Rev. Mat. Iberoam. 42 (2026), no. 1, 1–29
DOI 10.4171/RMI/1594

The existence of solutions to a k-sum equation arising from conformal geometry

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Abstract. On compact manifolds (M, g), we derive the existence of metrics in a given conformal class $[g]$ with prescribed negative partial curvature. This curvature corresponds to a fully nonlinear equation derived from conformal geometry. For manifolds with boundary, we demonstrate the solvability of equations involving prescribed negative partial curvature within M, coupled with mean curvature along ∂M.

1. Introduction

On a given compact smooth manifold (M^n, g) where $n \geq 3$, we study the existence of the metric $g_u = e^{-2u} g \in [g]$ such that, for $1 \leq k \leq n-1$ and $0 \leq \tau \leq 1$,

$$\min_{1 \leq i_1 < \cdots < i_k \leq n} [\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}](g_u^{-1} A_{g_u}^{\tau}) = -kQ. \tag{1.1}$$

Here Q is a prescribed positive function, $\lambda(g_u^{-1} A_{g_u}^{\tau}) = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of $g_u^{-1} A_{g_u}^{\tau}$, Ric_{g_u} is the Ricci tensor of g_u, R_{g_u} is the scalar curvature of g_u, and

$$A_{g_u}^{\tau} = \frac{1}{n-2} \Big(\mathrm{Ric}_{g_u} - \frac{\tau R_{g_u}}{2(n-1)}\, g_u \Big).$$

The fully nonlinear operator

$$\mathcal{P}_k^-(X) = \sum_{i=1}^{k} \lambda_i(X) \tag{1.2}$$

has been studied extensively. Here, $\lambda_1(X) \leq \cdots \leq \lambda_n(X)$ are the eigenvalues of $X \in \mathcal{S}^n$, and $\mathcal{S}^n$ is the space of $n \times n$ real symmetric matrices. The operator $\mathcal{P}_k^-(X)$ can be viewed as a degenerate elliptic Bellman operator.

Mathematics Subject Classification 2020: 35J60 (primary); 53C21 (secondary).
Keywords: C^2 estimates, manifolds with boundary, Monge–Ampère type equation, Neumann boundary condition.

Arising from the partial positivity of the curvature on the manifolds, this type of operator (1.2) has emerged in geometric analysis. In the 1970s, the concept of k-pseudo-convexity was introduced to explore complex manifolds [20, 45]. In the 1980s, Sha [41] proposed the concept of k-convexity for ∂M and classified the topology of M. Wu in [46] investigated the similar concept of k-positivity for sectional curvature of M, resulting in further classification of the topology of M.

In [25–27], Harvey–Lawson studied the p-geometry and the corresponding equations. As in [27], for any integer $1 \le k \le n$, we denote

$$\mathcal{P}_k(\mathbb{R}^n) := \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \lambda_1(A) + \cdots + \lambda_k(A) \ge 0\},$$

where $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ are the eigenvalues of the matrix A. In [25], Harvey–Lawson studied the Dirichlet problem $\mathcal{P}_k^-(\nabla^2 u) = 0$ in bounded domains, and obtained uniqueness and existence of continuous solutions in the convex cone $\mathcal{P}_k(\mathbb{R}^n)$. In [2,3], Caffarelli–Li–Nirenberg explored the maximum principle and the singularities of the solutions to $\mathcal{P}_k^-(\nabla^2 u) = 0$. Oberman–Silvestre in [40] studied the interior $C^{1,\alpha}$ regularity of solutions to $\mathcal{P}_1^-(D^2 u) = 0$ in Ω, subject to the boundary condition $u(x) = g(x)$ on $\partial\Omega$, where $g(x) \in C^{1,\alpha}$. In [1], Birindelli–Galise–Ishii studied a more general class of fully nonlinear degenerate elliptic equations and established Lipschitz regularity under the assumptions of convexity for the domain. Additional relevant works and references can be found in [15–17] and the references therein.

The fully nonlinear equation has played an essential role in conformal geometry. See the milestone work [5] by Chang–Gursky–Yang. For $1 \le p \le n-1$, Guan–Wang in [22] investigated

$$W_p(g_u) := (n-p)\sum_{i \le p} \lambda_i(g_u^{-1} A^1_{g_u}) + p \sum_{i > p} \lambda_i(g_u^{-1} A^1_{g_u}) = -1, \tag{1.3}$$

and $W_1(g_u) = \min \mathrm{Ric}_{g_u}$, where $\lambda_1 \le \cdots \le \lambda_n$ are the eigenvalues of $g_u^{-1} A^1_{g_u}$. Gursky in [23] discussed the smallest eigenvalue of the Schouten tensor and proved the existence of the solution u to $\min_{1 \le i \le n} \lambda_i(g_u^{-1} A^1_{g_u}) = -1/2$ in closed manifolds.

Motivated by p-geometry and conformal geometry, we want to explore equation (1.1) in two aspects: the first consists of studying the equation on closed manifolds, and the second is to obtain the existence of solutions to the equation with prescribed mean curvature on compact manifolds.

On $(M, g, \partial M)$, sometimes $\partial M = \varnothing$, for $g_u = e^{-2u} g$ and $0 \le \tau \le 1$, we denote

$$A^\tau_{g_u} = \nabla^2 u + \frac{1-\tau}{n-2} \Delta u g + \nabla u \otimes \nabla u - (2-\tau)\frac{|\nabla u|^2}{2} g + A^\tau_g,$$

and

$$L_{g_u} e^u = \frac{\partial u}{\partial \vec{n}} g + L_g,$$

where $\vec{n}$ is the unit inner normal vector on ∂M and L_g is the second fundamental form of boundary. We have that

$$h_g = \frac{1}{n-1} \mathrm{Tr}_g L_g$$

is the mean curvature of ∂M with respect to g.

The equation (1.1) is fully nonlinear degenerate elliptic for $\tau = 1$. Although the regularity theory for this equation is not yet fully resolved, the method of approximation introduced in [23] enables us to establish the following theorem, by introducing a Monge–Ampère type equation from [4] which has attracted growing attention and will be introduced in Section 2.

Theorem 1.1. *Given a smooth positive function Q on a closed manifold (M, g), suppose that the scalar curvature $R_g < 0$ and that $g^{-1}(A_g^\tau + Qg) \in \mathcal{P}_k(\mathbb{R}^n)$. There exists a conformal factor $u \in C^{1,1}(M)$ such that*

$$\min_{1\le i_1<\cdots<i_k\le n}[\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}](g_u^{-1} A_{g_u}^\tau) = -kQ$$

for any $0 \le \tau \le 1$ and $1 \le k \le n-1$.

From Lohkamp's classical theorem in [38], we know that every compact manifold has a metric g with negative Ricci curvature (with totally geodesic boundary if boundary exists). With proper rescaling, for any positive function Q, there always exists a metric satisfying the assumption of this theorem. We point out that for the equation related with $A_{g_u}^\tau$ for $0 \le \tau < 1$, equation (1.1) is uniform elliptic and the solution u is smooth. As equation (1.1) is uniformly elliptic for $0 \le \tau < 1$, from the gradient estimates and C^0 estimates, $C^{2,\alpha}$ estimates hold. For $\tau = 1$, instead, the equation is partial degenerate and in Euclidean space; it seems that no interior $C^{1,\alpha}$ estimates for equation (1.1) are known without boundary condition.

For compact manifolds with boundary, analogous results can be established for equations involving prescribed mean curvature, which is equivalent to the Neumann boundary condition. We state the boundary case in the following theorem.

Theorem 1.2. *Given a smooth positive function Q on a smooth manifold $(M, g, \partial M)$, suppose that the scalar curvature $R_g < 0$ on M, and also that $h_g \le 0$ on ∂M and $g^{-1}(A_g^\tau + Qg) \in \mathcal{P}_k(\mathbb{R}^n)$. Then for any smooth function $c(x) \le 0$ and $0 \le \tau < 1$, there exists a smooth conformal factor u such that*

$$\begin{cases} \min\limits_{1\le i_1<\cdots<i_k\le n}[\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}](g_u^{-1} A_{g_u}^\tau) = -kQ(x) & \text{in } M, \\ h_{g_u} = c(x) & \text{on } \partial M. \end{cases} \tag{1.4}$$

For $\tau = 1$, there may exist a Lipschitz solution to (1.4) with a uniform C^1 estimate for u with respect to τ. The prescribed mean curvature of the boundary appears both in the context of the classical Yamabe problem and the σ_k-Yamabe problem. For the classical Yamabe problem with boundary, significant contributions can be found in the seminal works of Escobar [13, 14]. Meanwhile, the σ_k-Yamabe problem with boundary has been a subject of study over the past two decades, with references including [8–11, 28–34]. Most of references focused on locally conformally flat manifolds with umbilic boundary. In [10, 11], the authors investigated the σ_2-Yamabe equation on manifolds with boundary in a so-called Γ_2^+ cone, in which the metric has Yamabe constant $Y(M, \partial M, [g]) > 0$. Here, Theorem 1.2 discusses the metric with $Y(M, \partial M, [g]) < 0$ on manifolds with boundary. We can relax the assumption of R_g further to $R_g \le 0$, as stated in Theorem 1.3. Differing from the case $R_g < 0$, we cannot have C^0 estimates directly. Inspired by Gursky [23], we first obtain the C^1 and C^2 estimates depending on $\inf_M u$, and then we get $\sup_M u \le C$ for $R_g \le 0$ under the assumption that g is not Ricci-flat.

Theorem 1.3. *Given a smooth positive function Q on a manifold (M, g), suppose that the scalar curvature $R_g \le 0$, that $g^{-1}(A_g^\tau + Qg) \in \mathcal{P}_k(\mathbb{R}^n)$, and that (M, g) is not Ricci-flat. Then the following hold:*

(1) *For $0 \le \tau \le 1$, there exists a $C^{1,1}$ solution u to*

$$\min_{1\le i_1<\cdots<i_k\le n}[\lambda_{i_1}+\lambda_{i_2}+\cdots+\lambda_{i_k}](g_u^{-1}A_{g_u}^\tau) = -kQ. \tag{1.5}$$

(2) *Assume that M has boundary ∂M with $h_g \le 0$. For $0 \le \tau < 1$, given any smooth non-positive function $c(x)$ on ∂M, there exists a smooth solution u to*

$$\begin{cases} \min\limits_{1\le i_1<\cdots<i_k\le n}[\lambda_{i_1}+\lambda_{i_2}+\cdots+\lambda_{i_k}](g_u^{-1}A_{g_u}^\tau) = -kQ & \text{in } M, \\ h_{g_u} = c(x) & \text{on } \partial M. \end{cases} \tag{1.6}$$

The proof of Theorem 1.2 and Theorem 1.3 involves the establishment of a Monge–Ampère type equation with Neumann boundary condition on Riemannnian manifolds, which has its own interest. See Theorem 5.2 in Section 5. In the context of Neumann boundary problems for fully nonlinear equations, substantial progress has been made, inspired in part by Lions–Trudinger–Urbas [37]. See Ma–Qiu [39] for recent developments on Neumann boundary problems. Previous studies concerning fully nonlinear equations with Neumann boundary predominantly focused on Euclidean space and manifolds with umbilic boundary. Typically, the solvablity of the fully nonlinear equations with boundary relies on the convexity of the boundary. We extend the methods to manifolds with some delicate computations involving boundary curvature.

