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The Dirichlet-to-Neumann map for Poincaré–Einstein fillings

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Abstract. We study the non-linear Dirichlet-to-Neumann map for the Poincaré–Einstein filling problem. For even-dimensional manifolds, the range of this non-local map is described in terms of a rank-two "Dirichlet-to-Neumann tensor" along the boundary determined by the Poincaré–Einstein metric. This tensor is proportional to the variation of renormalized volume along a path of Poincaré–Einstein metrics. We construct natural "Dirichlet-to-Neumann hypersurface invariants" that are conformally invariant and recover all Dirichlet-to-Neumann tensors. We give an explicit formula for these hypersurface invariants and use a new vanishing result for odd order T-curvatures to show that they are the unique, natural conformal hypersurface invariant of transverse order equalling the boundary dimension. We also construct such conformally invariant Dirichlet-to-Neumann hypersurface invariants for Poincaré–Einstein fillings for odd-dimensional manifolds with conformally flat boundary.

1. Introduction

A smooth Riemannian d-manifold (M_+, g^o) is said to be conformally compact if M_+ is the interior of a smooth manifold with boundary M, and the *compactified metric*

$$g_s = s^2 g^o$$

extends smoothly to a metric on $M = \bar{M}_+$ for some (and so any) smooth defining function $s \in C^\infty M$. By the latter we mean that $\mathrm{d}s$ is nowhere vanishing along $\Sigma := s^{-1}(0) = \partial M$. Conformally compact manifolds for which the trace-free Ricci tensor vanishes, i.e.,

$$\mathrm{Ric}^{g^o}_{(ab)\circ} = 0,$$

are termed *Poincaré–Einstein*. Necessarily, Poincaré–Einstein manifolds have constant negative scalar curvature, which by convention, we choose to be $-d(d-1)$. A classical example is the Poincaré ball $M_+ = \{\vec{x} \in \mathbb{R}^d \mid |\vec{x}|^2 < 1\}$ with

$$g^o = \frac{4|d\vec{x}|^2}{(1-|\vec{x}|^2)^2}.$$

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In addition to myriad mathematical applications, Poincaré–Einstein and conformally compact structures have played a central rôle in the geometry of physical models relating boundary field theories to bulk gravitational ones (see, for example, [15, 16, 27, 30, 37] and [29, 31, 34], respectively).

Since $s^2 g^o$ extends to the boundary for any defining function s, a conformally compact structure canonically determines a conformal class of boundary metrics $\boldsymbol{c}_\Sigma$ that are induced by a corresponding conformal class of metrics $\boldsymbol{c}$ on M. It turns out that the formal asymptotics of a Poincaré–Einstein metric g^o provide an effective tool for the study of the conformal manifold $(\Sigma, \boldsymbol{c}_\Sigma)$ (see [15, 16]). We say that a conformally compact structure, for which

$$\mathrm{Ric}^{g^o}_{(ab)\circ} = s^k Q_{(ab)\circ}$$

for some smooth tensor Q_{ab} on M^d, is *asymptotically Poincaré–Einstein of order* k and denote such data by APE^d_k. We often drop the label d when context makes it clear. We also adopt the notation $\mathcal{O}(s^k)$ for the right-hand side of the above display and other such quantities. The case $k = d - 2$ is distinguished in the sense that g_s, modulo terms of order $\mathcal{O}(s^d)$, is no longer determined solely by the boundary conformal manifold $(\Sigma, \boldsymbol{c}_\Sigma)$. This is exactly when the asymptotics of a global solution first probe the interior structure of M_+.

A key global conformal invariant of an even-dimensional Poincaré–Einstein structure (M_+, g^o) is its renormalized volume [23, 25, 31], which arises as follows. Firstly, a key result of Graham and Lee [28] (valid for both even or odd dimension parity) is that, for each choice of boundary metric representative $\bar{g}$ for $\boldsymbol{c}_\Sigma$, on a collar neighbourhood of Σ, there exists a unique defining function s_{GL} such that this function returns the (minimal) geodesic distance r to the boundary Σ, as measured by the compactified metric $g_r = r^2 g^o$. Now consider the one-parameter $\varepsilon \in \mathbb{R}_{>0}$ family of Riemannian manifolds M_ε determined by the metric $\bar{g}$ according to

$$M_\varepsilon := \{p \in M_+ \mid r(p) > \varepsilon\},$$

with Riemannian metric given by the restriction of the metric g^o. Then the *renormalized volume* is given, independently of the original choice of boundary metric $\bar{g}$ required to determine s_{GL} and when d is even, by

$$\mathrm{Vol}^{\mathrm{ren}}(M_+, g^o) := \frac{1}{(d-1)!} \frac{\mathrm{d}^{d-1}}{\mathrm{d}\varepsilon^{d-1}} \Big(\varepsilon^{d-1} \int_{M_\varepsilon} \mathrm{dVol}^{g^o} \Big)\Big|_{\varepsilon=0}.$$

For four-manifolds, a result of Anderson [5] is that

$$\mathrm{Vol}^{\mathrm{ren}} = \frac{4\pi^2}{3}\chi - \frac{1}{24}\int_{M_+} |W^{g^o}|^2 \, \mathrm{dVol}^{g^o},$$

where W denotes the Weyl tensor and χ is the Euler characteristic of M_+. Moreover, Anderson also established that the functional gradient of the renormalized volume, along a path of Poincaré–Einstein metrics, is given by

$$\frac{1}{6} \mathring{\top} C^g_{\hat{n}(ab)}.$$

Here C denotes the Cotton tensor, $\hat{n}$ is the inward unit normal of $\Sigma \hookrightarrow (M, g)$, and $\mathring{\top}$ denotes evaluation along Σ as well as projection of $\odot^2 T^*M|_\Sigma$ to directions tangential to Σ and removal of the trace part. The resulting section space is isomorphic to that of $\odot^2_\circ T^*\Sigma$. Note that, since conformal transformations preserve angles (and metric tracelessness), this projection is independent of any choice of $g \in \boldsymbol{c}$. Throughout, we use a $\circ$ to denote trace-free parts of a tensor.

As discussed above, picking a metric $\bar{g} \in \boldsymbol{c}_\Sigma$ and any p in a suitable collar neighbourhood of Σ, we may write the corresponding *Graham–Lee defining function* $s_{\mathrm{GL}}(p) = r(p)$, where $r(p)$ is the minimal geodesic distance between p and Σ, as measured by the metric $g_r = r^2 g^o$. In coordinates $(r, \vec{x})$ for which $g_r(\partial/\partial r, \partial/\partial\vec{x}) = 0$, series in powers of r are termed *Fefferman–Graham expansions*, following their fruitful employment for the description of conformal invariants in [15]. In d dimensions, terms of order r^{d-1} in the expansion of the *Graham–Lee compactified metric* g_r are of particular interest. In particular, in dimension $d = 4$, the third Lie derivative of the compactified metric g_r extracts the term of order r^3 and defines a tensor

$$\mathrm{DN}^{(4)}_{ab} := (\mathcal{L}^3_{\partial/\partial r} g_r)_{ab}\big|_\Sigma .$$

Remarkably, the above tensor satisfies

$$\mathrm{DN}^{(4)}_{ab} \propto \mathring{\top} C^{g_r}_{\hat{n}(ab)},$$

where $\propto$ denotes equality up to a non-zero constant. This tensor is interesting, especially for general relativity (see, for example, [19, 32]), and because it is the first odd-order term in the Fefferman–Graham expansion of g_r. As shown in [15, 25], the tensor $\mathrm{DN}^{(4)}$ changes covariantly when computed with respect to an expansion in a new geodesic coordinate r' corresponding to a different choice of $\bar{g}' \in \boldsymbol{c}_\Sigma$. On the other hand, under a general conformal transformation of the metric

$$\mathring{\top} C^{\Omega^2 g}_{\hat{n}'(ab)} = \bar{\Omega}^{-1} \mathring{\top} C^{g}_{\hat{n}(ab)}, \tag{1.1}$$

where $\hat{n}'_a = \bar{\Omega}\,\hat{n}_a$, for $0 < \Omega \in C^\infty M$ and $\bar{\Omega} := \Omega|_\Sigma$, so long as $g \in \boldsymbol{c}$, where $\boldsymbol{c}$ is a conformal class of metrics defined by a Poincaré–Einstein structure (see [9] for the generalization to arbitrary conformal structures). It therefore constitutes an example of what we shall later term a conformal hypersurface invariant of the conformal embedding $\Sigma \hookrightarrow (M, \boldsymbol{c})$, while the tensor $\mathrm{DN}^{(4)}$ defines a section of $\odot^2_\circ T^*\Sigma[-1]$. (This is the space of symmetric, rank-two, trace-free tensor-valued, weight -1 conformal densities; the latter are explained in Section 2 but here it suffices to view a weight w tensor-valued density T on a conformal manifold M as an equivalence class $T = [g, t] = [\Omega^2 g, \Omega^w t]$ of metric-tensor pairs with respect to conformal rescalings.) In the same vein, the tensor $\mathring{\top} C^g_{\hat{n}(ab)}$ defines a section of $\mathring{\top} \odot^2 T^*M[-1]|_\Sigma$ that encodes the information of $\mathrm{DN}^{(4)}$.

Since the section of $\odot^2_\circ T^*\Sigma[-1]$ defined by the tensor $\mathrm{DN}^{(4)}$ is determined in terms of g^o, it is an invariant of the Poincaré–Einstein structure. However, it is not determined by the local data of the boundary conformal class $\boldsymbol{c}_\Sigma$, again see [15]. Knowledge of $\mathrm{DN}^{(4)}$ requires information about the global Poincaré–Einstein metric g^o. Our point is that in four dimensions, the tensor $\mathring{\top} C^g_{\hat{n}(ab)}$ captures the range of the global, non-linear, Dirichlet-to-Neumann map for Poincaré–Einstein metrics introduced by Graham in [26].

A natural tensor, such as $\mathring{\top} C^{g_r}_{\hat{n}(ab)}$, which depends on g_r through three (and possibly fewer) derivatives with respect to a coordinate r transverse to Σ, is an example of what will be termed a natural hypersurface tensor of transverse order 3 (see Section 3). The natural hypersurface tensor $\mathring{\top} C^{g_r}_{\hat{n}(ab)}$ is special in that it obeys the conformal covariance property of equation (1.1). Broadly, our aim is to construct conformally covariant, natural hypersurface tensors of transverse order $d-1$ in even dimensions d, that capture the Dirichlet-to-Neumann data $\mathrm{DN}^{(d)}$.

In arbitrary even dimensions d, the coefficient $(\mathcal{L}_{\partial/\partial r})^{d-1} g_r|_\Sigma$ in the Fefferman–Graham expansion of a Poincaré–Einstein metric also changes covariantly when computed with respect to a different choice of boundary metric representative $\bar{g} \in \boldsymbol{c}_\Sigma$, but once again it is not determined by the boundary data $(\Sigma, \boldsymbol{c}_\Sigma)$, see [25]. It therefore defines an invariant of the Poincaré–Einstein structure, which we term a (*Poincaré–Einstein*) *Dirichlet-to-Neumann tensor* $\mathrm{DN}^{(d)}$. When it exists (meaning when the data of $(\Sigma, \boldsymbol{c}_\Sigma) \hookrightarrow M^d$ uniquely determines a corresponding Poincaré–Einstein structure, for example, for boundary metrics suitably close to the round sphere [28]), the map

$$\boldsymbol{c}_\Sigma \mapsto \mathrm{DN}^{(d)} \in \Gamma(\odot^2_\circ T^*\Sigma[3-d])$$

is termed the (*Poincaré–Einstein*) *Dirichlet-to-Neumann map*; see [26]. The linearization of this map is studied in [36]. The data of the boundary conformal class of metrics $\boldsymbol{c}_\Sigma$ and the tensor $\mathrm{DN}^{(d)}$ determines the formal Fefferman–Graham asymptotics of g_r to all orders [15].

A first result here is the existence of a unique natural transverse order 5 tensor for dimension 6 Poincaré–Einstein structures that captures the Dirichlet-to-Neumann tensor.

Theorem 1.1. *Let (M^6_+, g^o) be a Poincaré–Einstein structure with conformal infinity Σ. Then the unique natural transverse order 5 section of $\mathring{\top} \odot^2 T^*M[-3]|_\Sigma$ that is an invariant of the Poincaré–Einstein structure is given, up to a non-zero constant multiple, for a choice of g in the conformal class $\boldsymbol{c}$ determined by g^o, by*

$$\mathring{\mathrm{VI}}^{\mathrm{DN}}_{ab} := \mathring{\top}((\nabla_{\hat{n}} + 2H)B_{ab}) - 4\bar{C}_{c(ab)}\bar{\nabla}^c H.$$

Furthermore, $\mathring{\mathrm{VI}}^{\mathrm{DN}} \simeq \mathrm{DN}^{(6)}$.

The detailed notions of transverse order and natural tensors are described in Section 3. In the above, B_{ab} denotes the Bach tensor, H is the mean curvature of the embedding $\Sigma \hookrightarrow (M, g)$, and bars are used to denote objects intrinsic to the hypersurface Σ. Furthermore, since $\mathring{\mathrm{VI}}^{\mathrm{DN}}$ and $\mathrm{DN}^{(6)}$ live in differing section spaces, namely, those of $\mathring{\top} \odot^2 T^*M[-3]|_\Sigma$ and $\mathring{\top} \odot^2 T^*\Sigma[-3]$, respectively, we have employed the notation $\simeq$. For $A \in \mathring{\top} \odot^2 T^*M[3-d]|_\Sigma$ and $B \in \mathring{\top} \odot^2 T^*\Sigma[3-d]$, we write

$$A \simeq B$$

if when given any $\bar{g} \in \boldsymbol{c}_\Sigma$ and its corresponding Graham–Lee compactified metric g_r, then their respective evaluations obey

$$A^{g_r} \propto B^{\bar{g}}.$$

Note that, as bundles, $\mathring{\top} \odot^2 T^*M[-3]|_\Sigma$ and $\mathring{\top} \odot^2 T^*\Sigma[-3]$ are in fact isomorphic. However, on the one hand, the tensor $\mathrm{DN}^{(6)}$ is defined invariantly with respect to choices of $\bar{g} \in \boldsymbol{c}_\Sigma$, while $\mathring{\mathrm{VI}}^{\mathrm{DN}}$ is invariant with respect to choices of $g \in \boldsymbol{c}$. Therefore, we will use both notations, rather than invoking this isomorphism. Also, strictly speaking, since $\mathring{\top}$ includes restriction to Σ in its definition, we could drop the moniker $|_\Sigma$, but shall not do so for reasons of emphasis. Theorem 1.1 can be proved by exhaustion; key details are provided in Appendix A.

