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The atomic Leibniz rule

Ben Elias, Hankyung Ko, Nicolás Libedinsky and Leonardo Patimo

Abstract. The Demazure operator associated to a simple reflection satisfies the twisted Leibniz rule. In this paper, we introduce a generalization of the twisted Leibniz rule for the Demazure operator associated to any atomic double coset. We prove that this atomic Leibniz rule is equivalent to a polynomial forcing property for singular Soergel bimodules.

1. Introduction

We introduce a generalization of the twisted Leibniz rule which applies to Demazure operators associated with certain double cosets. We call it the atomic Leibniz rule, and it should play an important role in an eventual description of the singular Hecke category by generators and relations. This result is the algebraic heart which we extract from singular Soergel bimodules and transplant to their diagrammatic calculus.

1.1.

In this paper, we work with general Coxeter systems and quite general actions of these groups on polynomial rings. For ease of exposition, in this section, we assume that $R = \mathbb{Q}[x_1, \dots, x_n]$, with the standard action of the symmetric group $W = \mathbf{S}_n$. Let s_i be the simple reflection swapping i and $i+1$, and let $S = \{s_1, \dots, s_{n-1}\}$ be the set of simple reflections. There is a *Demazure operator* $\partial_{s_i} \colon R \to R$ defined by the formula

$$\partial_{s_i}(f) = \frac{f - s_i f}{x_i - x_{i+1}}. \tag{1.1}$$

This operator famously satisfies a *twisted Leibniz rule*

$$\partial_{s_i}(fg) = s_i(f)\,\partial_{s_i}(g) + \partial_{s_i}(f)\,g. \tag{1.2}$$

Because the operators ∂_{s_i} satisfy the braid relations, one can unambiguously define an operator ∂_w associated with any $w \in \mathbf{S}_n$ by composing the operators ∂_{s_i} along a reduced expression. When computing $\partial_w(fg)$, one could apply the twisted Leibniz rule repeatedly to obtain a complicated generalization of (1.2) for ∂_w. We discuss this in §2.2.

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What we provide in this paper is the natural generalization of the twisted Leibniz rule to the setting of double cosets. In double coset combinatorics, the analogues of simple reflections are called atomic cosets. We prove a version of (1.2) for Demazure operators attached to atomic cosets.

1.2.

We recall some definitions so as to precisely state our first theorem. Let (W, S) be any Coxeter system. For any subset $I \subset S$, let W_I be the parabolic subgroup of W generated by I. We assume that I is *finitary*, i.e., that W_I is a finite group. Let $R^I \subset R$ be the subring of polynomials invariant under W_I. Let w_I denote the longest element of W_I. For two subsets $I, J \subset S$, a double coset $p \in W_I \setminus W / W_J$ will be called an (I, J)*-coset*. Any (I, J)-coset p has a unique minimal element $\underline{p} \in W$ and a unique maximal element $\overline{p} \in W$ with respect to the Bruhat order.

In Section 3.4 of [7], we introduced a Demazure operator $\partial_p \colon R^J \to R^I$ for any (I, J)-coset p. In fact, ∂_p is equal to ∂_w for $w = \overline{p} w_J \in W$. Normally one views ∂_w as a map $R \to R$, but when the source of this map is restricted from R to R^J, then the image is contained in R^I. When $I = J = \emptyset$, so that $p = \{w\}$ for some $w \in W$, then $\partial_p = \partial_w$.

In [9], we introduced the notion of an *atomic* (I, J)-coset. Briefly stated, $\mathtt{a}$ is atomic if

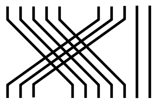
- there exist $s, t \in S$ (possibly equal) with $s \notin I$ and $t \notin J$ such that

$$I \cup \{s\} = J \cup \{t\} =: M.$$

- W_M is a finite group, $\bar{\mathtt{a}} = w_M$ and $w_M s = t w_M$.

In particular, the subsets I and J are conjugate under both $\bar{\mathtt{a}}$ and $\underline{\mathtt{a}}$. If $f \in R^J$, then $\underline{\mathtt{a}}(f) \in R^I$. We also have $\underline{\mathtt{a}} = \bar{\mathtt{a}} w_J$, so that $\partial_{\mathtt{a}}$ is the restriction of $\partial_{\underline{\mathtt{a}}}$.

When $I = J = \emptyset$, the atomic (I, J)-cosets have the form $\{s\}$ for $s \in S$. Atomic cosets in $\mathbf{S}_n$ can be described using cabled crossings. For example, the permutation

$\underline{\mathtt{a}} =$  $\cdots$,

or (678123459...) in one-line notation, is a prototypical example of $\underline{\mathtt{a}}$ for an atomic coset $\mathtt{a}$; in this example $W_I \cong \mathbf{S}_5 \times \mathbf{S}_3 \times \cdots$ and $W_J \cong \mathbf{S}_3 \times \mathbf{S}_5 \times \cdots$ and $W_M \cong \mathbf{S}_8 \times \cdots$.

1.3.

Our story began with the defining representation of $\mathbf{S}_n$ over $\mathbb{Q}$. Then we considered the polynomial ring R, its invariant subrings, and the structure derived therefrom. Alternatively, one can start with a *realization* of a Coxeter system (W, S), which is effectively a "reflection representation" of W over a commutative base ring $\Bbbk$, equipped with a choice of roots and coroots. From this representation we construct its polynomial ring R, Demazure operators, etcetera. See §3.1 for details.

For each realization[1], Soergel defined a monoidal category of R-bimodules called Soergel bimodules, and proved that for certain realizations[2] (that we call *SW-realizations*) one obtains a categorification of the Hecke algebra. Soergel bimodules and related categorifications of the Hecke algebra are objects of critical importance in geometric representation theory. The following is the first main result of this paper (Theorem 5.10).

Theorem A (The atomic Leibniz rule). *Let (W, S) be any Coxeter system equipped with an SW-realization, or the symmetric group $\mathbf{S}_n$ with $R = \mathbb{Z}[x_1, \dots, x_n]$. Let $\mathtt{a}$ be an atomic (I, J)-coset and let $\leq$ be the Bruhat order for double cosets[3]. For every $q < \mathtt{a}$, there is a unique $R^{I \cup J}$-linear operator on polynomials denoted $T_q^{\mathtt{a}}$, satisfying the equation*

$$\partial_{\mathtt{a}}(f \cdot g) = \underline{\mathtt{a}}(f)\, \partial_{\mathtt{a}}(g) + \sum_{q < \mathtt{a}} \partial_q(T_q^{\mathtt{a}}(f) \cdot g) \tag{1.3}$$

for all $f, g \in R^J$. Polynomials in the image of $T_q^{\mathtt{a}}$ are appropriately invariant, see Definition 3.8 for details. When $\mathtt{a}$ is fixed, we will often write $T_q := T_q^{\mathtt{a}}$.

The motivation for this result will take some setup, so please bear with us.

Example 1.1. We let e denote the identity element. Let s be a simple reflection and let $\mathtt{a}$ be the $(\emptyset, \emptyset)$-coset $\{s\}$. There is only one coset less than $\mathtt{a}$, namely $q = \{e\}$. We set $T_q^{\mathtt{a}}(f) = \partial_s(f)$. Note that $\partial_{\mathtt{a}} = \partial_s$ and $\partial_q = \mathrm{id}$. Then (1.3) becomes

$$\partial_s(fg) = s(f)\, \partial_s(g) + \partial_s(f)\, g,$$

which recovers the twisted Leibniz rule.

For more examples, see Section 2. There we also give an example of a non-atomic coset p for which no equality of the form (1.3) can hold (with p replacing $\mathtt{a}$).

Remark 1.2. Though we can prove the existence and uniqueness of such a formula, we do not have an explicit description of the operators T_q. We consider this an interesting open problem. A more accessible problem is to compute T_q on a (carefully chosen) set of generators of R^J, which we accomplish in type A in Theorem 6.3. This is useful computationally because it gives enough information to apply the atomic Leibniz rule for any pair of elements in R^J (see the discussion of multiplicativity in §1.6).

Remark 1.3. In §3.3, we prove that the atomic Leibniz rule for one realization implies the atomic Leibniz rule for related realizations (e.g., obtained via base change, enlargement, or quotient). The atomic Leibniz rule also depends only on the restriction of the realization to the finite parabolic subgroup W_M associated to $\mathtt{a}$. Thus Theorem A implies the atomic Leibniz rule in broad generality for realizations of Coxeter systems in both finite and affine type A, and in finite characteristic as well, see Example 3.19.

[1] Soergel's construction depends only on a reflection representation of W, and not a choice of roots and coroots. Elias and Williamson gave a presentation of Soergel's category in [14] which depends on a choice of realization. The Demazure operators also depend on the choice of roots and coroots.

[2] Specifically, the base ring $\Bbbk$ should be an infinite field of characteristic not equal to 2, and the representation should be (faithful and) reflection-faithful.

[3] The Bruhat order on (I, J)-cosets can be defined by $q \leq \mathtt{a}$ if and only if $\underline{q} \leq \underline{\mathtt{a}}$. See Theorem 2.16 in [8] for equivalent definitions. Note that q need not be atomic, so ∂_q need not equal (the restriction of) $\partial_{\underline{q}}$.

1.4.

Our second main result is a connection between the atomic Leibniz rule and the theory of singular Soergel bimodules [19]. Like Soergel bimodules, singular Soergel bimodules are ubiquitous in geometric representation theory, appearing, e.g., in the geometric Satake equivalence and other situations where partial flag varieties play a role. Specifically, we connect the atomic Leibniz rule to a property called polynomial forcing, whose motivation we postpone a little longer.

Let us first be more precise. Singular Soergel bimodules are graded bimodules over graded rings, but we ignore all gradings in this paper. To an atomic coset $\mathtt{a}$ as above, one can associate the (R^I, R^J)-bimodule $B_{\mathtt{a}} := R^I \otimes_{R^M} R^J$. This is an indecomposable bimodule; there are also indecomposable (R^I, R^J)-bimodules B_q associated to any (I, J)-coset q, which we will not try to describe here. Then $B_{\mathtt{a}}$ has a submodule called the *submodule of lower terms*, spanned by the images of all bimodule endomorphisms of $B_{\mathtt{a}}$ which factor through B_q for $q < \mathtt{a}$.

Atomic polynomial forcing is the statement that $1 \otimes f$ and $\underline{\mathtt{a}}(f) \otimes 1$ are equal modulo lower terms in $B_{\mathtt{a}}$, for any $f \in R^J$. Williamson has proven for SW-realizations (see Theorem 6.4 in [19]) that singular Soergel bimodules have "standard filtrations", which implies atomic polynomial forcing for SW-realizations. *Polynomial forcing* (see Definition 5.7) is a generalization of atomic polynomial forcing for a bimodule B_p associated with an arbitrary double coset p. Using [8, 9], we prove Theorem 5.13, which says that polynomial forcing for any double coset is a consequence of atomic polynomial forcing.

The restrictions imposed on SW-realizations are significant as they rule out some examples of great importance to modular representation theory, e.g., affine Weyl groups in finite characteristic. An *almost SW-realization* (Definition 4.21) is a realization over a domain $\Bbbk$ such that one obtains an SW-realization after base change to the fraction field of $\Bbbk$. An example is $\mathbb{Z}[x_1, \ldots, x_n]$ with its action of $\mathbf{S}_n$. The second main result of this paper is the following equivalence (Theorem 5.5).

Theorem B. *Let (W, S) be a Coxeter system equipped with an almost SW-realization. Then the atomic Leibniz rule is equivalent to atomic polynomial forcing.*

We deduce the SW-realization case of Theorem A from Theorem B and Williamson's theory of standard filtrations.

1.5.

Let us now explain the motivation for our results.

The diagrammatic presentation for the category of Soergel bimodules, developed by Elias, Khovanov, Libedinsky, and Williamson [4, 5, 14, 16], has proven to be an important tool for both abstract and computational reasons. This diagrammatically constructed category is typically called the *Hecke category*, and is equivalent to the category of Soergel bimodules for SW-realizations. For non-SW-realizations the category of Soergel bimodules need not behave well, but the diagrammatic category does behave well, and continues to provide a categorification of the Hecke algebra. The ability to compute within the Hecke category has led to advances such as [2, 13, 15, 17, 18, 20].

The authors are in the midst of a concerted effort to define the *singular Hecke category*, a diagrammatic presentation of singular Soergel bimodules. The framework for such a presentation was developed in [12], but this framework lacks some of the relations needed. In [10] we described what should be the basis for morphisms in the diagrammatic category, called the *double leaves basis*, and proved that it descends to a basis for morphisms between singular Soergel bimodules for SW-realizations.

Once the remaining relations are understood, proving that the diagrammatic category is correctly presented reduces to proving that double leaves span. An important part of that proof will be *diagrammatic polynomial forcing*, which is like polynomial forcing except that the submodule of lower terms is replaced by the span of double leaves associated to smaller elements in the Bruhat order. We abbreviate diagrammatic polynomial forcing to DL-*forcing*. It is DL-forcing which is our true goal in this paper.

For an atomic coset $\mathtt{a}$, one can identify $\mathrm{End}(B_{\mathtt{a}})$ with $B_{\mathtt{a}}$ as bimodules. In §4.2.3, we explicitly compute the submodule $\mathrm{DL}_{<\mathtt{a}} \subset B_{\mathtt{a}}$ corresponding to the $\mathbb{k}$-linear subspace of $\mathrm{End}(B_{\mathtt{a}})$ spanned by double leaves factoring through $q < \mathtt{a}$. The following theorem (Corollary 4.33 and Theorem 5.5) is proven by direct computation and is the reason we discovered the atomic Leibniz rule in the first place.

Theorem C. *Let (W, S) be a Coxeter system equipped with an almost SW-realization. Then the atomic Leibniz rule is equivalent to atomic* DL-*forcing.*

In other words, for a given $f \in R^J$, we prove that $1 \otimes f - \underline{\mathtt{a}}(f) \otimes 1 \in \mathrm{DL}_{<\mathtt{a}}$ if and only if the atomic Leibniz rule holds for f and any $g \in R^J$.

For an SW-realization, the results of [10] can be used to prove that $\mathrm{DL}_{<\mathtt{a}}$ agrees precisely with the submodule of lower terms (see Corollary 4.19). In §4.4, we use some novel localization tricks to extend this statement to almost-SW-realizations. Therefore, DL-forcing agrees with polynomial forcing for almost-SW-realizations, thus Theorem C implies Theorem B.

1.6.

One is still motivated to prove DL-forcing (or equivalently, the atomic Leibniz rule) for almost-SW-realizations and more general realizations. We now discuss how this process is simplified by Theorem B.

Let us say the atomic Leibniz rule *holds for* $f \in R^J$ if (1.3) holds for that specific f and all $g \in R^J$. It is obvious that if the rule holds for f_1 and f_2 then it holds for $f_1 + f_2$. It is not obvious that if it holds for f_1 and f_2 then it holds for $f_1 \cdot f_2$, thus the formula (1.3) is not obviously *multiplicative* in f. However, atomic polynomial forcing is obviously multiplicative in f. Our equivalence in Theorem B is proven on an element-by-element basis, from which we conclude that the atomic Leibniz rule is actually multiplicative.

As a consequence, the atomic Leibniz rule can be proven without relying on Williamson's results, by checking it on each generator f of R^J. We perform this computation in type A in Section 6, for carefully chosen generators f, via a direct and elementary proof. The operators T_q are simplified dramatically when applied to these generators.

On the one hand, this proves Theorem A for the symmetric group over $\mathbb{Z}[x_1, \ldots, x_n]$, and consequently for other realizations (see Remark 1.3). Via Theorem C, this implies DL-forcing in the same generality. On the other hand, it also gives a computationally effective way to apply the atomic Leibniz rule (or polynomial forcing), even if one does not know the operators T_q in general: decompose f as a linear combination of products of generators, and apply the atomic Leibniz rule for generators one term at a time.

2. Examples and remarks

In this section, we give some examples of atomic Leibniz rules, in the relatively small Coxeter group $W = \mathbf{S}_4$. The reader is not expected, and in fact discouraged, from attempting to check these examples immediately. In a later section, we introduce methods that will considerably simplify such computations. Rather, we present these examples only to illustrate the level of complexity that arises even in small cases.

The reader will not miss out on anything important if they skip directly to Section 3.

2.1. Examples

Our examples take place in $W = \mathbf{S}_4$. We let $s = (12)$, $t = (23)$, and $u = (34)$ denote the simple reflections. We give the examples without justification first, and then discuss the verification afterwards. We require that $T_q(f) \in R^K$ for some $K \subset S$ depending on q (precisely: $K = \underline{q}^{-1} I \underline{q} \cap J$).

Notation 2.1. Let $I \subset S$ and $s \in S$. Whenever $s \notin I$, we write Is for the disjoint union $I \cup \{s\}$. When the meaning is unambiguous, we denote subsets of S by juxtaposition: for example, su stands for $\{s,u\}$ and sut for $\{s,u,t\}$. Accordingly, we write $R^{st} := R^{\{s,t\}}$, and by an (su, st)-coset we mean an $(\{s,u\},\{s,t\})$-coset.

Example 2.2. There are three (su, su)-cosets in $\mathbf{S}_4$: the maximal one p which is atomic, the submaximal coset q with $\underline{q} = t$, and the minimal coset r containing the identity. We claim that for $f, g \in R^{su}$, we have

$$\partial_{tsut}(fg) = tsut(f) \cdot \partial_{tsut}(g) + \partial_{sut}(T_q(f) \cdot g) + T_r(f) \cdot g, \tag{2.1}$$

or in other words,

$$\partial_p(fg) = \underline{p}(f) \cdot \partial_p(g) + \partial_q(T_q(f) \cdot g) + \partial_r(T_r(f) \cdot g),$$

where

$$T_q(f) = su\partial_t(f) \quad \text{and} \quad T_r(f) = \partial_{tsut}(f) - \partial_{sut}(T_q(f)).$$

Note that (obviously) $T_q(f) \in R = R^{\emptyset}$ while (less obviously) $T_r(f) \in R^{su}$. These are the invariance requirements.

Remark 2.3. Iterating the ordinary twisted Leibniz rule, it is not hard to deduce the existence of a formula of the form

$$\partial_{tsut}(fg) = tsut(f) \cdot \partial_{tsut}(g) + \sum_{x < tsut} \partial_x(T_x(f)g)$$

for some operators $T_x \colon R \to R$. This equality is generalized in Lemma 2.7 below.