The paper is structured as follows. In Section 2, we introduce a particular type of Monge–Ampère equation, which will be subsequently explored in Sections 3 and 4. At the end of Section 2, we explain the strategy for the proof of the main theorem. Section 3 is devoted to providing a priori estimates C^0, C^1 by constructing some auxiliary functions. In Section 4, utilizing the method of Lions–Trudinger–Urbas [37] in Euclidean space, we investigate a priori C^2 global estimates on manifolds, including the second tangential derivatives and the second normal derivatives on boundary. Meanwhile, we adopt the auxiliary function from [7, 21] to obtain the global C^1, C^2 estimates on closed manifold. In Section 5, we employ the method of continuity and approximation to demonstrate the main theorem.

2. Preliminaries

In local coordinates, consider the $(1,1)$-tensor $A = A_j^i dx^i \otimes \partial/\partial x^j$. Additionally, for each $1 \le \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_k \le n$ and $1 \le \beta_1 < \beta_2 < \cdots < \beta_k \le n$, we introduce

$$W_{\beta_1\cdots\beta_k}^{\alpha_1\cdots\alpha_k} = \sum_{i=1}^k\sum_{j=1}^n A_j^{\alpha_i}\delta\begin{pmatrix} \alpha_1\cdots & \alpha_{i-1} & j & \alpha_{i+1} & \cdots\alpha_k \\ \beta_1\cdots & \beta_{i-1} & \beta_i & \beta_{i+1} & \cdots\beta_k \end{pmatrix}. \tag{2.1}$$

In local coordinates, define

$$W = W^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k} dx^{\beta_1} \otimes \cdots \otimes dx^{\beta_k} \otimes \frac{\partial}{\partial x^{\alpha_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\alpha_k}}.$$

In particular,

$$W^{\alpha_1\cdots\alpha_k}_{\alpha_1\cdots\alpha_k} = \sum_{i=1}^{k} A^{\alpha_i}_{\alpha_i},$$

and if $\{\alpha_1, \ldots, \alpha_k\}$ and $\{\beta_1, \ldots, \beta_k\}$ differ in two elements, then

$$W^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_k} = 0.$$

If $\{\alpha_1, \ldots, \alpha_k\}\backslash\{\alpha_j\} = \{\beta_1, \ldots, \beta_k\}\backslash\{\beta_i\}$ and $\alpha_j \neq \beta_i$, then

$$W^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k} = (-1)^{i-j} A^{\alpha_j}_{\beta_i}. \tag{2.2}$$

If $A^{\beta}_{\alpha} = \delta^{\beta}_{\alpha} A^{\alpha}_{\alpha}$, then

$$W^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k} = \sum_{i=1}^{k} A^{\alpha_i}_{\alpha_i} \delta\begin{pmatrix} \alpha_1\cdots & \alpha_{i-1} & \alpha_i & \alpha_{i+1} & \cdots\alpha_k \\ \beta_1\cdots & \beta_{i-1} & \beta_i & \beta_{i+1} & \cdots\beta_k \end{pmatrix}.$$

We have C^k_n choices of $(\alpha_1, \ldots, \alpha_k)$, where $\alpha_1 < \cdots < \alpha_k$, and use N_α to denote the order number of $(\alpha_1, \ldots, \alpha_k)$. Thus, we denote $W^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k} := W^{N_\alpha}_{N_\beta}$. For example, we denote $\alpha = (1, 2, \ldots, k)$ as N_1 and $W^{(1,2,\ldots,k)}_{(1,2,\ldots,k)} = W^{N_1}_{N_1}$.

We have C^k_n of k-sums of eigenvalues of $\lambda_i(A)$. For example, for $n = 4$ and $k = 2$, we have $\lambda_1+\lambda_2, \lambda_1+\lambda_3, \lambda_1+\lambda_4, \lambda_2+\lambda_3, \lambda_2+\lambda_4$ and $\lambda_3+\lambda_4$, and we denote them as $\mu_1, \ldots, \mu_6$. In general, we denote the k-sums of eigenvalues of $\lambda_i(A)$ by $\mu_1, \ldots, \mu_{C^k_n}$, which are also the eigenvalues of W. It is easy to check that

$$\det W = \det \mu = \frac{1}{C^k_n!} \sum_{\substack{i_1,\ldots,i_{C^k_n}=1 \\ j_1,\ldots,j_{C^k_n}=1}}^{C^k_n} \delta\begin{pmatrix} i_1 & \cdots & i_{C^k_n} \\ j_1 & \cdots & j_{C^k_n} \end{pmatrix} W^{j_1}_{i_1} \cdots W^{j_{C^k_n}}_{i_{C^k_n}}. \tag{2.3}$$

The operator (2.3) was originally introduced by Caffarelli–Nirenberg–Spruck in [4]. The Monge–Ampère type equation (2.3) for $k = n-1$ emerges naturally in the "form-type" Calabi–Yau equation, as discussed by Fu–Wang–Wu in [18] and Tosatti–Weinkove in [43]. Equation (2.3) for $k = n-1$ is also important in the investigation of the Gauduchon conjecture, explored further by Szélyhidi–Tosatti–Weinkove in [42] and Tosatti–Weinkove in [44]. For further insights into the generalized operator, we refer to [6, 12, 19, 24].

In the following paragraphs, we take A as $A^{\tau}_{g_u} + Q(x)e^{-2u}g$ in (2.1). Let λ be the eigenvalue of $A^{\tau}_{g_u} + Q(x)e^{-2u}g$ with respect to g and let μ be the corresponding k-sum of the eigenvalue of λ, which is also the eigenvalue of $\{W_{N_\alpha N_\beta}\}_{C^k_n \times C^k_n}$. We denote

$$\begin{aligned} f(\lambda) = G(\mu) = (\det(\mu))^{1/C^k_n} &= \Big(\prod_{j=1}^{C^k_n} \mu_j\Big)^{1/C^k_n} \\ &=: f(\lambda(g^{-1}(A^{\tau}_{g_u} + Q(x)g_u)) =: F(g^{-1}(A^{\tau}_{g_u} + Q(x)g_u)). \end{aligned} \tag{2.4}$$

Below we summarize some basic properties of this operator.

Lemma 2.1. *For* $\mu \in \Gamma^+_{C_n^k} = \{(\mu_1, \ldots, \mu_{C_n^k}), \mu_i > 0$ *for* $i = 1, \ldots, C_n^k\}$,

$$\frac{\partial G}{\partial \mu_j} > 0, \quad \textit{for } j = 1, \ldots, C_n^k,$$

$$\frac{\partial f}{\partial \lambda_i} = \sum_{j=1}^{C_n^k} \frac{\partial G}{\partial \mu_j} \frac{\partial \mu_j}{\partial \lambda_i} > 0, \quad \textit{for } i = 1, \ldots, n,$$

$$\sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i} = \sum_{i=1}^{n} \sum_{j=1}^{C_n^k} \frac{\partial G}{\partial \mu_j} \frac{\partial \mu_j}{\partial \lambda_i} = k \sum_{j=1}^{C_n^k} \frac{\partial G}{\partial \mu_j} \geq k, \tag{2.5}$$

and

$$\sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i} \lambda_i = \sum_{j=1}^{C_n^k} \frac{\partial G}{\partial \mu_j} \sum_{i=1}^{n} \frac{\partial \mu_j}{\partial \lambda_i} \lambda_i = \sum_{j=1}^{C_n^k} \frac{\partial G}{\partial \mu_j} \mu_j = G. \tag{2.6}$$

Also,

$$\left\{ \frac{\partial^2 G}{\partial \mu_i \partial \mu_j} \right\}_{C_n^k \times C_n^k} \leq 0$$

and

$$\left\{ \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \right\}_{n \times n} = \left\{ \sum_{p,q=1}^{C_n^k} \frac{\partial^2 G}{\partial \mu_p \partial \mu_q} \frac{\partial \mu_p}{\partial \lambda_i} \frac{\partial \mu_q}{\partial \lambda_j} \right\}_{n \times n} \leq 0, \tag{2.7}$$

where $f(\lambda)$ *and* $G(\mu)$ *are defined in* (2.4).

To ensure the ellipticity of the operator, we always assume that

$$g^{-1}(A^\tau_{g_u} + Q(x) g_u) \in \mathcal{P}_k(\mathbb{R}^n).$$

For simplicity, we call u (k, n, Q)-*convex* if $g^{-1}(A^\tau_{g_u} + Q(x) g_u) \in \mathcal{P}_k(\mathbb{R}^n)$. When u is (k, n, Q)-*convex*, we have $\mu \in \Gamma^+_{C_n^k}$ and then

$$\mathrm{Tr}\, W = \mathrm{Tr}\, \mu = \sum_{j=1}^{C_n^k} \mu_j = C_n^k \frac{k}{n} \sum_{i=1}^{n} \lambda_i > 0,$$

and this yields

$$\begin{aligned} &\mathrm{Tr}_g (A^\tau_{g_u} + Q(x) g_u) \\ &\quad = \frac{2n - 2 - n\tau}{n-2} \left(\Delta u - \frac{n-2}{2} |\nabla u|^2 \right) + \mathrm{Tr}_g A^\tau_g + n Q e^{-2u} \geq 0. \end{aligned} \tag{2.8}$$

Lemma 2.2. *Suppose u is (k,n,Q)-convex. For any $1 \le i, j \le n$, it holds that*

$$|\nabla^2 u|_g \le C_1(\Delta u + |\nabla u|^2) + C_2(1 + nQ\,e^{-2u}), \tag{2.9}$$

where C_1 and C_2 are positive constants depending on k and n.

Proof. For any fixed x_0, for simplicity, we assume that the metric g is geodesic normal at x_0. We write A for $A^{\tau}_{g_u} + Q(x)g_u$. At x_0, with $\{\alpha_1,\dots,\alpha_k\}\backslash\{\alpha_j\} = \{\beta_1,\dots,\beta_k\}\backslash\{\beta_i\}$ and $\alpha_j \neq \beta_i$, we have

$$(A_{\alpha_j\beta_i})^2 = W^2_{N_\alpha N_\beta} \le W_{N_\alpha N_\alpha}\, W_{N_\beta N_\beta} \le (\operatorname{Tr} W)^2, \tag{2.10}$$

where the last inequality holds because $\mu_i \ge 0$ for any $1 \le i \le C_n^k$.

