Rev. Mat. Iberoam. 42 (2026), no. 2, 477–488
DOI 10.4171/RMI/1568

Characterizing symplectic capacities on ellipsoids

Jean Gutt and Vinicius G. B. Ramos

Abstract. It is a long-standing conjecture that all symplectic capacities which are equal to the Gromov width for ellipsoids coincide on a class of convex domains in $\mathbb{R}^{2n}$. It is known that they coincide for monotone toric domains in all dimensions. In this paper, we study whether requiring a capacity to be equal to the kth Ekeland–Hofer capacity for all ellipsoids can characterize it on a class of domains. We prove that for $k = n = 2$, this holds for convex toric domains, but not for all monotone toric domains. We also prove that, for $k = n \geq 3$, this does not hold even for convex toric domains.

1. Introduction

Since Darboux's theorem, it is known that all symplectic manifolds are locally "the same". In particular, any symplectic invariant has to be of a global nature. This prompted the difficult quest in symplectic geometry of finding such invariants. The first nontrivial invariant was a so-called width defined by Gromov in [13]. Inspired by Gromov's work, Ekeland and Hofer defined the concept of a symplectic capacity in [10]. If X and X' are domains in $\mathbb{R}^{2n}$, a symplectic embedding from X to X' is a smooth embedding $\varphi\colon X \to X'$ such that $\varphi^*\omega = \omega$, where $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ is the standard symplectic form on $\mathbb{R}^{2n}$. A *symplectic capacity* is a function c which assigns to each subset in $\mathbb{R}^{2n}$ a number $c(X) \in [0, \infty]$ satisfying the following axioms:

(Monotonicity) If $X, X' \subset \mathbb{R}^{2n}$, and if there exists a symplectic embedding $X \hookrightarrow X'$, then $c(X) \leq c(X')$.

(Conformality) If r is a positive real number, then $c(rX) = r^2 c(X)$.

Various examples of symplectic capacities have emerged, such as the Hofer–Zehnder capacity c_{HZ} defined in [20] and the Viterbo capacity c_{SH} defined in [25]. There are also useful families of symplectic capacities parametrized by a positive integer k, including the Ekeland–Hofer capacities c_k^{EH} defined in [10, 11] using calculus of variations, and the capacities c_k^{CH} defined in [14] using positive equivariant symplectic homology. The latter coincide with the Ekeland–Hofer capacities on all star-shaped domains [17]. Another family of capacities in the four-dimensional case is the ECH capacities c_k^{ECH} defined in [21]

Mathematics Subject Classification 2020: 53D05 (primary); 53D35 (secondary).
Keywords: symplectic capacities, Viterbo conjecture, toric domains.

using embedded contact homology. For more about symplectic capacities in general, we refer to [6, 23] and the references therein. In view of all these different constructions and the desire for symplectic invariants to be defined axiomatically, a very natural question is the following.

Question 1.1. *Is there an axiomatic characterization of some symplectic capacities?*

One motivation for this question is a well-known conjecture which asserts that a certain normalization condition uniquely characterizes c on the set of convex domains. This conjecture was recently disproved in the most general setting [19], but it is still open for convex domains with $\mathbb{Z}/2$-symmetry. For $a_1, \dots, a_n \in (0, \infty]$, define the ellipsoid

$$E(a_1, \dots, a_n) := \left\{ z \in \mathbb{C}^n \;\middle|\; \sum_{\substack{i=1 \\ a_i \neq \infty}}^{n} \frac{\pi |z_i|^2}{a_i} \leq 1 \right\}.$$

A symplectic capacity c is *ball normalized* if we have $c(E(a_1, \dots, a_n)) = \min\{a_1, \dots, a_n\}$ for all $a_1 \dots, a_n$. Notice that this condition is equivalent to the more usual one requiring that

$$c(B^{2n}(1)) = c(Z^{2n}(1)) = 1,$$

where

$$B^{2n}(r) = E(r, \dots, r) \quad \text{and} \quad Z^{2n}(r) = E(r, \infty, \dots, \infty) = B^2(r) \times \mathbb{C}^{n-1}.$$

This normalization does characterize a capacity on the set of monotone toric domains, as was proven in [8, 15]. This class is related but is not the same as the class of convex sets. In particular, there are monotone toric domains which are not symplectomorphic to convex domains, see [4, 9]. Conversely, there exist convex domains not symplectomorphic to toric domains [22].

Another normalization, called the cube-normalization, have been introduced by the authors and M. Pereira [16]. The authors proved that cube-normalized capacities coincide on all monotone toric domains in any dimensions, and it is still unknown if they coincide or not on convex domains. The example in [19] does not give a counterexample in this case. A natural step towards answering Question 1.1 is the following question.

