Ordinals arising as residual finiteness depths

Martin R. Bridson

Abstract. For every natural number n, there exist finitely presented groups with residual finiteness depths $\omega \cdot n$ and $\omega \cdot n + 1$. The ordinals that arise as the residual finiteness depth of a finitely generated group (equivalently, a countable group) are 0, 1, the countable limit ordinals, and the successors of these limit ordinals.

Introduction

The *residual core* $\mathrm{Core}_{\mathrm{RF}}(G)$ of a group G is the normal subgroup consisting of those elements of G that have trivial image in every finite quotient of G, i.e. the kernel of the natural map from G to its profinite completion $\hat{G}$. If an infinite group G is residually finite, then its *residual finiteness depth* is defined as $\mathrm{Depth}_{\mathrm{RF}}(G) = \omega$, because the shortest chain of finite-index subgroups interpolating between G and $\{1\}$ has length ω. If G is not residually finite, then one can continue beyond ω by examining the finite quotients of $\mathrm{Core}_{\mathrm{RF}}(G)$. In [3], Brody and Jankiewicz define a countable group G to have residual finiteness depth $\omega \cdot n$ if there is a sequence of infinite groups $G = C_0 > C_1 > \cdots > C_{n-1}$ such that C_{n-1} is residually finite and $C_i = \mathrm{Core}_{\mathrm{RF}}(C_{i-1})$ for $i = 1, \ldots, n-1$. They also define what it means for $\mathrm{Depth}_{\mathrm{RF}}(G)$ to be equal to other ordinals (see Section 1). For example, $\mathrm{Depth}_{\mathrm{RF}}(G) = \omega \cdot n + 1$ if, instead of C_{n-1} being residually finite, $\mathrm{Core}_{\mathrm{RF}}(C_{n-1})$ is a non-trivial finite group. And $\mathrm{Depth}_{\mathrm{RF}}(G) = \omega^2$ if there is a sequence of infinite groups $G = C_0 > C_1 > \ldots$ such that each $C_i = \mathrm{Core}_{\mathrm{RF}}(C_{i-1})$ is infinite but $\mathrm{Core}_{\mathrm{RF}}{}^{\omega}(G) := \bigcap_i C_i = 1$. If $\mathrm{Core}_{\mathrm{RF}}{}^{\omega}(G)$ is a non-trivial finite group, then $\mathrm{Depth}_{\mathrm{RF}}(G) = \omega^2 + 1$.

Using a wreath product construction, Brody and Jankiewicz proved ([3], Theorem 1.1) that for every integer $n > 0$, there exist finitely generated groups G_n with $\mathrm{Depth}_{\mathrm{RF}}(G_n) = \omega \cdot n$. Their groups are not finitely presented when $n \geq 2$ and they ask (Question 5.3 in [3]) if there might exist an alternative construction that gives *finitely presented* groups Γ_n with $\mathrm{Depth}_{\mathrm{RF}}(\Gamma_n) = \omega \cdot n$. The first purpose of this note is to describe such a construction. Brody and Jankiewicz also ask (Question 5.4 in [3]) if there might exist groups M_n with $\mathrm{Depth}_{\mathrm{RF}}(M_n) = \omega \cdot n + 1$.

Theorem A. *For every positive integer n, there exist a finitely presented group Γ_n with $\mathrm{Depth}_{\mathrm{RF}}(\Gamma_n) = \omega \cdot n$ and a finitely presented group M_n with $\mathrm{Depth}_{\mathrm{RF}}(M_n) = \omega \cdot n + 1$.*

The groups in Theorem A will be constructed in an inductive manner as amalgamated free products; the proof relies on Bass–Serre theory. The construction is quite flexible, allowing one to impose extra conditions on the groups constructed; for example, one can require the Γ_n to have finite classifying spaces. But in an effort to make the ideas as clear as possible, I shall concentrate on specific families of examples, leaving the reader to consider variations.

A third question of Brody and Jankiewicz (Question 5.5 in [3]) concerns the existence of groups whose depths are ω^n and other ordinals. They also ask about the difference between finitely generated and finitely presented groups in this context.

The following theorem gives a complete classification of the ordinals that arise as the residual finiteness depths of finitely generated groups. We shall see in Section 3 that the answer is the same for countable groups.

Theorem B. *Let α be an ordinal. There exists a finitely generated group J_α for which* $\mathrm{Depth}_{\mathrm{RF}}(J_\alpha) = \alpha$ *if and only if α is 0, 1, a countable limit ordinal, or the successor of a countable limit ordinal.*

There are uncountably many countable limit ordinals but only countably many finitely presented groups, so we obtain the following consequence.

Corollary C. *There are uncountably many ordinals that arise as the residual finiteness depth of a finitely generated group but not as the residual finiteness depth of a finitely presented group.*

For ordinals $\alpha \geq \omega^2$, none of the groups constructed in the proof of Theorem B are finitely presented, and for the moment I do not know how to realise any of these ordinals as the residual finiteness depth of a finitely presented group.

This paper is organised as follows. In Section 1, we gather basic definitions, relate depth to cores for countable ordinals, and prove some elementary lemmas describing the interplay of these notions with Bass–Serre theory. In Section 2, we prove that every limit ordinal less than ω^2 arises as the residual finiteness depth of a finitely presented group. In Section 3, we prove that every countable limit ordinal arises as the residual finiteness depth of a countable group, and deduce some related facts concerning successor ordinals; the constructions here involve a combination of wreath products and free products. In Section 4, we embed the countable groups of the previous section into finitely generated groups in a careful manner, enabling us to show that every countable limit ordinal is the residual finiteness depth of a finitely generated group. In Section 5, we complete the proof of Theorem A by dealing with successor ordinals, and in Section 6, we complete the proof of Theorem B by combining ideas from previous sections.

1. $\mathrm{Depth}_{\mathrm{RF}}$ and residual cores

Brody and Jankiewicz [3] introduce and explore extensions of the following notion of *residual finiteness depth*; for our purposes, this simplified definition will suffice. A similar notion is implicit in [1].

Definition 1.1. Let α be an ordinal. A group G is α-*residually finite* if there exist subgroups $\{G_\beta\}_{\beta \leq \alpha}$ of G so that

- $G_0 = G$ and $G_\alpha = \{1\}$,
- $G_\gamma \leq G_\beta$ if $\gamma > \beta$,
- $[G_\beta : G_{\beta+1}]$ is finite for all $\beta < \alpha$,
- $G_\lambda = \bigcap_{\beta < \lambda} G_\beta$ for all limit ordinals $\lambda \leq \alpha$.

If α is the least ordinal such that G is α-residually finite, then $\text{Depth}_{\text{RF}}(G) := \alpha$.

Note that there are finitely generated groups which are not α-residually finite for any α, for example groups that have no non-trivial finite quotients. Note too that if $\text{Depth}_{\text{RF}}(G)$ is not 0 (the trivial group) or 1 (finite groups), then it is either a limit ordinal or the successor of a limit ordinal: if G is $(\alpha + n)$-residually finite, with n finite, then in a filtration $\{G_\beta\}$ witnessing this, one can replace $G_\alpha \geq G_{\alpha+1} \geq \cdots \geq G_{\alpha+n}$ by $G_\alpha \geq G_{\alpha+n}$ to see that $\text{Depth}_{\text{RF}}(G) \leq \alpha + 1$.

In the introduction we defined $\text{Depth}_{\text{RF}}(G)$ in terms of cores, and it will be convenient to use the language of cores again in our proofs, so we make the following definition.

Definition 1.2. Let G be a group. The *residual core* $\text{Core}_{\text{RF}}(G)$ is the normal subgroup consisting of those elements of G that have trivial image in every finite quotient of G. Starting with $\text{Core}_{\text{RF}}^0(G) = G$ and $\text{Core}_{\text{RF}}^1(G) = \text{Core}_{\text{RF}}(G)$, for successor ordinals (e.g., natural numbers) we define $\text{Core}_{\text{RF}}^{\alpha+1}(G) := \text{Core}_{\text{RF}}(\text{Core}_{\text{RF}}^\alpha(G))$, and for limit ordinals, $\text{Core}_{\text{RF}}^\alpha(G) := \bigcap \{\text{Core}_{\text{RF}}^\beta(G) \mid \beta < \alpha\}$.

