



Partial scalar curvatures and topological obstructions for submanifolds

Christos-Raent Onti, Kleanthis Polymerakis and Theodoros Vlachos

Abstract. We investigate specific intrinsic curvatures ρ_k , $1 \leq k \leq n$, that interpolate between the minimum Ricci curvature ρ_1 and the normalized scalar curvature $\rho_n = \rho$ of n -dimensional Riemannian manifolds. For n -dimensional submanifolds in space forms, these curvatures satisfy an inequality involving the mean curvature H and the normal scalar curvature ρ^\perp , which reduces to the well-known DDVV inequality when $k = n$. We derive topological obstructions for compact n -dimensional submanifolds based on universal lower bounds of the $L^{n/2}$ -norms of certain functions involving ρ_k , H and ρ^\perp . These obstructions are expressed in terms of the Betti numbers. Our main result applies for any $1 \leq k \leq n - 1$, but it generally fails for $k = n$, where the involved norm vanishes precisely for Wintgen ideal submanifolds. We demonstrate this by providing a method of constructing new compact 3-dimensional minimal Wintgen ideal submanifolds in even-dimensional spheres. Specifically, we prove that such submanifolds exist in \mathbb{S}^6 with arbitrarily large first Betti number.

1. Introduction

A fundamental problem in differential geometry is to investigate the intricate relationship between the geometry and topology of Riemannian manifolds. From the perspective of submanifold theory, there is a profound interplay between the intrinsic and extrinsic geometries of a submanifold, significantly influencing its topological properties. In particular, the intrinsic geometry, governed by metrics and curvature tensors intrinsic to the submanifold itself, interacts with the extrinsic geometry, characterized by how the submanifold is embedded within a higher-dimensional ambient space. This interaction can lead to profound insights into the topological structure of the submanifold.

One of the key questions in this area involves understanding how intrinsic curvature invariants, such as the Ricci curvature and scalar curvature, relate to extrinsic ones, including the mean curvature and normal curvature. These relationships often manifest through inequalities that bind intrinsic quantities to extrinsic ones, providing constraints that influence the topological classification of the submanifolds.

Mathematics Subject Classification 2020: 53C42 (primary); 53C40 (secondary).

Keywords: eigenvalues of Ricci tensor, partial/intermediate scalar curvatures, normal curvature, DDVV inequality, Wintgen ideal submanifolds, Betti numbers, Morse theory, Eells–Kuiper manifolds.

In the case of surfaces $f: M^2 \rightarrow \mathbb{Q}_c^{2+m}$ isometrically immersed into the complete simply connected space form \mathbb{Q}_c^{2+m} of constant curvature c , such a pointwise relation is Wintgen’s inequality:

$$K \leq c + H^2 - |K_N|.$$

This inequality, first proved by Wintgen [28] for surfaces in \mathbb{R}^4 , provides an extrinsic upper bound for the Gaussian curvature K of the surface in terms of the length H of the mean curvature vector field and the normal curvature K_N of f , which is related to the area of the curvature ellipse. Furthermore, Wintgen demonstrated that equality holds precisely at the points where the ellipse of curvature is a circle. Guadalupe and Rodríguez [19] extended Wintgen’s inequality in the aforementioned form and straightforwardly derived an inequality for compact surfaces. This inequality establishes a relationship between the integral of $c + H^2 - |K_N|$ and the Euler–Poincaré characteristic of the surface.

Wintgen’s inequality has been generalized to any isometric immersion $f: M^n \rightarrow \mathbb{Q}_c^{n+m}$ with arbitrary dimension into space forms, resulting in what is known as the DDVV inequality

$$\rho \leq c + H^2 - \rho^\perp$$

(see [15, 18, 22]). This pointwise inequality provides an upper bound for the normalized scalar curvature ρ of M^n in terms of the length H of the mean curvature vector field, and of the normal scalar curvature of f given by $\rho^\perp = \|R^\perp\|/n(n-1)$, where R^\perp is the normal curvature tensor of f . Submanifolds for which the DDVV inequality holds as equality at every point are termed Wintgen ideal submanifolds. These submanifolds often exhibit highly symmetric structures, and their classification remains an area of active research, especially in higher dimensions (see [30] and references therein). In the case $n = 2$, these precisely correspond to superconformal surfaces, which are surfaces with curvature ellipses becoming circles at every point. However, in dimensions $n \geq 3$, the classification of Wintgen ideal submanifolds remains an intriguing yet challenging problem.

In this paper, we are interested in a type of partial scalar curvatures essentially introduced by Wolfson in [29]. More precisely, let M^n be a Riemannian manifold and denote by $\lambda_1 \leq \dots \leq \lambda_n$ the eigenvalues of the normalized Ricci tensor at each point. The k -th scalar curvature of M^n is defined pointwise as

$$\rho_k = \frac{1}{k} \sum_{i=1}^k \lambda_i, \quad 1 \leq k \leq n.$$

At any point, the k -th scalar curvature interpolates between the minimum ρ_1 of the normalized Ricci curvature for $k = 1$, and the normalized scalar curvature $\rho = \rho_n$. In fact, ρ_k is the minimum of the average of the sum of Ricci curvatures in k orthonormal vectors, and it is a continuous function on M^n for every k .

Using the DDVV inequality and its equality case (cf. [18, 22]), one can show that for every n -dimensional submanifold $f: M^n \rightarrow \mathbb{Q}_c^{n+m}$, $n \geq 3$, the inequality

$$\rho_k \leq c + H^2 - \rho^\perp, \quad 1 \leq k \leq n - 1,$$

holds at every point of M^n , and the equality holds precisely at the umbilical points.

The aim of the present paper is to derive topological obstructions for compact n -dimensional submanifolds based on universal lower bounds of the $L^{n/2}$ -norms of certain

functions involving ρ_k, H and ρ^\perp . These obstructions are expressed in terms of the Betti numbers. For compact submanifolds into space forms \mathbb{Q}_c^{n+m} with $c \geq 0$, we prove the following theorem, that can be regarded as a generalization of the result of Guadalupe-Rodríguez [19] in the context of higher-dimensional submanifolds. Throughout this paper, we assume all considered manifolds to be connected without boundary and oriented, with i -th Betti number $b_i(M^n; \mathbb{F})$ over an arbitrary coefficient field \mathbb{F} .

Theorem 1.1. *Given integers $n \geq 3$ and $m \geq 1$, there exists for every $k \in \{1, \dots, n - 1\}$ a positive constant $\varepsilon_k(n, m)$, depending only on n and m , such that if M^n is a compact Riemannian manifold that admits an isometric immersion into \mathbb{Q}_c^{n+m} with $c \geq 0$, then*

$$\int_{M^n} (c + H^2 - \rho^\perp - \rho_k)^{n/2} dM \geq \varepsilon_k(n, m) \sum_{i=1}^{n-1} b_i(M^n; \mathbb{F})$$

for any coefficient field \mathbb{F} . Moreover, if

$$\int_{M^n} (c + H^2 - \rho^\perp - \rho_k)^{n/2} dM < 2\varepsilon_k(n, m)$$

for some $k \in \{1, \dots, n - 1\}$, then M^n is either homeomorphic to S^n , or it is an Eells–Kuiper manifold. In particular, M^n is homeomorphic to S^n if

$$\int_{M^n} (c + H^2 - \rho^\perp - \rho_k)^{n/2} dM < \varepsilon_k(n, m).$$

We recall that an Eells–Kuiper manifold [16] of dimension n is a compactification of \mathbb{R}^n by a sphere of dimension $n/2$, where $n = 2, 4, 8$, or 16 . For $n \geq 4$, it is simply connected and has the integral cohomology structure of the complex ($n = 4$), quaternionic ($n = 8$), or the Cayley ($n = 16$) projective plane.

We point out that the assumption $n \geq 3$ in the above theorem is essential, and the result does not hold in the case $n = 2$. In fact, the involved integral vanishes for every compact superconformal surface, and there exists an abundance of such surfaces with arbitrary genus in any codimension $m \geq 2$.

Chen and Wei in [8] obtained a geometric rigidity result for compact submanifolds $f: M^n \rightarrow \mathbb{Q}_c^{n+m}$, $n \geq 4$, with parallel mean curvature vector field, in terms of the lowest eigenvalue of the Ricci tensor. More precisely, for $c \geq 0$, they proved that such a submanifold is totally umbilical if the $L^{n/2}$ -norm of the nonnegative part of the function $\lambda - (n - 1)\rho_1$ is sufficiently pinched, where λ is a constant satisfying $(n - 2)(c + H^2) < \lambda \leq (n - 1)(c + H^2)$. The following immediate consequence of Theorem 1.1 is the analogous topological rigidity result for submanifolds of dimension $n \geq 3$, without imposing any assumption on the mean curvature.

Corollary 1.2. *Let $f: M^n \rightarrow \mathbb{Q}_c^{n+m}$, $n \geq 3$, $c \geq 0$, be an isometric immersion of a compact Riemannian manifold. If*

$$\int_{M^n} (c + H^2 - \rho^\perp - \rho_1)^{n/2} dM < \varepsilon_1(n, m),$$

then M^n is homeomorphic to S^n .

The following result is a consequence of Theorem 1.1 that applies to Einstein submanifolds.

