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Noncommutative endpoint maximal estimates for spherical means

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Abstract. We establish the noncommutative restricted weak-type endpoint maximal estimates for both the continuous and the discrete spherical means, extending results of Bourgain (1985) and Ionescu (2004) to the noncommutative setting. As an application, we obtain the noncommutative restricted weak-type maximal ergodic inequalities for spherical averages, which were the natural questions left open after Chen and Hong (2024) and Hong (2013).

1. Introduction

Spherical maximal functions and relevant estimates have been playing a significant role in harmonic analysis and related topics since their inception by Stein [35]. Recall that, given $d \geq 2$, the spherical maximal function on $\mathbb{R}^d$ is defined by

$$A_*^{\mathbb{R}^d} f(x) = \sup_{r>0} |f| * d\sigma_r(x),$$

where $f:\mathbb{R}^d \to \mathbb{C}$ is a suitable function and $d\sigma_r$ denotes the normalized invariant measure on the sphere $\mathbb{S}_r := \{x \in \mathbb{R}^d : |x|^2 = r^2\}$. Stein's spherical maximal theorem (cf. [35]) asserts that if $d \geq 3$ and $p > d/(d-1)$, then

$$\|A_*^{\mathbb{R}^d} f\|_{L_p(\mathbb{R}^d)} \lesssim_{p,d} \|f\|_{L_p(\mathbb{R}^d)}, \quad \forall f \in L_p(\mathbb{R}^d), \tag{1.1}$$

where the notation $A \lesssim_\theta B$ stands for that there exists a constant $C_\theta > 0$, depending only on the parameter θ, such that $A \leq C_\theta B$. It was proved in the same paper that this result fails to hold for $d = 1$ and $p < \infty$, or $d \geq 2$ and $p \leq d/(d-1)$. Bourgain [3] fixed the gap by establishing the estimate (1.1) in the challenging case $d = 2$ and $p > 2$. Almost at the same time, Bourgain [2] established the restricted weak-type estimate for $A_*^{\mathbb{R}^d}$ at the endpoint $p = d/(d-1)$ for $d \geq 3$; that is,

$$\|A_*^{\mathbb{R}^d} \mathbb{1}_E\|_{L_{d/(d-1),\infty}(\mathbb{R}^d)} \lesssim_d |E|^{d/(d-1)} \tag{1.2}$$

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for any measurable set E with finite measure, where $\mathbb{1}_E$ denotes the characteristic function of the set E, and $|E|$ its Lebesgue measure. By the general theory of Lorentz spaces, this is equivalent to saying that $A_*^{\mathbb{R}^d}$ extends to a bounded operator from $L_{d/(d-1),1}(\mathbb{R}^d)$ to $L_{d/(d-1),\infty}(\mathbb{R}^d)$.

The above mentioned results and their extensions have been found extremely abundant in their connections with many important problems in harmonic analysis, partial differential equations, ergodic theory, etc. For instance, with the purpose of obtaining Bourgain's circular maximal inequality on manifolds, Sogge et al. [31, 34] discovered its close connection to local smoothing phenomenon, Bochner–Riesz means, Fourier restriction and Kakeya problem, etc. This remains a quite active research direction nowadays, see, e.g., [18, 25] for more information. On the other hand, the discrete analogues of spherical means are also subtle and involved, and the associated maximal inequalities are closely related to some interesting problems arising in number theory, for example, the celebrated Waring problem.

The first study of discrete spherical maximal function appeared in Magyar's paper [27]. Given $d \geq 2$ and $r > 0$, let $N_d(r)$ denote the number of lattice points on the discrete sphere $\mathbb{Z}^d \cap \mathbb{S}_r$. Obviously, the set $\mathbb{Z}^d \cap \mathbb{S}_r$ is non-empty precisely when $r^2 \in \mathbb{N}$. We denote by $\mathcal{R}$ the set of all positive radii r such that $\mathbb{Z}^d \cap \mathbb{S}_r$ is non-empty. Note that for $d \geq 4$, the inclusion $\mathbb{N} \subsetneq \mathcal{R}$ follows from Lagrange's four square theorem. Then the discrete analogue of spherical maximal function is defined by

$$A_*^{\mathbb{Z}^d} f(n) = \sup_{r \in \mathcal{R}} \frac{1}{N_d(r)} \sum_{m \in \mathbb{Z}^d \cap \mathbb{S}_r} |f(n-m)|,$$

where $f : \mathbb{Z}^d \to \mathbb{C}$ is a function. Magyar [27] established the discrete spherical maximal inequalities under the restriction that r runs over some 'dyadic' block. Based on this work, Magyar, Stein and Wainger [28] further proved that if $d \geq 5$ and $p > d/(d-2)$, then

$$\|A_*^{\mathbb{Z}^d} f\|_{\ell_p(\mathbb{Z}^d)} \lesssim_{p,d} \|f\|_{\ell_p(\mathbb{Z}^d)}, \quad \forall f \in \ell_p(\mathbb{Z}^d). \tag{1.3}$$

Their proof mainly relies upon certain sampling technique and the so-called circle method rooted in number theory. This estimate is sharp in the sense that it fails to hold for $d \geq 5$ and $p \leq d/(d-2)$. Moreover, as one can recognize from the result, the discrete spherical maximal function behaves essentially different from its continuous counterpart in the property of boundedness. In the subsequent paper [19], Ionescu established a restricted weak-type inequality for $A_*^{\mathbb{Z}^d}$, which provides an endpoint version of (1.3). More precisely, it was shown by Ionescu [19] that if $d \geq 5$, then

$$\|A_*^{\mathbb{Z}^d} \mathbb{1}_F\|_{\ell_{d/(d-2),\infty}(\mathbb{Z}^d)} \lesssim_d |F|^{d/(d-2)} \tag{1.4}$$

for any finite set F. For readers interested in recent development of discrete spherical maximal inequalities, we recommend to consult the papers [1, 7, 16, 17, 23, 24, 26, 30, 32].

The primary objective of this paper is to study restricted weak-type maximal inequalities for both the continuous and discrete spherical averages in the noncommutative setting. Our motivation comes from recent advances on maximal type inequalities in noncommutative analysis. This is a newly emerging research field, although the investigation of this topic can be traced back to Cuculescu [8] and Yeadon [38], where noncommutative Doob's inequality and noncommutative maximal ergodic inequality for $p = 1$ were formulated. By using the idea of vector-valued noncommutative L_p-spaces due to Pisier [33], Junge [20] established noncommutative Doob's inequalities for $1 < p < \infty$. Subsequently, Junge and Xu [21] developed the Marcinkiewicz type interpolation theorem in noncommutative maximal spaces, providing a useful tool for the investigation of noncommutative maximal inequalities. These fundamental work lead to a rapid development of noncommutative maximal inequalities. We refer to the paper [29] for the noncommutative version of Hardy–Littlewood maximal inequalities, to the writings [14, 21] for noncommutative maximal ergodic inequalities, to the work [13] for weak-type maximal inequalities of noncommutative singular integrals, and to the articles [4, 11] for the noncommutative counterparts of (1.1) and (1.3). Motivated by [4, 11], it is natural to consider the noncommutative analogues of (1.2) and (1.4).

To better describe our main results, let us fix some notations. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ, and let $\mathcal{S}_{\mathcal{M}}$ consist of elements in $\mathcal{M}$ with supports of finite trace. For any $r > 0$, we define the noncommutative continuous spherical average as follows:

$$\mathcal{A}_r f(x) = \int_{\mathbb{S}_1} f(x - ry)\, d\sigma(y), \quad \forall x \in \mathbb{R}^d,$$

where $f \in \mathcal{S}(\mathbb{R}^d) \otimes \mathcal{S}_{\mathcal{M}}$, with $\mathcal{S}(\mathbb{R}^d)$ denoting the Schwartz class on $\mathbb{R}^d$, and where $d\sigma$ abbreviates the normalized surface measure $d\sigma_1$ on the unit sphere $\mathbb{S}_1$.