A main point of this article is to both give a simple characterization of these maps and, in particular, compute conformal hypersurface invariants, denoted $\mathring{\underline{d}}^{\mathrm{DN}}$ and termed *Dirichlet-to-Neumann hypersurface invariants*, that each capture the Dirichlet-to-Neumann tensor $\mathrm{DN}^{(d)}$ for dimensions $d \geq 4$. Critically, Dirichlet-to-Neumann tensors are natural and of transverse order $d-1$; see Proposition 3.1. That is, we shall establish higher-dimensional analogues of Theorem 1.1.

These results dovetail nicely with a seminal physics conjecture of Deser and Schwimmer [14] that was proved by Alexakis [2, 3], as well as work on renormalized volumes by Anderson [5] and Chang, Qing and Yang [37]. The latter work builds on the former ones to show that general even-dimensional renormalized volumes are a sum of an Euler characteristic term plus an integral over M_+ whose integrand is a natural local conformal invariant. In principle, following Anderson, even-dimensional Dirichlet-to-Neumann tensors could be computed by varying renormalized volumes [5]. The latter are intimately related to Q-curvature invariants, whose complexity in terms of Riemannian invariants explodes in eight and higher dimensions [20], so a more powerful method is required. Luckily, tractor calculus methods and, in particular, recent results on conformally invariant normal operators [21] and conformal fundamental forms [9] yield a rather simple solution to this problem. This is a main theorem:

Theorem 1.2. *Let (M_+^d, g^o) be an even-dimensional Poincaré–Einstein structure with conformal infinity Σ and $d \geq 6$. For $d = 6$, we have that*

$$\mathring{\mathrm{VI}}^{\mathrm{DN}} = \bar{q}^* \mathring{\top} \delta_R W.$$

Also, for $d \geq 8$, set

$$\mathring{\underline{d}}^{\mathrm{DN}} := \bar{q}^* \mathring{\top} \delta_{(d-6)/2,(d-4)/2} W.$$

Then, for $d \geq 6$,

$$\mathring{\underline{d}}^{\mathrm{DN}} \simeq \mathrm{DN}^{(d)}.$$

Moreover, $\mathring{\underline{d}}^{\mathrm{DN}}$ has leading transverse order term

$$\mathring{\top} : \nabla_{\hat{n}}^{d-5} P : B_{ab},$$

and is the unique (up to a non-zero constant multiple) natural conformal hypersurface invariant of transverse order $(d-1)$ in $\Gamma(\mathring{\top} \odot^2 T^*M[3-d]|_\Sigma)$ determined by the Poincaré–Einstein structure. Thus, it is the functional gradient of the renormalized volume along a path of Poincaré–Einstein metrics.

Here we have employed a normal ordering notation $: \nabla_{\hat{n}}^{\ell} :$ for $\hat{n}^{a_1} \cdots \hat{n}^{a_\ell} \nabla_{a_1} \cdots \nabla_{a_\ell}$ (the composition of the operators $\nabla_{\hat{n}} \circ \nabla_{\hat{n}}$ is ill-defined) while the above tractor calculus notions and notations, including the W-tractor W, are explained in Section 2. The normal operator $\delta_{(d-6)/2,(d-4)/2}$ acting on the W-tractor is an example of those constructed in [21] and generalized to act on tractors in [7], see equation (2.5).

It is interesting to note that, in the context of minimal submanifolds of Poincaré–Einstein manifolds, there is a natural analogue of the Dirichlet-to-Neumann map studied here [4, 35], that is again linked to variations of renormalized volume. We expect there to be a version of our construction applicable to that setting.

The recently developed conformal fundamental forms of [7, 9], designed for the study of conformal hypersurface embeddings $\Sigma \hookrightarrow (M, \boldsymbol{c})$, play a key rôle in establishing our main theorems for the case where d is even. Given such an embedding, a k-th *conformal fundamental form* $\mathring{\underline{k}}$ is any section of $\mathring{\top} \odot^2 T^*M[3-k]|_\Sigma$ that is a natural conformal hypersurface invariant of transverse order $k-1$ (in many contexts the integer k is denoted by Roman numerals). In the cases $2 \leq k \leq d-1$, k-th conformal fundamental forms are obstructions to a d-dimensional conformally compact manifold being an APE_{d-3} structure. This generalizes an older result [18, 33] that the hypersurface embedding $\Sigma \hookrightarrow (M, s^2 g^o)$ (for any defining function s) of a Poincaré–Einstein structure (M_+, g^o) is necessarily umbilic – so the trace-free second fundamental form $\mathring{\mathrm{II}}$ vanishes. In fact, vanishing of all conformal fundamental forms for $2 \leq k \leq \ell$ with $\ell \leq d-1$ is both a necessary and sufficient condition for a conformally compact structure to be $\mathrm{APE}_{\ell-2}$. Moreover, we have denoted Dirichlet-to-Neumann hypersurface invariants by $\mathring{\underline{d}}^{\mathrm{DN}}$ because they may be viewed as critical order conformal fundamental forms, whose job is no longer to measure obstructions to the Poincaré–Einstein condition but rather to extract Dirichlet-to-Neumann data.

In dimension four, a conformally invariant fourth fundamental form is explicitly given by (see [9, 19])

$$\mathring{\mathrm{IV}}_{ab} := C^\top_{\hat{n}(ab)} - \bar{\nabla}^c W^\top_{c(ab)\hat{n}} + H W_{\hat{n}ab\hat{n}}.$$

For conformal embeddings arising from an APE_1 structure, the conformal hypersurface invariant $\mathring{\mathrm{IV}}$ restricts to $\mathring{\mathrm{IV}}^{\mathrm{DN}} \simeq \mathrm{DN}^{(4)}$. Importantly, unlike the tensor $\mathrm{DN}^{(4)}$, the fourth fundamental form is a natural conformal hypersurface invariant for *any* conformal hypersurface embedding, rather than only those embeddings arising as the boundary of a Poincaré–Einstein structure. An interesting question is whether there are d-th conformal fundamental forms for more general conformal embeddings.

The case of odd-dimensional Poincaré-Einstein structures is subtle. Firstly, note that no Anderson-type formula for the renormalized volume is available, and in general this quantity depends on a choice of boundary metric representative $\bar{g} \in \boldsymbol{c}_\Sigma$. Concordantly, there is an obstruction to smoothness of the Fefferman–Graham expansion, termed the obstruction tensor [15]. For general boundary conformal structures, the r^{d-1} coefficient in the Fefferman–Graham expansion of a Poincaré–Einstein metric does not define a tensor-density valued in $\odot^2_\circ T^*\Sigma[3-d]$ independently of the choice of $\bar{g} \in \boldsymbol{c}_\Sigma$. Indeed, the quantity $(\mathcal{L}_{\partial/\partial r})^{d-1} g_r$ may not extend smoothly to the boundary Σ. When the Fefferman–Graham obstruction tensor vanishes, we may at least define $\mathrm{DN}^{(d)} := (\mathcal{L}_{\partial/\partial r})^{d-1} g_r|_\Sigma$ for odd d. However, this does not, in general, transform conformally covariantly as one changes the choice of boundary metric $\bar{g}$ required to determine the Graham–Lee compactified metric. But, when the boundary $(\Sigma, \boldsymbol{c}_\Sigma)$ is conformally flat, we can instead define a canonical section $\mathring{\underline{d}}^{\mathrm{DN}^\flat}$ of $\mathring{\top} \odot^2 T^*M[3-d]|_\Sigma$, which is an invariant of the Poincaré–Einstein structure.

Theorem 1.3. *Let (M^5_+, g^o) be a Poincaré–Einstein structure with (locally) conformally flat conformal infinity Σ. Then the unique (up to a non-zero constant multiple) natural transverse order 4 section of $\mathring{\top} \odot^2_\circ T^*M[-2]|_\Sigma$ that is an invariant of the Poincaré–Einstein structure, is given, for a choice of $g \in \boldsymbol{c}$, by*

$$\mathring{\mathrm{V}}^{\mathrm{DN}^\flat} := B^\top_{(ab)\circ}.$$

Moreover, evaluated on the Graham–Lee compactified metric g_r corresponding to a (locally) flat boundary metric,

$$\mathring{\mathrm{V}}^{\mathrm{DN}^\flat} \simeq \mathrm{DN}^{(5)}.$$

The above is an odd-dimensional analogue of Theorem 1.1, and the following is that of Theorem 1.2.

Theorem 1.4. *Let (M^d_+, g^o) be an odd-dimensional Poincaré–Einstein structure with a (locally) conformally flat conformal infinity and $d \geq 5$. If $d = 5$, then we have that*

$$\mathring{\mathrm{V}}^{\mathrm{DN}^\flat} = \mathring{\top} \circ q^*([\hat{n}^a : \nabla_{\hat{n}} : F_{ab}{}^A{}_B]|_\Sigma).$$

For $d \geq 7$, set

$$\mathring{\underline{d}}^{\mathrm{DN}^\flat} := \mathring{\top} \circ q^*([\hat{n}^a : \nabla^{d-4}_{\hat{n}} : F_{ab}{}^A{}_B]|_\Sigma).$$

Then, evaluated on the Graham–Lee compactified metric g_r corresponding to a (locally) flat boundary metric,

$$\mathring{\underline{d}}^{\mathrm{DN}^\flat} \simeq \mathrm{DN}^{(d)}.$$

Moreover, $\mathrm{DN}^{(d)} := (\mathcal{L}_{\partial/\partial r})^{d-1} g_r|_\Sigma$ *defines an element of* $\Gamma(\odot^2_\circ T^*\Sigma[3-d])$.

The article is structured as follows. Section 2 introduces the main technologies we require for handling conformal geometries and hypersurfaces embedded therein (tractor and hypersurface *cognoscenti* might safely leapfrog this section). Our main results are proved in Section 3.

1.1. Conventions

Throughout, we take M to be a smooth, for simplicity, oriented, d-dimensional manifold, with $d \geq 3$, and throughout, unless otherwise specified, all structures are taken to be smooth. The canonical d-form determined by a metric g is denoted dVol^g and may be used when integrating over M; this is normalized so that in local coordinates it gives the measure $\sqrt{\det g}\, \mathrm{d}x^1 \cdots \mathrm{d}x^d$, where g denotes the matrix of metric components. The tangent, cotangent, and tensor bundles of M are denoted, respectively, by TM, T^*M, and $\mathbb{T}M$. Sections are often handled using an abstract index notation; for example, $t^{ab}{}_c \in \Gamma(\otimes^2 TM \otimes T^*M)$, and the contraction $v(\omega)$ of a vector field v and a one-form field ω is denoted by $v^a\omega_a = v_\omega = \omega_v \in C^\infty M$. The inverse of a metric $g_{ab} \in \Gamma(\odot^2 T^*M)$ is denoted g^{ab}, and can be used to "raise indices" in the standard way, for example, $\omega^a = g^{ab}\omega_b$. The symmetric trace-free part of a tensor X_{ab} is denoted

$$X_{(ab)\circ} := \frac{1}{2}(X_{ab} + X_{ba}) - \frac{1}{d}\, g_{ab} X_c{}^c \in \Gamma(\odot^2_\circ T^*M).$$

Also, for a tensor $X_{abc\cdots}$, we denote

$$|X|_g^2 := X_{abc\cdots} X^{abc\cdots}.$$

While we work solely in Riemannian signature, many results carry over *mutatis mutandis* to indefinite metric signatures.

The Levi-Civita connection of a metric is denoted ∇^g (or simply ∇ when it is clear from the context; we will similarly drop the superscript g on other Riemannian tensors). The Riemann tensor R of ∇^g is defined by

$$(R(x,y)z)^a := ((\nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]})z)^a = x^c y^d R_{cd}{}^a{}_b z^b = R_{xy}{}^a{}_z \in \Gamma(TM),$$

where $x, y, z \in \Gamma(TM)$ and $[x, y]$ is their Lie bracket. The Weyl, Cotton and Bach tensors are, respectively, given by

$$\begin{cases} W_{abcd} := R_{abcd} - g_{ac} P_{bd} + g_{ad} P_{bc} + g_{bc} P_{ad} - g_{bd} P_{ac}, \\ C_{abc} := \nabla_a P_{bc} - \nabla_b P_{ac}, \\ B_{ab} := \Delta P_{ab} - \nabla^c \nabla_a P_{bc} + P^{cd} W_{acbd}, \end{cases} \tag{1.2}$$

where the Schouten tensor P_{ab} and its trace $J = P_a{}^a$ are defined by the equation

$$\mathrm{Ric}_{ab} := R_{ca}{}^c{}_b = (d-2) P_{ab} + g_{ab} J,$$

and $\Delta := \nabla^a \nabla_a$ is the (negative) rough Laplacian. We will often place a symbol above an equals sign to qualify its domain of applicability; for example, $A \overset{\Sigma}{=} B$ implies equality along the hypersurface Σ. We also rely on the canonical isomorphism between the projection of the bulk tangent bundle along a hypersurface and the hypersurface tangent bundle, to employ the same abstract indices for hypersurface tensors as for their bulk counterparts.

2. Conformal hypersurface calculus

A *conformal manifold* $(M, \boldsymbol{c})$ is a smooth manifold equipped with a conformal class of metrics $\boldsymbol{c}$, meaning that if $g, g' \in \boldsymbol{c}$, then $g' = \Omega^2 g$ for some $0 < \Omega \in C^\infty M$. A *conformal hypersurface embedding* $\Sigma \hookrightarrow (M, \boldsymbol{c})$ is a conformal manifold equipped with a smoothly embedded codimension 1 submanifold Σ. The data $(M, \boldsymbol{c})$ may be viewed as a ray subbundle of $\odot^2 T^*M$ and in turn as an $\mathbb{R}_+$-principal bundle. The group action $t \mapsto s^{-w} t$ on $t \in \mathbb{R}$ (corresponding to $g(p) \mapsto s^2 g(p)$) for $w \in \mathbb{R}$ and $s \in \mathbb{R}_+$ determines an associated line bundle $\mathcal{E}M[w]$ over M, called a *weight* w *conformal density bundle*. Given any vector bundle $\mathcal{V}M$ over M, we denote $\mathcal{V}M[w] := \mathcal{V}M \otimes \mathcal{E}M[w]$. It is also useful to define the operator $\underline{w}$, which acts by multiplication by w on tensor-valued densities Φ of weight w, so that $\underline{w}\Phi = w\Phi$.

