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Local Poincaré constants and mean oscillation functionals for BV functions

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Abstract. We introduce the concept of local Poincaré constant of a BV function as a tool to understand the relation between its mean oscillation and its total variation at small scales. This enables us to study a variant of the BMO-type seminorms on ε-size cubes introduced by Ambrosio, Bourgain, Brezis, and Figalli. More precisely,we relax the size constraint by considering a family of functionals that allow cubes of sidelength smaller than or equal to ε. These new functionals converge, as ε tends to zero, to a local functional defined on BV, which can be represented by integration in terms of the local Poincaré constant and the total variation. This contrasts with the original functionals, whose limit is defined on SBV and may not exist for functions with a non-trivial Cantor part. Moreover, we characterize the local Poincaré constant of a function with a cell-formula given by the maximum mean oscillation of its BV blow-ups. As a corollary of this characterization, we show that the new limit functional extends the original one to all BV functions. Finally, we discuss rigidity properties and other challenging questions relating the local Poincaré constant of a function to its fine properties.

1. Introduction

The Poincaré–Wirtinger inequality for a locally integrable function $f\colon\Omega\to\mathbb{R}$ on a (sufficiently regular) domain $\Omega\subset\mathbb{R}^n$ reads

$$\operatorname{Osc}(f,\Omega)\le C(\Omega)|Df|(\Omega).$$

This inequality establishes a quantitative relationship between the mean oscillation

$$\operatorname{Osc}(f,\Omega):=\fint_\Omega\Big|f(x)-\fint_\Omega f\Big|\,dx,$$

and the total variation

$$|Df|(\Omega):=\sup\Big\{\int_\Omega f(x)\operatorname{div}\varphi(x)\,dx:\varphi\in C^1_c(\Omega;\mathbb{R}^n),\ \|\varphi\|_\infty\le 1\Big\}.$$

The optimal Poincaré constant for characteristic functions on the unit open cube $Q_0:=(-1/2,1/2)^n$ is $1/2$, as shown in equation (2.2) of [2]. This result can be generalized to all

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functions of bounded variation using a simple convexity argument based on the coarea formula. In particular, for an open cube $Q \subset \mathbb{R}^n$, a rescaling argument yields the optimal bound

$$\ell(Q)^{n-1}\,\mathrm{Osc}(f, Q) \le \frac{1}{2}|Df|(Q), \tag{1.1}$$

where $\ell(Q)$ is the sidelength of Q. We record (cf. Lemma 7.1) that equality in this bound is attained only by functions that exhibit a jump-type discontinuity across a hyperplane that bisects the cube and is parallel to one of its sides.

In view of (1.1), one may wonder if it is possible to extract information about the total variation in terms of the mean oscillation (or related difference quotients). With this paradigm in mind, building on pioneering work of Bourgain et al. [10] (see also [9,12,13]), there has been a growing interest in characterizing Sobolev and BV functions through BMO-type functionals. In this vein, Ambrosio et al. [1, 2] proposed to study the limiting behavior of the functionals

$$\mathsf{K}_\varepsilon(f) := \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{Q \in \mathcal{H}_\varepsilon} \mathrm{Osc}(f, Q), \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n), \tag{1.2}$$

which measure the mean oscillation of a function f over a collection $\mathcal{H}_\varepsilon$ of disjoint ε-size cubes with arbitrary orientation (an isotropic variant of the original functionals introduced in [9, 10]). They showed that these functionals provide a characterization of sets of finite perimeter and suggested that they *could* be extended to study certain BV functions. This suggestion was particularly influential in sparking several contributions (e.g., [3,7,11,16, 19,20,22,24,25]) concerning the limiting behavior of (1.2) as ε approaches zero. In this regard, Ponce and Spector [25] and Fusco et al. [20] established a limiting lower bound for K_ε, which together with the upper bound given by the Poincaré inequality yields

$$\frac{1}{4}|Df|(\mathbb{R}^n) \le \liminf_{\varepsilon \to 0} \mathsf{K}_\varepsilon(f) \le \limsup_{\varepsilon \to 0} \mathsf{K}_\varepsilon(f) \le \frac{1}{2}|Df|(\mathbb{R}^n). \tag{1.3}$$