Similarly, given $r \in \mathcal{R}$, the noncommutative discrete spherical average is defined by

$$A_r f(n) = \frac{1}{N_d(r)} \sum_{m \in \mathbb{Z}^d \cap \mathbb{S}_r} f(n - m), \quad \forall n \in \mathbb{Z}^d,$$

where $f \in \mathcal{S}(\mathbb{Z}^d) \otimes \mathcal{S}_{\mathcal{M}}$, and $\mathcal{S}(\mathbb{Z}^d)$ is the Schwartz class on $\mathbb{Z}^d$. Let $\mathcal{U}$ denote the tensor von Neumann algebra $L_\infty(\mathbb{R}^d) \overline{\otimes} \mathcal{M}$ endowed with the usual tensor trace $\phi = \int \otimes \tau$, and let $\mathcal{N}$ denote the tensor von Neumann algebra $\ell_\infty(\mathbb{Z}^d) \overline{\otimes} \mathcal{M}$ equipped with the usual tensor trace $\nu = \Sigma \otimes \tau$. The main results of this paper can be stated as follows (see Section 2 for the definition of the weak-type maximal space $\Lambda_{d/(d-1),\infty}(\mathcal{U}; \ell_\infty)$):

Theorem A. *Assume that $d \geq 3$. Then the continuous spherical averages $(\mathcal{A}_r)_{r>0}$ are of restricted weak-type $(d/(d-1), d/(d-1))$. That is, we have*

$$\|(\mathcal{A}_r e)_{r>0}\|_{\Lambda_{d/(d-1),\infty}(\mathcal{U};\ell_\infty)} \lesssim_d \phi(e)^{d/(d-1)}$$

for all finite projections e in $\mathcal{U}$.

Theorem B. *Assume that $d \geq 5$. Then the discrete spherical averages $(A_r)_{r\in\mathcal{R}}$ are of restricted weak-type $(d/(d-2), d/(d-2))$. That is, we have*

$$\|(A_r e)_{r\in\mathcal{R}}\|_{\Lambda_{d/(d-2),\infty}(\mathcal{N};\ell_\infty)} \lesssim_d \nu(e)^{d/(d-2)}$$

for all finite projections e in $\mathcal{N}$.

Theorem A and Theorem B extend (1.2) and (1.4), respectively, to the noncommutative setting. The idea of the proof of Theorem A is essentially due to Bourgain [2]: given $r > 0$, decompose each $\mathcal{A}_r$ into two parts; one of them can be handled by using the noncommutative Hardy–Littlewood averages, and the other relies upon a lemma from [15] (see Lemma 3.1 below). In contrast, the proof of Theorem B is more involved. The strategy can be summarized as follows. Given $r \in \mathcal{R}$, we first split each spherical average A_r into five pieces $A_{r,i}$, $i = 1, 2, 3, 4, 5$. According to this decomposition, the desired assertion is reduced to the verifications of the weak-type $(1, 1)$ estimate of $(A_{r,i})_{r\in\mathcal{R}}$ for $i \in \{1, 4\}$ and the strong type $(2, 2)$ estimate of $(A_{r,i})_{r\in\mathcal{R}}$ for $i \in \{2, 3, 5\}$. The estimates of $(A_{r,1})_{r\in\mathcal{R}}$ and $(A_{r,4})_{r\in\mathcal{R}}$ are easy to handle since both of them can be controlled by the discrete Hardy–Littlewood averages. To deal with $(A_{r,2})_{r\in\mathcal{R}}$ and $(A_{r,3})_{r\in\mathcal{R}}$, we need to introduce a new family of operators $(T_{m_r})_{r\in\mathcal{R}}$ and apply two relevant estimates (namely, Proposition 4.1 and Proposition 4.2). For $(A_{r,5})_{r\in\mathcal{R}}$, besides the noncommutative sampling principle, another key ingredient is Lemma 3.1. This strategy germinated in Bourgain's work [2] and Ionescu's article [19]. However, due to the noncommutativity, we have to modify certain classical techniques to fit our situation.

As an application of Theorems A and B, we may establish the restricted weak-type maximal inequalities for noncommutative ergodic spherical averages. Since in both the continuous and discrete cases the statements can be formulated similarly and the transference techniques can be conducted in a similar fashion, we will only state the result and give a proof in the discrete case. More precisely, let γ_i, $1 \le i \le d$, be a commuting family of trace-preserving automorphisms on $\mathcal{M}$, that is, $\tau = \tau \circ \gamma_i$ and $\gamma_i \circ \gamma_j = \gamma_j \circ \gamma_i$ for all $1 \le i, j \le d$. For any $r \in \mathcal{R}$, we introduce the noncommutative ergodic spherical averages associated with $\gamma = (\gamma_i)_{1\le i\le d}$ by

$$\mathfrak{A}_r x := \frac{1}{N_d(r)} \sum_{m\in\mathbb{Z}^d\cap\mathbb{S}_r} \gamma^m x, \quad \forall x \in \mathcal{S}_{\mathcal{M}},$$

where $m = (m_i)_{1\le i\le d} \in \mathbb{Z}^d$ and $\gamma^m := \gamma_1^{m_1}\gamma_2^{m_2}\cdots\gamma_d^{m_d}$. Then we have the following.

Theorem C. *Assume that $d \ge 5$. Then the ergodic spherical averages $(\mathfrak{A}_r)_{r\in\mathcal{R}}$ are of restricted weak-type $(d/(d-2), d/(d-2))$. That is, we have*

$$\|(\mathfrak{A}_r x)_{r\in\mathcal{R}}\|_{\Lambda_{d/(d-2),\infty}(\mathcal{M};\ell_\infty)} \lesssim_d \tau(x)^{d/(d-2)}$$

for all finite projections x in $\mathcal{M}$.

This paper is structured as follows. The next section contains some preliminaries and basic facts regarding noncommutative L_p spaces and noncommutative maximal norms. Section 3 is dedicated to the proof of Theorem A. Section 4 is where we present the proof of Theorem B. In Section 5, we apply Theorem B to show Theorem C.

Notations. In the sequel, we will always adopt the following notations.

- We denote by C a positive constant independent of the main parameters, which may vary from line to line. The notation $A \lesssim B$ or $B \gtrsim A$ means that $A \le CB$ for some constant $C > 0$. If both $A \lesssim B$ and $B \lesssim A$ hold, then we write $A \simeq B$.
- The torus $\mathbb{T}^d$ may be identified with any box in $\mathbb{R}^d$ of sidelengths 1, for instance $[0, 1]^d$ or $[-1/2, 1/2]^d$.

- Set $\mathbb{R}^+ := (0,\infty)$ and $\mathbb{N} := \{1,2,\cdots\}$. For any $\alpha := (\alpha_1,\cdots,\alpha_d), \beta := (\beta_1,\cdots,\beta_d) \in \mathbb{R}^d$, we denote $|\alpha|^2 := \alpha_1^2 + \cdots + \alpha_d^2$ and $\langle \alpha, \beta \rangle := \alpha_1\beta_1 + \cdots + \alpha_d\beta_d$.
- We identify $\mathbb{Z}/q\mathbb{Z}$ with the set $\{1,2,\ldots,q\}$. For any finite set E, let $\mathbb{1}_E$ denote its characteristic function.
- Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Denote by $\mathcal{U} := L_\infty(\mathbb{R}^d) \overline{\otimes} \mathcal{M}$ the tensor von Neumann algebra equipped with the tensor trace $\phi = \int \otimes \tau$. Similarly, we set $\mathcal{N} := \ell_\infty(\mathbb{Z}^d) \overline{\otimes} \mathcal{M}$ with the tensor trace $\nu = \sum \otimes \tau$.
- Given a function $f \in \mathcal{S}(\mathbb{R}^d) \otimes \mathcal{S}_{\mathcal{M}}$, we define $\widehat{f}$ (or $\mathcal{F} f$) and $\check{f}$, the Fourier transform and the inverse Fourier transform of f, by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i\langle x,\xi\rangle}\, dx \quad \text{and} \quad \check{f}(\xi) := \int_{\mathbb{R}^d} f(x)\, e^{2\pi i\langle x,\xi\rangle}\, dx, \quad \forall \xi \in \mathbb{R}^d.$$

  Similarly, given a suitable function f on $\mathbb{T}^d$ (respectively, $\mathbb{Z}^d$), the Fourier transform of f is defined by

$$\widehat{f}(m) := \int_{\mathbb{T}^d} f(x)\, e^{2\pi i\langle x,m\rangle}\, dx, \quad \forall m \in \mathbb{Z}^d$$

$$\text{(respectively, } \widehat{f}(\xi) := \sum_{n\in\mathbb{Z}^d} f(n)\, e^{-2\pi i\langle n,\xi\rangle}, \quad \forall \xi \in \mathbb{T}^d).$$

- Let δ be the Dirac measure and let φ denote a radial Schwartz function on $\mathbb{R}^d$ such that $\widehat{\varphi}(\xi) = 1$ for all $|\xi| \le 1$.
- Let B_1 and B_2 be (quasi-)Banach spaces. We denote by $\|T\|_{B_1\to B_2}$ the operator norm of the operator T from the space B_1 to the space B_2.
- With no risk of confusion, we use T_a to denote the multiplier operator with symbol a.

2. Preliminaries

In this preliminary section, we introduce some basic definitions and well-known results concerning noncommutative L_p spaces and noncommutative maximal norms. We shall use standard notation for operator algebras, which can be found in the monographs [22] and [36].