To get the estimate for A_{11}, taking all the choices of $\gamma_i = (1, \gamma_{i,2},\dots,\gamma_{i,k})$, where $(\gamma_{i,2},\dots,\gamma_{i,k})$ is the ordered number of $2 \le \gamma_{i,2} < \cdots < \gamma_{i,k} \le n$, it holds that

$$W_{N_{\gamma_i}N_{\gamma_i}} = A_{11} + A_{\gamma_{i,2}\gamma_{i,2}} + \cdots + A_{\gamma_{i,k}\gamma_{i,k}} \ge 0. \tag{2.11}$$

Summing all the choices of γ_i above, we have

$$C_{n-1}^{k-1}\Big(1 - \frac{k-1}{n-1}\Big)A_{11} + \frac{k-1}{n-1}\,C_{n-1}^{k-1}\operatorname{Tr}_g A = \sum_{\gamma_i} W_{N_{\gamma_i}N_{\gamma_i}},$$

and then,

$$\begin{aligned}
-\frac{k-1}{n-1}\,C_{n-1}^{k-1}\operatorname{Tr}_g A \le C_{n-1}^{k-1}\Big(1 - \frac{k-1}{n-1}\Big)A_{11} &= \sum_{\gamma_i} W_{N_{\gamma_i}N_{\gamma_i}} - \frac{k-1}{n-1}\,C_{n-1}^{k-1}\operatorname{Tr}_g A \\
&\le \operatorname{Tr}_g W - \frac{k-1}{n-1}\,C_{n-1}^{k-1}\operatorname{Tr}_g A \le \Big(C_{n-1}^{k-1}\frac{k}{n} - \frac{k-1}{n-1}\,C_{n-1}^{k-1}\Big)\operatorname{Tr}_g A.
\end{aligned}$$

Thus we get

$$|u_{11}| \le C_1(\Delta u + |\nabla u|^2) + C_2(1 + nQ\,e^{-2u}), \tag{2.12}$$

and for any $i = 2,\dots,n$, the same argument holds for u_{ii}. By (2.10) and (2.12), we have proved (2.9). ■

The strategy of the proof of Theorems 1.1 and 1.2 is inspired by [23]. We explain it now. We first show the existence of the solution u_i to the following equation:

$$F(g^{-1}(A^{\tau}_{g_u} + Q(x)g_u)) = \frac{1}{i}\,e^{-2u} \quad \text{in } M. \tag{2.13}$$

By uniform a priori estimates for (2.13) with respect to i, we have a sequence of solutions u_i converging to u_∞ such that

$$F(g^{-1}(A^{\tau}_{g_{u_\infty}} + Q(x)g_{u_\infty})) = 0 \quad \text{in } M.$$

Therefore,

$$\min_{1\le l\le C_n^k} \mu_l = 0.$$

In lieu of equation (2.13) by the following equation (2.14):

$$\begin{cases} f(\lambda) = \dfrac{1}{i}\, e^{-2u} & \text{in } M, \\ \dfrac{\partial u}{\partial \vec{n}_g} = c(x) e^{-u} & \text{on } \partial M, \end{cases} \tag{2.14}$$

we have a similar argument for Theorem 1.2.

To prove the existence of solutions to equation (2.13), we use the method of continuity and consider the following equation for $0 \le t \le 1$:

$$F(g^{-1}(A^{\tau}_{g_u} + Q(x) g_u)) = tKe^{-2u} + (1-t)F(g^{-1}(A^{\tau}_{g} + Q(x) g)), \tag{2.15}$$

where K is a constant such that $K < k \min_M Q$.

For the boundary case, we consider the following equation:

$$\begin{cases} F(g^{-1}(A^{\tau}_{g_u} + Q(x) g_u)) = \phi(x,u) & \text{in } M, \\ \dfrac{\partial u}{\partial \vec{n}_g} + h_g = \varphi(x,u) + h_g & \text{on } \partial M, \end{cases} \tag{2.16}$$

where

$$\phi(x,u) := tKe^{-2u} + (1-t)F(g^{-1}(A^{\tau}_{g} + Q(x) g)) \quad \text{in } M,$$

and

$$\varphi(x,u) := (tc(x) + (1-t)h_g)e^{-u} - h_g \quad \text{on } \partial M.$$

Let

$$\mathbb{I} = \{t \in [0,1] : (2.15) \text{ has a } (k,n,Q)\text{-}convex \text{ solution } u\},$$

and

$$\mathbb{I}_b = \{t \in [0,1] : (2.16) \text{ has a } (k,n,Q)\text{-}convex \text{ solution } u\}.$$

Denote

$$F^{ij} = \frac{\partial F}{\partial r_{ij}} \quad \text{and} \quad F^{ij,kl} = \frac{\partial^2 F}{\partial r_{ij} \partial r_{kl}},$$

where

$$r := A^{\tau}_{g_u} + Q(x) g_u.$$

We first give the linearization of

$$F[u] := F(g^{-1}(A^{\tau}_{g_u} + Q(x) g_u)) - tKe^{-2u} - (1-t)F(A^{\tau}_{g} + Q(x) g)$$

as follows:

$$\begin{aligned} L_F[\varphi] &:= \frac{d}{ds}\Big|_{s=0} F[u + s\varphi] \\ &= \frac{\partial F}{\partial r_{ij}}\Big(\nabla_{ij}\varphi + \frac{1-\tau}{n-2}\Delta\varphi g_{ij} + u_i\varphi_j + u_j\varphi_i - (2-\tau)\langle\nabla u, \nabla\varphi\rangle g_{ij} \\ &\qquad - 2\varphi Q(x) e^{-2u} g_{ij}\Big) + 2tK\varphi e^{-2u} \\ &= a^{ij}\varphi_{ij} + b^i\varphi_i - (2c_0(x)Q - 2tK)\varphi e^{-2u} = a^{ij}\varphi_{ij} + b^i\varphi_i - l(x)\varphi, \end{aligned}$$

where

$$
\begin{aligned}
a^{ij} &:= \frac{\partial F}{\partial r_{ij}} + \sum_{p,q=1}^{n} \frac{\partial F}{\partial r_{pq}} g_{pq} \frac{1-\tau}{n-2} g^{ij} = F^{ij} + \frac{1-\tau}{n-2} g^{ij} \sum_{p=1}^{n} F_p^p,\\
b^i &:= 2\frac{\partial F}{\partial r_{ij}} u_j - (2-\tau) \sum_{p,q=1}^{n} \frac{\partial F}{\partial r_{pq}} g_{pq} u^i = 2F^{ij} u_j - (2-\tau) \sum_{p=1}^{n} F_p^p u^i,\\
c_0(x) &:= \sum_{p,q=1}^{n} \frac{\partial F}{\partial r_{pq}} g_{pq} = \sum_{p=1}^{n} F_p^p \geq k,
\end{aligned}
$$

and

$$
l(x) := (2c_0(x)Q - 2tK)e^{-2u}.
$$

Here $\{a^{ij}\}_{n\times n}$ is positive definite and $l(x)$ is positive. The openness of $\mathbb{I}$ (or $\mathbb{I}_b$) holds.

3. C^0 and C^1 estimates

In Section 3.1, we prove C^0-estimate for equations (2.15) and (2.16) under the assumption $R_g < 0$. In particular, we obtain a uniform lower bound of u with respect to $\sup_M R_g$. In Section 3.2, we establish the local C^1 estimate for equation (2.16) for $0 \leq \tau < 1$. We defer the C^2 estimate until Section 4.

3.1. The C^0 estimate for the solution to equations (2.15) and (2.16)

First, we obtain the upper C^0 estimate, and then the lower C^0 estimate.

Lemma 3.1. *Assume that (M, g) is a compact manifold (or with boundary ∂M) with $R_g < 0$ in M (or $h_g \leq 0$ on ∂M). Let u be a (k, n, Q)-convex solution to (2.15) (or to (2.16)) and let $c(x) \leq 0$ on ∂M. For $0 \leq \tau \leq 2 - 2/n$, we have*

$$
\max_M u \leq C,
$$

where C depends on $\sup_M R_g$ and other data.

Proof. We just consider the boundary case. Let ψ be a fixed solution to $\Delta\psi = -|\partial M|/|M|$ in M with boundary condition $\partial\psi/\partial\vec{n}_g|_{\partial M} = 1$ on ∂M. Let $v = u + B\psi$, where

$$
B = -\frac{n-2}{2(2n-2-n\tau)} \operatorname{Tr} A_g^\tau \frac{|M|}{|\partial M|}.
$$

Assume that $\max_M v = u(x_0) + B\psi(x_0)$. Then

$$
u(x) + B\psi \leq u(x_0) + B\psi(x_0) \quad \text{and} \quad u(x) \leq u(x_0) + B \cdot \operatorname{osc}_M \psi.
$$

If $x_0 \in \partial M$, then

$$
0 \geq \frac{\partial(u + B\psi)}{\partial \vec{n}_g}(x_0) = \frac{\partial u}{\partial \vec{n}_g} + B = (tc(x) + (1-t)h_g)e^{-u} - h_g + B.
$$

Now,

$$-B + h_g(x_0) \geq (tc(x_0) + (1-t)h_g)e^{-u(x_0)} \geq -Ce^{-u(x_0)}$$

for some positive constant C depending on h_g and c, and then

$$e^{-u(x_0)} \geq \frac{B - h_g(x_0)}{C} \geq \frac{B}{C}$$

for $h_g \leq 0$. It follows that $u(x_0) \leq -\ln(B/C)$. Now, if $x_0 \in \mathring{M}$, then $\nabla v(x_0) = 0$ and $\nabla^2 v(x_0) \leq 0$.

As u is (k, n, Q)-convex, we have

$$\begin{aligned}
0 &\leq \frac{n}{kC_n^k}\operatorname{Tr}\mu = \sum_{i=1}^{n}\lambda_i \\
&= -\frac{(2-\tau)n-2}{2}|\nabla u|^2 + \frac{2n-2-n\tau}{n-2}\Delta u + \operatorname{Tr}A_g^\tau + nQ\,e^{-2u} \\
&\leq -\frac{(2-\tau)n-2}{2}|\nabla(B\psi)|^2 + \frac{2n-2-n\tau}{n-2}\Delta(-B\psi) + \operatorname{Tr}A_g^\tau + nQ\,e^{-2u},
\end{aligned}$$

and then,

$$\begin{aligned}
nQ\,e^{-2u}|_{x_0} &\geq \frac{(2-\tau)n-2}{2}|\nabla(B\psi)|^2 + \frac{2n-2-n\tau}{n-2}\Delta(B\psi) - \operatorname{Tr}A_g^\tau \\
&\geq -B\frac{2n-2-n\tau}{n-2}\frac{|\partial M|}{|M|} - \operatorname{Tr}A_g^\tau > -\frac{1}{2}\operatorname{Tr}A_g^\tau > 0.
\end{aligned}$$

Therefore, $\max_M u \leq C$. ∎

Lemma 3.2. *Assume that (M, g) is a compact manifold (or with boundary ∂M, in which case we assume that $h_g \leq 0$). Let u be a (k, n, Q)-convex solution to (2.15) (or to (2.16) with $c(x) \leq 0$ on ∂M). For $0 \leq \tau \leq 1$, we have*

$$u(x) \geq \frac{1}{2}\ln(\min_M kQ(x) - K) - \frac{1}{2}\ln\big((a_0 + a)k + \max_M F(g^{-1}(A_g^\tau + Q(x)g))\big),$$

where a is a positive constant depending on g, n, k, τ and $a_0 = -\min_{x\in\partial M}\lambda_{min}[A_g^\tau(x)]$.

Proof. Taking

$$D = \sup_{\partial M}|h_g| + 1,$$

we define $w = u - D\psi$, where ψ is a fixed solution to $\Delta\psi = -|\partial M|/|M|$ in M with boundary condition $\partial\psi/\partial\vec{n}|_{\partial M} = 1$ on ∂M.