What is not obvious is that, when $f, g \in R^{su}$, this formula will simplify so that only the terms where $x = sut$ and $x = 1$ survive.

Example 2.4. There are two (tu, st)-cosets in $\mathbf{S}_4$: the maximal coset p which is atomic, and the minimal coset q containing the identity element. We claim that for $f, g \in R^{st}$, we have

$$\partial_{stu}(fg) = stu(f)\partial_{stu}(g) + \partial_{tu}(T_q(f) \cdot g), \tag{2.2}$$

or in other words,

$$\partial_p(fg) = \underline{p}(f)\partial_p(g) + \partial_q(T_q(f) \cdot g),$$

where

$$T_q(f) = st\partial_u(f).$$

Note also that $T_q(f) \in R^t$.

It helps to look at an example of a non-atomic coset, to see that Leibniz rules are not guaranteed.

Example 2.5. We return to the notation of Example 2.2. Note that r is the only coset less than q, and ∂_r is the identity map. Let us argue that there is no naive analogue of (1.3) for ∂_q. If there were, it would have to have the form

$$\partial_{sut}(f \cdot g) = sut(f)\partial_{sut}(g) + T(f) \cdot g \tag{2.3}$$

for some operator T. By evaluating at $g = 1$ we have $T(f) = \partial_{sut}(f)$.

Note that $f, g \in R^{su}$. Iterating the twisted Leibniz rule (cf. Lemma 2.7), we have

$$\begin{aligned}\partial_{sut}(f \cdot g) = sut(f)\partial_{sut}(g) + \partial_s(ut(f))\partial_{ut}(g) + \partial_u(st(f))\partial_{st}(g) \\ + \partial_{su}(t(f))\partial_t(g) + \partial_{sut}(f)g.\end{aligned} \tag{2.4}$$

Only two of these five terms are in (2.3). If g is linear and $\partial_t(g) = 1$, then the sum of the three missing terms is nonzero for some f. Thus (2.3) is false.

However, now suppose that $f \in t(R^{su})$ instead of R^{su}. Then many terms in (2.4) vanish, yielding

$$\partial_{sut}(f \cdot g) = sut(f)\partial_{sut}(g) + \partial_{sut}(f)g.$$

This is compatible with (2.3), with $T = \partial_{sut}$.

In view of this example, there is hope of finding some generalization of the atomic Leibniz rule to some non-atomic (I, J) cosets q, letting $f \in \underline{q}^{-1}(R^I)$. Be warned that the coset q considered in this example has various special properties, see Remark 3.12.

The examples above can easily be verified by computer (all the equations are R^{stu}-linear, so one need only check the result for f and g in a basis). They can also all be verified using (1.1) and (1.2), but this is a very tricky exercise. Doing this exercise may be very instructive for the reader, and emphasizes the difficult and subtlety in these formulae, so we encourage it, and provide some helping hands.

We begin with a few helpful general properties of Demazure operators, which hold in general when $m_{st} = 3$ and $m_{su} = 2$.

Lemma 2.6. *We have*

$$\partial_s(s(f)) = -\partial_s(f), \quad s\partial_s(f) = \partial_s(f), \quad \partial_s(\partial_s(f)) = 0, \tag{2.5}$$

$$st\partial_s(f) = \partial_t(stf), \quad s\partial_t(sf) = t\partial_s(tf), \quad s\partial_u(f) = \partial_u(sf), \tag{2.6}$$

$$\alpha_s\partial_s(f) = f - sf, \tag{2.7}$$

$$\partial_{st}(sf) + \partial_{ts}(f) = t\partial_{st}(f). \tag{2.8}$$

Note that $\alpha_{s_i} = x_i - x_{i+1}$ above.

Proof. The reader can verify these relations directly from (1.1). ■

Most of these formulae are well known. Meanwhile, we have not seen (2.8) before; we only use it in Example 2.10 below. With these relations in hand, one need not refer to the original definition (1.1) again, and need only use the twisted Leibniz rule.

Here are some example computations using (2.5) and (2.6):

$$\partial_s(t\partial_s(f)) = -\partial_s(st\partial_s(f)) = -\partial_s\partial_t(stf) = \partial_s\partial_t(tstf),$$
$$\partial_s(t\partial_s(f)) = \partial_s(ts\partial_s(f)) = ts\partial_t\partial_s(f).$$

Let us consider Example 2.4. It helps to observe that

$$\partial_{st}(u(f)) \in R^{st} \quad \text{and} \quad \partial_{stu}(f) \in R^{stu},$$

under the assumption that $f \in R^{st}$. One way to see the equality (2.2) is to expand both sides using the twisted Leibniz rule. Since $g \in R^{st}$, any terms containing $\partial_{st}(g)$ or $\partial_s(g)$ or $\partial_t(g)$ or $\partial_{su}(g)$ are zero. Now compare the coefficients of g, $\partial_u(g)$ and $\partial_{tu}(g)$ respectively. On the left, the coefficient of g is $\partial_{stu}(f)$, while on the right, the coefficient of g is

$$\partial_{tu}st\,\partial_u(f) = \partial_t s\,\partial_u tu\,\partial_u(f) = \partial_t stu\,\partial_{tu}(f) = st\,\partial_s u\,\partial_{tu}(f)$$
$$= stu\,\partial_{stu}(f) = \partial_{stu}(f).$$

We have applied (2.6) repeatedly and used that $\partial_{stu}(f) \in R^{stu}$. Thus the coefficients of g are the same on both sides of (2.2). We leave the other coefficients to the reader.

Example 2.2 is the hardest. One should first prove the following statements under the critical assumption that $f, g \in R^{su}$:

$$\partial_{tsut}(f) \in R^{stu}, \quad \partial_{ts}(ut(f)) \in R^{st},$$
$$\partial_{su}(tsu(\partial_t(f))) = tsut(\partial_{sut}(f)),$$
$$t(\partial_{sut}(f)) = tsu(\partial_{sut}(f)),$$
$$tsu[\partial_{sut}(f) - t\partial_{sut}(f)] = tsu(\alpha_t\partial_{tust}(f)) = (\alpha_s + \alpha_t + \alpha_u)\partial_{tsut}(f),$$
$$\partial_{tsu}(tf) = \partial_{tsu}(-\alpha_t\partial_t(f)) = -(\alpha_s + \alpha_t + \alpha_u)\partial_{tsut}(f).$$

We leave these verifications, and the deduction of (2.1) therefrom, to the ambitious reader.

2.2. Leibniz rules for permutations

A natural question is raised: for which double cosets p does one expect an equality of the form (1.3) to hold? Given the discussion in Example 2.5, one might ask instead: for which double cosets q does (1.3) hold under the alternate assumption that $f \in \underline{q}^{-1}(R^I)$? Note that $f \in \underline{q}^{-1}(R^I)$ is equivalent to $f \in R^J$ for atomic cosets, and also for more general cosets called *core cosets*. We believe these are interesting questions, even though these more general Leibniz rules currently lack a clear connection to the theory of singular Soergel bimodules.

A special case would be when $I = J = \emptyset$, so that double cosets are in bijection with elements $w \in W$; all such cosets are core. We start with the following well-known lemma.

Lemma 2.7. *Let $w \in W$ and $f, g \in R$. Let $w = s_1 \dots s_n$ be a reduced expression, and for $\mathbf{e} = \{0,1\}^n$, we define the element $w^{\mathbf{e}} = s_1^{e_1} \dots s_n^{e_n}$. For $\mathbf{e} = \{0,1\}^n$, let*

$$\theta_i^{\mathbf{e}} = \begin{cases} s_i & \text{if } e_i = 1, \\ \partial_i & \text{if } e_i = 0, \end{cases} \quad \text{and} \quad \Theta^{\mathbf{e}}(f) = \theta_1^{\mathbf{e}} \circ \theta_2^{\mathbf{e}} \circ \dots \circ \theta_n^{\mathbf{e}}(f).$$

Then we have

$$\partial_w(fg) = \sum_{x \le w} T'_x(f)\,\partial_x(g), \quad \text{where} \quad T'_x(f) = \sum_{\mathbf{e} \mid w^{\mathbf{e}} = x} \Theta^{\mathbf{e}}(f).$$

As a special case, $T'_w(f) = w(f)$.

Proof. This is just an iteration of the twisted Leibniz rule. ∎

To summarize, we obtain a generalized Leibniz rule of the form

$$\partial_w(fg) = w(f)\,\partial_w(g) + \sum_{x < w} T'_x(f)\,\partial_x(g), \tag{2.9}$$

for operators $T'_x : R \to R$ (depending on w). The *formula* for T'_x one derives in this way is seemingly dependent on the choice of reduced expression for w, though the *operator* only depends[4] on w. In practice, confirming the independence of reduced expression can be quite subtle. We are unaware of a formula for T'_x which is obviously independent of the choice of reduced expression.

Meanwhile, one can also deduce an equality of the form

$$\partial_w(fg) = w(f)\,\partial_w(g) + \sum_{x < w} \partial_x(T_x(f) \cdot g), \tag{2.10}$$

for some operators $T_x : R \to R$. We are unaware of any previous study of the operators T_x and the formula (2.10).

[4]Abstractly, the nilHecke algebra is the subalgebra of $\mathrm{End}(R)$ generated by R (i.e., multiplication by polynomials) and by Demazure operators. It is well known that the operators $\{\partial_w\}$ form a basis of the nilHecke algebra as a free left R-module. Letting m_f denote multiplication by f, (2.9) can be viewed as an equality

$$\partial_w \circ m_f = m_{w(f)} \circ \partial_w + \sum m_{T'_x(f)} \circ \partial_x$$

in the nilHecke algebra, which rewrites $\partial_w \circ m_f$ in this basis. Consequently, the coefficients $T'_x(f)$ in this linear combination depend only on w and f.

Remark 2.8. Later in the paper, we also discuss an atomic Leibniz rule similar to (2.9) rather than (2.10).

Here are some examples.

Example 2.9. Let $s = s_1$ and $t = s_2$. When $w = ts$, applying (1.2) twice gives

$$\partial_{ts}(fg) = ts(f)\partial_{ts}(g) + \partial_t(sf)\partial_s(g) + t\partial_s(f)\partial_t(g) + \partial_{ts}(f)g. \tag{2.11}$$

An equivalent formula is

$$\partial_{ts}(fg) = ts(f)\partial_{ts}(g) + \partial_t(\partial_s(f)\cdot g) + \partial_s(s\partial_t(sf)\cdot g) + \partial_{st}(sf)\cdot g. \tag{2.12}$$

By applying (1.2) to the second and third terms on the right-hand side of equation (2.12), and with a little help from (2.5), one obtains (2.11).

Example 2.10. With notation as above, when $w = sts$, applying (1.2) thrice gives

$$\begin{aligned}\partial_{sts}(fg) = {} & sts(f)\partial_{sts}(g) + \partial_s(tsf)\partial_{ts}(g) + \partial_t(stf)\partial_{st}(g) \\ & + \partial_s(t\partial_s(f))\partial_t(g) + \partial_t(s\partial_t(f))\partial_s(g) + \partial_{sts}(f)g.\end{aligned} \tag{2.13}$$

More honestly, applying (1.2) thrice gives the above except that the coefficient of $\partial_s(g)$ is $\partial_{st}(sf) + s\partial_{ts}(f)$. By applying s to (2.8), one obtains

$$\partial_{st}(sf) + s\partial_{ts}(f) = st\partial_{st}(f) = \partial_t(st\partial_t(f)) = \partial_t(s\partial_t(f)).$$

This is how one deduces (2.13).

An equivalent formula is

$$\begin{aligned}\partial_{sts}(fg) = {} & sts(f)\partial_{sts}(g) + \partial_{ts}(\partial_t(f)\cdot g) + \partial_{st}(\partial_s(f)\cdot g) \\ & - \partial_t(\partial_{st}(f)\cdot g) - \partial_s(\partial_{ts}(f)\cdot g) + \partial_{sts}(f)\cdot g.\end{aligned} \tag{2.14}$$

To verify that (2.14) and (2.13) agree, apply first (2.12) to the term $\partial_{ts}(\partial_t(f)\cdot g)$, and then apply (2.12) with s and t swapped to $\partial_{st}(\partial_s(f)\cdot g)$. After some additional massaging using (2.5) and (2.6), one will recover (2.13).

3. Atomic Leibniz rules

3.1. Realizations

We fix a Coxeter system (W, S). We let e denote the identity element of W. Recall the definition of a realization from Section 3.1 of [14].

Definition 3.1. A *realization* of (W, S) over $\Bbbk$ is the data $(\Bbbk, V, \Delta, \Delta^\vee)$ of a commutative ring $\Bbbk$, a free finite-rank $\Bbbk$-module V, a set $\Delta = \{\alpha_s\}_{s\in S}$ of *simple roots* inside V, and a set $\Delta^\vee = \{\alpha_s^\vee\}_{s\in S}$ of *simple coroots* inside $\mathrm{Hom}_{\Bbbk}(V, \Bbbk)$, satisfying the following properties. One has $\alpha_s^\vee(\alpha_s) = 2$ for all $s \in S$. The formula

$$s(v) := v - \alpha_s^\vee(v)\cdot\alpha_s$$

defines an action of W on V. Also, the technical condition (3.3) in [14] holds, which is redundant for most base rings $\Bbbk$.

For short, we often refer to the data of a realization simply by reference to the W-representation V.

Example 3.2. The *permutation realization* of $\mathbf{S}_n$ over $\mathbb{Z}$ has $V = \mathbb{Z}^n$ with basis $\{x_i\}_{i=1}^n$ and with dual bases $\{x_i^*\}$. For $1 \le i \le n-1$, one sets $\alpha_i = x_i - x_{i+1}$ and $\alpha_i^\vee = x_i^* - x_{i+1}^*$.

Example 3.3. For a Weyl group W, the *root realization* of (W, S) over $\mathbb{Z}$ is the free $\mathbb{Z}$-module with basis Δ. One defines $\Delta^\vee$ so that the pairings $\alpha_s^\vee(\alpha_t)$ agree with the usual Cartan matrix of W.

Example 3.4. Let $(\Bbbk, V, \Delta, \Delta^\vee)$ be a realization of (W, S), and $I \subset S$. Then $(\Bbbk, V, \Delta_I, \Delta_I^\vee)$ is also a realization of (W_I, I), the *restriction of the realization to a parabolic subgroup*. Here $\Delta_I = \{\alpha_s\}_{s \in I}$, and similarly for $\Delta_I^\vee$.

Given a realization, let R be the polynomial ring whose linear terms are V. We can associate Demazure operators $\partial_s \colon R \to R$ for $s \in S$, which agree with $\alpha_s^\vee$ on $V \subset R$, and are extended by the twisted Leibniz rule. For details, see Section 3.1 of [14]. For each finitary subset $I \subset S$, we also consider the subring R^I of W_I-invariants in R. The ring R^I is graded, and all R^I-modules will be graded, but we will not keep track of grading shifts in this paper as they will play no significant role. The background on this material in [7] should be sufficient.

Definition 3.5. A (*balanced*) *Frobenius realization* is a realization satisfying the following properties, see Section 3.1 of [7] for definitions:

- it is balanced;
- it satisfies generalized Demazure surjectivity;
- it is faithful when restricted to each finite parabolic subgroup W_I.

We assume tacitly throughout this paper that we work with a Frobenius realization. The main implication of these assumptions is that, when $I \subset S$ is finitary, the Demazure operator $\partial_{w_I} \colon R \to R^I$ is well defined and equips the ring extension $R^I \subset R$ with the structure of a Frobenius extension.

The left and right redundancy sets of an (I, J)-coset p are defined and denoted as

$$\mathrm{LR}(p) = I \cap \underline{p} J \underline{p}^{-1} \quad \text{and} \quad \mathrm{RR}(p) = \underline{p}^{-1} I \underline{p} \cap J.$$

An (I, J)-coset p is a *core coset* if $I = \mathrm{LR}(p)$ and $J = \mathrm{RR}(p)$.

For any (I, J)-coset q, in [7] we define a Demazure operator

$$\partial_q : R^J \to R^I.$$

By definition, ∂_q is the restriction of the ordinary Demazure operator

$$\partial_{\bar{q} w_J^{-1}} : R \to R$$

to the subring R^J. After restriction, the image is contained in R^I, see Lemma 3.9 in [7]. Note that $\bar{q} w_J^{-1} = \underline{q}$ if and only if q is a core coset.

As in [6], for $x, y \in W$, we write $x.y$ for a reduced composition, where $\ell(xy) = \ell(x) + \ell(y)$. We also use this notation for the reduced composition of reduced expressions, or the reduced composition of double cosets, see [6] for more details. Demazure operators compose well over reduced compositions: one has $\partial_{p.q} = \partial_p \circ \partial_q$, as proven in Corollary 3.19 of [7].

Remark 3.6. The element w_J is an involution, so $w_J = w_J^{-1}$. We write $\bar{q}w_J^{-1}$ above to emphasize that $\bar{q} = (\bar{q}w_J^{-1}).w_J$.

Remark 3.7. Some results from [7] require further that the realization is faithful, rather than just faithful upon restriction to each finite parabolic subgroup. In particular, the set $\{\partial_p\}$ as p ranges over (I, J)-cosets need not be linearly independent when the realization is not faithful.

A *multistep* (I, J)*-expression* is a sequence of finitary subsets

$$I_\bullet = [[I = I_0 \subset K_1 \supset I_1 \subset \cdots \subset K_m \supset I_m = J]].$$

The definition of a *reduced* multistep expression, and of the (I, J)-coset that it *expresses*, can be found in Definition 1.4 of [6]. When $I_\bullet$ is a (reduced) expression which expresses p, we write $p \leftrightharpoons I_\bullet$.

3.2. Precise statement of atomic Leibniz rules

Definition 3.8. Suppose M is finitary, $s \in M$, and $t = w_M s w_M$. Let $I = \hat{s} := M \setminus s$, and $J = \hat{t} := M \setminus t$. Let $\mathtt{a}$ be the (atomic) (I, J)-coset containing w_M. We say a (*rightward*) *atomic Leibniz rule* holds for $\mathtt{a}$ if there exist R^M-linear operators $T_q^{\mathtt{a}}$ from R^J to $R^{\mathrm{RR}(q)}$ for each (I, J)-coset $q < \mathtt{a}$, such that for any $f, g \in R^J$, we have

$$\partial_{\mathtt{a}}(f \cdot g) = \underline{\mathtt{a}}(f)\,\partial_{\mathtt{a}}(g) + \sum_{q<\mathtt{a}} \partial_{\bar{q}w_J^{-1}}(T_q^{\mathtt{a}}(f) \cdot g). \tag{3.1}$$

We encourage the reader to confirm that $T_q^{\mathtt{a}}(f) \in R^{\mathrm{RR}(q)}$ in the examples of Section 2. We continue to write T_q instead of $T_q^{\mathtt{a}}$ when $\mathtt{a}$ is understood.