The tautological section $\boldsymbol{g} \in \Gamma(\odot^2 T^*M[2])$ determined by $\boldsymbol{c}$ is called the *conformal metric*. We may equally well label a conformal manifold $(M, \boldsymbol{c})$ by the pair $(M, \boldsymbol{g})$. The *tractor bundle* is defined by (see [12])

$$\mathcal{T}M := \Big(\bigsqcup_{g \in \boldsymbol{c}} \mathcal{T}^g M \Big) \Big/ \sim,$$

where

$$\mathcal{T}_g M := \mathcal{E}M[1] \oplus T^*M[1] \oplus \mathcal{E}M[-1]. \tag{2.1}$$

The equivalence relation $\sim$ on direct sum bundles is defined by the following relation on sections:

$$\Gamma(\mathcal{T}_{g'}) \ni \Big(\tau, \mu + \Upsilon\tau, \rho - \Upsilon.\mu - \frac{1}{2}\Upsilon^2\tau\Big) \sim (\tau, \mu, \rho) \in \Gamma(\mathcal{T}_g),$$

for $\Upsilon := \mathrm{d}\log\Omega$. Here,

$$\Upsilon.\mu := \boldsymbol{g}^{-1}(\Upsilon, \mu) \quad \text{and} \quad \Upsilon^2 := \boldsymbol{g}^{-1}(\Upsilon, \Upsilon).$$

There is a canonical section X of $\mathcal{T}M[1]$ given by

$$(0, 0, 1) \in \mathcal{T}_g M[1],$$

for any $g \in \boldsymbol{c}$, which is termed the *canonical tractor*. Also, there is an indefinite, conformally invariant, bundle metric $h \in \Gamma(\odot^2 \mathcal{T}^* M)$ given, in an obvious matrix notation, by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}^{-1} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and termed the *tractor metric*. Sections $T \in \Gamma(\mathcal{T}M)$ can be denoted in an abstract index notation by T^A, so the tractor metric is then h_{AB} and will be used to raise and lower tractor indices in the standard way. For example, $h_{AB}X^AX^B$ is the zero section of $\mathcal{E}M[2]$.

On an oriented conformal hypersurface $\Sigma \hookrightarrow (M, \boldsymbol{c})$, there is a canonical section N of $\mathcal{T}M|_\Sigma$ given, for any $g \in \boldsymbol{c}$, by

$$(0, \hat{n}, -H) \in \mathcal{T}_g M,$$

where $\hat{n}$ is the inward unit conormal of g and H its mean curvature (i.e., the average of the eigenvalues of the second fundamental form). This section is called the *normal tractor* [6] and is central to the study of conformally embedded hypersurfaces. The normal tractor and tractor metric allow us to define the tractor analogue of the projection operator $\mathring{\top}$ in the obvious way; we recycle the same notation for this. There is a particularly useful relationship between N and defining functions for Σ, but we first must introduce further tractor technology.

The tractor bundle $\mathcal{T}M$ comes equipped with a canonical connection $\nabla^{\mathcal{T}}$, termed the *tractor connection*, defined for any $g \in \boldsymbol{c}$, and

$$T \overset{\mathcal{T}_g}{:=} (\tau, \mu, \rho) \in \Gamma(\mathcal{T}_g M)$$

by

$$\nabla^{\mathcal{T}} T \overset{\mathcal{T}_g}{=} (\nabla\tau - \mu, \nabla\mu + P^g\tau + \boldsymbol{g}\rho, \nabla\rho - P^g.\mu). \tag{2.2}$$

We will adorn equal signs with the symbol $\mathcal{T}_g$ to indicate that we have used the metric g to decompose sections of (tensor products of) the tractor bundle into a direct sum of density-valued sections according to equation (2.1).

The curvature of $\nabla^{\mathcal{T}}$ is denoted by F, or $F_{ab}{}^{A}{}_{B}$ in an abstract index notation. In the above,

$$P^{\boldsymbol{g}}.\mu := P(\boldsymbol{g}^{-1}(\mu,\cdot),\cdot).$$

Also, in the above, ∇ is the Levi-Civita connection ∇^g of g. It follows from our discussion of conformal densities that, given $v \in \Gamma(TM)$ and a weight w density $\varphi = [g; f]$, one has $\nabla^g_v \varphi = [g, (\mathrm{d}f)(v)]$. Note that a conformal density $0 < \tau \in \Gamma(\mathcal{E}M[1])$ defines a metric $g = \tau^{-2}\boldsymbol{g} \in \boldsymbol{c}$ and conversely a metric $g \in \boldsymbol{c}$ determines a strictly positive density

$$\tau_g := (\mathrm{dVol}^{\boldsymbol{g}}/\mathrm{dVol}^{g})^{1/d} \in \Gamma(\mathcal{E}M[1]).$$

We will term such a scale *true*. Thus, given $g \in \boldsymbol{c}$, we can define a canonical section

$$Z^A_a := \tau_g \nabla^{\mathcal{T}}_a (\tau_g^{-1} X^A) \in \Gamma(T^*M \otimes \mathcal{T}M[1]).$$

Tractor tensor bundles are defined in the standard way and sections of such can also be denoted by an abstract index notation, for example, $T^{AB}{}_C \in \Gamma(\otimes^2 \mathcal{T}M \otimes \mathcal{T}^*M)$; we will often label a generic tractor tensor bundle by $\mathcal{T}^\Phi M$ (in general, Φ is some representation of $\mathrm{SO}(d+1,1)$). Given a tractor $T \in \Gamma(\mathcal{T}^\Phi M[w])$, the *Thomas-D operator*

$$D : \Gamma(\mathcal{T}^\Phi M[w]) \to \Gamma(\mathcal{T}M \otimes \mathcal{T}^\Phi M[w-1])$$

is defined (see [6]) acting on T, for any $g \in \boldsymbol{c}$, as follows:

$$\begin{aligned} DT &= (w(d+2w-2)T, (d+2w-2)\nabla^{\mathcal{T}}T, -\Delta^{\mathcal{T}}T - w\boldsymbol{g}^{ab}P^g_{ab}T) \\ &\in \Gamma(\mathcal{T}_g M \otimes \mathcal{T}^\Phi M[w-1]), \end{aligned}$$

where

$$\Delta^{\mathcal{T}} := \boldsymbol{g}^{ab}\nabla^{\mathcal{T}}_a \nabla^{\mathcal{T}}_b$$

is the tractor Laplacian and $\nabla^{\mathcal{T}}$ denotes the tractor Levi-Civita coupled connection. When $w \neq 1 - d/2$, we define $\hat{D} := (d+2w-2)^{-1}D$.

There is a useful tractor analogue of the Weyl tensor: the *W-tractor* is a canonical section of $\otimes^4 \mathcal{T}^*M[-2]$, with Weyl-tensor symmetries, defined in dimensions $d \geq 5$, for any choice of $g \in \boldsymbol{c}$, by

$$\begin{aligned} W^{ABCD} &\stackrel{\mathcal{T}_g}{=} Z^A_a Z^B_b Z^C_c Z^D_d W^{abcd} \\ &\quad + 4 Z^{[A}_a Z^{B]}_b Z^{[C}_c X^{D]} C^{abc} + \frac{4}{d-4} Z^{[A}_a X^{B]} Z^{[C}_c X^{D]} B^{ac}, \end{aligned}$$

where W_{abcd}, C_{abc} and B_{ab} are, respectively, the Weyl, Cotton and Bach tensors of the metric g. Note that

$$F_{ab}{}^{CD} = Z^A_a Z^B_b W_{AB}{}^{CD} = Z^C_c Z^D_d W_{ab}{}^{cd} + 2 Z^{[C}_c X^{D]} C_{ab}{}^c,$$

see [9, 17, 20] for further details. Note that some authors define a W-tractor in dimensions $d \geq 5$ by multiplying the above definition by a factor $(d-4)$ so that it is defined in all dimensions $d \geq 3$.

Indeed, there is a general notion termed the *projecting part* of a tractor, of which we only need special cases. In particular, for a tractor $T \in \Gamma(\mathcal{T}^{\otimes 4}M[w])$ given in a choice of $g \in \boldsymbol{c}$ by

$$T^{ABCD} \stackrel{\mathcal{T}_g}{=} 4Z_a^{[A}X^{B]}Z_c^{[C}X^{D]}t^{ac}$$

such that $t^{ab} \neq 0$, we can extract the projecting part

$$q^*(T^{ABCD}) := [g; t_{ab}] \in \Gamma(\otimes^2 T^*M[w]).$$

Similarly, for a tractor $T^{AB} \stackrel{\mathcal{T}_g}{=} 2Z_b^{[A}X^{B]}t^b$, we have that $q^*(T^{AB}) = [g; t^b]$.

Given a conformal hypersurface embedding $\Sigma \hookrightarrow (M, \boldsymbol{c})$, we call $\sigma \in \Gamma(\mathcal{E}M[1])$ a *defining density* if $\sigma = s\tau$ for some $0 < \tau \in \Gamma(\mathcal{E}M[1])$ and s a defining function for Σ. Then we may define a metric

$$g^o = \sigma^{-2}\boldsymbol{g}$$

on $M_+ = M \setminus \Sigma$. Because g^o is not defined along $\Sigma = \partial M_+$, we refer to it as a *singular metric*. An important theorem of [18] is that (M_+, g^o) is Poincaré–Einstein precisely when the *scale tractor* $I_\sigma := \hat{D}\sigma$ is parallel with respect to the tractor connection $\nabla^{\mathcal{T}}$ and normalized, i.e.,

$$\nabla^{\mathcal{T}} I_\sigma = 0 = I_\sigma^2 - 1,$$

where $I_\sigma^2 := h(I_\sigma, I_\sigma)$. This allows us to label a Poincaré–Einstein structure by the data $(M, \boldsymbol{g}, \sigma)$, where the singular Einstein metric $g^o = \sigma^{-2}\boldsymbol{g}$. Note that, as was discussed in the introduction, the k-th order asymptotics of this data can be labelled by a representative triple (M^d, g_k, s), for which $\boldsymbol{c}_k = [g_k]$ and $\sigma = [g_k; s]$, and such that $(M_+, \sigma^{-2}g_k)$ is APE^d_k. We recall that for $\mathrm{APE}^d_{k\geq 0}$ structures, the normal tractor to Σ is given by (see [18])

$$N = I_\sigma|_\Sigma.$$

The final piece of conformal hypersurface technology that we need is a generalization of the scalar normal operators of [21]. For the simplest of these, recall that the *conformal-Robin operator* $\delta_R \colon \Gamma(\mathcal{E}M[w]) \to \Gamma(\mathcal{E}\Sigma[w-1])$ is defined along Σ, for any $g \in \boldsymbol{c}$ and $[g; f] \in \Gamma(\mathcal{E}M[w])$, by (see [11])

$$\delta_R f = (\nabla_{\hat{n}} - wH^g)f|_\Sigma.$$

This generalizes directly to tractors by replacing the Levi-Civita connection in the above with its tractor-coupled analogue.

If s is any defining function for Σ, we say that an operator

$$\mathsf{L} : \Gamma(\mathcal{T}^\Phi M[w]) \to \Gamma(\mathcal{T}^{\Phi'}M[w'])|_\Sigma$$

has *transverse order* k if

$$\mathsf{L}(s^{k+1}U) = 0 \neq \mathsf{L}(s^k V), \tag{2.3}$$

for every smooth tractor U and some smooth tractor V in its domain. Clearly, δ_R has transverse order 1. The *normal operators* δ_k defined by

(2.4) $\Gamma(\mathcal{T}^\Phi M[w]) \ni T$

$$\stackrel{\delta_k}{\longmapsto} \; : (N^A D_A)^k : T := N^{A_k} \cdots N^{A_1} D_{A_1} \cdots D_{A_k} T \in \Gamma(\mathcal{T}^\Phi M[w-k])|_\Sigma,$$

have transverse order k for generic weights $w \notin \{\frac{2k-d}{2}, \frac{2k-1-d}{2}, \ldots, \frac{k+1-d}{2}\}$, see [21]. An improvement of these operators acting on densities was also produced in Theorem 4.16 of [21], where the set of weights for which the transverse order is less than k was shrunk. In Theorem 3.4 of [7], these improved operators were generalized to act on arbitrary tractors. In particular, for even d and integers $0 < J$ and $0 < k < d/2$, it was shown that they give maps

$$\delta_{J,k} : \Gamma(\mathcal{T}^{\Phi} M[w]) \to \Gamma(\mathcal{T}^{\Phi} M[w-k-J])|_{\Sigma}, \tag{2.5}$$

with transverse order $J + k$ also at the special weights $w = k - d/2$. Importantly, these existence proofs are both constructive and algorithmic.

3. Dirichlet-to-Neumann tensors

Any Poincaré–Einstein structure (M_+, g^o) determines a conformal embedding

$$\Sigma \hookrightarrow (\bar{M}_+, \boldsymbol{c})$$

of its boundary, where $\boldsymbol{c}$ is the conformal class of metrics on $M := \bar{M}_+$ determined by g^o. Moreover, this embedding must be umbilic [18,33], so the corresponding trace-free second fundamental form $\mathring{\mathrm{II}} = 0$. This tensor is an example of a natural conformal hypersurface invariant (see below). In fact, a slew of other natural conformal hypersurface invariants are also forced to vanish by the Einstein condition; this underlies the conformal fundamental forms construction of [9].

We next need to define natural conformal hypersurface invariants. First, recall that a natural Riemannian invariant on a Riemannian manifold is any tensor-valued polynomial made from the metric, its inverse, the Riemann tensor, and covariant derivatives of the latter. Strictly, this is the class of "even invariants", meaning those that are unaffected by a change of orientation. It is clear that this type of invariant is all that will be required in our context. This definition extends in the obvious way to a Riemannian manifold equipped with any set of scalar functions. Extending these to natural conformal hypersurface invariants is discussed in detail in Section 2.4 of [24]. Here we are interested in the same notion with some restrictions.

Consider triples $(M, \boldsymbol{g}, \sigma)$, where σ is a defining density for a conformally embedded hypersurface $\Sigma \hookrightarrow (M, \boldsymbol{c})$. For any triple $(M, \boldsymbol{g}, \sigma)$, now consider a representative pair (g, s) for σ. Then we can construct the natural Riemannian invariants $i(g, s)$ built from the metric g and the scalar function $\hat{s} := s/|\mathrm{d}s|_g$ (in some collar neighbourhood of Σ where the denominator is non-vanishing), and then restrict these to Σ. In turn, we consider the special class of such objects $[g; i(g,s)|_\Sigma] \in \Gamma(\mathbb{T}M)|_\Sigma$ for which

$$\bar{\Omega}^{-w}\, i(\Omega^2 g, \Omega s)|_\Sigma = i(g,s)|_\Sigma \quad \text{and} \quad i(g, e^{\varpi} s)|_\Sigma = i(g,s)|_\Sigma,$$

where $0 < \Omega \in C^\infty M \ni \varpi$, $\bar{\Omega} = \Omega|_\Sigma$ and $w \in \mathbb{R}$. The equivalence class $[g, i(g,s)|_\Sigma]$ defines a (weight w) *natural conformal hypersurface invariant*. Upon dropping the conformal requirement imposed by the first equality displayed above, we employ the terminology *natural Riemannian hypersurface invariant*.