Notice that the equi-boundedness of either limit is equivalent to the finiteness of the total variation. Hence, a natural question that arises from this discussion is to understand those BV functions for which the limit exists. On the one hand, De Philippis et al. [15] (cf. [19]) proved that the limit exists for all functions $f \in \mathrm{SBV}(\mathbb{R}^n)$ of *special bounded variation* (functions with vanishing Cantor part $D^c f$). More precisely, they showed that if $f \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}^n)$, then

$$\mathsf{K}_0(f) := \lim_{\varepsilon \to 0} \mathsf{K}_\varepsilon(f) = \frac{1}{4}|D^a f|(\mathbb{R}^n) + \frac{1}{2}|D^j f|(\mathbb{R}^n), \tag{1.4}$$

where $D^a f$ and $D^j f$ denote, respectively, the absolutely continuous and the jump part of Df. Using a Cantor-type construction (see Appendix B), we show that there exist BV functions with non-trivial Cantor part for which the limit K_0 does not exist. Heuristically, this occurs because at a Cantor point $x \in \mathbb{R}^n$, the *localized ε-scale Poincaré constant*

$$\varepsilon \mapsto \sup\left\{\varepsilon^{n-1}\frac{\mathrm{Osc}(f, Q)}{|Df|(Q)} : x \in Q,\ \ell(Q) = \varepsilon\right\}$$

may oscillate wildly as ε tends to zero.

Introducing a new functional: a scale relaxation

To address the incompatibility of the scales that maximize the oscillation at Cantor points, we propose a scale-relaxation of the K_ε functionals to a 'less than or equal to ε' scale constraint. Informally, this is somehow reminiscent of the passage from the definition of Minkowski content (where one is allowed to use only balls of a fixed diameter) to the more flexible Hausdorff/spherical measure (which instead allows for balls of diameter less than a given threshold).

Let Ω be an open subset of $\mathbb{R}^n$. We define a functional on $L^1_{\mathrm{loc}}(\Omega)$ by setting

$$\mathsf{G}_\varepsilon(f,\Omega) := \sup_{\mathcal{H}_{\leq\varepsilon}(\Omega)} \sum_{Q\in\mathcal{H}_{\leq\varepsilon}(\Omega)} \ell(Q)^{n-1}\,\mathrm{Osc}(f,Q), \tag{1.5}$$

where $\mathcal{H}_{\leq\varepsilon}(\Omega)$ is a family of disjoint open cubes Q contained in Ω and having sidelength *at most* ε. Observe that the scale factor $\ell(Q)^{n-1}$ dwells now inside the sum, as it depends on each cube; the functional is free to choose the optimal scale at every location. This functional can be thought of as a geometric relaxation (from above) of K_ε, since by construction

$$\mathsf{G}_\varepsilon(f,\mathbb{R}^n) \geq \mathsf{K}_\varepsilon(f), \quad \text{for all } f\in L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{1.6}$$

It is straightforward to verify from the definition that $\varepsilon \mapsto \mathsf{G}_\varepsilon(f,\Omega)$ is non-decreasing. This allows us to define the pointwise limit

$$\mathsf{G}(f,\Omega) := \lim_{\varepsilon\to 0} \mathsf{G}_\varepsilon(f,\Omega) \quad f\in L^1_{\mathrm{loc}}(\Omega).$$

A localized version of (1.3) and (1.6), together with the Poincaré inequality, implies that $\mathsf{G}(f,\Omega)$ is finite if and only if $|Df|(\Omega) < \infty$. Thus, in studying $\mathsf{G}(f,\bullet)$, there is no loss of generality in requiring that $f\in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$.