Remark 3.9. We say “an atomic Leibniz rule” rather than “the atomic Leibniz rule” because we are defining a prototype for a kind of formula. If one specifies operators T_q such that the formula holds, then one has produced “the” atomic Leibniz rule for that coset $\mathtt{a}$ (indeed, we prove in Theorem 5.5 that such operators are unique for certain realizations).

The difference between (3.1) and (1.3) is subtle: we have written $\partial_{\bar{q}w_J^{-1}}$ instead of ∂_q. The difference between $\partial_{\bar{q}w_J^{-1}}$ and ∂_q is only a matter of the domain and codomain of the functions: the former is a function $R \to R$, while the latter is its restriction to a function $R^J \to R^I$. Meanwhile, $T_q(f)$ lives in $R^{\mathrm{RR}(q)}$. The inclusion $R^J \subset R^{\mathrm{RR}(q)}$ is proper unless q is core. It is therefore inappropriate to apply ∂_q to $T_q(f) \cdot g$. Having altered notation so that the domain of the operator is appropriate, we still need to worry about the codomain, which we address in the following lemma.

Lemma 3.10. *With notation as in Definition 3.8, we have* $\partial_{\bar{q}w_J^{-1}}(T_q(f) \cdot g) \in R^I$.

Proof. Recall from Proposition 4.28 in [6] that any (I, J)-coset q has a reduced expression of the form

$$q \leftrightharpoons [[I \supset \mathrm{LR}(q)]].q^{\mathrm{core}}.[[\mathrm{RR}(q) \subset J]]. \tag{3.2}$$

Let z be the $(I, \mathrm{RR}(q))$-coset with reduced expression

$$z \leftrightharpoons [[I \supset \mathrm{LR}(q)]].q^{\mathrm{core}}. \tag{3.3}$$

Since (3.2) is reduced, by Proposition 4.3 in [6], we have $\bar{q} = \bar{z}.(w_{\mathrm{RR}(q)}^{-1} w_J)$, so that

$$\bar{q} w_J^{-1} = \bar{z} w_{\mathrm{RR}(q)}^{-1}.$$

Consequently, the same operator $\partial_{\bar{q} w_J^{-1}} \colon R \to R$ restricts to both $\partial_q \colon R^J \to R^I$ and $\partial_z \colon R^{\mathrm{RR}(q)} \to R^I$. In particular, this operator sends $R^{\mathrm{RR}(q)}$ to R^I. ∎

Further elaboration will be helpful in subsequent chapters. By Lemma 3.17 in [7], the reduced expression (3.2) implies that the map ∂_q is a composition of three Demazure operators. Recall that $\partial_{[[I \supset \mathrm{LR}(q)]]}$ is the Frobenius trace map $R^{\mathrm{LR}(q)} \to R^I$, often denoted as $\partial_I^{\mathrm{LR}(q)}$. Recall also that $\partial_{[[\mathrm{RR}(q) \subset J]]}$ is the inclusion map $R^J \subset R^{\mathrm{RR}(q)}$. We denote this inclusion map $\iota_J^{\mathrm{RR}(q)}$ below. So we have

$$\partial_q = \partial_I^{\mathrm{LR}(q)} \circ \partial_{q^{\mathrm{core}}} \circ \iota_J^{\mathrm{RR}(q)}. \tag{3.4}$$

By (3.3), we have

$$\partial_z = \partial_I^{\mathrm{LR}(q)} \circ \partial_{q^{\mathrm{core}}},$$

which agrees with the restriction of $\partial_{\bar{q} w_J^{-1}}$ to $R^{\mathrm{RR}(q)}$. Thus one has the following reformulation of (3.1):

$$\partial_{\mathtt{a}}(f \cdot g) = \underline{\mathtt{a}}(f)\, \partial_{\mathtt{a}}(g) + \sum_{q < \mathtt{a}} \partial_I^{\mathrm{LR}(q)}\, \partial_{q^{\mathrm{core}}}(T_q(f) \cdot \iota_J^{\mathrm{RR}(q)}(g)).$$

Now the polynomial $T_q(f)$ appears more appropriately in the "middle" of this factorization of ∂_q. This discussion of the "placement" of the polynomial $T_q(f)$ will play a role in our diagrammatic proof of polynomial forcing.

We are now prepared to discuss another version of the atomic Leibniz rule, using the factorization (3.4). It should not be obvious that these two atomic Leibniz rules are related, though the equivalence with polynomial forcing will shed light on this issue.

Definition 3.11. Use the notation from Definition 3.8 and from (3.4). We say that a *leftward atomic Leibniz rule* holds for $\mathtt{a}$ if there exist R^M-linear operators T'_q from R^J to $R^{\mathrm{LR}(q)}$ for each (I, J)-coset $q < \mathtt{a}$, such that for any $f, g \in R^J$, we have

$$\partial_{\mathtt{a}}(f \cdot g) = \underline{\mathtt{a}}(f)\, \partial_{\mathtt{a}}(g) + \sum_{q < \mathtt{a}} \partial_I^{\mathrm{LR}(q)}(T'_q(f) \cdot \partial_{q^{\mathrm{core}}}(\iota_J^{\mathrm{RR}(q)}(g))).$$

Remark 3.12. The fact that $T_q(f)$ lives in $R^{\mathrm{RR}(q)}$ and not in R^J is easy to overlook, but overlooking it is dangerous. We have attempted to prove atomic Leibniz-style rules for more general families of cosets (core cosets, cosets whose core is atomic, etcetera). Each time what prevents one from bootstrapping from the atomic case to more general cases is the fact that $T_q(f)$ does not live in R^J. The generalization in Example 2.5 has the special feature that the lower cosets are all core, so that their right redundancy equals J. (It also has the special feature that q^{core} is atomic.)

3.3. Changing the realization

We argue that the atomic Leibniz rule for some realizations implies the atomic Leibniz rule for others. Given a realization, one can obtain another realization by applying base change $(-) \otimes_{\Bbbk} \Bbbk'$ to V, and choosing new roots and coroots in the natural way. We call this a *specialization*. Here are two other common ways to alter the realization.

Definition 3.13. Let $(V, \Delta, \Delta^\vee)$ be a realization of (W, S) over $\Bbbk$. Let N be a free $\Bbbk$-module acted on trivially by W. Then $(V \oplus N, \Delta \oplus 0, \Delta^\vee \oplus 0)$ is a realization, called a *W-invariant enlargement* of the original. More precisely, the new roots are the image of the old roots under the inclusion map, and the new coroots kill the summand N.

Definition 3.14. Let $(V, \Delta, \Delta^\vee)$ be a realization of (W, S) over $\Bbbk$. Suppose one has a decomposition $V = X \oplus Y$ of free $\Bbbk$-modules, such that W acts trivially on X, though W need not preserve Y. Note that the coroots necessarily annihilate X. Then $(Y, \overline{\Delta}, \Delta_Y^\vee)$ is a realization, called a *W-invariant quotient* of the original. Here, we identify Y as the quotient V/X, and $\overline{\Delta}$ represents the image of Δ under the quotient map. The functionals $\Delta^\vee$ kill X, so they descend to functionals $\Delta_Y^\vee$ on Y. We also make the technical assumption[5] that α_s induces a surjective map $Y^* \to \Bbbk$.

Example 3.15. Let (W, S) have type $\widetilde{A}_{n-1}$, with simple reflections s_i for $1 \le i \le n$. Let V be the free $\mathbb{Z}$-module spanned by $\{x_i\}_{i=1}^n$ and δ. Let $\{x_i^*\} \cup \{\delta^*\}$ denote the dual basis in $\mathrm{Hom}_{\Bbbk}(V, \Bbbk)$. With indices considered modulo n, let $\alpha_i = x_i - x_{i+1} + \delta$, and $\alpha_i^\vee = x_i^* - x_{i+1}^*$. This is a realization of (W, S) called the *affine permutation realization*. Note that $\sum_{i=0}^{n-1} \alpha_i = n\delta$, which is W-invariant. Let X be the span of δ, and Y be the span of $\{x_i\}_{i=1}^n$. Note that W does not preserve Y, since the roots are not contained in Y. There is a valid W-invariant quotient $(Y, \overline{\Delta}, \Delta_Y^\vee)$ which agrees, upon restriction to the parabolic subgroup $\mathbf{S}_n$ generated by $\{s_i\}_{i=1}^{n-1}$, with the permutation representation.

Example 3.16. Continuing the previous example, let $y_i = x_i - i\delta$. Then we can also view V as having basis $\{y_i\}_{i=1}^n \cup \{\delta\}$, and $\alpha_i = y_i - y_{i+1}$ for $i \ne n$. Upon restriction to the parabolic subgroup $\mathbf{S}_n$, we see that V is isomorphic to the W-invariant enlargement of the permutation representation of $\mathbf{S}_n$ (with basis $\{y_i\}$) by the W-invariant span of δ.

Indeed, W has n distinct maximal parabolic subgroups isomorphic to $\mathbf{S}_n$ as groups. A similar construction will show that the restriction of V to any maximal parabolic subgroup (a copy of $\mathbf{S}_n$) will be isomorphic to an invariant enlargement of its permutation representation.

Lemma 3.17. *If a rightward (respectively, leftward) atomic Leibniz rule holds for a Frobenius realization, then it also holds for specializations, W-invariant enlargements, and W-invariant quotients.*

Proof. Let R be the ring associated to the original realization, and R_{new} be the realization associated to the specialization, enlargement, or quotient. All three cases are united by the fact that R_{new} is a tensor product of the form $R \otimes_A B$, where $A \subset R$ is a subring on which W acts trivially, and B is a ring on which W acts trivially. For specializations,

[5]This assumption is required for the W-invariant quotient to satisfy Demazure surjectivity.

we have $R_{\mathrm{new}} = R \otimes_{\Bbbk} \Bbbk'$; for enlargements, we have $R_{\mathrm{new}} = R \otimes_{\Bbbk} R_N$, where R_N is the polynomial ring of N; for quotients, we have $R_{\mathrm{new}} = R \otimes_{R_X} \Bbbk$, where R_X is the polynomial ring of X, and $\Bbbk$ is its quotient by the ideal of positive degree elements. For $w \in W$, its action on R_{new} is given by $w \otimes \mathrm{id}$. The roots in R_{new} are given by $\alpha_s \otimes 1$, and the Demazure operators $\partial_s^{\mathrm{new}}$ on R_{new} have the form $\partial_s \otimes \mathrm{id}$.

The important point in all three cases is that for each $I \subset S$ finitary, we have

$$R^I_{\mathrm{new}} = R^I \otimes_A B.$$

We now prove this somewhat subtle point. There is an obvious inclusion $R^I \otimes_A B \subset R^I_{\mathrm{new}}$, so we need only show the other inclusion.

It is straightforward to verify that the new realization satisfies generalized Demazure surjectivity. A consequence is that the typical properties of Demazure operators are satisfied. For example, the kernel and the image of $\partial_s^{\mathrm{new}}$ are both equal to R^s_{new}, and $\partial_s^{\mathrm{new}}$ is R^s_{new}-linear. It follows that $\partial_I^{\mathrm{new}}$ is also R^I_{new}-linear.

Suppose that $g \in R^I_{\mathrm{new}}$, and write $g = \sum f_i \otimes b_i$. Choose some $P \in R$ with $\partial_I(P) = 1$, which exists by generalized Demazure surjectivity. Then

$$\partial_I^{\mathrm{new}}(P \otimes 1) = 1 \otimes 1 \quad \text{in } R_{\mathrm{new}}.$$

Thus

$$g = g\partial_I^{\mathrm{new}}(P \otimes 1) = \partial_I^{\mathrm{new}}(g \cdot (P \otimes 1)) = \sum \partial_I(f_i \cdot P) \otimes b_i,$$

and therefore, $g \in R^I \otimes_A B$.

The rest of the proof is straightforward. Fix an atomic coset a. For each $q < \mathsf{a}$, given operators T_q for the original realization satisfying (3.1), we define $T_{q,\mathrm{new}} := T_q \otimes \mathrm{id}$. By linearity, we need only check (3.1) for R_{new} on elements in R^J_{new} of the form $f \otimes b_1$ and $g \otimes b_2$ for $f, g \in R^J$. It is easy to verify (3.1) for R_{new} on such elements, since all operators (like $\underline{\mathsf{a}}$ or $\partial_{\bar{q}w_J^{-1}}$) are applied only to the first tensor factor, where we can use the atomic Leibniz rule from R. We conclude by noting that $T_{q,\mathrm{new}}$ has the appropriate codomain as well. ∎

We do not claim that any statements about the unicity of the operators T_q will extend from a realization to its specializations, enlargements, or quotients.

Lemma 3.18. *Let* $(\Bbbk, V, \Delta, \Delta^\vee)$ *be a realization of* (W, S). *If one can prove an atomic Leibniz rule for the restriction of* V *to* W_M, *for all (maximal) finitary subsets* $M \subset S$, *then an atomic Leibniz rule holds for* W.

Proof. Every atomic coset in W lives within W_M for some finitary M (which lives within a maximal finitary subset), and the same atomic Leibniz rule which works for W_M will work for W. ∎

Example 3.19. Suppose one can prove an atomic Leibniz rule for the permutation realization of $\mathbf{S}_n$ over $\mathbb{Z}$. Then by enlargement, one obtains an atomic Leibniz rule for the affine permutation realization restricted to any finite parabolic subgroup, see Example 3.16. By the previous lemma, an atomic Leibniz rule holds for the affine permutation realization of the affine Weyl group of type $\widetilde{A}_{n-1}$.

4. Lower terms

In this section, we give an explicit description of the ideal of lower terms for an atomic coset using the technology of singular light leaves.

4.1. Definition of lower terms

Definition 4.1. Let

$$I_\bullet = [[I = I_0 \subset K_1 \supset I_1 \subset \cdots \subset K_m \supset I_m = J]]$$

be a multistep (I, J) expression. To this expression, we associate a (*singular*) *Bott–Samelson bimodule*

$$\mathrm{BS}(I_\bullet) := R^{I_0} \otimes_{R^{K_1}} R^{I_1} \otimes_{R^{K_2}} \cdots \otimes_{R^{K_m}} R^{I_m}.$$

This is an (R^I, R^J)-bimodule. The collection of all Bott–Samelson bimodules is closed under tensor product, and forms (the set of 1-morphisms in) a full sub-2-category of the 2-category of bimodules. This sub-2-category is denoted **SBSBim**.

For two (R^I, R^J)-bimodules B and B', $\mathrm{Hom}(B, B')$ denotes the space of bimodule maps. Moreover, $\mathrm{Hom}(B, B')$ is itself an (R^I, R^J)-bimodule in the usual way.

Inside any linear category, given a collection of objects, their identity maps generate a two-sided ideal. This ideal consists of all morphisms which factor through one of those objects, and linear combinations thereof. In the context of (R^I, R^J)-bimodules, the actions of R^I and R^J commute with any morphism, and thus preserve the factorization of morphisms. Hence the morphisms within any such ideal form a sub-bimodule of the original Hom space.

Definition 4.2. Let p be an (I, J)-coset. Consider the set of reduced expressions $M_\bullet$ for any (I, J)-coset q with $q < p$. Let $\mathrm{Hom}_{<p}$ denote the ideal in the category of (R^I, R^J)-bimodules generated by the identity maps of $\mathrm{BS}(M_\bullet)$ for such expressions. Then $\mathrm{Hom}_{<p}$ is a two-sided ideal, the *ideal of lower terms* relative to p. The ideal $\mathrm{Hom}_{\le p}$ is defined similarly.

So $\mathrm{Hom}_{<p}(B, B')$ is a subset of $\mathrm{Hom}(B, B')$, and is a sub-bimodule for (R^I, R^J). We write $\mathrm{End}_{<p}(B)$ instead of $\mathrm{Hom}_{<p}(B, B) \subset \mathrm{End}(B)$.

We now focus on the case of atomic cosets. We use the letter a to denote an atomic coset and let $[[I \subset M \supset J]]$ denote the unique reduced expression of a. We let

$$B_{\mathsf{a}} := \mathrm{BS}([[I \subset M \supset J]]) = R^I \otimes_{R^M} R^J.$$

Because B_{a} is generated by $1 \otimes 1$ as a bimodule, any endomorphism is determined by where it sends this element. Thus

$$\mathrm{End}(B_{\mathsf{a}}) \cong R^I \otimes_{R^M} R^J$$

as (R^I, R^J)-bimodules, via the operations of left and right multiplication. Hence, $\mathrm{End}(B_{\mathtt{a}}) \cong B_{\mathtt{a}}$ as (R^I, R^J)-bimodules[6]. It is easy to deduce that $B_{\mathtt{a}}$ is indecomposable (when $\Bbbk$ is a domain) since there are no non-trivial idempotents in $\mathrm{End}(B_{\mathtt{a}}) \cong B_{\mathtt{a}}$.

4.2. Atomic double leaves

The goal of the section is to describe a large family of morphisms in $\mathrm{End}(B_{\mathtt{a}})$ called double leaves, most of which are in $\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$ by construction. We use the diagrammatic technology originally found in [12] and developed further in [10].

We assume a Frobenius realization, see Definition 3.5. In particular, the ring inclusions $R^I \subset R^J$ are Frobenius extensions. Under these assumptions, a diagrammatic 2-category **Frob** is constructed in [12], and it comes equipped with a 2-functor to **SBSBim**. This 2-functor is essentially surjective, but is not expected to be an equivalence; the category **Frob** is missing a number of relations.

Double leaves are to be constructed either as morphisms in **Frob**, or as their images in **SBSBim**, depending on the context.

The objects in **Frob** are indexed not by multistep expressions but by singlestep expressions. An (I, J) *singlestep expression* is a sequence

$$I_\bullet = [I = I_0, I_1, \dots, I_d = J]$$

where each I_i is a finitary subset of S, and each I_i and I_{i+1} differ by the addition or removal of a single simple reflection. We use single brackets for singlestep expressions, and double brackets for multistep expressions.

