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Schauder frames of discrete translates in $L^p(\mathbb{R})$

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Abstract. For every $p > (1+\sqrt{5})/2$, we construct a uniformly discrete real sequence $\{\lambda_n\}_{n=1}^\infty$ satisfying $|\lambda_n| = n + o(1)$, a function $g \in L^p(\mathbb{R})$, and continuous linear functionals $\{g_n^*\}_{n=1}^\infty$ on $L^p(\mathbb{R})$, such that every $f \in L^p(\mathbb{R})$ admits a series expansion

$$f(x) = \sum_{n=1}^{\infty} g_n^*(f)\, g(x - \lambda_n)$$

convergent in the $L^p(\mathbb{R})$ norm. We moreover show that g can be chosen nonnegative.

1. Introduction

1.1. A system of vectors $\{x_n\}_{n=1}^\infty$ in a Banach space X is called a *Schauder basis* if every $x \in X$ admits a unique series expansion $x = \sum_{n=1}^\infty c_n x_n$, where $\{c_n\}$ are scalars. It is well known that in this case there exist biorthogonal continuous linear functionals $\{x_n^*\}$ such that the coefficients of the series expansion are given by $c_n = x_n^*(x)$ (see, e.g., Section 1.6 of [23]). If the series converges unconditionally (i.e., if it converges for any rearrangement of its terms) for every $x \in X$, then $\{x_n\}$ is said to be an *unconditional* Schauder basis.

Given a function $g \in L^p(\mathbb{R})$, we denote its translates by

$$(T_\lambda g)(x) = g(x - \lambda), \quad \lambda \in \mathbb{R}. \tag{1.1}$$

There is a long-standing open problem, asking whether the space $L^p(\mathbb{R})$, $1 < p < \infty$, admits a Schauder basis formed by translates of a single function (see [20] and Problem 4.4 in [13]). It is known that *unconditional* Schauder bases consisting of translates do not exist in any of these spaces, see [5, 13, 20].

A sequence $\Lambda = \{\lambda_n\}_{n=1}^\infty$ of real numbers is said to be *uniformly discrete* if

$$\inf_{n \neq m} |\lambda_m - \lambda_n| > 0. \tag{1.2}$$

It was observed in Theorem 1 of [20] that the condition (1.2) is necessary for a system of translates $\{T_{\lambda_n} g\}_{n=1}^\infty$ to form a Schauder basis in $L^p(\mathbb{R})$. It is also known that in the space $L^1(\mathbb{R})$, a system of uniformly discrete translates cannot even be complete, see [3]. Hence, no Schauder bases of translates exist in $L^1(\mathbb{R})$.

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1.2. If X is a Banach space with dual space X^*, then a system $\{(x_n, x_n^*)\}_{n=1}^\infty$ in $X \times X^*$ is called a *Schauder frame* (or a quasi-basis) if every $x \in X$ has a series expansion

$$x = \sum_{n=1}^{\infty} x_n^*(x) x_n. \tag{1.3}$$

If the series (1.3) converges unconditionally for every $x \in X$, then $\{(x_n, x_n^*)\}$ is called an *unconditional* Schauder frame. We note that the series expansion (1.3) need not be unique and that the coefficient functionals $\{x_n^*\}$ need not be biorthogonal to $\{x_n\}$. Hence Schauder frames form a wider class of representation systems than Schauder bases.

It was shown in [5] that for every $p > 2$ there exists an unconditional Schauder frame in the space $L^p(\mathbb{R})$ consisting of translates, i.e., of the form $\{(T_{\lambda_n} g, g_n^*)\}$, where $g \in L^p(\mathbb{R})$, $\{\lambda_n\} \subset \mathbb{R}$ and $\{g_n^*\}$ are continuous linear functionals on $L^p(\mathbb{R})$. Moreover, $\{\lambda_n\}$ may be chosen to be *an arbitrary unbounded sequence*, and in particular, it may consist of integers, and may increase arbitrarily fast.

To the contrary, we proved recently [11] that if $1 \leqslant p \leqslant 2$, then the space $L^p(\mathbb{R})$ does not admit an unconditional Schauder frame consisting of translates.

In Section 4 of [6], a construction was given of Schauder frames (not unconditional) of translates in $L^p(\mathbb{R})$, $1 \leqslant p < \infty$. In fact, it was proved that whenever a system of translates $\{T_\lambda g\}_{\lambda \in \Lambda}$ is complete in the space $L^p(\mathbb{R})$, then there exists a Schauder frame $\{(T_{\lambda_n} g, g_n^*)\}_{n=1}^\infty$ such that $\{\lambda_n\} \subset \Lambda$. However, the Schauder frames obtained by this construction are *highly redundant*, as the sequence $\{\lambda_n\}$ is composed of countably many blocks of finite size, such that each block gets repeated a higher and higher number of times. The sequence $\{\lambda_n\}$ thus "runs back and forth" through the set Λ.

1.3. The following question was posed in Problem 4.4 of [13]: does there exist a Schauder frame in $L^p(\mathbb{R})$ formed by a *uniformly discrete* sequence of translates? The problem has remained open for $1 < p \leqslant 2$. We recently obtained [10] an affirmative answer for $p = 2$, i.e., we constructed in the space $L^2(\mathbb{R})$ a Schauder frame of the form $\{(T_{\lambda_n} g, g_n^*)\}_{n=1}^\infty$, where $\{\lambda_n\}_{n=1}^\infty$ is a uniformly discrete real sequence.

The proof in [10] was based on the fact that the Fourier transform is a unitary mapping $L^2(\mathbb{R}) \to L^2(\mathbb{R})$. This point breaks down for $L^p(\mathbb{R})$ spaces, $1 < p < 2$.

The main goal of the present paper is to extend the result from [10] to certain values of p within the range $1 < p < 2$. We will prove the following result.

Theorem 1.1. *Let $p > (1 + \sqrt{5})/2$. Then there exist a uniformly discrete real sequence $\{\lambda_n\}_{n=1}^\infty$ satisfying $|\lambda_n| = n + o(1)$, a function $g \in L^p(\mathbb{R})$ and a sequence $\{g_n^*\}_{n=1}^\infty$ in $(L^p(\mathbb{R}))^*$, such that every $f \in L^p(\mathbb{R})$ admits a series expansion*

$$f(x) = \sum_{n=1}^{\infty} g_n^*(f) g(x - \lambda_n) \tag{1.4}$$

convergent in the $L^p(\mathbb{R})$ norm.

The question whether the result holds for every $p > 1$, remains open.

Remarks. (1) The result can be strengthened in various directions; see Section 4 of [10], where several extensions of the result for $p = 2$ are given, some of which may still be valid for the result of the present paper.

(2) Motivated by recent interest in nonnegative coordinate systems in $L^p(\mathbb{R})$ spaces, see [6, 7, 21], one may ask whether there exists a Schauder frame in the space $L^p(\mathbb{R})$ formed by a uniformly discrete sequence of translates of some *nonnegative* function g. We will show that by a certain modification of our proof, this additional requirement of nonnegativity of g can be achieved in Theorem 1.1 (see Section 5).

2. Preliminaries

In this section, we present some necessary background and fix notation that will be used throughout the paper.

2.1. The *Schwartz space* $\mathcal{S}(\mathbb{R})$ consists of all infinitely smooth functions φ on $\mathbb{R}$ such that for each $n, k \geqslant 0$, the seminorm

$$\|\varphi\|_{n,k} := \sup_{x \in \mathbb{R}} (1 + |x|)^n \, |\varphi^{(k)}(x)|$$

is finite. A *tempered distribution* is a linear functional on the Schwartz space which is continuous with respect to the topology generated by this family of seminorms. We use $\alpha(\varphi)$ to denote the action of a tempered distribution α on a Schwartz function φ.