2.1. Noncommutative L_p spaces

Throughout, let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H equipped with a normal semifinite faithful trace τ. A closed and densely defined operator x on H is said to be affiliated with $\mathcal{M}$ if $ux = xu$ for all unitary operators u belonging to the commutant $\mathcal{M}'$ of $\mathcal{M}$. An operator x affiliated with $\mathcal{M}$ is called τ-measurable if there is $\lambda > 0$ such that

$$\tau(e_\lambda^\perp(|x|)) < \infty,$$

where $e_\lambda^\perp(|x|) = \mathbb{1}_{(\lambda,\infty)}(|x|)$ is the spectral decomposition of $|x|$. Let $L_0(\mathcal{M})$ denote the family of all τ-measurable operators. For any $x \in L_0(\mathcal{M})$, we define the distribution function of x by

$$\lambda_s(x) := \tau(e_s^\perp(|x|)), \quad \forall s > 0,$$

and the generalized singular numbers of x by

$$\mu_t(x) := \inf\{s > 0 : \lambda_s(x) \le t\}, \quad \forall t > 0.$$

For $0 < p < \infty$, we recall that the noncommutative L_p space associated with $(\mathcal{M},\tau)$ is defined by $L_p(\mathcal{M}) := \{x \in L_0(\mathcal{M}) : \tau(|x|^p) < \infty\}$, with

$$\|x\|_{L_p(\mathcal{M})} := \tau(|x|^p)^{1/p} = \Big(\int_0^\infty (\mu_t(x))^p \, dt\Big)^{1/p}.$$

Let $L_p^+(\mathcal{M})$ denote the positive part of $L_p(\mathcal{M})$. For any projection e in $\mathcal{M}$, it is easy to see that

$$\|e\|_{L_p(\mathcal{M})} = \tau(e)^{1/p}, \quad \text{for } 0 < p < \infty.$$

This basic fact will be frequently used in the sequel.

2.2. Noncommutative maximal norms

Let $1 \le p \le \infty$. Following Pisier [33] and Junge [20], we define the noncommutative maximal space $L_p(\mathcal{M};\ell_\infty)$ to be the set of all sequences $(x_n)_{n\in\mathbb{N}}$ in $L_p(\mathcal{M})$ which admit a factorization of the following form: there exist $a, b \in L_{2p}(\mathcal{M})$ and a bounded sequence $y = (y_n)_{n\in\mathbb{N}}$ such that

$$x_n = a y_n b, \quad \forall n \in \mathbb{N}.$$

Given $(x_n)_{n\in\mathbb{N}} \in L_p(\mathcal{M};\ell_\infty)$, define

$$\|(x_n)_{n\in\mathbb{N}}\|_{L_p(\mathcal{M};\ell_\infty)} := \inf\Big\{\|a\|_{L_{2p}(\mathcal{M})} \sup_{n\in\mathbb{N}} \|y_n\|_{\mathcal{M}} \, \|b\|_{L_{2p}(\mathcal{M})}\Big\},$$

where the infimum runs over all factorizations of x as above.

It is well known that $L_p(\mathcal{M};\ell_\infty)$ is a Banach space when equipped with the norm $\|\cdot\|_{L_p(\mathcal{M};\ell_\infty)}$. Moreover, if $(x_n)_{n\in\mathbb{N}}$ is a positive sequence, then x belongs to $L_p(\mathcal{M};\ell_\infty)$ if and only if there exists an element $a \in L_p^+(\mathcal{M})$ such that $0 < x_n \le a$, and

$$\|(x_n)_{n\in\mathbb{N}}\|_{L_p(\mathcal{M};\ell_\infty)} = \inf\{\|a\|_{L_p(\mathcal{M})} : 0 < x_n \le a \text{ for all } n \in \mathbb{N}\}.$$

Similarly, if $(x_n)_{n\in\mathbb{N}}$ is a self-adjoint sequence, then x belongs to $L_p(\mathcal{M};\ell_\infty)$ if and only if there exists an element $a \in L_p^+(\mathcal{M})$ such that $-a \le x_n \le a$, and

$$\|(x_n)_{n\in\mathbb{N}}\|_{L_p(\mathcal{M};\ell_\infty)} = \inf\{\|a\|_{L_p(\mathcal{M})} : -a \le x_n \le a \text{ for all } n \in \mathbb{N}\}. \tag{2.1}$$

For simplicity, we will always use the notation $\|{\sup_{n\in\mathbb{N}}}^+ x_n\|_{L_p(\mathcal{M})}$ to denote the norm $\|(x_n)_{n\in\mathbb{N}}\|_{L_p(\mathcal{M};\ell_\infty)}$. We emphasize that $\|{\sup_{n\in\mathbb{N}}}^+ x_n\|_{L_p(\mathcal{M})}$ is just a notation, since it makes no sense to take supremum over a sequence of operators.

For any index set I, we can similarly define the space $L_p(\mathcal{M};\ell_\infty(I))$ of families $(x_i)_{i\in I} \subseteq L_p(\mathcal{M})$. As shown in [21], $x = (x_i)_{i\in I} \in L_p(\mathcal{M};\ell_\infty(I))$ if and only if

$$\sup\Big\{\big\|{\sup_{i\in J}}^+ x_i\big\|_{L_p(\mathcal{M})} : J \subset I, J \text{ finite}\Big\} < \infty.$$

In this case, $\|{\sup_{i\in I}}^+ x_i\|_{L_p(\mathcal{M})}$ is equal to the above supremum. In the following, we will omit the index set I if there is no danger of confusion. For more properties on $L_p(\mathcal{M};\ell_\infty)$, we refer the reader to [20, 21, 33, 37].

We now introduce the weak version of noncommutative maximal spaces. Usually, there are two different formulations: one is in the same spirit as $L_p(\mathcal{M};\ell_\infty)$ and the other is rooted in Cuculescu's famous work [8]. In the present paper, we shall only use the latter; we recall it below. Given $1 \le p < \infty$, we define $\Lambda_{p,\infty}(\mathcal{M};\ell_\infty)$ to be set of all sequences $(x_n)_{n\in\mathbb{N}}$ in $L_{p,\infty}(\mathcal{M})$ such that

$$\|(x_n)_{n\in\mathbb{N}}\|_{\Lambda_{p,\infty}(\mathcal{M};\ell_\infty)} := \sup_{\lambda>0} \lambda \inf_{e\in\mathcal{P}(\mathcal{M})} \big\{(\tau(e^\perp))^{1/p} : \|ex_ne\|_\infty \le \lambda \text{ for all } n\in\mathbb{N}\big\}$$

is finite, where $\mathcal{P}(\mathcal{M})$ stands for the collection of all projections in $\mathcal{M}$. It is not hard to check that $\|\cdot\|_{\Lambda_{p,\infty}(\mathcal{M};\ell_\infty)}$ is a quasi-norm. If $(x_n)_{n\in\mathbb{N}} \in \Lambda_{p,\infty}(\mathcal{M};\ell_\infty)$ is a sequence of self-adjoint operators. Then, similar to (2.1), we have

$$\|(x_n)_{n\in\mathbb{N}}\|_{\Lambda_{p,\infty}(\mathcal{M};\ell_\infty)} = \sup_{\lambda>0} \lambda \inf_{e\in\mathcal{P}(\mathcal{M})} \big\{(\tau(e^\perp))^{1/p} : -\lambda \le ex_ne \le \lambda \text{ for all } n\in\mathbb{N}\big\}.$$

This definition and the relevant properties can be also extended to an arbitrary index set I. For simplicity, we still denote the corresponding space by $\Lambda_{p,\infty}(\mathcal{M};\ell_\infty)$. We now present a particularly useful lemma which is taken from Lemma 3.2 in [12].

Lemma 2.1 (Lemma 3.2 in [12]). *Let $1 \le p \le \infty$. Then for any positive $(x_t)_{t>0}$ in $L_p(\mathcal{M})$, we have*

$$\|(x_t)_{t>0}\|_{\Lambda_{p,\infty}(\mathcal{M};\ell_\infty)} \simeq \sup_N \|(x_t)_{0<t\le N}\|_{\Lambda_{p,\infty}(\mathcal{M};\ell_\infty)}.$$

For more information on $\Lambda_{p,\infty}(\mathcal{M};\ell_\infty)$, we refer the reader to [12, 13, 21].