Throughout this section, we fix $I \subset M = Is \subset S$ finitary, and let $t = w_M s w_M$ and $J = M \setminus t$, so that

$$\mathtt{a} \leftrightharpoons [I, M, J]$$

is an atomic coset. We also fix the (I, M)-coset

$$n = W_I e W_M.$$

4.2.1. Elementary light leaves for atomic Grassmannian pairs. By definition of atomic, $\bar{\mathtt{a}} = w_M$. Then for an (I, J)-coset q, the condition $q \le \mathtt{a}$ is equivalent (see Theorem 2.16 in [8]) to $\overline{q} \le w_M$, which in turn is equivalent to $q \subseteq n = W_M$.

For an (I, J)-coset q contained in W_M, the pair $q \subset n$ is Grassmannian in the sense of Definition 2.7 in [10]. Associated to such a pair, a distinguished map called an elementary light leaf is constructed in Section 7.3 of [10]. The map (and codomain of the map) depends on a choice we make now: we fix a reduced expression X_q of the form

$$X_q = [[I \supset \mathrm{LR}(q)]] \circ X_q^{\mathrm{core}} \circ [[\mathrm{RR}(q) \subset J]], \tag{4.1}$$

where X_q^{core} is a reduced expression of q^{core}.

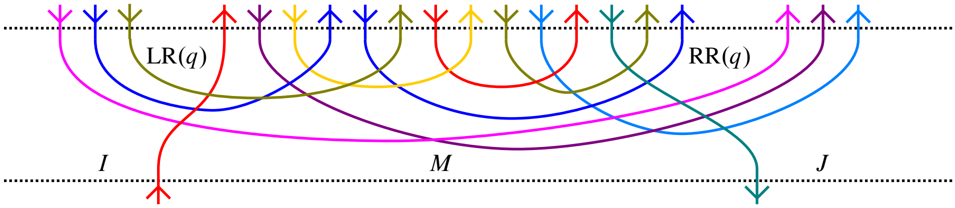
[6] We have ignored gradings in this paper. Using traditional grading conventions for Bott–Samelson bimodules, $\mathrm{End}(B_{\mathtt{a}})$ and $B_{\mathtt{a}}$ are only isomorphic up to shift. The identity map of $\mathrm{End}(B_{\mathtt{a}})$ is in degree zero, while $1 \otimes 1 \in B_{\mathtt{a}}$ is not.

Definition 4.3. Let q be an (I, J)-coset contained in n. The *elementary light leaf associated to* $[n, q]$ *(and* X_q*)* is the (R^I, R^J)-bimodule morphism

$$\mathbb{ELL}([n,q]) : \mathrm{BS}([I, M, J]) \to \mathrm{BS}(X_q)$$

sending the generator $1 \otimes 1 \in R^I \otimes_{R^M} R^J = \mathrm{BS}([I, M, J])$ to the element $1^{\otimes} := 1 \otimes \cdots \otimes 1 \in \mathrm{BS}(X_q)$. Equivalently, $\mathbb{ELL}([n,q])$ is defined by the diagrammatic construction in Section 7.3 of [10] (see also Lemma 8.10 in [10]).

We refer to [10] and [12] for a diagrammatic exhibition of morphisms between Bott–Samelson bimodules. The morphism $\mathbb{ELL}([n,q])$ is determined by the condition that its diagram consists only of counterclockwise cups and right-facing crossings, as in the following diagram.



In our examples, we color the simple reflection s in strawberry, and t in teal. Sometimes $s = t$, which will force us to change our convention.

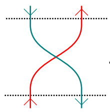
In type A, the expression X_q^{core} takes a simple form, and thus $\mathbb{ELL}([n,q])$ could be described more explicitly.

Example 4.4. Let $W_M = \mathbf{S}_{a+b}$ be a symmetric group, for some $a \neq b$, and let $I = \widehat{s}$ be such that $W_I = \mathbf{S}_b \times \mathbf{S}_a \subset \mathbf{S}_{a+b}$. Then $J = \widehat{t}$ is such that $W_J = \mathbf{S}_a \times \mathbf{S}_b \subset \mathbf{S}_{a+b}$. As will be explained in Section 6 (equation (6.1)), each (I, J)-coset q in W_M has a unique reduced expression X_q of the form (4.1).

(1) If $q = W_I e W_J$, then we have

$$q \leftrightharpoons X_q = [[\widehat{s} \supset \widehat{s}\,\widehat{t}\,]] \circ [\widehat{s}\,\widehat{t}\,] \circ [[\widehat{s}\,\widehat{t} \subset \widehat{t}\,]] = [\widehat{s} - t + s].$$

Here we have

$$\mathbb{ELL}([n,q]) = \quad$$  .

(2) If $q = W_I w_M W_J$, then we have

$$q = \mathsf{a} \leftrightharpoons X_q = [I, M, J].$$

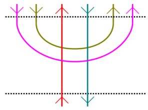
Here we have $\mathbb{ELL}([n,q]) = \mathrm{id}_{\mathrm{BS}([I,M,J])}$.

(3) Otherwise, we have

$$q \leftrightharpoons X_q = [[\widehat{s} \supset \widehat{s}\,\widehat{k}\,\widehat{\ell}\,]] \circ [\widehat{s}\,\widehat{k}\,\widehat{\ell} \subset \widehat{k}\,\widehat{\ell} \supset \widehat{t}\,\widehat{k}\,\widehat{\ell}\,] \circ [[\widehat{t}\,\widehat{k}\,\widehat{\ell} \subset \widehat{t}\,]]$$

for distinct $s, k, \ell, t \in M$. Here we have

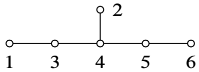
$$\mathbb{ELL}([n,q]) =$$



.

Here is a non-type A example.

Example 4.5. Let (W, S) be of type E_6, where S is indexed as in the Dynkin diagram



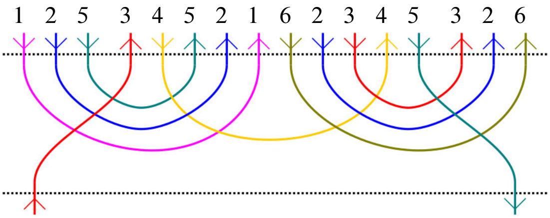
Let $M = S$ and $s = 3$. Then $w_M 3 w_M = 5$, and thus

$$\mathtt{a} \leftrightharpoons [\hat{3} + 3 - 5] = [\hat{3}, M, \hat{5}]$$

is an atom. For the $(\hat{3}, \hat{5})$-coset $q < \mathtt{a}$ with reduced expression

$$q \leftrightharpoons X_q = [[\hat{3} \supset \{4,6\}]] \circ [+3 - 4 + 5 + 2 + 1 - 6 - 2 - 3 + 4 - 5] \circ [[\{1,4\} \subset \hat{5}]],$$

the elementary light leaf $\mathbb{ELL}([n,q])$ is



4.2.2. Atomic double leaves. There is a contravariant (but monoidally-covariant) "duality" functor $\mathcal{D}$ from **Frob** to itself defined as follows:

- it preserves objects and 1-morphisms,
- on 2-morphisms, it flips each diagram upside-down and reverses all the orientations.

This functor is an involution.

Definition 4.6. Given $q \leq \mathtt{a}$ and $b \in R^{\mathrm{RR}(q)}$, the associated *right-sprinkled double leaf* $\mathbb{DLL}_r(q, b)$ is the composition

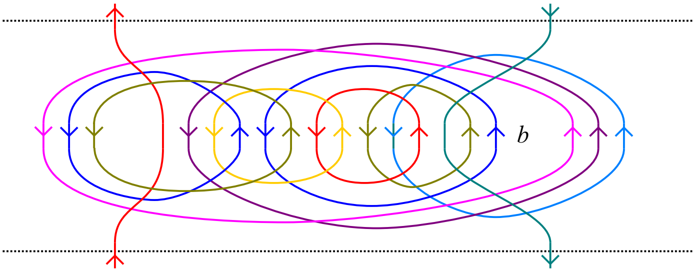
$$B_{\mathtt{a}} \xrightarrow{\mathbb{ELL}([n,q])} \mathrm{BS}(X_q) \xrightarrow{b} \mathrm{BS}(X_q) \xrightarrow{\mathcal{D}(\mathbb{ELL}([n,q]))} B_{\mathtt{a}}. \tag{4.2}$$

The middle map in (4.2) uses that X_q has the form (4.1): the map is multiplication by the element

$$1^{\otimes} \otimes b \otimes 1 \in \mathrm{BS}([[I \supset \mathrm{LR}(q)]] \circ X_q^{\mathrm{core}}) \otimes_{R^{\mathrm{RR}(q)}} R^{\mathrm{RR}(q)} \otimes_{R^J} R^J.$$

In diagrams, we have

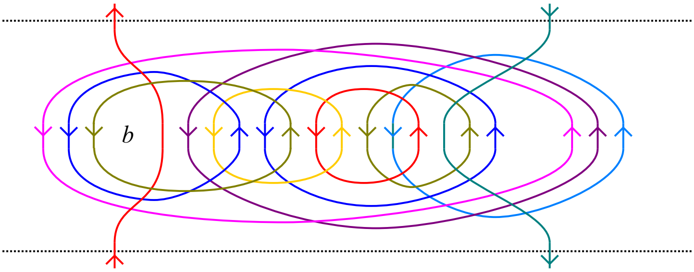
$$\mathbb{DLL}_r(q,b) =$$



Definition 4.7. Given $q \leq \mathtt{a}$ and $b \in R^{\mathrm{LR}(q)}$, the associated *left-sprinkled double leaf* $\mathbb{DLL}_l(q,b)$ is the composition

(4.3) $$B_{\mathtt{a}} \xrightarrow{\mathbb{ELL}([n,q])} \mathrm{BS}(X_q) \xrightarrow{b} \mathrm{BS}(X_q) \xrightarrow{\mathfrak{D}(\mathbb{ELL}([n,q]))} B_{\mathtt{a}},$$

whose diagram is



The maps provided here (when b ranges over a basis for $R^{\mathrm{RR}(q)}$ or $R^{\mathrm{LR}(q)}$) are the same as the "double leaves basis" from [10]. This is verified in the following remark, intended for a reader familiar with [10].

Remark 4.8. Let us verify that

$$\mathbb{DLL}_l(q,b) = \mathbb{DLL}(q,([p,n,q],1),([p,n,q],1),b),$$

where p is the identity (I,I)-coset, by following the stages in Chapter 7 of [10]. First, we construct $\mathbb{LL}(q,([p,n,q],1))$. The single step light leaf for the first step $[p,n]$ is the identity map since $[p,n]$ is reduced. Since the left redundancy for p and n are the same, In the second step, we have the coset pair $[n,q]$ which is already a Grassmannian pair, thus the single step light leaf is the elementary light leaf $\mathbb{ELL}([n,q])$.

Then we choose $X_q = Y_q$ and take the dual map for the upside-down light leaf from $\mathrm{BS}(X_q)$ to $B_{\mathtt{a}}$. Altogether we obtain the double leaf of the form (4.3). No rex moves were used at any stage of the process.

Moreover, for each $q \leq \mathtt{a}$, there is one subordinate path with terminus q, namely $[p,n,q]$. It follow that the morphisms $\mathbb{DLL}(q,([p,n,q],1),([p,n,q],1),b)$ form a double leaves basis in the sense of [10].

Remark 4.9. We know that $\mathrm{End}(B_{\mathsf{a}})$ is an (R^I, R^J)-bimodule, so it is natural to ask how the actions of R^I and R^J interact with the bases presented in Propositions 4.18 and 4.20. For $g \in R^J$, we claim that

$$\mathbb{DLL}_r(q,b)\cdot g = \mathbb{DLL}_r(q, b\cdot g). \tag{4.4}$$

Consider the diagram in Definition 4.6, and right-multiply by $g \in R^J$. Since g is also in $R^{\mathrm{RR}(q)}$, it can be slid from the right side to the region where b lives.

However, the left action of $f \in R^I$ is more mysterious. For fixed q, it does not preserve the span of $\{\mathbb{DLL}_r(q,b) \mid b \in R^{\mathrm{RR}(q)}\}$. Indeed, the comparison between the left action and the right action is controlled by polynomial forcing for $\mathrm{BS}(X_q)$, which involves lower terms.

Similarly, for $f \in R^I$, the left action on left-sprinkled double leaves is straightforward,

$$f\cdot \mathbb{DLL}_l(q,b) = \mathbb{DLL}_l(q, f\cdot b),$$

whereas the right action of R^J is mysterious.

4.2.3. Evaluation of double leaves. The following crucial computation links double leaves with the description $\mathrm{End}(B_{\mathsf{a}}) \cong R^I \otimes_{R^M} R^J$. Let $\Delta^J_{M,(1)}$ and $\Delta^J_{M,(2)}$ be dual bases of R^J over R^M, where we use Sweedler notation.

Lemma 4.10. *The double leaf $\mathbb{DLL}_r(q,b)$ coincides with multiplication by the element*

$$\partial_{\bar{q}w_J^{-1}}(b\cdot \Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}.$$

This can also be written as

$$\partial_I^{\mathrm{LR}(q)}\partial_{q^{\mathrm{core}}}(b\cdot \iota_J^{\mathrm{RR}(q)}\Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}.$$

Proof. The proof follows immediately from Algorithm 8.12 in [10]. It is proven exactly as Lemma 6.10 in [10]. ∎

A similar computation involving left-sprinkled double leaves gives the following.

Lemma 4.11. *The double leaf $\mathbb{DLL}_l(q,b)$ coincides with multiplication by the element*

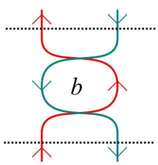
$$\partial_I^{\mathrm{LR}(q)}(b\cdot \partial_{q^{\mathrm{core}}}(\iota_J^{\mathrm{RR}(q)}\Delta^J_{M,(1)})) \otimes \Delta^J_{M,(2)}. \tag{4.5}$$

Example 4.12. If $q \le \mathsf{a}$ is the minimal (I,J)-coset, namely $q = W_I e W_J$, then we have two cases.

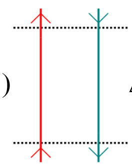
(1) When $t := w_M s w_M \neq s$, we have

$$q \leftrightharpoons [I,K,J] = [I - t + s]$$

for $K = I \cap J$. In this case, both left and right redundancies are K, and for $b \in R^K$, the double leaf $\mathbb{DLL}_r(q,b) = \mathbb{DLL}_l(q,b)$ is the left diagram in the equality



$$= \partial_I^K(b\cdot \Delta^J_{M,(1)})$$



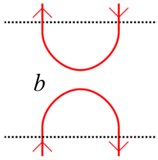
$$\Delta^J_{M,(2)}.$$

Thus we have

$$\mathbb{DLL}_r(q,b) = \mathbb{DLL}_l(q,b) = \partial_I^K(b \cdot \Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}$$

(see equations (89) and (91) in [10]).

(2) When $w_M\, s\, w_M = s$, we have $q \leftrightharpoons [I]$. In this case, we have $\mathrm{LR}(q) = I = \mathrm{RR}(q)$, and for $b \in R^I$, the double leaf has a capcup diagram



Thus we have $\mathbb{DLL}_r(q,b) = \mathbb{DLL}_l(q,b) = b \cdot \Delta^I_M = \Delta^I_M \cdot b$.

Definition 4.13. For an atomic coset $\mathtt{a} \leftrightharpoons [I, M, J]$, let $\mathrm{DL}_{<\mathtt{a}}$ denote the $\Bbbk$-linear subspace of $\mathrm{End}(B_\mathtt{a})$ spanned by right-sprinkled double leaves factoring through $q < \mathtt{a}$. More explicitly, we have

$$\begin{aligned}\mathrm{DL}_{<\mathtt{a}} &:= \mathrm{Span}_{q<\mathtt{a}}\{\mathbb{DLL}_r(q,b) \mid b \in R^{\mathrm{RR}(q)}\} \\ &= \mathrm{Span}_{q<\mathtt{a}}\left\{\partial_{\bar{q}w_J^{-1}}(R^{\mathrm{RR}(q)} \cdot \Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}\right\}.\end{aligned}$$

Lemma 4.14. *We have* $\mathrm{DL}_{<\mathtt{a}} \subset \mathrm{End}_{<\mathtt{a}}(B_\mathtt{a})$.

Proof. By construction, every double leaf associated to $q < \mathtt{a}$ factors through a reduced expression for q, and thus lives in $\mathrm{End}_{<\mathtt{a}}(B_\mathtt{a})$. ∎

We also note a consequence of Remark 4.9.

Corollary 4.15. *For each $g \in R^J$ and $b \in R^{\mathrm{RR}(q)}$, we have*

$$\partial_{\bar{q}w_J^{-1}}(b \cdot \Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)} \cdot g = \partial_{\bar{q}w_J^{-1}}(b \cdot g \cdot \Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}.$$

In particular, $\mathrm{DL}_{<\mathtt{a}}$ *is a right R^J-module.*

Proof. This follows from (4.4) and Lemma 4.10. ∎

4.3. Double leaves and lower terms: part I

The main result of [10] is that double leaves form a basis for morphisms between Bott–Samelson bimodules. However, [10] relies on Williamson's theory of standard filtrations, which relies on several assumptions originally made by Soergel.

Definition 4.16. We call a realization a *Soergel–Williamson realization*, or an *SW-realization* for short, if it is a Frobenius realization (see Definition 3.5), and it also satisfies the following assumptions.

- The realization is *reflection faithful*, i.e., it is faithful, and the reflections in W are exactly those elements that fix a codimension-one subspace.
- The ring $\Bbbk$ is an infinite field of characteristic not equal to 2.

Remark 4.17. Abe [1] has recently developed a theory of singular Soergel bimodules that works for Frobenius realizations, without the extra restrictions of an SW-realization. One expects that the results of [10] can be straightforwardly generalized to Abe's setting. For simplicity and because of the current state of the literature, we will work with Williamson's category of bimodules.

Proposition 4.18. *Assume an SW-realization. Let $\mathbb{B}_q$ be a $\Bbbk$-basis of $R^{\mathrm{RR}(q)}$, for each $q \leq \mathtt{a}$. Then*

$$\{\mathbb{DLL}_r(q,b)\}_{b\in\mathbb{B}_q}$$

gives a basis of $\mathrm{End}_{\leq q}(B_{\mathtt{a}})/\mathrm{End}_{<q}(B_{\mathtt{a}})$ *over* $\Bbbk$. *In particular,*

$$\{\mathbb{DLL}_r(q,b) \mid q \leq \mathtt{a}, b \in \mathbb{B}_q\}$$

is a $\Bbbk$*-basis of* $\mathrm{End}(B_{\mathtt{a}})$*, and the subset indexed by* $q < \mathtt{a}$ *is a basis for* $\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$.