Remarks 1.3. (1) If H is a countable group, then one can construct a sequence of finite-index subgroups $H = H_0 > H_1 > \ldots$ with $\bigcap_i H_i = \text{Core}_{\text{RF}}(H)$ by choosing coset representatives $\{c_0 = 1, c_1, c_2, \ldots\}$ for $\text{Core}_{\text{RF}}(H)$ in H and defining H_i to be the intersection of H_{i-1} with the kernel of a map from H to a finite group that maps c_i non-trivially.

(2) If G is countable and $\text{Depth}_{\text{RF}}(G) = \alpha$, then α is countable: if α were not countable, then by choosing an element $g_\beta \in \text{Core}_{\text{RF}}^\beta(G) \smallsetminus \text{Core}_{\text{RF}}^{\beta+1}(G)$ for each countable ordinal β, we would obtain an uncountable subset of G.

The reader will recall that all limit ordinals have the form $\omega \cdot \beta$. (Throughout, it is assumed that the reader is familiar with elementary facts about ordinals and their arithmetic, as found in Chapter 8 of [5] for example.)

Lemma 1.4. *Let G be a countable group and let α and β be ordinals.*

(1) $\text{Core}_{\text{RF}}^\beta(G) = 1$ *if and only if G is $(\omega \cdot \beta)$-residually finite.*

(2) *When $\text{Core}_{\text{RF}}^\beta(G)$ is non-trivial, it is finite if and only if G is $(\omega \cdot \beta + 1)$-residually finite.*

(3) $\text{Depth}_{\text{RF}}(G) = \omega \cdot \alpha + 1$ *if and only if $\text{Core}_{\text{RF}}^\alpha(G)$ is finite but non-trivial.*

(4) $\text{Depth}_{\text{RF}}(G) = \omega \cdot \alpha$ *if and only if $\text{Core}_{\text{RF}}^\alpha(G) = 1$ and α is the least ordinal such that $\text{Core}_{\text{RF}}^\alpha(G)$ is finite.*

Proof. For (1), assuming $\text{Core}_{\text{RF}}^\beta(G)$ to be trivial, we must define a family of subgroups $\{G_\gamma\}_{\gamma \leq \omega \cdot \beta}$ showing that G is $\omega \cdot \beta$-residually finite. First, we use Remark 1.3 to construct a sequence of finite-index subgroups $H_i(0)$ in G that intersect in $\text{Core}_{\text{RF}}(G)$. Next, for

limit ordinals $\omega \cdot \delta \leq \omega \cdot \beta$, we define $G_{\omega \cdot \delta} = \mathrm{Core}_{\mathrm{RF}}{}^{\delta}(G)$. To finish, for successor ordinals we use Remark 1.3 again to construct a sequence of finite-index subgroups $H_i(\delta)$ in $\mathrm{Core}_{\mathrm{RF}}{}^{\delta}(G)$ that intersect in $\mathrm{Core}_{\mathrm{RF}}{}^{\delta+1}(G)$ and define $G_{\omega \cdot \delta + i} = H_i(\delta)$.

For the reverse implication in (1), we take $\{G_{\gamma}\}_{\gamma \leq \omega \cdot \beta}$ to be a set of subgroups witnessing the fact that G is $\omega \cdot \beta$-residually finite. Since $\mathrm{Core}_{\mathrm{RF}}(K) \leq \mathrm{Core}_{\mathrm{RF}}(H)$ for all groups $K < H$, an obvious transfinite induction shows that $\mathrm{Core}_{\mathrm{RF}}{}^{\delta}(G) \leq G_{\omega \cdot \delta}$ for all $\delta \leq \beta$.

The passage from (1) to (2) is trivial. Items (3) and (4) follow immediately from (1) and (2). ∎

1.1. Cores, depth, and graphs of groups

I shall assume that the reader is familiar with the rudiments of Bass–Serre theory [6]. If Γ is the fundamental group of a graph of groups $\mathbb{G}$, the universal cover $\widetilde{\mathbb{G}}$ (in the category of graphs of groups) is the Bass–Serre tree for this splitting. Associated to each subgroup $H < \Gamma$, one has the graph of groups $\widetilde{\mathbb{G}} /\!/ H$ covering $\mathbb{G}$. We shall be particularly interested in the case where the underlying graph $|\widetilde{\mathbb{G}} /\!/ H|$ is a tree, which it will be if and only if H is generated by elliptic elements, i.e., by $\bigcup(H \cap G_v^{\gamma})$, where the union is taken over all conjugates of the vertex groups G_v of $\mathbb{G}$. Given a normal subgroup $K \trianglelefteq \Gamma$, we also consider the graph of groups $\mathbb{G}_K$ (thought of as "$\mathbb{G}$ mod K") that is obtained from $\mathbb{G}$ without changing the underlying graph by replacing each of local groups G with $G/(G \cap K)$, taking the induced inclusions of edge groups. The kernel of the natural epimorphism $\Gamma = \pi_1 \mathbb{G} \to \pi_1 \mathbb{G}_K$ is the subgroup of K generated by elliptic elements. The quotient map $\Gamma \to \Gamma/K$ factors through $\Gamma \to \pi_1 \mathbb{G}_K$ giving a natural epimorphism $\pi_1 \mathbb{G}_K \to \Gamma/K$ that is inclusion on the local groups; this epimorphism is not injective in general.

Lemma 1.5. *Let $\mathbb{G}$ be a tree of groups with $\Gamma = \pi_1 \mathbb{G}$. If all of the edge groups of $\mathbb{G}$ lie in $\mathrm{Core}_{\mathrm{RF}}(\Gamma)$, then $|\widetilde{\mathbb{G}} /\!/ \mathrm{Core}_{\mathrm{RF}}(\Gamma)|$ is a tree.*

Proof. We apply the preceding general considerations with $K = \mathrm{Core}_{\mathrm{RF}}(\Gamma)$. The vertex groups of $\mathbb{G}_K$ are residually finite, since the cores of the vertex groups of $\mathbb{G}$ are contained in K, and the edge groups are trivial by hypothesis. Thus $\pi_1 \mathbb{G}_K$ is a free product of residually finite groups. In particular, it is residually finite, so the natural epimorphism $\pi_1 \mathbb{G}_K \to \Gamma/\mathrm{Core}_{\mathrm{RF}}(\Gamma)$ splits and $\mathrm{Core}_{\mathrm{RF}}(\Gamma) = \ker(\Gamma \to \pi_1 \mathbb{G}_K)$, which is generated by elliptic elements. ∎

Remark 1.6. When we interpolate between Γ and $\mathrm{Core}_{\mathrm{RF}}(\Gamma)$ with finite-index subgroups $\Gamma = \Gamma_0 > \Gamma_1 > \ldots$, the graphs $|\widetilde{\mathbb{G}} /\!/ \Gamma_i|$ need not be trees. For example, in the infinite dihedral group $\Gamma = (\mathbb{Z}/2) * (\mathbb{Z}/2)$, if Γ_1 is the infinite cyclic subgroup of index two, then $|\widetilde{\mathbb{G}} /\!/ \Gamma_1|$ has two vertices and two edges.

Lemma 1.7. *If a countable group Γ is the free product of non-trivial groups A_i, then*

(1) *$A_i \cap \mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(\Gamma) = \mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(A_i)$ for any ordinal α, and*

(2) *$\mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(\Gamma)$ is a free product of copies of the groups $\mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(A_i)$.*

(3) *If $\sup_i \mathrm{Depth}_{\mathrm{RF}}(A_i)$ is a limit ordinal, then $\mathrm{Depth}_{\mathrm{RF}}(\Gamma) = \sup_i \mathrm{Depth}_{\mathrm{RF}}(A_i)$;*

(4) *if $\sup_i \mathrm{Depth}_{\mathrm{RF}}(A_i)$ is a successor ordinal, $\mathrm{Depth}_{\mathrm{RF}}(\Gamma) = \sup_i \mathrm{Depth}_{\mathrm{RF}}(A_i) + \omega$.*

Proof. Assertion (1) holds more generally for retracts A of Γ: for $\alpha = 1$, it is the obvious assertion that an element of A survives in a finite quotient of A if and only if it survives in some finite quotient of Γ; for arbitrary α, the result follows by transfinite induction, noting repeatedly that a retraction $\Gamma \to A$ restricts to a retraction $\mathrm{Core}_{\mathrm{RF}}(\Gamma) \to \mathrm{Core}_{\mathrm{RF}}(A)$.