2.3. Noncommutative sampling principle

Throughout, we set Q_0 to be the fundamental cube given by $Q_0 := [-1/2, 1/2]^d$, and for any $a \in \mathbb{R}^+$, let aQ_0 denote the cube with the same center as Q_0 but with side length $a\ell(Q_0)$. Suppose that m is a smooth function on $\mathbb{R}^d$ supported in Q_0. Let T_m denote the multiplier operator associated with the symbol m, given by

$$T_m f(x) := \int_{\mathbb{R}^d} m(\xi)\,\widehat{f}(\xi)\,e^{2\pi i\langle x,\xi\rangle}\,d\xi, \quad x\in\mathbb{R}^d,$$

where $f:\mathbb{R}^d \to \mathcal{M}$ is an operator-valued function. Moreover, for any $q\in\mathbb{N}$, let μ be a smooth function on $\mathbb{R}^d$ that is supported on Q_0/q. Let μ^q_{per} be the periodic extension of μ, that is,

$$\mu^q_{\mathrm{per}}(\xi) := \sum_{\ell\in\mathbb{Z}^d} \mu(\xi - \ell/q), \quad \xi\in\mathbb{T}^d. \tag{2.2}$$

Let $(T^q_\mu)_{\rm dis}$ be the multiplier operator associated with symbol $\mu^q_{\rm per}$, which means that for suitable f,

$$\sum_{n\in\mathbb{Z}^d}((T^q_\mu)_{\rm dis} f)(n)\, e^{-2\pi i\langle n,\xi\rangle} = \mu^q_{\rm per}(\xi)\sum_{n\in\mathbb{Z}^d} f(n)\, e^{-2\pi i\langle n,\xi\rangle}.$$

The following noncommutative sampling principle constitutes a key ingredient in the proof of Theorem B. For the proof, we refer to Lemma 5.2 in [4], see also Lemma 3.5 in [5] for the one-dimensional case.

Lemma 2.2. *Let $\{m_\alpha\}_{\alpha\in I}$ be a collection of smooth functions on $\mathbb{R}^d$ supported in the cube Q_0, and let T_{m_α} be the multiplier operator acting on functions on $\mathbb{R}^d$ with the symbol m_α. For $1\le p\le\infty$, if the estimate*

$$\big\|{\sup_{\alpha\in I}}^+ T_{m_\alpha} g\big\|_{L_p(\mathcal{U})} \lesssim \|g\|_{L_p(\mathcal{U})}, \quad \forall g\in L_p(\mathcal{U}),$$

holds, then we have

$$\big\|{\sup_{\alpha\in I}}^+ (T^q_{\mathfrak{m}_\alpha})_{\rm dis} f\big\|_{L_p(\mathcal{N})} \lesssim \|f\|_{L_p(\mathcal{N})}, \quad \forall f\in L_p(\mathcal{N}),$$

where $\mathfrak{m}_\alpha(\xi) := m_\alpha(q\xi)$.

3. Proof of Theorem A: continuous spherical averages

This section is devoted to proving Theorem A. It suffices to show that for any finite projection $f\in\mathcal{U}$ and any $\lambda>0$, there is a projection $e\in\mathcal{U}$ such that

$$\sup_{r>0}\|e(\mathcal{A}_r f)e\|_\infty \le \lambda \quad \text{and} \quad \phi(e^\perp)\lesssim \lambda^{-d/(d-1)}\phi(f). \tag{3.1}$$

Here, the implicit constant depends only on d, but not on f and λ. To prove (3.1), the following useful lemma, which follows from Lemma 3.9 in [15], will be required.

Lemma 3.1. *Let $K\in L_1(\mathbb{R}^d)$ and assume that $\widehat{K}$ is differentiable. Let $K_r(x)=\frac{1}{r^d}K(x/r)$ for any $x\in\mathbb{R}^d$ and $r>0$. Define the following quantities for $j\in\mathbb{Z}$:*

$$\alpha_j := \sup_{2^j\le|\xi|\le 2^{j+2}} |\widehat{K}(\xi)| \quad \textit{and} \quad \beta_j := \sup_{2^j\le|\xi|\le 2^{j+2}} |\langle\nabla\widehat{K}(\xi),\xi\rangle|.$$

Then for any $f\in L_2(\mathcal{U})$, we have

$$\big\|{\sup_{r>0}}^+ f * K_r\big\|_{L_2(\mathcal{U})} \le C\,\Gamma(K)\,\|f\|_{L_2(\mathcal{U})},$$

where C is an absolute positive constant and

$$\Gamma(K) = \sum_{j\in\mathbb{Z}} \alpha_j^{1/2}(\alpha_j^{1/2}+\beta_j^{1/2}).$$

Proof of (3.1). Recall that $d\sigma$ is the normalized surface measure on the unit sphere $\mathbb{S}_1$. For $r > 0$ and $g \in \mathcal{S}(\mathbb{R}^d) \otimes \mathcal{S}_{\mathcal{M}}$, the operator $\mathcal{A}_r g$ can be expressed as

$$\mathcal{A}_r g = g * d\sigma_r.$$

Recall that δ is the Dirac measure and φ is a radial Schwartz function on $\mathbb{R}^d$ such that $\widehat{\varphi}(\xi) = 1$ when $|\xi| \le 1$. Then by using the fact that

$$d\sigma = d\sigma * \delta = d\sigma * \varphi_t + d\sigma * (\delta - \varphi_t),$$

where $\varphi_t(x) := t^{-d}\varphi(x/t)$ for $t > 0$ and $x \in \mathbb{R}^d$, we can split the operator $\mathcal{A}_r$ into the sum of the following two operators:

$$\mathcal{A}_{r,1} g := g * (d\sigma * \varphi_t)_r \quad \text{and} \quad \mathcal{A}_{r,2} g := g * [d\sigma * (\delta - \varphi_t)]_r.$$

We claim that $(\mathcal{A}_{r,1})_{r>0}$ is of weak-type $(1,1)$ with constant t^{-1}. That is, for any $\lambda > 0$, there is a projection $e_1 \in \mathcal{U}$ such that for any $g \in L_1(\mathcal{U})$, the following estimates hold:

$$\sup_{r>0} \|e_1(\mathcal{A}_{r,1} g)e_1\|_\infty \le \lambda \quad \text{and} \quad \phi(e_1^\perp) \lesssim t^{-1} \frac{\|g\|_{L_1(\mathcal{U})}}{\lambda}. \tag{3.2}$$

Also, we claim that $(\mathcal{A}_{r,2})_{r>0}$ is of strong-type $(2,2)$ with constant $t^{d/2-1}$, meaning that it satisfies the estimate

$$\Big\|{\sup_{r>0}}^+ \mathcal{A}_{r,2} g\Big\|_{L_2(\mathcal{U})} \lesssim t^{d/2-1} \|g\|_{L_2(\mathcal{U})}, \quad \forall g \in L_2(\mathcal{U}). \tag{3.3}$$

Once (3.2) and (3.3) were established, we can finish the proof of (3.1) as follows. Fix $\lambda > 0$ and let f be any finite projection in $\mathcal{U}$. From (3.2), we obtain

$$\sup_{r>0} \|e_1(\mathcal{A}_{r,1} f)e_1\|_\infty \le \frac{\lambda}{2} \quad \text{and} \quad \phi(e_1^\perp) \lesssim t^{-1} \frac{\phi(f)}{\lambda}. \tag{3.4}$$

Similarly, it follows from (3.3) that there exists a projection e_2 such that

$$\sup_{r>0} \|e_2(\mathcal{A}_{r,2} f)e_2\|_\infty \le \frac{\lambda}{2} \quad \text{and} \quad \phi(e_2^\perp) \lesssim t^{d-2} \frac{\phi(f)}{\lambda^2}. \tag{3.5}$$

Let $e = e_1 \wedge e_2$. We deduce from (3.4) and (3.5) that

$$\sup_{r>0} \|e(\mathcal{A}_r f)e\|_\infty \le \sup_{r>0} \|ee_1(\mathcal{A}_{r,1} f)e_1 e\|_\infty + \sup_{r>0} \|ee_2(\mathcal{A}_{r,2} f)e_2 e\|_\infty \le \lambda,$$

and

$$\phi(e^\perp) \le \phi(e_1^\perp) + \phi(e_2^\perp) \lesssim \lambda^{-1} t^{-1} \phi(f) + \lambda^{-2} t^{d-2} \phi(f).$$

Choosing $t = \lambda^{1/(d-1)}$, we find that

$$\phi(e^\perp) \lesssim \lambda^{-d/(d-1)} \phi(f).$$

Thus, the desired estimate (3.1) is proved.