Proof. This is a special case of Lemma 8.35 and Theorem 7.49 in [10], as confirmed in Remark 4.8. ■

Corollary 4.19. *Assume an SW-realization. We have*

$$\mathrm{DL}_{<\mathtt{a}} = \mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}}) = \bigoplus_{q<\mathtt{a}} \partial_{\bar{q}w_J^{-1}}(R^{\mathrm{RR}(q)} \cdot \Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}.$$

In particular, $\mathrm{DL}_{<\mathtt{a}}$ *is an* (R^I, R^J)*-bimodule.*

Proof. As $\mathrm{DL}_{<\mathtt{a}}$ is the span of double leaves factoring through $q < \mathtt{a}$, the first equality follows from Proposition 4.18. The second equality follows from the linear independence of double leaves. Since $\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$ is an (R^I, R^J)-bimodule, so is $\mathrm{DL}_{<\mathtt{a}}$. ■

Similarly, double leaves provide a left-sprinkled basis.

Proposition 4.20. *Assume an SW-realization. Let $\mathbb{B}_q$ be a $\Bbbk$-basis of $R^{\mathrm{LR}(q)}$, for each $q \leq \mathtt{a}$. Then*

$$\{\mathbb{DLL}_l(q,b)\}_{b\in\mathbb{B}_q}$$

gives a basis of $\mathrm{End}_{\leq q}(B_{\mathtt{a}})/\mathrm{End}_{<q}(B_{\mathtt{a}})$ *over* $\Bbbk$. *In particular,*

$$\{\mathbb{DLL}_l(q,b) \mid q \leq \mathtt{a}, b \in \mathbb{B}_q\}$$

is a $\Bbbk$*-basis of* $\mathrm{End}(B_{\mathtt{a}})$*, and the subset indexed by* $q < \mathtt{a}$ *is a basis for* $\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$.

4.4. Double leaves and lower terms: part II

Definition 4.21. An *almost-SW realization* is a Frobenius realization, together with the following assumptions.

- The ring $\Bbbk$ is a domain with fraction field $\mathbb{F}$.
- After base change to $\mathbb{F}$, the result is an SW-realization.
- Finitely-generated projective modules over $\Bbbk$ are free.

Example 4.22. The defining representation of $\mathbf{S}_n$ over $\mathbb{Z}$ is an almost-SW realization, see Lemma 5 in [3].

Example 4.23. The root realization of a Weyl group is almost-SW when defined over $\Bbbk = \mathbb{Z}[1/N]$ for small N ($N = 30$ will suffice for all Weyl groups by Proposition 8 in [3]).

Lemma 4.24. *Let $I \subset S$ be finitary. Then R^I is a free $\Bbbk$-module.*

Proof. As a polynomial ring over a free $\Bbbk$-module V, R is a free $\Bbbk$-module. By the assumption of generalized Demazure surjectivity, R is free as an R^I-module when $I \subset S$ is finitary. Thus R^I is a direct summand of R, and is therefore projective as a $\Bbbk$-module. Since both R and R^I are finitely-generated as $\Bbbk$-modules in each graded degree, we deduce that R^I is also a free module over $\Bbbk$. ■

Our goal in this section is to generalize the results of the previous section to almost-SW realizations. In all the lemmas in this section, we assume an almost-SW realization. First we note the compatibility of base change with most of the constructions above.

We let R be the polynomial ring of the realization over $\Bbbk$, and let $R_{\mathbb{F}} := R \otimes_{\Bbbk} \mathbb{F}$ be the polynomial ring of the realization after base change. Let $R^I_{\mathbb{F}} \subset R_{\mathbb{F}}$ be the invariant subring.

Lemma 4.25. *We have $R^I_{\mathbb{F}} \cong R^I \otimes_{\Bbbk} \mathbb{F}$.*

Proof. There is a natural map $R^I \otimes_{\Bbbk} \mathbb{F} \to R_{\mathbb{F}}$, and since scalars are W-invariant, the image lies within $R^I_{\mathbb{F}}$. The map is injective since $\mathbb{F}$ is flat over $\Bbbk$. We now argue that the map $R^I \otimes_{\Bbbk} \mathbb{F} \to R^I_{\mathbb{F}}$ is surjective. If $f \in R^I_{\mathbb{F}}$, then there is some $c \in \Bbbk$ such that $cf \in R$ (e.g., letting c be the product of the denominators of each monomial in f). Clearly, $cf \in R^I$, whence f is the image of $cf \otimes \frac{1}{c}$. ■

Let $B_{\mathrm{a},\mathbb{F}} = R^I_{\mathbb{F}} \otimes_{R^M_{\mathbb{F}}} R^J_{\mathbb{F}}$. If $I_\bullet = [[I = I_0 \subset K_1 \supset I_1 \subset \cdots \subset K_m \supset I_m = J]]$ is a multistep (I, J)-expression, let

$$\mathrm{BS}_{\mathbb{F}}(I_\bullet) := R^{I_0}_{\mathbb{F}} \otimes_{R^{K_1}_{\mathbb{F}}} R^{I_1}_{\mathbb{F}} \otimes_{R^{K_2}_{\mathbb{F}}} \cdots \otimes_{R^{K_m}_{\mathbb{F}}} R^{I_m}_{\mathbb{F}}.$$

Lemma 4.26. *The natural inclusion map $\mathrm{BS}(I_\bullet) \to \mathrm{BS}(I_\bullet) \otimes_{\Bbbk} \mathbb{F}$ is injective. We have $\mathrm{BS}_{\mathbb{F}}(I_\bullet) \cong \mathrm{BS}(I_\bullet) \otimes_{\Bbbk} \mathbb{F}$. As a consequence, we have an injective map*

$$\mathrm{Hom}(\mathrm{BS}(I_\bullet), \mathrm{BS}(I'_\bullet)) \to \mathrm{Hom}(\mathrm{BS}_{\mathbb{F}}(I_\bullet), \mathrm{BS}_{\mathbb{F}}(I'_\bullet)). \tag{4.6}$$

Proof. By our assumptions from §3.1, R^I is free over R^K whenever $I \subset K$. We fix a basis $\{b^{I,K}_i\}$ for this extension. Hence the Bott–Samelson bimodule $\mathrm{BS}(I_\bullet)$ is free as a right R^{I_m}-module with basis

$$\left\{b^{I_0,K_1}_{i_1} \otimes \cdots \otimes b^{I_{m-1},K_m}_{i_m} \otimes 1\right\}. \tag{4.7}$$

It is also free as a right $\Bbbk$-module by Lemma 4.24. So base change is injective on bimodules. Notice that $\{b^{I,K}_i \otimes 1\}$ is a basis of $R^I_{\mathbb{F}}$ over $R^K_{\mathbb{F}}$. Hence (4.7) gives also a basis of $\mathrm{BS}_{\mathbb{F}}(I_\bullet)$ over $R^{I_m}_{\mathbb{F}}$. Since it sends a basis to a basis, we deduce that the natural map $\mathrm{BS}(I_\bullet) \otimes_{\Bbbk} \mathbb{F} \to \mathrm{BS}_{\mathbb{F}}(I_\bullet)$ is an isomorphism.

The localization functor gives the map in (4.6). For a morphism ϕ between bimodules over $\Bbbk$, let $\phi \otimes 1$ denote its image, a morphism between bimodules over $\mathbb{F}$. The restriction of $\phi \otimes 1$ to the subset $\mathrm{BS}(I_\bullet) \subset \mathrm{BS}_{\mathbb{F}}(I_\bullet)$ is the original morphism ϕ. Hence if $\phi \otimes 1$ is the zero morphism, so is ϕ. ∎

Lemma 4.27. *For $q \leq \mathsf{a}$ and $b \in R^{\mathrm{RR}(q)}$, let us temporarily write $\mathbb{DLL}_{r,\Bbbk}(q, b)$ for the double leaf as a morphism between Bott–Samelson bimodules over $\Bbbk$, and $\mathbb{DLL}_{r,\mathbb{F}}(q, b)$ for the double leaf as a morphism between Bott–Samelson bimodules over $\mathbb{F}$, where for the latter we identify b with its image in $R_{\mathbb{F}}^{\mathrm{RR}(q)}$. Under the map (4.6), we have*

$$\mathbb{DLL}_{r,\Bbbk}(q, b) \mapsto \mathbb{DLL}_{r,\mathbb{F}}(q, b).$$

Proof. The calculus from [12] for interpreting diagrams is invariant under base change. Alternatively, B_{a} is generated as a bimodule by $1 \otimes 1$, and the elementary light leaves $\mathbb{ELL}([n, q])$ are determined uniquely in their Hom space by the fact that they send $1 \otimes 1$ to $1 \otimes \cdots \otimes 1$. This property is preserved by base change. ∎

Lemma 4.28. *Let $\mathbb{B}_q$ be a basis of $R^{\mathrm{RR}(q)}$ over $\Bbbk$. Then the set $\{\mathbb{DLL}_{r,\Bbbk}(q, b) \mid q \leq \mathsf{a}, b \in \mathbb{B}_q\}$ is linearly independent. Let $\mathrm{DL}_{<\mathsf{a},\Bbbk}$ and $\mathrm{DL}_{<\mathsf{a},\mathbb{F}}$ be defined as before for their respective realizations. The map (4.6) induces an isomorphism*

$$\mathrm{DL}_{<\mathsf{a},\Bbbk} \otimes_{\Bbbk} \mathbb{F} \xrightarrow{\sim} \mathrm{DL}_{<\mathsf{a},\mathbb{F}}\,. \tag{4.8}$$

Proof. By definition, $\mathrm{DL}_{<\mathsf{a}}$ is the span (over $\Bbbk$ or $\mathbb{F}$) of the double leaf morphisms. The map (4.6) restricts to a map $\mathrm{DL}_{<\mathsf{a},\Bbbk} \to \mathrm{Hom}(\mathrm{BS}_{\mathbb{F}}(I_\bullet), \mathrm{BS}_{\mathbb{F}}(I'_\bullet))$. By the previous lemma, the image of this map is contained in $\mathrm{DL}_{<\mathsf{a},\mathbb{F}}$. Thus one has an induced map $\mathrm{DL}_{<\mathsf{a},\Bbbk} \otimes_{\Bbbk} \mathbb{F} \to \mathrm{DL}_{<\mathsf{a},\mathbb{F}}$.

Note that the basis $\mathbb{B}_q$ is sent by base change to a basis of $R_{\mathbb{F}}^{\mathrm{RR}(q)}$ over $\mathbb{F}$. The elements $\{\mathbb{DLL}_{r,\Bbbk}(q, b) \otimes 1\}$ (ranging over the appropriate index set) form an $\mathbb{F}$-spanning set for the left-hand side of (4.8), and are sent to $\{\mathbb{DLL}_{r,\mathbb{F}}(q, b)\}$, which form a $\mathbb{F}$-basis for $\mathrm{DL}_{<\mathsf{a},\mathbb{F}}$ by Proposition 4.18. Thus the map (4.8) is an isomorphism, and the elements $\{\mathbb{DLL}_{r,\Bbbk}(q, b) \otimes 1\}$ are linearly independent over $\mathbb{F}$. Consequently, $\{\mathbb{DLL}_{r,\Bbbk}(q, b)\}$ are linearly independent over $\Bbbk$. ∎

Lemma 4.29. *We have*

$$\mathrm{DL}_{<\mathsf{a},\Bbbk} = \bigoplus_{q<\mathsf{a}} \partial_{\bar{q}w_J^{-1}}(R^{\mathrm{RR}(q)} \cdot \Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}.$$

Proof. The span in Definition 4.13 is indeed a direct sum of subspaces, by the linear independence shown in the previous lemma. ∎

Henceforth we return to the $\Bbbk$-linear setting by default (i.e., in the absence of a subscript). Now the question remains: is the inclusion $\mathrm{DL}_{<\mathsf{a}} \subset \mathrm{End}_{<\mathsf{a}}(B_{\mathsf{a}})$ an equality over $\Bbbk$, knowing that the result holds over $\mathbb{F}$? In Corollary 4.33 below, we prove that the answer is yes.

Lemma 4.30. *For any $\phi \in \mathrm{End}_{<\mathsf{a}}(B_{\mathsf{a}})$, there is some $n \in \Bbbk$ such that $n\phi \in \mathrm{DL}_{<\mathsf{a}}$.*

Proof. In view of Lemma 4.26 and (4.6), we can regard both $\mathrm{DL}_{<\mathtt{a}}$ and $\mathrm{End}(B_{\mathtt{a}})$ as $\Bbbk$-submodules of $\mathrm{End}(B_{\mathtt{a},\mathbb{F}})$, which we identify with $B_{\mathtt{a},\mathbb{F}}$. We have that $\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}}) \subset \mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a},\mathbb{F}})$ by definition. Meanwhile, Corollary 4.19 holds over $\mathbb{F}$ and thus

$$\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a},\mathbb{F}}) \cong \bigoplus_{q<\mathtt{a}} \partial_{\bar{q}w_J^{-1}}(R_{\mathbb{F}}^{\mathrm{RR}(q)} \cdot \Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}.$$

So any $\phi \in \mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$ is a $\mathbb{F}$-multiple of an element of $\mathrm{DL}_{<\mathtt{a}}$. Multiplying by the denominator, there exists $n \in \Bbbk$ such that $n\phi \in \mathrm{DL}_{<\mathtt{a}}$. ∎

We continue with a divisibility lemma on Demazure operators, which ensures that Demazure operators do not "produce" additional divisibility by elements of $\Bbbk$.

Lemma 4.31. *Let $b \in R$ and let $n \in \Bbbk$. If $n \mid \partial_{w_M w_J^{-1}}(bg)$ for all $g \in R^J$, then $n \mid b$.*

Proof. Assume that $n \nmid b$. We need to find $g \in R^J$ such that $n \nmid \partial_{w_M w_J^{-1}}(bg)$.

Recall from Lemma 2.7 that for $w \in W$ and $f, g \in R$, we have

$$\partial_w(fg) = \sum_{x \le w} T'_x(f)\, \partial_x(g)$$

for certain operators T'_x defined over $\Bbbk$, where $T'_w(f) = w(f)$. Let W^J be the subset of elements in W which are minimal (for the Bruhat order) in their right W_J-coset. We have $\partial_x(g) = 0$ when $x \notin W^J$ and $g \in R^J$, so we have

$$\partial_w(fg) = \sum_{x \le w,\, x \in W^J} T'_x(f)\, \partial_x(g).$$

We apply this formula when $w = w_M w_J^{-1}$. Let $W^J_M = W^J \cap W_M$. Since $n \nmid b$, then also $n \nmid w_M w_J^{-1}(b) = T'_{w_M w_J^{-1}}(b)$. So there exists some $y \in W^J_M$ (not necessarily unique) which is minimal with respect to the property that $n \nmid T'_y(b)$. Now we have

$$\partial_{w_M w_J^{-1}}(bg) = \sum_{x \in W^J_M} T'_x(b)\, \partial_x(g) \equiv \sum_{x \not< y,\, x \in W^J_M} T'_x(b)\, \partial_x(g) \pmod{n}. \tag{4.9}$$

Recall that $\ell(x w_M) = \ell(w_M) - \ell(x)$ for all $x \in W_M$. Let $z = y^{-1} w_M$ so that $y.z = w_M$. We have

$$\ell(w_J y^{-1} w_M) = \ell(w_M) - \ell(w_J y^{-1}) = \ell(w_M) - \ell(y^{-1}) - \ell(w_J) = \ell(z) - \ell(w_J),$$

thus the left descent set of z contains J. By Lemma 3.2 in [7], we have $\mathrm{Im}(\partial_z) \subset R^J$. By our assumption of generalized Demazure surjectivity, we can choose $P_M \in R$ such that $\partial_{w_M}(P_M) = 1$. Set $g = \partial_z(P_M) \in R^J$ and note that $\partial_y(g) = 1$.

Let $x \in W_M$. If $x.z$ is not reduced, then $\partial_x(g) = \partial_x \partial_z(P_M) = 0$ by equation (27) in [7]. If $x.z$ is reduced, then

$$\ell(w_M) - \ell(y x^{-1}) = \ell(x.y^{-1} w_M) = \ell(x) + \ell(w_M) - \ell(y).$$

It follows that $\ell(yx^{-1}) + \ell(x) = \ell(y)$, so $yx^{-1}.x = y$ and, in particular, $x \leq y$. This means that $\partial_x(g) \neq 0$ only if $x \leq y$.

Finally, we plug $g = \partial_z(P_M)$ into (4.9) and observe that $\partial_{w_M w_J^{-1}}(bg) \equiv T'_y(b) \not\equiv 0 \pmod{n}$. ∎

Now we can prove that $\mathrm{DL}_{<\mathrm{a}}$, as a submodule of $\mathrm{End}(B_\mathrm{a})$, is closed under division by elements in $\Bbbk$ (when that makes sense).

Proposition 4.32. *Let $\phi \in \mathrm{DL}_{<\mathrm{a}}$ and assume there exists $n \in \Bbbk$ such that $\frac{1}{n}\phi \in \mathrm{End}(B_\mathrm{a})$. Then $\frac{1}{n}\phi \in \mathrm{DL}_{<\mathrm{a}}$.*

Proof. If n is a unit in $\Bbbk$ the result is trivial, so assume otherwise.

Let $\phi \in \mathrm{DL}_{<\mathrm{a}}$ and assume that $\frac{1}{n}\phi \in \mathrm{End}(B_\mathrm{a}) \cong R^I \otimes_{R^M} R^J$. We can write

$$\phi = \sum_{q<\mathrm{a}} \partial_{\bar{q}w_J^{-1}}(b_q \cdot \Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}$$

for some unique $b_q \in R^{\mathrm{RR}(q)}$. For clarity we choose to unravel Sweedler's notation. Choose dual bases $\{c_i\}$ and $\{d_i\}$ for R^J over R^M relative to the Frobenius trace map ∂^J_M. We have

$$\phi = \sum_i \sum_{q<\mathrm{a}} \partial_{\bar{q}w_J^{-1}}(b_q \cdot c_i) \otimes d_i.$$

Since d_i is a basis of R^J over R^M, any element of $R^I \otimes_{R^M} R^J$ is uniquely expressible as $\sum_i f_i \otimes d_i$ for $f_i \in R^I$. In particular, if n divides $\sum_i f_i \otimes d_i$, we have

$$\frac{1}{n}\sum_i f_i \otimes d_i = \sum_i f'_i \otimes d_i$$

for some unique f'_i. Then $\sum f_i \otimes d_i = \sum_i n f'_i \otimes d_i$, and by the unicity mentioned before, this implies that $n f'_i = f_i$ for all i.