Corollary 1.3. *Let $f: M^n \rightarrow \mathbb{Q}_c^{n+m}$, $n \geq 3$, $c \geq 0$, be an isometric immersion of a compact Einstein manifold with Ricci curvature ρ . Then*

$$\int_{M^n} (c + H^2 - \rho^\perp - \rho)^{n/2} dM \geq \varepsilon_1(n, m) \sum_{i=1}^{n-1} b_i(M^n; \mathbb{F})$$

for any coefficient field \mathbb{F} . Moreover, if

$$\int_{M^n} (c + H^2 - \rho^\perp - \rho)^{n/2} dM < 2\varepsilon_1(n, m),$$

then M^n is either homeomorphic to S^n , or it is an Eells–Kuiper manifold. In particular, M^n is homeomorphic to S^n if

$$\int_{M^n} (c + H^2 - \rho^\perp - \rho)^{n/2} dM < \varepsilon_1(n, m).$$

The above result demonstrates that Theorem 1.1 is also valid for $k = n$ in the case of any Einstein submanifold, as $\rho_k = \rho$ for any $1 \leq k \leq n$ in this scenario. However, Theorem 1.1 does not hold for $k = n$ in general, in which case the involved integral vanishes for Wintgen ideal submanifolds. We note that apart from superconformal surfaces and totally umbilical submanifolds, there are only a few known compact Wintgen ideal submanifolds (cf. [15, 30]) and all of them have dimension $n = 3$. In Section 5, we provide counterexamples to Theorem 1.1 for $k = n = 3$. In fact, we present a method to generate new compact Wintgen ideal submanifolds with positive first Betti number, which is an interesting result in its own right. These Wintgen ideal submanifolds are unit bundles of plane subbundles of the normal bundle of appropriate compact minimal surfaces in even-dimensional spheres. This class of minimal surfaces encompasses pseudoholomorphic curves in the nearly Kähler sphere S^6 . This justifies the inclusion of the additional term in the following result when considering the case $k = n$.

Theorem 1.4. *Given integers $n \geq 2$ and $m \geq 1$, there exists for every $\lambda \in [0, 1)$ a positive constant $\varepsilon_\lambda(n, m)$, depending only on n and m , such that if M^n is a compact Riemannian manifold that admits an isometric immersion into \mathbb{Q}_c^{n+m} , then*

$$\int_{M^n} (c + H^2 - \lambda\rho^\perp - \rho)^{n/2} dM \geq \varepsilon_\lambda(n, m) \sum_{i=1}^{n-1} b_i(M^n; \mathbb{F})$$

for any coefficient field \mathbb{F} . Moreover, if

$$\int_{M^n} (c + H^2 - \lambda\rho^\perp - \rho)^{n/2} dM < 2\varepsilon_\lambda(n, m)$$

for some $\lambda \in [0, 1)$, then M^n is either homeomorphic to S^n , or it is an Eells–Kuiper manifold. In particular, M^n is homeomorphic to S^n if

$$\int_{M^n} (c + H^2 - \lambda\rho^\perp - \rho)^{n/2} dM < \varepsilon_\lambda(n, m).$$

It is worth noting that the integral in Theorem 1.4 is conformally invariant, rendering Theorem 1.4 applicable to submanifolds in any space form. Moreover, the above theorem may be regarded as an enhancement of a topological result by Shiohama and Xu [25] for submanifolds in any space form, where the case $\lambda = 0$ is considered.

The paper is organized as follows. In Section 2, we prove an auxiliary lemma that is essential for the proofs of our results and may have potential applications in other contexts. In Section 3, we derive algebraic inequalities regarding symmetric bilinear forms, which are crucial to our proofs. Additionally, we provide examples that suggest Theorem 1.1 may not hold in general for $k = n > 3$. In Section 4, we give the detailed proofs of Theorems 1.1 and 1.4. The final Section 5 is devoted to the construction of the aforementioned new compact 3-dimensional Wintgen ideal submanifolds with arbitrarily large first Betti number, providing counterexamples to Theorem 1.1 for $k = n = 3$.

2. An auxiliary lemma

The aim of this section is to prove an algebraic lemma essential to our results. In the following, let V and W be finite-dimensional real vector spaces with dimensions $n \geq 2$ and $m \geq 1$, respectively. Both spaces are equipped with positive definite inner products, which, for convenience, we denote by the same symbol $\langle \cdot, \cdot \rangle$. We denote the space of all symmetric bilinear forms from V to W by $\text{Sym}(V \times V, W)$, and the space of all self-adjoint endomorphisms of V by $\text{End}(V)$. We recall that the *index* of an element $A \in \text{End}(V)$ is the number of the negative eigenvalues of A , and it is denoted by $\text{Index } A$. Given $\beta \in \text{Sym}(V \times V, W)$, we associate to each $\xi \in W$ the formal shape operator $A_\xi(\beta) \in \text{End}(V)$, defined by

$$\langle \beta(x, y), \xi \rangle = \langle A_\xi(\beta)x, y \rangle, \quad x, y \in V.$$

The space $\text{Sym}(V \times V, W)$ is a complete metric space with respect to the norm $\|\cdot\|$ defined by

$$\|\beta\|^2 = \sum_{i,j=1}^n \|\beta(e_i, e_j)\|^2 = \sum_{a=1}^m \|A_{\xi_a}(\beta)\|^2,$$

where $\{e_i\}_{1 \leq i \leq n}$ and $\{\xi_a\}_{1 \leq a \leq m}$ are orthonormal bases of V and W , respectively.

A function $\varphi: \text{Sym}(V \times V, W) \rightarrow \mathbb{R}$ is called *homogeneous of degree* $d \in \mathbb{R}^+$ if

$$\varphi(t\beta) = t^d \varphi(\beta) \quad \text{for every } t \geq 0 \text{ and any } \beta \in \text{Sym}(V \times V, W).$$

The following lemma, that may be useful for other purposes, is crucial for the proofs of our results.

Lemma 2.1. *Let $n \geq 2, m \geq 1$ and $0 \leq p < n/2$ be integers, and let $d \in \mathbb{R}^+$. Suppose that there is a map that assigns to each pair V and W of vector spaces of dimensions n and m respectively, and equipped with positive definite inner products, a nonnegative continuous function $\varphi_{V,W}: \text{Sym}(V \times V, W) \rightarrow \mathbb{R}$ that is homogeneous of degree d and satisfies the following conditions:*

(i) *For any isometries $i: \tilde{V} \rightarrow V$ and $j: \tilde{W} \rightarrow W$, we have*

$$\varphi_{V,W}(\beta) = \varphi_{\tilde{V},\tilde{W}}(j^{-1} \circ \beta \circ (i \times i)) \quad \text{for all } \beta \in \text{Sym}(V \times V, W).$$

(ii) *If $\varphi_{V,W}(\beta) = 0$ for some $\beta \in \text{Sym}(V \times V, W)$, then the endomorphism $A_u(\beta)$ has an eigenvalue of multiplicity at least $n - p$ for every unit vector $u \in W$, the vanishing of which implies that $A_u(\beta) = 0$.*

Then there exists a positive constant $\delta(n, m)$, depending only on n and m , such that the following inequality holds for any vector spaces V, W and all $\beta \in \text{Sym}(V \times V, W)$:

$$\varphi_{V,W}(\beta) \geq \delta(n, m) (\psi_p(\beta))^{d/n},$$

where the function $\psi_p: \text{Sym}(V \times V, W) \rightarrow \mathbb{R}$ is given by

$$\psi_p(\beta) = \int_{\Lambda_p(\beta)} |\det A_u(\beta)| dS_u, \quad \Lambda_p(\beta) = \{u \in \mathbb{S}^{m-1} : p < \text{Index } A_u(\beta) < n - p\},$$

and dS_u stands for the volume element¹ of the unit $(m - 1)$ -sphere \mathbb{S}^{m-1} in W .

To simplify the proof of the above lemma, we first prove the following result.

Sublemma 2.2. *Let $\{\beta_r\}_{r \in \mathbb{N}}$ be a sequence in $\psi_p^{-1}(1) \subset \text{Sym}(V \times V, W)$ and let $\{s_r\}_{r \in \mathbb{N}}$ be a sequence of positive real numbers. Assume that the sequence $\gamma_r = s_r \beta_r$ converges to some $\gamma \in \text{Sym}(V \times V, W)$ with $\varphi_{V,W}(\gamma) = 0$. Then $\gamma = 0$.*

Proof. For every $r \in \mathbb{N}$, since $\beta_r \in \psi_p^{-1}(1)$ and $\Lambda_p(\beta_r) = \Lambda_p(\gamma_r)$, there exists an open subset $\mathcal{U}_r \subset \mathbb{S}^{m-1} \subset W$ such that $\mathcal{U}_r \subset \Lambda_p(\gamma_r)$ and $\det A_\xi(\gamma_r) \neq 0$ for all $\xi \in \mathcal{U}_r$. Let $\{u_r\}$ be any convergent sequence such that $u_r \in \mathcal{U}_r$ for all $r \in \mathbb{N}$, and set $u = \lim_{r \rightarrow \infty} u_r$. We claim that

$$(2.1) \quad \lim_{r \rightarrow \infty} A_{u_r}(\gamma_r) = 0.$$

Since $\varphi_{V,W}(\gamma) = 0$, by our assumption $A_u(\gamma)$ has an eigenvalue $\mu(u)$ with multiplicity at least $n - p$.

We argue that $\mu(u) = 0$. Using that $\lim_{r \rightarrow \infty} A_{u_r}(\gamma_r) = A_u(\gamma)$ and since $u_r \in \mathcal{U}_r$, we obtain $\text{Index } A_u(\gamma) < n - p$ (see [1]). Assuming that $\mu(u) < 0$, since its multiplicity is at least $n - p$, it follows that $\text{Index } A_u(\gamma) \geq n - p$, and this is a contradiction. Therefore $\mu(u) \geq 0$, and this implies that $\text{Index } A_u(\gamma) \leq p$. Assuming now that $\mu(u) > 0$, from $\lim_{r \rightarrow \infty} A_{u_r}(\gamma_r) = A_u(\gamma)$ it follows that $\text{Index } A_{u_r}(\gamma_r) \leq p$ for r large enough, and this contradicts the fact that $u_r \in \Lambda_p(\gamma_r)$. Therefore, $\mu(u) = 0$ and the second condition for $\varphi_{V,W}$ yields that $\lim_{r \rightarrow \infty} A_{u_r}(\gamma_r) = A_u(\gamma) = 0$.