Our main goal is to establish an integral representation for $\mathsf{G}(f,\Omega)$ in terms of a quantity that we call the *local Poincaré constant* of f, which we introduce next.

Local Poincaré constants of BV functions

Let $f\in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. To better understand the structural properties of $\mathsf{G}(f,\Omega)$, let us express the sum appearing in (1.5) as

$$\sum_{Q\in\mathcal{H}_{\leq\varepsilon}} P_f(Q)\,|Df|(Q), \quad \text{where} \quad P_f(Q) := \frac{\ell(Q)^{n-1}\,\mathrm{Osc}(f,Q)}{|Df|(Q)}. \tag{1.7}$$

Here $P_f(Q)$ can be thought of as the Poincaré quotient of f on Q. By the definition of G_ε as a supremum and G as a limit of G_ε, it is natural to consider the optimal behavior of P_f amongst all cubes Q, of sidelength at most ε (as $\varepsilon\to 0$) and containing a given point x. In fact, as we shall see next, we will need to consider a hierarchy of infinitesimal Poincaré quotients containing additional information about the position of the point: Given $x\in\Omega$ and $\tau\in[0,1]$, we define the *local τ-Poincaré constant of f at x* as

$$p_f^\tau(x) := \begin{cases} \lim_{\varepsilon\to 0} P_f^\tau(x,\varepsilon) & \text{if } x\in \mathrm{spt}(|Df|),\\ 0 & \text{if } x\notin \mathrm{spt}(|Df|),\end{cases}$$

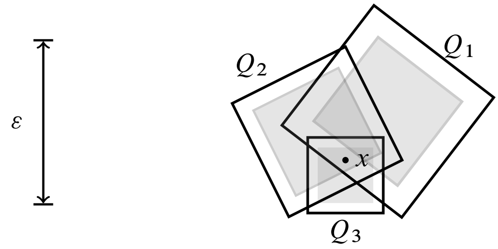

Figure 1. The shaded area depicts the τ-contraction τQ_i of each cube Q_i ($i = 1, 2, 3$). Only Q_2 and Q_3 are admissible for $P_f^\tau(x, \varepsilon)$.

where τQ denotes the concentric τ-contraction of Q for $\tau > 0$ (see Figure 1), and the center point of Q in the case $\tau = 0$, and

$$P_f^\tau(x, \varepsilon) := \sup\{P_f(Q) : x \in \tau Q, \ \ell(Q) \leq \varepsilon\}. \tag{1.8}$$

Observe that $p_f^\tau \geq p_f^\eta$ whenever $\tau \geq \eta$. Moreover, if $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$, then p_f^τ defines a non-negative Borel function on $\mathrm{spt}(|Df|)$.

The integral representation problem

A heuristic reasoning, thinking of (1.7) as a Riemann sum, suggests the validity of the following integral representation: if $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, then

$$\mathsf{G}(f, \Omega) = \int_\Omega p_f^1(x) \, d|Df|(x). \tag{1.9}$$

Our first main result demonstrates that such an integral representation holds in terms of τ-Poincaré constants for τ sufficiently close to 1. The precise statement is contained in the following theorem.

Theorem 1.1. *There exists a dimensional constant $\tau(n) \in (0, 1)$ with the following property: if $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, then*

$$\mathsf{G}(f, \Omega) = \int_\Omega p_f^\tau(x) \, d|Df|(x)$$

for all $\tau(n) \leq \tau < 1$.