Consequently, ϕ is divisible by n if and only if, for all i, we have

$$n \mid \sum_{q<\mathrm{a}} \partial_{\bar{q}w_J^{-1}}(b_q \cdot c_i).$$

We want to show that all the b_q are actually divisible by n, so that $\frac{1}{n}\phi \in \mathrm{DL}_{<\mathrm{a}}$.

Assume for contradiction that there exists a minimal r such that $n \nmid b_r$. Let z be such that $z.\bar{r}w_J^{-1} = w_M w_J^{-1}$. Then by similar arguments to the previous lemma, we have

$$0 \equiv \partial_z\Big(\sum_{q<\mathrm{a}} \partial_{\bar{q}w_J^{-1}}(b_q \cdot c_i)\Big) \equiv \partial_{w_M w_J^{-1}}(b_r \cdot c_i) \pmod{n}$$

for each i. The subset of R^J consisting of those c for which $\partial_{w_M w_J^{-1}}(b_r \cdot c) \equiv 0 \pmod{n}$ is evidently an R^M-submodule. Since this submodule contains a basis $\{c_i\}$ for R^J over R^M, it must contain all of R^J. Thus

$$n \mid \partial_{w_M w_J^{-1}}(b_r \cdot g)$$

for all $g \in R^J$. By Lemma 4.31, we deduce $n \mid b_r$, leading to a contradiction. ∎

Corollary 4.33. *For an almost SW-realization we have* $\mathrm{DL}_{<\mathtt{a}} = \mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$. *In particular,* $\mathrm{DL}_{<\mathtt{a}}$ *is an* (R^I, R^J)*-bimodule.*

Proof. The containment $\mathrm{DL}_{<\mathtt{a}} \subset \mathrm{End}_{\mathtt{a}}(B_{\mathtt{a}})$ was already shown in Lemma 4.14. Now pick an arbitrary element $\psi \in \mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$. By Lemma 4.30, there is some $n \in \Bbbk$ such that $n\psi \in \mathrm{DL}_{<\mathtt{a}}$. Then $\psi = \frac{1}{n}(n\psi) \in \mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$, so by Proposition 4.32, we deduce that $\psi \in \mathrm{DL}_{<\mathtt{a}}$. ■

5. Polynomial forcing and atomic Leibniz

5.1. Polynomial forcing for atomic cosets

Now we explain the concept of polynomial forcing. We consider first the case of an atomic coset

$$\mathtt{a} \leftrightharpoons [[I \subset M \supset J]].$$

Recall that $\underline{\mathtt{a}}J = I\underline{\mathtt{a}}$ and $\mathrm{LR}(\mathtt{a}) = I$, since $\mathtt{a}$ is core. Thus there is an isomorphism

$$R^J \to R^I, \quad f \mapsto \underline{\mathtt{a}}(f).$$

Definition 5.1. Let $\mathtt{a}$ be an atomic coset with reduced expression $[[I \subset M \supset J]]$ and let $f \in R^J$. We say that *polynomial forcing* holds for f and $\mathtt{a}$ if we have

$$1 \otimes f - \underline{\mathtt{a}}(f) \otimes 1 \in \mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}}). \tag{5.1}$$

We say that *polynomial forcing* holds for $\mathtt{a}$ if (5.1) holds for all $f \in R^J$.

Lemma 5.2. *Suppose that* (5.1) *holds for* f_1 *and for* f_2*, with both* $f_1, f_2 \in R^J$. *Then it holds for* $f_1 + f_2$ *and* $f_1 f_2$.

Proof. Additivity is trivial, because $\mathrm{Hom}_{<\mathtt{a}}$ is closed under addition. Now consider the following:

$$1 \otimes f_1 \cdot f_2 - \underline{\mathtt{a}}(f_1 \cdot f_2) \otimes 1 = (1 \otimes f_1 - \underline{\mathtt{a}}(f_1) \otimes 1) \cdot f_2 + \underline{\mathtt{a}}(f_1) \cdot (1 \otimes f_2 - \underline{\mathtt{a}}(f_2) \otimes 1).$$

Since $\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$ is closed under right and left multiplication, both terms on the right-hand side above are in $\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$, and the result is proven. ■

Before continuing, let us contrast polynomial forcing with an a priori different notion.

Definition 5.3. Consider $\mathrm{DL}_{<\mathtt{a}}$ from Definition 4.13. We say that DL-*forcing* holds for $\mathtt{a}$ and f if $1 \otimes f - \underline{\mathtt{a}}(f) \otimes 1 \in \mathrm{DL}_{<\mathtt{a}} \subset \mathrm{End}(B_{\mathtt{a}})$. We say that DL-*forcing* holds for $\mathtt{a}$ if it holds for $\mathtt{a}$ and f, for all $f \in R^J$.

For an almost-SW realization, DL-forcing is equivalent to polynomial forcing, since $\mathrm{DL}_{<\mathtt{a}} = \mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$ by Corollary 4.33. In general, it is not obvious that DL-forcing is multiplicative. The proof of multiplicativity in Lemma 5.2 relied on the fact that $\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$ is an (R^I, R^J)-bimodule, whereas $\mathrm{DL}_{<\mathtt{a}}$ is only a priori a right R^J-module.

5.2. Equivalence

Now we prove the equivalence between atomic Leibniz rules and polynomial forcing. To formulate an intermediate condition in the proof, which is also of importance for the next section, we agree to say the following. Given an atomic (I, J)-coset $\mathtt{a}$ and an element $f \in R^J$, *an atomic Leibniz rule for* $\mathtt{a}$ *and* f is said to hold if there exist elements $T_q(f)$ such that equation (3.1) is satisfied for all $g \in R^J$. Since this condition is stated for one polynomial f at a time, there is no requirement that T_q is an R^M-linear operator.

Proposition 5.4. *Let* $\mathtt{a} \leftrightharpoons [I, M, J]$ *be an atomic* (I, J)*-coset, and let* $f \in R^J$. *We have a rightward atomic Leibniz rule for* $\mathtt{a}$ *and* f *if and only if* DL*-forcing holds for* $\mathtt{a}$ *and* f. *Moreover, for an almost-SW realization, if* DL*-forcing holds for* $\mathtt{a}$ *and* f, *then the atomic Leibniz rule is unique, i.e., the elements* $T_q(f) \in R^{\mathrm{RR}(q)}$ *in* (3.1) *are uniquely determined.*

Proof. Note that $R^J \subset R^M$ is a Frobenius extension, see Section 24.3.2 of [11]. The trace map is

$$\partial^J_M := \partial_{w_M w_J^{-1}} = \partial_{\mathtt{a}}.$$

Let $\Delta^J_M \in R^J \otimes_{R^M} R^J$ denote the coproduct element (the image of $1 \in R^J$ under the coproduct map), which we often denote using Sweedler notation. Then equation (2.2) with $f = 1$ in [12] implies that

$$1 \otimes 1 = \partial_{\mathtt{a}}(\Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}.$$

Multiplying by $\underline{\mathtt{a}}(f)$ on the left, we get

$$\underline{\mathtt{a}}(f) \otimes 1 = \underline{\mathtt{a}}(f) \cdot \partial_{\mathtt{a}}(\Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}.$$

Meanwhile, equation (2.2) in [12] implies that

$$1 \otimes f = \partial_{\mathtt{a}}(f \cdot \Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}.$$

Thus we have

$$1 \otimes f - \underline{\mathtt{a}}(f) \otimes 1 = \left[\partial_{\mathtt{a}}(f \cdot \Delta^J_{M,(1)}) - \underline{\mathtt{a}}(f) \cdot \partial_{\mathtt{a}}(\Delta^J_{M,(1)})\right] \otimes \Delta^J_{M,(2)}. \tag{5.2}$$

Letting $g = \Delta^J_{M,(1)}$, a rightward atomic Leibniz rule for $\mathtt{a}$ and f gives

$$\partial_{\mathtt{a}}(f \cdot g) - \underline{\mathtt{a}}(f) \cdot \partial_{\mathtt{a}}(g) = \sum_{q<\mathtt{a}} \partial_{\bar{q} w_J^{-1}}(T_q(f) \cdot g).$$

Thus we have

$$1 \otimes f - \underline{\mathtt{a}}(f) \otimes 1 = \sum_{q<\mathtt{a}} \partial_{\bar{q} w_J^{-1}}(T_q(f) \cdot \Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}.$$

which lies in $\mathrm{DL}_{<\mathtt{a}}$ by definition.

We prove now the other direction. We have

$$1 \otimes f - \underline{\mathtt{a}}(f) \otimes 1 \in \mathrm{DL}_{<\mathtt{a}}\,.$$

By (5.2), we obtain

$$\left[\partial_{\mathtt{a}}(f\cdot\Delta^J_{M,(1)}) - \underline{\mathtt{a}}(f)\cdot\partial_{\mathtt{a}}(\Delta^J_{M,(1)})\right]\otimes\Delta^J_{M,(2)}\in\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}}).$$

By Corollary 4.33, we have that $\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}}) = \mathrm{DL}_{<\mathtt{a}}$, and by definition of $\mathrm{DL}_{<\mathtt{a}}$, we deduce that

$$\begin{aligned}&\partial_{\mathtt{a}}(f\cdot\Delta^J_{M,(1)})\otimes\Delta^J_{M,(2)}\\ &\qquad= \underline{\mathtt{a}}(f)\cdot\partial_{\mathtt{a}}(\Delta^J_{M,(1)})\otimes\Delta^J_{M,(2)} + \sum_{q<\mathtt{a}}\partial_{\overline{q}w_J^{-1}}(T_q(f)\cdot\Delta^J_{M,(1)})\otimes\Delta^J_{M,(2)}\end{aligned}\tag{5.3}$$

for some $T_q(f)\in R^{\mathrm{RR}(q)}$. For an almost-SW realization, Lemma 4.29 implies that the $T_q(f)$ are unique.

Note that $\Delta^J_{M,(1)}$ and $\Delta^J_{M,(2)}$ run over dual bases of R^J over R^M. The elements $\mathbb{B} = \{1\otimes\Delta^J_{M,(2)}\}$ form a basis for $B_{\mathtt{a}}$, when viewed as a left R^I-module. Thus in order for the equation (5.3) to hold, it must be an equality for each coefficient with respect to the basis $\mathbb{B}$. Hence we conclude

$$\partial_{\mathtt{a}}(f\cdot\Delta) = \underline{\mathtt{a}}(f)\cdot\partial_{\mathtt{a}}(\Delta) + \sum_{q<\mathtt{a}}\partial_q(T_q(f)\cdot\Delta)\tag{5.4}$$

for all Δ ranging through a basis of R^J over R^M.

Using the linearity of (5.4) over R^M, we deduce that it continues to hold when Δ is replaced by any element $g\in R^J$. Thus the atomic Leibniz rule for f is proven. ■

Theorem 5.5. *Assume an almost-SW realization (see Definition 4.21). Let* $\mathtt{a}\leftrightharpoons[I,M,J]$ *be an atomic* (I,J)*-coset. Then the following are equivalent.*

1. *A rightward atomic Leibniz rule holds for* $\mathtt{a}$.
2. *A leftward atomic Leibniz rule holds for* $\mathtt{a}$.
3. *For a set of generators* $\{c_i\}$ *of the* R^M*-algebra* R^J*, a rightward atomic Leibniz rule holds for* $\mathtt{a}$ *and each* c_i.
4. *For a set of generators* $\{c_i\}$ *of the* R^M*-algebra* R^J*, a leftward atomic Leibniz rule holds for* $\mathtt{a}$ *and each* c_i.
5. *Polynomial forcing holds for* $\mathtt{a}$.

Moreover, in this case, there are unique operators T_q, T'_q *that satisfy atomic Leibniz rules.*

Proof. First, we observe that polynomial forcing holds for $f\in R^M$. Clearly $1\otimes f = f\otimes 1$. Moreover, $\mathtt{a}\subset W_M$ so $\underline{\mathtt{a}}(f) = f$.

That (1) implies (3) is clear.

Suppose that (3) holds. By Proposition 5.4, DL forcing holds for all c_i. By Corollary 4.33, DL-forcing for c_i is equivalent to polynomial forcing for c_i. By Lemma 5.2, the subset of R^J consisting of those f for which polynomial forcing holds is a subring. As explained above, this subring includes R^M, so if it includes $\{c_i\}$ then it must be all of R^J. In this way, (3) implies (5).

Suppose that (5) holds. Once again, Proposition 5.4 and Corollary 4.33 imply that, for each $f\in R^J$, a rightward atomic Leibniz rule holds for f, with the elements $T_q(f)\in$

$R^{\mathrm{RR}(q)}$ being unique. To prove that a rightward atomic Leibniz rule holds, it remains to prove that the operators $T_q \colon R^J \to R^{\mathrm{RR}(q)}$ are R^M-linear. We do this below, finishing the proof that (5) implies (1).

Let $g \in R^M$. Multiplying both sides of equation (5.3) on the left by g, and pulling g into various R^M-linear operators (namely $\partial_{\mathtt{a}}$ and $\underline{\mathtt{a}}$ and $\partial_{\bar{q} w_J^{-1}}$), we obtain

$$\partial_{\mathtt{a}}(gf \cdot \Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)} = \underline{\mathtt{a}}(gf) \cdot \partial_{\mathtt{a}}(\Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)} + \sum_{q<\mathtt{a}} \partial_{\bar{q} w_J^{-1}}(gT_q(f) \cdot \Delta^J_{M,(1)}) \otimes \Delta^J_{M,(2)}.$$

This is exactly (5.3) with gf replacing f, except that $gT_q(f)$ appears instead of $T_q(gf)$. By uniqueness, we deduce that $T_q(gf) = gT_q(f)$.

We have thus shown the equivalence of (1), (3) and (5). A similar argument will imply the equivalence of (2) and (4) and (5), and the uniqueness of T'_q. This similar argument replaces $\mathbb{DLL}_r(q,b)$ with $\mathbb{DLL}_l(q,b)$, using Proposition 4.20 and Lemma 4.11. The left analogue of the remaining arguments (e.g., Proposition 5.4 and Corollary 4.33) is left to the reader. ∎

Remark 5.6. The intermediate conditions (3) and (4) do not play a significant role in the proof. We have included them to make it easier to prove the atomic Leibniz rule by establishing it on a set of generators.

5.3. Polynomial forcing for general cosets

Now let p be an arbitrary (I,J)-coset, with a reduced expression $I_\bullet$. We wish to avoid the technicalities of changing the reduced expression $I_\bullet$ in this paper. Instead we focus on the special case when $I_\bullet$ is an *atomic-factored reduced expression*, i.e., it has the following form:

$$I_\bullet = [[I \supset \mathrm{LR}(p)]] \circ I'_\bullet \circ [[\mathrm{RR}(p) \subset J]] \tag{5.5}$$

where $I'_\bullet$ is an *atomic reduced expression* (see below) for p^{core}.

An atomic reduced expression for a core coset p^{core} is a reduced expression of the form

$$I'_\bullet = [[\mathrm{LR}(p) = N_0 \subset M_1 \supset N_1 \subset \cdots \subset M_m \supset N_m = \mathrm{RR}(p)]],$$

where each $[[N_i \subset M_{i+1} \supset N_{i+1}]]$ is a reduced expression for an atomic coset $\mathtt{a}_{i+1}$. In particular, $p = \mathtt{a}_1.\mathtt{a}_2.\cdots.\mathtt{a}_m$. Any core coset has an atomic reduced expression, see Corollary 2.17 in [9], and thus any coset has an atomic-factored reduced expression by Proposition 4.28 in [6].

We have

$$\mathrm{BS}(I_\bullet) = R^{\mathrm{LR}(p)} \otimes_{R^{M_1}} R^{N_1} \otimes_{R^{M_2}} \cdots \otimes_{R^{M_m}} R^{\mathrm{RR}(p)} \tag{5.6}$$

viewed as an (R^I, R^J)-bimodule. Meanwhile, $\mathrm{BS}(I'_\bullet)$ is the same abelian group, but is viewed as an $(R^{\mathrm{LR}(p)}, R^{\mathrm{RR}(p)})$-bimodule. There is an action of each R^{N_i} on $\mathrm{BS}(I_\bullet)$ by multiplication in the i-th tensor factor of (5.6). Indeed, this induces an injective map

$$\mathrm{BS}(I_\bullet) \to \mathrm{End}(\mathrm{BS}(I_\bullet)) \tag{5.7}$$

which is not surjective in general.

An arbitrary reduced expression for p might never factor through the subset $\mathrm{LR}(p)$ or $\mathrm{RR}(p)$. The first advantage of an atomic-factored expression is that there is an obvious action of $R^{\mathrm{LR}(p)}$ on $\mathrm{BS}(I_\bullet)$ by left-multiplication, and an obvious action of $R^{\mathrm{RR}(p)}$ by right-multiplication. The goal is to prove that these two actions agree up to a twist by $\underline{p}$, modulo lower terms.

We denote by $\mathrm{Id}_{I_\bullet}$ the identity morphism of $\mathrm{BS}(I_\bullet)$.

Definition 5.7. Let $I_\bullet$ be an atomic-factored reduced expression as in (5.5). We say that *polynomial forcing* holds for $I_\bullet$ if for all $f \in R^{\mathrm{RR}(p)}$, within $\mathrm{End}(\mathrm{BS}(I_\bullet))$ as described in (5.6) and (5.7), we have

$$\underline{p}(f) \cdot \mathrm{Id}_{I_\bullet} \equiv \mathrm{Id}_{I_\bullet} \cdot f \quad \text{modulo } \mathrm{End}_{<p}(\mathrm{BS}(I_\bullet)). \tag{5.8}$$

We say that *polynomial forcing* holds for a double coset p if it holds for all atomic-factored reduced expressions $I_\bullet$ satisfying $I_\bullet \leftrightharpoons p$.

This definition generalizes Definition 5.1 because atomic cosets have only one reduced expression.

Let p be an arbitrary (I, J)-coset. Since $\mathrm{LR}(p) \subset I$, there is an inclusion of rings $R^I \subset R^{\mathrm{LR}(p)}$, and $R^{\mathrm{LR}(p)}$ is naturally an R^I-module. Similarly, $R^{\mathrm{RR}(p)}$ is an R^J-module. If $f \in R^{\mathrm{RR}(p)}$, then $\underline{p}(f) \in R^{\mathrm{LR}(p)}$. We can identify the rings $R^{\mathrm{LR}(p)}$ and $R^{\mathrm{RR}(p)}$ via $\underline{p}$. In this way, $R^{\mathrm{LR}(p)}$ becomes an (R^I, R^J)-bimodule.