We may choose convergent sequences $\{u_r^{(1)}\}, \dots, \{u_r^{(m)}\}$ in \mathcal{U}_r such that the vectors $u_r^{(1)}, \dots, u_r^{(m)}$ span W for all $r \in \mathbb{N}$. Using the Gram–Schmidt process, we obtain sequences $\{\xi_r^{(1)}\}, \dots, \{\xi_r^{(m)}\}$ with $\xi_r^{(a)} \in \text{span}\{u_r^{(1)}, \dots, u_r^{(a)}\}$ for each $1 \leq a \leq m$. Then $\xi_r^{(a)} = \sum_{\ell=1}^a x_r^{(\ell)} u_r^{(\ell)}$, where $\{x_r^{(\ell)}\}$ are convergent sequences for each $1 \leq \ell \leq m$. Consequently, we have that

$$(2.2) \quad \gamma_r(\cdot, \cdot) = \sum_{a=1}^m \langle A_{\xi_r^{(a)}}(\gamma_r) \cdot, \cdot \rangle \xi_r^{(a)}, \quad \text{where} \quad A_{\xi_r^{(a)}}(\gamma_r) = \sum_{\ell=1}^a x_r^{(\ell)} A_{u_r^{(\ell)}}(\gamma_r).$$

Using (2.1), we obtain

$$\lim_{r \rightarrow \infty} A_{\xi_r^{(a)}}(\gamma_r) = \lim_{r \rightarrow \infty} \sum_{\ell=1}^a x_r^{(\ell)} A_{u_r^{(\ell)}}(\gamma_r) = 0 \quad \text{for all } 1 \leq a \leq m.$$

From (2.2) and the above, it follows that $\lim_{r \rightarrow \infty} \gamma_r = 0$ and thus, $\gamma = 0$. ■

¹In case $m = 1$, integration reduces to summation.

Proof of Lemma 2.1. Let V and W be vector spaces as in the statement of the lemma. We first show that the set $\psi_p^{-1}(1)$ is nonempty. Indeed, since the function ψ_p is homogeneous of degree n , it follows that for an arbitrary $\beta \in \text{Sym}(V \times V, W)$ with $\psi_p(\beta) \neq 0$, we have that $\beta/(\psi_p(\beta))^{1/n} \in \psi_p^{-1}(1)$.

Let $\{\beta_r\}$ be a sequence in $\psi_p^{-1}(1)$ such that

$$\lim_{r \rightarrow \infty} \varphi_{V,W}(\beta_r) = \inf \varphi_{V,W}(\psi_p^{-1}(1)) \geq 0.$$

We claim that the sequence $\{\beta_r\}$ is bounded. Arguing indirectly, assume that there exists a subsequence, which by abuse of notation is again denoted by $\{\beta_r\}$, such that $\lim_{r \rightarrow \infty} \|\beta_r\| = \infty$. Since $\beta_r \in \psi_p^{-1}(1)$ implies that $\beta_r \neq 0$ for all $r \in \mathbb{N}$, we can consider the sequence $\gamma_r = s_r \beta_r$, where $s_r = 1/\|\beta_r\|$. Then $\{\gamma_r\}$ is bounded with $\|\gamma_r\| = 1$ and, by passing to a subsequence if necessary, we may assume that it converges to some $\gamma \in \text{Sym}(V \times V, W)$ with $\|\gamma\| = 1$. From $\varphi_{V,W}(\gamma_r) = \varphi_{V,W}(\beta_r)/\|\beta_r\|^d$, it follows that $\lim_{r \rightarrow \infty} \varphi_{V,W}(\gamma_r) = 0$. Then, the continuity of $\varphi_{V,W}$ yields that $\varphi_{V,W}(\gamma) = 0$ and Sublemma 2.2 implies that $\gamma = 0$. This contradicts the fact that $\|\gamma\| = 1$, and the proof of the claim follows.

Thus we may assume that $\{\beta_r\}$ converges to some $\beta_\infty \in \text{Sym}(V \times V, W)$. We argue that $\varphi_{V,W}(\beta_\infty) > 0$. Suppose to the contrary that $\varphi_{V,W}(\beta_\infty) = 0$. Then, Sublemma 2.2 applied to $\gamma_r = \beta_r$ yields that $\beta_\infty = 0$. Therefore $A_u(\beta_\infty) = 0$ for every $u \in S^{m-1}$. On the other hand, since $\beta_r \in \psi_p^{-1}(1)$, the mean value theorem implies that there exists $\xi_r \in S^{m-1}$ such that

$$1 = \psi_p(\beta_r) \leq \int_{S^{m-1}} |\det A_u(\beta_r)| dS_u = |\det A_{\xi_r}(\beta_r)| \text{Vol}(S^{m-1}) \quad \text{for all } r \in \mathbb{N}.$$

Since the sequence $\{\xi_r\}$ is bounded, we may assume that it converges to some $\xi \in S^{m-1}$. Then, by letting $r \rightarrow \infty$ in the above, we conclude that $\det A_\xi(\beta_\infty) \neq 0$, which contradicts the fact that $A_u(\beta_\infty) = 0$ for every $u \in S^{m-1}$.

Thus, the function $\varphi_{V,W}$ attains a positive minimum $\delta(n, m) = \varphi_{V,W}(\beta_\infty)$ on $\psi_p^{-1}(1)$, which by condition (i) depends only on n and m . Therefore, for an arbitrary $\beta \in \text{Sym}(V \times V, W)$ with $\psi_p(\beta) \neq 0$, since $\gamma = \beta/(\psi_p(\beta))^{1/n} \in \psi_p^{-1}(1)$, it follows that $\varphi_{V,W}(\gamma) \geq \delta(n, m)$. Then, the homogeneity of $\varphi_{V,W}$ implies the desired inequality. ■

3. Algebraic preliminaries

Let V and W be real vector spaces of dimensions $n \geq 2$ and $m \geq 1$ respectively, equipped with positive definite inner products.

The *Kulkarni–Nomizu product* of two bilinear forms $\phi, \psi \in \text{Hom}(V \times V, \mathbb{R})$ is the $(0, 4)$ -tensor $\phi \otimes \psi: V \times V \times V \times V \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \phi \otimes \psi(x_1, x_2, x_3, x_4) &= \phi(x_1, x_3) \psi(x_2, x_4) + \phi(x_2, x_4) \psi(x_1, x_3) \\ &\quad - \phi(x_1, x_4) \psi(x_2, x_3) - \phi(x_2, x_3) \psi(x_1, x_4). \end{aligned}$$

Using the inner product of W , we extend the Kulkarni–Nomizu product to bilinear forms $\beta, \gamma \in \text{Hom}(V \times V, W)$, as the $(0, 4)$ -tensor $\beta \otimes \gamma: V \times V \times V \times V \rightarrow \mathbb{R}$ defined by

$$\beta \otimes \gamma(x_1, x_2, x_3, x_4) = \langle \beta(x_1, x_3), \gamma(x_2, x_4) \rangle + \langle \beta(x_2, x_4), \gamma(x_1, x_3) \rangle - \langle \beta(x_1, x_4), \gamma(x_2, x_3) \rangle - \langle \beta(x_2, x_3), \gamma(x_1, x_4) \rangle.$$

For any $c \in \mathbb{R}$, we define the formal Ricci tensor as the map $\text{Ric}_c: \text{Sym}(V \times V, W) \rightarrow \text{Sym}(V \times V, \mathbb{R})$ given by

$$\text{Ric}_c(\beta)(x, y) = \text{tr } \mathbf{R}_c(\beta)(\cdot, x, \cdot, y), \quad x, y \in V,$$

where

$$\mathbf{R}_c(\beta) = \frac{1}{2} (c \langle \cdot, \cdot \rangle \otimes \langle \cdot, \cdot \rangle + \beta \otimes \beta).$$

For any $\beta \in \text{Sym}(V \times V, W)$, we denote by $\lambda_{c,1}(\beta) \leq \dots \leq \lambda_{c,n}(\beta)$ the eigenvalues of the self-adjoint operator $\mathbf{T}_c(\beta) \in \text{End}(V)$ determined by

$$\langle \mathbf{T}_c(\beta)x, y \rangle = \frac{1}{n-1} \text{Ric}_c(\beta)(x, y).$$

For each $1 \leq k \leq n$, we consider the functions $\rho_{c,k}: \text{Sym}(V \times V, W) \rightarrow \mathbb{R}$ defined by

$$\rho_{c,k}(\beta) = \frac{1}{k} \sum_{i=1}^k \lambda_{c,i}(\beta).$$

For $k = n$, we set $\rho_c(\beta) = \rho_{c,n}(\beta)$. Clearly we have

$$\rho_c(\beta) = \frac{1}{n(n-1)} \text{tr } \text{Ric}_c(\beta).$$

Furthermore, we define the function $\rho^\perp: \text{Sym}(V \times V, W) \rightarrow \mathbb{R}$ by

$$\rho^\perp(\beta) = \frac{1}{n(n-1)} \|\mathbf{R}^\perp(\beta)\| = \frac{1}{n(n-1)} \left(\sum_{i,j=1}^n \sum_{a,b=1}^m (\mathbf{R}^\perp(\beta)(e_i, e_j, \xi_a, \xi_b))^2 \right)^{1/2},$$

where $\{e_i\}_{1 \leq i \leq n}$ and $\{\xi_a\}_{1 \leq a \leq m}$ are orthonormal bases of V and W , respectively. Here, the tensor $\mathbf{R}^\perp(\beta): V \times V \times W \times W \rightarrow \mathbb{R}$ is given by

$$\mathbf{R}^\perp(\beta)(x, y, \xi, \eta) = \langle [A_\xi(\beta), A_\eta(\beta)]x, y \rangle, \quad x, y \in V, \xi, \eta \in W,$$

being $[A, B] = A \circ B - B \circ A$ the commutator of $A, B \in \text{End}(V)$.