Question 1.2 (Abbondandolo and Hutchings). *Does imposing the value of a symplectic capacity on all ellipsoids characterize it on a large family of domains?*

Remark 1.3. The original question formulated by Hutchings is whether the axioms of Theorem 1.1 in [14] uniquely characterize symplectic capacities for convex domains. These axioms are somewhat stronger than assuming that the capacities take certain values on ellipsoids. We will focus on Question 1.2 in this paper.

The goal of this paper is to give some answers to Question 1.2. In order to state our results, we recall some definitions. Let $\mu\colon \mathbb{R}^{2n} \to [0, \infty)^n$ be the standard momentum map, i.e.,

$$\mu(z_1, \dots, z_n) = (\pi |z_1|^2, \dots, \pi |z_n|^2).$$

A toric domain is a set of the form $X_\Omega = \mu^{-1}(\Omega)$, where $\Omega \subset [0, \infty)^n$ is the closure of a non-empty relatively open set. Let $\partial_+\Omega = \partial\Omega \cap (0, \infty)^n$. We assume henceforth that $\partial_+\Omega$ is piecewise smooth.

Definition 1.4. A toric domain X_Ω is said to be:

- monotone, if every outward-pointing normal vector[1] $(v_1, \dots, v_n)$ to $\partial_+\Omega$ is such that $v_i \geq 0$, for all i;
- concave, if Ω is compact and $\mathbb{R}^n_{\geq 0} \setminus \Omega$ is convex;
- convex, if
$$\widehat{\Omega} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (|x_1|, \dots, |x_n|) \in \Omega\}$$
is compact and convex.

Remark 1.5. We note that every concave or convex toric domain is monotone. If $\Omega' \subset [0,\infty)^{n-1}$ is a monotone domain (i.e., $X_{\Omega'}$ is monotone) and $f\colon \Omega' \subset [0,\infty)^{n-1} \to \mathbb{R}$ is a smooth function such that $\partial f/\partial x_i \leq 0$, for every i, then $X_{\mathrm{Gr}(f)}$ is a monotone toric domain, where Gr(f) is the region bounded above by the graph of f, that is, Gr(f) $= \{(x,y) \in \Omega' \times [0,\infty) \mid y \leq f(x)\}$. Moreover, if $X_{\Omega'}$ and f are simultaneously convex or concave, then $X_{\mathrm{Gr}(f)}$ is concave or convex, respectively. We also observe that every monotone toric domain can be approximated (in the Hausdorff topology) by domains of the form $X_{\mathrm{Gr}(f)}$. Finally we note that if $\partial_+\Omega$ is smooth and X_Ω is monotone, then X_Ω is dynamically convex, see Proposition 1.8 in [15].

For $a_1, \dots, a_n > 0$, let $N_k(a_1, \dots, a_n)$ denote the k-th smallest number in the multiset $\{ma_i \mid i = 1, \dots, n \text{ and } m \in \mathbb{Z}_{>0}\}$. "Multiset" means that repetition of numbers is allowed. These correspond to the Ekeland–Hofer or the Gutt–Hutchings capacities of an ellipsoid:

$$c_k^{\mathrm{EH}}(E(a_1, \dots, a_n)) = c_k^{\mathrm{CH}}(E(a_1, \dots, a_n)) = N_k(a_1, \dots, a_n). \tag{1.1}$$

Definition 1.6. A symplectic capacity c is called *k-normalized* if, for every $a_1, \dots a_n > 0$,
$$c(E(a_1, \dots, a_n)) = N_k(a_1, \dots, a_n).$$

To the best of our knowledge, this definition first appeared in [2], and in [1].

Remark 1.7. Note that $N_1(a_1, \dots, a_n) = \min(a_1, \dots, a_n)$. So being 1-normalized is equivalent to being ball normalized and thus by [8, 15], 1-normalized capacities coincide for all monotone toric domains.

The "max" and "min" k-normalized capacities are defined as:
$$c_k^{\max}(X) := \inf\{N_k(a_1, \dots, a_n) \mid X \hookrightarrow E(a_1, \dots, a_n), \text{ for some } a_1, \dots, a_n > 0\}.$$
$$c_k^{\min}(X) := \sup\{N_k(a_1, \dots, a_n) \mid E(a_1, \dots, a_n) \hookrightarrow X, \text{ for some } a_1, \dots, a_n > 0\}.$$
In fact, it is easy to see that if c is a k-normalized capacity and $X \subset \mathbb{R}^{2n}$,
$$c_k^{\min}(X) \leq c(X) \leq c_k^{\max}(X).$$
It follows that all k-normalized capacities coincide for X if, and only if, $c_k^{\min}(X) = c_k^{\max}(X)$. One could then ask whether k-normalized capacities coincide for any class of convex domains. In fact, it was recently proven by Abbondandolo, Benedetti and Edtmair that this does hold for n-normalized capacities in a neighborhood of the round ball.

[1] This definition includes domains for which $\partial_+\Omega$ is not smooth where the normal vector at a point is not uniquely defined, but where we can still define a normal cone.