We use the same language to also refer to the larger class of natural Riemannian hypersurface invariants $i(g,s)$ that may only obey the requirements above when restricting to the subset of triples $(M, \boldsymbol{g}, \sigma)$ such that $(M_+, \sigma^{-2}\boldsymbol{g})$ is Poincaré–Einstein. An example of such a natural conformal hypersurface invariant for Poincaré–Einstein structures is $\mathring{\top} C_{\hat{n}(ab)}|_\Sigma \in \Gamma(\mathring{\top} \odot^2 T^*M[-1])|_\Sigma$ (see [9, 19]). For brevity, we often refer to such tensors as "natural" and list the relevant section spaces and underlying structures. In some contexts, we additionally require that the boundary $(\Sigma, \bar{\boldsymbol{c}})$ is conformally Einstein or conformally flat.

We employ the terminology *preinvariant* for any representative $i(g,s)$ of a natural conformal hypersurface invariant I (see [24]). Examples of natural conformal hypersurface invariants include the unit conormal $\hat{n}$ and trace-free second fundamental form $\mathring{\mathrm{II}}$. Sample preinvariants for these are, respectively,

$$\mathrm{d}\hat{s} = \mathrm{d}(s/|\mathrm{d}s|_g), \quad (\mathrm{Id} - \mathrm{d}\hat{s} \otimes g^{-1}(\mathrm{d}\hat{s}, \cdot)) \circ \nabla \mathrm{d}\hat{s}.$$

Consider a conformal hypersurface invariant I that is represented by $i(g,s)$ and obeys

$$i(g + s^k h, s)|_\Sigma \neq i(g,s)|_\Sigma = i(g + s^{k+1} h', s)|_\Sigma, \quad k \in \mathbb{Z}_{\geq 0},$$

for some h and any $h' \in \Gamma(\odot^2 T^*M)$ such that $g + s^k h, g + s^{k+1} h'$ are metrics on M. The number k does not depend on the choice of (g,s) used to construct I. Thus (in line with definition of equation (2.3)), we then say that I has *transverse order* k.

In fact, Dirichlet-to-Neumann tensors are natural hypersurface invariants:

Proposition 3.1. *Let (M^d_+, g^o) be a Poincaré–Einstein structure. In the case that d is odd, also assume that the boundary Σ is conformally flat. Then the Dirichlet-to-Neumann tensor $\mathrm{DN}^{(d)}$ is a natural hypersurface invariant.*

Proof. First, recall that, given a Riemannian hypersurface embedding $\Sigma \hookrightarrow (M, g)$ and any defining function s, it is always possible to improve s to $s_1 = fs$ such that

$$|ds_1|^2_g = 1 + \mathcal{O}(s^\ell),$$

for any integer ℓ, and the smooth function f is some local formula in terms of s and g; see [24] (this is an asymptotic solution to the well-known eikonal problem). To (arbitrarily) high order, the improved defining function s_1 gives the geodesic distance to the hypersurface.

Now pick $\bar{g} \in \boldsymbol{c}_\Sigma$. This determines the corresponding Graham–Lee compactified metric g_r. Then applying the above eikonal construction to g_r, starting with any defining function s, at least to sufficiently high order for our purposes, we obtain an asymptotic formula $s^{(\ell)}_{\mathrm{GL}}$ for the Graham–Lee defining function s_{GL} as a natural formula in terms of s and g_r (this follows directly from the construction and uniqueness statement given in Proposition 2.5 of [24]). Moreover, by picking large enough ℓ, we have that $\hat{s} = s^{(\ell)}_{\mathrm{GL}}/|\mathrm{d}s^{(\ell)}_{\mathrm{GL}}|_{g_r}$ equals the geodesic normal coordinate function r to any desired order. Thus, we can ignore higher-order corrections in what follows.

Now recall that, given a vector $x \in \Gamma(TM)$ and a tensor $T \in \Gamma(\odot^2 T^*M)$, one has

$$\mathcal{L}_x T_{ab} = x^c \nabla_c T_{ab} + (\nabla_a x^c) T_{cb} + (\nabla_b x^c) T_{ac}.$$

Note that the vector $\partial/\partial r$ determined by the normal form of the Graham–Lee compactified metric is natural because it solves the equation $g_r(\partial/\partial r, \cdot) = \mathrm{d}s_{\mathrm{GL}}$ and s_{GL} has a natural formula in terms of s. Because $\hat{s} = r$, we have $\mathcal{L}_{\partial/\partial r} g_r = 2\nabla^{g_r} \mathrm{d}\hat{s}$. Similarly, higher Lie derivatives with respect to the vector $\partial/\partial r$ can also be expressed naturally in terms of $\mathrm{d}\hat{s}$, the metric g_r, its inverse and covariant derivatives. Hence, we have by now expressed such Lie derivatives of the Graham–Lee compactified metric g_r as a natural formula in s and the metric g_r. The result follows upon remembering that for Poincaré–Einstein structures, the tensor $\mathrm{DN}^{(d)}$ is defined through $d-1$ Lie derivatives of the Graham–Lee compactified metric g_r with respect to the vector $\partial/\partial r$. ■

3.1. *T*-curvatures

Before proving our main results, we need the notion of a T-curvature and a result of [21]. Recall that the mean curvature H^g behaves as follows under conformal rescalings:

$$H^{\Omega^2 g} = \Omega^{-1}(H^g + \delta_1 \log \Omega),$$

where $\delta_1 = \delta_R$ is the first of the sequence of operators defined in equation (2.4), in the case where $w = 0$ and Φ is the trivial representation (and also in the scale g). Each *T-curvature* is a higher-order generalization of the mean curvature to a natural Riemannian hypersurface invariant T_k^g with conformal variation

$$T_k^{\Omega^2 g} = \Omega^{-k}(T_k^g + N_k \log \Omega), \tag{3.1}$$

where N_k is any conformally invariant hypersurface operator with differential order k and with leading symbol containing a term arising from an operator proportional to $: \nabla_{\hat{n}}^k :$ for some $k \geq 2$. The integer k equals the transverse order of the T-curvature T_k, so we shall call T_k a *T-curvature of order k*. For example, $T_1 = H$ is (up to an overall non-zero coefficient) the first T-curvature and the scalar

$$T_2 := P_{\hat{n}\hat{n}} - \frac{1}{2} H^2 - \frac{1}{d-3} \bar{J}$$

is also a T-curvature, see Section 7.1 of [21]. Existence of T-curvatures of arbitrary order on any even-dimensional conformal manifold with boundary was established in Theorem 4.16 of [21]. Moreover, in that work, the following result was proved.

Proposition 3.2 (Proposition 6.15 in [21]). *Let $\Sigma \hookrightarrow (M^d, \boldsymbol{c})$ be a conformal hypersurface embedding with d even. Then, for any finite set of T-curvatures $\{T_1^g, \dots, T_k^g\}$, there exists a metric $g' \in \boldsymbol{c}$ such that*

$$T_i^{g'} = 0, \quad \textit{for all } i \in \{1, \dots, k\}.$$

An example of how we plan to use the above proposition is as follows: The usual definition of the dimension $d \geq 4$ *Fialkow tensor* is (for some choice of $g \in \boldsymbol{c}$)

$$F_{ab} := P_{ab}^{\top} - \bar{P}_{ab} + H \mathring{\mathrm{I\!I}}_{ab} + \frac{1}{2} \bar{g}_{ab} H^2 \stackrel{\Sigma}{=} \frac{1}{d-3}\Big(W_{\hat{n}a\hat{n}b} + \mathring{\mathrm{I\!I}}_{ac} \mathring{\mathrm{I\!I}}^c{}_b - \frac{1}{2(d-2)} \bar{g}_{ab} |\mathring{\mathrm{I\!I}}|^2\Big).$$

The second equality above is a standard hypersurface identity establishing that the Fialkow tensor defines a conformally invariant section of $\mathring{\top} \odot^2 T^*M[0]|_\Sigma$. This identity can be equivalently rewritten as

$$R^{\top}_{\hat{n}a\hat{n}b} \stackrel{\Sigma}{=} \bar{P}_{ab} - \frac{1}{d-3}\,\bar{g}_{ab}\bar{J} + (d-2)F_{ab} - \mathring{\mathrm{II}}_{ac}\mathring{\mathrm{II}}^{c}{}_{b} + \frac{1}{2(d-2)}\,\bar{g}_{ab}|\mathring{\mathrm{II}}|^2 - H\,\mathring{\mathrm{II}}_{ab} + \bar{g}_{ab}T_2. \tag{3.2}$$

Importantly, both the trace-free Fialkow tensor and trace-free second fundamental form vanish for APE structures of sufficiently high order; they are the first two conformal fundamental forms discussed above (note also that the trace $F_a{}^a = \frac{1}{2(d-2)}|\mathring{\mathrm{II}}|^2$). Thus, for APE_1 structures, the right-hand side of equation (3.2) reduces to the sum of an intrinsic tensor plus a T-curvature. Hence, Proposition 3.2 can be used to establish that there exists a metric in the conformal class $\boldsymbol{c}$ such that $R^{\top}_{\hat{n}a\hat{n}b}|_\Sigma$ is expressible in terms of intrinsic quantities alone when the structure is Poincaré–Einstein. The above decomposition into intrinsic curvatures, conformal fundamental forms, and T-curvatures applies generally to bulk curvatures and their derivatives; see [8] where a classification of certain hypersurface invariants was given (note that we, in particular, rely on Theorem 2.5 of that work, which applies for d even). It also underlies the strategy for the proof of Theorem 3.7, where it is used to show that derivatives of ambient curvatures can be re-expressed in terms of intrinsic invariants, conformal fundamental forms and T-curvatures. The last of these can be eliminated using Proposition 3.2.

In the case of Poincaré–Einstein structures, there is a particularly simple T-curvature construction. This involves log densities given by $\log\tau$, where τ is a true scale. These are defined in [22], but the details are not important here. The key point is that the *Laplace–Robin operator* $I_\sigma.D$ acting on a log density, returns a density of weight -1, according to

$$I_\sigma.D\log\tau = \frac{d-2}{\tau}\,I_\sigma.\hat{D}\tau + \frac{\sigma}{\tau^2}(\hat{D}\tau)^2 \in \Gamma(\mathcal{E}M[-1]). \tag{3.3}$$

Then, following [21], a T-curvature of order $k \le \lfloor d/2 \rfloor$ is given by

$$t^g_k := \delta_R(I_\sigma.D)^{k-1}\log\tau|_\Sigma.$$

Notice that $t_1 = \delta_R\log\tau|_\Sigma = -H$, because δ_R can here be replaced by $(d-2)^{-1}I_\sigma.D$ restricted to Σ.

When d is even, a version of the above construction applies to $k > d/2$. This is based on a polynomial weight continuation of the operator $\delta_R(I_\sigma.D)^{k-1}$ acting on densities of weight w; again see [21]. Let us now explain; for simplicity, take k odd. Then one has

$$\delta_R(I_\sigma.D)^{k-1} = \hat{P}_k(d+2w-2k+2)(d+2w-2k+4)\cdots(d+2w-k-1), \tag{3.4}$$

where the operator $\hat{P}_k$ is conformally invariant, even when acting on weights w such that any of the factors to its right vanish. It can be expressed in terms of bulk covariant derivatives, curvatures, the scale σ, and depends polynomially on w.

Remark 3.3. The operator $\hat{P}_k$ is proportional to the operator

$$\delta_{J,k-J} : \Gamma(\mathcal{E}M[k-J-d/2]) \to \Gamma(\mathcal{E}\Sigma[-J-d/2]),$$

for some $J \in \{1, \dots, (k-1)/2\}$ and odd $k > d/2$, constructed in [21] (see equation (2.5) above). This corresponds to removing a factor $(d + 2w - 2k + 2J)$ from the right-hand side of equation (3.4) and then continuing in w to the critical value $w = k - J - d/2$.

The action of $\hat{P}_k$ on a log density is defined by replacing appearances of w by the operator $\underline{w}$ placed on the far right of any term. Note that $\underline{w} \log \tau = 1$. Based on [21], it is easy to show that

$$t_k := \hat{P}_k \log \tau|_\Sigma \tag{3.5}$$

defines a T-curvature for any odd $k > d/2$. (A very similar construction applies to even k, but is not important here.) We now characterize these T-curvatures for the Graham–Lee scale of a Poincaré–Einstein structure.

Proposition 3.4. *Let $(M^d, \boldsymbol{g}, \sigma)$ be Poincaré–Einstein. If g_r is the Graham–Lee compactified metric for some $\bar{g} \in \boldsymbol{c}_\Sigma$, then for any odd integer $k \le \lfloor d/2 \rfloor$,*

$$t_k^{g_r} = 0.$$

Proof. It is well known that the mean curvature of $\Sigma \hookrightarrow (M, g_r)$ vanishes, so $t_1^{g_r} = 0$.

Now let τ be any true scale, let g be the corresponding compactified metric and call $s = \sigma/\tau$. Then the quantity appearing in equation (3.3) (divided by $d-2$) can be expressed as

$$\frac{I_\sigma.\hat{D}\tau}{\tau} + \frac{1}{d-2}\,\frac{\sigma}{\tau^2}(\hat{D}\tau)^2 \overset{g}{=} -\frac{\Delta^g s + J^g s}{d} - \frac{s}{d-2}\,J^g.$$

Here we have placed a g above the equality sign to indicate that we have trivialized the density bundle $\mathcal{E}M[-1]$ using the choice of metric g to obtain the right-hand side of the above display. This notational device will appear again below. Now, if we specialize the above formula to the Graham–Lee compactified metric g_r, then $s = r$, the geodesic distance to the boundary. Moreover, the Graham–Lee metric has the normal form $g_r = dr^2 + h(r)$. We shall say that a polyhomogeneous expansion in the variable r is even/odd to order m if there are no odd/even terms of order m or less and there are no terms involving $r^k \log r$ for $k \le m$. For a Poincaré–Einstein structure, $h(r)$ has an even expansion in the coordinate r up to order $d-2$, see [15]. Hence, the above display has an odd expansion in r to order $d-3$ (a slightly stronger statement is available when d is even, see the proof of Proposition 3.5). Both the Laplace–Robin and conformal-Robin operator are odd operators with respect to r to some order. Indeed, acting on a density of weight w and employing r as a coordinate in a collar neighbourhood of Σ, one has

$$I_\sigma.D \overset{g_r}{=} (d + 2w - 2)\partial_r - r\partial_r^2 + \operatorname{odd}(r),$$

where (the operator) $\operatorname{odd}(r)$ stands for terms that are odd to order $d-3$. Thus, acting at weight $w = 0$ and in the Graham–Lee scale,

$$\begin{aligned}\delta_R (I_\sigma.D)^{k-2} \\ &= (\partial_r + \operatorname{odd}(r))\,\underbrace{((d-2k+4)\partial_r - r\partial_r^2 + \operatorname{odd}(r))\cdots((d-2)\partial_r - r\partial_r^2 + \operatorname{odd}(r))}_{k-2 \text{ terms}}.\end{aligned}$$

Hence, $\delta_R(I_\sigma.D)^{k-1}\log\tau$ is odd to order $d-2-k$ and thus clearly vanishes along Σ when $k \leq \lfloor d/2 \rfloor$ and k is odd. ∎

Specializing to even dimensions d, we have a stronger statement.

