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Remarks on the construction of K_σ sets associated to trees not satisfying a separation condition

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Abstract. K_σ sets involving sticky maps σ have been used in the theory of differentiation of integrals to probabilistically construct Kakeya-type sets that imply certain types of directional maximal operators are unbounded on $L^p(\mathbb{R}^2)$ for all $1 \le p < \infty$. We indicate limits to this approach by showing that, given $\varepsilon > 0$ and a natural number N, there exists a tree $\mathcal{T}_{N,\varepsilon}$ of finite height that is lacunary of order N but such that, for *every* sticky map $\sigma\colon \mathcal{B}^{h(\mathcal{T}_{N,\varepsilon})} \to \mathcal{T}_{N,\varepsilon}$, one has $|K_\sigma \cap ((1,2) \times \mathbb{R})| \ge 1 - \varepsilon$.

1. Introduction

Let Ω be a nonempty subset of $[0,1]$. Associated to Ω is the *directional maximal operator M_Ω* acting on measurable functions on $\mathbb{R}^2$, defined by

$$M_\Omega f(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f|,$$

where the supremum is taken over all rectangles in $\mathbb{R}^2$ containing x with longest side having slope in Ω.

If $\Omega = [0,1]$, the maximal operator M_Ω is unbounded on $L^p(\mathbb{R}^2)$ for all $1 \le p < \infty$, see [3, 7]. If Ω is the lacunary set $\{2^{-j} : j \in \mathbb{N}\}$, then M_Ω is bounded on $L^p(\mathbb{R}^2)$ for $1 < p \le \infty$, see [4, 6, 9]. More generally, if Ω is N-lacunary, then M_Ω is bounded on $L^p(\mathbb{R}^2)$ for $1 < p \le \infty$, see [8].

In [2], Bateman and Katz utilized probabilistic methods involving *sticky maps* to show that the maximal operator $M_{\mathcal{C}}$ associated to the ternary Cantor set $\mathcal{C}$ is unbounded on $L^p(\mathbb{R}^2)$ for $1 \le p < \infty$. Bateman subsequently announced a result in [1] that the maximal operator M_Ω is bounded on $L^p(\mathbb{R}^2)$ for all $1 < p \le \infty$ if and only if Ω is a finite union of sets of finite lacunary order. Bateman's clever argument involved using probabilistic methods to show that, if Ω were not of finite lacunary order, then for every $N \in \mathbb{N}$ there would exist a sticky map σ and associated sets

$$K_\sigma = K_{\sigma,1} \cup K_{\sigma,2}$$

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(where the $K_{\sigma,j}$ are of positive measure and with a structure that we will detail in the next section) such that

$$|K_{\sigma,1}| \gtrsim (\ln N)|K_{\sigma,2}|, \quad \text{with} \quad M_\Omega \chi_{K_{\sigma,2}} \gtrsim \frac{1}{2} \quad \text{on } K_{\sigma,1}.$$

Unfortunately, we recently discovered a subtle gap in the proof of this statement in the case that the set Ω, although not of finite lacunary order, fails to satisfy a *separation condition*, although the proof does hold with some minor modification if the separation condition is satisfied [5]. We now recognize that there exist certain non-separated non-finite lacunary sets Ω for which the desired K_σ sets simply do not exist. The purpose of this paper is to show this is the case.

In Section 2, we define the appropriate terminology, largely following that of Bateman in [1]. In Section 3, we construct a set of directions $\Omega \subset [0, 1]$ for which the desired K_σ sets do not exist. In Section 4, we suggest further directions for research in this area.

2. Terminology

In this section we, largely following the terminology and setup of Bateman in [1], define sets K_σ associated to sticky maps σ mapping a truncated binary tree to itself.

We first define the binary tree $\mathcal{B}$. We fix a vertex v_0, called the origin, and define $\mathcal{B}_0 = \{v_0\}$. Suppose $\mathcal{B}_n$ has been defined. To each vertex $v \in \mathcal{B}_n$ we associate two new vertices $c_0(v)$ and $c_1(v)$, and define

$$\mathcal{B}_{n+1} = \bigcup_{v \in \mathcal{B}_n} \{c_0(v), c_1(v)\}.$$

We define the binary tree $\mathcal{B}$ to be the graph with vertices in $\bigcup_{n=0}^{\infty} \mathcal{B}_n$ and edges connecting a vertex v with each of its children $c_0(v)$ and $c_1(v)$. We say the vertices in $\mathcal{B}_n$ are of height n. If the vertex v is of height n, we may write this as $h(v) = n$.

