1 & \xrightarrow{\pi_M} & M \end{array}$$

with rank $(\mathrm{rank}(E_1), \mathrm{rank}(E_0))$.

In analogy with the classical construction for principal bundles with structural groups/groupoids, $P[E]_2 \rightrightarrows P[E]_1$ is called the *associated bundle*.

Proof. It is known that $P[E]_2 \to \mathcal{G}$ and $P[E]_1 \to M$ are vector bundles of ranks, respectively, equal to the ranks of $E_1 \times_M E_0 \to X$ and $E_0 \to X$. The groupoid structure on $P[E]_2 \rightrightarrows P[E]_1$ is induced by that of the product Lie groupoid

$$P_{\mathcal{G}} \times_X (E_1 \times_X E_0) \rightrightarrows (P_M \times_X E_0),$$

the free and proper action of $\mathcal{H}_2$ and of $\mathcal{H}_1$, and Proposition 3.15. ∎

4. From VB-groupoids to PB-groupoids

4.1. Frames adapted to a VB-groupoid

The entire frame bundle of a VB-groupoid is too big to support the structure of a PB-groupoid (or even only of a Lie groupoid). In this section, we are going to identify the class of frames which are "adapted" to the VB-groupoid structure, and which will give rise to a PB-groupoid.

We write here, for later use, the following formula of the "right-translation" by elements $g \in \mathcal{G}$ in an arbitrary VB-groupoid $E_{\mathcal{G}} \rightrightarrows E_M$:

(4.1) $$\widetilde{R}_g := \widetilde{m}(\cdot, 0_g) : \ker(\tilde{s})|_{s^{-1}(t(g))} \to \ker(\tilde{s})|_{s^{-1}(s(g))}.$$

This is simply the vector bundle isomorphism which generalises the differential of the right translation $R_g : s^{-1}(t(g)) \to s^{-1}(s(g))$ for the tangent VB-groupoid $T\mathcal{G} \rightrightarrows TM$.

The main ingredient we will use to define adapted frames (and to prove that they have a groupoid structure) is the notion of fat groupoid, introduced in Section 8 of [25] and briefly recalled below.

Definition 4.1. Let $E_{\mathcal{G}} \rightrightarrows E_M$ be a VB-groupoid of rank (l, k) over $\mathcal{G} \rightrightarrows M$. Its *fat groupoid* $\tilde{\mathcal{G}}(E_{\mathcal{G}})$ consists of all the pairs (g, H_g), with $g \in \mathcal{G}$ and $H_g \subseteq (E_{\mathcal{G}})_g$ any (necessarily k-dimensional) subspace such that

$$H_g \oplus \ker(\tilde{s}_g) = (E_{\mathcal{G}})_g \quad and \quad H_g \oplus \ker(\tilde{t}_g) = (E_{\mathcal{G}})_g. \quad \blacklozenge$$

The set $\tilde{\mathcal{G}}(E_{\mathcal{G}})$ has a natural Lie groupoid structure over M, with structure maps

$$s(g, H_g) = s(g), \quad t(g, H_g) = t(g),$$
$$m((g, H_g), (h, H_h))$$
$$= \left(gh, \tilde{m}(H_g, H_h) := \{\tilde{m}(v_g, v_h) \mid v_g \in H_g, v_h \in H_h, \tilde{s}(v_g) = \tilde{t}(v_h)\}\right),$$
$$u(x) = (1_x, H_{1_x} := \tilde{u}(E_M)_x), \quad i(g, H_g) = (g^{-1}, H_{g^{-1}} := \tilde{\tau}(H_g)).$$

Moreover, there is a natural representation of $\tilde{\mathcal{G}}(E_{\mathcal{G}}) \rightrightarrows M$ on the vector bundle $E_M \to M$ given by

$$(4.1) \qquad\qquad (g, H_g) \cdot e_x := \tilde{t}_g (\tilde{s}_g |_{H_g})^{-1}(e_x),$$

and a similar one on $C \to M$, given by

$$(4.2) \qquad (g, H_g) \cdot c_x := \alpha_g \circ_{E_{\mathcal{G}}} c_x \circ_{E_{\mathcal{G}}} 0_{g^{-1}} = \tilde{m}(\alpha_g, \tilde{R}_{g^{-1}}(c_x)),$$

for $\alpha \in H_g$ such that $\tilde{s}_g(\alpha) = \tilde{t}_{1_x}(c_x)$.

Example 4.2 (Trivial base and trivial core). If $l = 0$, then the Lie groupoid $\tilde{\mathcal{G}}(E_{\mathcal{G}}) \rightrightarrows M$ is isomorphic to $\mathcal{G} \rightrightarrows M$: indeed, since $E_{\mathcal{G}}$ has rank $l + k$ and the spaces H_g are k-dimensional, we are forced to impose $H_g = (E_{\mathcal{G}})_g$. Note also that the representation (4.1) coincides precisely with the representation of $\mathcal{G}$ on E_M from Example 2.7.

Similarly, if $k = 0$, then $\tilde{\mathcal{G}}(E_{\mathcal{G}}) \cong \mathcal{G}$ as well, since each H_g is forced to be the zero vector space. Also in this case, the representation (4.2) boils down to the representation of $\mathcal{G}$ on C from Example 2.8. $\diamond$

We can finally define the frames adapted to a VB-groupoid.

Definition 4.3. Let $E_{\mathcal{G}} \rightrightarrows E_M$ be a VB-groupoid of rank (l, k) over $\mathcal{G} \rightrightarrows M$. A frame $\phi_g \in \mathrm{Fr}(E_{\mathcal{G}})$ is called an *s-bisection frame* if the following two conditions holds:

(1) $\phi_g |_{\mathbb{R}^l \times \{0\}}$ takes values in $\ker(\tilde{s}_g)$;

(2) the image $H_g \subseteq (E_{\mathcal{G}})_g$ of $\phi_g |_{\{0\} \times \mathbb{R}^k}$ is an element of the fat groupoid $\tilde{\mathcal{G}}(E_{\mathcal{G}})$ (Definition 4.1).

The collection

$$\mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}} := \{\phi_g : \mathbb{R}^{l+k} \to (E_{\mathcal{G}})_g \mid \phi_g \ s\text{-bisection frame}\} \subseteq \mathrm{Fr}(E_{\mathcal{G}}).$$

of s-bisection frames of $E_{\mathcal{G}}$ is called the *s-bisection frame bundle* of $E_{\mathcal{G}}$. ♦

Remark 4.4. The term "s-bisection frames" in Definition 4.3 is motivated by the following reasons.

- The "bisection" part refers to the second condition and points to an analogy with the definition of bisections. Indeed, a bisection of a Lie groupoid $\mathcal{G}$ can be equivalently described as a submanifold $\Sigma \subseteq T\mathcal{G}$ such that $(g, T_g\Sigma)$ belongs to the fat groupoid $\tilde{\mathcal{G}}(E_{\mathcal{G}})$ for every $g \in \Sigma$.

- The symbol s refers to the first condition. Of course, the choice of $\ker(\tilde{s})$ instead of $\ker(\tilde{t})$ is purely a convention; indeed, later on we will also use t-bisection frames (see Definition 5.10).

Notice also that the first condition in Definition 4.3 implies that H_g is automatically transverse to $\ker(\tilde{s})$; therefore, condition (2) can be simplified to H_g being transverse to $\ker(\tilde{t})$. ◊

The key feature of s-bisection frames is the fact that they induce two frames of the base vector bundle $E_M \to M$ (Lemma 4.5) and two frames of the core $C \to M$ (Lemmas 4.6 and 4.8).

Lemma 4.5. *Let $E_{\mathcal{G}} \rightrightarrows E_M$ be a VB-groupoid of rank (l, k) over $\mathcal{G} \rightrightarrows M$ and let $\varphi_g : \mathbb{R}^k \to (E_{\mathcal{G}})_g$ be an injective linear map. The following two properties are equivalent:*

(1) *the image $\mathrm{Im}(\varphi_g)$ is transverse to $\ker(\tilde{s})$, i.e., $\ker(\tilde{s}_g) \oplus \mathrm{Im}(\varphi_g) = (E_{\mathcal{G}})_g$;*

(2) *the composition*

$$\tilde{s} \circ \varphi_g : \mathbb{R}^k \to (E_M)_{s(g)}$$

is a frame of $E_M \to M$.

The same statement holds replacing $\tilde{s}$ with $\tilde{t}$ everywhere.

In particular, any s-bisection frame $\phi_g : \mathbb{R}^{l+k} \to E_{\mathcal{G}}$ induces the following frames of $E_M \to M$:

$$\mathbf{s}(\phi_g)^b : \mathbb{R}^k \to (E_M)_{s(g)}, \quad \mathbf{s}(\phi_g)^b(v) := \tilde{s}(\phi_g(0, v)),$$
$$\mathbf{t}(\phi_g)^b : \mathbb{R}^k \to (E_M)_{t(g)}, \quad \mathbf{t}(\phi_g)^b(v) := \tilde{t}(\phi_g(0, v)).$$

Proof. $(1) \Rightarrow (2)$. Take $v_1, v_2 \in \mathbb{R}^k$ such that $\tilde{s}(\varphi_g(v_1)) = \tilde{s}(\varphi_g(v_2))$; then the hypothesis yields

$$\varphi_g(v_1 - v_2) \in \mathrm{Im}(\varphi_g) \cap \ker(\tilde{s}_g) = \{0\}.$$

As ϕ_g is injective, this implies that $v_1 = v_2$, i.e., $\tilde{s} \circ \varphi_g$ is injective. Since $\mathrm{rank}(E_M) = k = \dim(\mathrm{Im}(\phi_g))$, we conclude that $\tilde{s} \circ \phi_g$ is a linear isomorphism, i.e., a frame of $E_M \to M$.

$(2) \Rightarrow (1)$. Consider any element $\varphi_g(v)$ in the intersection $\ker(\tilde{s}_g) \cap \mathrm{Im}(\varphi_g)$. Since $\tilde{s} \circ \phi_g$ is injective, the condition $\tilde{s}(\varphi_g(v)) = 0$ implies $v = 0$, that is, the intersection $\ker(\tilde{s}_g) \cap \mathrm{Im}(\varphi_g)$ contains only the zero vector. It follows that

$$\dim(\ker(\tilde{s}_g) \oplus \mathrm{Im}(\varphi_g)) = \dim(\ker(\tilde{s}_g)) + \dim(\mathrm{Im}(\varphi_g)) = l + k = \mathrm{rank}(E_{\mathcal{G}}),$$

hence $\ker(\tilde{s}_g) \oplus \mathrm{Im}(\varphi_g) = (E_{\mathcal{G}})_g$. ∎

Lemma 4.6. *Let $E_{\mathcal{G}} \rightrightarrows E_M$ be a VB-groupoid of rank (l, k) over $\mathcal{G} \rightrightarrows M$ and let $\varphi_g : \mathbb{R}^l \to (E_{\mathcal{G}})_g$ be an injective linear map. The following two properties are equivalent:*

(1) *φ_g takes values in $\ker(\tilde{s}_g)$;*

(2) *the composition*

$$\widetilde{R}_{g^{-1}} \circ \varphi_g : \mathbb{R}^l \to C_{t(g)} \subseteq (E_{\mathcal{G}})_{1_{t(g)}},$$

with $\widetilde{R}$ given by (4.1), is well-defined and is a frame of $C \to M$.

The same holds replacing $\tilde{s}$ with $\tilde{t}$ everywhere.

In particular, any s-bisection frame $\phi_g : \mathbb{R}^{l+k} \to E_{\mathcal{G}}$ induces the following frame of $C \to M$:

$$\mathbf{t}(\phi_g)^c : \mathbb{R}^l \to C_{t(g)}, \quad \mathbf{t}(\phi_g)^b(v) := \widetilde{R}_{g^{-1}}(\phi_g(w, 0)).$$

Proof. $(1) \Rightarrow (2)$. Since φ_g is an injective linear map, taking values in the l-dimensional space $\ker(\tilde{s}_g)$, it is a linear isomorphism. Then its composition with the linear isomorphism $\widetilde{R}_{g^{-1}}$ is well defined and is a frame of $C \to M$.