Proposition 3.5. *Let $(M^d, \boldsymbol{g}, \sigma)$ be Poincaré–Einstein and let d be even. If g_r is the Graham–Lee compactified metric for some $\bar{g} \in \boldsymbol{c}_\Sigma$, then for any odd integer $k \leq d-1$,*

$$t_k^{g_r} = 0.$$

Proof. The case when $k \leq d/2$ was treated in Proposition 3.4. Hence, we consider the T-curvatures defined in equation (3.5). The Laplace–Robin operator is given, for some choice of $g \in \boldsymbol{c}$, by

$$I_\sigma.D = (d+2w-2)(\nabla_n + w\rho) - \sigma(\Delta + wJ),$$

where $\rho := -\frac{1}{d}(\Delta\sigma + J\sigma)$. Hence, we see that $\hat{P}_k$ can be expressed as a sum of words of length k built partly from the following four letters:

$$\alpha := \nabla_n, \quad \beta := \rho, \quad \gamma := \sigma\Delta, \quad \delta := \sigma J.$$

We must also allow a fifth letter

$$\varepsilon := \underline{w}$$

that may appear as many as $2k-1$ times at the far right of any word. It is given by the weight operator $\underline{w}$, which returns a factor of w, the weight of a density, when acting on such. Also, recall that $\underline{w}\log\tau = 1$, see [22]. As an example, $I_\sigma.D$ is expressed as a sum of the words α, $\alpha\varepsilon$, $\beta\varepsilon$, $\beta\varepsilon^2$, γ, and δ. Viewed as operators, in the Graham–Lee scale, each of the four letters $\alpha, \beta, \gamma, \delta$ is odd up to high order, or more precisely can be expressed as a sum of terms

$$\partial_r, \quad r\partial_r^2, \quad \mathrm{odd}(r),$$

where $\mathrm{odd}(r)$ stands for terms odd in r to order $d-3$. Also, in the Graham–Lee scale, one has $\log\tau = 0$, $\nabla_n \log\tau = 0$, $\Delta\log\tau = 0$, while in general $\underline{w}\log\tau = 1$ and $\underline{w}^\ell \log\tau = 0$ for $\ell \geq 2$. One can now apply the same parity argument as in the proof of Proposition 3.4 to establish that t_k vanishes for odd $k < d-1$. For the case t_{d-1}, we must also carefully examine the highest r-derivatives of the terms $\mathrm{odd}(r)$. It not difficult to see that along Σ (where $r = 0$), the highest order term is $\partial_r^{d-2}\,\mathrm{odd}(r)|_{r=0}$. Since this quantity is a scalar, a simple analysis of the operators appearing in $\alpha, \beta, \gamma, \delta$ shows that it is proportional to

$$\mathrm{tr}_{h(r)}\,\partial_r^{d-1} h(r)|_{r=0}.$$

This is precisely the boundary trace of the image of the Dirichlet-to-Neumann map. The latter is trace-free (see [16]). ∎

A uniqueness argument establishes a vanishing result for odd-order T-curvatures.

Theorem 3.6. *Let $(M^d, \boldsymbol{g}, \sigma)$ be Poincaré–Einstein and d be even. If g_r is the Graham–Lee compactified metric for some $\bar{g} \in \boldsymbol{c}_\Sigma$, then any T-curvature of odd-order $k \leq d-1$ vanishes.*

Proof. We have established that the odd-order T-curvatures t_k vanish, so it only remains to establish the uniqueness of odd-order T-curvatures $T_{k \le d-1}$ in the current setting.

On an even-dimensional Riemannian manifold, every natural hypersurface invariant with transverse order k may be expressed as a (partial) contraction polynomial in

$$\{\bar{g}, \bar{g}^{-1}, \bar{\nabla}, \bar{R}, \hat{n}, \mathring{\mathrm{II}}, \mathring{\top} : \nabla_{\hat{n}} : P, \ldots, \mathring{\top} : \nabla_n^{k-2} : P, H, J|_\Sigma, \ldots, : \nabla_{\hat{n}}^{k-2} : J|_\Sigma\},$$

see Theorem 2.5 in [8]. Notably, on a Poincaré–Einstein manifold, when $k \le d-2$, this family reduces to

$$\{\bar{g}, \bar{g}^{-1}, \bar{\nabla}, \bar{R}, \hat{n}, H, J|_\Sigma, \ldots, : \nabla_{\hat{n}}^{k-2} : J|_\Sigma\}.$$

To see this, firstly note that conformal fundamental forms exist up to $\underline{\overset{\circ}{\overline{d-1}}}$ for embeddings in even-dimensional conformal manifolds, see Proposition 3.6 in [7]. Moreover, normal derivatives of P may be expressed in terms of these conformal fundamental forms, as well as terms from the list directly above, see Corollary 3.3 in [8]. However, on Poincaré–Einstein manifolds, all conformal fundamental forms up to and including $\underline{\overset{\circ}{\overline{d-1}}}$ (which has transverse order $d-2$) vanish, see Theorem 1.8 in [10]. We wish to consider transverse order as high as $k = d-1$, and so we must, in principle, also include $\mathring{\top} : \nabla_{\hat{n}}^{d-3} : P$ in the above family of terms.

By the transverse order requirement for T-curvatures, the log term of equation (3.1) must take the form

$$N_k \log \Omega = \Omega^{-1} : \nabla_{\hat{n}}^k : \Omega + \text{ltots}, \tag{3.6}$$

where "ltots" stands for terms involving only lower transverse order operators acting on Ω. Also, it follows from the conformal transformation rule for the Schouten tensor P that

$$J^{\Omega^2 g} \overset{\Sigma}{=} J^g - \Omega^{-1} : \nabla_{\hat{n}}^2 : \Omega + \text{ltots}.$$

And so, for $2 \le k \le d-2$, the only term in our family with the correct transformation law (matching equation (3.6)) at leading transverse order is $: \nabla_{\hat{n}}^{k-2} : J|_\Sigma$. (Note that at order $k = d-1$, the term $\mathring{\top} : \nabla_{\hat{n}}^{d-3} : P$ is precluded from appearing in the leading transverse order term of our putative T-curvature by a combination of its tensor structure and a homogeneity argument.) This establishes that, on a Poincaré–Einstein manifold, any T-curvature T_k has leading transverse-order term proportional to $: \nabla_{\hat{n}}^{k-2} : J|_\Sigma$.

Under constant conformal transformations $g \mapsto \lambda^2 g$, any T-curvature T_k is homogeneous of weight $-k$. But $\bar{g}$, $\bar{g}^{-1}$, $\bar{\nabla}$, $\bar{R}$, and all even-order T-curvatures have *even* homogeneity. Thus, homogeneity implies that each subleading term in any *odd* T-curvature can be expressed in a form that contains at least one (lower-order) odd T-curvature. (Here we used a tensor plus weight argument to preclude terms involving $\hat{n}$.)

The proof is completed by induction in the order of T-curvatures. The base case relies on the fact that any first T-curvature $T_1 \propto H$ vanishes in the Graham–Lee scale. Also, we have just shown that distinct odd-order T-curvatures differ only by terms that can be expressed in terms of lower-order T-curvatures. Consulting Proposition 3.5 completes the proof. ∎

3.2. Even-dimensional bulk

Here we establish Theorem 1.2 concerning even-dimensional Poincaré–Einstein structures. The following theorem is key to the proof.

Theorem 3.7. *Let $\Sigma \hookrightarrow (M^d, \boldsymbol{g})$ be a conformal hypersurface embedding and let $(M^d, \boldsymbol{g}, \sigma) \in \mathrm{APE}_{k-2}$ with $2 \le k \le d-1$ and d even. Moreover, let $I \in \Gamma(\mathbb{T}\Sigma[w])$, with $w \in 2\mathbb{Z}+1$, be a natural conformal hypersurface invariant with transverse order $0 \le m \le k-1$. Then, $I = 0$.*

Proof. We rely on the decomposition discussed above. By definition, natural hypersurface invariants are built from polynomials in Riemann curvatures, unit conormals, their (covariant and possibly tangential) derivatives, and metric contractions thereof. Since we are studying sections I of $\mathbb{T}\Sigma$, we may always deal with expressions such that all (undifferentiated) unit conormals are contracted into Riemann curvatures and their derivatives (a pair of unit conormals can always be re-expressed as $\hat{n}_a \hat{n}_b = g_{ab}|_\Sigma - \bar{g}_{ab}$, while no invariant proportional to an overall factor of a single $\hat{n}_a$ can live in $\Gamma(\mathbb{T}\Sigma)$).

We now analyse when I can have a given transverse order. Consider first the tensor

$$X^{(\ell)}_{ab} := \hat{n}^c \hat{n}^d : \nabla^\ell_{\hat{n}} : R_{cadb}|_\Sigma \in \Gamma(\mathbb{T}\Sigma). \tag{3.7}$$

Clearly, $X^{(\ell)}_{ab}$ has transverse order at most $\ell+2$; this fits with equation (3.2) because T_2 and the trace-free Fialkow tensor have transverse order 2. To see that the transverse order is exactly $\ell+2$, we study how $X^{(\ell)}_{ab}$ behaves upon replacing g by $g + s^q h$. For that, we consider the linearization of the Riemann tensor around g for the perturbation $g + s^q h$, namely,

$$\begin{aligned}(3.8)\quad & R^{g+s^q h}_{abcd} - R^g_{abcd} \\ &= -\frac{1}{2}(\nabla^g_a \nabla^g_c (s^q h_{bd}) - \nabla^g_b \nabla^g_c (s^q h_{ad}) - \nabla^g_a \nabla^g_d (s^q h_{bc}) + \nabla^g_b \nabla^g_d (s^q h_{ac})) + \mathcal{O}(s^{q-1}) \\ &= -\frac{q(q-1)}{2} s^{q-2} (\hat{n}_a \hat{n}_c h_{bd} - \hat{n}_b \hat{n}_c h_{ad} - \hat{n}_a \hat{n}_d h_{bc} + \hat{n}_b \hat{n}_d h_{ac}) + \mathcal{O}(s^{q-1}),\end{aligned}$$

where the notation $\hat{n}$ has been recycled to denote any smooth extension of $\hat{n}$ to M; similarly, below we will employ the notation $\top$ to denote any extension of the corresponding projector to M. In turn, we have

$$\top\big(\hat{n}^c \hat{n}^d : \nabla^\ell_{\hat{n}} : (R^{g+s^{\ell+2}h}_{cadb} - R^g_{cadb})\big) = -\frac{(\ell+2)!}{2} h^\top_{ab} + \mathcal{O}(s).$$

This establishes that $X^{(\ell)}_{ab}$ has transverse order $\ell+2$. Along similar lines, inspecting the (suitably manipulated) Codazzi and Gauß relations,

$$R^\top_{abc\hat{n}} \stackrel{\Sigma}{=} \bar{\nabla}_a \mathring{\mathrm{II}}_{bc} - \bar{\nabla}_b \mathring{\mathrm{II}}_{ac} + \bar{g}_{bc} \bar{\nabla}_a H - \bar{g}_{ac} \bar{\nabla}_b H, \tag{3.9}$$

and

$$\begin{aligned}(3.10)\quad R^\top_{abcd} \stackrel{\Sigma}{=} {} & \bar{R}_{abcd} - \mathring{\mathrm{II}}_{ac} \mathring{\mathrm{II}}_{bd} + \mathring{\mathrm{II}}_{ad} \mathring{\mathrm{II}}_{bc} \\ & - H(\bar{g}_{ac} \mathring{\mathrm{II}}_{bd} - \bar{g}_{bc} \mathring{\mathrm{II}}_{ad} - \bar{g}_{ad} \mathring{\mathrm{II}}_{ba} + \bar{g}_{bd} \mathring{\mathrm{II}}_{ac}) - (\bar{g}_{ac}\bar{g}_{bd} - \bar{g}_{bc}\bar{g}_{cd}) H^2,\end{aligned}$$

shows that

$$Y^{(\ell)}_{abc} := (\hat{n}^d : \nabla^{\ell}_{\hat{n}} : R_{abcd})^{\top} \quad \text{and} \quad Z^{(\ell)}_{abcd} := (: \nabla^{\ell}_{\hat{n}} : R_{abcd})^{\top}$$

both have at most transverse order $\ell + 1$. Indeed, a similar linearized Riemann argument to the above establishes that their transverse orders are exactly $\ell + 1$; the tensors X, Y, Z, and the second fundamental form II are the basic atoms from which the invariant I is produced via contractions and hypersurface derivatives. Equations (3.9) and (3.10) satisfy the same type of decomposition as outlined for equation (3.2), namely, into intrinsic, vanishing-for-APE$_k$-structures (k sufficiently high), and (hypersurface derivatives of) T-curvatures.

We next need to show that the above decomposition property holds upon suitable application of normal derivatives $: \nabla^{\ell}_{\hat{n}} :$ required to construct X, Y and Z. This has been established in a slightly different context in [8]. The argument is as follows: first we need to show that normal derivatives of the ambient Schouten tensor, or its trace, can be traded for conformal fundamental forms and T-curvatures at the cost of lower transverse order terms. Therefore, we must express X, Y and Z in these terms. Indeed, a Bianchi identity manoeuvre can be employed to reduce expressions involving Y and Z (at leading transverse order) to ones involving X. Consider, for example (referring to [8] for the Z case),

$$\begin{aligned} Y^{(\ell)}_{abc} &= (\hat{n}^d \hat{n}^e : \nabla^{\ell-1}_{\hat{n}} : \nabla_e R_{abcd})^{\top} \\ &= -(\hat{n}^d \hat{n}^e : \nabla^{\ell-1}_{\hat{n}} : (\nabla_a R_{becd} + \nabla_b R_{eacd}))^{\top} \\ &= -(\nabla^{\top}_a (\hat{n}^d \hat{n}^e : \nabla^{\ell-1}_{\hat{n}} : R_{becd}) - \nabla^{\top}_b (\hat{n}^d \hat{n}^e : \nabla^{\ell-1}_{\hat{n}} : R_{eacd}))^{\top} + \text{ltots} \\ &= -\bar{\nabla}_a X^{(\ell-1)}_{bc} + \bar{\nabla}_b X^{(\ell-1)}_{ac} + \text{ltots}. \end{aligned}$$

Here "ltots" stands for lower transverse order tensors.