Definition 5.8. Let p be an (I, J)-coset. The *standard bimodule associated to* p, denoted R_p, is $R^{\mathrm{LR}(p)}$ as a left R^I-module. If $f \in R^J$ and $m \in R_p$, then

$$m \cdot f := \underline{p}(f) m.$$

We identify R_p with either $R^{\mathrm{LR}(p)}$ (with right action twisted) or $R^{\mathrm{RR}(p)}$ (with left action twisted), as is more convenient.

Let

$$\mathrm{qu} : \mathrm{End}(\mathrm{BS}(I_\bullet)) \to \mathrm{End}(\mathrm{BS}(I_\bullet)) / \mathrm{End}_{<p}(\mathrm{BS}(I_\bullet))$$

denote the quotient map.

Lemma 5.9. *Let p be a core (I, J)-coset and let $I_\bullet$ be a reduced expression for p. The bimodule map*

$$R_p \to \mathrm{End}(\mathrm{BS}(I_\bullet)) / \mathrm{End}_{<p}(\mathrm{BS}(I_\bullet)), \quad 1 \mapsto \mathrm{qu}(\mathrm{Id}_{I_\bullet}), \tag{5.9}$$

is well defined if and only if polynomial forcing holds for $I_\bullet$.

Proof. The right action of $f \in R^J$ on $1 \in R^J = R_p$ yields $f \in R^J$, and the right action on $\mathrm{qu}(\mathrm{Id}_{I_\bullet})$ yields $\mathrm{qu}(\mathrm{Id}_{I_\bullet} \cdot f)$. The left action of $\underline{p}(f) \in R^I$ on $1 \in R^J = R_p$ yields $f \in R^J$, and the left action on $\mathrm{qu}(\mathrm{Id}_{I_\bullet})$ yields $\mathrm{qu}(\underline{p}(f) \cdot \mathrm{Id}_{I_\bullet})$. These agree if and only if the bimodule map is well defined, and if and only if (5.8) holds. ∎

In conclusion, we have shown the equivalence of three ideas (for almost SW-realizations) for an atomic coset a: the well-definedness of the morphism (5.9) when $p = \mathsf{a} \leftrightharpoons I_\bullet$, atomic polynomial forcing, and the atomic Leibniz rule.

For SW-realizations, (5.9) is an isomorphism by the theory of singular Soergel bimodules. We can thus prove one of our main theorems.

Theorem 5.10. *For an SW-realization, atomic polynomial forcing and atomic Leibniz hold. Moreover, the operators T_q and T'_q in the Leibniz formulas are unique.*

Proof. Assume $\mathtt{a} \leftrightharpoons [I, M, J]$ is an atomic coset. Then $B_{\mathtt{a}} \cong \mathrm{BS}([I, M, J])$.

Recall from Section 4.5 of [19] the definition of the submodule $\Gamma_{<\mathtt{a}} B_{\mathtt{a}}$ of elements supported on lower cosets. By Lemma 3.31 in [8], we have a short exact sequence

$$0 \to \mathrm{End}(B_{\mathtt{a}}, \Gamma_{<\mathtt{a}} B_{\mathtt{a}}) \to \mathrm{End}(B_{\mathtt{a}}) \to \mathrm{Hom}(B_{\mathtt{a}}, B_{\mathtt{a}}/\Gamma_{<\mathtt{a}} B_{\mathtt{a}}) \to 0 \tag{5.10}$$

and, by Theorem 3.30 in [8], the first term in (5.10) is isomorphic to $\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$. Moreover, since $B_{\mathtt{a}}$ is indecomposable, by Theorem 7.10 in [19], we have $B_{\mathtt{a}}/\Gamma_{<\mathtt{a}} B_{\mathtt{a}} \cong R_{\mathtt{a}}$. The Soergel–Williamson hom formula (Theorem 7.9 in [19]) implies that we have an isomorphism[7] $\mathrm{Hom}(B_{\mathtt{a}}, R_{\mathtt{a}}) \cong R_{\mathtt{a}}$ given by $f \mapsto f(1 \otimes 1)$. Putting all together, we obtain an isomorphism

$$\mathrm{End}(B_{\mathtt{a}})/\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}}) \xrightarrow{\sim} R_{\mathtt{a}}$$

which sends $\mathrm{id}_{\mathtt{a}}$ to $1 \in R_{\mathtt{a}}$

By Lemma 5.9, the existence of the isomorphism implies polynomial forcing for $\mathtt{a}$. By Theorem 5.5, this is in turn equivalent to the atomic Leibniz rule for $\mathtt{a}$. Moreover, as proven in Theorem 5.5, the operators T_q and T'_q in the Leibniz formulas are unique. ■

Remark 5.11. For an SW-realization, there is an equivalent module-theoretic (rather than morphism-theoretic) version of polynomial forcing. We first recall from Definition 3.23 in [9] the filtration on Soergel bimodules

$$N_{<p}(B) = \sum_{f \in \mathrm{Hom}(\mathrm{BS}(I_\bullet), B),\, I_\bullet \leftrightharpoons q < p} \mathrm{Im}(f).$$

In Proposition 3.25 of [9], we have showed that this coincides with the support filtration $\Gamma_{<p}$ introduced in [19].

Let $\mathtt{a} \leftrightharpoons [I, M, J]$ be an atomic coset. We say that (*module-theoretic*) *polynomial forcing* holds for $\mathtt{a}$ and f if

$$\underline{\mathtt{a}}(f) \otimes 1 - 1 \otimes f \in N_{<\mathtt{a}}(B_{\mathtt{a}}). \tag{5.11}$$

There is an isomorphism $B_{\mathtt{a}} \cong \mathrm{End}(B_{\mathtt{a}})$, where $b \otimes b' \in B_{\mathtt{a}}$ is sent to multiplication by $b \otimes b'$. Moreover, by Theorem 3.30 in [9], we have

$$\mathrm{Hom}(B_{\mathtt{a}}, N_{<\mathtt{a}} B_{\mathtt{a}}) \cong \mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}}).$$

Hence, (5.11) holds if and only if multiplication by $\underline{\mathtt{a}}(f) \otimes 1 - 1 \otimes f$ induces a morphism in $\mathrm{End}_{<\mathtt{a}}(B_{\mathtt{a}})$, that is, if an only if (morphism-theoretic) polynomial forcing holds for f.

[7] As in the rest of this paper, we are ignoring degrees here.

5.4. Polynomial forcing: atomic and general

In the diagrammatic category, we intend to use the atomic Leibniz rule to prove polynomial forcing, and not vice versa. In that context, polynomial forcing is to be interpreted as the morphism-theoretic statement that (5.9) is a well-defined morphism, when $I_\bullet$ is an atomic-factored reduced expression. The goal of this section is to prove that atomic polynomial forcing implies general polynomial forcing.

In [8], a compatibility between the Bruhat order and concatenation of reduced expression is proven, which implies the following result.

Proposition 5.12 (Proposition 3.7 in [8]). *Let $P_\bullet \leftrightharpoons p$ and $Q_\bullet \leftrightharpoons q$ and $R_\bullet \leftrightharpoons r$ be reduced expressions such that $P_\bullet \circ Q_\bullet \circ R_\bullet \leftrightharpoons p.q.r$ is reduced. Then*

$$\mathrm{id}_{P_\bullet} \otimes \mathrm{End}_{<q}(Q_\bullet) \otimes \mathrm{id}_{R_\bullet} \subset \mathrm{End}_{<p.q.r}(P_\bullet \circ Q_\bullet \circ R_\bullet).$$

Theorem 5.13. *Assume an SW-realization. Then polynomial forcing holds for all double cosets.*

Proof. We first treat the case where $p = p^{\mathrm{core}}$ is a core (I, J)-coset. Consider an atomic reduced expression $I_\bullet$ for p, yielding atomic cosets a_i such that $p = \mathsf{a}_1.\mathsf{a}_2.\cdots.\mathsf{a}_m$. Since a_i are core cosets, by Lemma 2.10 in [9], we have $\underline{p} = \underline{\mathsf{a}_1} \cdot \underline{\mathsf{a}_2} \cdots \underline{\mathsf{a}_m}$. Now within $\mathrm{BS}(I_\bullet)$ we have

$$\begin{aligned}\mathrm{Id}_{\mathsf{a}_1} \otimes \cdots \otimes \mathrm{Id}_{\mathsf{a}_m} f &\equiv \mathrm{Id}_{\mathsf{a}_1} \otimes \cdots \otimes \underline{\mathsf{a}_m}(f)\, \mathrm{Id}_{\mathsf{a}_m} \\ &\equiv \cdots \equiv \underline{\mathsf{a}_1}(\cdots(\underline{\mathsf{a}_m}(f)))\, \mathrm{Id}_{\mathsf{a}_1} \otimes \cdots \otimes \mathrm{Id}_{\mathsf{a}_m},\end{aligned}$$

where $\equiv$ indicates equality modulo lower terms. At each step we applied polynomial forcing for an atomic coset as proved in Theorem 5.10, and used Proposition 5.12 to argue that lower terms for a_i embed into lower terms for p. Thus polynomial forcing holds for p.

If p is not a core coset, let $I_\bullet$ be a special reduced expression for p as in (5.5). Polynomial forcing for p^{core} implies that

$$\underline{p}(f) \cdot \mathrm{Id}_{I_\bullet} \equiv \mathrm{Id}_{I_\bullet} \cdot f$$

modulo

$$\mathrm{id}_{[[I \supset \mathrm{LR}(p)]]} \otimes \mathrm{End}_{<p^{\mathrm{core}}}(I'_\bullet) \otimes \mathrm{id}_{[[\mathrm{RR}(p) \subset J]]}\,.$$

By Proposition 5.12, they are also equivalent modulo $\mathrm{End}_{<p}(I_\bullet)$, as desired. ∎

The reader familiar with Soergel bimodules might be familiar with the following example, which showcases how atomic polynomial forcing implies the general case. It also relates our new concept of polynomial forcing to the concept previously in the literature.

Example 5.14. Consider the $(\emptyset, \emptyset)$-coset $p = \{s\}$ for a simple reflection s, and the reduced expression $I_\bullet = [\emptyset, s, \emptyset]$. Inside $B_s := \mathrm{BS}(I_\bullet) = R \otimes_{R^s} R$, we have

$$1 \otimes f - s(f) \otimes 1 = \partial_s(f) \cdot \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s).$$

The term on the right-hand side is in $\mathrm{End}_{<s}(I_\bullet)$, a consequence of the so-called *polynomial forcing relation* in the Hecke category, see, e.g., equation (5.2) in [14].

Now consider the $(\emptyset, \emptyset)$ coset $p = \{w\}$ for some $w \in W$, and a reduced expression $I_\bullet = [\emptyset, s_1, \emptyset, s_2, \ldots, \emptyset, s_d, \emptyset]$. By applying the polynomial forcing relation for B_{s_d}, we see that $1 \otimes \cdots \otimes 1 \otimes f \equiv 1 \otimes \cdots \otimes s_d(f) \otimes 1$ modulo maps which factor through $[\emptyset, s_1, \emptyset, \ldots, \emptyset, s_{d-1}, \emptyset]$. Continuing, we can apply polynomial forcing for each B_{s_i} to force f across all the tensors, at the cost of maps which factor through subexpressions of $s_1 \cdots s_d$. By the subexpression property of the Bruhat order, such maps consist of lower terms.

6. Atomic Leibniz rule in type A

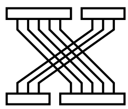
In this section, we explicitly prove condition (3) in Theorem 5.5 in type A. We establish this result for $\Bbbk = \mathbb{Z}$ in this section, rather than over a field. In addition to extending our results over $\mathbb{Z}$, we feel the ability to be explicit in a key example is its own reward.

In order to achieve this, we first prove in Theorem 6.3 explicitly an atomic Leibniz rule for a specific set of generators when W_M is the entire symmetric group. In §6.3, we extend our results to the case when W_M is a product of symmetric groups. This handles all atomic cosets in type A.

6.1. Notation in type A

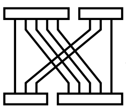
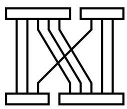
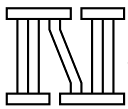
We fix notation under the assumption that W_M is an irreducible Coxeter group of type A. For $s, t \in M$, we write $\widehat{s} := M \setminus \{s\}$ and $\widehat{s}\widehat{t} := M \setminus \{s, t\}$, etcetera.

Fix $a, b \geq 1$ and let $n = a + b$. Let $W_M = \mathbf{S}_n$. Let $t = s_a$ and $s = s_b = w_M t w_M$, so that $W_J = \mathbf{S}_a \times \mathbf{S}_b$ and $W_I = \mathbf{S}_b \times \mathbf{S}_a$. Let $\mathtt{a}$ be the $(\mathbf{S}_b \times \mathbf{S}_a, \mathbf{S}_a \times \mathbf{S}_b)$-coset containing w_M. The coset $\mathtt{a}$ is depicted as follows, with its minimal element $\underline{\mathtt{a}}$ being the string diagram visible:

$\mathtt{a} =$  .

Drawn is the example $a = 3$ and $b = 5$.

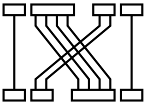
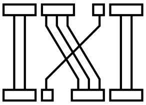
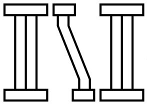
For each $0 \leq k \leq \min(a, b)$, there is an $(\widehat{s}, \widehat{t})$-coset q_k depicted as follows:

$q_1 =$  , $q_2 =$  , $q_3 =$  .

Then $\mathtt{a} = q_0$, and $\{q_k\}_{0 \leq k \leq \min(a,b)}$ is an enumeration of all the $(\widehat{s}, \widehat{t})$-cosets. The Bruhat order is a total order in this case:

$$q_0 > q_1 > q_2 > \cdots > q_{\min(a,b)}.$$

The left redundancy subgroup (see Section 1.2 of [9] to see how to calculate redundancies and cores) of q_k is $\mathbf{S}_k \times \mathbf{S}_{b-k} \times \mathbf{S}_{a-k} \times \mathbf{S}_k$. For brevity, let $\ell := n - k$. Then (with one exception) $\mathrm{LR}(q_k) = \widehat{b}\widehat{k}\widehat{\ell} := M \setminus \{b, k, \ell\}$ and $\mathrm{RR}(q_k) = \widehat{a}\widehat{k}\widehat{\ell}$. The core of q_k is the double coset depicted as

$q_1^{\text{core}} =$ , $q_2^{\text{core}} =$ , $q_3^{\text{core}} =$ .

Note that the core of q_k is itself atomic, except for $k = \min(a, b)$ when the core is an identity coset.

The case $k = a = b = \ell$ is relatively special. We denote this special coset as $q_{a=b}$. We have $\mathrm{LR}(q_{a=b}) = \mathrm{RR}(q_{a=b}) = I = J$. Unlike other q_i, $q_{a=b}$ is core.

Remark 6.1. In type A, the following statement is always true: if $\mathtt{a}$ is atomic and $q < \mathtt{a}$ then q^{core} is either atomic or an identity coset. We do not know for which atomic cosets $\mathtt{a}$ this property holds in other types.

A reduced expression for q_k, which factors through the core, is

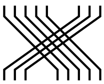
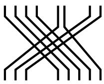
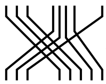
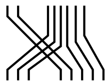
(6.1) $$q_k \leftrightharpoons [[\widehat{b} \supset \widehat{b}\widehat{k}\widehat{\ell} \subset \widehat{k}\widehat{\ell} \supset \widehat{a}\widehat{k}\widehat{\ell} \subset \widehat{a}]].$$

The exception is when $a = b = k$, in which case

(6.2) $$q_{a=b} \leftrightharpoons [\widehat{b}],$$

that is, the identity expression of $I = \widehat{b}$ is a reduced expression for the length zero coset $q_{a=b}$.

Finally, let us write $y_k = \overline{q_k} w_J^{-1}$. Then $\partial_{q_k} = \partial_{y_k}$. Here are examples of y_k:

(6.3) $y_0 =$ , $y_1 =$ , $y_2 =$ , $y_3 =$ .

Of course, $y_0 = \underline{\mathtt{a}}$. Meanwhile, y_k is obtained from y_0 by removing a $k \times k$ square of crossings from the bottom. In the special case of the coset $q_{a=b}$, we have $y_{a=b} = e$, the identity of W.

6.2. Complete symmetric polynomials

Fix $a, b \geq 1$ and continue to use the notation from the previous section. The standard action of $\mathbf{S}_n$ on $\mathbb{Z}^n$ (with the standard choice of roots and coroots) we call the *permutation realization*. Let $R = \mathbb{Z}[x_1, \ldots, x_n]$. All Demazure operators preserve R. By a result of Demazure (see Lemme 5 in [3]), Frobenius surjectivity holds in type A over $\mathbb{Z}$, that is, for any $I \subset M$ we can find $P_I \in R$ such that $\partial_I(P_I) = 1$.

Moreover, for any $J \subset I$ the ring R^I is Frobenius over R^J and we can choose dual bases $\Delta^J_{I,(1)}$ and $\Delta^J_{I,(1)}$ accordingly.

It is well known that the subring $R^J = R^{\mathbf{S}_a \times \mathbf{S}_b}$ is generated over $R^{\mathbf{S}_n}$ by the complete symmetric polynomials

$$h_i(x_1, \ldots, x_a) = \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_i \leq a} x_{k_1} x_{k_2} \cdots x_{k_i}$$

in the first a variables. In this section, we directly prove the atomic Leibniz rule for a when $f = h_i(x_1, \ldots, x_a)$.

One of the great features of complete symmetric polynomials is their behavior under Demazure operators. For example, we have

$$\partial_3(h_i(x_1, x_2, x_3)) = h_{i-1}(x_1, x_2, x_3, x_4).$$

As a consequence, $\partial_2\partial_3(h_i(x_1, x_2, x_3)) = 0$, a fact which is false if h_i is replaced by some general polynomial inside $R^{\mathbf{S}_3 \times \mathbf{S}_{n-3}}$. Indeed, the only elements $w \le s_1 s_2 s_3$ for which $\partial_w(h_i(x_1, x_2, x_3)) \neq 0$ are $w = s_3$ and $w = e$. This will simplify the computation considerably.