We get

$$\frac{\partial w}{\partial\vec{n}_g} = \frac{\partial u}{\partial\vec{n}_g} - D = (tc(x) + (1-t)h_g)e^{-u} - h_g - D < 0 \quad \text{on } \partial M,$$

and $\min_M w$ is achieved at an interior point x_0 of M. Now at x_0,

$$\nabla w(x_0) = 0 \quad \text{and} \quad \nabla^2 w(x_0) \geq 0.$$

We have

$$A^{\tau}_{g_u}(x_0)+Q(x_0)e^{-2u(x_0)}g \tag{3.1}$$
$$\geq A^{\tau}_{g}(x_0)+Q(x_0)e^{-2u(x_0)}g+\nabla^2(D\psi)(x_0)+\frac{1-\tau}{n-2}\Delta(D\psi)(x_0)g$$
$$+\nabla(D\psi)(x_0)\otimes\nabla(D\psi)(x_0)-(2-\tau)\frac{|\nabla(D\psi)|^2}{2}g(x_0)$$
$$\geq A^{\tau}_{g}(x_0)+Q(x_0)e^{-2u(x_0)}g-ag\geq Q(x_0)e^{-2u(x_0)}g-(a+a_0)g,$$

where a is a positive constant depending on D and $|\psi|_{C^2(M)}$.

If

$$Q(x_0)e^{-2u(x_0)}g-(a+a_0)g\leq 0,$$

then

$$Q(x_0)e^{-2u(x_0)}\leq a+a_0,$$

and

$$\min_M u=u(x_0)\geq\frac{1}{2}\ln Q(x_0)-\frac{1}{2}\ln(a_0+a)\geq\frac{1}{2}\ln(kQ(x_0))-\frac{1}{2}\ln k-\frac{1}{2}\ln(a_0+a)$$
$$\geq\frac{1}{2}\ln(k\min_M Q-K)-\frac{1}{2}\ln k-\frac{1}{2}\ln(a_0+a).$$

Also, if

$$Q(x_0)e^{-2u(x_0)}g-(a+a_0)g>0,$$

then by (3.1),

$$tKe^{-2u(x_0)}+(1-t)F(g^{-1}(A^{\tau}_g+Q(x)g))(x_0)$$
$$\geq f(\lambda[(Q(x_0)e^{-2u(x_0)}g-(a+a_0))g(x_0)])$$
$$=f(\lambda[g])(x_0)(Q(x_0)e^{-2u(x_0)}-a-a_0)=k(Q(x_0)e^{-2u(x_0)}-a-a_0),$$

and

$$u(x_0)\geq\frac{1}{2}\ln(kQ(x_0)-tK)-\frac{1}{2}\ln\big((a_0+a)k+(1-t)F(g^{-1}(A^{\tau}_g+Q(x)g))(x_0)\big).$$

∎

The lower bound of u is independent of $R_g\leq 0$, and plays an essential role in the proof of Theorem 1.3.

3.2. The a priori C^1 estimates for equation (2.16)

Theorem 3.3. *Let u be a (k,n,Q)-convex solution to equation (2.16). For $\tau<1$ and any $\mathcal{O}\subset\mathcal{O}_1\subset\overline{M}$, assume that there exists a positive constant C_1 such that $u\geq -C_1$ in $\mathcal{O}_1$. Then*

$$|\nabla u|_{C^0(\mathcal{O})}\leq C,$$

where C depends on K, g, τ, n, k, $|Q|_{C^2(\mathcal{O}_1)}$, $d_g(\partial\mathcal{O}_1,\partial\mathcal{O})^{-1}$ and C_1.

Proof. We only need to prove the theorem near the boundary. Let $\mathcal{O}_1$ be the domain near ∂M such that $3/2 > 1 - \varphi_z(u)d > 1/2$ and

$$G = \rho|\nabla(u - \varphi d)|^2 e^{Bd},$$

where $d = \mathrm{dist}_g(x, \partial M)$, ρ is a cut-off function in $\mathcal{O}_1$ such that

$$\rho = 1 \text{ on } \mathcal{O}, \quad |\nabla^2\rho| \le C, \quad |\nabla\rho| \le C\rho^{1/2} \quad \text{and} \quad \partial\rho/\partial\vec{n}_g = 0 \text{ on } \partial M \cap \mathcal{O}.$$

Here

$$B = 4 \sup_{\mathcal{O}_1 \cap \partial M} (|L_g|_g + |\varphi_x| + |\varphi_z|) + 1$$

is chosen to exclude the maximum point of G on the boundary. We sometimes denote $w = u - \varphi d$ for simplicity. We naturally extend $c(x)$ and h_g to the interior of M near the boundary, and φ is well defined near the boundary.

Assume that $\max_{\mathcal{O}_1} G = G(x_0)$. Without loss of generality, we assume that $G(x_0)$ is large and

$$\frac{1}{2}|\nabla u|^2(x_0) \le |\nabla(u - \varphi d)|^2(x_0) \le 2|\nabla u|^2(x_0);$$

otherwise, the theorem is proved.

Case 1. $x_0 \in \partial M$. Then, as $u_n - (\varphi d)_n = 0$ on ∂M, we obtain

$$\begin{aligned} 0 \ge \frac{\partial \log G}{\partial \vec{n}_g}(x_0) &= B + \frac{2w_k(u_{kn} - (\varphi d)_{kn})}{|\nabla(u - \varphi d)|^2} = B + \frac{2w_\alpha(\overline{\nabla}_\alpha(u_n) - L_{\alpha\beta}u_\beta - \varphi_\alpha)}{|\nabla(u - \varphi d)|^2} \\ &\ge B - \frac{2w_\alpha L_{\alpha\beta}u_\beta}{|\nabla(u - \varphi d)|^2} \ge B - 4 \sup_{\mathcal{O}_1 \cap \partial M} (|L_g| + |\varphi_x| + |\varphi_z|) > 0. \end{aligned}$$

Case 2. $x_0 \in M$. We take normal coordinates at x_0 such that $g_{ij}(x_0) = \delta_{ij}$ and then

$$\nabla \log G(x_0) = 0 \quad \text{and} \quad \nabla^2 \log G(x_0) \le 0. \tag{3.2}$$

Thus,

$$\begin{aligned} 0 &\ge a^{ij}[\log G]_{ij} + b^i(\log G)_i \\ &= a^{ij}(\log\rho)_{ij} + b^i(\log\rho)_i + a^{ij}\Big(\frac{|\nabla w|^2_{ij}}{|\nabla w|^2} - \frac{|\nabla w|^2_i|\nabla w|^2_j}{|\nabla w|^4}\Big) \\ &\quad + b^i\frac{|\nabla w|^2_i}{|\nabla w|^2} + a^{ij}(Bd)_{ij} + b^i(Bd)_i \\ &\ge -C\sum F^i_i\Big(\frac{|\nabla^2\rho|}{\rho} + \Big|\frac{\nabla\rho}{\rho}\Big|^2 + 1\Big) + a^{ij}\frac{2w_{kj}w_{ki} + 2w_k w_{kij}}{|\nabla w|^2} + b^i\frac{2w_k w_{ki}}{|\nabla w|^2} \\ &\ge -C\sum F^i_i\Big(\frac{|\nabla^2\rho|}{\rho} + |\frac{\nabla\rho}{\rho}|^2 + 1\Big) + 2\Big(F^{ij} + \frac{1-\tau}{n-2}g^{ij}\sum_{p=1}^n F^p_p\Big)\frac{w_{kj}w_{ki}}{|\nabla w|^2} \\ &\quad + 2w_k\frac{\{a^{ij}(u_{kij} - (\varphi d)_{kij}) + b^i(u_{ki} - (\varphi d)_{ki})\}}{|\nabla w|^2}. \end{aligned}$$

Differentiating the equation once,

$$a^{ij}u_{ijk}+b^iu_{ki}=-F^{ij}\big(A^{\tau}_{ij,k}+(Q(x)e^{-2u}g_{ij})_k\big)+\phi_k. \tag{3.3}$$

Also,

$$\begin{aligned}-2w_k(a^{ij}(\varphi(x,u)d)_{kij}+b^i(\varphi d)_{ki})\ge{}&-2w_k\varphi_u d(a^{ij}u_{kij}+b^iu_{ki})\\&-CF^{ii}(|\nabla w||\nabla^2u|+|\nabla u|^3+|\nabla w|^4d+|\nabla w|^2|\nabla^2u|d).\end{aligned} \tag{3.4}$$

Noting that $\mathrm{Tr}\,\mu>0$, we have

$$\Big(1+\frac{n(1-\tau)}{n-2}\Big)\Delta u\ge\Big((2-\tau)\frac{n}{2}-1\Big)|\nabla u|^2-\mathrm{Tr}\,A^{\tau}_g-nQe^{-2u}, \tag{3.5}$$

yielding that, for $3/2>1-\varphi_z(u)d>1/2$ and small d,

$$g^{ij}w_{kj}w_{ki}\ge\frac12|\nabla^2w|^2+\frac{1}{2n}(\Delta w)^2\ge\frac14|\nabla^2w|^2+C(n,\tau)|\nabla u|^4, \tag{3.6}$$

where $C(n,\tau)$ is a positive constant depending on τ and n.

With $|u_{ijk}-u_{kij}|\le|R_m||\nabla u|$, and (3.3), (3.4), (3.5) and (3.6), we have, for sufficiently small d,

$$\begin{aligned}0\ge{}&a^{ij}[\log G]_{ij}+b^i(\log G)_i|_{x=x_0}\\
\ge{}&-C\sum F^i_i\Big(\frac{|\nabla^2\rho|}{\rho}+\Big|\frac{\nabla\rho}{\rho}\Big|^2\Big)\\
&-CF^i_i\Big(|\nabla u||\frac{\nabla\rho}{\rho}|+|\nabla u|+\frac{|\nabla^2u|}{|\nabla u|}+1+|\nabla^2u|d+|\nabla u|^2d\Big)\\
&+2\frac{1-\tau}{n-2}\sum_{p=1}^{n}F^p_p\frac{\frac14|\nabla^2w|^2+\frac12C(n,\tau)|\nabla(u-\varphi d)|^4}{|\nabla w|^2}\\
\ge{}&\frac{1-\tau}{n-2}\sum_{p=1}^{n}F^p_p\frac{\frac14|\nabla^2w|^2+\frac12C(n,\tau)|\nabla(u-\varphi d)|^4}{|\nabla w|^2}-C\sum_{i=1}^{n}F^i_i\Big(\frac{|\nabla^2\rho|}{\rho}+\Big|\frac{\nabla\rho}{\rho}\Big|^2\Big).\end{aligned}$$

Thus, we have proved $\rho|\nabla(u-\varphi d)|^2\le C$. ■

4. C^2 estimates

In this section, we begin by providing C^2 estimates for equation (2.15) on closed manifolds. Subsequently, we take a two-step approach to derive a C^2 estimate for equation (2.16). First, we reduce the C^2 estimates to the normal second derivatives u_{nn} on ∂M. Second, constructing a proper auxiliary function, we obtain a estimate for u_{nn} by the maximum principle. The two-step method to obtain the C^2 estimates was initially introduced by Lions–Trudinger–Urbas [37] for the Monge–Ampère equation with Neumann boundary condition in Euclidean space.

4.1. The a priori C^2 estimates for equation (2.15)

In this subsection, we derive a priori estimates for equation (2.15) using the auxiliary function introduced in [21] and [7]. The procedure for obtaining these estimates follows a fairly standard approach, and we present it here in detail to effectively track the impact of $Q\,e^{-2u}$.