$(2) \Rightarrow (1)$. Since the composition $\widetilde{R}_{g^{-1}} \circ \varphi_g$ is well defined, φ_g must necessarily takes values in $\ker(\tilde{s}_g) \subseteq (E_{\mathcal{G}})_g$. ∎

Definition 4.7. For any s-bisection frame $\phi_g \in \mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$, we denote by

$$d_{\phi_g} : \mathbb{R}^l \to \mathbb{R}^k$$

the unique linear map satisfying the following diagram:

$$\begin{array}{ccc}
C_{t(g)} & \xrightarrow{\ \rho_{t(g)}\ } & (E_M)_{t(g)} \\
{\scriptstyle \mathbf{t}(\phi_g)^c} \Big\uparrow \cong & & \cong \Big\uparrow {\scriptstyle \mathbf{t}(\phi_g)^b} \\
\mathbb{R}^l & \dashrightarrow{\ d_{\phi_g}\ } & \mathbb{R}^k.
\end{array}$$

Equivalently, since $\mathbf{t}(\phi_g)^b$ is an isomorphism by Lemma 4.5, d_{ϕ_g} is simply the composition of linear maps

$$d_{\phi_g} := (\mathbf{t}(\phi_g)^b)^{-1} \circ \rho_{t(g)} \circ \mathbf{t}(\phi_g)^c. \qquad \blacklozenge$$

In particular, d_{ϕ_g} vanishes if and only if $\phi_g(0, w) \in \ker(\tilde{t}_g)$ for every $w \in \mathbb{R}^l$: the map d_{ϕ_g} measures then how non-zero the element $\tilde{t}(\phi_g(w, 0))$ can be.

Lemma 4.8. *Any s-bisection frame $\phi_g : \mathbb{R}^{l+k} \to (E_{\mathcal{G}})_g$ induces the following frame of $C \to M$:*

$$\mathbf{s}(\phi_g)^c : \mathbb{R}^l \to C_{s(g)}, \quad \mathbf{s}(\phi_g)^c(w) := \widetilde{R}_g(\tilde{\tau}(\phi_g(-w, d_{\phi_g}(w)))),$$

with $\widetilde{R}_g$ given by equation (4.1) and d_{ϕ_g} from Definition 4.7.

Proof. We note first that the vector $\mathbf{s}(\phi_g)^c(w) \in (E_{\mathcal{G}})_g$ takes values in $\ker(\tilde{s}_{1_{s(g)}}) = C_{s(g)}$. Indeed, Lemma 4.10 yields $\phi_g(-w, d_{\phi_g}(w)) \in \ker(\tilde{t}_g)$, and after applying $\tilde{\tau}$ and $\widetilde{R}_g$ one obtains an element in $\ker(\tilde{s}_{1_{s(g)}})$.

Moreover, we claim that $\mathbf{s}(\phi_g)^c$ is a linear isomorphism. Since C has rank l, it is enough to check its injectivity. If $\mathbf{s}(\phi_g)^c(w) = 0$, then condition (1) of Definition 4.3 implies that

$$0 = \tilde{t}(\mathbf{s}(\phi_g)^c(w)) = \tilde{t}(\tilde{\tau}(\phi_g(-w, d_\phi(w))) = \tilde{s}(\phi_g(-w, d_\phi(w))) = \tilde{s}(\phi_g(0, d_\phi(w))).$$

As a consequence, the element $\phi_g(0, d_\phi(w)) \in \mathrm{Im}(\phi_g|_{\{0\} \times \mathbb{R}^k})$ belongs also to $\ker(\tilde{s})$ and is therefore zero by Lemma 4.5, as the image of $\phi_g|_{\{0\} \times \mathbb{R}^k}$ is an element of the fat groupoid. Since ϕ_g is injective, one has $d_\phi(w) = 0$. Plugging this in $\mathbf{s}(\phi_g)^c(w) = 0$, we get

$$0 = \widetilde{R}_g(\tilde{\tau}(\phi_g(w, 0)))$$

which implies $\phi_g(w, 0) = 0$; by the injectivity of ϕ_g, we conclude that $w = 0$. ∎

4.2. The groupoid structure of the s-bisection frames

The first part of the main result of this section is the Lie groupoid structure on the space of adapted frames:

$$\mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}} \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M).$$

We will focus here on the algebraic part (i.e., the groupoid structure), postponing the smoothness to Theorem 4.15 later on.

Theorem 4.9. *Given a VB-groupoid $E_{\mathcal{G}} \rightrightarrows E_M$ of rank (l,k), its s-bisection frame bundle $\mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}}$ is a groupoid over $\mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$, where $C \to M$ is the core of $E_{\mathcal{G}} \rightrightarrows E_M$ (Definition 2.2), with structure maps induced by those of $E_{\mathcal{G}} \rightrightarrows E_M$:*

$\mathbf{s} : \mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}} \to \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M),$

$\quad \mathbf{s}(\phi_g)(w,v) = \big(\mathbf{s}(\phi_g)^c(w) := \tilde{R}_g(\tilde{\tau}(\phi_g(-w, d_{\phi_g}(w)))), \mathbf{s}(\phi_g)^b(v) := \tilde{s}(\phi_g(0,v))\big);$

$\mathbf{t} : \mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}} \to \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M),$

$\quad \mathbf{t}(\phi_g)(w,v) = \big(\mathbf{t}(\phi_g)^c(w) := \tilde{R}_{g^{-1}}(\phi_g(w,0)), \mathbf{t}(\phi_g)^b(v) := \tilde{t}(\phi_g(0,v))\big);$

$\mathbf{m} : \mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}} {}_{\mathbf{s}}\times_{\mathbf{t}} \mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}} \to \mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}},$

$\quad \mathbf{m}(\phi_g, \phi_h)(w,v) := \tilde{m}(\phi_g(0,v), \phi_h(0,v)) + \tilde{R}_h(\phi_g(w,0));$

$\mathbf{u} : \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M) \to \mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}},$

$\quad \mathbf{u}(\varphi_x^c, \varphi_x^b)(w,v) := \tilde{u}(\varphi_x^b(v)) + \varphi_x^c(w);$

$\boldsymbol{\tau} : \mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}} \to \mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}},$

$\quad \boldsymbol{\tau}(\phi_g)(w,v) := \tilde{\tau}(\phi_g(-w, v + d_{\phi_g}(w))).$

Lemma 4.10. *The map d_{ϕ_g} from Definition 4.7 satisfies the following properties:*

(1) $d_{\phi_g} = d_{\mathbf{ut}\phi_g} = d_{\mathbf{us}\phi_g}$;

(2) *if ϕ_g and ϕ_h are composable, then $d_{\phi_g} = d_{\phi_h} = d_{\mathbf{m}(\phi_g, \phi_h)}$.*

Proof. It is also clear that (1) implies (2); moreover, by definition of d_{ϕ_g}, one has $d_{\phi_g} = d_{\mathbf{ut}\phi_g}$. We only need to prove $d_{\phi_g} = d_{\mathbf{us}\phi_g}$:

$$\begin{aligned}
d_{\mathbf{us}\phi_g}(w) &= \big((\mathbf{s}(\phi_g)^b)^{-1} \circ \tilde{t} \circ \mathbf{s}(\phi_g)^c\big)(w) \\
&= (\mathbf{s}(\phi_g)^b)^{-1}\big(\tilde{t}(\tilde{R}_{g^{-1}}(\tilde{\tau}(\phi_g(-w, d_{\phi_g}(w)))))\big) \\
&= (\mathbf{s}(\phi_g)^b)^{-1}\big(\tilde{s}(\phi(-w, d_{\phi_g}(w)))\big) \\
&= (\mathbf{s}(\phi_g)^b)^{-1}\big(\tilde{s}(\phi_g(0, d_{\phi_g}(w))) - \tilde{s}(\phi_g(w,0))\big) = d_{\phi_g}(w). \qquad \blacksquare
\end{aligned}$$

The proof of Theorem 4.9 is based on the fact that $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$ can be described by means of its fat groupoid $\tilde{\mathcal{G}}(E_{\mathcal{G}})$ (Definition 4.1). Notice first that the representations (4.1) and (4.2) induce a left action of $\tilde{\mathcal{G}}(E_{\mathcal{G}})$ on $\mathrm{Fr}(C) \to M$ and $\mathrm{Fr}(E_M) \to M$, hence an action (which we denote act) of $\tilde{\mathcal{G}}(E_{\mathcal{G}})$ on $\mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M) \to M$. One can therefore consider the action groupoid

$$\tilde{\mathcal{G}}(E_{\mathcal{G}}) \times_M (\mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)) \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M),$$

with source pr and target act.

Lemma 4.11. *Let $E_{\mathcal{G}}$ be a VB-groupoid. The function*

$$F\colon \mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \to \tilde{\mathcal{G}}(E_{\mathcal{G}}) \times_M (\mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)),$$

$$\phi_g \mapsto \big((g, H_g := \mathrm{Im}(\phi_g(0, \cdot))), (\mathbf{s}(\phi_g)^c, \mathbf{s}(\phi_g)^b)\big),$$

is a bijection such that

$$\mathbf{s}(\phi_g) = \mathrm{pr}(F(\phi_g)) \quad and \quad \mathbf{t}(\phi_g) = \mathrm{act}(F(\phi_g)), \quad \forall \phi_g \in \mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}).$$

Proof. The function F is well defined: indeed, the pair (g, H_g) belongs to the fat groupoid by definition of bisection frame, while $\mathbf{s}(\phi_g)^c$ and $\mathbf{s}(\phi_g)^b$ are frames of C and E_M, respectively, by Lemmas 4.8 and 4.5. Furthermore, F is a bijection, with inverse given by

$$F^{-1}\big((g, H_g), (\varphi^c, \varphi^b)\big)(w, v) := (\tilde{s}|_{H_g})^{-1}(\varphi^b(v)) + (\tilde{s}|_{H_g})^{-1}(\tilde{t}(\varphi^c(w)))$$
$$- \tilde{\tau}(\widetilde{R}_g)^{-1}(\varphi^c(w)).$$

The definition of pr and of F yields directly the first condition, i.e.,

$$\mathrm{pr}(F(\phi_g)) = (\mathbf{s}(\phi_g)^c, \mathbf{s}(\phi_g)^b) = \mathbf{s}(\phi_g)$$

To check the second condition, we have to compute

$$\mathrm{act}(F(\phi_g)) = \big((g, H_g) \cdot \mathbf{s}(\phi_g)^c, (g, H_g) \cdot \mathbf{s}(\phi_g)^b\big).$$

Using (4.1), the second component becomes

$$(g, H_g) \cdot \mathbf{s}(\phi_g)^b(\cdot) = (g, H_g) \cdot \tilde{s}_g(\phi_g(0, \cdot)) = \tilde{t}_g \circ (\tilde{s}_g|_{\mathrm{Im}(\phi_g(0, \cdot))})^{-1} \circ \tilde{s}_g \circ \phi_g|_{\{0\} \times \mathbb{R}^k}(\cdot)$$
$$= \tilde{t}_g(\phi_g(0, \cdot)) = \mathbf{t}(\phi_g)^b(\cdot).$$

For the first component, first we notice that, for every $w \in \mathbb{R}^l$,

$$\alpha_g := \phi_g(0, d_{\phi_g}(w))$$

belongs to $H_g = \mathrm{Im}(\phi_g|_{\{0\} \times \mathbb{R}^k})$ and satisfies

$$\tilde{t}(\mathbf{s}(\phi_g)^c(w)) = \tilde{t}(\widetilde{R}_g \circ \tilde{\tau} \circ \phi_g(-w, d_{\phi_g} w)) = \tilde{t}(\tilde{\tau} \circ \phi_g(-w, d_{\phi_g} w))$$
$$= \tilde{s}(\phi_g(-w, d_{\phi_g} w)) = \tilde{s}(\phi_g(0, d_{\phi_g} w)) + \underline{\tilde{s}(\phi_g(-w, 0))} = \tilde{s}(\alpha_g).$$