We denote by $\operatorname{supp}(\alpha)$ the closed support of a tempered distribution α.

If φ is a Schwartz function on $\mathbb{R}$, then we define its Fourier transform by

$$\widehat{\varphi}(x) = \int_{\mathbb{R}} \varphi(t) \, e^{-2\pi i x t} \, dt.$$

The Fourier transform of a tempered distribution α is defined by $\widehat{\alpha}(\varphi) = \alpha(\widehat{\varphi})$.

If α is a tempered distribution and if φ is a Schwartz function, then the product $\alpha \cdot \varphi$ is a tempered distribution defined by $(\alpha \cdot \varphi)(\psi) = \alpha(\varphi \cdot \psi)$, for $\psi \in \mathcal{S}(\mathbb{R})$. The convolution $\alpha * \varphi$ of a tempered distribution α and a Schwartz function φ is an infinitely smooth function which is also a tempered distribution, and whose Fourier transform is $\widehat{\alpha} \cdot \widehat{\varphi}$.

2.2. Let $A^p(\mathbb{T})$, $1 \leqslant p < \infty$, denote the Banach space of Schwartz distributions α on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ whose Fourier coefficients $\{\widehat{\alpha}(n)\}$, $n \in \mathbb{Z}$, belong to $\ell^p(\mathbb{Z})$, endowed with the norm $\|\alpha\|_{A^p(\mathbb{T})} := \|\widehat{\alpha}\|_{\ell^p(\mathbb{Z})}$. For $p = 1$, this is the classical Wiener algebra $A(\mathbb{T})$ of continuous functions with an absolutely convergent Fourier series.

We also use $A^p(\mathbb{R})$, $1 \leqslant p < \infty$, to denote the Banach space of tempered distributions α on $\mathbb{R}$ whose Fourier transform $\widehat{\alpha}$ is in $L^p(\mathbb{R})$, with the norm $\|\alpha\|_{A^p(\mathbb{R})} := \|\widehat{\alpha}\|_{L^p(\mathbb{R})}$.

Note that $A^1(\mathbb{T})$ and $A^1(\mathbb{R})$ are function spaces, continuously embedded in $C(\mathbb{T})$ and $C_0(\mathbb{R})$ respectively. Similarly, for $1 < p \leqslant 2$, the space A^p (on either $\mathbb{T}$ or $\mathbb{R}$) is a function space, continuously embedded in L^q, $q = p/(p-1)$, by the Hausdorff–Young inequality. On the other hand, A^p is not a function space for $p > 2$.

2.3. For $0 < h < 1/2$, we denote by Δ_h the "triangle function" on $\mathbb{T}$ vanishing outside $(-h, h)$, linear on $[-h, 0]$ and on $[0, h]$, and satisfying $\Delta_h(0) = 1$. Then $\widehat{\Delta}_h(0) = h$, and

$$\|\Delta_h\|_{A^p(\mathbb{T})} \leqslant h^{(p-1)/p}, \quad 1 \leqslant p < \infty. \tag{2.1}$$

Indeed, to obtain the estimate (2.1) one can use the fact that Fourier coefficients of Δ_h are real and nonnegative, hence $\|\Delta_h\|_{A(\mathbb{T})} = \sum_n \widehat{\Delta}_h(n) = \Delta_h(0) = 1$. Moreover, we have $\widehat{\Delta}_h(n) \leqslant \int_{\mathbb{T}} \Delta_h(t)dt = h$ for every $n \in \mathbb{Z}$, and so $\|\Delta_h\|^p_{A^p(\mathbb{T})} = \sum_n \widehat{\Delta}_h(n)^p \leqslant h^{p-1}$.

For $0 < h < 1/4$, we also use τ_h to denote the "trapezoid function" on $\mathbb{T}$ which vanishes outside $(-2h, 2h)$, is equal to 1 on $[-h, h]$, and is linear on $[-2h, -h]$ and on $[h, 2h]$. Then $\widehat{\tau}_h(0) = 3h$, and

$$\|\tau_h\|_{A^p(\mathbb{T})} \leqslant 3h^{(p-1)/p}, \quad 1 \leqslant p < \infty, \tag{2.2}$$

which follows from (2.1) and the fact that $\tau_h(t) = \Delta_h(t+h) + \Delta_h(t) + \Delta_h(t-h)$.

2.4. By a *trigonometric polynomial* we mean a finite sum of the form

$$P(t) = \sum_j a_j \, e^{2\pi i \sigma_j t}, \quad t \in \mathbb{R}, \tag{2.3}$$

where $\{\sigma_j\}$ are distinct real numbers, and $\{a_j\}$ are complex numbers.

By the *spectrum* of P we mean the set $\operatorname{spec}(P) := \{\sigma_j : a_j \neq 0\}$. We observe that if P has integer spectrum, $\operatorname{spec}(P) \subset \mathbb{Z}$, then P is 1-periodic, that is, $P(t+1) = P(t)$. In this case, P may be considered also as a function on $\mathbb{T}$.

By the *degree* of P we mean the number $\deg(P) := \min\{r \geqslant 0 : \operatorname{spec}(P) \subset [-r, r]\}$.

The (symmetric) *partial sum* $S_r(P)$ of a trigonometric polynomial (2.3) is defined by

$$S_r(P)(t) = \sum_{|\sigma_j| \leqslant r} a_j \, e^{2\pi i \sigma_j t}.$$

We say that P is *analytic* if we have $\operatorname{spec}(P) \subset [0, +\infty)$.

2.5. For a trigonometric polynomial (2.3), we use the notations $\|\widehat{P}\|_p = (\sum_j |a_j|^p)^{1/p}$ and $\|\widehat{P}\|_\infty = \max_j |a_j|$. If $f \in A^p(\mathbb{R})$ and P is a trigonometric polynomial, then

$$\|f \cdot P\|_{A^p(\mathbb{R})} \leqslant \|\widehat{P}\|_1 \cdot \|f\|_{A^p(\mathbb{R})}, \tag{2.4}$$

which follows by an application of the triangle inequality in the space $L^p(\mathbb{R})$, using the fact that the Fourier transform of $f \cdot P$ is given by $\sum_j a_j \widehat{f}(x - \sigma_j)$.

In a similar way, one can establish the inequality

$$\|f \cdot g\|_{A^p(\mathbb{T})} \leqslant \|f\|_{A(\mathbb{T})} \cdot \|g\|_{A^p(\mathbb{T})}, \quad f \in A(\mathbb{T}), \ g \in A^p(\mathbb{T}).$$

2.6. We introduce an auxiliary norm $\|\cdot\|_*$ on the Schwartz space $\mathcal{S}(\mathbb{R})$, defined by

$$\|u\|_* := 10 \cdot \sup_{x \in \mathbb{R}} (1 + x^2) |\widehat{u}(x)|, \quad u \in \mathcal{S}(\mathbb{R}).$$

If $u \in \mathcal{S}(\mathbb{R})$ and $f \in A^p(\mathbb{T})$, $1 \leqslant p < \infty$, then f may be considered also as a 1-periodic tempered distribution on $\mathbb{R}$, so the product $u \cdot f$ makes sense and is well defined.