Our result has the following implications for local Poincaré constants: Firstly, the local τ-Poincaré constants *do not depend* on the particular choice of $\tau \in (0, 1)$, provided that τ is sufficiently close to 1. Secondly, since $\liminf_{\varepsilon \to 0} \mathsf{K}_\varepsilon(f) \leq \mathsf{G}(f, \mathbb{R}^n)$, then

$$\frac{1}{4} \leq p_f^\tau \leq p_f^1 \leq \frac{1}{2} \quad |Df|\text{-almost everywhere}. \tag{1.10}$$

In Theorem 1.1, the restriction $\tau \in [\tau(n), 1)$ stems mostly from the techniques we adopt, e.g., the need of suitable covering theorems with *uncentered* cubes (see Remark 5.2

for further details). Unfortunately, at the moment we are unable to tell if this restriction is purely technical. In view of this discussion one may wonder if for all $\tau \geq \tau(n)$,

$$p_f^\tau = p_f^1 \quad |Df|\text{-almost everwhere in } \Omega. \tag{1.11}$$

In dimension one, we are able to establish this identity through the validity of (1.9):

Theorem 1.2. *Let $f \in \mathrm{BV}_{\mathrm{loc}}(I)$, where $I \subset \mathbb{R}$ is open. Then*

$$\mathsf{G}(f, I) = \int_I p_f^1(x)\, d|Df|(x).$$

In particular, for all $\tau(1) \leq \tau < 1$,

$$p_f^\tau = p_f^1 \quad |Df|\text{-almost everywhere.}$$

The validity of (1.9) and (1.11) for dimensions $n > 1$ remains an open question.

Tangent oscillations: a representation formula

The original motivation behind our work was to find a way to retrieve information about the pointwise limit $\mathsf{K}_0(f)$ from the behavior of the mean oscillation of f at infinitesimally small scales. Our idea was to study the mean oscillation of the BV blow-ups (see p. 187 of [4]) of f, at points in the support of $|Df|$.

Delving into this and motivated by the definition of G, we introduce a suitable concept of BV rescaling that takes into account the geometry of cubes: given an open cube Q with $|Df|(Q) > 0$ and an affine bijection $T_Q\colon Q_0 \to Q$ we consider the normalized function (see Section 6 for the precise definition, which uses *oriented* cubes)

$$f_Q(y) := \frac{1}{|Df|(Q)} \Big(f(T_Q y) - \fint_Q f \Big)\, \ell(Q)^{n-1}, \quad y \in Q_0.$$

Now, in order to analyze the infinitesimal behavior of p_f^τ, we also introduce a notion of uncentered BV blow-up tailored for cubes: Given $x \in \operatorname{spt}(|Df|)$, we define the set $\operatorname{Tan}^\tau(f, x)$, of *uncentered τ-tangents of f at x*, as the set of all limit points in $L^1(Q_0)$ of the family $\{f_Q : x \in \tau Q\}$ as $\ell(Q) \to 0$.

With these basic concepts at hand, we are now in the position to state our second main result, which gives a description of the Poincaré constant in terms of the largest mean oscillations of τ-tangents for $0 \leq \tau < 1$.

Theorem 1.3. *Let $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and let $\tau \in [0, 1)$. Then, for every $x \in \operatorname{spt}(|Df|)$,*

$$p_f^\tau(x) = \sup_{u \in \operatorname{Tan}^\tau(f,x)} \operatorname{Osc}(u, Q_0),$$

and the supremum in the right-hand side is attained.

For $f \in \mathrm{SBV}_{\mathrm{loc}}$, the largest mean oscillation over τ-tangents is 1/4 at $|D^a f|$-a.e. point and 1/2 at $|D^j f|$-a.e. point, regardless of the choice of τ (see Section 6.5). Combining this observation with Theorem 1.1 and Theorem 1.3 we discover that the functional $\mathsf{G}(\bullet, \mathbb{R}^n)$ coincides with K_0 on $\mathrm{SBV}_{\mathrm{loc}}$ functions and can, therefore, be considered as a relaxation from above of K_0:

Corollary 1.4. *Let* $f \in \mathrm{SBV}_{\mathrm{loc}}(\Omega)$. *Then*

$$\mathsf{G}(f, \Omega) = \mathsf{K}_0(f) = \frac{1}{4}|D^a f|(\Omega) + \frac{1}{2}|D^j f|(\Omega).$$