It remains to prove the claim. Without loss of generality, we will assume that g denotes a positive operator-valued function in either $L_1(\mathcal{U})$ or $L_2(\mathcal{U})$. Let us first deal with (3.2). For any $r > 0$, it is easy to see that

$$\mathcal{A}_{r,1} g(x) \lesssim \|d\sigma * \varphi_t\|_{L_\infty(\mathbb{R}^d)} \mathbf{M}_r g(x), \quad \forall x \in \mathbb{R}^d, \tag{3.6}$$

where $\mathbf{M}_r$ is the Hardy–Littlewood average given by

$$\mathbf{M}_r h(x) := \frac{1}{r^d} \int_{|x-y|\le r} h(y)\, dy, \quad \forall x \in \mathbb{R}^d. \tag{3.7}$$

A standard argument (see p. 1415 in [19] or inequality (39) in [16]) shows that

$$\|d\sigma * \varphi_t\|_{L_\infty(\mathbb{R}^d)} \lesssim t^{-1}, \tag{3.8}$$

and therefore, by the noncommutative Hardy–Littlewood maximal inequalities (see Theorem 3.3 (i) in [29]), we deduce from (3.6) and (3.8) that there exists a projection $e_1 \in \mathcal{N}$ such that for any $\lambda > 0$,

$$\sup_{r>0} \|e_1(A_{r,1} g)e_1\|_\infty \lesssim t^{-1} \sup_{r>0} \|e_1(\mathbf{M}_r g)e_1\|_\infty \le t^{-1} \cdot t\lambda = \lambda,$$

and

$$\phi(e_1^\perp) \lesssim t^{-1} \frac{\|g\|_{L_1(\mathcal{U})}}{\lambda}.$$

This gives the estimate (3.2).

Now let us turn to show (3.3). According to Lemma 3.1, we immediately obtain

$$\Big\|{\sup_{r>0}}^+ \mathcal{A}_{r,2}\, g\Big\|_{L_2(\mathcal{U})} \lesssim \sum_{j\in\mathbb{Z}} \alpha_j^{1/2}(\alpha_j^{1/2} + \beta_j^{1/2}) \cdot \|g\|_{L_2(\mathcal{U})}, \tag{3.9}$$

where

$$\alpha_j = \sup_{2^j \le |\xi| \le 2^{j+2}} |\mathcal{F}(d\sigma * (\delta - \varphi_t))(\xi)|,$$

and

$$\beta_j = \sup_{2^j \le |\xi| \le 2^{j+2}} |\langle \nabla \mathcal{F}(d\sigma * (\delta - \varphi_t))(\xi), \xi\rangle|.$$

Since[1] $\widehat{\delta}(\xi) = 1$ for any $\xi \in \mathbb{R}^d$ and $\widehat{\varphi}(\xi) = 1$ when $|\xi| \le 1$, it follows that $\alpha_j = \beta_j = 0$ if $2^j < t^{-1}$. Thus, we only need to consider j such that $2^j \ge t^{-1}$. Additionally, by the estimates (11) and (12) in [2], we have

$$|\widehat{d\sigma}(\eta)| \lesssim (1 + |\eta|)^{-(d-1)/2}, \quad |\langle \nabla \widehat{d\sigma}(\eta), \eta\rangle| \lesssim (1 + |\eta|)^{-(d-3)/2}, \quad \forall \eta \in \mathbb{R}^d. \tag{3.10}$$

This implies that

$$\alpha_j \lesssim 2^{-j(d-1)/2} \quad \text{and} \quad \beta_j \lesssim 2^{-j(d-3)/2} \tag{3.11}$$

when $2^j \ge t^{-1}$. Plugging (3.11) into (3.9), we conclude that

$$\Big\|{\sup_{r>0}}^+ \mathcal{A}_{r,2}\, g\Big\|_{L_2(\mathcal{U})} \lesssim \sum_{2^j \ge t^{-1}} 2^{-j(d-1)/4}\, 2^{-j(d-3)/4} \cdot \|g\|_{L_2(\mathcal{U})} \lesssim t^{d/2-1} \cdot \|g\|_{L_2(\mathcal{U})},$$

which establishes (3.3). The proof is complete. ∎

[1] Here the Fourier transform of δ should be understood in the sense of tempered distributions (see, e.g., Section 2.3.3 of [10]).

4. Proof of Theorem B: discrete spherical averages

Similar to the discussion in the previous section, we need to show that for any finite projection $f \in \mathcal{N}$ and for any $\lambda > 0$, there exists a projection $e \in \mathcal{N}$ such that the following estimate holds:

$$\sup_{r\in\mathcal{R}} \|e(A_r f)e\|_\infty \le \lambda \quad \text{and} \quad \nu(e^\perp) \lesssim \lambda^{-d/(d-2)}\,\nu(f). \tag{4.1}$$

Here, the implicit constant depends only on d, but not on f or λ. In contrast to the continuous case, the discrete one is more involved. Let $q \in \mathbb{N}$ and $1 \le a \le q$ with $(a,q) = 1$. We define the smooth function $m_r^{a/q}$ by

$$m_r^{a/q}(\xi) := \sum_{\ell\in\mathbb{Z}^d} G(a/q,\ell)\,\Psi_q(\xi - \ell/q)\,\widehat{d\sigma}_r(\xi - \ell/q).$$

In the above, $G(a/q,\ell)$ is the normalized Gauss sum

$$G(a/q,\ell) := q^{-d} \sum_{n\in(\mathbb{Z}/q\mathbb{Z})^d} e^{2\pi i(|n|^2 a/q + \langle n,\ell/q\rangle)},$$

while

$$\Psi_q(\eta) := \Psi(q\eta),$$

with Ψ being a smooth function supported in the cube $Q_0/2$ and identically equal to 1 in the cube $Q_0/4$. Finally, $\widehat{d\sigma_r}$ denotes the Fourier transform of the normalized surface measure $d\sigma_r$.

The following proposition provides a maximal norm estimate for $(T_{m_r^{a/q}})_{r\in\mathcal{R}}$. We refer to Proposition 5.1 in [4] for the proof.

Proposition 4.1. *Assume that $d \ge 3$. Then the following estimate holds:*

$$\Big\|\sup_{r\in\mathcal{R}}{}^+ T_{m_r^{a/q}} f\Big\|_{L_2(\mathcal{N})} \lesssim q^{-d/2}\,\|f\|_{L_2(\mathcal{N})}, \quad \forall f \in L_2(\mathcal{N}).$$

Here the implicit constant depends only on d.

Let m_r denote the infinite sum

$$m_r := C_d \sum_{q\ge 1} \sum_{\substack{1\le a\le q\\ (a,q)=1}} e^{-2\pi i r^2 a/q}\, m_r^{a/q},$$

where the sum is taken over all reduced fractions a/q with $1 \le a \le q$, $(a,q) = 1$, and the constant C_d is given by $C_d = \pi^{d/2}/\Gamma(d/2)$. The lemma below shows that T_{m_r} serves as a proper approximation of A_r. This assertion simply follows from the combination of (1) and (2) in Proposition 4.1 of [4].

Proposition 4.2. *Assume that $d \ge 5$. If $R > 0$, then we have*

$$\Big\|\sup_{R<r\le 2R}{}^+ (A_r f - T_{m_r} f)\Big\|_{L_2(\mathcal{N})} \lesssim R^{2-d/2}\,\|f\|_{L_2(\mathcal{N})}, \quad \forall f \in L_2(\mathcal{N}).$$

Here the implicit constant does not depend on R.

We now split each A_r into the sum of five operators. For this purpose, define the smooth functions $m_{r,1}^{a/q}$ and $m_{r,2}^{a/q}$ by

$$\sum_{\ell\in\mathbb{Z}^d} G(a/q,\ell)\,\Psi_q(\xi-\ell/q)\,\widehat{d\sigma}_r(\xi-\ell/q)\,\Psi_{rq/\kappa}(\xi-\ell/q),$$

and

$$\sum_{\ell\in\mathbb{Z}^d} G(a/q,\ell)\,\Psi_q(\xi-\ell/q)\,\widehat{d\sigma}_r(\xi-\ell/q)\,(1-\Psi)_{rq/\kappa}(\xi-\ell/q),$$

where $\kappa>0$ is a fixed positive constant. Clearly, we have

$$m_r^{a/q}=m_{r,1}^{a/q}+m_{r,2}^{a/q}.$$

Set

$$\mathcal{R}_1:=\{r\in\mathcal{R}:r\le\kappa\}\quad\text{and}\quad\mathcal{R}_2:=\{r\in\mathcal{R}:r>\kappa\}.$$