Lemma 2.1. *Let $u \in \mathcal{S}(\mathbb{R})$ and $f \in A^p(\mathbb{T})$, $1 \leqslant p < \infty$. Then*

$$\|u \cdot f\|_{A^p(\mathbb{R})} \leqslant \|u\|_* \|f\|_{A^p(\mathbb{T})}. \tag{2.5}$$

Indeed, an application of Hölder's inequality yields

$$\begin{aligned}\|u\cdot f\|^p_{A_p(\mathbb{R})} &= \int \Big|\sum_{n\in\mathbb{Z}} \widehat{f}(n)\,\widehat{u}(x-n)\Big|^p\,dx \\ &\leqslant \int \Big(\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^p\,|\widehat{u}(x-n)|\Big)\Big(\sum_{n\in\mathbb{Z}} |\widehat{u}(x-n)|\Big)^{p-1} dx \\ &\leqslant \|u\|_*^{p-1} \int \Big(\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^p\,|\widehat{u}(x-n)|\Big)\,dx \\ &\leqslant \|u\|_*^{p-1}\,\|u\|_* \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^p = \|u\|_*^p\,\|f\|^p_{A^p(\mathbb{T})}.\end{aligned}$$

If $f\in A^p(\mathbb{T})$ and ν is a positive integer, then we use f_ν to denote the element of the space $A^p(\mathbb{T})$ whose Fourier series is given by $\sum_{n\in\mathbb{Z}} \widehat{f}(n)\,e^{2\pi i n\nu t}$.

Lemma 2.2. *Let $u\in\mathcal{S}(\mathbb{R})$ and $f\in A^p(\mathbb{T})$, $1\leqslant p<\infty$. Then*

$$\lim_{\nu\to\infty} \|u\cdot f_\nu\|_{A^p(\mathbb{R})} = \|u\|_{A^p(\mathbb{R})}\,\|f\|_{A^p(\mathbb{T})}. \tag{2.6}$$

This is obvious if u has a compactly supported Fourier transform $\widehat{u}$. In the general case, (2.6) can be proved by approximating u in the $\|\cdot\|_*$ norm by a Schwartz function v with a compactly supported Fourier transform $\widehat{v}$, and using the inequality (2.5) to estimate the error.

2.7. If α is a tempered distribution on $\mathbb{R}$ and P is a trigonometric polynomial, then the product $\alpha\cdot P$ is a tempered distribution defined by $(\alpha\cdot P)(\varphi)=\alpha(P\cdot\varphi)$, $\varphi\in\mathcal{S}(\mathbb{R})$.

Lemma 2.3. *Let $\alpha\in A^p(\mathbb{R})$, $1\leqslant p<\infty$, and suppose that P is a nonzero trigonometric polynomial on $\mathbb{R}$. If $\alpha\cdot P=0$, then $\alpha=0$.*

Proof. The condition $\alpha\cdot P=0$ implies that $\operatorname{supp}(\alpha)$ is contained in the set of zeros of P, which is a discrete closed set in $\mathbb{R}$. Let χ be a Schwartz function on $\mathbb{R}$ with $\chi(t)=1$ in a neighborhood of a point $a\in\operatorname{supp}(\alpha)$, and $\chi(t)=0$ in a neighborhood of $\operatorname{supp}(\alpha)\setminus\{a\}$. The distribution $\alpha\cdot\chi$ is then supported at the point a and coincides with α in a neighborhood of a. It is well known that a distribution supported at a single point a is a finite linear combination of derivatives of Dirac's measure at the point a. In turn, this implies that $(\widehat{\alpha}*\widehat{\chi})(x)=e^{-2\pi i a x}q(x)$, where q is a polynomial. But $\widehat{\alpha}*\widehat{\chi}\in L^p(\mathbb{R})$, so this is possible only if $\widehat{\alpha}*\widehat{\chi}$ is zero, and hence $\alpha\cdot\chi$ is zero. We conclude that α must vanish in a neighborhood of any point $a\in\operatorname{supp}(\alpha)$, a contradiction unless $\alpha=0$. ∎

2.8. We will use the known fact that in the space $L^p(\mathbb{R})$, $p>1$, there exist *complete systems* formed by uniformly discrete translates of a single function.

It was proved in [1] that for every $p>2$, there is a function $g\in L^p(\mathbb{R})$ whose translates by the positive integers $\{g(x-n)\}$, $n=1,2,3,\dots$, span the whole space $L^p(\mathbb{R})$, that is, these translates are complete in $L^p(\mathbb{R})$. A similar result was proved also in the space $C_0(\mathbb{R})$. On the other hand, no system of integer translates can be complete in $L^p(\mathbb{R})$ for $1\leqslant p\leqslant 2$ (see, e.g., Example 11.2 and Corollary 12.26 in [17]).

However, it was proved in [14] that for any "small perturbation" of the integers,

$$\lambda_n = n + \alpha_n, \quad 0 \neq \alpha_n \to 0 \quad (|n| \to +\infty) \tag{2.7}$$

there exists $g \in L^2(\mathbb{R})$ such that the system $\{g(x - \lambda_n)\}$, $n \in \mathbb{Z}$, is complete in $L^2(\mathbb{R})$. It was moreover shown in [16] that if the perturbations are exponentially small, i.e., if

$$0 < |\alpha_n| < C r^{|n|}, \quad n \in \mathbb{Z}, \tag{2.8}$$

for some $0 < r < 1$ and $C > 0$, then g can be chosen in the Schwartz class.

More recently, by a development of the approach from [16], the latter result was extended to $L^p(\mathbb{R})$ spaces [19], namely, it was proved that there is a Schwartz function g such that if the sequence $\{\lambda_n\}$, $n \in \mathbb{Z}$, satisfies (2.7) and (2.8) then the system $\{g(x - \lambda_n)\}$, $n \in \mathbb{Z}$, is complete in $L^p(\mathbb{R})$ for every $p > 1$ (see also [18]).

In a recent paper [9], a different approach was given for constructing a function g which spans the space $L^p(\mathbb{R})$, $p > 1$, by uniformly discrete translates. In fact, this approach allows to use only positive translates, and moreover, the completeness remains true for any subsystem obtained by the removal of a finite number of elements.

Theorem 2.4 (see Theorem 1.1 in [9]). *There is a real sequence $\{\lambda_n\}_{n=1}^{\infty}$ satisfying $\lambda_n = n + o(1)$, and there is a Schwartz function g on $\mathbb{R}$, such that for any N, the system*

$$\{g(x - \lambda_n)\}, \quad n > N,$$

is complete in the space $L^p(\mathbb{R})$ for every $p > 1$.

3. Localization lemma

3.1. A key ingredient in our proof of Theorem 1.1 is the following lemma.

Lemma 3.1. *Let $p > (1 + \sqrt{5})/2$. Given any $\varepsilon > 0$, there exist two real trigonometric polynomials P and γ with integer spectrum, such that*

(i) $\widehat{P}(0) = 0$, $\|\widehat{P}\|_\infty < \varepsilon$;

(ii) $\|\gamma - 1\|_{A^p(\mathbb{T})} < \varepsilon$;

(iii) $\|\gamma \cdot P - 1\|_{A^p(\mathbb{T})} < \varepsilon$;

(iv) $\max_l \|\gamma \cdot S_l(P)\|_{A^p(\mathbb{T})} < C_p$,

where C_p is a constant depending only on p.

Remarks. (1) We note that the conditions (i) and (ii)+(iii) "go against" each other, as (i) says that P is "small", while (ii)+(iii) imply that P should be "nearly" one. For instance, it is easy to see that the lemma fails for $p = 1$, since in this case (ii)+(iii) imply that P must be uniformly close to one, and hence $\widehat{P}(0)$ cannot be small.