Furthermore, using the interchange law,

$$\mathbf{s}(\phi_g)^c(w) = \widetilde{R}_g(\tilde{\tau}(\phi_g(-w, d_{\phi_g}(w)))) = \tilde{m}\big(\tilde{\tau}(\phi_g(-w, d_{\phi_g}(w))), 0_g\big)$$
$$= \tilde{m}\big(\tilde{\tau}(\phi_g(0, d_{\phi_g}(w))) + \tilde{\tau}(\phi_g(-w, 0)), \phi_g(w, 0) + \phi_g(-w, 0)\big)$$
$$= \tilde{m}\big(\tilde{\tau}(\phi_g(0, d_{\phi_g}(w))), \phi_g(w, 0)\big) + \tilde{m}\big(\tilde{\tau}(\phi_g(-w, 0)), \phi_g(-w, 0)\big)$$
$$= \tilde{\tau}(\alpha_g) \circ \phi_g(w, 0) + \tilde{u}(\tilde{s}(\phi_g(-w, 0)))$$
$$= \tilde{\tau}(\alpha_g) \circ \phi_g(w, 0) + 0_{1_{s(g)}} = \tilde{\tau}(\alpha_g) \circ \phi_g(w, 0).$$

Therefore, using (4.2),

$$(g, H_g) \cdot \mathbf{s}(\phi_g)^c(w) = \alpha_g \circ \mathbf{s}(\phi_g)^c(w) \circ 0_{g^{-1}} = \alpha_g \circ \tilde{\tau}(\alpha_g) \circ \phi_g(w, 0) \circ 0_{g^{-1}}$$
$$= 0_{1_{t(g)}} \circ \phi_g(w, 0) \circ 0_{g^{-1}} = \widetilde{R}_{g^{-1}} \circ \phi_g(w, 0) = \mathbf{t}(\phi_g)^c(w). \quad \blacksquare$$

Proof of Theorem 4.9. In general, given three sets $\mathcal{G}_1$, $\mathcal{G}_2$ and M, and given four maps $s_1, t_1 \colon \mathcal{G}_1 \to M$, $s_2, t_2 \colon \mathcal{G}_2 \to M$, if there is a bijection between $\mathcal{G}_1$ and $\mathcal{G}_2$ which preserves the maps s_1, s_2 and t_1, t_2, there we can transport a groupoid multiplication on $\mathcal{G}_2$ (with source s_2 and target t_2) to one on $\mathcal{G}_1$ (with source s_1 and target t_1). The statement of the theorem then follows from Lemma 4.11, by transporting the structure maps of the action groupoid to $\mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}}$. Indeed, the multiplication satisfies

$$
\begin{aligned}
F(\mathbf{m}(\phi_g, \phi_h)) &= F\big(\widetilde{m}(\phi_g|_{\{0\}\times\mathbb{R}^k}, \phi_h|_{\{0\}\times\mathbb{R}^k}) + \widetilde{R}_h(\phi_g|_{\mathbb{R}^l\times\{0\}})\big) \\
&= \big((gh, \mathrm{Im}(\widetilde{m}(\phi_g|_{\{0\}\times\mathbb{R}^k}, \phi_h|_{\{0\}\times\mathbb{R}^k}))), (\mathbf{s}(\mathbf{m}(\phi_g, \phi_h))^c, \mathbf{s}(\mathbf{m}(\phi_g, \phi_h))^b)\big) \\
&= \big((gh, \widetilde{m}(H_g, H_h)), (\mathbf{s}(\phi_h)^c, \mathbf{s}(\phi_h)^b)\big) \\
&= F(\phi_g)F(\phi_h),
\end{aligned}
$$

while in the last line we used the multiplication of the action groupoid. Similarly, for the unit map one has

$$
\begin{aligned}
F(\mathbf{u}(\varphi_x^c, \varphi_x^b)) &= \big((1_x, \mathrm{Im}(\widetilde{u}(\varphi_x^b))), \mathbf{s}(\mathbf{u}(\varphi_x^c, \varphi_x^b))^c, \mathbf{s}(\mathbf{u}(\varphi_x^c, \varphi_x^b))^b\big) \\
&= \big((1_x, \widetilde{u}(E_M)_x), \varphi_x^c, \varphi_x^b\big) = u(\varphi_x^c, \varphi_x^b),
\end{aligned}
$$

and for the inversion,

$$
\begin{aligned}
F(\boldsymbol{\tau}(\phi_g)) &= \big((g^{-1}, \mathrm{Im}(\widetilde{\tau}(\phi_g(0, \cdot)))), (\mathbf{s}(\boldsymbol{\tau}(\phi_g))^c, \mathbf{s}(\boldsymbol{\tau}(\phi_g))^b)\big) \\
&= \big((g^{-1}, \widetilde{\tau}(\mathrm{Im}(\phi_g(0, \cdot)))), (\mathbf{t}(\phi_g)^c, \mathbf{t}(\phi_g))\big) = i(F(\phi_g)). \quad \blacksquare
\end{aligned}
$$

Remark 4.12 (PB-groupoids and transitivity). It is known that, given a VB-groupoid $E_{\mathcal{G}} \rightrightarrows E_M$ over $\mathcal{G} \rightrightarrows M$, if the groupoid $E_{\mathcal{G}} \rightrightarrows E_M$ is transitive, then $\mathcal{G} \rightrightarrows M$ is transitive as well (but the converse does not hold). For PB-groupoids, the issue is more intricate, since there are more groupoid structures in play (including those of the structural Lie 2-groupoid). In particular, it is not clear if/when the groupoid structure on $\mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}} \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$ from Theorem 4.9 is transitive; we are currently looking into it, with the goal of providing an equivalent description in term of a "gauge PB-groupoid". This could shed lights on the relation between our correspondence VB/PB-groupoids, and the correspondence DVB/DPB discussed in [31]. $\diamond$

4.3. The (set-theoretical) 2-action on the s-bisection frames

As anticipated, after having shown that $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$ is a set-theoretical groupoid (Theorem 4.9), we describe now the (free) action of the general linear 2-groupoid $\mathrm{GL}(l, k)$ on $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$. In the next section we will deal with the smoothness of these spaces, and such action will turn out to be smooth and principal.

To this end, recall (e.g., from [29, 43]) that, given any Lie subgroup $G \subseteq \mathrm{GL}(l + k)$, a G-*structure* on a vector bundle $E \to M$ of rank $l + k$ is a principal G-subbundle of the frame bundle $\mathrm{Fr}(E) \to M$. In particular, let $\mathrm{GL}^l(l + k)$ denote the group of block matrices $\left(\begin{smallmatrix} A_1 & A_2 \\ 0 & A_4 \end{smallmatrix}\right)$, where $A_1 \in \mathrm{GL}(l)$ and $A_4 \in \mathrm{GL}(k)$. Then there is a 1-1 correspondence

$$
\{\mathrm{GL}^l(l + k)\text{-structures on } E\} \Longleftrightarrow \{\text{rank } l \text{ vector subbundle } D \subseteq E \text{ over } M\}.
$$

Indeed, to any such D one associates the subbundle $\mathrm{Fr}(E, D) \subseteq \mathrm{Fr}(E)$ consisting of frames $\phi_x : \mathbb{R}^{l+k} \to E_x$ preserving D_x, i.e., such that $\phi_x(\mathbb{R}^k \times \{0\}) \subseteq D_x$. The right $\mathrm{GL}(l + k)$-action on $\mathrm{Fr}(E)$, i.e., $(\phi_x, A) \mapsto \phi_x \circ A$, restricts to a $\mathrm{GL}^l(l + k)$-action on $\mathrm{Fr}(E, D)$, since $\phi_x(A(w, 0)) = \phi_x(A_1 w, 0)$. We conclude that $\mathrm{Fr}(E, D)$ is a principal $\mathrm{GL}^l(l + k)$-subbundle of $\mathrm{Fr}(E)$. Conversely, given any principal $\mathrm{GL}^l(l + k)$-subbundle $P \subseteq \mathrm{Fr}(E)$, one defines uniquely a rank k subbundle $D \subseteq E$ by taking D_x as the union of the subspaces $\phi_x(\mathbb{R}^l \times \{0\}) \subseteq E_x$, for all $\phi_x \in P_x$, so that $P = \mathrm{Fr}(E, D)$.

Theorem 4.13. *Let $E_{\mathcal{G}} \rightrightarrows E_M$ be a VB-groupoid of rank (l, k) over a Lie groupoid $\mathcal{G} \rightrightarrows M$. There the (principal) right Lie group actions of $\mathrm{GL}(l + k)$ on $\mathrm{Fr}(E_{\mathcal{G}})$ and of $\mathrm{GL}(l) \times \mathrm{GL}(k)$ on $\mathrm{Fr}(C) \times \mathrm{Fr}(E_M)$ induce a canonical right free 2-action of the (Lie) 2-groupoid $\mathrm{GL}(l, k)$ (Definition 2.24) on the (set-theoretical) groupoid $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \rightrightarrows \mathrm{Fr}(C) \times \mathrm{Fr}(E_M)$ from Theorem 4.9, with moment map given by Lemma 4.10 as*

$$\mu : \mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \to \mathrm{GL}(l, k)_0, \quad \phi_g \mapsto d_{\phi_g}.$$

More precisely, the induced 2-action is defined as follows:
- *the right groupoid action of $\mathrm{GL}(l, k)_2 \rightrightarrows \mathrm{GL}(l, k)_0$ on $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$ is given by*

$$\phi_g \cdot \left(d, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix} \right) := \phi_g \cdot \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix};$$

- *the right groupoid action of $\mathrm{GL}(l, k)_1 \rightrightarrows \mathrm{GL}(l, k)_0$ on $\mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$ is given by*

$$(\varphi_x^c, \varphi_x^b) \cdot (d, A, B) := (\varphi_x^c \cdot A, \varphi_x^b \cdot B).$$

Proof. First of all, notice that $\ker(\tilde{s}) \subseteq E_{\mathcal{G}}$ is a rank l vector subbundle, and the corresponding $\mathrm{GL}^l(l + k)$-structure $\mathrm{Fr}(E_{\mathcal{G}}, \ker(\tilde{s}))$ (see the discussion above) coincides precisely with the set of all frames $\phi_g \in \mathrm{Fr}(E_{\mathcal{G}})$ satisfying the first condition of Definition 4.3. Accordingly, the $\mathrm{GL}(l + k)$-action on $\mathrm{Fr}(E_{\mathcal{G}})$ restricts to a $\mathrm{GL}^l(l + k)$-action on $\mathrm{Fr}(E_{\mathcal{G}}, \ker(\tilde{s}))$.