Then we may write

$$A_rf=A_{r,1}f+A_{r,2}f+C_d(A_{r,3}f+A_{r,4}f+A_{r,5}f)$$

with

$$\begin{aligned}
A_{r,1}f&:=A_r(f)\,\mathbb{1}_{\mathcal{R}_1}(r),\\
A_{r,2}f&:=[A_rf-T_{m_r}f]\,\mathbb{1}_{\mathcal{R}_2}(r),\\
A_{r,3}f&:=\Big(\sum_{q=\kappa}^{\infty}\sum_{\substack{1\le a\le q\\(a,q)=1}}e^{-2\pi ir^2a/q}\,T_{m_r^{a/q}}f\Big)\,\mathbb{1}_{\mathcal{R}_2}(r),\\
A_{r,4}f&:=\Big(\sum_{q=1}^{\kappa}\sum_{\substack{1\le a\le q\\(a,q)=1}}e^{-2\pi ir^2a/q}\,T_{m_{r,1}^{a/q}}f\Big)\,\mathbb{1}_{\mathcal{R}_2}(r),\\
A_{r,5}f&:=\Big(\sum_{q=1}^{\kappa}\sum_{\substack{1\le a\le q\\(a,q)=1}}e^{-2\pi ir^2a/q}\,T_{m_{r,2}^{a/q}}f\Big)\,\mathbb{1}_{\mathcal{R}_2}(r).
\end{aligned}$$

Unlike Ionescu's proof, where the analysis of A_r was eventually divided into two parts (following the idea of the continuous case, see p. 1413 of [19]), we here split each A_r into five pieces and deal with them separately. We should emphasize that in the case $\kappa\in(0,1)$,

$$A_{r,1}f=A_{r,4}f=A_{r,5}f=0.$$

We claim that for $i\in\{1,4\}$, there exists a projection $e_i\in\mathcal{N}$ such that for any $g\in L_1(\mathcal{N})$,

$$\sup_{r\in\mathcal{R}}\|e_i(A_{r,i}\,g)e_i\|_\infty\le\lambda\quad\text{and}\quad\nu(e_i^\perp)\lesssim\kappa^2\,\frac{\|g\|_{L_1(\mathcal{N})}}{\lambda};\tag{4.2}$$

and for $i\in\{2,3,5\}$, $(A_{r,i})_{r\in\mathcal{R}}$ satisfies the following estimate:

$$\Big\|{\sup_{r\in\mathcal{R}}}^+A_{r,i}\,g\Big\|_{L_2(\mathcal{N})}\lesssim\kappa^{2-d/2}\,\|g\|_{L_2(\mathcal{N})},\quad\forall g\in L_2(\mathcal{N}).\tag{4.3}$$

Before verifying the claim, let us finish the proof of (4.1). Fix $\lambda > 0$ and let f be any finite projection in $\mathcal{N}$. By (4.3), we may find three projections e_2, e_3 and e_5 such that

$$\sup_{r\in\mathcal{R}} \|e_i(A_{r,i}f)e_i\|_\infty \le \frac{\lambda}{5} \quad \text{and} \quad \nu(e_i^\perp) \lesssim \kappa^{4-d}\,\frac{\nu(f)}{\lambda^2}, \quad \text{for } i\in\{2,3,5\}. \tag{4.4}$$

Similarly, it follows from (4.2) that there exist two projections e_1 and e_4 such that

$$\sup_{r\in\mathcal{R}} \|e_i(A_{r,i}f)e_i\|_\infty \le \frac{\lambda}{5} \quad \text{and} \quad \nu(e_i^\perp) \lesssim \kappa^{2}\,\frac{\nu(f)}{\lambda}, \quad \text{for } i\in\{1,4\}. \tag{4.5}$$

Let

$$e = e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5.$$

We deduce from (4.4) and (4.5) that

$$\begin{aligned}\|e(A_r f)e\|_\infty &= \|e[A_{r,1}f + A_{r,2}f + C_d(A_{r,3}f + A_{r,4}f + A_{r,5}f)]e\|_\infty \\ &\lesssim \sum_{i=1}^{5} \|ee_i(A_{r,i}f)e_ie\|_\infty \le \lambda, \quad \forall r\in\mathcal{R},\end{aligned}$$

and

$$\begin{aligned}\nu(e^\perp) &\le \nu(e_1^\perp) + \nu(e_2^\perp) + \nu(e_3^\perp) + \nu(e_4^\perp) + \nu(e_5^\perp) \\ &\lesssim \lambda^{-1}\kappa^2\nu(f) + \lambda^{-2}\kappa^{4-d}\,\nu(f).\end{aligned}$$

Choosing $\kappa = \lambda^{1/(2-d)}$, the latter implies

$$\nu(e^\perp) \lesssim \lambda^{-d/(d-2)}\,\nu(f).$$

The desired estimate (4.1) (and so Theorem B) is proved.

It remains to check the claim. This will be accomplished by establishing the estimate (4.2) for $A_{r,1}$ and $A_{r,4}$, and the estimate (4.3) for $A_{r,2}$, $A_{r,3}$ and $A_{r,5}$, separately. Below, without loss of generality, we always assume that g represents a positive operator-valued function either in $L_1(\mathcal{N})$ or in $L_2(\mathcal{N})$.

Estimate for $A_{r,1}$. The estimate for $A_{r,1}$ directly follows from the fact that $(A_{r,1})_{r\in\mathcal{R}}$ is controlled by the discrete Hardy–Littlewood averages. More precisely, for any $h > 0$, let $\mathcal{M}_h$ denote the discrete Hardy–Littlewood average, which is given by

$$\mathcal{M}_h f(n) := \frac{1}{\#B_h} \sum_{m\in\mathbb{Z}^d:|m|\le h} f(n-m), \quad n\in\mathbb{Z}^d,$$

where B_h is the ball with the origin as its center and h as its radius, $\#B_h$ is the number of lattice points in the ball B_h, which is almost the volume of B_h in d dimension, that is,

$$\#B_h = |B_h| + O(h^{d-1}).$$

Given a positive operator-valued function g in $L_1(\mathcal{N})$, we extend it to a function G defined on $\mathbb{R}^d$ by setting

$$G(x) := \sum_{n\in\mathbb{Z}^d} g(n)\,\mathbb{1}_{Q_0}(x-n), \quad x\in\mathbb{R}^d.$$

Clearly, G belongs to $L_1(\mathcal{U})$ precisely when $g\in L_1(\mathcal{N})$, and moreover,

$$\|G\|_{L_1(\mathcal{U})} = \|g\|_{L_1(\mathcal{N})}.$$

Following the argument used in the classical setting (see [32], p. 36), we know that there exists a constant $C>0$ such that

$$\mathcal{M}_h g \le C\,\mathbf{M}_h G,$$

where $\mathbf{M}_h$ is the continuous Hardy–Littlewood averaging operator defined in (3.7). Therefore, benefiting from Mei's Hardy–Littlewood maximal inequality established in Theorem 3.3 (i) of [29], there exists a projection $e\in\mathcal{N}$ such that

$$\sup_h \|e(\mathcal{M}_h g)e\|_\infty \lesssim \sup_h \|e(\mathbf{M}_h G)e\|_\infty \le \frac{\lambda}{\kappa^2}$$

and

$$\nu(e^\perp) \lesssim \kappa^2\,\frac{\|G\|_{L_1(\mathcal{U})}}{\lambda} = \kappa^2\,\frac{\|g\|_{L_1(\mathcal{N})}}{\lambda}.$$

Note that

$$N_d(r)\simeq r^{d-2} \quad \text{for } d\ge 5$$

(see Theorem 4.1 in [9]). Therefore, using the discrete Hardy–Littlewood averages to bound the operators $A_{r,1}$, we arrive at

$$\|e(A_{r,1}g)e\|_\infty \le r^2\,\|e(\mathcal{M}_r g)e\|_\infty \lesssim r^2\,\frac{\lambda}{\kappa^2} \le \lambda, \quad \forall r\in\mathcal{R},$$

and

$$\nu(e_1^\perp) \lesssim \kappa^2\,\frac{\|g\|_{L^1(\mathcal{N})}}{\lambda}.$$

This gives the estimate (4.2) for $A_{r,1}$.