Theorem 1.8 ([1]). *There exists a C^2-neighborhood of the ball $B^{2n}(1)$ in which all n-normalized capacities coincide.*

Our first result is that this holds in dimension 4 for all convex toric domains, but not for all monotone domains.

Theorem 1.9. *If $X \subset \mathbb{R}^4$ is a convex toric domain, then $c_2^{\min}(X) = c_2^{\max}(X)$. Moreover, there exists a concave toric domain $V \subset \mathbb{R}^4$ such that $c_2^{\min}(V) < c_2^{\max}(V)$.*

Our second result says that $k = 2$ is unusual. Let $P(a_1, \dots, a_n)$ be the polydisk defined by

$$P(a_1, \dots, a_n) = \{z \in \mathbb{C}^n \mid \pi |z_i|^2 \le a_i\}.$$

Theorem 1.10. *For all $a_1, \dots, a_n > 0$,*

$$c_2^{\min}(P(a_1, \dots, a_n)) = c_2^{\max}(P(a_1, \dots, a_n)).$$

If $k \ge \max(n, 3)$, then

$$c_k^{\min}(P(1, \dots, 1)) < c_k^{\max}(P(1, \dots, 1)).$$

It follows from Theorem 1.10 that n-normalized capacities cannot coincide for all convex sets if $n \ge 3$.

The rest of the paper is organized in three sections. In Section 2, we prove the first part of Theorem 1.9. In Section 3, we construct a concave toric domain for which 2-normalized capacities do not coincide. Finally, in Section 4, we prove Theorem 1.10.

2. 2-normalized capacities for 4-dimensional convex toric domains

In this section, we prove the first part of Theorem 1.9, namely that for a convex toric domain $X_\Omega \subset \mathbb{R}^4$, we have

$$c_2^{\min}(X_\Omega) = c_2^{\max}(X_\Omega). \tag{2.1}$$

Before proving (2.1) in all generality, we prove the special case of a polydisk.

Lemma 2.1. *For any 2-normalized symplectic capacity c, we have*

$$c(P(a,b)) = 2\min(a,b).$$

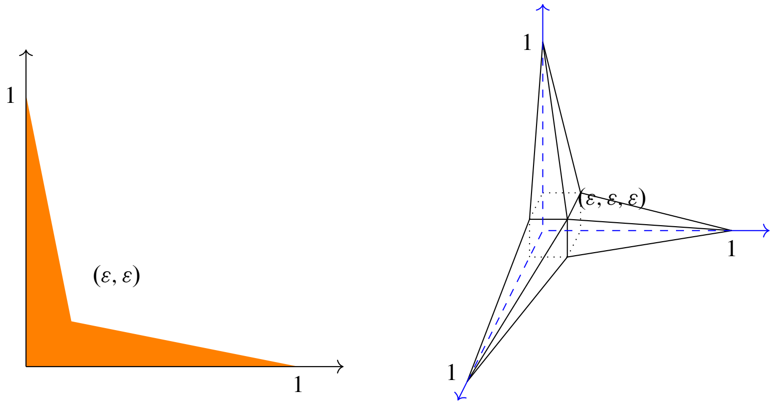
Proof. We can assume without loss of generality that $a \le b$. Let $\varepsilon > 0$ such that $\varepsilon < b/2$. It follows from Theorem 1.3 in [12] that $\operatorname{int}(E(a,2a)) \hookrightarrow P(a,a)$. So

$$\operatorname{int}(E(a,2a)) \hookrightarrow P(a,a) \subset P(a,b) \subset E\Big(a+\varepsilon, \frac{(a+\varepsilon)b}{\varepsilon}\Big). \tag{2.2}$$

The last inclusion above is a simple analytic geometry exercise. If c is any 2-normalized capacity, it follows from Definition 1.6, (2.2) and (1.1) that

$$2a = c(E(a,2a)) \le c(P(a,b)) \le c\Big(E\Big(a+\varepsilon, \frac{(a+\varepsilon)b}{\varepsilon}\Big)\Big) = 2(a+\varepsilon).$$

Taking the limit as $\varepsilon \to 0$, we conclude that $c(P(a,b)) = 2a$. ∎



(a) $n = 2$. **(b)** $n = 3$.

Figure 1. The domain Q_ε.