Given a vertex $v \in \mathcal{B}$, we define a ray R rooted at v to be an ordered set of vertices $v_1 = v, v_2, v_3, \ldots$ such that v_{j+1} is a child of v_j for $j = 1, 2, \ldots$. Given a subtree $\mathcal{T}$ of $\mathcal{B}$ and a vertex $v \in \mathcal{T}$, we set $\mathfrak{R}_{\mathcal{T}}(v)$ to be the collection of all rays rooted at v with vertices in $\mathcal{T}$.

If $u \in R$ for some $R \in \mathfrak{R}_{\mathcal{T}}(v)$, we say that u is a descendant of v or that v is an ancestor of u.

Given a subtree $\mathcal{T}$ of $\mathcal{B}$ and $h \in \mathbb{N}$, by $\mathcal{T}^h$ we denote the induced subtree of $\mathcal{T}$ associated to its vertices of height less than or equal to h.

Given a subtree $\mathcal{T}$ of $\mathcal{B}$, we say a vertex $v \in \mathcal{T}$ splits, or we say v is a splitting vertex, if v has two children in $\mathcal{T}$. We define the splitting number split(R) of a ray R in $\mathcal{T}$ to be the number of splitting vertices in $\mathcal{T}$ on R. The splitting number of a vertex v with respect to a tree $\mathcal{S}$ rooted at v is defined as

$$\text{split}_{\mathcal{S}}(v) = \min_{R \in \mathfrak{R}_{\mathcal{S}}(v)} \text{split}(R),$$

and the splitting number of v is defined as

$$\operatorname{split}(v) = \sup_{\mathcal{S}} \operatorname{split}_{\mathcal{S}}(v),$$

where the supremum is taken over all subtrees $\mathcal{S}$ of $\mathcal{T}$ rooted at v. For a tree $\mathcal{T}$, we set

$$\operatorname{split}(\mathcal{T}) = \sup_{v \in \mathcal{T}} \operatorname{split}(v),$$

where the supremum is taken over all the vertices v in $\mathcal{T}$.

A tree $\mathcal{T} \subset \mathcal{B}$ is said to be lacunary of order 0 if it consists of a single ray (possibly truncated to be of finite height) rooted at the origin of $\mathcal{B}$. For $N \geq 1$, $\mathcal{T}$ is said to be lacunary of order N if all of the splitting vertices of $\mathcal{T}$ lie on a lacunary tree of order $N-1$ and moreover that $\mathcal{T}$ is not lacunary of order $N-1$.

Let $\mathcal{T} \subset \mathcal{B}$ be lacunary of order N. We say that $\mathcal{T}$ is pruned provided, for every ray $R \in \mathfrak{R}_{\mathcal{T}}(v_0)$ and every $j = 1, 2, \ldots, N$, R contains exactly one vertex v_j such that split $v_j = j$.

Let v be a vertex in $\mathcal{B}$ of height k. Let $(j_1, \ldots, j_k)$ be a sequence of 0's and 1's such that, letting v_0 denote the origin, v lies on the ray $v_0, v_1, v_2, \ldots, v_k = v, \ldots$ in $\mathcal{B}$ such that $v_i = c_{j_i}(v_{i-1})$ for $i = 1, \ldots, k$. For notational convenience, we will on occasion denote v by the $(k+1)$-string $0j_1 \ldots j_k$, with v_0 itself being denoted simply by the 1-string 0.

Let $\sigma\colon \mathcal{B}^M \to \mathcal{B}^M$. σ is said to be a *sticky map* if $h(\sigma(v)) = h(v)$ for all $v \in \mathcal{B}^M$ and $\sigma(u)$ is an ancestor of $\sigma(v)$ whenever $u, v \in \mathcal{B}^M$ and u is an ancestor of v.

For each $\sigma\colon \mathcal{B}^M \to \mathcal{B}^M$, we may construct a set $K_\sigma \subset \mathbb{R}^2$ as follows.