When imposing also the second condition of Definition 4.3, i.e., when restricting $\mathrm{Fr}(E_{\mathcal{G}}, \ker(\tilde{s}))$ to $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$, then the action of the Lie *group* $\mathrm{GL}^l(l + k) \subseteq \mathrm{GL}(l + k)$ induces an action of the Lie 2-*groupoid* $\mathrm{GL}(l, k)$ from Definition-Example 2.24. Indeed, by Lemma 4.10, the map μ satisfies

$$\mu(\phi_g) = \mu(\mathbf{u}\mathbf{s}\phi_g) = \mu(\mathbf{u}\mathbf{t}\phi_g)$$

for any *s*-bisection frame ϕ_g, hence this is a Lie 2-groupoid action as a direct consequence of Proposition 4.14. ∎

Lemma 4.14. *Let $E_{\mathcal{G}} \rightrightarrows E_M$ be a VB-groupoid of rank (l, k) over $\mathcal{G} \rightrightarrows M$ and consider two composable arrows $g, h \in \mathcal{G}$, three frames $\phi_g, \phi_g', \phi_g'' \in \mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$, and two frames $\phi_h, \phi_h' \in \mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$ such that ϕ_g, ϕ_h and ϕ_g', ϕ_h' are composable with respect to the groupoid structure on $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(A)$ from Theorem 4.9.*

(1) *The unique matrix $M_g \in \mathrm{GL}(l + k)$ satisfying $\phi_g M_g = \phi_g'$ also satisfies $(d_{\phi_g}, M_g) \in \mathrm{GL}(l, k)_2$ and $d_{\phi_g'} = \mathbf{s}_{20}(d_{\phi_g}, M_g)$.*

(2) *If M_g', $M_g'' \in \mathrm{GL}(l + k)$ are the unique matrices satisfying*

$$\phi_g' M_g' = \phi_g'' \quad and \quad \phi_g M_g'' = \phi_g'',$$

then

$$(d_{\phi_g}, M_g'') = (d_{\phi_g}, M_g) \circ_{20} (d_{\phi_g'}, M_g').$$

(3) *If $C, A \in \mathrm{GL}(l)$ and $D, B \in \mathrm{GL}(k)$ are the unique matrices satisfying*

$$\mathbf{s}(\phi_g)_c C = \mathbf{s}(\phi_g')_c, \quad \mathbf{s}(\phi_g)_b B = \mathbf{s}(\phi_g')_b,$$
$$\mathbf{t}(\phi_g)_c A = \mathbf{t}(\phi_g')_c \quad and \quad \mathbf{t}(\phi_g)_b D = \mathbf{t}(\phi_g')_b,$$

then

$$(d_{\phi_g}, C, B) = s_{21}(d_{\phi_g}, M_g) \in \mathrm{GL}(l, k)_1$$
$$and \quad (d_{\phi_g}, A, D) = t_{21}(d_{\phi_g}, M_g) \in \mathrm{GL}(l, k)_1.$$

(4) *If M_h is the unique matrix satisfying $\phi_h M_h = \phi_h'$, and M_{gh} is the unique matrix satisfying $(\phi_g \circ \phi_h) M_{gh} = (\phi_g' \circ \phi_h')$, then*

$$(d_{\phi_g}, M_{gh}) = (d_{\phi_g}, M_g) \circ_{21} (d_{\phi_h}, M_h).$$

Proof. This is mostly a calculation. We will only show part (3); the other parts can be proved with similar arguments.

Given any $\phi_g, \phi_g' \in \mathrm{Fr}(E_{\mathcal{G}})_g$, there exist unique matrices A, B, C, D and M_g (interpreted as "changes of coordinates") such that

$$
\begin{array}{ccccc}
(\mathbf{t}(\phi_g)^c, \mathbf{t}(\phi)^b) & \longleftarrow & \phi_g & \longrightarrow & (\mathbf{s}(\phi_g)^c, \mathbf{s}(\phi_g)^b) \\
\downarrow {\scriptstyle A \times D} & & \downarrow {\scriptstyle M_g} & & \downarrow {\scriptstyle C \times B} \\
(\mathbf{t}(\phi_g')^c, \mathbf{t}(\phi_g')^b) & \longleftarrow & \phi_g' & \longrightarrow & (\mathbf{s}(\phi_g')^c, \mathbf{s}(\phi_g')^b).
\end{array}
$$

Because of the definitions of $\mathbf{s}$ and $\mathbf{t}$ for the s-bisection frames, in particular the first l coordinates in $\mathbf{t}$ and the last k coordinates on $\mathbf{t}$, the matrix $M_g \in \mathrm{GL}(l + k)$ must be of the form

$$M_g = \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix},$$

for some $l \times k$ matrix J. Spelling out what this means, one gets that:

- A is the unique matrix such that $\phi_g(Aw, 0) - \phi_g'(w, 0) = 0$ for all $w \in \mathbb{R}^l$;
- B is the unique matrix such that $\tilde{s}\phi_g(0, Bv) - \tilde{s}\phi_g'(0, v) = 0$ for all $v \in \mathbb{R}^k$;
- J is the unique $l \times k$ matrix such that $\phi_g(JBv, Bv) = \phi_g'(0, v)$ for every $v \in \mathbb{R}^k$, equivalently,

$$J = \phi_g^{-1}(\phi_g'(0, B^{-1}\cdot) - \phi_g(0, \cdot));$$

- $\phi_g(Aw + JBv, Bv) = \phi_g'(w, v)$ for all $(w, v) \in \mathbb{R}^{l+k}$.

Now we want to calculate D and C in terms of A, B, J and d_{ϕ_g}. By definition, D is the matrix such that

$$\mathbf{t}\phi_g(0, Dv) = \mathbf{t}\phi'_g(0, v).$$

Using the definition of d_{ϕ_g} for the equality $(\star)$ and that $\phi_g M_g = \phi'_g$ in the equality $(\star\star)$, we get

$$\mathbf{t}\phi_g(0, d_{\phi_g} JBv + Bv) \overset{(\star)}{=} \mathbf{t}\phi_g(JBv, Bv) \overset{(\star\star)}{=} \mathbf{t}\phi'_g(0, v) = \mathbf{t}\phi_g(0, Dv),$$

therefore we conclude

$$D = d_{\phi_g} JB + B = (I + d_{\phi_g} J)B.$$

The value for D implies part of statement (1) of this lemma, i.e.,

$$s_{20}(d_{\phi_g}, M_g) = ((I + d_{\phi_g} J)D)^{-1} d_{\phi_g} A = d_{\phi'_g}.$$

Later in this argument we will use this fact in the form of $d_{\phi_g} A = (I + d_{\phi_g} J)D d_{\phi'_g}$.

Last, C is the unique $l \times l$ matrix satisfying, for any $w \in \mathbb{R}^l$,

$$\mathbf{s}(\phi_g)^c(Cw) = \mathbf{s}(\phi'_g)^c(w) \;\Rightarrow\; \widetilde{R}_g \tilde{\tau}\big(\phi_g(Cw, -d_{\phi_g} Cw)\big) = \widetilde{R}_g \tilde{\tau}\big(\phi'_g(w, -d_{\phi'_g} w)\big),$$

so that

$$\phi_g(Cw, -d_{\phi_g} Cw) = \phi'_g(w, -d_{\phi'_g} w) = \phi_g(Aw - JD d_{\phi'_g} w, -D d_{\phi'_g} w).$$

Now we use that $d_{\phi_g} A = (I + d_{\phi_g} J)D d_{\phi'_g}$ and obtain

$$\phi_g(Cw, -d_{\phi_g} Cw) = \phi_g(Aw - J(I + d_{\phi_g} J)^{-1} d_{\phi_g} Aw, -(I + d_{\phi_g} J)^{-1} d_{\phi_g} Aw).$$

Therefore we have

$$C = (I - J(I + d_{\phi_g} J)^{-1} d_{\phi_g})A.$$

To conclude the proof, we need to show that

$$I - J(I + d_{\phi_g} J)^{-1} d_{\phi_g} = (I + J d_{\phi_g})^{-1}.$$

Indeed,

$$\begin{aligned}
(I - J(I &+ d_{\phi_g} J)^{-1} d_{\phi_g})(I + J d_{\phi_g}) \\
&= I - J(I + d_{\phi_g} J)^{-1} d_{\phi_g} + J d_{\phi_g} - J(d_{\phi_g} J + I)^{-1} d_{\phi_g} J d_{\phi_g} \\
&= I - J\big((I + d_{\phi_g} J)^{-1} - I + (d_{\phi_g} J + I)^{-1} d_{\phi_g} J\big) d_{\phi_g} \\
&= I - J\big((d_{\phi_g} J + I)^{-1}(I + d_{\phi_g} J) - I\big) d_{\phi_g} = I,
\end{aligned}$$

and similarly,

$$(I + J d_{\phi_g})(I - J(I + d_{\phi_g} J)^{-1} d_{\phi_g}) = I.$$

The discussion above is summarised by the source and target as in Definition 2.24:

$$\big(d_{\phi_g}, A, (I + d_{\phi_g} J)B = D\big) \xleftarrow{\ t_{21}\ } \left(d_{\phi_g}, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix}\right)$$

$$\xmapsto{\ s_{21}\ } \big(d_{\phi_g}, C = (I + J d_{\phi_g})^{-1} A, B\big). \qquad \blacksquare$$

4.4. The smooth structure of the s-bisection frames

As anticipated, in this section we take care of the smoothness issues, concluding the proof that $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$ defines a PB-groupoid.

Theorem 4.15. *Let $E_{\mathcal{G}} \rightrightarrows E_M$ be a VB-groupoid of rank (l, k) over a Lie groupoid $\mathcal{G} \rightrightarrows M$. Then*

- *the total space of the set-theoretical groupoid $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$ from Theorem 4.9 is a submanifold of $\mathrm{Fr}(E_{\mathcal{G}})$;*
- *the set-theoretical 2-action from Theorem 4.13 is a smooth principal action.*

Proof. We begin by proving the existence, around any point of $\mathcal{G}$, of smooth local sections of $\mathrm{Fr}(E_{\mathcal{G}}) \to \mathcal{G}$ taking values in $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$. First of all, since $\tilde{s}$ and $\tilde{t}$ are submersions, around any $g \in \mathcal{G}$ there are local sections taking values in $\mathrm{Fr}(\ker(\tilde{s}))$ and in $\mathrm{Fr}(\ker(\tilde{t}))$. Let us think of frames as a local ordered basis: $(e_1, \ldots, e_l)$ is an ordered basis for $\ker(\tilde{s})$ and $(f_1, \ldots, f_l)$ an ordered basis for $\ker(\tilde{t})$. We now describe an algorithm to get s-bisection frames.

Set a counter $i = 1$ and take non-vanishing local sections h_i that are not in the span of $(e_1, \ldots, e_{l+i-1})$ and h'_i that are not in the span of $(f_1, \ldots, f_{l+i-1})$. Then let

$$e_{l+i} = f_{l+i} := h_i + h'_i$$

and redefine $i = i + 1$. Repeating this process until $i = k + 1$ we get that for i between $l + 1$ and $l + k$, the elements $e_i = f_i$ are linearly independent and not in $\ker(\tilde{s})$ nor in $\ker(\tilde{t})$, therefore the ordered basis of sections $(e_1, \ldots, e_{l+k})$ is equivalent to an s-bisection frame.

Recall now that, since $\mathrm{Fr}(E_{\mathcal{G}}) \to \mathcal{G}$ is a principal bundle, the smooth structure of $\mathrm{Fr}(E_{\mathcal{G}})$ is given by smooth local sections $\phi_i \colon U_i \to \mathrm{Fr}(E_{\mathcal{G}})$, which we can pack together as a section $\phi \colon \mathcal{U}_{\mathcal{G}} \to \mathrm{Fr}(E_{\mathcal{G}})$, where

$$\mathcal{U}_{\mathcal{G}} := \bigsqcup_{i \in I} U_i$$

and $(U_i)_{i \in I}$ is an open cover of $\mathcal{G}$. This structure is the only one making the local trivialisation

$$\varphi : \mathcal{U}_{\mathcal{G}} \times \mathrm{GL}(l + k) \to \mathrm{Fr}(E_{\mathcal{G}}), \quad (g, A) \mapsto \phi(g) \cdot A$$

a local diffeomorphism. By the first part of the proof, we can assume that ϕ takes values in $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$. Since being an immersed submanifold is a local property, we check that $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$ is smooth using the local trivialisation φ. By part (1) of Lemma 4.14, the subset $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \subset \mathrm{Fr}(E_{\mathcal{G}})$ locally looks like

$$\varphi^{-1}(\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})) = \mathcal{U}_{\mathcal{G}\ d_\phi} \times_{\mathbf{t}_{20}} \mathrm{GL}(l, k)_2.$$

Since $\mathbf{t}_{20}$ is a submersion, and for any $g \in \mathcal{U}_{\mathcal{G}}$ the natural inclusion $\mathrm{GL}(l, k)_2|_{d_{\phi(g)}} \hookrightarrow \mathrm{GL}(l + k)$ is a submanifold, then

$$\mathcal{U}_{\mathcal{G}\ d_\phi} \times_{\mathbf{t}_{20}} \mathrm{GL}(l, k)_2 \hookrightarrow \mathcal{U}_{\mathcal{G}} \times \mathrm{GL}(l + k)$$

is a submanifold and so is $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \subset \mathrm{Fr}(E_{\mathcal{G}})$.