Now, let X_Ω be a 4-dimensional convex toric domain whose moment map image intercepts with the axes are a and b. We can assume without loss of generality that $a \leq b$. Since $X_\Omega \subset P(a, b)$, it follows from Lemma 2.1 that $c_2^{\max}(X_\Omega) \leq 2a$. Now let w be the minimum of $r > 0$ such that $X_\Omega \subset B(r)$. Since $X_\Omega \subset B(w)$, it follows from Definition 1.6 that $c_2^{\max}(X_\Omega) \leq w$. Therefore,

$$c_2^{\max}(X_\Omega) \leq \min(2a, w). \tag{2.3}$$

Now we observe that there exists a point $(x, y) \in \partial_+\Omega$ such that $x + y = w$. Let Q denote the convex hull of $(0, 0)$, $(a, 0)$, (x, y) and $(0, b)$. Since Ω is convex, $X_Q \subset X_\Omega$. The first author and Usher proved in Proposition 3.5 of [18] that $\operatorname{int}(E(a, w)) \hookrightarrow X_Q$ and so $\operatorname{int}(E(a, w)) \hookrightarrow X_\Omega$. Using Definition 1.6 again, we obtain

$$c_2^{\min}(X_\Omega) \geq \min(2a, w). \tag{2.4}$$

Combining (2.3) and (2.4), we conclude that $c_2^{\min}(X_\Omega) = c_2^{\max}(X_\Omega)$.

3. A family of concave toric domains

We now prove the second statement in Theorem 1.9. In fact, we will construct a family of concave toric domains V_ε such that $c_2^{\min}(V_\varepsilon) < c_2^{\max}(V_\varepsilon)$ for ε sufficiently small. For $\varepsilon \in (0, 1/2)$, let $V_\varepsilon = \mu^{-1}(Q_\varepsilon)$, where Q_ε is the quadrilateral with vertices $(0, 0)$, $(0, 1)$, $(\varepsilon, \varepsilon)$ and $(1, 0)$, see Figure 1(a).

The following result is enough to prove the second statement in Theorem 1.9, although it proves quite a bit more.

Theorem 3.1. *Let $\varepsilon \in (0, 1/2)$. Then $c_2^{\min}(V_\varepsilon) = c_2^{\max}(V_\varepsilon)$ if and only if $\varepsilon \geq 2/9$.*

Proof. Let $\varepsilon \in [2/9, 1/2)$. We will show that $c_2^{\min}(V_\varepsilon) = c_2^{\max}(V_\varepsilon)$.

First suppose that $\varepsilon \in [1/3, 1/2)$. Then

$$E(1/2, 1) \subset V_\varepsilon \subset B^4(1) = E(1, 1).$$

Since $N_2(1/2, 1) = N_2(1, 1) = 1$, it follows that

$$c_2^{\max}(V_\varepsilon) \le N_2(1, 1) = N_2(1/2, 1) \le c_2^{\min}(V_\varepsilon).$$

Therefore $c_k^{\min}(V_\varepsilon) = c_k^{\max}(V_\varepsilon)$.

Before proceeding to the case when $\varepsilon \in [2/9, 1/3)$, we now recall how one can associate to a concave toric domain X_Ω a natural ball packing $\bigsqcup_{j=1}^{\infty} \operatorname{int}(B^4(w_j)) \hookrightarrow X_\Omega$. For a more thorough explanation, see [5]. Let $T(w) \subset \mathbb{R}^2_{\ge 0}$ be the triangle with vertices $(0, 0)$, $(w, 0)$ and $(0, w)$. So $\mu^{-1}(T(w)) = B^4(w)$. Let w_1 the supremum of w such that $T(w) \subset \Omega$. So $\Omega \setminus T(w_1)$ has of two connected components Ω_1 and Ω_1'. We translate Ω_1 and Ω_1' so that the corners lie at the origin and we apply the linear transformations $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, respectively. By taking the closure of the sets above, we obtain two domains $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_1'$ in $\mathbb{R}^2_{\ge 0}$. Now we define the next weights w_2 and w_3 to be, respectively, the suprema of w such that $T(w) \subset \widetilde{\Omega}_1$ and $T(w) \subset \widetilde{\Omega}_1'$, arranged in decreasing order. We continue this process by induction and we obtain a decreasing sequence $w_1, w_2, \dots$. It follows from the "Traynor trick" (Proposition 5.2 in [24]), followed by a transformation in $\mathrm{SL}(2, \mathbb{Z})$, as explained in Lemma 1.8 of [5], that

$$\bigsqcup_{j=1}^{\infty} \operatorname{int}(B(w_j)) \hookrightarrow X_\Omega.$$

In [7], Cristofaro–Gardiner proved that

$$\operatorname{int}(X_\Omega) \hookrightarrow B(a) \quad \Longleftrightarrow \quad \bigsqcup_{j=1}^{\infty} \operatorname{int}(B^4(w_j)) \hookrightarrow B^4(a). \tag{3.1}$$