For every $(M+1)$-string $v = 0j_1 \ldots j_M$ consisting of 0's and 1's, let $(k_1, \ldots, k_M)$ be such that $\sigma(v) = 0k_1 \ldots k_M$. Let ρ_v denote the parallelogram with vertices at the points

$$\Big(0, \sum_{i=1}^{M} 2^{-i} j_i\Big), \quad \Big(0, 2^{-M} + \sum_{i=1}^{M} 2^{-i} j_i\Big), \quad \Big(2, \sum_{i=1}^{M} 2^{-i} j_i + 2\sum_{i=1}^{M} 2^{-i} k_i\Big),$$
$$\text{and} \quad \Big(2, 2^{-M} + \sum_{i=1}^{M} 2^{-i} j_i + 2\sum_{i=1}^{M} 2^{-i} k_i\Big).$$

Define the set $K_\sigma \subset \mathbb{R}^2$ by

$$K_\sigma = \bigcup_{v \in \mathcal{B} : h(v) = M} \rho_v.$$

Note that if v is a vertex of height M in $\mathcal{B}^M$, the parallelogram $\rho_v \subset \mathbb{R}^2$ has a left-hand side that is an interval on the y-axis of length 2^{-M} whose lowest point has a y-coordinate in $[0, 1]$ with binary expansion given by v.

The primary result of this paper is the following.

Theorem 1. *Let $N \in \mathbb{N}$ and $\varepsilon > 0$. There exists a pruned tree $\mathcal{P}$ of finite height that is lacunary of order N such that, for* every *sticky map $\sigma\colon \mathcal{B}^{h(\mathcal{P})} \to \mathcal{P}$, we have*

$$|K_\sigma \cap ([1, 2] \times \mathbb{R})| \geq 1 - \varepsilon.$$

In contrasting this result with Claim 7 (B) of [1], it is helpful to recognize that, as indicated in [5], the proof of Claim 7 (B) implicitly relies on an assumption that $\mathcal{P}$ satisfies a separation condition. Theorem 1 indicates what can happen if such a separation condition is not satisfied.

3. Proof of Theorem 1

Proof of Theorem 1. Let $N \in \mathbb{N}$ and $\varepsilon > 0$. Let $k_1, \dots, k_N$ be a sequence of natural numbers, all greater than 2, such that

$$8\,[2^{-k_1} + \cdots + 2^{-k_N}] < \varepsilon.$$

For $1 \leq i \leq N$, let a_i denote the k_i-string $0111\cdots1$ and b_i denote the k_i-string $1000\cdots0$. For example, if $k_1 = 5$, then $a_1 = 01111$ and $b_1 = 10000$. Let $\mathcal{P}$ be the pruned tree consisting of all of the vertices of the form $0s_1 \dots s_n$ for $1 \leq n \leq N$, where each s_i is either the string a_i or b_i, together with the ancestors of these vertices. Note that $\mathcal{P}$ is lacunary of order N, and $\mathcal{P}$ is a tree of height $k_1 + \cdots + k_N$.

Let $\sigma\colon \mathcal{B}^{h(\mathcal{P})} \to \mathcal{P}$. Note K_σ is the union of $2^{h(\mathcal{P})}$ parallelograms of the form ρ_v indicated above, where v is an element of $\mathcal{B}$ of height $h(\mathcal{P})$.

Let γ denote the number of parallelograms ρ_v for which there is a $w \neq v$ such that $|\rho_v \cap \rho_w| > 0$. Since σ is a sticky map and by the structure of $\mathcal{P}$, we have that, for $1 \leq l \leq N-1$, if ρ_u and ρ_v are parallelograms whose left-hand sides lie in a dyadic interval of length $2^{-k_1-k_2-\cdots-k_l}$ on the y-axis, their slopes are within $2^{-k_1-k_2-\cdots-k_l} \cdot 2 \cdot 2^{-k_{l+1}}$ of each other. Also, all of the ρ_v have slopes within $2 \cdot 2^{-k_1}$ of each other. Accordingly we have the bound

$$\begin{aligned}\gamma &\leq 2^{h(\mathcal{P})}\, 8\,[2^{-k_1} + 2^{k_1}2^{-k_1-k_2} + 2^{k_1+k_2}2^{-k_1-k_2-k_3} + \cdots + 2^{k_1+\cdots+k_{N-1}}2^{-k_1-\cdots-k_N}] \\ &= 2^{h(\mathcal{P})}\, 8\,[2^{-k_1} + \cdots + 2^{-k_N}].\end{aligned}$$

A few remarks justifying the above inequality are in order. Note that if $|\rho_u \cap \rho_v| \neq 0$, then u and v are in different halves of [0,1] or are in different halves of a dyadic interval of length $2^{-k_1-\cdots-k_j}$ for some $1 \leq j \leq N-1$ (as otherwise ρ_u and ρ_v would be parallel.) The number of parallelograms ρ_u whose interior can intersect the interior of a parallelogram ρ_v where u and v are on opposite halves of $[0, 1]$ is $2^{h(\mathcal{P})} \cdot 8 \cdot 2^{-k_1}$. Similarly, given a dyadic interval in $[0, 1]$ of length $2^{-k_1-\cdots-k_{j-1}}$ (and there are $2^{k_1+\cdots+k_{j-1}}$ of these), there are $2^{h(\mathcal{P})} \cdot 8 \cdot 2^{-k_1-\cdots-k_j}$ parallelograms ρ_u that can intersect the interior of a parallelogram ρ_v where u and v are on opposite halves of the interval. Summing over the initial interval $[0, 1]$ and all of the dyadic intervals in $[0, 1]$ with length of the form $2^{-k_1-\cdots-k_j}$ for $1 \leq j \leq N-1$ yields the desired estimate.