For (2), we consider the action of $\mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(\Gamma)$ on the Bass–Serre tree for the splitting $\mathbb{G}$ of Γ as a free product of the A_i. The vertex groups in the quotient graph-of-groups are conjugates of $A_i \cap \mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(\Gamma)$, which by (1) are copies of $\mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(A_i)$. Proceeding by transfinite induction, with Lemma 1.5 as the base step, we shall argue that the underlying graph $|\widetilde{\mathbb{G}} /\!/ \mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(\Gamma)|$ is a tree. In the inductive step, we assume that this is the case for ordinals $\beta < \alpha$.

At successor ordinals $\alpha = \alpha_0 + 1$, we can apply Lemma 1.5 with $\mathrm{Core}_{\mathrm{RF}}{}^{\alpha_0}(\Gamma)$ in place of Γ, since $\widetilde{\mathbb{G}} /\!/ \mathrm{Core}_{\mathrm{RF}}{}^{\alpha_0}(\Gamma)$ (which is a *tree* of groups, by induction) has the same universal cover as $\mathbb{G}$. To clarify the argument at limit ordinals, we first note that an arbitrary normal subgroup N in $\Gamma = *_i A_i$ will be generated by its elliptic elements $\bigcup_{i,\gamma}(N \cap A_i^{\gamma})$ if and only if there does *not* exist an element $v \in N$ whose free-product normal form is $v = a_1 \dots a_m$, with all $a_j \in A_{i(j)} \smallsetminus N$. (Such an element v determines a reduced loop in the quotient graph of groups $\mathbb{G} /\!/ N$.) We apply this observation with $N = \mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(\Gamma)$: if there were to exist $v \in \mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(\Gamma)$ with $v = a_1 \dots a_m$ and all $a_j \in A_{i(j)} \smallsetminus \mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(\Gamma)$ then, because $\mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(\Gamma) = \bigcap_{\beta < \alpha} \mathrm{Core}_{\mathrm{RF}}{}^{\beta}(\Gamma)$, this would imply that $a_j \in A_{i(j)} \smallsetminus \mathrm{Core}_{\mathrm{RF}}{}^{\beta}(\Gamma)$ for all a_j when $\beta < \alpha$ is sufficiently large; but this cannot happen, because it implies that $\mathrm{Core}_{\mathrm{RF}}{}^{\beta}(\Gamma)$ is not generated by its elliptic elements, contrary to the inductive hypothesis. This completes the proof of (2).

Towards proving (3) and (4), we consider $\sigma = \sup_i \mathrm{Depth}_{\mathrm{RF}}(A_i)$. If σ is a limit ordinal $\sigma = \omega \cdot \alpha$, then $\mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(A_i) = 1$ for all i, so from (2) we have $\mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(\Gamma) = 1$, hence $\mathrm{Depth}_{\mathrm{RF}}(\Gamma) \leq \omega \cdot \alpha = \sigma$. And by applying to $A_i < \Gamma$ the observation that $H < K$ implies $\mathrm{Depth}_{\mathrm{RF}}(H) \leq \mathrm{Depth}_{\mathrm{RF}}(K)$, we have $\sigma \leq \mathrm{Depth}_{\mathrm{RF}}(\Gamma)$, so (3) is proved. For (4), we suppose $\sigma = \omega \cdot \beta + 1$, which means that $\mathrm{Core}_{\mathrm{RF}}{}^{\beta}(A_i)$ is finite for all indices i but $\mathrm{Core}_{\mathrm{RF}}{}^{\beta}(A_i)$ is non-trivial for some i. In this case, $\mathrm{Core}_{\mathrm{RF}}{}^{\beta}(\Gamma)$, which has more than one free factor isomorphic to $\mathrm{Core}_{\mathrm{RF}}{}^{\beta}(A_i)$ for each i, is a free product of finite groups and is infinite. As any free product of finite groups is residually finite, we conclude that $\mathrm{Core}_{\mathrm{RF}}{}^{\beta}(\Gamma)$ is infinite but $\mathrm{Core}_{\mathrm{RF}}{}^{\beta+1}(\Gamma) = 1$, hence $\mathrm{Depth}_{\mathrm{RF}}(\Gamma) = \omega \cdot (\beta + 1) = \sigma + \omega$. ∎

1.2. Direct products

For completeness, we record the analogue of Lemma 1.7 for direct sums. The proof in this case is straightforward, so we omit it.

Lemma 1.8. *If a countable group Γ is the direct sum of non-trivial groups A_i, then*

(1) $A_i \cap \mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(\Gamma) = \mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(A_i)$ *for any ordinal α, and*

(2) $\mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(\Gamma)$ *is the direct sum of the groups $\mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(A_i)$.*

(3) *If $\sup_i \mathrm{Depth}_{\mathrm{RF}}(A_i)$ is a limit ordinal, then $\mathrm{Depth}_{\mathrm{RF}}(\Gamma) = \sup_i \mathrm{Depth}_{\mathrm{RF}}(A_i)$;*

(4) *if $\sup_i \mathrm{Depth}_{\mathrm{RF}}(A_i)$ is a successor ordinal, then $\mathrm{Depth}_{\mathrm{RF}}(\Gamma) = \sup_i \mathrm{Depth}_{\mathrm{RF}}(A_i)$ if there are only finitely many A_i that realise the supremum, and $\mathrm{Depth}_{\mathrm{RF}}(\Gamma) = \sup_i \mathrm{Depth}_{\mathrm{RF}}(A_i) + \omega$ if there are infinitely many.*

1.3. Some more basic facts

The following observations will be useful throughout and we shall often use them without comment.

Lemma 1.9. *For all ordinals α and groups G,*

(1) *if $H \leq G$ then $\mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(H) \leq \mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(G)$;*

(2) *if $H \leq \mathrm{Core}_{\mathrm{RF}}{}^{\beta}(G)$ for all $\beta < \alpha$ then $\mathrm{Core}_{\mathrm{RF}}(H) \leq \mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(G)$;*

(3) *if $p\colon G \twoheadrightarrow Q$ is a surjection with kernel K, then $\mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(G) \leq p^{-1}(\mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(Q))$, with equality if $K \leq \mathrm{Core}_{\mathrm{RF}}{}^{\alpha}(G)$.*

2. Finitely presented groups realising the ordinals $\omega \cdot n$

We want to prove that $\omega \cdot n$ is the residual finiteness depth of a finitely presented group, proceeding by induction on n. In order to make the induction run smoothly, it is convenient to make the inductively-constructed groups Γ_n have the following properties:

(1) Γ_n is finitely presented;

(2) $\mathrm{Depth}_{\mathrm{RF}}(\Gamma_n) = \omega \cdot n$;

(3) Γ_n has a generating set $\{a_0, \ldots, a_m\}$ for which there is an epimorphism $h_n\colon \Gamma_n \to \mathbb{Z}$ with $h_n(a_i) = h_n(a_0)$ for $i = 1, \ldots, m$. (The integer m depends on n.)

In the inductive step, we construct a group Γ_n^0 satisfying (1) and (2), and then define $\Gamma_n = \Gamma_n^0 * \mathbb{Z}$, invoking the following observation to get condition (3).

Lemma 2.1. *If Γ_n^0 satisfies (1) and (2), then $\Gamma_n = \Gamma_n^0 * \mathbb{Z}$ satisfies (1), (2) and (3).*

Proof. If Γ_n^0 is generated by $b_1, \ldots, b_m$ and ζ generates $\mathbb{Z}$, define $a_0 = \zeta$ and $a_i = b_i\zeta$. Lemma 1.7 assures us that $\Gamma_n * \mathbb{Z}$ satisfies condition (2), and killing Γ_n^0 yields h_n. ∎

Remark 2.2. The key point about property (3) is that it ensures that all of the cyclic subgroups $\langle a_i \rangle < \Gamma_n$ are retracts of Γ_n. To see why this might be useful, recall from [2] that if $\mathbb{G}$ is a tree of groups where each vertex group is residually finite and each edge group is a retract of the vertex groups into which it includes, then $\pi_1\mathbb{G}$ is residually finite.