Now suppose that $\varepsilon \in [2/9, 1/3)$. Let $w_1 \ge w_2 \ge w_3 \ge \dots$ be the weights of V_ε as defined above. It follows from a simple calculation that $w_1 = 2\varepsilon$ and $w_2 = w_3 = \varepsilon$. The triangles $T(w_1)$, $T(w_2)$ and $T(w_3)$ correspond to the red and the two blue triangles in Figure 2(a), respectively. Moreover, the domains

$$\Omega_2 := \operatorname{cl}(\widetilde{\Omega}_1 \setminus B^4(w_2)) \quad \text{and} \quad \Omega_2' := \operatorname{cl}(\widetilde{\Omega}_1' \setminus B^4(w_3))$$

are triangles which are affinely equivalent under $\mathrm{SL}(2, \mathbb{Z})$ to the yellow triangles in Figure 2(a). It follows that Ω_2 and Ω_2' are affinely equivalent to a right triangle $T(\varepsilon, 1 - 3\varepsilon)$ with sides ε and $1 - 3\varepsilon$. Note that $X_{T(\varepsilon, 1-3\varepsilon)} = E(\varepsilon, 1 - 3\varepsilon)$. It follows that

$$\begin{aligned} &\bigsqcup_{k=2}^{\infty} \operatorname{int}(B^4(w_{2k})) \hookrightarrow \operatorname{int}(E(\varepsilon, 1 - 3\varepsilon)), \\ &\bigsqcup_{k=2}^{\infty} \operatorname{int}(B^4(w_{2k+1})) \hookrightarrow \operatorname{int}(E(\varepsilon, 1 - 3\varepsilon)). \end{aligned} \tag{3.2}$$

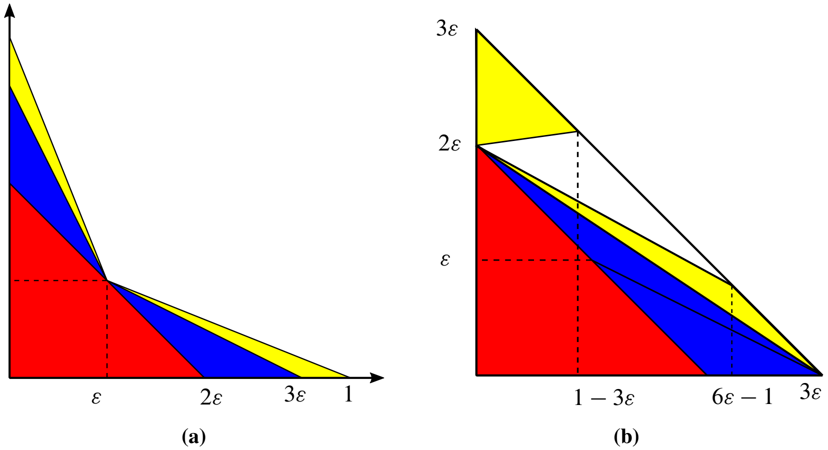


Figure 2. Ball packings.

Now we can find an explicit packing of the interiors of affine copies of $T(2\varepsilon)$, two copies of $T(\varepsilon)$ and two copies of $T(\varepsilon, 1-3\varepsilon)$ into $T(3\varepsilon)$, as shown in Figure 2(b). Note that the right corners of $T(\varepsilon, 1-3\varepsilon)$ are taken to $(3\varepsilon, 0)$ and $(0, 3\varepsilon)$. To make sure that the yellow triangles do not overlap, we use the fact that $1-3\varepsilon \leq 6\varepsilon - 1$, which is equivalent to $\varepsilon \geq 2/9$. In particular, when $1-3\varepsilon = 6\varepsilon - 1$, the ball packing is volume filling. It follows from the "Traynor trick" that

$$\text{int}(B^4(w_1)) \sqcup \text{int}(B^4(w_2)) \sqcup \text{int}(B^4(w_3)) \sqcup \text{int}(X_{T(\varepsilon)}) \sqcup \text{int}(X_{T(\varepsilon)}) \hookrightarrow T(3\varepsilon). \tag{3.3}$$

Using (3.1), (3.2) and (3.3), we conclude that

$$V_\varepsilon \hookrightarrow B^4(3\varepsilon).$$

Moreover, since $\varepsilon < 1/3$, it follows that $E(3\varepsilon/2, 3\varepsilon) \subset V_\varepsilon$. Therefore

$$c_2^{\max}(V_\varepsilon) \leq N_2(3\varepsilon, 3\varepsilon) = N_2(3\varepsilon/2, 3\varepsilon) \leq c_2^{\min}(V_\varepsilon).$$

Hence $c_2^{\min}(V_\varepsilon) = c_2^{\max}(V_\varepsilon)$.

Now we suppose that $\varepsilon < 2/9$. Using the explicit formula for c_2^{CH} for any concave toric domain (Theorem 1.14 in [14]), we have that

$$c_2^{\text{CH}}(V_\varepsilon) = 3\varepsilon.$$

On the other hand, we claim that, $c_2^{\max}(V_\varepsilon) > 3\varepsilon$. To prove the claim, it is enough to show that there exists $\delta > 0$ such that, for any ellipsoid $E(a,b)$ such that $V_\varepsilon \hookrightarrow E(a,b)$, we have $N_2(a,b)) \geq 3\varepsilon + \delta$. We assume, without loss of generality, that $a \leq b$. Since $B^4(2\varepsilon) \subset V_\varepsilon \hookrightarrow E(a,b)$, it follows that

$$2\varepsilon = N_1(2\varepsilon, 2\varepsilon) \leq N_1(a,b) = a.$$

We also have

$$\varepsilon = \mathrm{Vol}(V_\varepsilon) \le \mathrm{Vol}\big(E(a,b)\big) = \frac{ab}{2} \le \frac{b^2}{2}.$$