Now we can focus on $X^{(\ell)}$. The case $\ell = 1$ has the desired decomposition by virtue of equation (3.2). For higher ℓ, we examine equation (3.7), which can be re-expressed as

$$X^{(\ell)}_{ab} = (\hat{n}^c \hat{n}^d : \nabla^{\ell}_{\hat{n}} : W_{cadb} : \nabla^{\ell}_{\hat{n}} : P_{ab})^{\top} + \bar{g}_{ab} \hat{n}^c \hat{n}^d : \nabla^{\ell}_{\hat{n}} : P_{cd} + \text{ltots}.$$

Next (remembering that identically $\nabla^e W_{caeb} = (d-3)(\nabla_c P_{ab} - \nabla_a P_{cb})$), notice that

$$\begin{aligned} (\hat{n}^c \hat{n}^d : \nabla^{\ell}_{\hat{n}} : W_{cadb})^{\top} &= (\hat{n}^c (g^{de} - \bar{g}^{de}) : \nabla^{\ell-1}_{\hat{n}} : \nabla_e W_{cadb})^{\top} \\ &= (d-3)(\hat{n}^c : \nabla^{\ell-1}_{\hat{n}} : (\nabla_c P_{ab} - \nabla_a P_{cb}))^{\top} + \text{ltots} \\ &= (d-3)(: \nabla^{\ell}_{\hat{n}} : P_{ab})^{\top} + \text{ltots}, \end{aligned}$$

so that

$$X^{(\ell)}_{ab} = (d-2)(: \nabla^{\ell}_{\hat{n}} : P_{ab})^{\top} + \bar{g}_{ab} \hat{n}^c \hat{n}^d : \nabla^{\ell}_{\hat{n}} : P_{cd} + \text{ltots}.$$

Now, the hypersurface trace-free part of the first term on the right-hand side above can be re-expressed as an $(\ell + 3)$-th conformal fundamental form plus lower transverse order terms – again, see Corollary 3.3 in [8]. The trace, as well as the second term above, can be written in terms of T-curvatures modulo lower transverse order terms. To see that, we may focus on the hypersurface trace

$$\begin{aligned}
\bar{g}^{ab} X^{(\ell)}_{ab} &= (d-2)(g^{ab} - \hat{n}^a \hat{n}^b) : \nabla^{\ell}_{\hat{n}} : P_{ab} + (d-1)\hat{n}^c \hat{n}^d : \nabla^{\ell}_{\hat{n}} : P_{cd} + \text{ltots} \\
&= (d-2) : \nabla^{\ell}_{\hat{n}} : J + \hat{n}^a \hat{n}^b : \nabla^{\ell}_{\hat{n}} : P_{ab} + \text{ltots} \\
&= (d-2) : \nabla^{\ell}_{\hat{n}} : J + \hat{n}^a g^{bc} : \nabla^{\ell-1}_{\hat{n}} : \nabla_c P_{ab} + \text{ltots} \\
&= (d-1) : \nabla^{\ell}_{\hat{n}} : J + \text{ltots}.
\end{aligned}$$

In the above, we used $\nabla^a P_{ab} = \nabla_b J$. We have by now shown, at leading transverse order, that any tensor built from X, Y, Z, and $\mathrm{I\!I}$ (as well as gradients thereof, and the hypersurface metric) can be expressed in terms of H, normal derivatives of J, conformal fundamental forms, and possibly intrinsic tensors. It remains to focus on normal derivatives of J. We know that T-curvatures are Riemannian hypersurface invariants built from X, Y, Z, $\mathrm{I\!I}$, covariant derivatives thereof, and hypersurface metrics. So from the above display, it must be that normal derivatives of J can be expressed in terms of T-curvatures, conformal fundamental forms, and intrinsic tensors. Hence, X, Y, and Z themselves (and any tensor built therefrom) are also expressible (at leading order) in terms of conformal fundamental forms, T-curvatures, and possibly intrinsic tensors. By descent in the transverse order, it follows that this statement holds to all orders.

Recall that $X^{(\ell)}_{ab}$ has transverse order $\ell + 2$ and that I has transverse order at most $m = k - 1$. So there exists an expression for I such that for any appearance of X (from above we have that Y and Z can be reduced to terms involving only X at leading order) one has that ℓ is not greater than $k - 3$. But for APE_{k-2} structures, the k-th fundamental form (which has transverse order $k - 1$) and all lower-order fundamental forms vanish. Hence, I is expressible in terms of T-curvatures and intrinsic tensors alone. But there exists a scale for which these T-curvatures vanish (see Theorem 6.15 in [21] or Proposition 3.2). Moreover, natural intrinsic curvatures are composed of sums of products of the Riemann tensor, metrics as well as covariant derivatives and traces thereof, all of which have even homogeneity under constant metric rescalings. Thus, because the weight of I is odd, there are no such intrinsic tensors available as these are taken to be (polynomially) built from $\bar{g}$, $\bar{g}^{-1}$, $\bar{R}$, $\bar{\nabla}$ and contractions thereof. This establishes that I vanishes for one choice of scale, and hence, by its conformal invariance, $I = 0$. ∎

Remark 3.8. Note that another approach to establishing this result is to examine the tensor structures that can appear as coefficients in the Fefferman–Graham expansion of an APE_{k-2} metric; this approach might be used to write a more general result for d odd.

We will need explicit formulæ for Dirichlet-to-Neumann tensors in even dimensions in order to prove Theorem 1.2.

Lemma 3.9. *Let (M^d_+, g^o) be a Poincaré–Einstein structure with $d \geq 6$ even. Then*

$$\text{DN}^{(d)}_{ab} \propto \mathring{\top} : \nabla^{d-5}_{\hat{n}} : B^{g_r}_{ab},$$

where the right-hand side is evaluated on the Graham–Lee compactified metric g_r used to define the left-hand side.

Proof. By definition, the image of the Dirichlet-to-Neumann map of a metric representative $\bar{g} \in \boldsymbol{c}_\Sigma$ is given by $(\mathcal{L}_{\partial/\partial r})^{d-1} g_r|_\Sigma$, where g_r is the corresponding Graham–Lee compactified metric. Calling $n = \mathrm{d}r$, $n^\sharp := g_r^{-1}(n, \cdot)$, and using that $|n| = 1$, it follows that

$$\mathrm{DN}^{(d)}_{ab} = (\mathcal{L}_{n^\sharp})^{d-1} g_{ab} \stackrel{\Sigma}{=} (\mathcal{L}_{n^\sharp})^{d-2} (\nabla_a n_b + \nabla_b n_a).$$

Now, evidently, the leading derivative term in the above is the same as that of

$$2\nabla^{d-2}_{n^\sharp} \nabla_{(a} n_{b)}.$$

Moreover, the natural hypersurface invariant $\mathrm{DN}^{(d)}_{ab} - 2\nabla^{d-2}_{n^\sharp}\nabla_{(a}n_{b)}$ has transverse order no larger than $d-2$. So, from [8], it is expressible entirely in terms of Riemannian invariants intrinsic to Σ, conformal fundamental forms ranging from $\mathring{\mathrm{II}}$ to $\overset{\circ}{\overline{d-1}}$, and T-curvatures up to order $d-2$. However, as (M^d_+, g^o) is Poincaré–Einstein, it follows from [10] that all conformal fundamental forms from $\mathring{\mathrm{II}}$ to $\overset{\circ}{\overline{d-1}}$ vanish. Hence, by the hypersurface invariant decomposition of type (3.2) established earlier in this section, the difference in question can be expressed solely in terms of intrinsic invariants and T-curvatures. Now, because this difference has a definite *odd* homogeneity of $3-d$ under constant rescalings of the metric $g \mapsto \lambda^2 g$, and because intrinsic invariants always have even homogeneity, it follows that every summand in any such decomposition must involve at least one occurrence of an odd-order T-curvature. But, as g is a Graham–Lee compactified metric, it follows from Theorem 3.6 that all odd-order T-curvatures up to order $d-1$ inclusive vanish, and so

$$\mathrm{DN}^{(d)}_{ab} = (\mathcal{L}_{n^\sharp})^{d-1} g_{ab}|_\Sigma = 2\nabla^{d-2}_n \nabla_{(a} n_{b)}|_\Sigma.$$

Now, using the Ricci identity, we have the following identity:

$$\begin{aligned}\nabla_n \nabla_a n_b &= n^c \nabla_c \nabla_a \nabla_b r = n^c R_{cabd} n^d + n^c \nabla_a \nabla_b n_c \\ &= R_{nabn} - (\nabla_a n^c)(\nabla_b n_c) + \frac{1}{2} \nabla_a \nabla_b n^2 = R_{nabn} - (\nabla_a n^c)(\nabla_b n_c),\end{aligned}$$

where the last identity follows because $n^2 = 1$. Now, using this identity in the formula for $\mathrm{DN}^{(d)}$, and throwing away lower transverse order terms because they either involve (vanishing) odd-order T-curvatures or conformal fundamental forms, we have that

$$\mathrm{DN}^{(d)}_{ab} = 2\,\nabla^{d-3}_n R_{nabn}.$$

Now, applying the same reasoning as directly above to the decomposition of Riemann into its Weyl and Schouten tensor pieces, we may in turn express this as

$$\mathrm{DN}^{(d)}_{ab} = -2(d-2)\mathring{\top} : \nabla^{d-3}_{\hat{n}} : P_{ab}.$$

Finally, using equation (1.2), we have $\mathring{\top} B = \mathring{\top} \nabla^2_n P$ modulo lower-order terms (which again vanish by similar reasoning to above), we have that

$$\mathrm{DN}^{(d)}_{ab} \propto \mathring{\top} : \nabla^{d-5}_{\hat{n}} : B_{ab},$$

thus completing the proof. ∎

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. To unify the $d = 6$ and ≥ 8 formulæ, let us denote $\delta_{0,1} := \delta_R$. We shall begin by checking that $\bar{q}^* \mathring{\top} \delta_{(d-6)/2,(d-4)/2} W$ has the required leading transverse order and that the tractor $\delta_{(d-6)/2,(d-4)/2} W$ has a projecting part with the correct tensor structure. This will establish that the leading transverse order term in $\underline{\mathring{d}}^{\mathrm{DN}}$ is as required by the theorem. (Note that by its construction, $\underline{\mathring{d}}^{\mathrm{DN}}$ is clearly a natural conformal hypersurface invariant.)

From Theorem 3.4 in [8], we have that $\delta_{(d-6)/2,(d-4)/2}$ acting on weight -2 tractors has transverse order $d - 5$, and thus

$$\delta_{(d-6)/2,(d-4)/2} W \propto: \nabla_{\hat{n}}^{d-5} : W + \text{ltots}.$$

The highest transverse order tensor component of the W-tractor is the Bach tensor, so

$$(d-4) W_{ABCD} \stackrel{\mathcal{T}_g}{=} 4 X_{[A} Z_{B]}^a X_{[C} Z_{D]}^b B_{ab}^g + \text{ltots}.$$

(Here $g \in \boldsymbol{c}$ is any choice of metric representative, and we employ this choice for the remainder of this proof.) Thus, we have that

$$\mathring{\top} \delta_{(d-6)/2,(d-4)/2} W_{ABCD} \propto \mathring{\top} X_{[A} Z_{B]}^a X_{[C} Z_{D]}^b : \nabla_{\hat{n}}^{d-5} : B_{ab}^g + \text{ltots}_{d-2}, \tag{3.11}$$

where ltots_{d-2} are tensor-valued tractors with transverse order less than or equal to $d - 2$. The $d \geq 4$ Bach tensor

$$B_{bd} = \Big(\frac{1}{d-3} \nabla^a \nabla^c + P^{ac}\Big) W_{abcd}$$

has transverse order 4. To see this, we apply equation (3.8) to the Weyl tensor and find

$$\begin{aligned} W_{abcd}^{g+s^\ell h} - W_{abcd}^g = -\frac{\ell(\ell-1)}{2(d-2)} s^{\ell-2} &([(d-3)\hat{n}_a \hat{n}_c - \bar{g}_{ac}] h_{bd}^{\mathring{\top}} - [(d-3)\hat{n}_b \hat{n}_c - \bar{g}_{bc}] h_{ad}^{\mathring{\top}} \\ &- [(d-3)\hat{n}_a \hat{n}_d - \bar{g}_{ad}] h_{bc}^{\mathring{\top}} + [(d-3)\hat{n}_b \hat{n}_d - \bar{g}_{bd}] h_{ac}^{\mathring{\top}}) + \mathcal{O}(s^{\ell-1}), \end{aligned}$$

where $\bar{g}_{ab}$ is any extension of $g_{ab} - \hat{n}_a \hat{n}_b$ to M, and $h_{ab}^{\mathring{\top}}$ is any such extension of $\bar{g}_{ac} \bar{g}^{cd} h_{db} - \frac{1}{d-1} \bar{g}_{ab} \bar{g}^{cd} h_{cd}$. Hence, for the Bach tensor, we obtain

$$B_{ab}^{g+s^\ell h} - B_{ab}^g = -\frac{\ell(\ell-1)(\ell-2)(\ell-3)}{2(d-2)} s^{\ell-4} h_{ab}^{\mathring{\top}} + \mathcal{O}(s^{\ell-3}).$$

This establishes the claimed transverse order result. In turn, it follows that the tractor in equation (3.11) has transverse order $d - 1$. It is also now clear that, for generic structures, the coefficient of $\mathring{\top} X_{[A} Z_{B]}^a X_{[C} Z_{D]}^b$ in $\mathring{\top} \delta_{(d-6)/2,(d-4)/2} W$ is non-zero.

Now suppose the projecting part of $\mathring{\top} \delta_{(d-6)/2,(d-4)/2} W_{ABCD}$ is not proportional to $\mathring{\top}(X_{[A} Z_{B]}^a X_{[C} Z_{D]}^b)$. In that case, it must consist of terms appearing among those labelled ltots_{d-2} in (3.11). But then the putative projecting part of $\mathring{\top} \delta_{(d-6)/2,(d-4)/2} W_{ABCD}$ would have both odd weight and transverse order strictly less than $d-1$, and thus vanish by Theorem 3.7. Thus, we must now have

$$\mathring{\top}\delta_{(d-6)/2,(d-4)/2}W_{ABCD} \propto \mathring{\top} X_{[A}Z^a_{B]}X_{[C}Z^b_{D]}(:\nabla_{\hat{n}}^{d-5}:B^g_{ab} + \text{ltots}_{d-2}).$$

So, by construction, we have that

$$\bar{q}^*\mathring{\top}\delta_{(d-6)/2,(d-4)/2}W \propto \mathring{\top}:\nabla_{\hat{n}}^{d-5}:B^g + \text{ltots}_{d-2} \in \Gamma(\mathring{\top}\odot^2 T^*M[3-d]|_\Sigma). \tag{3.12}$$

We next must show that $\mathring{\underline{d}}^{\rm DN}$ is the unique (up to multiplication by a constant and natural) conformal hypersurface invariant of transverse order $d-1$ in $\Gamma(\mathring{\top}\odot^2 T^*M[3-d]|_\Sigma)$ determined by the Poincaré–Einstein structure. The $d=6$ case is handled in Theorem 1.1, whose proof is given in Appendix A. By similar considerations as those given there, the leading transverse order term of $\mathring{\underline{d}}^{\rm DN}$ for $d \geq 8$ is unique. Now, suppose that there exist two distinct conformal hypersurface invariants

$$\mathring{\top}:\nabla_{\hat{n}}^{d-5}:B+Q \quad \text{and} \quad \mathring{\top}:\nabla_{\hat{n}}^{d-5}:B+Q',$$

where $Q \neq Q'$ are tensors with transverse order less than or equal to $d-2$. Necessarily, $Q-Q'$ is conformally invariant, has transverse order at most $d-2$, and weight $3-d$ (an odd integer) so it vanishes by dint of Theorem 3.7. This establishes the uniqueness of $\mathring{\underline{d}}^{\rm DN}$.