For every $\beta \in \text{Sym}(V \times V, W)$, we set $\mathcal{H}_\beta = \frac{1}{n} \text{tr } \beta$ and $H_\beta = \|\mathcal{H}_\beta\|$, and we denote by $\hat{\beta} = \beta - \langle \cdot, \cdot \rangle \mathcal{H}_\beta$ the traceless part of β . The form β is called *umbilical* if $\hat{\beta} = 0$. It is clear that β is umbilical if and only if $A_\xi(\beta) = \langle \mathcal{H}_\beta, \xi \rangle \text{Id}_V$ for every $\xi \in W$.

Lemma 3.1. *For every $\beta \in \text{Sym}(V \times V, W)$ and any $c \in \mathbb{R}$, the following hold:*

(i) *The formal Ricci tensor is given by*

$$\text{Ric}_c(\beta)(x, y) = c(n-1)\langle x, y \rangle + n\langle \mathcal{H}_\beta, \beta(x, y) \rangle - \sum_{i=1}^n \langle \beta(x, e_i), \beta(y, e_i) \rangle,$$

for $x, y \in V$, where $\{e_i\}_{1 \leq i \leq n}$ is an orthonormal basis of V .

(ii) *The DDVV inequality is valid:*

$$\rho_c(\beta) \leq c + H_\beta^2 - \rho^\perp(\beta).$$

Equality holds in the above inequality if and only if there exist orthonormal bases $\{e_i\}_{1 \leq i \leq n}$ of V and $\{\xi_a\}_{1 \leq a \leq m}$ of W such that

$$A_{\xi_1}(\beta) = \text{diag}(\lambda_1 + \mu, \lambda_1 - \mu, \lambda_1, \dots, \lambda_1),$$

$$A_{\xi_2}(\beta) = \text{diag}\left(\begin{pmatrix} \lambda_2 & \mu \\ \mu & \lambda_2 \end{pmatrix}, \lambda_2, \dots, \lambda_2\right),$$

and $A_{\xi_a}(\beta) = \lambda_a I_n$ for $3 \leq a \leq m$, where $\mu, \lambda_1, \dots, \lambda_m$ are real numbers.

Proof. Part (i) follows by a straightforward computation. For the second part, we choose an orthonormal basis $\{u_a\}_{1 \leq a \leq m}$ of W such that $u_1 = \mathcal{H}_\beta / H_\beta$ if $\mathcal{H}_\beta \neq 0$. Let $B_{u_a}(\beta) = A_{u_a}(\beta) - \langle \mathcal{H}_\beta, u_a \rangle \text{Id}_V$ be the traceless part of $A_{u_a}(\beta)$ for $1 \leq a \leq m$. Using part (i) and taking into account that $\|\mathring{\beta}\|^2 = \|\beta\|^2 - nH_\beta^2$, a direct computation yields that

$$(3.1) \quad c + H_\beta^2 - \rho_c(\beta) = \frac{\|\mathring{\beta}\|^2}{n(n-1)} = \frac{1}{n(n-1)} \sum_{a=1}^m \|B_{u_a}(\beta)\|^2.$$

On the other hand, it follows easily that

$$(3.2) \quad \rho^\perp(\beta) = \frac{\|R^\perp(\beta)\|}{n(n-1)} = \frac{1}{n(n-1)} \left(\sum_{a,b=1}^m \|[B_{u_a}(\beta), B_{u_b}(\beta)]\|^2 \right)^{1/2}.$$

Therefore, the desired inequality is equivalent to

$$\sum_{a,b=1}^m \|[B_{u_a}(\beta), B_{u_b}(\beta)]\|^2 \leq \left(\sum_{a=1}^m \|B_{u_a}(\beta)\|^2 \right)^2.$$

The last inequality, along with its equality case, has been proved in [18,22]. The remainder of the proof follows the same steps as the proof of Corollary 1.2 in [18], with $T_p M$ replaced by V and $T_p^\perp M$ replaced by W . ■

Proposition 3.2. *The following assertions hold for any vector spaces V and W , any $\beta \in \text{Sym}(V \times V, W)$ and every $c \in \mathbb{R}$.*

(i) *If $n \geq 3$, then for every $k \in \{1, \dots, n-1\}$, the following inequality holds:*

$$(3.3) \quad \rho_{c,k}(\beta) \leq c + H_\beta^2 - \rho^\perp(\beta).$$

Equality holds for some $k \in \{1, \dots, n-1\}$ if and only if β is umbilical.

(ii) *For every $\lambda \in [0, 1)$, the following inequality holds:*

$$(3.4) \quad \rho_c(\beta) \leq c + H_\beta^2 - \lambda \rho^\perp(\beta).$$

Equality holds for some $\lambda \in [0, 1)$ if and only if β is umbilical.

Proof. (i) Assume that $n \geq 3$. For every $k \in \{1, \dots, n - 1\}$, we have that

$$(3.5) \quad \rho_{c,k}(\beta) \leq \rho_{c,k+1}(\beta),$$

and equality holds if and only if $\lambda_{c,1}(\beta) = \dots = \lambda_{c,k+1}(\beta)$. The proof of (3.3) then follows from the DDVV inequality in Lemma 3.1(ii).

Assume now that for some $k \in \{1, \dots, n - 1\}$, inequality (3.3) holds as equality. From (3.5), it follows that all inequalities in

$$\rho_{c,k}(\beta) \leq \dots \leq \rho_{c,n}(\beta) \leq c + H_\beta^2 - \rho^\perp(\beta)$$

hold as equalities. This implies that all eigenvalues of $T_c(\beta)$ are equal and thus, $\text{Ric}_c(\beta) = (n - 1)\rho\langle \cdot, \cdot \rangle$, where $\rho = \rho_{c,i}(\beta)$, $1 \leq i \leq n$.

Moreover, the DDVV inequality holds as equality for β . Let $\{e_i\}_{1 \leq i \leq n}$ and $\{\xi_a\}_{1 \leq a \leq m}$ be orthonormal bases of V and W , respectively, as in Lemma 3.1(ii). Using Lemma 3.1 and taking into account that $\mathcal{H}_\beta = \sum_{a=1}^m \lambda_a \xi_a$, a straightforward computation yields that

$$\begin{aligned} \text{Ric}_c(\beta)(e_1, e_1) &= (n - 1)(c + H_\beta^2) + \mu((n - 2)\lambda_1 - 2\mu), \\ \text{Ric}_c(\beta)(e_2, e_2) &= (n - 1)(c + H_\beta^2) - \mu((n - 2)\lambda_1 + 2\mu), \\ \text{Ric}_c(\beta)(e_i, e_i) &= (n - 1)(c + H_\beta^2), \quad \text{if } i \geq 3. \end{aligned}$$

Since $\text{Ric}_c(\beta) = (n - 1)\rho\langle \cdot, \cdot \rangle$, the above yield that $\mu = 0$. Therefore, we obtain that $A_{\xi_a}(\beta) = \lambda_a I_n$, $1 \leq a \leq m$, and this implies that β is umbilical. The converse is obvious.

(ii) The desired inequality follows immediately from the DDVV inequality. If (3.4) holds as equality, then

$$c + H_\beta^2 - \rho_c(\beta) - \rho^\perp(\beta) = (\lambda - 1)\rho^\perp(\beta).$$

By virtue of Lemma 3.1(ii), the left-hand side of the above is nonnegative. Thus $\rho^\perp(\beta) = 0$ and the DDVV inequality holds as equality for β . Now we choose the orthonormal bases of V and W as in Lemma 3.1(ii). Then a direct computation yields that $\|\mathbf{R}^\perp(\beta)\| = 4\mu^2$. Since $\rho^\perp(\beta) = 0$, it follows that $\mu = 0$. Then, as in the proof of part (i), we conclude that β is umbilical. ■

Proposition 3.3. *Given integers $n \geq 2$ and $m \geq 1$, the following assertions hold.*

- (i) *If $n \geq 3$, then for any $c \in \mathbb{R}$ and $k \in \{1, \dots, n - 1\}$, there exists a constant $\delta_{c,k}(n, m) > 0$ depending only on n and m , such that the inequality*

$$c + H_\beta^2 - \rho^\perp(\beta) - \rho_{c,k}(\beta) \geq \delta_{c,k}(n, m) (\psi_0(\beta))^{2/n}$$

holds for any vector spaces V and W and all $\beta \in \text{Sym}(V \times V, W)$, where ψ_0 is the function defined in Lemma 2.1.

- (ii) *For every $c \in \mathbb{R}$ and $\lambda \in [0, 1)$, there exists a constant $\delta_{c,\lambda}(n, m) > 0$ depending only on n and m , such that the inequality*

$$c + H_\beta^2 - \lambda \rho^\perp(\beta) - \rho_c(\beta) \geq \delta_{c,\lambda}(n, m) (\psi_0(\beta))^{2/n},$$

holds for any vector spaces V and W and all $\beta \in \text{Sym}(V \times V, W)$.

Proof. For each pair of vector spaces V and W of dimensions n and m , respectively, $c \in \mathbb{R}$, $\lambda \in [0, 1]$ and $k \in \{1, \dots, n\}$, we consider the function $\varphi_{V,W}^{k,\lambda}: \text{Sym}(V \times V, W) \rightarrow \mathbb{R}$ given by

$$\varphi_{V,W}^{k,\lambda}(\beta) = c + H_\beta^2 - \lambda \rho^\perp(\beta) - \rho_{c,k}(\beta).$$

We claim that the function $\varphi_{V,W}^{k,1}$ for any $k \in \{1, \dots, n - 1\}$, $n \geq 3$, as well as the function $\varphi_{V,W}^{n,\lambda}$ for any $\lambda \in [0, 1)$, fulfill the requirements in Lemma 2.1.