Theorem 4.1. *For $\tau < 2 - 2/n$, suppose that u is the (k, n, Q)-convex solution to equation (2.15). There exists a positive constant C, depending on K, $|Q|_{C^2(M)}$, k, n, τ and g, such that*

$$|\nabla^2 u| + |\nabla u|^2 \le C\left(\sup_M e^{-2u} + 1\right).$$

Proof. Let

$$G := \Delta u + |\nabla u|^2$$

and assume that $G(x_0) = \max_M G$.

Noting that $\operatorname{Tr}\mu > 0$, we have

$$\left(1 + \frac{n(1-\tau)}{n-2}\right)\Delta u \ge \left((2-\tau)\frac{n}{2} - 1\right)|\nabla u|^2 - \operatorname{Tr} A_g^\tau - nQ\,e^{-2u}. \tag{4.1}$$

At x_0, we choose a normal coordinate such that $g_{ij}(x_0) = \delta_{ij}$. We know

$$u_{ijk} = u_{kij} + R_{ljik}u_l, \tag{4.2}$$

and

$$u_{kkij} = u_{ijkk} + 2R_{mikj}u_{mk} - R_{mj}u_{mi} - R_{mi}u_{mj} - R_{mi,j}u_m + R_{mikj,k}u_m. \tag{4.3}$$

At x_0, we have

$$G_i(x_0) = 0, \quad G_{ij}(x_0) \le 0,$$

and

$$\begin{aligned} 0 &\ge a^{ij}G_{ij} + b^i G_i \\ &= a^{ij}(u_{kkij} + 2u_{ki}u_{kj} + 2u_k u_{kij}) + b^i(u_{kki} + 2u_k u_{ki}). \end{aligned} \tag{4.4}$$

Using normal coordinates at x_0, it holds that

$$\begin{aligned} r_{ij,k} = {} & u_{ijk} + \frac{1-\tau}{n-2}(\Delta u)_k g_{ij} - \frac{2-\tau}{2}(|\nabla u|^2)_k g_{ij} + u_{ik}u_j + u_{jk}u_i \\ & + A^\tau_{ij,k} + (Q(x)e^{-2u}g_{ij})_k, \end{aligned}$$

and

$$\begin{aligned} r_{ij,kk} = {} & u_{ijkk} + \frac{1-\tau}{n-2}(\Delta u)_{kk} g_{ij} - \frac{2-\tau}{2}(|\nabla u|^2)_{kk} g_{ij} \\ & + u_{ikk}u_j + u_{jkk}u_i + 2u_{ik}u_{jk} + A^\tau_{ij,kk} + (Q(x)e^{-2u}g_{ij})_{kk}. \end{aligned} \tag{4.5}$$

Differentiating the equation once, we get

$$\phi_k = F^{ij}\Big(u_{ijk} + \frac{1-\tau}{n-2}(\Delta u)_k g_{ij} - \frac{2-\tau}{2} 2u_p u_{pk} g_{ij} + u_{ik}u_j + u_{jk}u_i + A^{\tau}_{ij,k} + (Q(x)e^{-2u}g_{ij})_k\Big),$$

and then

$$a^{ij}u_{ijk} + b^i u_{ki} = -F^{ij}\big(A^{\tau}_{ij,k} + (Q(x)e^{-2u}g_{ij})_k\big) + \phi_k. \tag{4.6}$$

From

$$\phi_{kk} = F^{ij} r_{ij,kk} + F^{ij,rs} r_{ij,k} r_{rs,k},$$

we obtain

$$a^{ij}u_{ijkk} + b^i u_{ikk} = \phi_{kk} - F^{ij,rs} r_{ij,k} r_{rs,k} - F^{ij}\big(A^{\tau}_{ij,kk} + (Q(x)e^{-2u}g_{ij})_{kk}\big) + (2-\tau)u_{pk}^2 F^{ii} - 2u_{ik}u_{jk}F^{ij}. \tag{4.7}$$

From Lemma 2.1,

$$-F^{ij,rs} r_{ij,k} r_{rs,k} \geq 0.$$

By (4.2) and (4.3), we have

$$|u_{ijk} - u_{kij}| \leq |R_m||\nabla u|,$$

and

$$|u_{ijkk} - u_{kkij}| \leq C_1|\nabla R_m||\nabla^2 u| + C|R_m||\nabla u|.$$

Thus by (4.4), (4.6) and (4.7), we obtain that

$$0 \geq F^{ii}u_{kl}^2\Big(2-\tau+\frac{2(1-\tau)}{n-2}\Big) - 2F^{ij}\big(A^{\tau}_{ij,k} + (Q(x)e^{-2u}g_{ij})_k\big)u_k + 2\phi_k u_k - F^{ij}A^{\tau}_{ij,kk} - F^{ij}(Q e^{-2u}g_{ij})_{kk} + \phi_{kk} - F^{ij,rs}r_{ij,k}r_{rs,k} - CF^{ii}|\nabla^2 u|.$$

By Lemma 2.2, Lemma 2.1 and (4.1), we learn that

$$|\nabla u|^2 + |\nabla^2 u| \leq C\big(\sup_M e^{-2u} + 1\big),$$

where C depends on K, $|Q|_{C^2(M)}$ and $|g|_{C^4(M)}$, τ, n and k. ∎

4.2. The a priori C^2 estimates for equation (2.16)

In this subsection, we first reduce the global second derivatives to the normal second derivatives u_{nn} of u on ∂M, which can be obtained by constructing a proper auxiliary function and applying the maximum principle.

Lemma 4.2. *Suppose $0 \leq \tau \leq 1$ and let u be a (k,n,Q)-convex solution to (2.16). Then,*

$$\max_{x\in B_{r/2}\cap M} \max_{\substack{\zeta\in T_xM \\ |\zeta|=1}} \nabla^2 u(\zeta,\zeta) \leq C + \max_{x\in B_r\cap\partial M} \nabla^2 u(\vec{n},\vec{n}),$$

where C is a positive constant depending on $|g|_{C^4(B_r)}$, K, $|Q|_{C^2(B_r)}$, r, $|c|_{C^3(B_r\cap\partial M)}$, $\sup_{B_r} e^{-2u}$, $\sup_{B_r}|\nabla u|$, τ, n and k.

Proof. Let η be a smooth cut-off function in B_r such that $\eta = 1$ on $B_{r/2}$ and $\langle \nabla\eta, \nabla d\rangle = 0$ on ∂M. Let $\mathbb{S}\overline{M}$ denote the spherical tangent bundle on $\overline{M}$. Given $(x,\zeta) \in \mathbb{S}\overline{M}$, we define

$$G(x,\zeta) : = \eta e^{ad}\big[\nabla^2 u(\zeta,\zeta) - 2\langle \zeta, \nabla d\rangle\big(\langle \zeta^{\mathsf{T}}, \nabla\varphi\rangle - \nabla^2 d(\nabla u, \zeta^{\mathsf{T}})\big) + |\nabla u|^2\big],$$

where $a \in \mathbb{R}^+$ is to be determined later, $d(x) := d_g(x, \partial M)$, $\zeta^{\mathsf{T}} = \zeta - \langle \zeta, \nabla d\rangle \nabla d$ and

$$B = 2\Big(\sup_{B_r\cap\partial M} |ce^{-u}| + n \sup_{B_r\cap\partial M} |L|\Big).$$

Here G is a global function and thus is independent of the selection of local coordinates. We naturally extend $c(x)$ and h_g to the interior of M near the boundary, and φ is well defined near the boundary.

By definition of ζ^{T}, we have

$$\langle \zeta^{\mathsf{T}}, \nabla\varphi\rangle - \nabla^2 d(\nabla u, \zeta^{\mathsf{T}}) = \langle \zeta, \nabla\varphi\rangle - \langle \zeta, \nabla d\rangle\langle \nabla d, \nabla\varphi\rangle - \nabla^2 d(\nabla u, \zeta).$$

Suppose

$$\max_{x\in\overline{B_r}} \max_{\substack{\zeta\in T_x\overline{M}\\ |\zeta|=1}} G(x,\zeta) = G(x_1,\zeta_1), \quad \text{for some } (x_1,\zeta_1) \in \mathbb{S}\overline{M}.$$

Without loss of generality, we assume that $G(x_1,\zeta_1) \gg 1$, otherwise we are done.

Case 1. $x_1 \in \overline{B_r} \cap \partial M$.

Case 1.1: $\zeta_1 = \alpha\nabla d + \beta\tau$ for some $\tau \in T_{x_1}(\partial M)$ with $|\tau| = 1$, where $\alpha = \langle \zeta_1, \nabla d\rangle$ and $\beta = \langle \zeta_1, \tau\rangle$ satisfy $\alpha^2 + \beta^2 = 1$.

As $\vec{n} = \nabla d$ on ∂M and $u_n = \varphi$ on ∂M, we have

$$\nabla^2 u(\tau, \vec{n}) = \tau(u_n) - (\nabla_\tau \vec{n})u = \langle \tau, \nabla\varphi\rangle - \nabla^2 d(\nabla u, \tau) \quad \text{on } \partial M.$$

Thus, we obtain

$$\begin{aligned} G(x_1,\zeta_1) &= \eta e^{ad}\big[\alpha^2\nabla^2 u(\vec{n},\vec{n}) + \beta^2\nabla^2 u(\tau,\tau) + 2\alpha\beta\nabla^2 u(\tau,\vec{n}) \\ &\qquad - 2\alpha\beta(\langle \tau, \nabla\varphi\rangle - \nabla^2 d(\nabla u, \tau)) + |\nabla u|^2\big]\big|_{x_1} \\ &= \eta e^{ad}\big[\alpha^2\nabla^2 u(\vec{n},\vec{n}) + \beta^2\nabla^2 u(\tau,\tau) + |\nabla u|^2\big]\big|_{x_1} \\ &= \alpha^2 G(x_1,\vec{n}) + \beta^2 G(x_1,\tau) \le \alpha^2 G(x_1,\vec{n}) + \beta^2 G(x_1,\zeta_1), \end{aligned}$$

which yields that

$$G(x_1,\zeta_1) \le G(x_1,\vec{n}).$$

Now we just need to consider ζ_1 to be either $\vec{n}$ or a unit vector in $T_{x_1}(\partial M)$.

Case 1.2. $\zeta_1 \in T_{x_1}(\partial M)$.