It will be convenient to have a finitely presented group Λ for which $\mathrm{Core}_{\mathrm{RF}}(\Lambda) \cong \mathbb{Z}$. A famous example of such a group is Deligne's central extension [4] of $\mathrm{Sp}(2g, \mathbb{Z})$, any $g \geq 2$, for which $\mathrm{Core}_{\mathrm{RF}}(\Lambda)$ is the subgroup of index 2 in the kernel of the central extension

$$(2.1) \qquad 1 \to \mathbb{Z} \to \Lambda \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1.$$

Henceforth, Λ will always denote this central extension. Later on, we shall also need a central extension of a residually finite group that has finite centre and is not residually finite. Such a group can be obtained from Λ by killing a subgroup of odd index $d > 2$ in the kernel of the extension (2.1),

$$(2.2) \qquad 1 \to \mathbb{Z}/d\mathbb{Z} \to \Delta \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1.$$

The value of $d > 2$ will not matter, but we fix $g \geq 3$ to ensure that Δ is perfect (which follows immediately from the fact that $\mathrm{Sp}(2g, \mathbb{Z})$ is perfect, since the central $\mathbb{Z}/d\mathbb{Z}$ dies in every finite quotient of Δ).

Theorem 2.3. *For every positive integer n, there exists a finitely presented group Γ_n with* $\mathrm{Depth}_{\mathrm{RF}}(\Gamma_n) = \omega \cdot n$.

Proof. We argue by induction on n. For Γ_1, we can take $\mathbb{Z}$, for example. For Γ_2, we can take $\Gamma_2^0 * \mathbb{Z}$, where $\Gamma_2^0 \cong \Lambda$ is the group from (2.1). In the inductive step, we assume $n \geq 2$ and that Γ_n satisfies the conditions (1) to (3) described above, and we must construct Γ_{n+1}^0 satisfying (1) and (2). Define Γ_{n+1}^0 to be the group obtained by amalgamating $m + 1$ copies of Λ with Γ_n, where the i-th copy Λ_i of Λ is amalgamated by identifying a generator ζ_i of $\mathrm{Core}_{\mathrm{RF}}(\Lambda_i) \cong \mathbb{Z}$ with the generator $a_i \in \Gamma_n$.

In any finite image of Γ_{n+1}^0, each ζ_i must "die" (i.e., map trivially), so Γ_n must die, since each of its generators has been identified with some ζ_i. Thus every map from Γ_{n+1}^0 to a finite group will factor through the epimorphism $\rho \colon \Gamma_{n+1}^0 \to *_{i=0}^m \Lambda_i / \langle \zeta_i \rangle$. This free product of copies of $\Lambda / \langle \zeta \rangle$ is residually finite, so the kernel of ρ is $\mathrm{Core}_{\mathrm{RF}}(\Gamma_{n+1}^0)$. Bass–Serre theory, as in the proof of Lemma 1.7, tells us that this kernel is the fundamental group of a graph of groups $\mathcal{T}$ some of whose vertex groups are conjugates of Γ_n while the others (from the Λ_i vertices) are infinite cyclic groups; each of the latter type is generated by each of the edge groups incident at that vertex, since each is a conjugate of $\langle \zeta_i \rangle$ and $\langle \zeta_i \rangle$ is normal in Λ_i. Moreover, Lemma 1.5 tells us that the underlying graph of $\mathcal{T}$ is a tree.

The map h_n in condition (3) provides a map onto $\mathbb{Z}$ from each vertex group of type Γ_n in the decomposition $\mathcal{T}$, and these maps agree on intersections (the edge groups). Thus we obtain a map h_{n+1} from $\pi_1 \mathcal{T} = \mathrm{Core}_{\mathrm{RF}}(\Gamma_{n+1}^0)$ onto $\mathbb{Z}$ that restricts to an isomorphism on each edge group and on each of the vertex groups that is cyclic. It follows that the subgroup of $\mathrm{Core}_{\mathrm{RF}}(\Gamma_{n+1}^0)$ consisting of elements that die in every finite quotient of $\mathrm{Core}_{\mathrm{RF}}(\Gamma_{n+1}^0)$ intersects the edge groups and cyclic vertex groups trivially, and therefore this subgroup (which is $\mathrm{Core}_{\mathrm{RF}}^2(\Gamma_{n+1}^0)$) is the fundamental group of a graph of groups whose edge groups are trivial and whose vertex groups are either trivial or else conjugates in $\mathrm{Core}_{\mathrm{RF}}(\Gamma_{n+1}^0)$ of $I_n := \Gamma_n \cap \mathrm{Core}_{\mathrm{RF}}^2(\Gamma_{n+1}^0)$, which we claim is $\mathrm{Core}_{\mathrm{RF}}(\Gamma_n)$.

It is obvious that I_n contains $\mathrm{Core}_{\mathrm{RF}}(\Gamma_n)$, so to prove the claim it is enough to argue that each $\gamma \in \Gamma_n \smallsetminus \mathrm{Core}_{\mathrm{RF}}(\Gamma_n)$ survives in some finite quotient of $\mathrm{Core}_{\mathrm{RF}}(\Gamma_{n+1}^0)$. By definition, there is a finite quotient $p_\gamma \colon \Gamma_n \to Q$ with $p_\gamma(\gamma) \neq 1$, and we will be done if we can extend p_γ to the whole of $\mathrm{Core}_{\mathrm{RF}}(\Gamma_{n+1}^0)$. For this, we consider the components of the graph-of-groups obtained when we delete the base vertex (where the local group is Γ_n) from the tree of groups $\mathcal{T}$ described above. There is one such component C_i of $\mathcal{T}$ for each of the generators $a_0, \ldots, a_m$ of Γ_n, and $\pi_1 C_i < \pi_1 \mathcal{T} = \mathrm{Core}_{\mathrm{RF}}(\Gamma_{n+1}^0)$ intersects Γ_n in $\langle a_i \rangle$. If $p_\gamma(a_i)$ has order e_i, we extend p_γ to $\pi_1 C_i$ by composing the surjection $h_{n+1} \colon \pi_1 C_i \to \mathbb{Z}$ (defined above) with $\mathbb{Z} \to \mathbb{Z}/e_i\mathbb{Z}$. This completes the proof of the claim.

At this point, we have proved that $\mathrm{Core}_{\mathrm{RF}}^2(\Gamma_{n+1}^0)$ is a free product of (infinitely many) copies of $I_n = \mathrm{Core}_{\mathrm{RF}}(\Gamma_n)$ and possibly a free group. By induction, $\mathrm{Depth}_{\mathrm{RF}}(\Gamma_n) = \omega \cdot n$, so Lemma 1.7 tells us that $\mathrm{Depth}_{\mathrm{RF}}(\mathrm{Core}_{\mathrm{RF}}^2(\Gamma_{n+1}^0)) = \omega \cdot (n - 1)$. Thus $\mathrm{Depth}_{\mathrm{RF}}(\Gamma_{n+1}^0) = \omega \cdot (n + 1)$. This completes the induction. ∎

Later, we shall need the following additional property of our construction.