First, we argue that $\varphi_{V,W}^{k,\lambda}$ is continuous. Let $\{\beta_r\}$ be a sequence in $\text{Sym}(V \times V, W)$ with $\lim_{r \rightarrow \infty} \beta_r = \beta \in \text{Sym}(V \times V, W)$ and consider orthonormal bases $\{e_i\}_{1 \leq i \leq n}$ and $\{\xi_a\}_{1 \leq a \leq m}$ of V and W , respectively. From Lemma 3.1(i), it follows that the Ricci tensors satisfy

$$\lim_{r \rightarrow \infty} \text{Ric}_c(\beta_r)(e_i, e_j) = \text{Ric}_c(\beta)(e_i, e_j), \quad \text{for } 1 \leq i, j \leq n.$$

Therefore $\lim_{r \rightarrow \infty} T_c(\beta_r) = T_c(\beta)$ for the associated self-adjoint operators. Then the ordered eigenvalues of these operators satisfy $\lim_{r \rightarrow \infty} \lambda_{c,i}(\beta_r) = \lambda_{c,i}(\beta)$, $1 \leq i \leq n$ (see for instance Theorem 1 in [1]). Therefore,

$$\lim_{r \rightarrow \infty} \rho_{c,k}(\beta_r) = \rho_{c,k}(\beta) \quad \text{for every } k \in \{1, \dots, n\}.$$

Given that $\lim_{r \rightarrow \infty} A_{\xi_a}(\beta_r) = A_{\xi_a}(\beta)$ for any $1 \leq a \leq m$, we obtain $\lim_{r \rightarrow \infty} \rho^\perp(\beta_r) = \rho^\perp(\beta)$. It is clear that $\lim_{r \rightarrow \infty} H_{\beta_r}^2 = H_\beta^2$. Hence the above imply that $\lim_{r \rightarrow \infty} \varphi_{V,W}^{k,\lambda}(\beta_r) = \varphi_{V,W}^{k,\lambda}(\beta)$ and thus the function $\varphi_{V,W}^{k,\lambda}$ is continuous.

We next show that $\varphi_{V,W}^{k,\lambda}$ is a homogeneous function of degree $d = 2$. Lemma 3.1(i) yields that

$$\text{Ric}_c(\beta)(x, y) = c(n - 1)\langle x, y \rangle + \text{Ric}_0(\beta)(x, y) \quad \text{for every } x, y \in V.$$

This implies that $T_c(\beta) = c \text{Id}_V + T_0(\beta)$ and the corresponding eigenvalues are related by $\lambda_{c,i}(\beta) = c + \lambda_{0,i}(\beta)$, $1 \leq i \leq n$. Therefore, $\rho_{c,k}(\beta) = c + \rho_{0,k}(\beta)$, and thus

$$\varphi_{V,W}^{k,\lambda} = H_\beta^2 - \lambda \rho^\perp(\beta) - \rho_{0,k}(\beta), \quad \text{for } 1 \leq k \leq n.$$

From Lemma 3.1(i), it follows that $\text{Ric}_0(t\beta) = t^2 \text{Ric}_0(\beta)$ for every $t \in \mathbb{R}$. Then a straightforward computation shows that $\varphi_{V,W}^{k,\lambda}$ is a homogeneous function of degree $d = 2$.

The fact that the function $\varphi_{V,W}^{k,\lambda}$ satisfies condition (i) in Lemma 2.1 follows directly from its definition. Moreover, Proposition 3.2 implies that condition (ii) is fulfilled for $p = 0$. The proof now follows from Lemma 2.1. ■

The following example shows that the inequality in part (i) of Proposition 3.3 fails for $k = n$. In fact, this example shows that $\liminf_{\lambda \rightarrow 1^-} \delta_{c,\lambda}(n, m) = 0$.

Example 3.4. Given $\mu > 0$ and a sequence $\sigma_r > 0$ such that $\lim_{r \rightarrow \infty} \sigma_r = 0$, we define the sequence $\beta_r \in \text{Sym}(V \times V, W)$, $r \in \mathbb{N}$, given by $\beta_r = A_{\xi_1}(\beta_r)\xi_1 + A_{\xi_2}(\beta_r)\xi_2$, where

$$A_{\xi_1}(\beta_r) = \text{diag}(\mu, -\mu, -\sigma_r, \sigma_r, \dots, \sigma_r), \quad A_{\xi_2}(\beta_r) = \text{diag}\left(\begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, -\sigma_r, \sigma_r, \dots, \sigma_r\right),$$

with respect to orthonormal bases $\{e_i\}_{1 \leq i \leq n}$ and $\{\xi_a\}_{1 \leq a \leq m}$ of $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, respectively, with $n \geq 3$ and $m \geq 2$.

Then, for any unit vector $u = \sum_{a=1}^m u_a \xi_a \in W$, the eigenvalues of $A_u(\beta_r)$ are

$$\begin{aligned} \tau_1 &= \mu(u_1^2 + u_2^2)^{1/2}, & \tau_2 &= -\mu(u_1^2 + u_2^2)^{1/2}, \\ \tau_3 &= -\sigma_r(u_1 + u_2), & \text{and } \tau_i &= \sigma_r(u_1 + u_2), \quad \text{for } 4 \leq i \leq n. \end{aligned}$$

Thus $\text{Index } A_u(\beta_r) = 2$ if $u_1 + u_2 > 0$, whereas $\text{Index } A_u(\beta_r) = n - 2$ if $u_1 + u_2 < 0$. Therefore, $\Lambda_1(\beta_r) = U$ for every $r \in \mathbb{N}$, where $U = \{u \in \mathbb{S}^{m-1} : u_1 + u_2 \neq 0\}$ and, consequently,

$$(3.6) \quad \psi_1(\beta_r) = I \sigma_r^{n-2}, \quad \text{where } I = \mu^2 \int_U (u_1^2 + u_2^2) |u_1 + u_2|^{n-2} dS_u.$$

On the other hand, a straightforward computation using (3.1) and (3.2) yields that

$$c + H_{\beta_r}^2 - \rho^\perp(\beta_r) - \rho_{c,n}(\beta_r) = D(n) \sigma_r^2, \quad \text{where } D(n) = \frac{4(3n - 8)}{n^2(n - 1)}.$$

From (3.6) and the above, it follows that the quotient

$$\frac{c + H_{\beta_r}^2 - \rho^\perp(\beta_r) - \rho_{c,n}(\beta_r)}{(\psi_1(\beta_r))^{2/n}} = \frac{D(n)}{I^{2/n}} \sigma_r^{4/n}$$

tends to zero. Therefore, the function $(c + H^2 - \rho^\perp - \rho_{c,n})/\psi_1^{2/n}$ is not bounded from below by a positive constant. Since $\psi_0(\beta) \geq \psi_1(\beta)$ for every $\beta \in \text{Sym}(V \times V, W)$, we conclude that this also holds for the function $(c + H^2 - \rho^\perp - \rho_{c,n})/\psi_0^{2/n}$.

4. Proofs of the results

Let $f: M^n \rightarrow \mathbb{R}^{n+m}$ be an isometric immersion of a compact Riemannian manifold into the Euclidean space \mathbb{R}^{n+m} equipped with the usual inner product $\langle \cdot, \cdot \rangle$. We denote by $N_f M$ the normal bundle of f and by $\alpha_f \in \Gamma(\text{Hom}(TM \times TM, N_f M))$ its second fundamental form.

First, let us recall some well-known facts about total curvature and how Morse theory imposes constraints on the Betti numbers. The unit normal bundle of f is defined as the set

$$UN_f = \{(p, \xi) \in N_f M : \|\xi\| = 1\}.$$

The *generalized Gauss map* $v: UN_f \rightarrow \mathbb{S}^{n+m-1}$ is given by $v(p, \xi) = \xi$. Now, for each $u \in \mathbb{S}^{n+m-1}$, we consider the height function h_u defined by $h_u(p) = \langle f(p), u \rangle$, $p \in M^n$. Since h_u has a degenerate critical point if and only if u is a critical value of v , by Sard's theorem, there exists a subset $E \subset \mathbb{S}^{n+m-1}$ of measure zero such that h_u is a Morse function for all $u \in \mathbb{S}^{n+m-1} \setminus E$. We denote by $\mu_i(u)$ the number of critical points of h_u of index i for each $u \in \mathbb{S}^{n+m-1} \setminus E$ and set $\mu_i(u) = 0$ for every $u \in E$. Following Kuiper [21], we define the *total curvature of index i of f* by

$$\tau_i(f) = \frac{1}{\text{Vol}(\mathbb{S}^{n+m-1})} \int_{\mathbb{S}^{n+m-1}} \mu_i(u) dS,$$

where dS denotes the volume element of the sphere \mathbb{S}^{n+m-1} . From the weak Morse inequalities (see Theorem 5.2 on p. 29 of [23]), we have

$$(4.1) \quad \mu_i(u) \geq b_i(M^n; \mathbb{F}) \quad \text{for all } u \in \mathbb{S}^{n+m-1} \setminus E,$$

where $b_i(M^n; \mathbb{F})$ is the i -th Betti number of M^n over an arbitrary coefficient field \mathbb{F} . By integrating over \mathbb{S}^{n+m-1} , we obtain

$$(4.2) \quad \tau_i(f) \geq b_i(M^n; \mathbb{F}).$$

There is a natural volume element $d\Sigma$ on the unit normal bundle UN_f . In fact, if dV is a $(m - 1)$ -form on UN_f such that its restriction to a fiber of the unit normal bundle at (p, ξ) is the volume element of the unit $(m - 1)$ -sphere of the normal space of f at p , then $d\Sigma = dM \wedge dV$, where dM is the volume element of M^n . Shiohama and Xu (see [24], p. 381) refined a well-known integral formula due to Chern–Lashof [10, 11], and proved that

$$(4.3) \quad \int_{U^i N_f} |\det A_\xi| d\Sigma = \int_{\mathbb{S}^{n+m-1}} \mu_i(u) dS,$$

where $U^i N_f$ is the subset of the unit normal bundle of f defined by

$$U^i N_f = \{(p, \xi) \in UN_f : \text{Index } A_\xi = i\}, \quad 0 \leq i \leq n,$$

and A_ξ the shape operator of f with respect to ξ , where $(p, \xi) \in UN_f$.