Below we shall use letters like X, Y, and Z to denote subsets of $\{1, \ldots, n\}$. We write $h_i(X)$ for the i-th complete symmetric polynomial in the variables x_j for $j \in X$.

Lemma 6.2. *We have*

$$\partial_j(h_i(X)) = \begin{cases} h_{i-1}(X \cup \{j+1\}) & \text{if } j \in X \text{ and } j+1 \notin X, \\ -h_{i-1}(X \cup \{j\}) & \text{if } j+1 \in X \text{ and } j \notin X, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

Proof. Clearly, $h_i(X)$ is s_j-invariant if both j and $j+1$ are inside or outside X, so $\partial_j(h_i(X)) = 0$. Assume now that $j \in X$ and $j+1 \notin X$. We have

$$h_i(X) = h_i(X \setminus \{j\}) + h_{i-1}(X) x_j.$$

Hence,

$$\partial_j(h_i(X)) = \partial_j(h_{i-1}(X) x_j).$$

Applying the twisted Leibniz rule, by induction on $|X|$, we obtain

$$\begin{aligned} \partial_j(h_i(X)) &= h_{i-2}(X \cup \{j+1\}) x_j + s_j(h_{i-1}(X)) \partial_j(x_j) \\ &= h_{i-2}(X \cup \{j+1\}) x_j + h_{i-1}(X \cup \{j+1\} \setminus \{j\}) \\ &= h_{i-1}(X \cup \{j+1\}). \end{aligned}$$

The case where $j+1 \in X$ and $j \notin X$ follows because $\partial_j(s_j(f)) = -\partial_j(f)$, so we have

$$\partial_j(h_i(X)) = -\partial_j(h_i(X \cup \{j\} \setminus \{j+1\})) = -h_{i-1}(X \cup \{j\}).$$ ■

Theorem 6.3. *Use the notation of §6.1. Fix $a, b \ge 1$, and let $n = a + b$. Let $X = \{1, \ldots, a\}$ and $Y = \{n, n-1, \ldots, n+1-a\}$. Recall $y_0, y_1 \in \mathbf{S}_n$ from (6.3).*

Then for any $i \ge 0$ and any $g \in R^{\mathbf{S}_a \times \mathbf{S}_b} = R^J$, we have

$$\partial_{y_0}(h_i(X) \cdot g) = h_i(Y) \cdot \partial_{y_0}(g) + \partial_{y_1}(h_{i-1}(X \cup n) \cdot g).$$

As $\partial_p = \partial_{y_0}$, $\underline{p} = y_0$, and $y_0(h_i(X)) = h_i(Y)$, this is compatible with (3.1), where $T_{q_1}(h_i(X)) = h_{i-1}(X \cup n)$, and $T_{q_k}(h_i(X))$ is zero for all $k > 1$. Most of the terms in (3.1) are zero for complete symmetric polynomials, making the formula much easier than the general case.

Proof. We will do a proof by example, for the example[8] $a = 3$ and $b = 4$. The general proof is effectively the same only the notation is more cumbersome.

In this proof, we write ∂_{123} for $\partial_1 \circ \partial_2 \circ \partial_3$ (and not the Frobenius trace associated to the longest element w_{123}). We use parenthesization for emphasis, so that $\partial_{(12)3}$ is the same thing as ∂_{123}, but emphasizes that $\partial_{(12)3} = \partial_{12} \circ \partial_3$.

Remember that g is invariant under anything except s_3, so $\partial_j(g) = 0$ if $j \neq 3$. This implies, for example, that ∂_3 kills $\partial_{23}(g)$, and that ∂_2 kills $\partial_{3243}(g)$, etcetera.

We claim that

$$\partial_{123}(h_i(123)\,g) = h_{i-1}(1234)\,\partial_{23}(g) + h_i(234)\,\partial_{123}(g). \tag{6.5}$$

One proof is to apply the ordinary twisted Leibniz rule repeatedly using (6.4). After one step, we obtain

$$\partial_{123}(h_i(123)\,g) = \partial_{12}(h_{i-1}(1234)\,g + h_i(124)\,\partial_3(g)).$$

The first term on the right-hand side is invariant under s_2, so it is killed by ∂_2. Thus we have

$$\partial_{123}(h_i(123)\,g) = \partial_1(\partial_2(h_i(124)\,\partial_3(g))) = \partial_1(h_{i-1}(1234)\,\partial_3(g) + h_i(134)\,\partial_{23}(g)).$$

Again the first term on the right-hand side is invariant under s_1, so it is killed by ∂_1. A final application of the twisted Leibniz rule to $\partial_1(h_i(134)\,\partial_{23}(g))$ gives (6.5). Essentially, this proof is by iterating the twisted Leibniz rule and arguing that the first term vanishes in every application but the last, because the first term is appropriately invariant. We call this the *easy invariance argument*.

Note that $\partial_{123}(g)$ is invariant under everything but s_4. More generally, $\partial_{12\ldots a}(g)$ is killed by ∂_j for all $j \neq a+1$. This is true when $j = 1$ since $\partial_1\partial_1 = 0$. This is true when $j > a+1$ since $\partial_j\partial_{12\ldots a} = \partial_{12\ldots a}\partial_j$ and $\partial_j(g) = 0$. This is true when $2 \leq j \leq a$ because $\partial_j\partial_{12\ldots a} = \partial_{12\ldots a}\partial_{j-1}$ and $\partial_{j-1}(g) = 0$.

By the easy invariance argument again, but with indices shifted and g replaced by $\partial_{123}(g)$, we have

$$\partial_{234}(h_i(234)\,\partial_{123}(g)) = h_{i-1}(2345)\,\partial_{34123}(g) + h_i(345)\,\partial_{234123}(g). \tag{6.6}$$

This is how we treat the second term in the right side of (6.5).

Note that all computations above are unchanged by adding new variables to our complete symmetric polynomials which are untouched by any of the simple reflections used by the formula. For example, adding 7 to every h_i in (6.5), we get

$$\partial_{123}(h_i(1237)\,g) = h_{i-1}(12347)\,\partial_{23}(g) + h_i(2347)\,\partial_{123}(g).$$

In this example, we call 7 an *irrelevant index*.

Now we examine the first term in the right side of (6.5). Note that $\partial_{23}(g)$ is invariant under all simple reflections except s_1 and s_4. For the next computation, the index 1 is irrelevant. The easy invariance argument again implies that

$$\partial_{234}(h_{i-1}(1234)\,\partial_{23}(g)) = h_{i-2}(12345)\,\partial_{3423}(g) + h_{i-1}(1345)\,\partial_{23423}(g).$$

[8] Our chosen example is slightly different from the example $a = 3$ and $b = 5$ from the previous section.

However, as $23423 = 32434$, we have $\partial_{23423}(g) = 0$, so one has the simpler formula

$$\partial_{234}(h_{i-1}(1234)\,\partial_{23}(g)) = h_{i-2}(12345)\,\partial_{3423}(g). \tag{6.7}$$

Overall, we see that

$$\begin{aligned}\partial_{(234)(123)}(h_i(123)g) = {} & h_i(345)\,\partial_{(234)(123)}(g) \\ & + h_{i-1}(2345)\,\partial_{(34)(123)}(g) \\ & + h_{i-2}(12345)\,\partial_{(34)(23)}(g).\end{aligned} \tag{6.8}$$

The pattern is relatively straightforward. Here is the next one in the pattern:

$$\begin{aligned}\partial_{(345)(234)(123)}(h_i(123)g) = {} & h_i(456)\,\partial_{(345)(234)(123)}(g) \\ & + h_{i-1}(3456)\,\partial_{(45)(234)(123)}(g) \\ & + h_{i-2}(23456)\,\partial_{(45)(34)(123)}(g) \\ & + h_{i-3}(123456)\,\partial_{(45)(34)(23)}(g).\end{aligned} \tag{6.9}$$

The word whose Demazure is applied to g is obtained from the concatenation of triples (345)(234)(123) by removing the first index from some of the triples; more specifically, from a prefix of the set of triples. The indices that get removed are instead added to the complete symmetric polynomial. The reason triples appear is because $a = 3$.

The inductive proof of this pattern is the same as above. First, one takes (6.8) and applies ∂_{345}. The first term splits in two, giving the first two terms of (6.9), similar to (6.5) or (6.6). Each other term contributes one term in (6.9), similar to (6.7).

Note that we could have added the irrelevant index 7 to every set in sight within (6.9), without any issues. This will be important later.

Repeating until one applies ∂_{y_0}, we calculate $\partial_{y_0}(h_i(X) \cdot g)$:

$$\begin{aligned}\partial_{(456)(345)(234)(123)}(h_i(123)g) = {} & h_i(567)\,\partial_{(456)(345)(234)(123)}(g) \\ & + h_{i-1}(4567)\,\partial_{(56)(345)(234)(123)}(g) \\ & + h_{i-2}(34567)\,\partial_{(56)(45)(234)(123)}(g) \\ & + h_{i-3}(234567)\,\partial_{(56)(45)(34)(123)}(g) \\ & + h_{i-4}(1234567)\,\partial_{(56)(45)(34)(23)}(g).\end{aligned} \tag{6.10}$$

The fact that there were four triples is because $b = 4$. Note that the first term in the right-hand side is $h_i(Y)\partial_{y_0}(g)$.

Let us compute $\partial_{y_1}(h_{i-1}(1237) \cdot g)$. Note that $y_1 = (56)(345)(234)(123)$. To compute $\partial_{(345)(234)(123)}(h_{i-1}(1237)\,g)$, we take (6.9), add the irrelevant index 7 to all variable lists, and reduce i by one. Now we need only apply $\partial_{(56)}$ to the result. The key thing to note here is that each $h_\bullet(\cdots 567)$ is invariant already under s_5 and s_6. Thus both operators in $\partial_{(56)}$ simply apply to the g term. From this, we can compute $\partial_{y_1}(h_{i-1}(X \cup n) \cdot g)$:

$$\begin{aligned}\partial_{(56)(345)(234)(123)}(h_{i-1}(1237)g) = {} & h_{i-1}(4567)\,\partial_{(56)(345)(234)(123)}(g) \\ & + h_{i-2}(34567)\,\partial_{(56)(45)(234)(123)}(g) \\ & + h_{i-3}(234567)\,\partial_{(56)(45)(34)(123)}(g) \\ & + h_{i-4}(1234567)\,\partial_{(56)(45)(34)(23)}(g).\end{aligned}$$

This exactly matches all terms from (6.10) except the first term. Thus the theorem is proven! ∎

Theorem 6.4. *The atomic Leibniz rule and atomic polynomial forcing both hold for atomic cosets* $\mathsf{a} \leftrightharpoons [I, M, J]$ *when* $W_M = \mathbf{S}_n$ *when* $R = \mathbb{Z}[x_1, \ldots, x_n]$.

Proof. Theorem 6.3 proved property (3) from Theorem 5.5 in this case. Thus conditions (1) and (5) also hold in this case. ∎

6.3. Reduction to the connected case

The previous section proves an atomic Leibniz rule under the assumption $W_M = \mathbf{S}_n$. Now we do the general case.

Let $W = \mathbf{S}_n$. An arbitrary atomic coset in W contains the longest element of the reducible Coxeter group $W_M = \mathbf{S}_{n_1} \times \cdots \times \mathbf{S}_{n_k}$ where $\sum n_i = n$. It is a coset for $(\widehat{s}, \widehat{t})$, where s and t are simple reflections in the same irreducible component $\mathbf{S}_{n_i}$ of W_M. We can prove polynomial forcing for an arbitrary atomic coset in type A if we can bootstrap the result from $\mathbf{S}_{n_i}$ to W_M.

In this discussion, there is no difference between type A and a general Coxeter type. Thus let M be finitary, with $s, t \in M$ and $w_M s w_M = t$. Let a be the atomic $(\widehat{s}, \widehat{t})$-coset containing w_M.

Now suppose that $M = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k$ is a disjoint union of connected components (the simple reflections in C_i commute with those in C_j for $i \neq j$). Suppose without loss of generality that $s \in C_1$, and let $D = C_2 \sqcup \ldots \sqcup C_k$. Then $t \in C_1$ as well, and $t = w_{C_1} s w_{C_1}$. Let a' denote the atomic $(C_1 \setminus s, C_1 \setminus t)$-coset containing w_{C_1}. Then a and a' are related by the operation $+D$ described in Section 4.10 of [6].

Lemma 6.5. *With the notation as above, polynomial forcing holds for* a *if and only if it holds for* a'.

Lemma 6.6. *With the notation as above, an atomic Leibniz rule holds for* a *if and only if it holds for* a'.

Proof. The proof is straightforward and left to the reader, but we wish to point out the available ingredients. Many basic properties of the operator $+D$ are given in Section 4.10 of [6]. There is a bijection between cosets $q < \mathsf{a}$ and cosets $q' < \mathsf{a}'$, and also a bijection between their reduced expressions. Note that

$$\partial_{\overline{q} w_{\widehat{t}}^{-1}} = \partial_{\overline{q}' w_{C_1 \setminus t}^{-1}}$$

as operators $R \to R$. Finally, dual bases for the Frobenius extension $R^M \subset R^{M \setminus t}$ can also be chosen as dual bases for the Frobenius extension $R^{C_1} \subset R^{C_1 \setminus t}$. ∎

Theorem 6.7. *The atomic Leibniz rule and polynomial forcing hold for any atomic coset in type* A_{n-1} *when* $R = \mathbb{Z}[x_1, \ldots, x_n]$.

Proof. The restriction of the permutation realization to any $\mathbf{S}_{n_i} \subset \mathbf{S}_n$ is a W-invariant enlargement of the permutation realization of $\mathbf{S}_{n_i}$. Thus the result follows from Lemma 6.5, Lemma 6.6, Theorem 6.4, and Lemma 3.17. ∎

Applying Lemma 3.17, we also deduce the atomic Leibniz rule for a host of other realizations, including when $R = \Bbbk[x_1, \dots, x_n]$ for any commutative ring $\Bbbk$.

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References

[1] Abe, N.: Singular Soergel bimodules for realizations. *Int. Math. Res. Not. IMRN* (2025), no. 1, article no. rnae274, 29 pp. Zbl 1560.20012 MR 4846616

[2] Achar, P. N., Makisumi, S., Riche, S. and Williamson, G.: Koszul duality for Kac–Moody groups and characters of tilting modules. *J. Amer. Math. Soc.* **32** (2019), no. 1, 261–310. Zbl 1450.20011 MR 3868004

[3] Demazure, M.: Invariants symétriques entiers des groupes de Weyl et torsion. *Invent. Math.* **21** (1973), 287–301. Zbl 0269.22010 MR 0342522

[4] Elias, B.: The two-color Soergel calculus. *Compos. Math.* **152** (2016), no. 2, 327–398. Zbl 1382.20006 MR 3462556

[5] Elias, B. and Khovanov, M.: Diagrammatics for Soergel categories. *Int. J. Math. Math. Sci.* (2010), article no. 978635, 58 pp. Zbl 1219.18003 MR 3095655

[6] Elias, B. and Ko, H.: A singular Coxeter presentation. *Proc. Lond. Math. Soc. (3)* **126** (2023), no. 3, 923–996. Zbl 1527.20057 MR 4563864

[7] Elias, B., Ko, H., Libedinsky, N. and Patimo, L.: Demazure operators for double cosets. Preprint 2023, arXiv:2307.15021.

[8] Elias, B., Ko, H., Libedinsky, N. and Patimo, L.: Subexpressions and the Bruhat order for double cosets. Preprint 2023, arXiv:2307.15726.

[9] Elias, B., Ko, H., Libedinsky, N. and Patimo, L.: On reduced expressions for core double cosets. *Math. Res. Lett.* **32** (2026), no. 6, 1845–1876. MR 5026240

[10] Elias, B., Ko, H., Libedinsky, N. and Patimo, L.: Singular light leaves. To appear in *J. Eur. Math. Soc.* (2025), published online first, DOI 10.4171/JEMS/1715.

[11] Elias, B., Makisumi, S., Thiel, U. and Williamson, G.: *Introduction to Soergel bimodules*. RSME Springer Ser. 5, Springer, Cham, 2020. Zbl 1507.20001 MR 4220642

[12] Elias, B., Snyder, N. and Williamson, G.: On cubes of Frobenius extensions. In *Representation theory – Current trends and perspectives*, pp. 171–186. EMS Ser. Congr. Rep., European Mathematical Society, Zürich, 2017. Zbl 1369.19003 MR 3644793

[13] Elias, B. and Williamson, G.: The Hodge theory of Soergel bimodules. *Ann. of Math. (2)* **180** (2014), no. 3, 1089–1136. Zbl 1326.20005 MR 3245013

[14] Elias, B. and Williamson, G.: Soergel calculus. *Represent. Theory* **20** (2016), 295–374. Zbl 1427.20006 MR 3555156

[15] Elias, B. and Williamson, G.: Relative hard Lefschetz for Soergel bimodules. *J. Eur. Math. Soc. (JEMS)* **23** (2021), no. 8, 2549–2581. Zbl 1475.20006 MR 4269421

[16] Libedinsky, N.: Presentation of right-angled Soergel categories by generators and relations. *J. Pure Appl. Algebra* **214** (2010), no. 12, 2265–2278. Zbl 1252.20002 MR 2660912

[17] Lusztig, G. and Williamson, G.: Billiards and tilting characters for SL_3. *SIGMA Symmetry Integrability Geom. Methods Appl.* **14** (2018), article no. 015, 22 pp. Zbl 1447.20007 MR 3766576

[18] Riche, S. and Williamson, G.: Tilting modules and the p-canonical basis. *Astérisque* (2018), no. 397, ix+184. Zbl 1437.20001 MR 3805034

[19] Williamson, G.: Singular Soergel bimodules. *Int. Math. Res. Not. IMRN* (2011), no. 20, 4555–4632. Zbl 1236.18009 MR 2844932

[20] Williamson, G.: Schubert calculus and torsion explosion. *J. Amer. Math. Soc.* **30** (2017), no. 4, 1023–1046. Zbl 1380.20015 MR 3671935

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Ben Elias
Department of Mathematics, University of Oregon
Fenton Hall, Eugene, OR 97403-1222, USA;
belias@uoregon.edu

Hankyung Ko
Department of Mathematics, Uppsala University
Box 480, 75106 Uppsala, Sweden;
hankyung.ko@math.uu.se

Nicolás Libedinsky
Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile
Santiago, Chile;
nlibedinsky@gmail.com

Leonardo Patimo
Dipartimento di Matematica, Università di Pisa
Pisa, Italy;
leonardo.patimo@unipi.it