Estimates for $A_{r,2}$ and $A_{r,3}$. Using the approximation formula presented in Proposition 4.2, we obtain

$$\begin{aligned}
\Big\|\sup_{r\in\mathcal{R}}{}^+ A_{r,2}\,g\Big\|_{L_2(\mathcal{N})} &= \Big\|\sup_{r\in\mathcal{R}_2}{}^+ (A_r g - T_{m_r} g)\Big\|_{L_2(\mathcal{N})}\\
&\le \sum_{j=0}^\infty \Big\|\sup_{\kappa\cdot 2^j<r\le\kappa\cdot 2^{j+1}}{}^+ (A_r g - T_{m_r} g)\Big\|_{L_2(\mathcal{N})}\\
&\lesssim \sum_{j=0}^\infty (\kappa\cdot 2^j)^{2-d/2}\,\|g\|_{L_2(\mathcal{N})} \lesssim \kappa^{2-d/2}\,\|g\|_{L_2(\mathcal{N})}.
\end{aligned}$$

This completes the proof of (4.3) for $A_{r,2}$. On the other hand, summing the bounds in Proposition 4.1 over $q \geq \kappa$ and $1 \leq a \leq q$ with $(a,q) = 1$, we conclude

$$\big\| \sup_{r\in\mathcal{R}_2}{}^{+} A_{r,3}\, g \big\|_{L_2(\mathcal{N})} \leq \sum_{q=\kappa}^{\infty} \sum_{\substack{1\leq a\leq q\\ (a,q)=1}} \big\| \sup_{r\in\mathcal{R}}{}^{+} T_{m_r^{a/q}}\, g \big\|_{L_2(\mathcal{N})}$$
$$\lesssim \sum_{q=\kappa}^{\infty} \sum_{\substack{1\leq a\leq q\\ (a,q)=1}} q^{-d/2}\, \|g\|_{L_2(\mathcal{N})} \lesssim \kappa^{2-d/2}\, \|g\|_{L_2(\mathcal{N})},$$

which verifies the estimate (4.3) for $A_{r,3}$.

Estimate for $A_{r,4}$. Let $K_{r,1}^{a/q}$ denote the kernel of the operator $T_{m_{r,1}^{a/q}}$. Then we have

$$A_{r,4}\, g(n) = \sum_{q=1}^{\kappa} \sum_{\substack{1\leq a\leq q\\ (a,q)=1}} g * (e^{-2\pi i r^2 a/q}\, K_{r,1}^{a/q})(n).$$

A standard argument used in p. 1415 of [19] shows that

$$|e^{-2\pi i r^2 a/q}\, K_{r,1}^{a/q}(n)| \lesssim \frac{1}{r^d}\Big(1 + \frac{|n|}{r}\Big)^{-(d+1)} \frac{\kappa}{q}, \quad \text{for any } n \in \mathbb{Z}^d.$$

Moreover, we have

$$\sum_{q=1}^{\kappa} \sum_{\substack{1\leq a\leq q\\ (a,q)=1}} |e^{-2\pi i r^2 a/q}\, K_{r,1}^{a/q}(n)| \lesssim \sum_{q=1}^{\kappa} \sum_{\substack{1\leq a\leq q\\ (a,q)=1}} \frac{1}{r^d}\Big(1 + \frac{|n|}{r}\Big)^{-(d+1)} \frac{\kappa}{q} \tag{4.6}$$
$$\lesssim \frac{\kappa^2}{r^d}\Big(1 + \frac{|n|}{r}\Big)^{-(d+1)}.$$

Since g is positive (as we assumed in the paragraph before the estimate for $A_{r,1}$), it follows from (4.6) that

$$\mathrm{Re}(A_{r,4}\, g)(n) \leq \sum_{q=1}^{\kappa} \sum_{\substack{1\leq a\leq q\\ (a,q)=1}} g * |\mathrm{Re}(e^{-2\pi i r^2 a/q}\, K_{r,1}^{a/q})|(n)$$
$$\leq \sum_{q=1}^{\kappa} \sum_{\substack{1\leq a\leq q\\ (a,q)=1}} g * |e^{-2\pi i r^2 a/q}\, K_{r,1}^{a/q}|(n)$$
$$\lesssim \kappa^2 g * \Big(\frac{1}{r^d}\Big(1 + \frac{|\cdot|}{r}\Big)^{-(d+1)}\Big)(n)$$
$$\lesssim \kappa^2 g * \Big(\frac{1}{r^d}\Big(\frac{|\cdot|}{r}\Big)^{-(d+1)}\Big)(n) =: \kappa^2 g * \phi_r(n).$$

Then, by the proof of Theorem 4.3 in [6], we know that there exists a projection $e_1 \in \mathcal{N}$ such that

$$\|e_1(\operatorname{Re} A_{r,4}\, g)e_1\|_\infty \lesssim \kappa^2\, \|e_1(\phi_r * g)e_1\|_\infty \lesssim \kappa^2\, \frac{\lambda}{\kappa^2} = \lambda, \quad \forall r > 0,$$

and

$$\nu(e_1^\perp) \lesssim \kappa^2\, \frac{\|g\|_{L_1(\mathcal{N})}}{\lambda}.$$

Similarly, we may find a projection $e_2 \in \mathcal{N}$ such that the same holds true for $\operatorname{Im} A_{r,4}$. Then, the desired estimate (4.2) for $A_{r,4}$ follows by taking $e = e_1 \wedge e_2$.

Estimate for $A_{r,5}$. To verify the estimate (4.3) for $A_{r,5}$, we will adopt an argument similar to that of Proposition 3.1 in [28]. Let Υ be a smooth function, supported in the cube Q_0, with $\Upsilon(\xi) = 1$ for $\xi \in Q_0/2$. Recall that Ψ is a smooth function supported in the cube $Q_0/2$ and identically equal to 1 in the cube $Q_0/4$. Then, it is easy to see that $\Psi = \Psi \cdot \Upsilon$. Let the functions $s^{a/q}$ and $t^q_{r,\kappa}$ be respectively defined by

$$s^{a/q}(\xi) := \sum_{\ell \in \mathbb{Z}^d} G(a/q, \ell)\, \Upsilon_q(\xi - \ell/q)$$

and

$$t^q_{r,\kappa}(\xi) := \sum_{\ell \in \mathbb{Z}^d} \Psi_q(\xi - \ell/q)\, (1 - \Psi)_{rq/\kappa}(\xi - \ell/q)\, \widehat{d\sigma}_r(\xi - \ell/q).$$

Then, clearly, we have

$$m^{a/q}_{r,2} = s^{a/q} \cdot t^q_{r,\kappa}.$$

Note that $T_{s^{a/q}}$ does not depend on the radius $r \in \mathcal{R}$. Then it is easy to see that

$$\left\|(T_{m^{a/q}_{r,2}})_{r\in\mathcal{R}}\right\|_{L_2(\mathcal{N})\to L_2(\mathcal{N};\ell_\infty)} \leq \left\|(T_{t^q_{r,\kappa}})_{r\in\mathcal{R}}\right\|_{L_2(\mathcal{N})\to L_2(\mathcal{N};\ell_\infty)} \cdot \|T_{s^{a/q}}\|_{L_2(\mathcal{N})\to L_2(\mathcal{N})}.$$

From inequality (5.4) in [4], we know that

$$\|T_{s^{a/q}}\|_{L_2(\mathcal{N})\to L_2(\mathcal{N})} \lesssim q^{-d/2}.$$

Therefore, by the definition of $A_{r,5}$ and the above two estimates, we have

$$\begin{aligned}\|(A_{r,5})_{r\in\mathcal{R}}\|_{L_2(\mathcal{N})\to L_2(\mathcal{N};\ell_\infty)} &\lesssim q \cdot \kappa \left\|(T_{m^{a/q}_{r,2}})_{r\in\mathcal{R}}\right\|_{L_2(\mathcal{N})\to L_2(\mathcal{N};\ell_\infty)} \\ &\lesssim q^{1-d/2} \cdot \kappa \left\|(T_{t^q_{r,\kappa}})_{r\in\mathcal{R}}\right\|_{L_2(\mathcal{N})\to L_2(\mathcal{N};\ell_\infty)}.\end{aligned}$$

Thus, the proof for $A_{r,5}$ will be complete if one can show that

$$\left\|(T_{t^q_{r,\kappa}})_{r\in\mathcal{R}}\right\|_{L_2(\mathcal{N})\to L_2(\mathcal{N};\ell_\infty)} \lesssim q^{d/2-1} \cdot \kappa^{1-d/2}. \tag{4.7}$$

Let $u^q_{r,\kappa}$ be the smooth function defined by

$$u^q_{r,\kappa}(\xi) := \Psi_q(\xi)\, (1 - \Psi)_{rq/\kappa}(\xi)\, \widehat{d\sigma}_r(\xi).$$

Note that the multiplier $u^q_{r,\kappa}$ is supported in Q_0/q. It follows from Lemma 2.2 that

$$\big\|(T_{t^q_{r,\kappa}})_{r\in\mathcal{R}}\big\|_{L_2(\mathcal{N})\to L_2(\mathcal{N};\ell_\infty)} \lesssim \big\|(T_{u^q_{r,\kappa}})_{r\in\mathcal{R}}\big\|_{L_2(\mathcal{U})\to L_2(\mathcal{U};\ell_\infty)}.$$