(2) Moreover, according to a recent preprint [2], the lemma fails for $p < (1 + \sqrt{5})/2$. While this leaves open the question of whether Theorem 1.1 holds for every $p > 1$, it indicates that proving this would require new ideas.

(3) The case $p = 2$ is simpler since we have $A^2(\mathbb{T}) = L^2(\mathbb{T})$ by Parseval's theorem; in this case, the lemma follows from Lemma 2.2 in [15]. However, for $p < 2$, the Parseval

theorem is no longer available, and our proof below requires some additional ideas and a more careful analysis of the $A^p(\mathbb{T})$ norm.

(4) The proof of Lemma 3.1 given below establishes condition (iv) with an absolute constant C_p which in fact does not depend on p.

3.2. The following assertion will be used in our proof of Lemma 3.1.

Lemma 3.2. *Let $P_0, \dots, P_{N-1}$ be trigonometric polynomials with integer spectrum, and let ν be a positive integer, $\nu > 2\deg(P_j)$, $0 \leqslant j \leqslant N-1$. Define*

$$P(t) := \prod_{j=0}^{N-1} P_j(\nu^j t).$$

Then we have

$$\widehat{P}(0) = \prod_{j=0}^{N-1} \widehat{P}_j(0) \quad \textit{and} \quad \|P\|_{A^p(\mathbb{T})} = \prod_{j=0}^{N-1} \|P_j\|_{A^p(\mathbb{T})}.$$

Indeed, expanding each P_j as a Fourier sum yields

$$P(t) = \sum \Big[\prod_{j=0}^{N-1} \widehat{P}_j(k_j) \Big] e^{2\pi i t \sum_{j=0}^{N-1} k_j \nu^j} \tag{3.1}$$

where the sum goes through all integer vectors $(k_0, k_1, \dots, k_{N-1})$ with $|k_j| \leqslant \deg(P_j)$. The condition $\nu > 2\deg(P_j)$, $0 \leqslant j \leqslant N-1$, ensures that the exponentials in (3.1) have distinct frequencies, so that (3.1) is the Fourier expansion of P. The conclusion of the lemma now follows in a straightforward manner.

3.3 Proof of Lemma 3.1. The idea of the proof is inspired by the "separation of spectra" technique, see Section 2.3 of [8]. We decompose the polynomial P as a sum of small elementary pieces, whose Fourier spectra are localized on disjoint intervals. In turn, the polynomial γ is obtained as a Riesz-type product, which ensures that the set where P attains large values is essentially localized away from the support of γ.

We now turn to the details of the proof. It is divided into several steps.

3.3.1. Due to monotonicity of the A^p norms, we may assume that $p < 2$. Let us choose $\delta = \delta(\varepsilon, p) > 0$ small enough, to be specified later. We then choose and fix $0 < h < 1/3$ and a positive integer N (both depending on ε, δ and p) such that

$$N > \varepsilon^{-1}, \quad (1 + 3^p h^{p-1})^N < 1 + \delta, \quad (1 - 3h)^N > 1 - \delta, \quad 4^p N^{-p} h^{-1} < 1 - \delta. \tag{3.2}$$

We show that such a choice of h and N exists. Indeed, denote $\eta := 3^{-p} \log(1 + \delta)$, and let $h = h(N, \eta, p)$ be defined by the condition $N = \eta \cdot h^{1-p}$. Then for N sufficiently large we have the inequalities

$$N > \varepsilon^{-1}; \quad (1 + 3^p h^{p-1})^N < \exp(3^p \eta) = 1 + \delta; \quad (1 - 3h)^N > 1 - 3hN > 1 - \delta,$$

and lastly, using the assumption $p > (1 + \sqrt{5})/2$, we can also ensure that

$$N^{-p} h^{-1} = \eta^{-p} h^{p(p-1)-1} < 4^{-p}(1 - \delta).$$

3.3.2. Now observe that $(1-\tau_h)\cdot(1-h^{-1}\Delta_h) = 1-\tau_h$. Let f and g be Fourier partial sums of $1-\tau_h$ and $1-h^{-1}\Delta_h$, respectively, of sufficiently high order such that

$$\|f\cdot g - f\|_A < \delta. \tag{3.3}$$

Next, choose a positive integer ν satisfying

$$\nu > 2(\deg(f)+\deg(g)), \tag{3.4}$$

and define

$$\gamma(t) := \prod_{j=0}^{N-1} f(\nu^j t) \quad \text{and} \quad P(t) := \frac{1}{N}\sum_{j=0}^{N-1} g(\nu^j t).$$

We will check that the conditions (i)–(iv) are satisfied.

3.3.3. First we note that $\widehat{P}(0)=0$. Also, due to (3.4),

$$\|\widehat{P}\|_\infty = \frac{1}{N}\|\widehat{g}\|_\infty \leqslant \frac{1}{N}\sup_{n\neq 0} h^{-1}\widehat{\Delta}_h(n) \leqslant \frac{1}{N}\int_{\mathbb{T}} h^{-1}\Delta_h(t)dt = \frac{1}{N} < \varepsilon,$$

where the last inequality is due to (3.2). Thus we obtain condition (i).

3.3.4. Next, due to (2.2), we have

$$\|f\|_{A^p}^p \leqslant \|1-\tau_h\|_{A^p}^p = (1-3h)^p + \sum_{n\neq 0}|\widehat{\tau}_h(n)|^p < 1 + 3^p h^{p-1}, \tag{3.5}$$

hence using (3.2) and (3.4), we obtain

$$\|\gamma\|_{A^p}^p = \big(\|f\|_{A^p}^p\big)^N < (1+3^p h^{p-1})^N < 1+\delta. \tag{3.6}$$

Also, again using (3.2) and (3.4), we have

$$\widehat{\gamma}(0) = \widehat{f}(0)^N = (1-3h)^N > 1-\delta, \tag{3.7}$$

and so it follows from (3.6) and (3.7) that

$$\|\gamma - 1\|_{A^p}^p = \|\gamma\|_{A^p}^p - \widehat{\gamma}(0)^p + (1-\widehat{\gamma}(0))^p < (1+\delta) - (1-\delta)^p + \delta^p < (\varepsilon/2)^p, \tag{3.8}$$

provided that $\delta = \delta(\varepsilon, p)$ is sufficiently small. So condition (ii) follows.

3.3.5. Next, we have

$$\gamma(t)(P(t)-1) = \frac{1}{N}\sum_{j=0}^{N-1}(f(\nu^j t)g(\nu^j t) - f(\nu^j t))\prod_{k\neq j} f(\nu^k t), \tag{3.9}$$

and thus, recalling (3.3), (3.4), (3.5) and (3.6), this implies

$$\|\gamma\cdot(P-1)\|_{A^p} \leqslant \|f\cdot g - f\|_A \cdot \big(\|f\|_{A^p}\big)^{N-1} < \delta(1+\delta)^{1/p} < \varepsilon/2, \tag{3.10}$$

for $\delta = \delta(\varepsilon, p)$ small enough. We conclude from (3.8) and (3.10) that

$$\|\gamma\cdot P - 1\|_{A^p} \leqslant \|\gamma\cdot(P-1)\|_{A^p} + \|\gamma - 1\|_{A^p} < \varepsilon/2 + \varepsilon/2 = \varepsilon, \tag{3.11}$$

and thus condition (iii) holds.