Thus $b \ge \sqrt{2\varepsilon} > 3\varepsilon + \delta$, for some small $\delta > 0$, since $\varepsilon < 2/9$. Moreover,

$$N_2(a,b) = \min\{2a, b\} \ge \min(4\varepsilon, 3\varepsilon + \delta) \ge 3\varepsilon + \delta.$$

It follows that

$$c_2^{\max}(V_\varepsilon) \ge 3\varepsilon + \delta = c_2^{\mathrm{CH}}(V_\varepsilon) + \delta > c_2^{\min}(V_\varepsilon). \qquad \blacksquare$$

Remark 3.2. Using a similar argument to that in Theorem 1.2 of [9], it can be shown that V_ε is not symplectomorphic to a convex set for ε sufficiently small.

Remark 3.3. The family V_ε can be generalized to higher dimensions, see Figure 1 (b). Using the same argument as in the proof of Theorem 3.1, we can show that if $\varepsilon < n!/(2n-1)^n$, then $c_n^{\min}(V_\varepsilon) < c_n^{\max}(V_\varepsilon)$. However, we do not know what happens when $\varepsilon \ge n!/(2n-1)^n$, since the techniques used in this case do not generalize to higher dimensions.

4. Proof of Theorem 1.10

We start with the case $k = 2$. Generalizing Lemma 2.1, we claim that if c is 2-normalized, then

$$c(P(a_1,\dots,a_n)) = 2\min(a_1,\dots,a_n). \tag{4.1}$$

We assume without lack of generality that $a_1 \le \cdots \le a_n$. Again, from a simple calculation, we have

$$P(a_1,a_2,\dots,a_n) \subset E\Big(a_1 + \varepsilon, \frac{(n-1)(a_1+\varepsilon)a_2}{\varepsilon}, \dots, \frac{(n-1)(a_1+\varepsilon)a_n}{\varepsilon}\Big).$$

It follows that $c(P(a_1,\dots,a_n)) \le 2(a_1+\varepsilon)$ for all sufficiently small $\varepsilon > 0$, and hence

$$c(P(a_1,\dots,a_n)) \le 2a_1.$$

Now, noting that $P(a_1,\dots,a_1) \subset P(a_1,\dots,a_n)$, we can reduce the proof of

$$c(P(a_1,\dots,a_n)) \ge 2a_1$$

to demonstrating that

$$\mathrm{int}\, E(1,2,\dots,2) \hookrightarrow P(1,\dots,1). \tag{4.2}$$

We shall proceed by induction on the dimension and use the family method from Section 2.1 of [3]. For ease of readability, we shall add the complex dimension of the ellipsoid in superscript.

By Theorem 1.3 in [12], the embedding (4.2) exists for $n = 2$. Now we suppose the embedding (4.2) exists in complex dimension $n-1$. We have

$$\begin{aligned}
&\operatorname{int} E^n(1,2,\ldots,2) \\
&= \bigcup_{z\in \operatorname{int} E^{n-2}(2,\ldots,2)} \left\{(w,z)\in\mathbb{C}^2\times\mathbb{C}^{n-2} \,\middle|\, w\in \sqrt{1-\pi|z|^2/2}\operatorname{int} E^2(1,2)\right\} \\
&\hookrightarrow \bigcup_{z\in \operatorname{int} E^{n-2}(2,\ldots,2)} \left\{(w,z)\in\mathbb{C}^2\times\mathbb{C}^{n-2} \,\middle|\, w\in \sqrt{1-\pi|z|^2/2}\operatorname{int} P^2(1,1)\right\} \\
&\subset \bigcup_{z\in \operatorname{int} E^{n-2}(2,\ldots,2)} \operatorname{int} B^2(1)\times\left\{(w_2,z)\in\mathbb{C}\times\mathbb{C}^{n-1} \,\middle|\, w_2\in \sqrt{1-\pi|z|^2/2}\operatorname{int} B^2(1)\right\} \\
&\subset \operatorname{int} B^2(1)\times \operatorname{int} E^{n-1}(1,2,\ldots,2)
\end{aligned}$$

We now use our induction assumption to obtain an embedding

$$\operatorname{int} E^n(1,2,\ldots,2)\hookrightarrow B^2(1)\times P^{n-1}(1,\ldots,1) = P^n(1,\ldots,1).$$

For the second part of Theorem 1.10, we fix $k, n \geq 2$ such that $k \geq \max(n,3)$. We shall prove for the polydisk $P(1,\ldots,1)\subset\mathbb{C}^n$, we have $c_k^{\min}(P(1,\ldots,1)) < c_k^{\max}(P(1,\ldots,1))$.