Lemma 4.1. *Let $f: M^n \rightarrow \mathbb{R}^{n+m}$, with $n \geq 2$, be an isometric immersion of a compact Riemannian manifold such that $\sum_{i=1}^{n-1} \tau_i(f) < 2$. Then, for any coefficient field \mathbb{F} , the Betti numbers satisfy $\sum_{i=1}^{n-1} b_i(M^n; \mathbb{F}) \in \{0, 1\}$. In particular,*

- (i) *if $b_i(M^n; \mathbb{F}) = 0$ for any $1 \leq i \leq n - 1$, then M^n is homeomorphic to \mathbb{S}^n ;*
- (ii) *if $\sum_{i=1}^{n-1} b_i(M^n; \mathbb{F}) = 1$, then M^n is an Eells–Kuiper manifold.*

Proof. The assumption is equivalent to

$$\int_{\mathbb{S}^{n+m-1}} \sum_{i=1}^{n-1} \mu_i(u) dS < 2 \text{Vol}(\mathbb{S}^{n+m-1}).$$

This implies that there exists a unit vector u_0 such that the height function h_{u_0} is a Morse function satisfying $\sum_{i=1}^{n-1} \mu_i(u_0) < 2$ or, equivalently,

$$(4.4) \quad \sum_{i=1}^{n-1} \mu_i(u_0) \leq 1.$$

The above and (4.1) yield that $\sum_{i=1}^{n-1} b_i(M^n; \mathbb{F}) \in \{0, 1\}$ for every coefficient field \mathbb{F} .

(i) Suppose that

$$(4.5) \quad b_i(M^n; \mathbb{F}) = 0 \quad \text{for every } 1 \leq i \leq n - 1.$$

We claim that the homology groups of M^n over the integers satisfy $H_k(M^n; \mathbb{Z}) = 0$ for any $1 \leq k \leq n - 1$, which in view of (4.5), is equivalent to the claim that M^n has no torsion. Indeed, if $H_k(M^n; \mathbb{Z})$ contains torsion for some $1 \leq k \leq n - 1$, then $H_k(M^n; \mathbb{Z}_p) \neq 0$ for some prime p , which contradicts (4.5). Hence M^n is a homology sphere over the integers.

We now prove that M^n is simply connected. This is clear if $n = 2$. Assume that $n \geq 3$ and suppose, to the contrary, that the fundamental group is $\pi_1(M^n) \neq 0$. It follows from Proposition 4.5.7 on p. 90 of [2] that $\mu_1(u_0) \neq 0$. Thus, (4.4) yields that $\mu_1(u_0) = 1$ and $\mu_i(u_0) = 0$ for any $2 \leq i \leq n - 1$. Then, by Morse theory, it follows that the manifold M^n has the homotopy type of a CW-complex with no cells of dimension $2 \leq i \leq n - 1$. In particular, there are no 2-cells, and thus, by the cellular approximation theorem, the inclusion of the 1-skeleton $X^{(1)} \hookrightarrow M^n$ induces isomorphism between the fundamental groups. Therefore, $\pi_1(M^n)$ is a free group on $b_1(M^n; \mathbb{Z}) = 0$ elements. Hence, $\pi_1(M^n) = 0$, which is a contradiction.

Thus, M^n is a simply connected homology sphere over the integers and therefore a homotopy sphere. By the (generalized) Poincaré conjecture (Smale $n \geq 5$, Freedman $n = 4$, Perelman $n = 3$), M^n is homeomorphic to S^n .

(ii) Assume that $\sum_{i=1}^{n-1} b_i(M^n; \mathbb{F}) = 1$. From Poincaré duality, it follows that n is even and that the Betti numbers are

$$b_{n/2}(M^n; \mathbb{F}) = 1 \quad \text{and} \quad b_i(M^n; \mathbb{F}) = 0, \quad \text{for } 1 \leq i \leq n - 1, i \neq n/2.$$

Jointly with (4.1) and (4.4), the above implies that

$$(4.6) \quad \mu_{n/2}(u_0) = 1 \quad \text{and} \quad \mu_i(u_0) = 0, \quad \text{for } 1 \leq i \leq n - 1, i \neq n/2.$$

Since $\mu_1(u_0) = 0$, it follows from Lemma 4.11 on p. 85 of [7] that $\mu_0(u_0) \leq 1$. Hence, from (4.1) we obtain

$$(4.7) \quad \mu_0(u_0) = 1.$$

The Euler–Poincaré characteristic of M^n is given by

$$\chi(M^n) = \sum_{i=0}^n (-1)^i b_i(M^n; \mathbb{F}) = 3.$$

On the other hand, using (4.6) and (4.7), from Theorem 5.2 on p. 29 of [23] we have

$$\chi(M^n) = \sum_{i=0}^n (-1)^i \mu_i(u_0) = 2 + \mu_n(u_0),$$

and thus $\mu_n(u_0) = 1$. Taking into account (4.6) and (4.7), this implies that the height function h_{u_0} is a Morse function with three critical points. Therefore, M^n is an Eells–Kuiper manifold. ■

Proof of Theorem 1.1. Let $f: M^n \rightarrow \mathbb{Q}_c^{n+m}$, $n \geq 3$, be an isometric immersion with second fundamental form α_f , mean curvature H_f and shape operator A_ξ with respect to ξ , where $(p, \xi) \in UN_f$. Assume first that the ambient space is the Euclidean space \mathbb{R}^{n+m} , that is $c = 0$. Since $H_f(p) = H_{\alpha_f(p)}$, $\rho_k(p) = \rho_{0,k}(\alpha_f(p))$ and $\rho^\perp(p) = \rho^\perp(\alpha_f(p))$, it follows from Proposition 3.3(i) that

$$(H_f^2 - \rho^\perp - \rho_k)^{n/2}(p) \geq (\delta_{0,k}(n, m))^{n/2} \int_{\Lambda_0(\alpha_f(p))} |\det A_\xi| dV_\xi$$

for all $p \in M^n$. Integrating over M^n and using (4.3), we have

$$\int_{M^n} (H_f^2 - \rho^\perp - \rho_k)^{n/2} dM \geq (\delta_{0,k}(n, m))^{n/2} \text{Vol}(\mathbb{S}^{n+m-1}) \sum_{i=1}^{n-1} \tau_i(f).$$

Thus, from the above and (4.2), we obtain

$$(4.8) \quad \int_{M^n} (H_f^2 - \rho^\perp - \rho_k)^{n/2} dM \geq \varepsilon_k(n, m) \sum_{i=1}^{n-1} \tau_i(f) \geq \varepsilon_k(n, m) \sum_{i=1}^{n-1} b_i(M; \mathbb{F}),$$

where $\varepsilon_k(n, m) = (\delta_{0,k}(n, m))^{n/2} \text{Vol}(\mathbb{S}^{n+m-1})$.

Now, assume that

$$\int_{M^n} (H_f^2 - \rho^\perp - \rho_k)^{n/2} dM < 2\varepsilon_k(n, m).$$

Then, it follows directly from (4.8) that $\sum_{i=1}^{n-1} \tau_i(f) < 2$, and the proof follows from Lemma 4.1. In particular, if

$$\int_{M^n} (H_f^2 - \rho^\perp - \rho_k)^{n/2} dM < \varepsilon_k(n, m),$$

then $\sum_{i=1}^{n-1} \tau_i(f) < 1$ and (4.2) implies that only the first case in Lemma 4.1 can occur.

Suppose now that $c > 0$. We consider the isometric immersion $\tilde{f}: M^n \rightarrow \mathbb{R}^{n+m+1}$ given by $\tilde{f} = i \circ f$, where i is an umbilical inclusion of the sphere \mathbb{S}_c^{n+m} of radius $R = 1/\sqrt{c}$ into the Euclidean space \mathbb{R}^{n+m+1} . Clearly, $H_{\tilde{f}}^2 = H_f^2 + c$, and the normal curvature $\tilde{\rho}^\perp$ of \tilde{f} is given by $\tilde{\rho}^\perp = \rho^\perp$. Then the proof follows from the above argument applied to \tilde{f} with $\varepsilon_k(n, m) = (\delta_{0,k}(n, m + 1))^{n/2} \text{Vol}(\mathbb{S}^{n+m})$. ■

Remark 4.2. It is immediate from (3.5) that the constants in Proposition 3.3(i) satisfy $\delta_{c,1}(n, m) \geq \dots \geq \delta_{c,n-1}(n, m) > 0$. Then, it follows from the proof of Theorem 1.1 that the constants $\varepsilon_k(n, m)$, $1 \leq k \leq n - 1$, satisfy $\varepsilon_1(n, m) \geq \dots \geq \varepsilon_{n-1}(n, m) > 0$.