3.3.6. Let us finally check that also condition (iv) is satisfied. Any partial sum $S_l(P)$ can be decomposed as $S_l(P)(t) = A(t) + B(t)$, where

$$A(t) := \frac{1}{N}\sum_{j=0}^{s-1} g(\nu^j t) \quad \text{and} \quad B(t) := \frac{1}{N} S_m(g)(\nu^s t).$$

Using the same argument as in (3.9) and (3.10), we can obtain

$$\|\gamma \cdot (A - s/N)\|_{A^p} \leqslant \frac{s}{N} \cdot \|f \cdot g - f\|_A \cdot \big(\|f\|_{A^p}\big)^{N-1} < \delta(1+\delta)^{1/p} < \varepsilon/2,$$

and consequently,

$$\|\gamma \cdot A\|_{A^p} \leqslant \|\gamma \cdot (A - s/N)\|_{A^p} + \frac{s}{N} \cdot \|\gamma\|_{A^p} < \varepsilon/2 + (1+\varepsilon) < 2.$$

Next, we have

$$\gamma(t)B(t) = \frac{1}{N}(f \cdot S_m(g))(\nu^s t)\prod_{k \neq s} f(\nu^k t),$$

and therefore, due to (3.4),

$$\|\gamma \cdot B\|_{A^p} = \frac{1}{N}\|f \cdot S_m(g)\|_{A^p} \cdot \big(\|f\|_{A^p}\big)^{N-1}. \tag{3.12}$$

We note using (2.2) that

$$\|f\|_A \leqslant \|1 - \tau_h\|_A \leqslant 1 + \|\tau_h\|_A \leqslant 4, \tag{3.13}$$

and due to (2.1),

$$\|S_m(g)\|_{A^p}^p \leqslant \|g\|_{A^p}^p \leqslant \|1 - h^{-1}\Delta_h\|_{A^p}^p = h^{-p}\sum_{n \neq 0} \hat{\Delta}_h(n)^p < h^{-p}h^{p-1} = h^{-1}. \tag{3.14}$$

Therefore, using (3.12), (3.13), (3.14) and (3.2),

$$\|\gamma \cdot B\|_{A^p}^p \leqslant 4^p N^{-p} h^{-1}(1+\delta) < (1-\delta)(1+\delta) < 1.$$

We conclude that

$$\|\gamma \cdot S_l(P)\|_{A^p} \leqslant \|\gamma \cdot A\|_{A^p} + \|\gamma \cdot B\|_{A^p} < 2 + 1 = 3.$$

Thus condition (iv) is established and the lemma is proved. ■

4. Schauder frames of weighted exponentials

In this section, we prove Theorem 1.1. First we note that the Fourier transform is an isometric isomorphism $A^p(\mathbb{R}) \to L^p(\mathbb{R})$, which allows us to reformulate Theorem 1.1 as a result about Schauder frames of weighted exponentials in $A^p(\mathbb{R})$.

Theorem 4.1. *Let $p > (1+\sqrt{5})/2$. There exist $w \in A^p(\mathbb{R})$, $\{h_j^*\} \subset (A^p(\mathbb{R}))^*$, and a uniformly discrete real sequence $\{\lambda_j\}_{j=1}^\infty$ satisfying $|\lambda_j| = n_j + o(1)$, where n_j are positive integers, $0 < n_1 < n_2 < \dots$, such that every $f \in A^p(\mathbb{R})$ admits a series expansion*

$$f(t) = \sum_{j=1}^{\infty} h_j^*(f)\, w(t)\, e^{2\pi i \lambda_j t} \tag{4.1}$$

convergent in the $A^p(\mathbb{R})$ norm.

Note that $\{n_j\}$ is allowed to be a *subsequence* of the positive integers, since we may add more elements with zeros as coefficient functionals, and the series expansion (4.1) will remain valid. Hence Theorem 1.1 follows from Theorem 4.1.

The restriction $p > (1+\sqrt{5})/2$ is needed so that we can invoke Lemma 3.1, while otherwise we only use the assumption $p > 1$ in the proof.

4.1. We begin the proof by an application of Theorem 2.4, which implies the existence of a function $u_0 \in \mathcal{S}(\mathbb{R})$ and a real sequence $\{\sigma(n)\}_{n=1}^\infty$ satisfying

$$\sigma(n) = n + o(1), \quad n \to +\infty, \tag{4.2}$$

such that for every N the system $\{u_0(t) e^{2\pi i \sigma(n) t}\}$, $n > N$, is complete in $A^p(\mathbb{R})$.

Next, we choose a normalized Schauder basis $\{\varphi_k\}_{k=1}^\infty$ for the space $A^p(\mathbb{R})$. (For example, one may take $\{\widehat{\varphi}_k\}_{k=1}^\infty$ to be the normalized basis of Haar functions in $L^p(\mathbb{R})$).

We will now construct by induction a sequence of Schwartz functions $\{u_k\}$ on $\mathbb{R}$. We will perform the construction in such a way that for each k and every N, the system

$$\{u_k(t) e^{2\pi i \sigma(n) t}\}, \quad n > N, \tag{4.3}$$

is complete in the space $A^p(\mathbb{R})$. The construction is done as follows.

At the kth step of the induction, given any $\eta_k > 0$ and any positive integer N_k, we use the completeness of the system $\{u_{k-1}(t)\, e^{2\pi i \sigma(n) t}\}$, $n > N_k$, to find a trigonometric polynomial

$$Q_k(t) = \sum_{N_k < n < N_k'} d_{n,k}\, e^{2\pi i \sigma(n) t}$$

such that

$$\|\varphi_k - u_{k-1} \cdot Q_k\|_{A^p(\mathbb{R})} < \eta_k. \tag{4.4}$$

We choose a small number $\varepsilon_k > 0$ so that

$$\varepsilon_k \cdot (1 + \|u_{k-1}\|_*) \Big(1 + \|\widehat{Q}_k\|_1 + \sum_{j=1}^{k-1} \|\widehat{P}_j\|_1 \cdot \|\widehat{Q}_j\|_1 \Big) < 2^{-k} \eta_k, \tag{4.5}$$

and apply Lemma 3.1 in order to find real trigonometric polynomials P_k and γ_k with integer spectrum, such that

(i) $\widehat{P}_k(0) = 0$, $\|\widehat{P}_k\|_\infty < \varepsilon_k$;

(ii) $\|\gamma_k - 1\|_{A^p(\mathbb{T})} < \varepsilon_k$;

(iii) $\|\gamma_k \cdot P_k - 1\|_{A^p(\mathbb{T})} < \varepsilon_k$;

(iv) $\max_l \|\gamma_k \cdot S_l(P_k)\|_{A^p(\mathbb{T})} < C_p$.

We now choose a large positive integer ν_k (to be specified later) and set

$$\tilde{P}_k(t) := P_k(\nu_k t) \quad \text{and} \quad \tilde{\gamma}_k(t) := \gamma_k(\nu_k t).$$

Define $u_k := u_{k-1} \cdot \tilde{\gamma}_k$, which is a Schwartz function on $\mathbb{R}$. We claim that for every N, the system (4.3) is complete in the space $A^p(\mathbb{R})$. Indeed, let α be a tempered distribution belonging to the dual space $(A^p(\mathbb{R}))^* = A^{p'}(\mathbb{R})$, $p' = p/(p-1)$, and suppose that α annihilates the system (4.3). This means that $\alpha \cdot \tilde{\gamma}_k$, which also lies in $A^{p'}(\mathbb{R})$, annihilates the system $\{u_{k-1}(t) e^{2\pi i \sigma(n) t}\}, n > N$. By the completeness of the latter system in $A^p(\mathbb{R})$, it follows that $\alpha \cdot \tilde{\gamma}_k = 0$. In turn, using Lemma 2.3, we conclude that $\alpha = 0$. Hence the system (4.3) is complete in $A^p(\mathbb{R})$.