We have, by Theorem 1.12 in [14],

$$c_k^{\mathrm{CH}}(P(1,\ldots,1)) = k.$$

We suppose that $c_k^{\min}(P(1,\ldots,1)) = k$. Let $0<\varepsilon<1$. Then there exist $0<a_1\leq\cdots\leq a_n$ such that

$$E(a_1,\ldots,a_n)\hookrightarrow P(1,\ldots,1), \tag{4.3}$$

$$N_k(a_1,\ldots,a_n)\geq k-\varepsilon. \tag{4.4}$$

We first suppose that $a_2\leq (k-1)a_1$. Under this condition, it follows from the definition of N_k that

$$N_k(a_1,\ldots,a_n)\leq (k-1)a_1. \tag{4.5}$$

From (4.3) we obtain $a_1 = c_1^{\mathrm{CH}}(E(a_1,\ldots,a_n)) \leq c_1^{\mathrm{CH}}(P(1,\ldots,1)) = 1$. So it follows from (4.5) that

$$N_k(a_1,\ldots,a_n)\leq k-1,$$

contradicting (4.4).

So we may assume that $a_2 > (k-1)a_1$. Under this condition, the definition of N_k implies that

$$N_k(a_1,\ldots,a_n)\leq \min(ka_1, a_2). \tag{4.6}$$

Combining (4.4) and (4.6), we obtain

$$a_1\geq \frac{k-\varepsilon}{k} \quad \text{and} \quad a_i\geq k-\varepsilon \quad \text{for } i\geq 2. \tag{4.7}$$

Since a symplectic embedding is volume preserving, it follows from (4.3) and (4.7) that

$$\frac{(k-\varepsilon)^n}{k\cdot n!} \le \frac{a_1\cdots a_n}{n!} = \mathrm{vol}(E(a_1,\dots,a_n)) \le \mathrm{vol}(P(1,\dots,1)) = 1. \tag{4.8}$$

Taking the limit as $\varepsilon \to 0$ in (4.7), we obtain

$$\frac{k^n}{k\cdot n!} \le 1. \tag{4.9}$$

But (4.9) does not hold under our assumptions. In fact, it is obvious that (4.9) is false for $k = 3$ and $n = 2$. Now if $k \ge n \ge 3$, then

$$\frac{k^n}{k\cdot n!} = \frac{k^{n-1}}{n!} \ge \frac{n^{n-2}}{(n-1)!} \ge \frac{n}{n-1} > 1,$$

contradicting (4.9). We conclude that our original assumption is false, and hence

$$c_k^{\min}(P(1,\dots,1)) < k = c_k^{\mathrm{GH}}(P(1,\dots,1)) \le c_k^{\max}(P(1,\dots,1)).$$

Remark 4.1. The conditions on k and n for which the conclusion of Theorem 1.10 holds can be relaxed. In particular, it holds for $1 \le k < n$ for which (4.9) is false.

Acknowledgments. We would like to thank Alberto Abbondandolo and Michael Hutchings for pointing us the question about k-normalized capacities. We are grateful to Alberto Abbondandolo, Oliver Edtmair and Gabriele Benedetti for sharing a preliminary version of their work [1] with us. We would also like to particularly thank Richard Hind for his insight on embedding ellipsoids into polydisks in higher dimensions and the argument of the proof of the first part of Theorem 1.10.

Funding. J. Gutt was partially supported by the ANR LabEx CIMI (grant ANR-11-LABX-0040) within the French State Programme "Investissements d'Avenir" and by the ANR COSY (ANR-21-CE40-0002) grant. V. G. B. Ramos is grateful for the hospitality of the Institute for Advanced Study, where part of this work was completed. V. G. B. Ramos was partially supported by NSF grant DMS-1926686, FAPERJ grant JCNE E-26/201.399/2021 and a Serrapilheira Institute grant.

References

[1] Abbondandolo, A., Benedetti, G. and Edtmair, O.: Symplectic capacities of domains close to the ball and Banach–Mazur geodesics in the space of contact forms. Prepint 2023, arXiv: 2312.07363v1.

[2] Baracco, L., Bernardi, D., Lange, C. and Mazzucchelli, M.: On the local maximizers of higher capacity ratios. To appear in *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, DOI 10.2422/2036-2145.202306_002.

[3] Buse, O. and Hind, R.: Symplectic embeddings of ellipsoids in dimension greater than four. *Geom. Topol.* **15** (2011), no. 4, 2091–2110. Zbl 1239.53107 MR 2860988

[4] Chaidez, J. and Edtmair, O.: The Ruelle invariant and convexity in higher dimensions. To appear in *J. Eur. Math. Soc. (JEMS)*, published online-first (2025), DOI 10.4171/JEMS/1671.