Proof of Corollary 1.2. It follows immediately from Theorem 1.1 for $k = 1$. ■

Proof of Corollary 1.3. Since M^n is Einstein and $n \geq 3$, it has constant normalized Ricci curvature equal to ρ . Therefore, $\rho_k = \rho$ for every $k \in \{1, \dots, n\}$. The proof follows immediately from Theorem 1.1. ■

Proof of Theorem 1.4. Let $f: M^n \rightarrow \mathbb{Q}_c^{n+m}$, $n \geq 2$, be an isometric immersion. We claim that the integral

$$\int_{M^n} (c + H_f^2 - \lambda\rho^\perp - \rho)^{n/2} dM$$

is invariant under conformal changes of the metric $\langle \cdot, \cdot \rangle$ of \mathbb{Q}_c^{n+m} . Indeed, from (3.1) and (3.2), it follows that

$$(4.9) \quad n(n - 1)(c + H_f^2 - \lambda\rho^\perp - \rho) = \|\Phi_f\|^2 - \lambda\|R^\perp\|,$$

where Φ_f is the traceless part of the second fundamental form of f and R^\perp its normal curvature tensor. Consider the conformal change $\widetilde{\langle \cdot, \cdot \rangle} = e^{2u} \langle \cdot, \cdot \rangle$ of the metric of \mathbb{Q}_c^{n+m} ,

where u is a smooth function, and let $\tilde{f}: \tilde{M}^n \rightarrow (\mathbb{Q}_c^{n+m}, \langle \cdot, \cdot \rangle)$ be the isometric immersion induced by f . Then, at corresponding points the normal spaces of f, \tilde{f} coincide and particularly, the second fundamental forms and the mean curvature vector fields are related by

$$\alpha_{\tilde{f}} = \alpha_f - \langle \cdot, \cdot \rangle (\text{grad } u)^\perp \quad \text{and} \quad \mathcal{H}_{\tilde{f}} = e^{-2u} (\mathcal{H}_f - (\text{grad } u)^\perp),$$

where $(\text{grad } u)^\perp$ is the normal component of the gradient of u with respect to the metric $\langle \cdot, \cdot \rangle$. The above imply that $\Phi_{\tilde{f}} = \Phi_f$ and thus, the traceless parts of the shape operators of \tilde{f} and f associated to a local orthonormal frame field $\{\xi_a\}_{1 \leq a \leq m}$ of $N_f M$ satisfy $\tilde{B}_{\xi_a} = B_{\xi_a}, 1 \leq a \leq m$. Any orthonormal tangent frame field $\{e_i\}_{1 \leq i \leq n}$ of $(M, \langle \cdot, \cdot \rangle)$ and any orthonormal normal frame field $\{\xi_a\}_{1 \leq a \leq m}$ of $N_f M$ give rise to orthonormal frame fields $\{\tilde{e}_i\}_{1 \leq i \leq n}$ of $(M, \langle \cdot, \cdot \rangle)$ with $\tilde{e}_i = e^{-u} e_i$ and $\{\tilde{\xi}_a\}_{1 \leq a \leq m}$ of $N_{\tilde{f}} M$ with $\tilde{\xi}_a = e^{-u} \xi_a$. Therefore, a direct computation using (3.1) and (3.2) yields that

$$\|\Phi_{\tilde{f}}\|_{\sim}^2 - \lambda \|\tilde{R}^\perp\|_{\sim} = e^{-2u} (\|\Phi_f\|^2 - \lambda \|R^\perp\|).$$

Now, by virtue of (4.9), the claim follows from the above equality and the fact that the volume element of $(M, \langle \cdot, \cdot \rangle)$ is $e^{nu} dM$.

Therefore, under a conformal change of the metric of (a part of) \mathbb{Q}_c^{n+m} , we may assume without loss of generality that the ambient space is the Euclidean space \mathbb{R}^{n+m} . Since $H_f(p) = H_{\alpha_f(p)}, \rho(p) = \rho_0(\alpha_f(p))$ and $\rho^\perp(p) = \rho^\perp(\alpha_f(p))$, it follows from Proposition 3.3 (ii) that

$$(H_f^2 - \lambda \rho^\perp - \rho)^{n/2}(p) \geq (\delta_{0,\lambda}(n, m))^{n/2} \int_{\Delta_0(\alpha_f(p))} |\det A_\xi| dV_\xi$$

for all $p \in M^n$. Then the proof follows arguing as in the proof of Theorem 1.1 with $\varepsilon_\lambda(n, m) = (\delta_{0,\lambda}(n, m))^{n/2} \text{Vol}(\mathbb{S}^{n+m-1})$. ■

Remark 4.3. Since $\lambda \in [0, 1)$, from (4.9) it follows that

$$n(n-1)(c + H_f^2 - \lambda \rho^\perp - \rho) \leq \|\Phi_f\|^2 = \|\alpha_f\|^2 - nH_f^2.$$

This shows that our integrand is smaller than the (normalized) one of Shiohama and Xu in [25], which corresponds to $\lambda = 0$.

5. Counterexamples to Theorem 1.1 for $k = n$

5.1. Minimal surfaces in spheres

In this subsection, we discuss some properties of the isotropic surfaces in spheres, which are the basic tool for the construction of new compact 3-dimensional Wintgen ideal submanifolds.

Let $f: L^2 \rightarrow \mathbb{S}^{n+2}$ denote an isometric immersion of a two-dimensional oriented Riemannian manifold into the sphere \mathbb{S}^{n+2} . The k^{th} -normal space of f at $x \in L^2$ for $k \geq 1$ is given by

$$N_k^f(x) = \text{span}\{\alpha_f^{k+1}(X_1, \dots, X_{k+1}) : X_1, \dots, X_{k+1} \in T_x L\}$$

where $\alpha_f^2 = \alpha_f: TL \times TL \rightarrow N_fL$ is the standard second fundamental form with values in the normal bundle and

$$\alpha_f^s: TL \times \cdots \times TL \rightarrow N_fL, \quad s \geq 3,$$

is the symmetric tensor called the s^{th} -fundamental form defined inductively by

$$\alpha_f^s(X_1, \dots, X_s) = (\nabla_{X_s}^\perp \dots \nabla_{X_3}^\perp \alpha_f(X_2, X_1))^\perp.$$

Here ∇^\perp is the induced connection in the normal bundle N_fL and $(\)^\perp$ means taking the projection onto the normal complement of $N_1^f \oplus \cdots \oplus N_{s-2}^f$ in N_fL .

Now suppose that $f: L^2 \rightarrow \mathbb{S}^{n+2}$ denotes an oriented minimal and substantial surface. The latter means that the codimension cannot be reduced, in fact, not even locally since minimal surfaces are real analytic. It is known (see [12, 26]) that the normal bundle N_fL of f splits along an open dense subset of L^2 as

$$N_fL = N_1^f \oplus \cdots \oplus N_m^f, \quad m = [(n + 1)/2],$$

since all higher normal bundles have rank two, except possibly the last one, that has rank one if n is odd. The orientation of L^2 induces an orientation on each plane vector bundle N_s^f given by the ordered pair

$$\xi_1^s = \alpha_f^{s+1}(X, \dots, X), \quad \xi_2^s = \alpha_f^{s+1}(JX, \dots, X)$$

where $0 \neq X \in TL$ and J is the complex structure of L^2 determined by the metric and orientation.

For each $1 \leq k \leq m$, the k^{th} -order curvature ellipse $\mathcal{E}_k^f(x) \subset N_k^f(x)$ is defined by

$$\mathcal{E}_k^f(x) = \{\alpha_f^{k+1}(Z^\varphi, \dots, Z^\varphi) : Z^\varphi = \cos \varphi Z + \sin \varphi JZ \text{ and } \varphi \in [0, 2\pi)\},$$

where $Z \in T_xL$ is any vector of unit length.

A surface $f: L^2 \rightarrow \mathbb{S}^{n+2}$ is called r -isotropic if it is minimal and at any $x \in L^2$ and for any $1 \leq k \leq r$, the ellipses of curvature $\mathcal{E}_k^f(x)$ contained in all two-dimensional N_k^f 's are circles. We point out that there are alternative ways to define isotropy for surfaces, for instance, in terms of the vanishing of higher order Hopf differentials [27].

We consider the open and dense subset L_1 of L^2 defined by

$$L_1 = \{x \in L^2 : \dim N_1^f(x) \text{ is maximal}\}.$$

Then the 1st normal spaces along L_1 form a subbundle $N_1^f|_{L_1}$ of the normal bundle N_fL . Inductively, we define the open and dense subset

$$L_s = \{x \in L_{s-1} : \dim N_s^f(x) \text{ is maximal}\}$$

and similarly the s -th normal spaces along L_s form a subbundle $N_s^f|_{L_s}$ of the normal bundle. The following result was proved in [9] (see also [14, 26]).

Proposition 5.1. *If the surface f is r -isotropic, then each $L^2 \setminus L_s$, $1 \leq s \leq r$, consists of isolated points and the vector bundle $N_s^f|_{L_s}$ smoothly extends to a vector bundle over L^2 still denoted by N_s^f .*

5.2. A class of compact 3-dimensional Wintgen ideal submanifolds

In the sequel, let $f: L^2 \rightarrow \mathbb{S}^{2n+2}$, $n \geq 2$, be a substantial $(n - 1)$ -isotropic surface. Then Proposition 5.1 implies that each plane bundle $N_s^f|_{L^2}$, $1 \leq s \leq n - 1$, smoothly extends to a plane bundle over L^2 . Clearly, N_n^f is the orthogonal complement of $N_1^f \oplus \dots \oplus N_{n-1}^f$ in $N_f L$. There are plenty of compact $(n - 1)$ -isotropic surfaces in \mathbb{S}^{2n+2} , $n \geq 2$. This is the case of pseudoholomorphic curves in the nearly Kähler \mathbb{S}^6 . In fact, it is known that they are 1-isotropic [6, 17, 27]. Moreover, there are compact substantial pseudoholomorphic curves in the nearly Kähler \mathbb{S}^6 of positive genus (see [6]). Additionally, all 1-isotropic tori in \mathbb{S}^6 were described in [3].

The following result provides a method to produce new compact Wintgen ideal submanifolds.

Theorem 5.2. *Let $f: L^2 \rightarrow \mathbb{S}^{2n+2}$, $n \geq 2$, be a compact oriented substantial $(n - 1)$ -isotropic surface of genus $g(L)$. Let M^3 be the total space of the unit bundle $p: UN_n^f \rightarrow L^2$ of the plane bundle N_n^f . Then $\phi_f: M^3 \rightarrow \mathbb{S}^{2n+2}$ given by $\phi_f(x, w) = w$ is a minimal Wintgen ideal submanifold, whose first Betti number satisfies*

$$b_1(M^3) \geq 2g(L).$$

Proof. It follows from Proposition 4.2 in [13] that the map $\phi_f: M^3 \rightarrow \mathbb{S}^{2n+2}$ is a minimal immersion of rank two with polar surface f . Moreover, Proposition 8 in [12] implies that the first curvature ellipse (as defined in [12]) of ϕ_f is a circle at any point. It follows from Corollary 1.2 in [18] that a minimal n -dimensional submanifold of a space form with relative nullity $n - 2$ is a Wintgen ideal submanifold if and only if its first curvature ellipse is a circle at any point. Hence ϕ_f is a Wintgen ideal submanifold.