Under Fermi coordinates around x_1, the metric can be expressed as

$$g = \mathrm{d}x_n^2 + g_{\alpha\beta}\mathrm{d}x_\alpha \mathrm{d}x_\beta,$$

where $(x_1, \ldots, x_{n-1})$ is the geodesic normal coordinates on ∂M and $\partial_{x_n} = \nabla d$ is the inward unit normal on ∂M. We choose the tangent vector field $\frac{1}{\sqrt{g_{11}}}\partial_{x_1}$ as an extension of $\zeta_1 = \partial/\partial x_1|_{x_1}$ to the interior and still denote by ζ_1. On ∂M, there hold

$$\Gamma^n_{\alpha\beta} = L_{\alpha\beta}, \quad \Gamma^\beta_{\alpha n} = -L_{\alpha\gamma}g^{\gamma\beta}, \quad \Gamma^n_{in} = 0 \quad \text{and} \quad \Gamma^i_{nn} = 0.$$

We have

$$u_{\alpha n} = \partial_\alpha u_n + L^\gamma_\alpha u_\gamma = \partial_\alpha \varphi + L^\gamma_\alpha u_\gamma,$$

and

$$\begin{aligned} u_{\alpha\beta n} &= L^\gamma_\beta u_{\gamma\alpha} + L^\gamma_\alpha u_{\gamma\beta} - L_{\alpha\beta} u_{nn} + \partial_{\alpha\beta}\varphi + L^\gamma_{\alpha;\beta} u_\gamma + L^\gamma_\alpha L_{\beta\gamma} u_n + R^i_{\alpha\beta n} u_i \\ &= L^\gamma_\beta u_{\gamma\alpha} + L^\gamma_\alpha u_{\gamma\beta} - L_{\alpha\beta} u_{nn} + L^\gamma_{\alpha;\beta} u_\gamma + L^\gamma_\alpha L_{\beta\gamma} u_n + R^i_{\alpha\beta n} u_i \\ &\quad + \varphi_z(u_{\alpha\beta} + L_{\alpha\beta} u_n) + \varphi_{zz} u_\alpha u_\beta + \varphi_{x_\alpha x_\beta} + \varphi_{x_\alpha z} u_\beta + \varphi_{z x_\beta} u_\alpha. \end{aligned}$$

In particular, we have

$$\begin{aligned} \partial_n(u_{11}) &= 2\Gamma^\alpha_{1n} u_{\alpha 1} + u_{11n} = u_{11n} - 2L^\alpha_1 u_{\alpha 1} \\ &= -L_{11} u_{nn} + L^\gamma_{1;1} u_\gamma + L^\gamma_1 L_{1\gamma} u_n + R^i_{11n} u_i \\ &\quad + \varphi_z(u_{11} + L_{11} u_n) + \varphi_{zz} u_1 u_1 + \varphi_{x_1 x_1} + \varphi_{x_1 z} u_1 + \varphi_{z x_1} u_1. \end{aligned}$$

Notice that

$$G(x, \zeta_1) = \eta e^{ad}\Big(\frac{u_{11}}{g_{11}} + |\nabla u|^2\Big) \quad \text{near } x_1.$$

Then we have

$$\begin{aligned} 0 &\geq G_n(x_1, \zeta_1) \\ &= \eta\big[a(u_{11} + |\nabla u|^2) + \partial_n(u_{11}) - u_{11}\partial_n(g_{11}) + 2u^\alpha u_{\alpha n} + 2u_n u_{nn}\big] \\ &\geq \eta\big[a(u_{11} + |\nabla u|^2) - L_{11} u_{nn} + \varphi_z u_{11} + 2u^\alpha(\partial_\alpha\varphi + L^\gamma_\alpha u_\gamma) + 2u_n u_{nn} - C\big]. \end{aligned} \tag{4.8}$$

Since (x_1, ζ_1) is a maximum point of G, for $\tau < 2 - 2/n$, by (2.8), we have

$$u_{nn} + (n-1)u_{11} \geq -C,$$

and then $u_{nn} \geq -(n-1)u_{11} - C$. Also we know that $u_{nn} \leq u_{11} + C$, and thus,

$$|u_{nn}|(x_1) \leq (n-1)u_{11}(x_1) + C.$$

Take a sufficiently large, for example,

$$a = \sup_{B_r \cap \partial M} (n-1)[2|L| + 2|\varphi| + 2|\varphi_z|].$$

Then, by (4.8), we have

$$u_{11}(x_1) \leq C,$$

where C depends on $\sup_{\overline{B}_r \cap \partial M}(\varphi + \varphi_z)$, $|g|_{C^2(\overline{B}_r)}$ and $\sup_{\overline{B}_r} |\nabla u|$.

Case 2. $x_1 \in B_r \cap M$.

We now choose geodesic normal coordinates around x_1 such that

$$g_{ij} = \delta_{ij}, \quad \Gamma_{ij}^k = 0 \quad \text{at } x_1.$$

We also use the extended tangent vector field $\zeta_1(x) = \frac{1}{\sqrt{g_{11}}}\partial_{x_1}$ near x_1 such that $\zeta_1 = \partial/\partial x_1$ at x_1.

With the above extended vector field ζ_1, we rewrite

$$G(x,\zeta_1) = \eta e^{ad}\Big(\frac{u_{11}}{g_{11}} - a^l u_l + 2\langle\zeta_1, \nabla d\rangle\langle\zeta_1^{\top}, \nabla h_g\rangle + |\nabla u|^2\Big) \quad \text{near } x_1,$$

where a^l depends on g, $d(x)$ and $c(x)e^{-u}$. Since G attains its maximum at (x_1, ζ_1) and by Lemma 2.1, at x_1 we have

$$|u_{ij}| \le C(\Delta u + 1) \le C(u_{11} + 1).$$

For brevity, we let

$$E := \frac{u_{11}}{g_{11}} - a^l u_l + z + |\nabla u|^2,$$

where $z = 2\langle\zeta_1, \nabla d\rangle\langle\zeta_1^{\top}, \nabla h_g\rangle$.

At x_1, we have

$$0 = \partial_i \log G = \frac{\eta_i}{\eta} + ad_i + \frac{\partial_i E}{E}, \quad \text{for all } 1 \le i \le n, \tag{4.9}$$

and $(\partial_j\partial_i \log G)$ is non-negative.

In what follows, all calculations are evaluated at x_1. A direct computation yields

$$\begin{aligned}\partial_j\partial_i u_k &= \partial_j(u_{ki} + \Gamma_{ik}^l u_l) = u_{kij} + \partial_j\Gamma_{ik}^l u_l,\\ \partial_j\partial_i(u_{11}) &= \partial_j(u_{11i} + 2\Gamma_{i1}^l u_{l1}) = u_{11ij} + 2\partial_j\Gamma_{i1}^l u_{l1},\end{aligned}$$

and

$$\partial_i E = \frac{\partial_i(u_{11})}{g_{11}} - \frac{u_{11}}{g_{11}^2}\partial_i(g_{11}) - \partial_i a^l u_l - a^l\partial_i\partial_l u + z_i + (|\nabla u|^2)_i. \tag{4.10}$$

Then we have

$$\begin{aligned}\partial_j\partial_i E = {}& u_{11ij} + 2\partial_j\Gamma_{i1}^l u_{l1} - u_{11}\partial_j\partial_i g_{11} - \partial_j\partial_i a^l u_l - a^l(u_{lij} + \partial_j\Gamma_{li}^k u_k)\\ & - \partial_i a^l u_{lj} - \partial_j a^l u_{li} + \partial_j\partial_i z + 2u_{kj}u_{ki} + 2u_k(u_{kij} + \partial_j\Gamma_{ik}^l u_l).\end{aligned}$$

It follows from (4.9) and (4.10) that

$$u_{11i} = -E\Big(\frac{\eta_i}{\eta} + ad_i\Big) + a^l u_{li} + \partial_i a^l u_l - z_i - 2u_k u_{ki}. \tag{4.11}$$

Thus, again by (4.9) we obtain

$$\begin{aligned}\partial_j\partial_i \log G &= \frac{\eta_{ij}}{\eta} - \frac{\eta_i\eta_j}{\eta^2} + ad_{ij} + \frac{\partial_j\partial_i E}{E} - \frac{\partial_i E\,\partial_j E}{E^2}\\ &= \frac{\eta_{ij}}{\eta} - \frac{\eta_i\eta_j}{\eta^2} + ad_{ij} + \frac{\partial_j\partial_i E}{E} - \Big(\frac{\eta_i}{\eta} + ad_i\Big)\Big(\frac{\eta_j}{\eta} + ad_j\Big).\end{aligned}$$

Hence, putting these facts together, we conclude that

$$\begin{aligned}0 &\ge Ea^{ij}\partial_i\partial_j(\log G) + Eb^i\partial_i(\log G)\\ &= Ea^{ij}\Big(\frac{\eta_{ij}}{\eta} - 2\frac{\eta_i\eta_j}{\eta^2}\Big) + aEa^{ij}d_{ij} - a^2Ea^{ij}d_id_j - 2aEa^{ij}\frac{\eta_i}{\eta}d_j + a^{ij}\partial_j\partial_i E\\ &\quad + Eb^i\Big(\frac{\eta_i}{\eta} + ad_i + \frac{\partial_i E}{E}\Big)\\ &\ge -CEF^{ii}\Big(\frac{|\nabla\eta|^2}{\eta^2} + \frac{|\nabla^2\eta|}{\eta^2}\Big) - CEF^{ii} + a^{ij}(u_{11ij} - a^l u_{lij} + 2u_k u_{kij})\\ &\quad + 2a^{ij}u_{kj}u_{ki} + a^{ij}\partial_j\partial_i z - 2a^{ij}\partial_i a^l u_{lj} - a^{ij}\partial_j\partial_i a^l u_l\\ &\quad + b^i(u_{11i} - \partial_i a^l u_l - a^l u_{il} + z_i + 2u_k u_{ki}).\end{aligned}$$

Differentiating the equation twice, we obtain

$$a^{ij}u_{ijk} + b^i u_{ki} = -F^{ij}\big(A^{\tau}_{ij,k} + (Q(x)e^{-2u}g_{ij})_k\big) + \phi_k, \tag{4.12}$$

and

$$\begin{aligned}F^{ij}\Big(&u_{ijkk} + \frac{1-\tau}{n-2}(\Delta u)_{kk}g_{ij} - \frac{2-\tau}{2}(|\nabla u|^2)_{kk}g_{ij}\\ &+ u_{ikk}u_j + u_{jkk}u_i + 2u_{ik}u_{jk}\Big)\\ &= \phi_{kk} - F^{ij,rs}r_{ij,k}r_{rs,k} - F^{ij}\big(A^{\tau}_{ij,kk} + (Q(x)e^{-2u}g_{ij})_{kk}\big),\end{aligned} \tag{4.13}$$

which in turn can be expressed as follows:

$$\begin{aligned}a^{ij}u_{ij11} + b^i u_{i11} &= \phi_{11} - F^{ij,rs}r_{ij,1}r_{rs,1} - F^{ij}\big(A^{\tau}_{ij,11} + (Q(x)e^{-2u}g_{ij})_{11}\big)\\ &\quad + (2-\tau)u_{p1}^2F^{ii} - 2u_{i1}u_{j1}F^{ij}\\ &\ge \phi_{11} - CEF^{ii} + (2-\tau)u_{p1}^2F^{ii} - 2u_{i1}u_{j1}F^{ij}.\end{aligned} \tag{4.14}$$

Using

$$u_{i11} = u_{1i1} = u_{11i} + R^m_{1i1}u_m,$$

and

$$\begin{aligned}a^{ij}u_{11ij} &= a^{ij}(u_{ij11} + R^m_{11i}u_{mj} + 2R^m_{i1j}u_{m1} + R^m_{11j}u_{mi} + R^m_{11i,j}u_m + R^m_{i1j,1}u_m)\\ &\ge a^{ij}u_{ij11} - CEF^{ii} - CF^{ii},\end{aligned}$$

with (4.12) and (4.14), we have

$$\begin{aligned}0 &\geq Ea^{ij}\partial_i\partial_j(\log G) + Eb^i\partial_i(\log G)\\ &\geq -CEF^{ii}\Big(\frac{|\nabla\eta|^2}{\eta^2} + \frac{|\nabla^2\eta|}{\eta^2}\Big) + 2a^{ij}u_{kj}u_{ki} + a^{ij}u_{ij11} + b^i u_{11i} + 2a^{ij}u_k u_{kij}\\ &\quad + 2b^i u_k u_{ki} - a^{ij}a^l u_{lij} - a^l b^i u_{il} - CEF^{ii} - CF^{ii}\\ &\geq -CEF^{ii}\Big(\frac{|\nabla\eta|^2}{\eta^2} + \frac{|\nabla^2\eta|}{\eta^2}\Big) + 2\sum_{k\neq 1} F^{ij}u_{kj}u_{ki}\\ &\quad + \Big[(2-\tau)\sum_p u_{p1}^2 + \frac{2(1-\tau)}{n-2}\sum_{k,j} u_{kj}^2\Big]\sum_i F^{ii} - CEF^{ii} - CF^{ii}.\end{aligned}$$

Then

$$u_{11}(x_1) \leq C.$$

Combining the above two cases, we obtain the desired estimate. ■

It remains to estimate the normal second derivative of u on the boundary.