Lastly, we record an interesting consequence of Theorem 1.3. In dimension $n = 1$, the local Poincaré constant p_f^1 can be expressed as the largest BMO-seminorm

$$\|u\|_{\mathrm{BMO}(Q_0)} := \sup\{\mathrm{Osc}(u, Q) : Q \subset Q_0 \text{ open cube}\},$$

amongst all 1-tangents:

Corollary 1.5. *Let* $I \subset \mathbb{R}$ *be an open interval and let* $f \in \mathrm{BV}_{\mathrm{loc}}(I)$. *Then,*

$$p_f^1(x) = \sup_{u \in \mathrm{Tan}^1(f,x)} \|u\|_{\mathrm{BMO}((-1/2,1/2))}$$

for $|Df|$*-almost every* $x \in I$.

Comments on other shapes

The fact that we work with cubes does not play a determining role in the essence of our statements. For instance, one could work with shapes – such as a fixed convex shape – that permit the use of covering theorems (see also [3, 16] for related results). It is worth remarking that, in stark contrast with cubes, the (local) Poincaré constants associated with other shapes might depend on the dimension n.

Structure of the paper

Let us briefly summarize the contents of this work. In Section 2, we prove that $\mathsf{G}(f, \cdot)$ extends to a Borel measure. In Section 3, we prove the inequality "$\geq$" in Theorem 1.1. In Section 4, we do some preparatory work for the other inequality, proving the existence of good families of cubes. In Section 5, we prove the inequality "$\leq$" in Theorem 1.1. In Section 6, we prove Theorem 1.3. In Section 7, we study the properties of sets where p_f^τ coincides with one of the extremal values $1/4$ and $1/2$. In Appendix A, we collect some measure-theoretic results and covering theorems. Finally, in Appendix B, we give an example showing that K_0 might fail to exist on BV functions with a non-trivial Cantor part.

Notation

Throughout the paper we adopt the following conventions. We let $n \geq 1$ be an integer and we always consider Ω an open subset of $\mathbb{R}^n$. We will generically denote (open) cubes in $\mathbb{R}^n$ by the letter Q, while Q_0 is the centered unit cube $(-1/2, 1/2)^n$. The sidelength of a cube Q will be denoted by $\ell(Q)$ and its center point by x_Q. The letter τ will denote a positive constant between 0 and 1, often times attaining the value 1. With a possible abuse of notation, we shall write τQ to denote the cube of sidelength $\tau\ell(Q)$, centered at x_Q, and whose faces are parallel to the ones of Q. As usual, we write $B(x, r)$ to denote an open ball of radius r centered at a point x. If A is a set, we denote by A^c its complement and by $\mathbb{1}_A$ its characteristic function. The n-dimensional Lebesgue measure of a set $A \subset \mathbb{R}^n$ is denoted by $\mathscr{L}^n(A)$ or, when no risk of confusion arises, simply by $|A|$. We employ the standard notation $1^* := n/(n-1)$ for $n \geq 2$ and $1^* := \infty$ if $n = 1$.

2. Integral representation

We start by defining a localized version of the main functionals. Let $\mathcal{O}(\Omega)$ be the family of all open sets contained in Ω, and let $\mathsf{G}_\varepsilon(f,\,\cdot\,)\colon \mathcal{O}(\Omega) \to [0,\infty]$ be defined as

$$\mathsf{G}_\varepsilon(f,U) := \sup_{\mathcal{H}_{\le\varepsilon}(U)} \sum_{Q\in\mathcal{H}_{\le\varepsilon}(U)} \ell(Q)^{n-1} \fint_Q \left| f(x) - \fint_Q f \right| dx, \quad U \in \mathcal{O}(\Omega),$$

where the supremum is taken among all families of essentially disjoint cubes of sidelength at most ε contained in U. We also define

$$\mathsf{G}(f,U) := \lim_{\varepsilon\to 0} \mathsf{G}_\varepsilon(f,U), \quad U \in \mathcal{O}(\Omega),$$

where the limit always exists by monotonicity. Next, we show that $\mathsf{G}(f,\bullet)$ is the restriction of a locally bounded Borel measure

Proposition 2.1. *If $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, the set function $\mathsf{G}(f,\,\bullet\,)\colon \mathcal{O}(\Omega) \to [0,\infty]$ can be extended to a Radon measure that is absolutely continuous with respect to $|Df|$. In particular, there exists a non-negative function $g_f\colon \Omega \to [0,\infty)$ such that*

$$\mathsf{G}(f,B) = \int_B g_f(x)\, d|Df|(x)$$

for every Borel set $B \subset \Omega$.