4.2. It follows from (ii) and (4.5) that

$$\|u_k - u_{k-1}\|_{A^p(\mathbb{R})} = \|u_{k-1}(\tilde{\gamma}_k - 1)\|_{A^p(\mathbb{R})} \leqslant \|u_{k-1}\|_* \|\gamma_k - 1\|_{A^p(\mathbb{T})} < 2^{-k},$$

hence the sequence u_k converges in the space $A^p(\mathbb{R})$ to some element $w \in A^p(\mathbb{R})$.

4.3. Next we claim that the estimate

$$\|\varphi_k - w \cdot \tilde{P}_k \cdot Q_k\|_{A^p(\mathbb{R})} < 3\eta_k \tag{4.6}$$

holds for all k. Indeed,

$$\|\varphi_k - w \cdot \tilde{P}_k \cdot Q_k\|_{A^p(\mathbb{R})} \leqslant \|\varphi_k - u_{k-1} \cdot Q_k\|_{A^p(\mathbb{R})} \tag{4.7}$$

$$+ \|u_{k-1} \cdot Q_k \cdot (1 - \tilde{\gamma}_k \cdot \tilde{P}_k)\|_{A^p(\mathbb{R})} + \|(u_k - w) \cdot \tilde{P}_k \cdot Q_k\|_{A^p(\mathbb{R})}. \tag{4.8}$$

The right-hand side of (4.7) is less than η_k, due to (4.4). To estimate the first term in (4.8), we apply inequalities (2.4) and (2.5), and use (iii) and (4.5), to obtain

$$\|u_{k-1} \cdot Q_k \cdot (1 - \tilde{\gamma}_k \cdot \tilde{P}_k)\|_{A^p(\mathbb{R})} \leqslant \|u_{k-1}\|_* \cdot \|\hat{Q}_k\|_1 \cdot \|1 - \gamma_k \cdot P_k\|_{A^p(\mathbb{T})} < \eta_k.$$

It remains to estimate the second term in (4.8). We observe that for any fixed k we have $u_j \cdot \tilde{P}_k \cdot Q_k \to w \cdot \tilde{P}_k \cdot Q_k$ as $j \to +\infty$ in the $A^p(\mathbb{R})$ norm, hence

$$\|(u_k - w) \cdot \tilde{P}_k \cdot Q_k\|_{A^p(\mathbb{R})} \leqslant \sum_{j=k+1}^{\infty} \|(u_{j-1} - u_j) \cdot \tilde{P}_k \cdot Q_k\|_{A^p(\mathbb{R})} \tag{4.9}$$

$$= \sum_{j=k+1}^{\infty} \|u_{j-1} \cdot (1 - \tilde{\gamma}_j) \cdot \tilde{P}_k \cdot Q_k\|_{A^p(\mathbb{R})} \tag{4.10}$$

$$\leqslant \sum_{j=k+1}^{\infty} \|u_{j-1}\|_* \cdot \|\hat{Q}_k\|_1 \cdot \|\hat{P}_k\|_1 \cdot \|1 - \gamma_j\|_{A^p(\mathbb{T})} \tag{4.11}$$

$$\leqslant \sum_{j=k+1}^{\infty} 2^{-j} \eta_j < \eta_k, \tag{4.12}$$

again using (ii) and (4.5), and assuming (as we may do) that the sequence $\{\eta_k\}$ is decreasing. The estimate (4.6) thus follows.

4.4. Let $\{\varphi_k^*\}_{k=1}^\infty$ be the sequence of continuous linear functionals on $A^p(\mathbb{R})$ which is biorthogonal to the Schauder basis $\{\varphi_k\}_{k=1}^\infty$. It follows from (4.6) that if we choose the sequence $\{\eta_k\}$ to satisfy $3\sum_{k=1}^\infty \|\varphi_k^*\| \cdot \eta_k < 1$, then the system

$$\{w \cdot \tilde{P}_k \cdot Q_k\}_{k=1}^\infty \tag{4.13}$$

forms another Schauder basis in the space $A^p(\mathbb{R})$ (see Section 1.9 of [23]). Hence every $f \in A^p(\mathbb{R})$ has a series expansion

$$f = \sum_{k=1}^\infty \psi_k(f)\, w \cdot \tilde{P}_k \cdot Q_k, \tag{4.14}$$

where $\{\psi_k\}$ are the continuous linear functionals on $A^p(\mathbb{R})$ which are biorthogonal to the system (4.13). Moreover, recall that we have chosen the Schauder basis $\{\varphi_k\}_{k=1}^\infty$ to be normalized, so it follows from (4.6) that the norms of the elements of the system (4.13) are bounded from below. This implies that

$$\sup_k |\psi_k(f)| \leqslant K\|f\|_{A^p(\mathbb{R})} \quad \text{and} \quad \lim_{k\to\infty} \psi_k(f) = 0, \tag{4.15}$$

where K is a constant not depending on f (see Section 1.6 of [23]).

4.5. Notice that we have

$$\tilde{P}_k(t)Q_k(t) = P_k(\nu_k t)\, Q_k(t) = \sum_{m\neq 0} \hat{P}_k(m)\, Q_{k,m}(t), \tag{4.16}$$

where $Q_{k,m}$ are trigonometric polynomials defined by

$$Q_{k,m}(t) := Q_k(t)\, e^{2\pi i m \nu_k t} = \sum_{N_k<n<N_k'} d_{n,k}\, e^{2\pi i(\sigma(n)+m\nu_k)t}. \tag{4.17}$$

If we choose the sequence $\{\nu_k\}$ increasing sufficiently fast, then the spectra of the polynomials $Q_{k,m}$ follow each other, meaning that

$$\max \operatorname{spec}(Q_{k,m_1}) < \min \operatorname{spec}(Q_{k,m_2}), \quad m_1 < m_2.$$

Moreover, there is a positive, increasing sequence $\{R_k\}$ such that

$$\operatorname{spec}(\tilde{P}_k \cdot Q_k) = \bigcup_{m\in\operatorname{spec}(P_k)} \operatorname{spec}(Q_{k,m}) \subset (-R_{k+1}, -R_k) \cup (R_k, R_{k+1}).$$

We now define

$$\Lambda := \bigcup_{k=1}^\infty \operatorname{spec}(\tilde{P}_k \cdot Q_k);$$

then each point $\lambda \in \Lambda$ has a unique representation as

$$\lambda = m \cdot \nu_k + \sigma(n), \quad k \geqslant 1, \quad m \in \operatorname{spec}(P_k), \quad N_k < n < N_k'. \tag{4.18}$$

We can use (4.2) to choose $\{N_k\}$ increasing fast enough, so that (say)

$$|\sigma(n) - n| < \frac{1}{10} \cdot k^{-1}, \quad n > N_k.$$

This implies that Λ is a uniformly discrete set. Moreover, if the elements of Λ are enumerated as $\{\lambda_j\}_{j=1}^{\infty}$ by increasing modulus, that is, $0 < |\lambda_1| \leqslant |\lambda_2| \leqslant \dots$, then there are positive integers n_j such that

$$|\lambda_j| = n_j + o(1), \quad j \to +\infty.$$

(We note that each λ_j may be either positive or negative, since the polynomials P_k have both positive and negative spectra).