[5] Choi, K., Cristofaro-Gardiner, D., Frenkel, D., Hutchings, M. and Ramos, V. G. B.: Symplectic embeddings into four-dimensional concave toric domains. *J. Topol.* **7** (2014), no. 4, 1054–1076. Zbl 1308.53125 MR 3286897

[6] Cieliebak, K., Hofer, H., Latschev, J. and Schlenk, F.: Quantitative symplectic geometry. In *Dynamics, ergodic theory, and geometry*, pp. 1–44. Math. Sci. Res. Inst. Publ. 54, Cambridge Univ. Press, Cambridge, 2007. Zbl 1143.53341 MR 2369441

[7] Cristofaro-Gardiner, D.: Symplectic embeddings from concave toric domains into convex ones. *J. Differential Geom.* **112** (2019), no. 2, 199–232. Zbl 1440.53084 MR 3960266

[8] Cristofaro-Gardiner, D. and Hind, R.: On the agreement of symplectic capacities in high dimension. Preprint 2023, arXiv:2307.12125v1.

[9] Dardennes, J., Gutt, J., Ramos, V. G. B. and Zhang, J.: Coarse distance from dynamically convex to convex. Preprint 2023, arXiv:2309.10912v1.

[10] Ekeland, I. and Hofer, H.: Symplectic topology and Hamiltonian dynamics. *Math. Z.* **200** (1989), no. 3, 355–378. Zbl 0641.53035 MR 0978597

[11] Ekeland, I. and Hofer, H.: Symplectic topology and Hamiltonian dynamics. II. *Math. Z.* **203** (1990), no. 4, 553–567. Zbl 0729.53039 MR 1044064

[12] Frenkel, D. and Müller, D.: Symplectic embeddings of 4-dim ellipsoids into cubes. *J. Symplectic Geom.* **13** (2015), no. 4, 765–847. Zbl 1339.53082 MR 3480057

[13] Gromov, M.: Pseudo holomorphic curves in symplectic manifolds. *Invent. Math.* **82** (1985), no. 2, 307–347. Zbl 0592.53025 MR 0809718

[14] Gutt, J. and Hutchings, M.: Symplectic capacities from positive S^1-equivariant symplectic homology. *Algebr. Geom. Topol.* **18** (2018), no. 6, 3537–3600. Zbl 1411.53062 MR 3868228

[15] Gutt, J., Hutchings, M. and Ramos, V. G. B.: Examples around the strong Viterbo conjecture. *J. Fixed Point Theory Appl.* **24** (2022), no. 2, article no. 41, 22 pp. Zbl 1497.53132 MR 4413022

[16] Gutt, J., Pereira, M. and Ramos, V. G. B.: Cube normalized symplectic capacities. Preprint 2022, arXiv:2208.13666v1.

[17] Gutt, J. and Ramos, V. G. B.: The equivalence of Ekeland–Hofer and equivariant symplectic homology capacities. Preprint 2024, arXiv:2412.09555v1.

[18] Gutt, J. and Usher, M.: Symplectically knotted codimension-zero embeddings of domains in $\mathbb{R}^4$. *Duke Math. J.* **168** (2019), no. 12, 2299–2363. Zbl 1481.53106 MR 3999447

[19] Haim-Kislev, P. and Ostrover, Y.: A counterexample to Viterbo's conjecture. Preprint 2024, arXiv:2405.16513v2.

[20] Hofer, H. and Zehnder, E.: *Symplectic invariants and Hamiltonian dynamics*. Mod. Birkhauser Class., Birkhäuser, Basel, 2011. Zbl 1223.37001 MR 2797558

[21] Hutchings, M.: Quantitative embedded contact homology. *J. Differential Geom.* **88** (2011), no. 2, 231–266. Zbl 1238.53061 MR 2838266

[22] Hutchings, M.: Zeta functions of dynamically tame Liouville domains. Preprint 2024, arXiv:2402.07003v3.

[23] Schlenk, F.: Symplectic embedding problems, old and new. *Bull. Amer. Math. Soc. (N.S.)* **55** (2018), no. 2, 139–182. Zbl 1464.53106 MR 3777016

[24] Traynor, L.: Symplectic packing constructions. *J. Differential Geom.* **42** (1995), no. 2, 411–429. Zbl 0861.52008 MR 1366550

[25] Viterbo, C.: Functors and computations in Floer homology with applications. I. *Geom. Funct. Anal.* **9** (1999), no. 5, 985–1033. Zbl 0954.57015 MR 1726235

Received October 24, 2024; revised March 13, 2025.

Jean Gutt
Institut de Mathématiques de Toulouse, Institut National Universitaire Champollion
Place de Verdun, 81000 Albi, France;
jean.gutt@math.univ-toulouse.fr

Vinicius G. B. Ramos
Instituto de Matemática Pura e Aplicada, IMPA
Estrada Dona Castorina, 110, 22460-320 Rio de Janeiro, Brazil;
vgbramos@impa.br