Similarly, using part (3) of Lemma 4.14 and the local trivialisation, one can check that the source and target maps of $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathscr{G}}) \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$ are submersions, and using part (4) of the same lemma, that the multiplication is smooth. We conclude that $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathscr{G}})$ is a Lie groupoid.

Last, the 2-action is smooth since, by part (2) of Lemma 4.14, under the local trivialisation φ it corresponds to the canonical (smooth) right action of $\mathrm{GL}(l,k)$ on $\mathcal{U}_{\mathscr{G}\ d_\phi} \times_{\mathbf{t}_{20}} \mathrm{GL}(l,k)_2$ with moment map

$$\mu : \mathcal{U}_{\mathscr{G}\ d_\phi} \times_{\mathbf{t}_{20}} \mathrm{GL}(l,k)_2 \to \mathrm{GL}(l,k)_0, \quad (g, d_{\phi(g)}, M_g) \mapsto \mathbf{s}_{20}(d_{\phi(g)}, M_g). \qquad \blacksquare$$

4.5. Second half of the correspondence

Now we move to the "inverse" of Theorem 3.22. Recall that, given any vector bundle $E \to M$ of rank k, the set $\mathrm{Fr}(E)$ of its frames is a principal $\mathrm{GL}(k)$-bundle over M.

Theorem 4.16. *Given any VB-groupoid $E_{\mathscr{G}} \rightrightarrows E_M$ of rank (l,k) over $\mathscr{G} \rightrightarrows M$, its s-bisection frame bundle $\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathscr{G}})$ (Definition 4.3) defines a PB-groupoid over $\mathscr{G} \rightrightarrows M$ (Definition 3.17), with structural Lie 2-groupoid $\mathrm{GL}(l,k)$.*

Proof. It follows by combining Theorems 4.9, 4.13, and 4.15. $\qquad \blacksquare$

Theorem 4.17. *Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid; then Theorem 3.22 and Theorem 4.16 give a 1-1 correspondence between*

- *VB-groupoids of rank (l,k) over $\mathscr{G} \rightrightarrows M$;*
- *PB-groupoids over $\mathscr{G} \rightrightarrows M$ with structural Lie 2-groupoid $\mathrm{GL}(l,k)$.*

More precisely, any VB-groupoid $E_{\mathscr{G}} \rightrightarrows E_M$ is isomorphic to the VB-groupoid associated to the s-bisection frame bundle of $E_{\mathscr{G}}$ and the canonical representation of $\mathrm{GL}(l,k)$ from Example 3.9, i.e.,

$$
\begin{array}{ccc}
\dfrac{\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathscr{G}}) \times_{\mathrm{GL}(l,k)_0} (\mathbb{R}^l \times \mathbb{R}^k \times \mathrm{GL}(l,k)_0)}{\mathrm{GL}(l,k)_2} & \xrightarrow{\ \pi_{\mathscr{G}}\ } & \mathscr{G} \\[4pt]
\tilde{t} \downdownarrows \tilde{s} & & t \downdownarrows s \\[4pt]
\dfrac{(\mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)) \times_{\mathrm{GL}(l,k)_0} (\mathbb{R}^k \times \mathrm{GL}(l,k)_0)}{\mathrm{GL}(l,k)_1} & \xrightarrow{\ \pi_M\ } & M.
\end{array}
$$

Conversely, any PB-groupoid $P_{\mathscr{G}} \rightrightarrows P_M$ is isomorphic to the s-bisection frame bundle of its associated VB-groupoid (Definition 4.3), i.e.,

$$
\begin{array}{ccc}
\mathrm{Fr}^{\mathrm{sbis}}\!\left(\dfrac{P_{\mathscr{G}} \times_{\mathrm{GL}(l,k)_0} (\mathbb{R}^k \times \mathbb{R}^l \times \mathrm{GL}(l,k)_0)}{\mathrm{GL}(l,k)_2}\right) & \xrightarrow{\qquad \Pi_{\mathscr{G}} \qquad} & \mathscr{G} \\[4pt]
\mathbf{t} \downdownarrows \mathbf{s} & & t \downdownarrows s \\[4pt]
\mathrm{Fr}\!\left(\dfrac{P_M \times_{\mathrm{GL}(l,k)_0} (\mathbb{R}^k \times \mathrm{GL}(l,k)_0)}{\mathrm{GL}(l,k)_1}\right) \times_M \mathrm{Fr}\!\left(\dfrac{P_{\mathscr{G}|M} \times_{\mathrm{GL}(l,k)_0} (\mathbb{R}^l \times \mathrm{GL}(l,k)_0)}{\mathrm{GL}(l,k)_2}\right) & \xrightarrow{\ \Pi_M\ } & M.
\end{array}
$$

Proof. For the first part, it is enough to check that

$$\frac{\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \times_{\mathrm{GL}(l,k)_0} (\mathbb{R}^l \times \mathbb{R}^k \times \mathrm{GL}(l,k)_0)}{\mathrm{GL}(l,k)_2} \to E_{\mathcal{G}}, \quad [\phi, (v, w, g)] \mapsto \phi(v, w)$$

and

$$\frac{(\mathrm{Fr}(E_M) \times \mathrm{Fr}(C)) \times_{\mathrm{GL}(l,k)_0} (\mathbb{R}^k \times \mathrm{GL}(l,k)_0)}{\mathrm{GL}(l,k)_1} \to E_M, \quad [(\varphi_x^b, \varphi_x^c), (v, g)] \mapsto \varphi_x^c(v)$$

are well-defined isomorphisms of vector bundles.

For the second part, it is enough to check that, for every $p \in P_{\mathcal{G}}$, the map

$$\phi_p : \mathbb{R}^k \times \mathbb{R}^l \times \mathrm{GL}(l,k)_0 \to \frac{P_{\mathcal{G}} \times_{\mathrm{GL}(l,k)_0} (\mathbb{R}^k \times \mathbb{R}^l \times \mathrm{GL}(l,k)_0)}{\mathrm{GL}(l,k)_2},$$
$$(v, w, g) \mapsto [p, (v, w, g)],$$

is an s-bisection frame, and therefore

$$P_{\mathcal{G}} \to \mathrm{Fr}^{\mathrm{sbis}}\left(\frac{P_{\mathcal{G}} \times_{\mathrm{GL}(l,k)_0} (\mathbb{R}^k \times \mathbb{R}^l \times \mathrm{GL}(l,k)_0)}{\mathrm{GL}(l,k)_2}\right), \quad p \mapsto \phi_p,$$

is an isomorphism of principal $(\mathrm{GL}(l,k)_2 \rightrightarrows \mathrm{GL}(l,k)_0)$-bundles. Similarly,

$$P_M \to \mathrm{Fr}\left(\frac{P_M \times_{\mathrm{GL}(l,k)_0} (\mathbb{R}^k \times \mathrm{GL}(l,k)_0)}{\mathrm{GL}(l,k)_1}\right) \times_M \mathrm{Fr}\left(\frac{P_{\mathcal{G}}|_M \times_{\mathrm{GL}(l,k)_0} (\mathbb{R}^l \times \mathrm{GL}(l,k)_0)}{\mathrm{GL}(l,k)_2}\right),$$
$$q \mapsto \mathbf{s}(\phi_{\mathbf{u}(q)}) = \mathbf{t}(\phi_{\mathbf{u}(q)}),$$

is an isomorphism of principal $(\mathrm{GL}(l,k)_1 \rightrightarrows \mathrm{GL}(l,k)_0)$-bundles. ∎

5. Examples

We illustrate the correspondence between VB- and PB-groupoids by describing the s-bisection frame bundle of several classes of VB-groupoids.

5.1. Special cases of VB-groupoids

Example 5.1 (Trivial core). Let $E_{\mathcal{G}} \rightrightarrows E_M$ be a VB-groupoid of rank $(0, k)$, i.e., with core $C = 0$ and $\mathrm{rank}(E_{\mathcal{G}} \to \mathcal{G}) = \mathrm{rank}(E_M \to M) = k$ (Example 2.7). Then its s-bisection frame bundle boils down to the Lie groupoid

$$\mathrm{Fr}(E_{\mathcal{G}}) \rightrightarrows \mathrm{Fr}(E_M)$$

with structure maps

$$\begin{aligned}
\mathbf{s} &: \mathrm{Fr}(E_{\mathcal{G}}) \to \mathrm{Fr}(E_M), \quad \mathbf{s}(\phi_g) = \tilde{s} \circ \phi_g; \\
\mathbf{t} &: \mathrm{Fr}(E_{\mathcal{G}}) \to \mathrm{Fr}(E_M), \quad \mathbf{t}(\phi_g) = \tilde{t} \circ \phi_g; \\
\mathbf{m} &: \mathrm{Fr}(E_{\mathcal{G}}) \,_{\mathbf{s}}\times_{\mathbf{t}} \mathrm{Fr}(E_{\mathcal{G}}) \to \mathrm{Fr}(E_{\mathcal{G}}), \quad \mathbf{m}(\phi_g, \phi_h)(v) := \tilde{m}(\phi_g(v), \phi_h(v)); \\
\mathbf{u} &: \mathrm{Fr}(E_M) \to \mathrm{Fr}(E_{\mathcal{G}}), \quad \mathbf{u}(\varphi_x) := \tilde{u} \circ \varphi_x; \\
\boldsymbol{\tau} &: \mathrm{Fr}(E_{\mathcal{G}}) \to \mathrm{Fr}(E_{\mathcal{G}}), \quad \boldsymbol{\tau}(\phi_g) := \tilde{\tau} \circ \phi_g.
\end{aligned}$$

Indeed, given a frame $\phi_g\colon \mathbb{R}^{0+k} \to (E_{\mathcal{G}})_g$, condition (1) in Definition 4.3 is trivially satisfied since $\phi_g|_{\mathbb{R}^0 \times \{0\}} = 0$, while the second condition holds true as $\phi_g|_{\{0\} \times \mathbb{R}^k} = \phi_g$, hence its image coincides with the entire $(E_{\mathcal{G}})_g$ since ϕ_g is a linear isomorphism.

Equivalently, since the fat groupoid $\widetilde{\mathcal{G}}(E_{\mathcal{G}})$ is isomorphic to $\mathcal{G}$ (Example 4.2), the action groupoid $\widetilde{\mathcal{G}}(E_{\mathcal{G}}) \times_M (\mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M))$ becomes simply $\mathcal{G} \times_M \mathrm{Fr}(E_M)$. In turn, this is an equivalent description of $\mathrm{Fr}(E_{\mathcal{G}})$ since $E_{\mathcal{G}} \cong \mathcal{G} \times_M E_M$.