Addendum 2.4. *For $n \geq 3$, the subgroup $\mathrm{Core}_{RF}^{n-1}(\Gamma_n)$ is a free product of a free group and conjugates of $\mathrm{Core}_{RF}(\Gamma_2)$.*

Proof. In the last paragraph of the preceding proof, we saw that $\mathrm{Core}_{RF}^2(\Gamma_{n+1}^0)$ was a free product of copies of $\mathrm{Core}_{RF}(\Gamma_n)$ and a free group. The addendum follows by a simple induction. For the inductive step, since $\Gamma_{n+1} = \Gamma_{n+1}^0 * \mathbb{Z}$, we use Lemma 1.7 to see that $\mathrm{Core}_{RF}(\Gamma_{n+1})$ is the free product of copies of $\mathrm{Core}_{RF}(\Gamma_{n+1}^0)$, hence $\mathrm{Core}_{RF}^2(\Gamma_{n+1})$ is a free product of copies of $\mathrm{Core}_{RF}^2(\Gamma_{n+1}^0)$, which is a free product of copies of $\mathrm{Core}_{RF}(\Gamma_n)$ and a free group. Repeated application of Lemma 1.7 then tells us that $\mathrm{Core}_{RF}^n(\Gamma_{n+1})$ is a free product of copies of $\mathrm{Core}_{RF}^{n-1}(\Gamma_n)$. ∎

Remark 2.5. In the proof of Theorem 2.3, it was convenient to name a specific Γ_1 and Γ_2 before starting the inductive step, but we could have made many other choices for these groups. If we chose as Γ_1 any residually finite group with generators $\{a_0, \ldots, a_m\}$ as in condition (3), then we could get Γ_2^0 by amalgamating $m + 1$ copies of Λ with Γ_1, as in the inductive step of the proof, identifying the generator ζ_i of the core of the i-th copy Λ_i with $a_i \in \Gamma_1$. Arguing as in the second paragraph of the proof, we would then get $\mathrm{Core}_{RF}(\Gamma_2)$ to be the fundamental group of a tree of groups with cyclic edge groups and vertex groups that are either cyclic or copies of Γ_1. The inclusions of edge groups in this tree of groups are either isomorphisms or else inclusions into copies of Γ_1 of one of the cyclic groups $\langle a_i \rangle$. In particular, all of the edge groups are retracts in the adjacent vertex groups, so the fundamental group $\mathrm{Core}_{RF}(\Gamma_2)$ is residually finite, by [2]. Hence $\mathrm{Depth}_{RF}(\Gamma_2) = \omega \cdot 2$, as required. The remainder of the proof then continues as before.

3. Residual finiteness depth for countable groups

The purpose of this section is to prove the following result, a step towards Theorem B.

Proposition 3.1. *For every countable limit ordinal α, there exists a countable group G with $\mathrm{Depth}_{RF}(G) = \alpha$.*

3.1. Wreath products

The results in this subsection can be gleaned from [3], but we present them in a way that is convenient to our aims. We use the standard notation $A \wr B$ for the restricted wreath product, i.e., the semidirect product $P_{A,B} \rtimes B$ where $P_{A,B} = \oplus_{b \in B} A_b$ is the direct sum of copies of A indexed by B, with B acting by left-multiplication on the index set. We also need to consider more general *permutational wreath products* $A \wr_I C = P_{A,I} \rtimes C$, where $P_{A,I}$ is the direct sum of copies of A indexed by a set I and C acts by a homomorphism $C \to \mathrm{Perm}(I)$. We shall only be interested in the setting where the action of C on I is free, which is case when C is a subgroup of B and the action is left multiplication on $B = I$; in this case, $A \wr_B C$ is the subgroup of $A \wr B$ generated by $P_{A,B}$ and C.

Lemma 3.2. *Let $A \wr_I C = P_{A,I} \rtimes C$ be a permutational wreath product where the action of C on I is free. If A is perfect and C is infinite, then $\mathrm{Core}_{RF}(A \wr_I C) = P_{A,I} \rtimes \mathrm{Core}_{RF}(C)$.*

Proof. For all $i \in I$ and $x, y \in A_i$ and any finite quotient $p: A \wr_I C \to Q$, we have $p(c) = 1$ for some non-trivial $c \in C$. Writing g^c for conjugation of g by c,

$$p([x, y]) = [p(x), p(y)] = [p(x), p(y)^{p(c)}] = [p(x), p(y^c)] = p([x, y^c]) = p(1) = 1,$$

where the penultimate equality comes from the fact that $y^c \in A_{c \cdot i}$ commutes with $x \in A_i$ since $c \cdot i \neq i$. Thus $[A_i, A_i] \subseteq \mathrm{Core}_{\mathrm{RF}}(A \wr_I C)$ for all i. But A_i is perfect, so $A_i = [A_i, A_i]$ and $P_{A,I} \subseteq \mathrm{Core}_{\mathrm{RF}}(A \wr_I C)$.

Conversely, if $g \notin P_{A,I} \rtimes \mathrm{Core}_{\mathrm{RF}}(C)$, then g projects to $C \smallsetminus \mathrm{Core}_{\mathrm{RF}}(C)$ and therefore survives in some finite quotient of C and hence of $A \wr_I C$. ∎

Arguing by transfinite induction, we deduce the following.

Corollary 3.3. *For all ordinals α, if A is perfect and $\mathrm{Core}_{\mathrm{RF}}{}^\beta(B)$ is infinite for all $\beta < \alpha$, then $\mathrm{Core}_{\mathrm{RF}}{}^\alpha(A \wr B) = P_{A,B} \rtimes \mathrm{Core}_{\mathrm{RF}}{}^\alpha(B)$.*

Proposition 3.4. *Let $A \neq 1$ be a countable group that is perfect and let B be a countable group that is infinite and β-residually finite for some β.*

(1) *If $\mathrm{Depth}_{\mathrm{RF}}(A)$ is a limit ordinal, then*

$$\mathrm{Depth}_{\mathrm{RF}}(A \wr B) = \mathrm{Depth}_{\mathrm{RF}}(B) + \mathrm{Depth}_{\mathrm{RF}}(A).$$

(2) *If $\mathrm{Depth}_{\mathrm{RF}}(A)$ is a successor ordinal, then*

$$\mathrm{Depth}_{\mathrm{RF}}(A \wr B) = \mathrm{Depth}_{\mathrm{RF}}(B) + \mathrm{Depth}_{\mathrm{RF}}(A) + \omega.$$

(Note that, in both cases, $\mathrm{Depth}_{\mathrm{RF}}(A \wr B)$ is a limit ordinal.)

Proof. If $\mathrm{Depth}_{\mathrm{RF}}(B) = \omega \cdot \alpha$ or $\omega \cdot \alpha + 1$, then $\mathrm{Core}_{\mathrm{RF}}{}^\alpha(B)$ is finite, so $\mathrm{Core}_{\mathrm{RF}}{}^\alpha(A \wr B)$ contains $P_{A,B}$ as a subgroup of finite index, by Corollary 3.3. Passing to a subgroup of finite index does not affect the residual finiteness depth of an infinite group, so we have $\mathrm{Depth}_{\mathrm{RF}}(A \wr B) = \mathrm{Depth}_{\mathrm{RF}}(B) + \mathrm{Depth}_{\mathrm{RF}}(P_{A,B})$.

Lemma 1.8 tells us that $\mathrm{Depth}_{\mathrm{RF}}(P_{A,B}) = \mathrm{Depth}_{\mathrm{RF}}(A)$ if $\mathrm{Depth}_{\mathrm{RF}}(A)$ is a limit ordinal, and $\mathrm{Depth}_{\mathrm{RF}}(P_{A,B}) = \mathrm{Depth}_{\mathrm{RF}}(A) + \omega$ otherwise. ∎

Brody and Jankiewicz [3] proved that there are finitely generated groups with residual finiteness depth $\omega \cdot n$ for any $n \in \mathbb{N}$ by repeatedly applying the following consequence of Proposition 3.4.