Lemma 4.3. *Suppose* $0 \leq \tau < 1$ *and let* u *be a* (k, n, Q)*-convex solution to* (2.16). *Then we have*

$$u_{nn} \leq C \quad \text{on } \partial M,$$

where C *is a positive constant depending on the quantities* $|g|_{C^4(M)}$, K, $|Q|_{C^2(M)}$, $|c|_{C^3(\partial M)}$, $\sup_M e^{-2u}$, τ, n *and* k.

Proof. Without loss of generality, we assume that

$$\sup_{\partial M} \nabla^2 u(\vec{n}, \vec{n}) = u_{nn}(x_0) := M^* \quad \text{for some } x_0 \in \partial M.$$

In

$$M_\mu := \{x \in \overline{M} : d_g(x, \partial M) \leq \mu\},$$

for a fixed small μ, we define

$$G(x) = \langle \nabla u, \nabla d_g(x, \partial M)\rangle - \varphi + a(\langle \nabla u, \nabla d_g(x, \partial M)\rangle - \varphi)^2 - \frac{1}{2}M^* d_g(x, \partial M).$$

We naturally extend $c(x)$ and h_g to the interior of M near the boundary, and φ is well defined near the boundary. We assume that M^* is sufficiently large so that $G(x) < 0$ on $\partial M_\mu \backslash \partial M$, otherwise we have an upper bound for M^* and the lemma is proved. Assume $\max_{M_\mu} G = G(x_1)$. For convenience, we still use Fermi coordinates around x_1, and thereby, the metric can be expressed as

$$g = \mathrm{d}x_n^2 + g_{\alpha\beta}\,\mathrm{d}x_\alpha\,\mathrm{d}x_\beta,$$

where $(x_1, \ldots, x_{n-1})$ are the geodesic normal coordinates on ∂M and $\partial_{x_n} = \nabla d$ is the inward unit normal on ∂M.

If $x_1 \in \partial M$, then with the fact $G|_{\partial M} = 0$, we have

$$\max_{M_\mu} G = 0 = G|_{\partial M} = G(x_0)$$

and

$$\begin{aligned} 0 \geq G_n(x_0) &= (u_{nn} - \varphi_n)(x_0)[1 + 2a(u_n - \varphi)(x_0)] - \frac{1}{2} M^* \\ &= (u_{nn} - \varphi_n)(x_0) - \frac{1}{2} M^* = \frac{1}{2} M^* - \varphi_n(x_0). \end{aligned}$$

This gives

$$u_{nn}(x_0) \leq 2\|\varphi\|_{C^1(\overline{M_r})}.$$

By contradiction, we assume

$$\max_{\overline{M}_\mu} G = G(x_1) > 0 \quad \text{for } x_1 \in \mathring{M}_\mu.$$

Let

$$a = \frac{1}{2}\Big[\sup_{M_\mu} |\langle \nabla u, \nabla d_g(x, \partial M)\rangle| + |\varphi| + 1\Big]^{-1} \quad \text{and} \quad B := u_n(x_1) - \varphi(x_1, u).$$

From $G(x_1) > 0$, we have

$$B + aB^2 > 0 \quad \text{and} \quad B > 0.$$

At x_1, a direct computation yields for all $1 \leq i \leq n$,

$$0 = G_i = (u_{ni} + u^\alpha (x_n)_{\alpha i} - \varphi_i)(1 + 2aB) - \frac{1}{2} M^* \delta_{ni},$$

and

$$\begin{aligned} 0 \geq G_{ij}(x_1) &= (u_{nij} + u_j^k (x_n)_{ki} + u^k (x_n)_{kij} + u_i^k (x_n)_{kj} - \varphi_{ij})(1 + 2aB) \\ &\quad + 2a(u_{ni} + u^k (x_n)_{ki} - \varphi_i)(u_{nj} + u^k (x_n)_{kj} - \varphi_j) - \frac{1}{2} M^* (x_n)_{ij} \\ &= (u_j^\beta (x_n)_{\beta i} + u^k (x_n)_{kij} + u_i^\beta (x_n)_{\beta j})(1+2aB) + (u_{nij} - \varphi_{ij})(1 + 2B) \\ &\quad + 2a(u_{ni} + u^k (x_n)_{ki} - \varphi_i)(u_{nj} + u^k (x_n)_{kj} - \varphi_j) - \frac{1}{2} M^* (x_n)_{ij}. \end{aligned} \tag{4.15}$$

In particular, for large M^*, we have

$$u_{nn}(x_1) = \frac{\frac{1}{2} M^*}{1 + 2aB} + \varphi_n \geq \frac{\frac{1}{4} M^*}{1 + 2aB} \tag{4.16}$$

and

$$u_{n\alpha} = -u^\beta (x_n)_{\beta\alpha} + \varphi_\alpha. \tag{4.17}$$

It follows from (2.9), Lemma 4.2, the choice of M^* and (4.16), that

$$|u_{ij}(x_1)| \le C(\Delta u + 1)(x_1) \le C u_{nn}(x_0) \le C u_{nn}(x_1) \le C M^*. \tag{4.18}$$

With

$$a^{ij} u_{ijn} + b^i u_{ni} = -F^{ij}\big(A^{\tau}_{ij,n} + (Q(x)e^{-2u} g_{ij})_n\big) + \phi_n, \tag{4.19}$$

from (4.15), (4.18) and (4.16), we get the following contradiction:

$$\begin{aligned}
0 &\ge a^{ij} G_{ij} + b^i G_i|_{x=x_1} \\
&= a^{ij}(u^{\beta}_j (x_n)_{\beta i} + u^k (x_n)_{kij} + u^{\beta}_i (x_n)_{\beta j})(1 + 2aB) \\
&\quad + (1 + 2aB)\big[a^{ij}(u_{nij} - \varphi_{ij}) + b^i(u_{ni} + u^{\alpha}(x_n)_{\alpha i} - \varphi_i)\big] \\
&\quad + 2a a^{ij}(u_{ni} + u^k (x_n)_{ki} - \varphi_i)(u_{nj} + u^k (x_n)_{kj} - \varphi_j) \\
&\quad - \frac{1}{2} a^{ij} M^* (x_n)_{ij} - \frac{1}{2} M^* b^i \delta_{ni} \\
&\ge -C_1 \sum_p F^p_p M^* + \frac{a}{32}\frac{1-\tau}{n-2} \sum_{p=1}^n F^p_p M^{*2} > 0,
\end{aligned}$$

where

$$a^{ij} u_{ni} u_{nj} \ge \frac{1-\tau}{n-2} \sum_{p=1}^n F^p_p \sum_i u_{ni} u^i_n \ge \frac{1-\tau}{n-2} \sum_{p=1}^n F^p_p u^2_{nn}$$

and C_1 is a positive constant depending on g and $\sup_{M_r}(\phi(x,u) + |\phi_x| + |\phi_z| + |\varphi| + |\varphi_x| + |\varphi_z| + |\varphi_{xx}| + |\varphi_{zz}| + |\varphi_{xz}|)$. ∎

From Lemmas 2.2, 4.2 and 4.3, and Theorem 3.3, we have

Theorem 4.4. *Suppose* $0 \le \tau < 1$ *and let* u *be a* (k,n,Q)*-convex solution to* (2.16). *Then we have*

$$|\nabla^2 u| \le C \quad \text{on } M,$$

where C *is a positive constant depending on* $|g|_{C^4(M)}$, K, $|Q|_{C^2(M)}$, $|c|_{C^3(\partial M)}$ *and* $\sup_M e^{-2u}$, τ, n *and* k.

5. Proof of the main theorem

In this section, we will prove the existence of solutions to equations (2.15) and (2.16), and then by approximation, Theorem 1.1 and Theorem 1.2 can be proved.

Theorem 5.1. *Suppose that* $R_g < 0$ *and that* $g^{-1}(A^{\tau}_g + Qg) \in \mathcal{P}_k(\mathbb{R}^n)$ *on* (M,g). *For* $0 < K < k \min_M Q$ *and* $0 \le \tau \le 1$, *there exists a smooth* (k,n,Q)*-convex solution* u *to*

$$F(g^{-1}(A^{\tau}_{g_u} + Q(x) g_u)) = K e^{-2u}. \tag{5.1}$$

At $t=0$, $u=0$ is the unique solution to (2.15) or (2.16). At $t=1$, the equation is (5.1) (or (5.1) with boundary). Sections 3 and 4 were dedicated to demonstrating a priori estimates including C^0, C^1, C^2 for equation (2.15) and (2.16). With these a priori estimates, $\mathbb{I}$ (or $\mathbb{I}_b$) is closed, subsequently leading to the existence of equation (5.1) (or (5.1) with boundary). Now Theorem 5.1 can be obtained using the method of continuity in conjunction with Theorem 4.1, Lemma 3.1 and Lemma 3.2. Setting $K=1/i$, we can derive a sequence of solutions u_i with uniform a priori estimates. Consequently, Theorem 1.1 and Theorem 1.2 are proved.

In the case where the manifold is not Ricci-flat, the assumption of R_g can be relaxed to $R_g \le 0$. We have the following theorem.

Theorem 5.2. *Let (M,g) be a non-Ricci-flat manifold with $R_g \le 0$ and $g^{-1}(A_g^\tau + Qg) \in \mathcal{P}_k(\mathbb{R}^n)$. We have the following:*

(a) *For $0 < K < k \min_M Q$ and $0 \le \tau \le 1$, there exists a smooth (k,n,Q)-convex solution u to*

$$F(g^{-1}(A_{g_u}^\tau + Q(x)g_u)) = Ke^{-2u}. \tag{5.2}$$

(b) *Further, if $h_g \le 0$ on ∂M, and for $0 < K < k\min_M Q$ and $0 \le \tau < 1$, given any non-positive function $c(x)$ on ∂M, there exists a smooth (k,n,Q)-convex solution u to*

$$\begin{cases} F(g^{-1}(A_{g_u}^\tau + Q(x)g_u)) = Ke^{-2u} & \text{in } M, \\ h_{g_u} = c(x) & \text{on } \partial M. \end{cases} \tag{5.3}$$

Proof. We use the method of continuity as in Theorem 5.1, and consider the solution $u(x,t)$ to the path equation (2.15) (or (2.16)). If there exists a positive constant C such that $\sup_M |u(x,t)| \le C$ for $0 \le t \le 1$, then with Theorem 3.3 and Theorem 4.4, the existence can be derived by the method of continuity with standard argument.