Note also that, due to $\mathrm{Hom}(\mathbb{R}^0, \mathbb{R}^k) = \{0\}$, the structural Lie 2-groupoid $\mathrm{GL}(0, k)$ boils down to a Lie 2-group (Definition 2.22), simply given by the unit groupoid $\mathrm{GL}(k) \rightrightarrows \mathrm{GL}(k)$. $\diamond$

Example 5.2 (Trivial base). Let $E_{\mathcal{G}} \rightrightarrows E_M$ be a VB-groupoid of rank $(l, 0)$, i.e., with $E_M = 0$ and $\mathrm{rank}(E_{\mathcal{G}} \to \mathcal{G}) = \mathrm{rank}(C \to M) = l$ (Example 2.8). Then its s-bisection frame bundle boils down to the Lie groupoid

$$\mathrm{Fr}(E_{\mathcal{G}}) \rightrightarrows \mathrm{Fr}(C)$$

with structure maps

$$\mathbf{s} : \mathrm{Fr}(E_{\mathcal{G}}) \to \mathrm{Fr}(C), \quad \mathbf{s}(\phi_g) := -\widetilde{R}_g \circ \tilde{\tau} \circ \phi_g;$$

$$\mathbf{t} : \mathrm{Fr}(E_{\mathcal{G}}) \to \mathrm{Fr}(C), \quad \mathbf{t}(\phi_g) = \widetilde{R}_{g^{-1}} \circ \phi_g;$$

$$\mathbf{m} : \mathrm{Fr}(E_{\mathcal{G}}) \,{}_{\mathbf{s}}\!\times_{\mathbf{t}} \mathrm{Fr}(E_{\mathcal{G}}) \to \mathrm{Fr}(E_{\mathcal{G}}), \quad \mathbf{m}(\phi_g, \phi_h)(w) := \widetilde{R}_h(\phi_g(w));$$

$$\mathbf{u} : \mathrm{Fr}(C) \to \mathrm{Fr}(E_{\mathcal{G}}), \quad \mathbf{u}(\varphi_x) := \varphi_x;$$

$$\boldsymbol{\tau} : \mathrm{Fr}(E_{\mathcal{G}}) \to \mathrm{Fr}(E_{\mathcal{G}}), \quad \boldsymbol{\tau}(\phi_g) := -\tilde{\tau} \circ \phi_g.$$

Indeed, given a frame $\phi_g\colon \mathbb{R}^{l+0} \to (E_{\mathcal{G}})_g$, both conditions in Definition 4.3 are trivially satisfied since $\tilde{s}$ and $\tilde{t}$ are the zero map, hence their kernels coincide with the entire $E_{\mathcal{G}}$.

Equivalently, since the fat groupoid $\widetilde{\mathcal{G}}(E_{\mathcal{G}})$ is isomorphic to $\mathcal{G}$ (Example 4.2), the action groupoid $\widetilde{\mathcal{G}}(E_{\mathcal{G}}) \times_M (\mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M))$ becomes simply $\mathcal{G} \times_M \mathrm{Fr}(C)$. In turn, this is an equivalent description of $\mathrm{Fr}(E_{\mathcal{G}})$ since $E_{\mathcal{G}} \cong \mathcal{G} \times_M C$.

Note also that, due to the fact that $\mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^0) = \{0\}$, the structural Lie 2-groupoid $\mathrm{GL}(l, 0)$ boils down to a Lie 2-group (Definition 2.22), simply given by the unit groupoid $\mathrm{GL}(l) \rightrightarrows \mathrm{GL}(l)$. $\diamond$

Example 5.3 (Pullback VB-groupoid). Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, let $\pi\colon E_M \to M$ be a vector bundle of rank k, and consider the pullback VB-groupoid

$$E_{\mathcal{G}} := E_M \,{}_\pi\!\times_t \mathcal{G} \,{}_s\!\times_\pi E_M \to \mathcal{G},$$

which has rank (k, k) (see Example 2.9). Then its s-bisection frame is isomorphic to a Lie subgroupoid of the pullback groupoid $P_{\mathcal{G}}$ of $\mathcal{G}$ with respect to the principal bundle $\mathrm{Fr}(E_M) \times_M \mathrm{Fr}(E_M) \to M$, i.e.,

$$P_{\mathcal{G}} := \big(\mathrm{Fr}(E_M) \times_M \mathrm{Fr}(E_M)\big) \,{}_\pi\!\times_t \mathcal{G} \,{}_s\!\times_\pi \big(\mathrm{Fr}(E_M) \times_M \mathrm{Fr}(E_M)\big) \rightrightarrows \mathrm{Fr}(E_M) \times_M \mathrm{Fr}(E_M);$$

more precisely,

$$\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \cong \big\{(\varphi^1_{t(g)}, \varphi^2_{t(g)}, g, \varphi^3_{s(g)}, \varphi^4_{s(g)}) \in P_{\mathcal{G}} \,|\, \varphi^3_{s(g)} = \varphi^4_{s(g)} \circ (\varphi^2_{t(g)})^{-1} \circ \varphi^1_{t(g)}\big\}$$
$$\subseteq P_{\mathcal{G}}.$$

Indeed, any frame ϕ_g of $E_{\mathcal{G}} \to \mathcal{G}$ at $g \in \mathcal{G}$ takes the form

$$\phi_g = (\phi_g^1, g, \phi_g^3) : \mathbb{R}^{2k} \to (E_{\mathcal{G}})_g = (E_M)_{t(g)} \times \{g\} \times (E_M)_{s(g)}.$$

Then the first condition from Definition 4.3 becomes

$$\phi_g^3(w, 0) = 0,$$

which implies that $\phi_g^1(\cdot, 0)$ is a frame of $E_M \to M$, while the second condition guarantees that $\phi_g^1(0, \cdot)$ and $\phi_g^3(0, \cdot)$ are frames of $E_M \to M$. Moreover, since the core-anchor $\rho : E_M \to E_M$ is the identity, the map $d_\phi : \mathbb{R}^k \to \mathbb{R}^k$ from Lemma 4.10 is defined by

$$\phi_g^1(w, 0) = \phi_g^1(0, d_\phi(w)), \quad \forall w \in \mathbb{R}^k,$$

i.e., it is the linear isomorphism

$$d_\phi = (\phi_g^1(0, \cdot))^{-1} \circ \phi_g^1(\cdot, 0).$$

This shows that any s-bisection frame ϕ_g is encoded in the arrow g together with a quadruple of frames of E_M, namely,

$$\left(\varphi_{t(g)}^1 := \phi_g^1(\cdot, 0), \varphi_{t(g)}^2 := \phi_g^1(0, \cdot), \varphi_{s(g)}^3 := \phi_g^3(0, d(\cdot)), \varphi_{s(g)}^4 := \phi_g^3(0, \cdot)\right),$$

where $\varphi_{s(g)}^3$ is by construction the composition

$$\varphi_{s(g)}^4 \circ d_\phi = \varphi_{s(g)}^4 \circ (\varphi_{t(g)}^2)^{-1} \circ \varphi_{t(g)}^1.$$

Moreover, the restrictions of the structure maps of the pullback groupoid

$$P_{\mathcal{G}} \rightrightarrows \mathrm{Fr}(E_M) \times_M \mathrm{Fr}(E_M)$$

coincide with those of Theorem 4.9.   ◇

Example 5.4 (Canonical VB-groupoid). The fact that the frame bundle of the canonical vector space $\mathbb{R}^n$ coincides with general linear group $\mathrm{GL}(n)$, with the principal action of $\mathrm{GL}(n)$ on itself, can be generalised to VB-groupoids. More precisely, we will show that the s-bisection frame bundle of the canonical VB-groupoid $\mathbb{R}^{(l,k)}$ (Example 2.10) coincides with the PB-groupoid associated, via Example 3.21, to the general linear 2-groupoid $\mathrm{GL}(l, k)$ (Example 2.24), i.e., with

$$\begin{array}{ccc}
\mathrm{GL}(l,k)_2 & \xrightarrow{\;t_{20}\;} & \mathrm{GL}(l,k)_0 \\
{\scriptstyle t_{21}}\big\downarrow\big\downarrow{\scriptstyle s_{21}} & & {\scriptstyle \mathrm{id}}\big\downarrow\big\downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{GL}(l,k)_1 & \xrightarrow{\;t_{10}\;} & \mathrm{GL}(l,k)_0
\end{array}$$

with the principal action of $\mathrm{GL}(l, k)$ on itself.

Indeed, one checks immediately that the frame bundle of the core of $\mathbb{R}^{(l,k)}$ is

$$\mathrm{Fr}(C) = \mathrm{Fr}\big(\mathbb{R}^l \times \mathrm{GL}(l,k)_0 \to \mathrm{GL}(l,k)_0\big) \cong \mathrm{GL}(l) \times \mathrm{GL}(l,k)_0,$$

while the frame bundle of the base of $\mathbb{R}^{(l,k)}$ is

$$\mathrm{Fr}(E_M) = \mathrm{Fr}\big(\mathbb{R}^k \times \mathrm{GL}(l,k)_0 \to \mathrm{GL}(l,k)_0\big) \cong \mathrm{GL}(k) \times \mathrm{GL}(l,k)_0,$$

so that their fibred product is

$$\mathrm{Fr}(C) \times_{\mathrm{GL}(l,k)_0} \mathrm{Fr}(E_M) \cong \mathrm{GL}(l,k)_0 \times \mathrm{GL}(l) \times \mathrm{GL}(k) = \mathrm{GL}(l,k)_1.$$

On the other hand, the frame bundle of the trivial vector bundle

$$\mathbb{R}_2^{(l,k)} = \mathbb{R}^l \times \mathbb{R}^k \times \mathrm{GL}(l,k)_0 \to \mathrm{GL}(l,k)_0$$

is the trivial bundle principal bundle

$$\mathrm{GL}(l+k) \times \mathrm{GL}(l,k)_0 \to \mathrm{GL}(l,k)_0.$$

To conclude that $\mathrm{Fr}^{\mathrm{sbis}}(\mathbb{R}_2^{(l,k)}) = \mathrm{GL}(l,k)_2$, it is enough to check that the conditions of Definition 4.3 coincide with those defining $\mathrm{GL}(l,k)_2 \subseteq \mathrm{GL}(l+k) \times \mathrm{GL}(l,k)_0$ in Example 2.24. To this end, consider an element $g \in \mathcal{G} = \mathrm{GL}(l,k)_0$ and a frame

$$\phi_g : \mathbb{R}^l \times \mathbb{R}^k \to (\mathbb{R}_2^{(l,k)})_g = \mathbb{R}^l \times \mathbb{R}^k \times \{g\},$$

which is identified with a block matrix in $\mathrm{GL}(l+k)$, i.e.,

$$\phi_g(w,v) = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}\begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} M_1 w + M_2 v \\ M_3 w + M_4 v \end{pmatrix}.$$

Then condition (1) of Definition 4.3 is equivalent to $M_3 = 0$, since

$$\ker(\tilde{s})_g = \mathbb{R}^l \times \{0\} \times \{g\}.$$

Similarly, from condition (2) of Definition 4.3, it follows that M_4 is invertible (hence M_1 is invertible as well, by the properties of block matrices). Setting

$$J := M_2(M_4)^{-1} : \mathbb{R}^k \to \mathbb{R}^l,$$

one can check the rest of the conditions defining $\mathrm{GL}(l,k)_2$ via computations completely analogous to those in part 3 of Lemma 4.14. $\diamondsuit$

5.2. Tangent VB-groupoids

Example 5.5 (Tangent VB-groupoids). Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider its tangent VB-groupoid $E_{\mathcal{G}} = T\mathcal{G} \rightrightarrows E_M = TM$, whose core C coincides with the Lie

algebroid $A = \text{Lie}(\mathcal{G})$ (Example 2.3). Then the Lie groupoid structure of the s-bisection frame bundle $\text{Fr}^{\text{sbis}}(\mathcal{G}) \rightrightarrows \text{Fr}(A) \times_M \text{Fr}(M)$ admits a simpler description as

$$\mathbf{s} : \text{Fr}(\mathcal{G})^{\text{sbis}} \to \text{Fr}(A) \times_M \text{Fr}(M),$$
$$\mathbf{s}(\phi_g)(w, v) = \big(d_{g^{-1}} R_g(d\tau(\phi_g(-w, d_{\phi_g}(w)))), d_g s(\phi_g(0, v))\big);$$
$$\mathbf{t} : \text{Fr}(\mathcal{G})^{\text{sbis}} \to \text{Fr}(A) \times_M \text{Fr}(M),$$
$$\mathbf{t}(\phi_g)(w, v) = \big(d_g R_{g^{-1}}(\phi_g(w, 0)), d_g t(\phi_g(0, v))\big);$$
$$\mathbf{m} : \text{Fr}(\mathcal{G})^{\text{sbis}} \,_\mathbf{s}\times_\mathbf{t} \text{Fr}(\mathcal{G})^{\text{sbis}} \to \text{Fr}(\mathcal{G})^{\text{sbis}},$$
$$\mathbf{m}(\phi_g, \phi_h)(w, v) := d_{(g,h)} m\big(\phi_g(0, v), \phi_h(0, v)\big) + d_g R_h(\phi_g(w, 0));$$
$$\mathbf{u} : \text{Fr}(A) \times_M \text{Fr}(M) \to \text{Fr}(\mathcal{G})^{\text{sbis}}, \quad \mathbf{u}(\varphi_x^a, \varphi_x^b)(w, v) := d_x u(\varphi_x^b(v)) + \varphi_x^a(w);$$
$$\boldsymbol{\tau} : \text{Fr}(\mathcal{G})^{\text{sbis}} \to \text{Fr}(\mathcal{G})^{\text{sbis}}, \quad \boldsymbol{\tau}(\phi_g)(w, v) := d_g \tau(\phi_g(-w, v + d_{\phi_g}(w))).$$

Equivalently, since $\tilde{\mathcal{G}}(T\mathcal{G})$ coincides with the jet groupoid $J^1\mathcal{G}$, i.e., the space of 1-jets of bisections of $\mathcal{G}$, one can interpret $\text{Fr}(\mathcal{G})^{\text{sbis}}$ as the action groupoid $J^1\mathcal{G} \times_M$ $(\text{Fr}(A) \times_M \text{Fr}(M))$, for the canonical representations of $J^1\mathcal{G}$ on A and TM.