Corollary 3.5. *If α is the residual finiteness depth of a finitely generated (resp. countable) group, then so is $\alpha + \omega$.*

Proof. The assertion is trivial if $\alpha \in \{0, 1\}$, so assume α is infinite. In Proposition 3.4, take $A \neq 1$ to be a finitely generated, perfect, residually finite group and choose B with $\mathrm{Depth}_{\mathrm{RF}}(B) = \alpha$. ∎

3.2. Proof of Proposition 3.1

The trivial group has residual finiteness depth 0 for non-trivial finite groups $\mathrm{Depth}_{\mathrm{RF}}(G) = 1$, and for residually-finite groups that are infinite $\mathrm{Depth}_{\mathrm{RF}}(G) = \omega$. For other limit ordinals α, we proceed by transfinite induction, assuming that for each limit ordinal $\beta < \alpha$, there

exists a group G_β with $\mathrm{Depth}_{\mathrm{RF}}(G_\beta) = \beta$. If $\alpha = \alpha_0 + \omega$, then we take a non-trivial finite perfect group A, define $G_\alpha = A \wr G_{\alpha_0}$ and appeal to Corollary 3.3. If α is not of this form, then it is the supremum of the (countably many) limit ordinals $\beta < \alpha$ and we define G_α to be either the free product or the direct sum of the inductively defined groups G_β corresponding to these ordinals. Lemma 1.7(3) and Lemma 1.8(3) assure us that $\mathrm{Depth}_{\mathrm{RF}}(G_\alpha) = \alpha$ in both cases.   ∎

3.3. Attaining some successor ordinals

Proposition 3.6. *If α is the residual finiteness depth of a finitely generated (resp. countable) group, then so is $\alpha + \omega + 1$.*

Proof. Let B be a finitely generated group with $\mathrm{Depth}_{\mathrm{RF}}(B) = \alpha$. (The countable case is entirely similar.) Let A be a finitely generated perfect group that has a finite central subgroup $Z = \mathrm{Core}_{\mathrm{RF}}(A) \neq 1$ – one of Deligne's group Δ from (2.2) will do nicely (with d odd and $g \geq 3$). Let $\overline{A} = A/Z$ and consider the central extension E of $\overline{A} \wr B$ that is obtained from $A \wr B$ by imposing relations that identify the copy of Z in each direct factor of $P_{A,B}$ with a single copy of Z. In other words, E is the quotient of $A \wr B$ by the relations $(b^{-1}zb = z \mid \forall b \in B, z \in Z)$. We have the central extension

$$1 \rightarrow Z \rightarrow E \rightarrow \overline{A} \wr B \rightarrow 1$$

and the preimage in E of each summand $\overline{A}_b$ in $P_{\overline{A},B}$ is a copy of A.

If $\alpha = \omega \cdot \gamma$, then as in Corollary 3.3, we see that $\mathrm{Core}_{\mathrm{RF}}{}^\gamma(E)$ is the preimage of $P_{\overline{A},B} < \overline{A} \wr B$. This preimage is the quotient of $P_{A,B}$ obtained by identifying the central $Z_b < A_b$ of the summands to a single Z. Thus, since $Z_b = \mathrm{Core}_{\mathrm{RF}}(A_b)$, the core of $\mathrm{Core}_{\mathrm{RF}}{}^\gamma(E)$ is Z. Therefore

$$\mathrm{Depth}_{\mathrm{RF}}(E) = \alpha + \mathrm{Depth}_{\mathrm{RF}}(\mathrm{Core}_{\mathrm{RF}}{}^\gamma(E)) = \alpha + \omega + 1.$$

If $\alpha = \omega \cdot \gamma + 1$, then we apply the preceding argument to $\omega \cdot \gamma$, noting that $\alpha + \omega + 1 = \omega \cdot \gamma + \omega + 1$.   ∎

Corollary 3.7. *If α is the residual finiteness depth of a finitely generated (resp. countable) group, then so is $\alpha + \omega \cdot n + 1$, for all positive $n \in \mathbb{N}$.*

Proof. Apply Proposition 3.6 repeatedly, noting that $(\alpha + \omega \cdot n + 1) + \omega + 1 = \alpha + \omega \cdot (n + 1) + 1$.   ∎

The central idea of the proof of Proposition 3.6 applies more generally: instead of looking for A and $Z = \mathrm{Core}_{\mathrm{RF}}(A)$ with A/Z residually finite, we could look for A and Z with $\mathrm{Depth}_{\mathrm{RF}}(A/Z) = \beta$ and $Z = \mathrm{Core}_{\mathrm{RF}}{}^\beta(A)$. Then, arguing in the same manner, we obtain:

Proposition 3.8. *If α is the residual finiteness depth of a finitely generated (resp. countable) group and $\beta + 1$ is the residual finiteness depth of a finitely generated (resp. countable) group A with $\mathrm{Core}_{\mathrm{RF}}{}^\beta(A) < A$ central, then there is a finitely generated (resp. countable) group with residual finiteness depth $\alpha + \beta + 1$.*

By applying the proposition to the groups from Theorem A, we obtain the following result, which will be subsumed into Theorem B.

Corollary 3.9. *If α is the residual finiteness depth of a finitely generated* (*resp. countable*) *group, then so is $\alpha + \omega \cdot n + 1$, for all positive $n \in \mathbb{N}$.*

4. Countable limit ordinals as residual finiteness depths

The purpose of this section is to prove the following result.

Theorem 4.1. *For every countable limit ordinal α, there exists a finitely generated group Γ with $\mathrm{Depth}_{\mathrm{RF}}(\Gamma) = \alpha$.*

Our proof divides into the case $\alpha < \omega^2$, which is covered by Theorem 2.3, and the case $\alpha \geq \omega^2$. A key technical point in the second case is that if $\mathrm{Depth}_{\mathrm{RF}}(G) = \alpha$, then $\mathrm{Depth}_{\mathrm{RF}}(G) = \mathrm{Depth}_{\mathrm{RF}}(\mathrm{Core}_{\mathrm{RF}}(G))$, because $\omega + \alpha = \alpha$.

4.1. Limit ordinals $\alpha \geq \omega^2$

In this regime, Theorem 4.1 is an immediate consequence of Proposition 3.1 and the following controlled-embedding lemma.

Proposition 4.2. *Let Γ be a countable group and suppose that $\mathrm{Depth}_{\mathrm{RF}}(\Gamma) \geq \omega^2$ is a limit ordinal. Then, there exist a 3-generator group Γ^+ and an embedding $\Gamma \hookrightarrow \Gamma^+$ such that $\mathrm{Depth}_{\mathrm{RF}}(\Gamma) = \mathrm{Depth}_{\mathrm{RF}}(\Gamma^+)$.*

At several points in the following proof, we shall use the obvious fact that $H < K$ implies $\mathrm{Depth}_{\mathrm{RF}}(H) \leq \mathrm{Depth}_{\mathrm{RF}}(K)$.

Proof. We list the elements of Γ without repetition, $\Gamma = \{g_0 = 1, g_1, g_2, \dots\}$. Let $\Gamma_0 = \Gamma * \mathbb{Z}$, fix a generator z for $\mathbb{Z}$, and for each $i \in \mathbb{N}$, define $a_i = zg_i$. Lemma 1.7 assures us that $\mathrm{Depth}_{\mathrm{RF}}(\Gamma) = \mathrm{Depth}_{\mathrm{RF}}(\Gamma_0)$. Note that each a_i has infinite order in Γ_0. Let Γ_1 be the HNN extension of Γ_0 with infinitely many stable letters t_i ($i \geq 1$), where t_i conjugates a_i to a_0:

$$\Gamma_1 = (\Gamma_0, t_1, t_2, \dots \mid t_i^{-1} a_i t_i = a_0 \text{ for } i \geq 1).$$

Note that Γ_1 is generated by $\{a_0, t_1, t_2, \dots\}$ and that the t_i freely generate a free group T onto which Γ_1 retracts with kernel $\langle\!\langle a_0 \rangle\!\rangle$. We claim that $\mathrm{Core}_{\mathrm{RF}}(\Gamma_1)$ is the free product of a free group and copies of $\Gamma_0 \cap \mathrm{Core}_{\mathrm{RF}}(\Gamma_1)$. Then, as the depth of $\Gamma_0 \cap \mathrm{Core}_{\mathrm{RF}}(\Gamma_1)$ is at most $\mathrm{Depth}_{\mathrm{RF}}(\Gamma_0)$, using Lemma 1.7 we have

$$\mathrm{Depth}_{\mathrm{RF}}(\Gamma_1) = \omega + \mathrm{Depth}_{\mathrm{RF}}(\mathrm{Core}_{\mathrm{RF}}(\Gamma_1))$$
$$\leq \omega + \mathrm{Depth}_{\mathrm{RF}}(\Gamma_0) = \omega + \mathrm{Depth}_{\mathrm{RF}}(\Gamma) = \mathrm{Depth}_{\mathrm{RF}}(\Gamma),$$

where the last equality holds because $\mathrm{Depth}_{\mathrm{RF}}(\Gamma) \geq \omega^2$. As $\Gamma < \Gamma_1$, we conclude that $\mathrm{Depth}_{\mathrm{RF}}(\Gamma_1) = \mathrm{Depth}_{\mathrm{RF}}(\Gamma)$.