We now observe that in fact we have $0 < n_1 < n_2 < \dots$. Indeed, the fact that the polynomials Q_k have positive spectra implies that if $\lambda \in \Lambda$ is given by (4.18), then $|\lambda| = |m| \cdot \nu_k + \operatorname{sign}(m) \cdot \sigma(n)$, where $\operatorname{sign}(m)$ is $+1$ or -1 according to whether m is positive or negative. Hence, if λ_j and λ_l are two distinct points of Λ then $n_j \neq n_l$.

4.6. For $\lambda \in \Lambda$ given by (4.18) we define $h_\lambda^* \in (A^p(\mathbb{R}))^*$ by

$$h_\lambda^* := d_{n,k} \widehat{P}_k(m) \psi_k.$$

We will prove that each $f \in A^p(\mathbb{R})$ admits a series representation

$$f(t) = \sum_{j=1}^{\infty} h_{\lambda_j}^*(f) w(t) e^{2\pi i \lambda_j t} \tag{4.19}$$

where the convergence is in the $A^p(\mathbb{R})$ norm. This will establish Theorem 4.1, and as a consequence, Theorem 1.1 will also be proved.

To prove the representation (4.19), we first observe that any partial sum of the series can be decomposed as $S' + S'' + S'''$ where

$$\begin{aligned} S'(t) &= \sum_{s=1}^{k-1} \psi_s(f) w(t) \tilde{P}_s(t) Q_s(t), \\ S''(t) &= \psi_k(f) w(t) Q_k(t) S_l(P_k)(\nu_k t), \\ S'''(t) &= \psi_k(f) w(t) \big[\widehat{P}_k(l+1) S_r(Q_{k,l+1})(t) + \widehat{P}_k(-(l+1)) S_r(Q_{k,-(l+1)})(t)\big], \end{aligned} \tag{4.20}$$

for some k, l and r. Indeed, S' consists of the series terms with λ_j belonging to entire "blocks" of the form $\operatorname{spec}(\tilde{P}_s \cdot Q_s)$, $1 \leqslant s \leqslant k-1$. Similarly, recalling (4.16) and (4.17), one can see that S'' consists of the terms with λ_j belonging to entire "sub-blocks" of the form $\operatorname{spec}(Q_{k,m})$, $|m| \leqslant l$. Finally, S''' consists of the remaining terms with λ_j belonging to a part of the last two sub-blocks $\operatorname{spec}(Q_{k,l+1})$ and $\operatorname{spec}(Q_{k,-(l+1)})$.

We have $\|f - S'\|_{A^p(\mathbb{R})} = o(1)$ as $k \to \infty$ due to (4.14).

In order to estimate $\|S''\|_{A^p(\mathbb{R})}$ we denote $\tilde{S}_{k,l}(t) := S_l(P_k)(\nu_k t)$. Then

$$\|w \tilde{S}_{k,l} Q_k\|_{A^p(\mathbb{R})} \leqslant \|(w - u_k) \tilde{S}_{k,l} Q_k\|_{A^p(\mathbb{R})} + \|u_{k-1} \tilde{\gamma}_k \tilde{S}_{k,l} Q_k\|_{A^p(\mathbb{R})}. \tag{4.21}$$

The first summand on the right-hand side can be estimated similarly to the inequalities (4.9)–(4.12), using the fact that $\|\widehat{S}_{k,l}\|_1 \leqslant \|\widehat{P}_k\|_1$. To estimate the second summand, we use Lemma 2.2 to conclude that if ν_k is chosen sufficiently large, then

$$\|u_{k-1}\,\tilde{\gamma}_k\,\tilde{S}_{k,l}\,Q_k\|_{A^p(\mathbb{R})} < \|u_{k-1}\,Q_k\|_{A^p(\mathbb{R})}\,\|\gamma_k\,S_{k,l}\|_{A^p(\mathbb{T})} + 1.$$

It follows from (4.4) that $\|u_{k-1}\,Q_k\|_{A^p(\mathbb{R})} < 2$, since the sequence $\{\varphi_k\}$ is normalized in $A^p(\mathbb{R})$, while according to (iv) we have $\|\gamma_k\,S_{k,l}\|_{A^p(\mathbb{T})} < C_p$. We conclude using (4.21) that $\|w\,\tilde{S}_{k,l}\,Q_k\|_{A^p(\mathbb{R})} = O(1)$. In turn, together with (4.15) and (4.20), this implies the desired estimate $\|S''\|_{A^p(\mathbb{R})} = o(1)$ as $k \to \infty$.

Finally, we estimate $\|S'''\|_{A^p(\mathbb{R})}$ using (i) and the inequalities (2.4), (4.5) and obtain

$$\|S'''\|_{A^p(\mathbb{R})} \leqslant 2\,|\psi_k(f)|\,\|\widehat{P}_k\|_\infty\,\|\widehat{Q}_k\|_1\,\|w\|_{A^p(\mathbb{R})} \leqslant 2\eta_k\,|\psi_k(f)|\,\|w\|_{A^p(\mathbb{R})},$$

which shows that $\|S'''\|_{A^p(\mathbb{R})} = o(1)$ as $k \to \infty$ as well.

We conclude that (4.19) indeed holds, which completes the proof of Theorem 4.1.

As a consequence, Theorem 1.1 is also established.

5. Schauder frames of translates and nonnegativity

5.1. In [21], the following question was considered: do there exist (unconditional or not) Schauder bases or Schauder frames in the space $L^p(\mathbb{R})$ consisting of nonnegative functions? It turns out that it is not difficult to construct a Schauder frame formed by nonnegative functions, while unconditional Schauder frames of nonnegative functions do not exist in any $L^p(\mathbb{R})$ space, see again [21].

In the paper [7] (which was written after [21]), some proofs from [21] were simplified, and also a Schauder basis consisting of nonnegative functions in $L^1(\mathbb{R})$ was constructed. A Schauder basis of nonnegative functions in $L^2(\mathbb{R})$ was constructed in [6]. The existence of such a basis in $L^p(\mathbb{R})$, $p \neq 1, 2$, remains open.

Motivated by the recent interest in nonnegative coordinate systems in $L^p(\mathbb{R})$ spaces, we consider the following question: does there exist a Schauder frame in the space $L^p(\mathbb{R})$ formed by a uniformly discrete sequence of translates of a *nonnegative* function?

In this section, our goal is to show that this additional requirement of nonnegativity can indeed be achieved in our main result, namely:

Theorem 5.1. *The function g in Theorem 1.1 can be chosen nonnegative.*

To establish this, we will make certain modifications to the proof of Theorem 1.1.

5.2. First, we recall that our construction in Section 4 began with an application of Theorem 2.4, which yields a real sequence $\{\lambda_n\}_{n=1}^{\infty}$ satisfying $\lambda_n = n + o(1)$, and a Schwartz function g on $\mathbb{R}$, such that for any N, the system

$$\{g(x - \lambda_n)\}, \quad n > N,$$

is complete in the space $L^p(\mathbb{R})$, $p > 1$. So first, we need the following result:

Lemma 5.2. *The function g in Theorem 2.4 can be chosen nonnegative.*

The proof, based on adapting the approach in [9], is given in Section 4 of [12].

5.3. The second ingredient, which enables us to construct a nonnegative function g in Theorem 1.1, is the following lemma.

Lemma 5.3. *Let a and h be two positive real numbers, $2h < a < 1/2 - 2h$. Then there is a nonnegative function $\varphi \in A(\mathbb{T})$ with the following properties*:

(i) $\widehat{\varphi}(0) = 1$, $\widehat{\varphi}(n) \geqslant 0$ *for all* $n \in \mathbb{Z}$;

(ii) φ *vanishes on the set* $[-a-h, -a+h] \cup [a-h, a+h]$;

(iii) $\|\varphi - 1\|_{A^p(\mathbb{T})} \leqslant 12 \cdot h^{(p-1)/p}$ *for every* $p \geqslant 1$.