One can also write down the structure maps of the s-bisection frame bundle of the cotangent VB-groupoid $T^*\mathcal{G}$ (Example 2.4) and describe a canonical isomorphism of PB-groupoids $\text{Fr}^{\text{sbis}}(T\mathcal{G}) \to \text{Fr}^{\text{sbis}}(T^*\mathcal{G})$ over $\text{Fr}(A) \times_M \text{Fr}(TM) \to \text{Fr}(T^*M) \times_M \text{Fr}(A^*)$. This will be discussed in greater generality in the next section. $\diamond$

We consider now a few special cases of Example 5.5.

Example 5.6 (Pair groupoids). Let $\mathcal{G} = M \times M \rightrightarrows M$ be the pair groupoid of an n-dimensional manifold M, and consider its tangent VB-groupoid $T\mathcal{G} = TM \times TM \rightrightarrows TM$, which is the pair groupoid of TM and has rank (n, n). A frame $\phi_{(x,y)}$ of $T\mathcal{G}$ at $g = (x, y) \in M \times M$ takes the form

$$\phi_{(x,y)} : \mathbb{R}^{2n} \to T_xM \times T_yM, \quad (w, v) \mapsto (\phi_x(w, v), \phi_y(w, v)).$$

Then the first condition from Definition 4.3 becomes

$$\phi_y(w, 0) = 0 \quad \text{for every } w \in \mathbb{R}^n,$$

which implies that $\phi_y(0, \cdot)$ is a frame of M, while the second condition guarantees that $\phi_x(\cdot, 0)$ and $\phi_x(0, \cdot)$ are frames of M. Last, since the core-anchor of $TM \times TM \rightrightarrows TM$ is the anchor of the Lie algebroid $TM \to M$, i.e., the identity id_{TM}, the map $d_\phi : \mathbb{R}^n \to \mathbb{R}^n$ from Lemma 4.10 is a linear isomorphism defined by

$$\phi_x(w, 0) = \phi_x(0, d_\phi(w)).$$

Using the definition of the structure maps of the pair groupoid, together with the fact that

$$d_{(x,y)} R_{(y,z)}(\alpha_x, 0_y) = (\alpha_x, 0_z),$$

one can see that the groupoid structure of $\mathrm{Fr}(M \times M)^{\mathrm{sbis}} \rightrightarrows \mathrm{Fr}(M) \times_M \mathrm{Fr}(M)$ is given by

$$\mathbf{s}(\phi_{(x,y)}) = (\phi_y(0, d_{\phi_g}(\cdot)), \phi_y(0, \cdot)), \quad \mathbf{t}(\phi_{(x,y)}) = (\phi_x(\cdot, 0), \phi_x(0, \cdot));$$
$$\mathbf{m}(\phi^1_{(x,y)}, \phi^2_{(y,z)})(w, v) := (\phi^1_x(0, v), \phi^2_z(0, v)) + (\phi^1_x(w, 0), 0_z);$$
$$\mathbf{u}(\varphi^a_x, \varphi^b_x)(w, v) := (\varphi^b_x(v), \varphi^b_x(v)) + (\varphi^a_x(w), 0_x),$$
$$\boldsymbol{\tau}(\phi_g)(w, v) := (\phi_y(0, v + d_{\phi_g}(w)), \phi_x(0, v)). \qquad \Diamond$$

Example 5.7 (Étale Lie groupoids). Let $\mathcal{G} \rightrightarrows M$ be an étale Lie groupoid, with $\dim(\mathcal{G}) = \dim(M) = k$, and consider its tangent VB-groupoid $E_{\mathcal{G}} = T\mathcal{G} \rightrightarrows E_M = TM$, which has rank $(0, k)$ and core $\mathrm{Lie}(\mathcal{G}) = 0$. As per Example 5.1, both conditions in Definition 4.3 are empty, therefore

$$\mathrm{Fr}(\mathcal{G})^{\mathrm{sbis}} = \mathrm{Fr}(\mathcal{G}),$$

and the groupoid structure of $\mathrm{Fr}(\mathcal{G}) \rightrightarrows \mathrm{Fr}(M)$ is given by

$$\mathbf{s}(\phi_g) = d_g s \circ \phi_g, \quad \mathbf{t}(\phi_g) = d_g t \circ \phi_g;$$
$$\mathbf{m}(\phi_g, \phi_h)(v) := d_{(g,h)} m(\phi_g(v), \phi_h(v));$$
$$\mathbf{u}(\varphi_x) := d_x u \circ \varphi_x, \qquad \boldsymbol{\tau}(\phi_g) := d_g \tau \circ \phi_g.$$

A particular instance of this example is given by the unit groupoid $\mathcal{G} = M \rightrightarrows M$: since $s = t = \mathrm{id}_M$, the formulae above show that its s-bisection frame bundle is precisely the unit groupoid of $\mathrm{Fr}(M)$, i.e., its multiplication reduces to

$$\mathbf{m}(\phi_g, \phi_h) = \phi_g = \phi_h. \qquad \Diamond$$

Example 5.8 (Lie groups). Let $G \rightrightarrows M = \{*\}$ be an l-dimensional Lie group and consider its tangent VB-groupoid $E_{\mathcal{G}} = TG \rightrightarrows E_M = \{*\}$, which has rank $(l, 0)$ and core $\mathfrak{g} = \mathrm{Lie}(G)$. As per Example 5.2, both conditions in Definition 4.3 are empty, therefore

$$\mathrm{Fr}(G)^{\mathrm{sbis}} = \mathrm{Fr}(G),$$

and the groupoid structure of $\mathrm{Fr}(G) \rightrightarrows \mathrm{Fr}(\mathfrak{g})$ is given by

$$\mathbf{s}(\phi_g) := -d_{g^{-1}} R_g \circ d_g \tau \circ \phi_g = d_g L_{g^{-1}} \circ \phi_g, \quad \mathbf{t}(\phi_g) = d_g R_{g^{-1}} \circ \phi_g;$$
$$\mathbf{m}(\phi_g, \phi_h)(w) := d_g R_h(\phi_g(w));$$
$$\mathbf{u}(\varphi_x) := \varphi_x, \quad \boldsymbol{\tau}(\phi_g) := -d_g \tau \circ \phi_g. \qquad \Diamond$$

Example 5.9 (Bundles of Lie groups). Let $\pi \colon \mathcal{G} \to M$ be a bundle of Lie groups, i.e., a Lie groupoid where source and target coincide, and consider its tangent VB-groupoid $d\pi \colon T\mathcal{G} \to TM$, which is a bundle of Lie groups as well. Then

$$\mathrm{Fr}^{\mathrm{sbis}}(\mathcal{G}) = \{\phi_g \in \mathrm{Fr}(\mathcal{G}) \,|\, \phi_g(w, 0) \in \ker(d_g \pi) \,\forall w \in \mathbb{R}^l\},$$

since the first condition in Definition 4.3 implies the second one by Lemma 4.4 and using $s = t = \pi$.

Moreover, the map d_ϕ from Lemma 4.10 is zero. Indeed, the element $d_\phi(w) \in \mathbb{R}^k$ is defined by

$$d\pi(\phi_g(0, d_\phi(w))) = d\pi(\phi_g(w, 0)) = 0,$$

i.e., by $\phi_g(0, d_\phi(w)) \in \ker(d\pi)$. However, $\phi_g(0, d_\phi(w))$ belongs also to $\mathrm{Im}(\phi_g|_{\{0\}\times\mathbb{R}^k})$, therefore it vanishes due to Lemma 4.5. Since ϕ_g is an isomorphism, it follows that $d_\phi(w) = 0$. Then the groupoid structure of $\mathrm{Fr}(\mathcal{G})^{\mathrm{sbis}} \rightrightarrows \mathrm{Fr}(A) \times \mathrm{Fr}(M)$ can be written as

$$\mathbf{s}(\phi_g)(w, v) = \big(d_{g^{-1}} R_g(d\tau(\phi_g(-w, 0))), d_g \pi(\phi_g(0, v))\big);$$
$$\mathbf{t}(\phi_g)(w, v) = \big(d_g R_{g^{-1}}(\phi_g(w, 0)), d_g \pi(\phi_g(0, v))\big);$$
$$\mathbf{m}(\phi_g, \phi_h)(w, v) := d_{(g,h)} m(\phi_g(0, v), \phi_h(0, v)) + d_g R_h(\phi_g(w, 0));$$
$$\mathbf{u}(\varphi_x^a, \varphi_x^b)(w, v) := d_x u(\varphi_x^b(v)) + \varphi_x^a(w), \quad \boldsymbol{\tau}(\phi_g)(w, v) := d_g \tau(\phi_g(-w, v)). \quad \diamond$$

5.3. PB-groupoid of the dual VB-groupoid

Recall that the fat groupoid of any VB-groupoid $E_\mathcal{G} \rightrightarrows E_M$ is canonically isomorphic to the fat groupoid of its dual VB-groupoid $E_\mathcal{G}^* \rightrightarrows C^*$ (Example 2.5); the isomorphism is given by

$$\widetilde{\mathcal{G}}(E_\mathcal{G}) \to \widetilde{\mathcal{G}}(E_\mathcal{G}^*), \quad (g, H_g) \to (g, H_g^\circ),$$

where $H_g^\circ \subseteq (E_\mathcal{G}^*)_g$ is the annihilator of $H_g \subseteq (E_\mathcal{G})_g$. Then one has the following chain of isomorphisms of Lie groupoids:

$$\mathrm{Fr}^{\mathrm{sbis}}(E_\mathcal{G}^*) \cong \widetilde{\mathcal{G}}(E_\mathcal{G}^*) \times_M (\mathrm{Fr}(E_M^*) \times_M \mathrm{Fr}(C^*)) \cong \widetilde{\mathcal{G}}(E_\mathcal{G}) \times_M (\mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M))$$
$$\cong \mathrm{Fr}^{\mathrm{sbis}}(E_\mathcal{G}).$$

The goal of this subsection is to describe the isomorphism $\mathrm{Fr}^{\mathrm{sbis}}(E_\mathcal{G}^*) \cong \mathrm{Fr}^{\mathrm{sbis}}(E_\mathcal{G})$ as a PB-groupoid isomorphism and provide an explicit formula (see Propositions 5.12 and 5.13 below) starting from the definition of s-bisection frame bundle of a VB-groupoid. To this end, as anticipated, we will have to consider a version of Definition 4.3 obtained by switching $\tilde{s}$ and $\tilde{t}$ and replacing $\mathbb{R}^l \times \mathbb{R}^k$ with $\mathbb{R}^k \times \mathbb{R}^l$. This will yield a PB-groupoid structure which is isomorphic to the one described in Theorems 4.9 and 4.15, and which will work as a bridge between the s-bisection and the t-bisection frame bundles.