Therefore, to prove (4.7), it suffices to show that

$$\big\|(T_{u^q_{r,\kappa}})_{r\in\mathcal{R}}\big\|_{L_2(\mathcal{U})\to L_2(\mathcal{U};\ell_\infty)} \lesssim q^{d/2-1}\cdot\kappa^{1-d/2}. \tag{4.8}$$

Let $m_r(\eta) := m(r\eta)$, where the function m is defined by

$$m(\eta) := \widehat{d\sigma}(\eta)\,(1-\Psi)(q\eta/\kappa), \quad \eta\in\mathbb{R}^d.$$

Then we have

$$u^q_{r,\kappa} = m_r\cdot\Psi_q.$$

Since Ψ_q is bounded, it follows from the vector-valued Plancherel theorem that

$$\begin{aligned}\|\check{\Psi}_q * f\|_{L_2(\mathcal{U})} = \|\Psi_q\cdot\widehat{f}\|_{L_2(\mathcal{U})} &\le \|\Psi_q\|_{L_\infty(\mathbb{R}^d)}\cdot\|\widehat{f}\|_{L_2(\mathcal{U})}\\ &\lesssim \|f\|_{L_2(\mathcal{U})}, \quad \forall f\in L_2(\mathcal{U}).\end{aligned} \tag{4.9}$$

Then, applying Lemma 3.1 and (4.9), we obtain

$$\begin{aligned}\big\|{\sup_{r\in\mathcal{R}}}^+ T_{u^q_{r,\kappa}} g\big\|_{L_2(\mathcal{U})} &= \big\|{\sup_{r\in\mathcal{R}}}^+\big(\check{m}_r * \check{\Psi}_q * g\big)\big\|_{L_2(\mathcal{U})}\\ &\lesssim \sum_{j\in\mathbb{Z}}\alpha_j^{1/2}(\alpha_j^{1/2}+\beta_j^{1/2})\,\|\check{\Psi}_q * g\|_{L_2(\mathcal{U})}\\ &\lesssim \sum_{j\in\mathbb{Z}}\alpha_j^{1/2}(\alpha_j^{1/2}+\beta_j^{1/2})\,\|g\|_{L_2(\mathcal{U})}, \quad g\in L_2(\mathcal{U}),\end{aligned} \tag{4.10}$$

where

$$\alpha_j := \sup_{2^j\le|\xi|\le 2^{j+2}}|m(\xi)| \quad\text{and}\quad \beta_j := \sup_{2^j\le|\xi|\le 2^{j+2}}|\langle\nabla m(\xi),\xi\rangle|.$$

Observe that the function m is supported on the set $\{\eta : |\eta|\ge\kappa/(8q)\}$ since $\Psi(\eta)=1$ when $|\eta|\le 1/8$. Then (3.10) implies that

$$\begin{cases}\alpha_j\lesssim 2^{-j(d-1)/2},\ \beta_j\lesssim 2^{-j(d-3)/2}, & \text{if } 2^j\ge\dfrac{\kappa}{16q},\\ \alpha_j=\beta_j=0, & \text{if } 2^j<\dfrac{\kappa}{16q}.\end{cases}$$

Hence, we have

$$\sum_{j\in\mathbb{Z}}\alpha_j^{1/2}(\alpha_j^{1/2}+\beta_j^{1/2})\lesssim\Big(\frac{q}{\kappa}\Big)^{d/2-1},$$

which combined with (4.10) yields the desired estimate (4.8). The proof is complete.

5. Proof of Theorem C: ergodic averages

In this section, we apply Theorem B to prove the restricted weak-type maximal inequalities for noncommutative ergodic averages (i.e., Theorem C). Fix a finite projection $x \in \mathcal{M}$. By Lemma 2.1, we have

$$\|(\mathfrak{A}_r x)_{r\in\mathcal{R}}\|_{\Lambda_{d/(d-2),\infty}(\mathcal{M};\ell_\infty)} \simeq \sup_N \|(\mathfrak{A}_r x)_{r\in\mathcal{R}_N}\|_{\Lambda_{d/(d-2),\infty}(\mathcal{M};\ell_\infty)},$$

where $\mathcal{R}_N := \mathcal{R} \cap [1, N]$ for $N \in \mathbb{N}$. Thus, to prove Theorem C, it suffices to establish the inequality

$$\|(\mathfrak{A}_r x)_{r\in\mathcal{R}_N}\|_{\Lambda_{d/(d-2),\infty}(\mathcal{M};\ell_\infty)} \lesssim \tau(x)^{d/(d-2)},$$

with the implicit constant independent of N. This is equivalent to showing that for any $\lambda > 0$, there is a projection $e \in \mathcal{M}$ such that

$$\sup_{r\in\mathcal{R}_N} \|e(\mathfrak{A}_r x)e\|_\infty \le \lambda \quad \text{and} \quad \tau(e^\perp) \lesssim \lambda^{-d/(d-2)}\tau(x). \tag{5.1}$$

Fix N to be a positive integer. For a large positive integer M with $M \gg N$, we introduce the cube $C(M)$ as follows:

$$C(M) := \{n \in \mathbb{Z}^d : |n_i| \le M, \text{ for } i \in 1, \dots, d\},$$

which can be viewed as a truncated set of lattice points. Let g denote the operator-valued function given by

$$g(n) := \gamma^n x\, \mathbb{1}_{C(M)}(n), \quad \forall n \in \mathbb{Z}^d.$$

It is easy to verify that g is a projection in $\mathcal{N}$, since γ is an automorphism on $\mathcal{M}$. Moreover, for all $n \in C(M-N)$, one can see that

$$A_r g(n) = \frac{1}{N_d(r)} \sum_{m\in\mathbb{Z}^d\cap\mathbb{S}_r} g(n+m) = \frac{1}{N_d(r)} \sum_{m\in\mathbb{Z}^d\cap\mathbb{S}_r} \gamma^{n+m} x = \gamma^n(\mathfrak{A}_r x). \tag{5.2}$$

Applying Theorem B to g, we may choose a projection $e \in \mathcal{N}$ such that for any $\lambda > 0$, the following hold:

$$\sup_{r\in\mathcal{R}_N} \|e(A_r g)e\|_\infty \le \lambda \tag{5.3}$$

and

$$\sum_{n\in\mathbb{Z}^d} \tau(e(n)^\perp) \lesssim \lambda^{-d/(d-2)} \sum_{n\in\mathbb{Z}^d} \tau(g(n)). \tag{5.4}$$

According to the definition of infimum, for any $\varepsilon > 0$, we may choose $n_0 \in C(M-N)$ and a projection

$$\tilde{e} := \gamma^{-n_0} e(n_0) \in \mathcal{M}$$

such that

$$\tau(\tilde{e}^\perp) \le \inf_{n\in C(M-N)} \tau(e(n)^\perp) + \varepsilon. \tag{5.5}$$

On the one hand, noting that γ^{-n_0} extends to an isometry on $\mathcal{M}$, a combination of (5.2) and (5.3) yields

$$\begin{aligned}\sup_{r\in\mathcal{R}_N}\|\tilde{e}(\mathfrak{A}_r x)\tilde{e}\|_\infty &= \sup_{r\in\mathcal{R}_N}\|(\gamma^{-n_0}e(n_0))(\gamma^{-n_0}(A_r g(n_0)))(\gamma^{-n_0}e(n_0))\|_\infty\\ &= \sup_{r\in\mathcal{R}_N}\|e(n_0)(A_r g(n_0))e(n_0)\|_\infty\\ &\le \sup_{r\in\mathcal{R}_N}\|e(A_r g)e\|_\infty \le \lambda.\end{aligned}$$

On the other hand, since γ is a τ-preserving automorphism of $\mathcal{M}$, it follows from (5.4) and (5.5) that

$$\begin{aligned}\#C(M-N)\,\tau(\tilde{e}^\perp) &\le \sum_{n\in C(M-N)}\tau(e(n)^\perp)+\varepsilon \le \lambda^{-d/(d-2)}\sum_{n\in\mathbb{Z}^d}\tau(g(n))+\varepsilon\\ &= \lambda^{-d/(d-2)}\sum_{n\in C(M)}\tau(\gamma^n x)+\varepsilon = \#C(M)\,\lambda^{-d/(d-2)}\,\tau(x)+\varepsilon,\end{aligned}$$

which implies

$$\frac{\#C(M-N)}{\#C(M)}\,\tau(\tilde{e}^\perp)\lesssim \lambda^{-d/(d-2)}\,\tau(x)+\varepsilon.$$

Note that ε is arbitrarily small. By sending $M\to\infty$, we conclude that

$$\tau(\tilde{e}^\perp)\lesssim\lambda^{-d/(d-2)}\,\tau(x).$$

Consequently, the estimate (5.1) is verified. The proof of Theorem C is complete.

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