Finally, we must check that $\mathring{\underline{d}}^{\rm DN} \simeq {\rm DN}^{(d)}$. From Lemma 3.9, this amounts to checking that the terms labelled ltots_{d-2} in equation (3.12) vanish when evaluated on a Graham–Lee compactified metric. This is achieved by recycling the argument used to prove Lemma 3.9. Because it has already been established that ${\rm DN}^{(d)}$ is the functional gradient of the renormalized volume – see Section 3.4 of [13], the discussion below Theorem 2.2 in [5], and Theorem 1.3 in [1]. ■

Remark 3.10. As discussed above, the tensor ${\rm DN}^{(d)}$ is a functional gradient along a path of Poincaré–Einstein metrics which can be labelled by a path of boundary metrics, so its boundary divergence must vanish,

$$\bar{\nabla}^a\,{\rm DN}^{(d)}_{ab} = 0.$$

In fact, both $\bar{\nabla}^a\,{\rm DN}^{(d)}_{ab}$ and $\bar{\nabla}^a\mathring{\underline{d}}^{\rm DN}_{ab}$ are conformally invariant. Therefore, Theorem 1.2 establishes the vanishing of the latter.

3.3. Odd-dimensional bulk

We first need to characterize conformal hypersurface invariants for (asymptotically) Poincaré–Einstein structures with conformally flat boundaries. The following is an analogue of Theorem 3.7.

Lemma 3.11. *Assume that $\Sigma \hookrightarrow (M^d, \boldsymbol{g})$ is a conformal hypersurface embedding, where $(M^d, \boldsymbol{g}, \sigma) \in {\rm APE}_{k-2}$ with $2 \leq k \leq d-1$. Moreover, suppose that the boundary $(\Sigma, \boldsymbol{c}_\Sigma)$ is conformally flat. Then any natural conformal hypersurface invariant $I \in \Gamma(\mathbb{T}\Sigma[w])$ with transverse order $1 \leq m \leq k-1$ vanishes.*

Proof. First, we choose a scale for which the boundary metric $\bar{g} = \bar{\delta}$ is flat. Then the corresponding Fefferman–Graham expansion of the compactified metric g_r of an APE_{k-2} structure is

$$g_r = \mathrm{d}r^2 + \bar{\delta} + r^k Y \tag{3.13}$$

for some smooth, rank-two, symmetric tensor Y, which is trace-free by virtue of the Poincaré–Einstein condition. The above expansion follows directly from the treatment of conformally flat and conformally Einstein spaces in Chapter 7 of [16]; see also equation (3.14) below. The statement now follows, since the only local conformal invariants for a flat metric are those built from the metric itself, and that the order r^k term cannot contribute to invariants of transverse order strictly less than k. Vanishing in one scale implies the same for all other scales. ∎

We are now ready to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. By Lemma 3.11, we only need to consider invariants of transverse order zero or four. Because the boundary conformal class is flat, the only intrinsic (and hence transverse order zero) conformal invariants of the correct homogeneity and tensor type are powers of the metric and its inverse (see also Remark 3.12). No trace-free tensors can be made this way. Then an exhaustive search establishes that the only Riemannian hypersurface invariant of the desired homogeneity, tensor type and transverse order four, for the metric of equation (3.13) (with $k = 4$), is $B^{\top}_{(ab)\circ}$. But the conformal transformation of the Bach tensor is

$$\Omega^2 B^{\Omega^2 g}_{ab} - B^{g}_{ab} = (d-4)(2C_{\Upsilon(ab)} - W_{\Upsilon ab \Upsilon}),$$

and the right-hand side vanishes when evaluated on the metric of equation (3.13), so we have that $B^{\top}_{(ab)\circ}$ is conformally invariant. Finally, a calculation of the $B^{\top}_{(ab)\circ}$ in the metric of equation (3.13) shows that this is proportional to Y along Σ, which establishes that $\mathring{\nabla}^{\mathrm{DN}^{\flat}} \simeq \mathrm{DN}^{(5)}$. ∎

Remark 3.12. Note that the only non-zero, transverse order zero, $d = 5$, natural, conformally invariant sections of $\odot^2_\circ T^*\Sigma[-2]$ are $\bar{B}_{ab}$ and $\bar{W}_{(a|cde}\bar{W}_{b)\circ}{}^{cde}$. Also, for APE_2 structures with a generic conformal class $\boldsymbol{c}_\Sigma$, there are no conformally-invariant transverse-order 2 or 4 tensors of this type. A simpler version of the exhaustive analysis of tensor structures and their conformal variations, given in Appendix A, can be used to establish this.

Proof of Theorem 1.4. We must verify the well-definedness of two definitions, perform a computation for $\mathrm{DN}^{(d)}$, and establish a proportionality statement. To start with, it is known (see [16]) that a $d \geq 7$ Poincaré–Einstein metric g^o with conformally flat boundary has the Fefferman–Graham expansion

$$g_r := r^2 g^o = \mathrm{d}r^2 + \bar{g} - \bar{P} r^2 + \frac{1}{4}\bar{P}^{\odot 2} r^4 + \mathcal{O}(r^{d-1}), \tag{3.14}$$

where $\bar{P}$ is the Schouten tensor of $\bar{g}$.

Importantly, the above expansion holds for *any* $\bar{g} \in \boldsymbol{c}_\Sigma$. In particular, by choosing (locally) $\bar{g}$ to be the flat metric $\bar{\delta}$, it follows that the Weyl tensor of g_r obeys

$$W_{ab}{}^{c}{}_{d} = \mathcal{O}(r^{d-3}).$$

In turn, for any $g \in \boldsymbol{c}$, we have that the tractor curvature

$$F_{ab}{}^{A}{}_{B} = W_{ab}{}^{cd} Z_c^A Z_{dB} + C_{ab}{}^{c}(Z_c^A X_B - Z_{cB} X^A) = \begin{pmatrix} 0 & 0 & 0 \\ \mathcal{O}(r^{d-4}) & \mathcal{O}(r^{d-3}) & 0 \\ 0 & \mathcal{O}(r^{d-4}) & 0 \end{pmatrix},$$

so that, for any $1 \leq k \leq d-4$ and $\sigma = [g; s]$,

$$: \nabla_{\hat{n}}^{k} : F_{ab}{}^{A}{}_{B} \overset{\mathcal{T}_g}{=} \begin{pmatrix} \mathcal{O}(s^{d-k-3}) & \mathcal{O}(s^{d-k-2}) & 0 \\ \mathcal{O}(s^{d-k-4}) & \mathcal{O}(s^{d-k-3}) & \mathcal{O}(s^{d-k-2}) \\ 0 & \mathcal{O}(s^{d-k-4}) & \mathcal{O}(s^{d-k-3}) \end{pmatrix}.$$

But any failure of $: \nabla_{\hat{n}}^{d-4} : F_{ab}{}^{A}{}_{B}$ to be conformally invariant necessarily involves terms proportional to the above display with $k \leq d-5$. These vanish along Σ; to see this, one can re-express the tractor connection (2.2) in a matrix notation and then explicitly examine the failure of the above to transform correctly.

Thus, we now observe that

$$\hat{n}^a : \nabla_{\hat{n}}^{d-4} : F_{ab}{}^{C}{}_{D} \overset{\Sigma}{=} \begin{pmatrix} 0 & 0 & 0 \\ \hat{n}^a : \nabla_{\hat{n}}^{d-4} : C_{ab}{}^{c} & 0 & 0 \\ 0 & -\hat{n}^a : \nabla_{\hat{n}}^{d-4} : C_{abd} & 0 \end{pmatrix}.$$

Note that

$$C_{n[bc]} = -\frac{1}{2} C_{bcn} = -n^a \nabla_{[b} P_{c]a},$$

so $C_{n[bc]} - n_{[b} C_{c]nn}$ (which restricts along Σ to $C^{\top}_{\hat{n}[bc]}$) vanishes to one higher order than its symmetric counterpart, and thus

$$(\hat{n}^a : \nabla_{\hat{n}}^{d-4} : C_{a[bc]})^{\top}|_\Sigma = 0.$$

Since $\hat{n}^a(: \nabla_{\hat{n}}^{d-4} : C_{abc})|_\Sigma$ appears as the projecting part of a tractor, and is symmetric, we then have that

$$\mathring{\top} \circ q^*([\hat{n}^a : \nabla_{\hat{n}}^{d-4} : F_{ab}{}^{C}{}_{D}]|_\Sigma) = \mathring{\top} \circ [\hat{n}^a : \nabla_{\hat{n}}^{d-4} : C_{abc}]|_\Sigma$$

is a well-defined element of $\Gamma(\mathring{\top} \odot^2 T^*M[3-d]|_\Sigma)$. In the case $d = 5$, a short computation shows that, in the current setting, the right-hand side above equals $B^{\top}_{(ab)\circ}$.

Unlike the even-dimensional case, no parity argument ensures that the relevant Fefferman–Graham expansion coefficient defining $\mathrm{DN}^{(d)}$ is covariant with respect to different choices of the boundary metric representative. We shall check that is it by hand. For that, we define a reference metric

$$\tilde{g}_r := \mathrm{d}r^2 + \bar{g} - \bar{P} r^2 + \frac{1}{4} \bar{P}^{\odot 2} r^4$$

on a collar neighbourhood of Σ. Note that the above is conformal to the hyperbolic metric [16]. Then we define a smooth tensor $X^{\bar{g}}$ in a collar of the boundary (and, by dint of smoothness, along Σ itself) by

$$r^{d-1}X^{\bar{g}} = g_r - \tilde{g}_r.$$

Upon choosing a different metric $\bar{g}' \in \boldsymbol{c}_\Sigma$, with respect to the new distance function $r' = \Omega(r)r$ (suppressing the dependence of $\Omega(r)$ on boundary directions), the new compactified metric g' then obeys (by virtue of equation (3.14))

$$g'_{r'} - \tilde{g}_{r'} = r'^{d-1}X^{\bar{g}'}.$$

But, again relying on [16],

$$g'_{r'} - \tilde{g}_{r'} = \Omega(r)^2(g_r - \tilde{g}_r) + \mathcal{O}(r^d).$$

Thus, we learn that

$$r'^{d-1}X^{\bar{g}'} = \Omega(r)^2 r^{d-1}X^{\bar{g}} + \mathcal{O}(r^d).$$

So, using $r' = \Omega(r)r$ and $0 < \Omega(0) = \bar{\Omega} \in C^\infty\Sigma$, we have that

$$X^{\bar{g}'} \stackrel{\Sigma}{=} \bar{\Omega}^{3-d}X^{\bar{g}}.$$

Clearly,

$$(d-1)!\,X^{\bar{g}}|_\Sigma = (\mathcal{L}_{\partial/\partial r})^{d-1}g_r|_\Sigma.$$

To show that $X|_\Sigma$ is trace-free, we first note that the Poincaré–Einstein condition implies that (see, for example, [18])

$$n^2 + 2\rho\sigma = 1,$$

where $n := \mathrm{d}r$ and $\rho := -(\nabla^a n_a + rJ)/d$, which in the scale $\sigma = [g_r; r]$ implies

$$0 = -d\rho = \nabla.n + rJ^{g_r}. \tag{3.15}$$

With impunity, we may once again compute in the boundary flat metric $\bar{\delta}$ scale for which

$$g_r = \mathrm{d}r^2 + \bar{\delta} + r^{d-1}X^{\bar{\delta}}. \tag{3.16}$$

Then simple computations show

$$\nabla.n = \frac{d-1}{2}r^{d-2}\operatorname{tr}_{\bar{\delta}} X^{\bar{\delta}} + \mathcal{O}(r^{d-1}) \quad \text{and} \quad rJ^{g_r} = -\frac{d-2}{2}r^{d-2}\operatorname{tr}_{\bar{\delta}} X^{\bar{\delta}} + \mathcal{O}(r^{d-1}).$$

Hence, the right-hand side of equation (3.15) equals $\frac{1}{2}\operatorname{tr}_{\bar{\delta}} X^{\bar{\delta}}$. Therefore, $X^{\bar{g}}|_\Sigma$ must be trace-free for any choice of boundary scale, so indeed

$$\mathrm{DN}^{(d)} := (\mathcal{L}_{\partial/\partial r})^{d-1}g_r|_\Sigma$$

defines an element of $\Gamma(\odot^2_\circ T^*\Sigma[3-d])$.

It only remains to check that $\mathrm{DN}^{(d)}$ and $\underline{\mathring{d}}^{\mathrm{DN}^\flat}$ are proportional when evaluated on a Graham–Lee compactified metric. By their respective conformal invariance, we may perform this computation in the metric choice of equation (3.16). It is now easy to verify the stated proportionality result. ∎

Remark 3.13. Fefferman and Graham [16] show that the expansion (3.14) also applies when the boundary class of metrics admits an Einstein metric when written in the Fefferman–Graham coordinate r corresponding to the boundary Einstein metric, and that these expansions are diffeomorphic if there happen to be distinct Einstein metrics in the boundary conformal class. This implies the existence of a Dirichlet-to-Neumann map that is an invariant of the Poincaré–Einstein structure for the boundary-Einstein case. It is likely that there are other such constructions for distinguished boundary conformal classes.