Definition 5.10. Let $E_\mathcal{G} \rightrightarrows E_M$ be a VB-groupoid of rank (l, k) over $\mathcal{G} \rightrightarrows M$. A frame $\phi_g \in \mathrm{Fr}(E_\mathcal{G})$ is called a *t-bisection frame* if the following two conditions holds:

(1) $\phi_g|_{\{0\}\times\mathbb{R}^l}$ takes values in $\ker(\tilde{t}_g)$;

(2) the image $H_g \subseteq (E_\mathcal{G})_g$ of $\phi_g|_{\{0\}\times\mathbb{R}^k}$ is an element of the fat groupoid $\widetilde{\mathcal{G}}(E_\mathcal{G})$ (Definition 4.1).

The collection

$$\mathrm{Fr}(E_\mathcal{G})^{\mathrm{tbis}} := \{\phi_g : \mathbb{R}^{l+k} \to (E_\mathcal{G})_g \mid \phi_g \text{ } t\text{-bisection frame}\} \subseteq \mathrm{Fr}(E_\mathcal{G}).$$

of t-bisection frames of $E_\mathcal{G}$ is called the *t-bisection frame bundle* of $E_\mathcal{G}$. $\blacklozenge$

Proposition 5.11. *Given a VB-groupoid $E_\mathcal{G} \rightrightarrows E_M$ of rank (l, k) over a Lie groupoid $\mathcal{G} \rightrightarrows M$, its t-bisection frame bundle $\mathrm{Fr}(E_\mathcal{G})^{\mathrm{tbis}}$ (Definition 5.10) is a Lie groupoid over $\mathrm{Fr}(E_M) \times_M \mathrm{Fr}(C)$, where $C \to M$ is the core of $E_\mathcal{G} \rightrightarrows E_M$ (Definition 2.2), with structure maps induced by those of $E_\mathcal{G} \rightrightarrows E_M$.*

Moreover, $\mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{tbis}}$ *is a* (left) *PB-groupoid over* $\mathcal{G} \rightrightarrows M$ *with structural Lie* 2-*group-oid* $\mathrm{GL}(l, k)$ *(Example* 2.24*), with principal left* 2-*action given by*

$$A \cdot \phi_g := \phi_g \circ T \circ A^T \circ T,$$

where T *is the map which flips the first and second group of coordinates in* $\mathbb{R}^{l+k}$:

$$T : \mathbb{R}^l \times \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^l, \quad (w_1, \dots, w_l, v_1, \dots, v_k) \mapsto (v_k, \dots, v_1, w_l, \dots, w_1).$$

Proof. The proof is formally identical to those of Theorems 4.9 and 4.15. In particular, note that, if $A \in \mathrm{GL}(l, k)_2$, then $T \circ A^T \circ T$ sends $\{0\} \times \mathbb{R}^l$ to $\{0\} \times \mathbb{R}^l$ and $\mathbb{R}^k \times \{0\}$ to $\mathbb{R}^k \times \{0\}$, hence it preserves the definition of t-bisection frame. The left action is therefore well-defined since $(AB)^T = B^T A^T$ and T is an involution. ∎

Given any VB-groupoid, the PB-groupoid structure of its s-bisection frame bundle and that of its t-bisection frame bundle are actually the same, up to the inversion $\tilde{\tau}$ of $E_{\mathcal{G}}$ and the flip T.

Proposition 5.12. *Let* $E_{\mathcal{G}} \rightrightarrows E_M$ *be a VB-groupoid of rank* (l, k) *over a Lie groupoid* $\mathcal{G} \rightrightarrows M$. *Then there is a canonical PB-groupoid isomorphism*

$$\psi : \mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \to \mathrm{Fr}^{\mathrm{tbis}}(E_{\mathcal{G}}), \quad \phi_g \mapsto \tilde{\tau} \circ \phi_g \circ T,$$

over the diffeomorphism

$$\mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M) \to \mathrm{Fr}(E_M) \times_M \mathrm{Fr}(C), \quad (\varphi^c, \varphi^b) \mapsto (\varphi^b, \varphi^c),$$

and with respect to the Lie 2-*groupoid isomorphism*

$$\mathrm{GL}(l, k) \to \mathrm{GL}(l, k), \quad A \mapsto A^T.$$

Proof. The flip T guarantees that the restrictions of a frame ϕ_g in Definition 4.3 are mapped precisely to the restrictions in Definition 5.10, while the inversion $\tilde{\tau}$ transforms $\ker(\tilde{s})$ into $\ker(\tilde{t})$, so that s-bisection frames are sent to t-bisection frames. This defines an isomorphism of Lie groupoids, with the inverse given by the same map $\phi_g \mapsto \tilde{\tau} \circ \phi_g \circ T$, since both $\tilde{\tau}$ and T are involutions.

Moreover, ψ is equivariant with respect to the right and the left principal 2- action of $\mathrm{GL}(l, k)$ and the isomorphism $A \mapsto A^T$, since

$$A^T \cdot \psi(\phi) = \psi(\phi) \circ T \circ (A^T)^T \circ T = \tilde{\tau} \circ \phi \circ T \circ T \circ A \circ T = \tilde{\tau} \circ \phi \circ A \circ T$$
$$= \tilde{\tau} \circ (\phi \cdot A) \circ T = \psi(\phi \cdot A). \qquad \blacksquare$$

Recall that, given a frame $\phi \colon \mathbb{R}^{l+k} \to V$ on a vector space V, its *dual frame* is the isomorphism $\phi^* \colon \mathbb{R}^{l+k} \to V^*$ satisfying

$$\phi^*(e_i)(\phi(e_j)) = \delta_{ij},$$

where $\{e_i\}$ is the canonical basis of $\mathbb{R}^{k+l}$. Accordingly, for every vector bundle $E \to M$, there is a diffeomorphism (actually an isomorphism of principal bundles)

$$\mathrm{Fr}(E) \to \mathrm{Fr}(E^*), \quad \phi \mapsto \phi^*.$$

Proposition 5.13. *Let $E_{\mathcal{G}} \rightrightarrows E_M$ be a VB-groupoid of rank (l, k) over a Lie groupoid $\mathcal{G} \rightrightarrows M$, and $E_{\mathcal{G}}^* \rightrightarrows C^*$ its dual VB-groupoid (Example 2.5). Then there is a canonical PB-groupoid isomorphism*

$$\Phi : \mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \to \mathrm{Fr}^{\mathrm{tbis}}(E_{\mathcal{G}}^*), \quad \phi_g \mapsto \phi_g^*,$$

over the diffeomorphism

$$\mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M) \to \mathrm{Fr}(C^*) \times_M \mathrm{Fr}(E_M^*), \quad (\varphi^c, \varphi^b) \mapsto ((\varphi^c)^*, (\varphi^b)^*),$$

and with respect to the Lie 2-groupoid isomorphism

$$\mathrm{GL}(l, k) \to \mathrm{GL}(l, k), \quad A \mapsto A^T.$$

Note that E_M^* is the core of $E_{\mathcal{G}}^*$ and C^* its base.

Proof. Given an element $\phi \in \mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$, its corresponding dual frame ϕ^* belongs to $\mathrm{Fr}^{\mathrm{tbis}}(E_{\mathcal{G}}^*)$ by Lemma 5.14. The fact that Φ is equivariant with respect to the right and the left principal 2-action of $\mathrm{GL}(l, k)$ and the isomorphism $A \mapsto A^T$ follows from

$$(\phi \cdot A)^* = A^T \cdot \phi^*. \qquad \blacksquare$$

Lemma 5.14. *Let $E_{\mathcal{G}} \rightrightarrows E_M$ be a VB-groupoid of rank (l, k) over a Lie groupoid $\mathcal{G} \rightrightarrows M$. Given any s-bisection frame $\phi \in \mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}})$, its dual frame $\phi^* \in \mathrm{Fr}(E_{\mathcal{G}}^*)$ is a t-bisection frame of the dual VB-groupoid $E_{\mathcal{G}}^* \rightrightarrows C^*$, which has rank (k, l).*

Proof. Recall from Example 2.5 that the source and target maps of $E_{\mathcal{G}}^* \rightrightarrows C^*$ are given by

$$\tilde{s}^*(\xi)(v) = -\xi(0_g \circ \tilde{\tau}(v)) \quad \text{and} \quad \tilde{t}^*(\xi)(v) = \xi(v \circ 0_g).$$

Their kernels at $g \in \mathcal{G}$ are

$$\{(v \circ 0_g) : v \in C_{\mathbf{s}(g)}\} = \ker(\tilde{s})_g \quad \text{and} \quad \{(0_g \circ \tilde{\tau}(v)) : v \in C_{\mathbf{t}(g)}\} = \ker(\tilde{t})_g,$$

so that

$$\ker(\tilde{t}^*) = (\ker(\tilde{s})^0) \quad \text{and} \quad \ker(\tilde{s}^*) = (\ker(\tilde{t})^0),$$

where 0 denotes the annihilator of those spaces. As a consequence, for any s-bisection frame ϕ of $E_{\mathcal{G}}$, its dual ϕ^* satisfies the first condition in Definition 5.10 by the analogue of Lemma 4.6, since $\phi^*|_{\{0\}\times\mathbb{R}^k}$ is a frame for $\ker(\tilde{s})^0 = \ker(\tilde{t}^*)$. Moreover, $\mathrm{Im}(\phi^*|_{\mathbb{R}^l\times\{0\}})$ is transverse to $\ker(\tilde{t}^*)$. We are left to show that $\phi^*|_{\mathbb{R}^l\times\{0\}}$ is transverse to $\ker(\tilde{s}^*) = (\ker(\tilde{t})^0)$; this is equivalent to the following linear map being injective:

$$\varphi : \ker(\tilde{t}) \to \mathbb{R}^l, \quad v \mapsto (\phi^*(e_1)(v), \dots, \phi^*(e_l)(v)).$$

We notice first that any $v \in \ker(\tilde{t})$ can be written as

$$v = \sum_{i=1}^{l}(\phi^*(e_i)(v))\phi(e_i) + \sum_{i=l+1}^{l+k}(\phi^*(e_i)(v))\phi(e_i).$$

If $v \in \ker(\varphi)$, then the first l coordinates vanish, so $v \in \mathrm{Span}(\phi(e_{l+1}), \dots, \phi(e_{l+k}))$. Since ϕ is an s-bisection frame, the last k coordinates must be transverse to $\ker(\tilde{t})$, and therefore $v = 0$. We conclude that ϕ^* satisfies the first condition in Definition 5.10, hence $\phi^* \in \mathrm{Fr}(E_{\mathcal{G}}^*)$. $\qquad \blacksquare$

Composing the isomorphism Ψ^{-1} from Proposition 5.12 and Φ from Proposition 5.13, one obtains the following conclusion.

Corollary 5.15. *Let $E_{\mathcal{G}} \rightrightarrows E_M$ be a VB-groupoid over a Lie groupoid $\mathcal{G} \rightrightarrows M$, and $E_{\mathcal{G}}^* \rightrightarrows C^*$ its dual VB-groupoid (Example 2.5). Then the PB-groupoid*

$$\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}) \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$$

is canonically isomorphic to the PB-groupoid

$$\mathrm{Fr}^{\mathrm{sbis}}(E_{\mathcal{G}}^*) \rightrightarrows \mathrm{Fr}(E_M^*) \times_M \mathrm{Fr}(C^*).$$

Example 5.16. Given a VB-groupoid $E_{\mathcal{G}} \rightrightarrows E_M$ with trivial core, its PB-groupoid

$$\mathrm{Fr}(E_{\mathcal{G}}) \rightrightarrows \mathrm{Fr}(E_M)$$

(Example 5.1) is canonically isomorphic to the PB-groupoid $\mathrm{Fr}(E_{\mathcal{G}}^*) \rightrightarrows \mathrm{Fr}(E_M^*)$ associated to the VB-groupoid $E_{\mathcal{G}}^* \rightrightarrows M \times \{0\}$ with trivial base and core E_M^* (Example 5.2). ◇

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