To prove the claim, consider the graph-of-groups decomposition $\mathbb{G}$ of Γ_1 corresponding to its HNN structure. Each edge group is a conjugate of $\langle a_0 \rangle$, so it intersects

$\mathrm{Core}_{\mathrm{RF}}(\Gamma_1)$ trivially because killing all of the stable letters t_i retracts Γ_1 onto $\langle a_0 \rangle$. Therefore, the covering of $\mathbb{G}$ with fundamental group $\mathrm{Core}_{\mathrm{RF}}(\Gamma_1)$ has trivial edge groups and has vertex groups that are conjugates of $\Gamma_0 \cap \mathrm{Core}_{\mathrm{RF}}(\Gamma_1)$.

For the next step in the construction, we choose a free group of infinite rank L in $F_2 = \mathrm{Free}(b, c)$ and identify this with T to form the amalgamated free product

$$\Gamma^+ := \Gamma_1 *_{T=L} F_2.$$

Note that Γ^+ is generated by $\{a_0, b, c\}$. The retraction of Γ_1 onto T extends to a retraction of Γ^+ onto F_2, which is residually finite, so $\mathrm{Core}_{\mathrm{RF}}(\Gamma^+)$ intersects each conjugate of F_2 trivially, and hence is the fundamental group of a graph of groups with trivial edge groups and vertex groups that are either trivial or conjugates of $\mathrm{Core}_{\mathrm{RF}}(\Gamma^+) \cap \Gamma_1$. Thus $\mathrm{Core}_{\mathrm{RF}}(\Gamma^+)$ is a free product of a free group and copies of $\mathrm{Core}_{\mathrm{RF}}(\Gamma^+) \cap \Gamma_1$. As above, this implies

$$\mathrm{Depth}_{\mathrm{RF}}(\Gamma^+) = \omega + \mathrm{Depth}_{\mathrm{RF}}(\mathrm{Core}_{\mathrm{RF}}(\Gamma^+)) \leq \omega + \mathrm{Depth}_{\mathrm{RF}}(\Gamma_1) = \mathrm{Depth}_{\mathrm{RF}}(\Gamma_1).$$

Hence $\mathrm{Depth}_{\mathrm{RF}}(\Gamma^+) = \mathrm{Depth}_{\mathrm{RF}}(\Gamma_1) = \mathrm{Depth}_{\mathrm{RF}}(\Gamma)$. ∎

5. Finitely presented groups realising the ordinals $\omega \cdot n + 1$

In Question 5.4 of [3], Brody and Jankiewicz ask if there exist finitely presented groups M_n with $\mathrm{Depth}_{\mathrm{RF}}(M_n) = \omega \cdot n + 1$. In this section, we will see that the proof of Theorem A can be modified to arrange this. For this construction, we need a central extension of a residually finite group that has finite centre and is not residually finite. We will use the group Δ from (2.2) with d odd and $g \geq 3$ fixed.

Theorem 5.1. *For every positive integer n and odd integer $d \geq 3$, there exists a finitely presented group M_n with centre $Z(M_n) \cong \mathbb{Z}/d\mathbb{Z}$ such that $\mathrm{Depth}_{\mathrm{RF}}(M_n) = \omega \cdot n + 1$.*

Proof. We follow the construction of Γ_n in Theorem 2.3 with a more careful choice of Γ_1 and Γ_2. Specifically, we take $\Gamma_1^0 = \mathrm{Sp}(2g, \mathbb{Z})$, for some fixed $g \geq 2$, then $\Gamma_1 = \Gamma_1^0 * \mathbb{Z}$; then we build Γ_2 as described in Remark 2.5. In the inductive step, we define Γ_{n+1} with $\mathrm{Depth}_{\mathrm{RF}}(\Gamma_{n+1}) = \omega \cdot (n + 1)$ by amalgamating Γ_n with $m + 1$ copies of Λ, as in the proof of Theorem 2.3, to obtain Γ_{n+1}^0 and then take $\Gamma_{n+1} = \Gamma_{n+1}^0 * \mathbb{Z}$. This family of groups comes with embeddings $\Gamma_n \hookrightarrow \Gamma_{n+1}$. Addendum 2.4 tells us that $\mathrm{Core}_{\mathrm{RF}}^{n-1}(\Gamma_n)$ is a free product of a free group and conjugates of $\mathrm{Core}_{\mathrm{RF}}(\Gamma_2)$, and in Remark 2.5 we saw that $\mathrm{Core}_{\mathrm{RF}}(\Gamma_2)$ is a residually finite group that contains conjugates of Γ_1. Thus $\mathrm{Core}_{\mathrm{RF}}^{n-1}(\Gamma_n)$ is a residually finite group that contains conjugates of Γ_1.

With this structure in hand, we will be done if we can describe a central extension

$$1 \to \mathbb{Z}/d\mathbb{Z} \to M_n \xrightarrow{p} \Gamma_n \to 1$$

that restricts over $\mathrm{Sp}(2g, \mathbb{Z}) = \Gamma_1^0 < \Gamma_n$ to Deligne's extension (2.2), because this will tell us that $\mathrm{Core}_{\mathrm{RF}}^{n-1}(M_n)$ is a central extension

$$1 \to \mathbb{Z}/d\mathbb{Z} \to \mathrm{Core}_{\mathrm{RF}}^{n-1}(M_n) \xrightarrow{p} \mathrm{Core}_{\mathrm{RF}}^{n-1}(\Gamma_n) \to 1$$

with $\mathrm{Core}_{\mathrm{RF}}^{n-1}(\Gamma_n)$ residually finite and $\mathbb{Z}/d\mathbb{Z} = \mathrm{Core}_{\mathrm{RF}}(p^{-1}(\Gamma_1^0)) = \mathrm{Core}_{\mathrm{RF}}^n(M_n)$.

To this end, we define M_1 by amalgamating the group Δ from (2.2) with $(\mathbb{Z}/d) \times \mathbb{Z}$, identifying the central subgroup $\mathbb{Z}/d\mathbb{Z} < \Delta$ with $(\mathbb{Z}/d) \times \{1\}$. Then, proceeding by induction, assuming that the desired central extension M_n has been constructed over Γ_n, we define M_{n+1}^0 to be the central extension of Γ_{n+1}^0 that restricts to M_n over Γ_n and restricts to the trivial extension $(\mathbb{Z}/d\mathbb{Z}) \times \Lambda$ over each of the $m+1$ copies of Λ attached to Γ_n in the proof of Theorem 2.3; we then identify the centre of M_{n+1}^0 with $\mathbb{Z}/d \times \{1\}$ to amalgamate M_{n+1}^0 with $(\mathbb{Z}/d\mathbb{Z}) \times \mathbb{Z}$. More explicitly, to obtain M_{n+1}^0, we fix a generator ζ for the central $\mathbb{Z}/d < M_n$, then amalgamate M_n with $m+1$ copies of $(\mathbb{Z}/d\mathbb{Z}) \times \Lambda$, along copies of $(\mathbb{Z}/d\mathbb{Z}) \times \mathbb{Z}$ identifying ζ with a generator of $(\mathbb{Z}/d\mathbb{Z}) \times \{1\}$ and $\tilde{a}_i$ with the generator of $\mathrm{Core}_{\mathrm{RF}}(\Lambda)$, where $\{\tilde{a}_0, \dots, \tilde{a}_m\}$ are preimages in M_n of the generators $a_i \in \Gamma_n$ used in the proof of Theorem 2.3. (An important point to observe here is that because $\langle a_i \rangle < \Gamma_n$ is infinite cyclic, its preimage $\langle \zeta, \tilde{a}_i \rangle$ in M_n is $(\mathbb{Z}/d\mathbb{Z}) \times \mathbb{Z}$.) Finally, we obtain M_{n+1} by amalgamating M_{n+1}^0 with $(\mathbb{Z}/d\mathbb{Z}) \times \mathbb{Z}$, again identifying ζ with a generator of $(\mathbb{Z}/d\mathbb{Z}) \times \{1\}$. By construction, $\langle \zeta \rangle \cong \mathbb{Z}/d\mathbb{Z}$ is central in M_{n+1} with quotient $\Gamma_{n+1} = \Gamma_{n+1}^0 * \mathbb{Z}$, which is centreless. ∎

6. Successors of limit ordinals

The use of the prime 3 in the following statement is arbitrary.