A. Proof of Theorem 1.1

The uniqueness part of the proof is by exhaustion. The key details are as follows. Given an embedded hypersurface $\Sigma \hookrightarrow (M, g)$, any natural diffeomorphism-invariant along Σ can be fully described by contractions of the conormal $\hat{n}$, its tangential derivatives, the metric and its inverse, and the bulk curvature R^g, along with its derivatives $\nabla^k R^g$. Note that the embedding $\Sigma \hookrightarrow (M_+, g^o)$ for Poincaré–Einstein structures is necessarily umbilic (see [18, 33]). Hence, tangential derivatives of $\hat{n}$ only produce the mean curvature, its hypersurface derivatives and the induced metric. The latter can be expressed in terms of the unit conormal and ambient metric. So, in a choice of metric representative $g \in \boldsymbol{c}$, any natural conformal hypersurface invariant of $\Sigma \hookrightarrow (M, \boldsymbol{c})$ is expressible by such a diffeomorphism-invariant. This is established in Proposition 2.2 of [8] (which is based on Proposition 2.7 in [24]). Thus, we construct all possible candidate diffeomorphism invariants and then show that only a single combination of these produces the desired, transverse order 5, conformal hypersurface invariant in $\Gamma(\odot^2_\circ T^*\Sigma[-3])$. The homogeneity under constant metric rescalings of the above set of "letters" is tabulated below:

Ingredient	Weight
g^{ab}	-2
$\hat{n}_a$	1
H^g	-1
$\bar{\nabla}_a$	0
∇_a	0
R_{abcd}	2

We note that, when building a rank-two, trace-free tensor with conformal weight -3 from the above letters, only the inverse metric can reduce the tensor rank. Hence, to find all words with the aforementioned properties, we are faced with a non-negative, integer-valued linear algebra problem, whose solutions are listed below:

$$\bar{\nabla}^2 H^3,\ g^{-1} H^3 R,\ (g^{-1})^2 \hat{n}^2 H^3 R,\ (g^{-1})^2 \hat{n}(\bar{\nabla} H^2) R,\ (g^{-1})^2 \hat{n} H^2 \nabla R,\ (g^{-1})^2 (\bar{\nabla}^2 H) R,$$
$$g^{-1}\bar{\nabla}^4 H,\ (g^{-1})^3 \hat{n}^2 (\bar{\nabla}^2 H) R,\ (g^{-1})^3 H R^2,\ (g^{-1})^4 \hat{n}^2 H R^2,\ (g^{-1})^2 (\bar{\nabla} H) \nabla R,$$
$$(g^{-1})^3 \hat{n}^2 (\bar{\nabla} H) \nabla R,\ (g^{-1})^2 H \nabla^2 R,\ (g^{-1})^3 \hat{n} H \nabla^2 R,\ (g^{-1})^4 \hat{n} \nabla R^2,\ (g^{-1})^3 \hat{n} \nabla^3 R.$$

Note that the condition $\hat{n}_a \hat{n}^a = 1$ gives an upper bound on the number of appearances of $\hat{n}$, conditioned on the other letters appearing. Also, since we are searching for rank-two, trace-free covariant tensors, all inverse metrics g^{-1} must be completely contracted.

The number of independent rank-two tensors built from linear combinations of the above list of words reduces significantly because (M^6_+, g^o) is Poincaré–Einstein, which allows these tensors to be re-expressed in terms of $\bar{g}$, $\bar{g}^{-1}$, $\bar{\nabla}$, $\bar{R}$, J, $\nabla_{\hat{n}} J$, H, and $\mathring{\top}\nabla_{\hat{n}} B$. The list of all weight -3, rank-two combinations of these letters is given by

$$\mathring{\top}\nabla_{\hat{n}} B,\ \bar{\nabla}^2\nabla_{\hat{n}} J,\ \bar{g}^{-1}\bar{R}\nabla_{\hat{n}} J,\ \bar{\nabla}^2 HJ,\ \bar{g}^{-1}HJ\bar{R},$$
$$\bar{g}^{-1}\bar{\nabla}^4 H,\ (\bar{g}^{-1})^2\bar{\nabla}H\bar{R},\ (\bar{g}^{-1})^3 H\bar{R}^2,\ \bar{\nabla}^2 H^3,\ \bar{g}^{-1}H^3\bar{R}.$$

These yield 24 linearly independent, symmetrized, and trace-free tensors of the correct weight and transverse orders:

$$\begin{aligned}\odot^2_\circ \{&\top\nabla_{\hat{n}} B_{ab},\ \bar{P}_{cd}\nabla_{\hat{n}} J, \bar{\nabla}_c\bar{\nabla}_d\nabla_{\hat{n}} J,\ (\bar{\nabla}_c H)\bar{\nabla}_d J,\ HJ\bar{P}_{cd},\ H\bar{\nabla}_c\bar{\nabla}_d J,\ \bar{W}^{acdb}\bar{\nabla}_a\bar{\nabla}_b H,\\ &\bar{C}_{acd}\bar{\nabla}^a H, H\bar{B}_{cd},\ H\bar{W}_c{}^{eab}\bar{W}_{dabe},\ H\bar{W}_c{}^{abe}\bar{W}_{dabe},\ H\bar{W}_c{}^{bae}\bar{W}_{dabe},\ H\bar{P}_{ca}\bar{P}^a_d,\\ &H\bar{P}^{ab}\bar{W}_{acdb},\ HJ\bar{P}_{cd},\ H^3\bar{P}_{cd},\ \bar{J}\bar{\nabla}_c\bar{\nabla}_d H,\ (\bar{\nabla}H_c)\bar{\nabla}_d\bar{J},\ H\bar{\nabla}_c\bar{\nabla}_d\bar{J},\ \bar{\nabla}_c\bar{\nabla}_d\bar{\Delta}H,\\ &\bar{P}^a_c\bar{\nabla}_a\bar{\nabla}_d H,\ H^2\bar{\nabla}_c\bar{\nabla}_d H,\ H(\bar{\nabla}_c H)(\bar{\nabla}_d H),\ \bar{P}_{cd}\bar{\Delta}H\}.\end{aligned}$$

Requiring that the conformal variation of an arbitrary linear combination of the above basis elements vanishes, we find a unique combination for which the coefficient of the (only) transverse order five tensor $\mathring{\top}\nabla_{\hat{n}} B$ is unity. In a choice of metric $g \in \boldsymbol{c}$, this is

$$\mathring{\top}\nabla_{\hat{n}} B_{ab} - 4\bar{C}_{c(ab)}\bar{\nabla}^c H + 4H\bar{B}_{ab}.$$

The Poincaré–Einstein condition in six dimensions gives $2\bar{B} = \mathring{\top} B|_\Sigma$, see [9], and in turn the expression quoted in the theorem. The conformal property ensures that the tensor defines an invariant of the Poincaré–Einstein structure.

Finally, when evaluated on the Graham–Lee compactified metric g_r associated with a boundary choice of metric representative $\bar{g} \in \boldsymbol{c}_\Sigma$, we must show that $\mathring{\mathrm{VI}}^{\mathrm{DN}}$ agrees with $\mathrm{DN}^{(6)}$. It has already been established in Lemma 3.9 that $\mathrm{DN}^{(6)} \propto \mathring{\top}\nabla_{\hat{n}} B$. Because g_r is a Graham-Lee compactified metric, it follows by virtue of Theorem 3.6 that the first, third, and fifth T-curvatures then vanish. In particular, the first T-curvature is the mean curvature, so it trivially follows that, in this metric representative,

$$\mathring{\mathrm{VI}}^{\mathrm{DN}} = \mathring{\top}\nabla_{\hat{n}} B.$$

The theorem follows.

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References

[1] Albin, P.: Renormalizing curvature integrals on Poincaré–Einstein manifolds. *Adv. Math.* **221** (2009), no. 1, 140–169. Zbl 1170.53017 MR 2509323

[2] Alexakis, S.: On the decomposition of global conformal invariants. II. *Adv. Math.***206** (2006), no. 2, 466–502. Zbl 1124.53013 MR 2263712

[3] Alexakis, S.: On the decomposition of global conformal invariants. I. *Ann. of Math. (2)* **170** (2009), no. 3, 1241–1306. Zbl 1190.53028 MR 2600873

[4] Alexakis, S. and Mazzeo, R.: Renormalized area and properly embedded minimal surfaces in hyperbolic 3-manifolds. *Comm. Math. Phys.* **297** (2010), no. 3, 621–651. Zbl 1193.53131 MR 2653898

[5] Anderson, M. T.: L^2 curvature and volume renormalization of AHE metrics on 4-manifolds. *Math. Res. Lett.* **8** (2001), no. 1-2, 171–188. Zbl 0999.53034 MR 1825268

[6] Bailey, T. N., Eastwood, M. G. and Gover, A. R.: Thomas's structure bundle for conformal, projective and related structures. *Rocky Mountain J. Math.* **24** (1994), no. 4, 1191–1217. Zbl 0828.53012 MR 1322223

[7] Blitz, S.: A sharp characterization of the Willmore invariant. *Internat. J. Math.* **34** (2023), no. 9, article no. 2350054, 32 pp. Zbl 07735448 MR 4631637

[8] Blitz, S.: Toward a classification of conformal hypersurface invariants. *J. Math. Phys.* **64** (2023), no. 8, article no. 082504, 10 pp. Zbl 1520.53013 MR 4630728

[9] Blitz, S., Gover, A. R. and Waldron, A.: Conformal fundamental forms and the asymptotically Poincaré–Einstein condition. *Indiana Univ. Math. J.* **72** (2023), no. 6, 2215–2284. Zbl 1536.53079 MR 4684302

[10] Blitz, S., Gover, A. R. and Waldron, A.: Generalized Willmore energies, Q-curvatures, extrinsic Paneitz operators, and extrinsic Laplacian powers. *Commun. Contemp. Math.* **26** (2024), no. 5, article no. 2350014, 50 pp. Zbl 1539.53043 MR 4731308

[11] Cherrier, P.: Problèmes de Neumann non linéaires sur les variétés riemanniennes. *J. Funct. Anal.* **57** (1984), no. 2, 154–206. Zbl 0552.58032 MR 749522

[12] Curry, S. N. and Gover, A. R.: An introduction to conformal geometry and tractor calculus, with a view to applications in general relativity. In *Asymptotic analysis in general relativity*, pp. 86–170. London Math. Soc. Lecture Note Ser. 443, Cambridge Univ. Press, Cambridge, 2018. Zbl 1416.83014 MR 3792084

[13] de Haro, S., Skenderis, K. and Solodukhin, S. N.: Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence. *Comm. Math. Phys.* **217** (2001), no. 3, 595–622. Zbl 0984.83043 MR 1822109

[14] Deser, S. and Schwimmer, A.: Geometric classification of conformal anomalies in arbitrary dimensions. *Phys. Lett. B* **309** (1993), no. 3-4, 279–284. MR 1227281

[15] Fefferman, C. and Graham, C. R.: Conformal invariants. In *The mathematical heritage of Élie Cartan (Lyon, 1984)*, 95–116. Astérisque S131, Société Mathématique de France, Paris, 1985. Zbl 0602.53007 MR 837196

[16] Fefferman, C. and Graham, C. R.: *The ambient metric*. Ann. of Math. Stud. 178, Princeton University Press, Princeton, NJ, 2012. Zbl 1243.53004 MR 2858236

[17] Gover, A. R.: Invariant theory and calculus for conformal geometries. *Adv. Math.* **163** (2001), no. 2, 206–257. Zbl 1004.53010 MR 1864834

[18] Gover, A. R.: Almost Einstein and Poincaré–Einstein manifolds in Riemannian signature. *J. Geom. Phys.* **60** (2010), no. 2, 182–204. Zbl 1194.53038 MR 2587388

[19] Gover, A. R. and Kopiński, J.: Higher fundamental forms of the conformal boundary of asymptotically de Sitter spacetimes. *Classical Quantum Gravity* **40** (2023), no. 1, article no. 015001, 25 pp. Zbl 1516.83012 MR 4516753

[20] Gover, A. R. and Peterson, L. J.: Conformally invariant powers of the Laplacian, Q-curvature, and tractor calculus. *Comm. Math. Phys.* **235** (2003), no. 2, 339–378. Zbl 1022.58014 MR 1969732

[21] Gover, A. R. and Peterson, L. J.: Conformal boundary operators, T-curvatures, and conformal fractional Laplacians of odd-order. *Pacific J. Math.* **311** (2021), no. 2, 277–328. Zbl 1472.35435 MR 4292932

[22] Gover, A. R. and Waldron, A.: Boundary calculus for conformally compact manifolds. *Indiana Univ. Math. J.* **63** (2014), no. 1, 119–163. Zbl 1310.53060 MR 3218267

[23] Gover, A. R. and Waldron, A.: Renormalized volume. *Comm. Math. Phys.* **354** (2017), no. 3, 1205–1244. Zbl 1372.53037 MR 3668619

[24] Gover, A. R. and Waldron, A.: Conformal hypersurface geometry via a boundary Loewner–Nirenberg–Yamabe problem. *Comm. Anal. Geom.* **29** (2021), no. 4, 779–836. Zbl 1511.53040 MR 4291291

[25] Graham, C. R.: Volume and area renormalizations for conformally compact Einstein metrics. In *The Proceedings of the 19th Winter School "Geometry and Physics" (Srní, 1999)*, pp. 31–42. Rend. Circ. Mat. Palermo (2), Suppl. no. 63, 2000. Zbl 0984.53020 MR 1758076

[26] Graham, C. R.: Dirichlet-to-Neumann map for Poincaré–Einstein metrics. *Oberwolfach Rep.* **2** (2005), 2200–2203.

[27] Graham, C. R., Jenne, R., Mason, L. J. and Sparling, G. A. J.: Conformally invariant powers of the Laplacian. I. Existence. *J. London Math. Soc. (2)* **46** (1992), no. 3, 557–565. Zbl 0726.53010 MR 1190438

[28] Graham, C. R. and Lee, J. M.: Einstein metrics with prescribed conformal infinity on the ball. *Adv. Math.* **87** (1991), no. 2, 186–225. Zbl 0765.53034 MR 1112625

[29] Graham, C. R. and Witten, E.: Conformal anomaly of submanifold observables in AdS/CFT correspondence. *Nuclear Phys. B* **546** (1999), no. 1-2, 52–64. Zbl 0944.81046 MR 1682674

[30] Graham, C. R. and Zworski, M.: Scattering matrix in conformal geometry. *Invent. Math.* **152** (2003), no. 1, 89–118. Zbl 1030.58022 MR 1965361

[31] Henningson, M. and Skenderis, K.: The holographic Weyl anomaly. *J. High Energy Phys.* (1998), no. 7, article no. 23, 12 pp. Zbl 0958.81083 MR 1644988

[32] Herfray, Y.: Tractor geometry of asymptotically flat spacetimes. *Ann. Henri Poincaré* **23** (2022), no. 9, 3265–3310. Zbl 1501.83004 MR 4469654

[33] LeBrun, C. R.: $\mathscr{H}$-space with a cosmological constant. *Proc. Roy. Soc. London Ser. A* **380** (1982), no. 1778, 171–185. Zbl 0549.53042 MR 652038

[34] Maldacena, J.: The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.* **2** (1998), no. 2, 231–252. Zbl 0914.53047 MR 1633016

[35] Marx-Kuo, J.: Variations of renormalized volume for minimal submanifolds of Poincaré-Einstein manifolds. *Comm. Anal. Geom.* **33** (2025), no. 1, 17–29. Zbl 08024128 MR 4870309

[36] Wang, F.: Dirichlet-to-Neumann map for Poincaré–Einstein metrics in even dimensions. *Calc. Var. Partial Differential Equations* **58** (2019), no. 1, article no. 9, 42 pp. Zbl 1404.58037 MR 3881880

[37] Yang, P., King, D. and Chang, S.-Y. A.: Renormalized volumes for conformally compact Einstein manifolds. *J. Math. Sci. (N.Y.)* **149** (2008), no. 6, 1755– 1769; translated from *Sovrem. Mat., Fundam. Napravl.* **17** (2006), 129–142. Zbl 1558.53043 MR 2336463

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