The lower bound follows from Lemma 3.2. The proof differs with Theorem 5.1 only in the upper bound for C^0. We establish the upper bound using a proof by contradiction. Let u^t be the solution to (2.15) or (2.16). Suppose there exist a $0 < t_0 \le 1$ and a sequence $t_i \to t_0$ such that

$$\max_M u^{t_i} = u^{t_i}(x_i) \to \infty, \quad \text{as } i \to +\infty.$$

Let

$$v_i(x) = u^{t_i}(x) - u^{t_i}(x_i) \le 0 \quad \text{in } M.$$

Now by Lemma 3.2, Theorem 3.3 and Theorem 4.4,

$$|\nabla v_i|^2 + |\nabla^2 v_i| \le C_1. \tag{5.4}$$

By (2.15) or (2.16), v_i satisfies the following equation:

$$F(g^{-1}(A_{g_{v_i}}^\tau + Q(x)g_{v_i + u^{t_i}(x_i)})) = t_i Ke^{-2(v_i + u^{t_i}(x_i))} + (1-t_i)h(x) \quad \text{in } M.$$

On ∂M,

$$\frac{\partial v_i}{\partial n_g} = (t_i c(x) + (1-t_i)h_g)\, e^{-(v_i + u^{t_i}(x_i))} - h_g.$$

By $v_i(x_i) = 0$ and (5.4), we obtain

$$\max_M |v_i| \le C,$$

and

$$\max_M |v_i| + |\nabla v_i| + |\nabla^2 v_i| \le C, \tag{5.5}$$

where C is independent of i. There exists a subsequence of v_i (still denoted as v_i) converging to a non-positive v_∞ in $C^{1,\alpha}$ sense. Meanwhile,

$$g_{v_i}^{-1}(A^\tau_{g_{v_i}} + Q(x)\, g_{v_i + u^{t_i}(x_i)}) \in \mathcal{P}_k(\mathbb{R}^n). \tag{5.6}$$

Also by (5.6) and $R_g \le 0$, we get that

$$\begin{aligned} \frac{(2-\tau)n-2}{2} |\nabla v_i|^2 &\le \frac{2n-2-n\tau}{n-2} \Delta v_i + \mathrm{Tr}\, A^\tau_g + nQ\, e^{-2(v_i + u^{t_i}(x_i))} \\ &\le \frac{2n-2-n\tau}{n-2} \Delta v_i + nQ\, e^{-2(v_i + u^{t_i}(x_i))}. \end{aligned}$$

By (5.5), as $i \to \infty$,

$$Q(x)\, g_{v_i + u^{t_i}(x_i)} = Q(x)\, e^{-2(v_i + u^{t_i}(x_i))} \to 0. \tag{5.7}$$

Therefore, as $i \to \infty$, we have $\Delta v_\infty \ge 0$ in the H^1 sense and $v_\infty \in C^{1,\alpha}$ ($\partial v_\infty / \partial \vec{n} = -h_g \ge 0$ on ∂M if ∂M exists). This yields that $\Delta v_\infty = 0$ (it also holds that $h_g = 0$ on ∂M if ∂M exists). As $v_\infty \le 0$, we have $v_\infty = 0$ in M. Therefore $v_i \to 0$ in the $C^{1,\alpha}$ sense.

From the definition of $\mathcal{P}_k(\mathbb{R}^n)$, we know the corresponding matrix $\{W^{t_i}_{N_\alpha N_\beta}\}_{C_n^k \times C_n^k}$ is non-negative, where

$$W^{t_i} = \Big\{ \sum_{l=1}^{k} \sum_{j=1}^{n} \big(A^\tau_{g_{v_i}} + Q(x)\, g_{v_i + u^{t_i}(x_i)}\big)_j^{\alpha_l} \delta\begin{pmatrix} \alpha_1 & \cdots & \alpha_{l-1} & j & \alpha_{l+1} & \cdots & \alpha_k \\ \beta_1 & \cdots & \beta_{l-1} & \beta_l & \beta_{l+1} & \cdots & \beta_k \end{pmatrix} \Big\}_{C_n^k \times C_n^k}.$$

The definition of the matrix W^{t_i} was given in the preliminaries.

For any smooth vector

$$X = X^{N_\beta} \frac{\partial}{\partial x^{\beta_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\beta_k}},$$

we have

$$\int_M (W^{t_i})^{N_\alpha}_{N_\beta} X_{N_\alpha} X^{N_\beta} \ge 0.$$

As $v_i \to 0$ in the $C^{1,\alpha}$ sense, we obtain

$$|\nabla v_i|_{C^0(M)} \to 0.$$

Furthermore,

$$
\begin{aligned}
0 &\le \lim_{i\to\infty}\int_M \left(W^{t_i}\right)^{N_\alpha}_{N_\beta} X_{N_\alpha} X^{N_\beta} \\
&\le \int_M W^{N_\alpha}_{N_\beta}(A^\tau_g) X_{N_\alpha} X^{N_\beta} + \lim_{i\to\infty} C\int_M \Delta v_i + |\nabla v_i|^2 + |\nabla v_i| + Q\, e^{-2(v_i+u^{t_i}(x_i))} \\
&\quad + \lim_{i\to\infty}\int_M \sum_{j=1}^n \Big[\sum_{l=1}^k {v_i}^{,\alpha_l}\delta\begin{pmatrix} \alpha_1 & \cdots & \alpha_{l-1} & j & \alpha_{l+1} & \cdots \alpha_k \\ \beta_1 & \cdots & \beta_{l-1} & \beta_l & \beta_{l+1} & \cdots \beta_k \end{pmatrix} X_{N_\alpha} X^{N_\beta}\Big]_{,j} \\
&\le \int_M W^{N_\alpha}_{N_\beta}(A^\tau_g) X_{N_\alpha} X^{N_\beta}.
\end{aligned}
$$

Thus, by (5.7)

$$
\int_M \sum_{l=1}^k \sum_{j=1}^n (A^\tau_g)^{\alpha_l}_j \delta\begin{pmatrix} \alpha_1\cdots & \alpha_{l-1} & j & \alpha_{l+1} & \cdots\alpha_k \\ \beta_1\cdots & \beta_{l-1} & \beta_l & \beta_{l+1} & \cdots\beta_k \end{pmatrix} X_{N_\alpha} X^{N_\beta} \ge 0,
$$

and now the following matrix $W(A^\tau_g)$ is non-negative:

$$
W(A^\tau_g) := \left\{\sum_{l=1}^k \sum_{j=1}^n (A^\tau_g)^{\alpha_l}_j \delta\begin{pmatrix} \alpha_1 & \cdots & \alpha_{l-1} & j & \alpha_{l+1} & \cdots \alpha_k \\ \beta_1 & \cdots & \beta_{l-1} & \beta_l & \beta_{l+1} & \cdots \beta_k \end{pmatrix}\right\}_{C^k_n\times C^k_n} \ge 0. \tag{5.8}
$$

According to $\mathrm{Tr}_g W(A^\tau_g) = 0$ and (5.8), we derive

$$
W(A^\tau_g) = 0 \quad \text{on } M.
$$

For any $x \in M$, assume that A^τ_g is diagonal and then $A^\tau_g = \frac{1}{n-2}\mathrm{Ric}_g$. For any ordered $\{\alpha_1,\dots,\alpha_k\} \subset \{1,2,\dots,n\}$,

$$
R_{\alpha_1\alpha_1} + R_{\alpha_2\alpha_2} + \cdots + R_{\alpha_k\alpha_k} = 0. \tag{5.9}
$$

From (5.9), we know that

$$
C^{k-1}_{n-1} R_{11} + C^{k-1}_{n-1}\frac{k-1}{n-1}(R_g - R_{11}) = 0.
$$

For $k < n$, we have $R_{11} = 0$. The same argument yields $R_{ii} = 0$ for any $i = 2,\dots,n$, resulting in a contradiction to the non-Ricci-flat assumption. ∎

Now we prove our main theorem using Theorem 5.2; the method is almost the same as above.

Proof of Theorem 1.3. Let $K = 1/i \to 0$. For each i, we have a sequence of solutions u_i to equation (5.2) or (5.3). According to Lemma 3.2, Theorem 3.3, and Theorem 4.4, it holds that

$$
\sup_M |\nabla u_i| + |\nabla^2 u_i| \le C \quad \text{for } u_i \ge -C,
$$

where C is a positive constant depending only on n, g and Q. To get the upper bound of u_i, we proceed as in Theorem 5.2. Suppose $\max_M u_i = u_i(x_i) \to +\infty$. Let

$$v_i(x) = u_i(x) - u_i(x_i) \le 0 \quad \text{in } M,$$

so that v_i satisfies the following equation:

$$F(g^{-1}(A^\tau_{g_{v_i}} + Q(x)\, g_{v_i+u_i(x_i)})) = \frac{1}{i}\, e^{-2(v_i+u_i(x_i))} \quad \text{in } M.$$

If ∂M exists, then

$$\frac{\partial v_i}{\partial n_g} = c(x)\, e^{-(v_i+u_i(x_i))} - h_g \quad \text{on } \partial M.$$

The rest of the proof is the same as that of Theorem 5.2. Therefore, u_i is bounded from above, and $|u_i|_{C^0} \le C$. Hence, there exists a subsequence of u_i such that $u_i \to u$ in $C^{1,\alpha}(M)$ for any $0 < \alpha < 1$, where $u \in C^{1,1}(M)$ satisfies equation (1.5) or (1.6). For $0 \le \tau < 1$, higher regularity follows from the uniform ellipticity of the operator. ∎

Remark 5.3. For $0 \le \tau \le 1$, Theorem 3.3 can be established by modifying the technique of proving Theorem 19 in Li [35], which combines Theorem 1.23 in [35] with a local gradient estimate depending on $\inf u$ and $\sup u$. The local gradient estimate related with two sides can be obtained by the auxiliary function: $\rho|\nabla(u - \varphi d)|^2 e^{Bd+\phi(u)}$, where ρ is a cut-off function, B is a positive constant and $\phi(u)$ is a function of u. With the uniform C^1 estimates with respect to τ, we may obtain a Lipschitz viscosity solution to equation (1.6) for $\tau = 1$. See also the argument in the proof Theorem 1.3 in [36]. As it is not our interest to introduce the viscosity solution here, we do not discuss the case $\tau = 1$ for equation (1.6).

Remark 5.4. Assuming that ∂M is totally geodesic, the C^2 estimates can also be obtained for $\tau = 1$ by mimicking the proof of [30] and [8].

Acknowledgements. W. Wei would like to thank Prof. X. Z. Chen for the helpful discussions on the boundary Yamabe problem. She also thanks the Institute of Mathematics, University of Freiburg, for its hospitality.

Funding. Research of C. Chen was supported by ZJNSF no. LRG25A010002 and NSFC no. 12571219, and research of W. Wei was supported by NSFC (no. 12201288, 12571218, 11771204), BK20220755 and the Alexander von Humboldt Foundation.

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Received January 4, 2024.

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