Theorem 6.1. *For every countable limit ordinal $\alpha = \omega \cdot \gamma$, there exists a finitely generated group $\tilde{J}_\alpha$ with $\mathrm{Depth}_{\mathrm{RF}}(\tilde{J}_\alpha) = \alpha + 1$ and $\mathrm{Core}_{\mathrm{RF}}^\gamma(\tilde{J}_\alpha) = Z(\tilde{J}_\alpha) \cong \mathbb{Z}/3$.*

Proof. We again proceed by transfinite induction. The limit ordinals $\alpha < \omega^2$ have the form $\omega \cdot n$, and for these we can take the groups M_n constructed in Theorem 5.1.

Thus we may assume that $\alpha \geq \omega^2$ and we may assume that $\tilde{J}_\beta$ has been constructed for limit ordinals $\beta < \alpha$. We shall again exploit the fact that $\alpha \geq \omega^2$ implies $\omega + \alpha = \alpha$ and hence $\mathrm{Core}_{\mathrm{RF}}^\alpha(G) = \mathrm{Core}_{\mathrm{RF}}^\alpha(\mathrm{Core}_{\mathrm{RF}}(G))$ for all countable groups G.

If $\alpha = \alpha_0 + \omega$, we take a finitely generated group with $\mathrm{Depth}_{\mathrm{RF}}(B) = \alpha_0$ and consider the quotient E of $\Delta \wr B$ constructed in our proof of Proposition 3.6. This group has centre $\mathbb{Z}/3$ and $\mathrm{Depth}_{\mathrm{RF}}(E) = \alpha + 1$.

If α is not of the form $\alpha_0 + \omega$, then α is the supremum of the limit ordinals $\beta < \alpha$, and we define $\tilde{\Gamma}^\alpha$ to be the amalgamated free product of the groups $\tilde{J}_\beta$ where the amalgamation identifies all of the centres $Z(\tilde{J}_\beta) \cong \mathbb{Z}/3$. Let J_β be $\tilde{J}_\beta$ modulo its centre and note that $\mathrm{Depth}_{\mathrm{RF}}(J_\beta) = \beta$. We have a central extension

$$1 \to \mathbb{Z}/3 \to \tilde{\Gamma}^\alpha \to \underset{\beta}{\ast} J_\beta \to 1.$$

For each β, the natural retraction $\tilde{\Gamma}^\alpha \to \tilde{J}_\beta$ maps $Z(\tilde{\Gamma}^\alpha)$ onto $Z(\tilde{J}_\beta)$, hence $Z(\tilde{\Gamma}^\alpha) \subseteq \mathrm{Core}_{\mathrm{RF}}^\beta(\tilde{\Gamma}^\alpha)$ for all $\beta < \alpha$, therefore $Z(\tilde{\Gamma}^\alpha) \subseteq \mathrm{Core}_{\mathrm{RF}}^\alpha(\tilde{\Gamma})$. On the other hand, $\tilde{\Gamma}^\alpha/Z(\tilde{\Gamma}^\alpha)$ is the free product of the J_β and $\mathrm{Depth}_{\mathrm{RF}}(J_\beta) = \beta$, so Lemma 1.7 tells us that the depth of $\tilde{\Gamma}^\alpha/Z(\tilde{\Gamma}^\alpha)$ is $\alpha = \omega \cdot \gamma$. Thus $\mathrm{Core}_{\mathrm{RF}}^\alpha(\tilde{\Gamma}^\gamma) = Z(\tilde{\Gamma}^\alpha) \cong \mathbb{Z}/3$. At this point, we have proved the weaker form of the theorem with "countable" in place of "finitely generated", and the remainder of the proof simply involves adapting the proof of Proposition 4.2 so as to upgrade to finite generation.

To this end, we let $\widetilde{\Gamma}_0 = \widetilde{\Gamma}^\alpha *_{\mathbb{Z}/3} ((\mathbb{Z}/3) \times \mathbb{Z})$, where the amalgamation identifies $Z(\widetilde{\Gamma}^\alpha)$ with $(\mathbb{Z}/3) \times 1$, and we fix generators $\{\zeta, a_0, a_1 \dots\}$ for $\widetilde{\Gamma}_0$ where ζ generates $Z(\widetilde{\Gamma}^\alpha)$ and the a_i have infinite order. We then take the HNN extension with infinitely many stable letters t_i, where t_i conjugates a_i to a_0 and fixes ζ:

$$\widetilde{\Gamma}_1 = (\widetilde{\Gamma}_0, t_1, t_2, \dots \mid t_i^{-1} a_i t_i = a_0, [t_i, \zeta] = 1 \ \forall i).$$

Note that $Z(\widetilde{\Gamma}_1) = \langle\zeta\rangle \cong \mathbb{Z}/3$, and $\widetilde{\Gamma}_1$ modulo its centre is the group Γ_1 we had constructed at this stage of the proof of Proposition 4.2. Finally, we define $\widetilde{J}_\alpha$ by taking a free group F of rank 2 and amalgamating $\widetilde{\Gamma}_1$ with $(\mathbb{Z}/3) \times F$ by identifying $\langle\zeta\rangle$ with $(\mathbb{Z}/3) \times 1$ and identifying an infinitely generated subgroup of F with the free subgroup $\langle t_1, t_2, \dots\rangle$. By construction, $Z(\widetilde{J}_\alpha) = \langle\zeta\rangle \cong \mathbb{Z}/3$ and $\widetilde{J}_\alpha / Z(\widetilde{J}_\alpha)$ is the group Γ^+ we had in proof of Proposition 4.2. In particular, $\mathrm{Depth}_{\mathrm{RF}}(\widetilde{J}_\alpha / Z(\widetilde{J}_\alpha)) = \alpha$. By construction, $\zeta \in \mathrm{Core}_{\mathrm{RF}}{}^\alpha(\widetilde{\Gamma}^\alpha)$, so $\mathrm{Core}_{\mathrm{RF}}(\widetilde{J}_\alpha) = \langle\zeta\rangle \cong \mathbb{Z}/3$ and $\mathrm{Depth}_{\mathrm{RF}}(\widetilde{J}_\alpha) = \alpha + 1$. This completes the induction. ∎

A new idea is needed to answer the following question.

Question 6.2. *Which countable ordinals arise as the residual finiteness depths of finitely presented groups? Might all limit ordinals less than ω^ω and their successors arise in this way, for example?*

References

[1] Baumslag, G.: Positive one-relator groups. *Trans. Amer. Math. Soc.* **156** (1971), 165–183. Zbl 0256.20043 MR 0274562

[2] Boler, J. and Evans, B.: The free product of residually finite groups amalgamated along retracts is residually finite. *Proc. Amer. Math. Soc.* **37** (1973), 50–52. Zbl 0255.55001 MR 0306329

[3] Brody, N. and Jankiewicz, K.: Generalized residual finiteness of groups. *New York J. Math.* **30** (2024), 583–592. Zbl 1542.20126 MR 4736002

[4] Deligne, P.: Extensions centrales non résiduellement finies de groupes arithmétiques. *C. R. Acad. Sci. Paris Sér. A-B* **287** (1978), no. 4, A203–A208. Zbl 0416.20042 MR 0507760

[5] Enderton, H. B.: *Elements of set theory*. Academic Press Harcourt Brace Jovanovich, New York-London, 1977. Zbl 0383.03034 MR 0439636

[6] Serre, J.-P.: *Trees*. Springer, Berlin-New York, 1980. Zbl 0548.20018 MR 0607504

Martin R. Bridson
Mathematical Institute, University of Oxford
Andrew Wiles Building, ROQ, Woodstock Road, Oxford OX2 6GG, UK;
bridson@maths.ox.ac.uk