Since for each parallelogram ρ_v we have $|\rho_v \cap ([1,2] \times \mathbb{R})| = 2^{-h(\mathcal{P})}$, we have that

$$|K_\sigma \cap ([1,2] \times \mathbb{R})| \geq 1 - \gamma 2^{-h(\mathcal{P})} \geq 1 - 8[2^{-k_1} + \cdots + 2^{-k_N}] > 1 - \varepsilon,$$

as desired. ■

4. Future directions

It is highly desirable to ascertain whether the maximal operator M_Ω is bounded on $L^p(\mathbb{R}^2)$ for all $1 < p \leq \infty$ if and only if Ω is a union of finitely many sets each of finite lacunary order. The proof of the above theorem suggests the following model case of consideration.

Let $k_1, k_2, \ldots$ be an infinite sequence of natural numbers greater than or equal to 2 such that

$$2^{-k_1} + 2^{-k_2} + 2^{-k_3} + \cdots < \infty$$

and let, as before, a_i denote the k_i-string $0111\cdots 1$ and b_i denote the k_i-string $1000\cdots 0$. Let $\Omega \subset [0,1]$ be the set of points with binary expansions corresponding to sequences of the form $0 s_1 s_2 s_3 \ldots s_N$, where N is arbitrary in $\mathbb{N}$ and each s_i is either of the form a_i or b_i. The set Ω is not finite lacunary, and hence one cannot use the Sjögren–Sjölin result in [8] to prove that M_Ω is bounded on $L^p(\mathbb{R}^2)$. However, Ω also does not satisfy the separation condition found in [5] that would imply that M_Ω is unbounded on $L^p(\mathbb{R}^2)$ for all $1 \leq p < \infty$. The set Ω having a rather straightforward structure, however, suggests that determining whether M_Ω is bounded on $L^p(\mathbb{R}^2)$ for all $1 < p \leq \infty$ provides a good starting point for investigating the above problem.

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References

[1] Bateman, M.: Kakeya sets and directional maximal operators in the plane. *Duke Math. J.* **147** (2009), no. 1, 55–77. Zbl 1165.42005 MR 2494456

[2] Bateman, M. and Katz, N. H.: Kakeya sets in Cantor directions. *Math. Res. Lett.* **15** (2008), no. 1, 73–81. Zbl 1160.42010 MR 2367175

[3] Busemann, H. and Feller, W.: Zur Differentiation der Lebesguesche Integrale. *Fund. Math.* **22** (1934), 226–256. Zbl 0009.10601

[4] Córdoba, A. and Fefferman, R.: On differentiation of integrals. *Proc. Nat. Acad. Sci. U.S.A.* **74** (1977), no. 6, 2211–2213. Zbl 0374.28002 MR 0476977

[5] Hagelstein, P., Radillo-Murguia, B. and Stokolos, A.: Probabilistic construction of Kakeya-type sets in $\mathbb{R}^2$ associated to separated sets of directions. *Duke Math. J.* **175** (2026), no. 2, 185–198. MR 5030464

[6] Nagel, A., Stein, E. M. and Wainger, S.: Differentiation in lacunary directions. *Proc. Nat. Acad. Sci. U.S.A.* **75** (1978), no. 3, 1060–1062. Zbl 0391.42015 MR 0466470

[7] Nikodym, O: Sur la mesure des ensembles plans dont tous les points sont rectilinéairement accessibles. *Fund. Math.* **10** (1927), 116–168. Zbl 53.0176.02

[8] Sjögren, P. and Sjölin, P.: Littlewood–Paley decompositions and Fourier multipliers with singularities on certain sets. *Ann. Inst. Fourier (Grenoble)* **31** (1981), no. 1, 157–175. Zbl 0437.42011 MR 0613033

[9] Strömberg, J.-O.: Weak estimates on maximal functions with rectangles in certain directions. *Ark. Mat.* **15** (1977), no. 2, 229–240. Zbl 0376.26007 MR 0487260

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