Proof. We consider the function

$$\psi(t) := 6 \cdot \Delta_h(t) - \tau_h(t+a) - \tau_h(t-a), \tag{5.1}$$

which is in $A(\mathbb{T})$. We observe that $\tau_h = \Delta_h * (\delta_{-h} + \delta_0 + \delta_h)$, where δ_t denotes the Dirac measure at a point $t \in \mathbb{T}$. Hence

$$\psi = 6 \cdot \Delta_h - \Delta_h * (\delta_{-h} + \delta_0 + \delta_h) * (\delta_{-a} + \delta_a) = \Delta_h * (6 \cdot \delta_0 - \nu), \tag{5.2}$$

where $\nu = (\delta_{-h} + \delta_0 + \delta_h) * (\delta_{-a} + \delta_a)$. Then ν is a positive measure on $\mathbb{T}$ with total mass $\widehat{\nu}(0) = 6$, and the Fourier coefficients $\widehat{\nu}(n)$ are all real valued. This implies that

$$-6 \leqslant \widehat{\nu}(n) \leqslant 6, \quad n \in \mathbb{Z}.$$

It now follows from (5.2) that $\widehat{\psi}(n) = \widehat{\Delta}_h(n)(6 - \widehat{\nu}(n))$, and since both terms in the product are nonnegative, we can conclude that $\widehat{\psi}(n) \geqslant 0$ for all $n \in \mathbb{Z}$. Furthermore,

$$\|\psi\|^p_{A^p(\mathbb{T})} = \sum_{n\in\mathbb{Z}} \widehat{\psi}(n)^p = \sum_{n\in\mathbb{Z}} \widehat{\Delta}_h(n)^p (6 - \widehat{\nu}(n))^p \leqslant 12^p \|\Delta_h\|^p_{A^p(\mathbb{T})} \leqslant 12^p h^{p-1} \tag{5.3}$$

for every $p \geqslant 1$, due to (2.1). Finally, set $\varphi := 1 + \psi$. Then it is obvious that condition (i) holds. It follows from (5.1) that the function φ is nonnegative and satisfies (ii). Moreover, we obtain (iii) as a consequence of (5.3). The lemma is thus proved. ∎

Remark 5.4. One can show that condition (ii) implies that $\|\varphi - 1\|_{A^p(\mathbb{T})} \geqslant h^{(p-1)/p}$ for every $p \geqslant 1$, so the estimate (iii) is sharp up to the numerical value of the constant.

5.4. We now use Lemma 5.3 to strengthen Lemma 3.1 as follows.

Lemma 5.5. *Let $p > (1 + \sqrt{5})/2$. Given any $\varepsilon > 0$, there exist two real trigonometric polynomials P and γ with integer spectrum, such that*

(i) $\widehat{\gamma}(0) = 1$, $\widehat{\gamma}(n) \geqslant 0$ *for all* $n \in \mathbb{Z}$;

(ii) $\widehat{P}(0) = 0$ *and* $\|\widehat{P}\|_\infty < \varepsilon$;

(iii) $\|\gamma - 1\|_{A^p(\mathbb{T})} < \varepsilon$;

(iv) $\|\gamma \cdot P - \mathbf{1}\|_{A^p(\mathbb{T})} < \varepsilon$;

(v) $\max_l \|\gamma \cdot S_l(P)\|_{A^p(\mathbb{T})} \leqslant 3$.

The novel part of this result compared to Lemma 3.1 is the addition of property (i), that is, the requirement that γ have nonnegative Fourier coefficients.

Proof of Lemma 5.5. We first choose a small $\delta = \delta(\varepsilon, p) > 0$, and then choose and fix $0 < h < 1/8$ and a positive integer N (both depending on ε, δ and p) such that

$$N > \varepsilon^{-1}, \quad (1 + 12^p h^{p-1})^N < 1 + \delta \quad \text{and} \quad 13^p N^{-p} h^{-1} < 1 - \delta.$$

To see that such a choice of h and N exists, it suffices to let $h = h(N, \delta, p)$ be defined by the condition $N = 12^{-p} h^{1-p} \log(1 + \delta)$, and take N to be sufficiently large.

We now choose an arbitrary number a satisfying $2h < a < 1/2 - 2h$. Let φ be the function of Lemma 5.3, and let $\psi = 1 - h^{-1} \Delta_h * \frac{1}{2}(\delta_{-a} + \delta_a)$, where again δ_t is the Dirac measure at a point $t \in \mathbb{T}$. We observe that $\varphi \cdot \psi = \varphi$. Let f and g be Fourier partial sums of φ and ψ respectively, of sufficiently high order such that $\|f \cdot g - f\|_A < \delta$.

Next, choose a positive integer ν satisfying $\nu > 2(\deg(f) + \deg(g))$, and define

$$\gamma(t) := \prod_{j=0}^{N-1} f(\nu^j t) \quad \text{and} \quad P(t) := \frac{1}{N} \sum_{j=0}^{N-1} g(\nu^j t).$$

The proof can now continue in the same way as in Lemma 3.1, and it follows that this choice of γ and P indeed satisfies all the conditions (i)–(v) of Lemma 5.5. ∎

5.5. We can now adjust the proof of Theorem 1.1 given in Section 4 as follows. Recall that in the proof we have constructed by induction a sequence of Schwartz functions u_k. Thanks to Lemma 5.2, we may now assume that the first Schwartz function u_0 has a nonnegative Fourier transform $\hat{u}_0$. Next, at the kth step of the induction we now use Lemma 5.5 instead of Lemma 3.1, which yields a trigonometric polynomial γ_k with nonnegative Fourier coefficients. As a consequence, it follows that the function $u_k := u_{k-1} \cdot \tilde{\gamma}_k$ is a Schwartz function whose Fourier transform $\hat{u}_k$ is nonnegative.

In turn, the sequence u_k converges in the space $A^p(\mathbb{R})$ to an element $w \in A^p(\mathbb{R})$ whose Fourier transform $\hat{w}$ is a nonnegative function in $L^p(\mathbb{R})$. We thus conclude that Theorem 4.1 holds with the extra condition that w have a nonnegative Fourier transform. Finally, this means that Theorem 1.1 holds with the function $g = \hat{w}$, which is nonnegative, and so Theorem 5.1 is established. ∎

Remarks. (1) In the space $L^p(\mathbb{R})$, $p > 2$, one can adapt the technique from Section 3 of [5] in order to construct a Schauder frame (not unconditional) formed by an *arbitrary unbounded* sequence of translates of a *nonnegative* function g.

(2) One may also consider the Banach space $C_0(\mathbb{R})$ of continuous functions on $\mathbb{R}$ vanishing at infinity, endowed with the norm $\|f\|_\infty = \sup |f(x)|$, $x \in \mathbb{R}$. First we note that this space *does not admit any unconditional Schauder frames*. Indeed, a Banach space with an unconditional Schauder frame is isomorphic to a complemented subspace of a Banach space with an unconditional Schauder basis [4], which is not the case for the space $C_0(\mathbb{R})$ (see Corollary 12 in Section II.D of [22]). On the other hand, again based on the technique from Section 3 of [5], one can show that the space $C_0(\mathbb{R})$ admits a Schauder frame (not unconditional) formed by an arbitrary unbounded sequence of translates of a function g, which can moreover be chosen nonnegative.

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