Now we argue that $b_1(M^3) \geq 2g(L)$. Clearly, the circle bundle $p: UN_n^f \rightarrow L^2$ is oriented (as defined in [4], p. 114). Then we have from Theorem 13.2 on p. 390 of [5] or [20], p. 438, that the cohomology rings of M^3 and L^2 are related by the following long exact sequence known as the Gysin sequence:

$$\dots \xrightarrow{\Psi_{i-1}} H^i(L; \mathbb{Z}) \xrightarrow{p_i^*} H^i(M; \mathbb{Z}) \xrightarrow{\Phi_i} H^{i-1}(L; \mathbb{Z}) \xrightarrow{\Psi_i} H^{i+1}(L; \mathbb{Z}) \xrightarrow{p_{i+1}^*} \dots$$

The exactness gives that p_1^* is a monomorphism and $\text{Im } p_1^* = \ker \Phi_1$. On the other hand, we have

$$\text{Im } p_1^* \cong H^1(L; \mathbb{Z}) / \ker p_1^* \cong H^1(L; \mathbb{Z}) \cong \mathbb{Z}^{2g(L)}.$$

Thus $\ker \Phi_1 \cong \mathbb{Z}^{2g(L)}$. The desired inequality follows from the fact that $H^1(M; \mathbb{Z})$ contains a subgroup isomorphic to $\mathbb{Z}^{2g(L)}$. ■

Theorem 5.2 shows that each compact oriented substantial $(n - 1)$ -isotropic surface $f: L^2 \rightarrow \mathbb{S}^{2n+2}$, $n \geq 2$, of positive genus $g(L)$ gives rise to a compact 3-dimensional Wintgen ideal submanifold with positive first Betti number. For instance, f can be chosen to be a compact substantial pseudoholomorphic curve in the nearly Kähler \mathbb{S}^6 of positive genus (see [6]), or a 1-isotropic torus in \mathbb{S}^6 as described in [3]. Clearly, these Wintgen ideal submanifolds violate the inequality in Theorem 1.1 for $k = n = 3$.

References

- [1] Amemiya, Y.: [On the convergence of the ordered roots of a sequence of determinantal equations](#). *Linear Algebra Appl.* **127** (1990), 531–542. Zbl [0696.15009](#) MR [1048819](#)
- [2] Audin, M. and Damian, M.: *Morse theory and Floer homology*. Universitext, Springer, London, 2014. Zbl [1281.57001](#) MR [3155456](#)
- [3] Bolton, J., Pedit, F. and Woodward, L.: [Minimal surfaces and the affine Toda field model](#). *J. Reine Angew. Math.* **459** (1995), 119–150. Zbl [0810.53048](#) MR [1319519](#)
- [4] Bott, R. and Tu, L. W.: *Differential forms in algebraic topology*. Grad. Texts in Math. 82, Springer, New York-Berlin, 1982. Zbl [0496.55001](#) MR [0658304](#)
- [5] Bredon, G. E.: *Topology and geometry*. Grad. Texts in Math. 139, Springer, New York, 1993. Zbl [0791.55001](#) MR [1224675](#)
- [6] Bryant, R. L.: [Submanifolds and special structures on the octonians](#). *J. Differential Geometry* **17** (1982), no. 2, 185–232. Zbl [0526.53055](#) MR [0664494](#)
- [7] Cheeger, J. and Ebin, D. G.: *Comparison theorems in Riemannian geometry*. North-Holland Mathematical Library 9, North-Holland, Amsterdam-Oxford; Elsevier, New York, 1975. Zbl [0309.53035](#) MR [0458335](#)
- [8] Chen, H. and Wei, G.: [Rigidity of minimal submanifolds in space forms](#). *J. Geom. Anal.* **31** (2021), no. 5, 4923–4933. Zbl [1466.53069](#) MR [4244891](#)
- [9] Chern, S.-S.: [On the minimal immersions of the two-sphere in a space of constant curvature](#). In *Problems in analysis (Sympos. in honor of Salomon Bochner, Princeton Univ., Princeton, NJ, 1969)*, pp. 27–40. Princeton Univ. Press, Princeton, NJ, 1970. Zbl [0217.47601](#) MR [0362151](#)
- [10] Chern, S.-S. and Lashof, R. K.: [On the total curvature of immersed manifolds](#). *Amer. J. Math.* **79** (1957), 306–318. Zbl [0078.13901](#) MR [0084811](#)
- [11] Chern, S.-S. and Lashof, R. K.: [On the total curvature of immersed manifolds. II](#). *Michigan Math. J.* **5** (1958), 5–12. Zbl [0095.35803](#) MR [0097834](#)
- [12] Dajczer, M. and Florit, L. A.: [A class of austere submanifolds](#). *Illinois J. Math.* **45** (2001), no. 3, 735–755. Zbl [0988.53004](#) MR [1879232](#)
- [13] Dajczer, M., Kasioumis, T., Savas-Halilaj, A. and Vlachos, T.: [Complete minimal submanifolds with nullity in Euclidean spheres](#). *Comment. Math. Helv.* **93** (2018), no. 3, 645–660. Zbl [1403.53048](#) MR [3854904](#)
- [14] Dajczer, M. and Vlachos, T.: [A class of complete minimal submanifolds and their associated families of genuine deformations](#). *Comm. Anal. Geom.* **26** (2018), no. 4, 699–721. Zbl [1397.53014](#) MR [3853925](#)
- [15] De Smet, P. J., Dillen, F., Verstraelen, L. and Vrancken, L.: [A pointwise inequality in submanifold theory](#). *Arch. Math. (Brno)* **35** (1999), no. 2, 115–128. Zbl [1054.53075](#) MR [1711669](#)
- [16] Eells, J., Jr. and Kuiper, N. H.: [Manifolds which are like projective planes](#). *Inst. Hautes Études Sci. Publ. Math.* (1962), no. 14, 5–46. Zbl [0109.15701](#) MR [0145544](#)
- [17] Eschenburg, J.-H. and Vlachos, T.: [Pseudoholomorphic curves in \$S^6\$ and \$S^5\$](#) . *Rev. Un. Mat. Argentina* **60** (2019), no. 2, 517–537. Zbl [1431.53062](#) MR [4049801](#)
- [18] Ge, J. and Tang, Z.: [A proof of the DDVV conjecture and its equality case](#). *Pacific J. Math.* **237** (2008), no. 1, 87–95. Zbl [1163.53038](#) MR [2415209](#)

- [19] Guadalupe, I. V. and Rodríguez, L.: [Normal curvature of surfaces in space forms](#). *Pacific J. Math.* **106** (1983), no. 1, 95–103. Zbl [0515.53044](#) MR [0694674](#)
- [20] Hatcher, A.: *Algebraic topology*. Cambridge University Press, Cambridge, 2002. Zbl [1044.55001](#) MR [1867354](#)
- [21] Kuiper, N. H.: [Minimal total absolute curvature for immersions](#). *Invent. Math.* **10** (1970), 209–238. Zbl [0195.51102](#) MR [0267597](#)
- [22] Lu, Z.: [Normal scalar curvature conjecture and its applications](#). *J. Funct. Anal.* **261** (2011), no. 5, 1284–1308. Zbl [1233.53008](#) MR [2807100](#)
- [23] Milnor, J.: *Morse theory*. Ann. of Math. Stud. 51, Princeton University Press, Princeton, NJ, 1963. Zbl [0108.10401](#) MR [0163331](#)
- [24] Shiohama, K. and Xu, H.: [Lower bound for \$L^{n/2}\$ curvature norm and its application](#). *J. Geom. Anal.* **7** (1997), no. 3, 377–386. Zbl [0960.53022](#) MR [1674797](#)
- [25] Shiohama, K. and Xu, H. W.: [Rigidity and sphere theorems for submanifolds. II](#). *Kyushu J. Math.* **54** (2000), no. 1, 103–109. Zbl [1005.53028](#) MR [1762795](#)
- [26] Vlachos, T.: [Minimal surfaces, Hopf differentials and the Ricci condition](#). *Manuscripta Math.* **126** (2008), no. 2, 201–230. Zbl [1165.53039](#) MR [2403186](#)
- [27] Vlachos, T.: [Exceptional minimal surfaces in spheres](#). *Manuscripta Math.* **150** (2016), no. 1-2, 73–98. Zbl [1408.53011](#) MR [3483170](#)
- [28] Wintgen, P.: [Sur l'inégalité de Chen–Willmore](#). *C. R. Acad. Sci. Paris Sér. A-B* **288** (1979), no. 21, A993–A995. Zbl [0421.53003](#) MR [0540375](#)
- [29] Wolfson, J.: [Manifolds with \$k\$ -positive Ricci curvature](#). In *Variational problems in differential geometry*, pp. 182–201. London Math. Soc. Lecture Note Ser. 394, Cambridge Univ. Press, Cambridge, 2012. Zbl [1246.53057](#) MR [2882775](#)
- [30] Xie, Z., Li, T., Ma, X. and Wang, C.: [Wintgen ideal submanifolds: new examples, frame sequence and Möbius homogeneous classification](#). *Adv. Math.* **381** (2021), article no. 107620, 31 pp. Zbl [1461.53050](#) MR [4206791](#)

Received July 31, 2024.

Christos-Raent Onti

Department of Mathematics and Statistics, University of Cyprus
1 Panepistimiou Avenue 2109 Aglantzia, 1678 Nicosia, Cyprus;
onti.christos-raent@ucy.ac.cy

Kleanthis Polymerakis

Department of Mathematics, University of Ioannina
University Campus - P.O. Box 1186, 45110 Ioannina, Greece;
kpolymerakis@uoi.gr, kleanthispol@gmail.com

Theodoros Vlachos

Department of Mathematics, University of Ioannina
University Campus - P.O. Box 1186, 45110 Ioannina, Greece;
tvlachos@uoi.gr