Proof. Let us fix $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and set $\mathsf{F}(\cdot) := \mathsf{G}(f,\cdot)$. The first inequality in (1.3) and Poincaré's inequality yield the (pointwise) bounds

$$\frac{1}{4}|Df|(U) \le \mathsf{F}(U) \le \frac{1}{2}|Df|(U), \quad \forall U \in \mathcal{O}(\Omega), \tag{2.1}$$

which implies that F vanishes on $|Df|$-null (open) sets. Notice that the assertions that F is absolutely continuous with respect to $|Df|$ and the existence of the density g_f will follow from (2.1) once we know that F is (the restriction of) a measure (to open sets). In order to verify this, we show that Theorem A.6 can be applied to the set function F.

Clearly, F is increasing and the condition $\mathsf{F}(\emptyset) = 0$ is satisfied. Let us prove that the three remaining assumptions of Theorem A.6 are satisfied.

(i) *Subadditivity.* We need to show that $\mathsf{F}(U \cup V) \le \mathsf{F}(U) + \mathsf{F}(V)$ for any open sets $U, V \in \mathcal{O}(\Omega)$. Without loss of generality, we may assume $\mathsf{F}(U) < +\infty$ and $\mathsf{F}(V) < +\infty$ so that, by (2.1), $|Df|(U \cup V) < +\infty$. Fix $\eta, \varepsilon > 0$ and let $\mathcal{Q} := \mathcal{H}_{\le\varepsilon}(U \cup V)$ be a family of disjoint cubes contained in $U \cup V$ with sidelength $\ell(Q) \le \varepsilon$. Let $U_\eta := \{x \in U : \operatorname{dist}(x,\partial U) > \eta\}$ and let V_η be defined analogously. We split the family $\mathcal{Q}$ into three subfamilies:

$$\begin{aligned}
\mathcal{Q}_{U_\eta} &:= \{Q \in \mathcal{Q} : Q \subset U_\eta\},\\
\mathcal{Q}_{V_\eta} &:= \{Q \in \mathcal{Q} \setminus \mathcal{Q}_{U_\eta} : Q \subset V_\eta\},\\
\mathcal{Q}_R &:= \mathcal{Q} \setminus (\mathcal{Q}_{U_\eta} \cup \mathcal{Q}_{V_\eta}).
\end{aligned}$$

Observe that $Q \in \mathcal{Q}_R$ implies

$$Q \subset (U \setminus U_{\eta+\varepsilon\sqrt{n}}) \cup (V \setminus V_{\eta+\varepsilon\sqrt{n}}).$$

Therefore we have

$$\begin{aligned}\sum_{Q\in\mathcal{Q}} \ell(Q)^{n-1} \operatorname{Osc}(f,Q) &\le \mathsf{G}_\varepsilon(f,U_\eta) + \mathsf{G}_\varepsilon(f,V_\eta) + \sum_{Q\in\mathcal{Q}_R} \ell(Q)^{n-1} \operatorname{Osc}(f,Q)\\ &\le \mathsf{G}_\varepsilon(f,U) + \mathsf{G}_\varepsilon(f,V) + \frac{1}{2}|Df|((U \setminus U_{\eta+\varepsilon\sqrt{n}}) \cup (V \setminus V_{\eta+\varepsilon\sqrt{n}})).\end{aligned}$$

Taking the supremum in the left-hand side and sending $\eta \to 0$ and $\varepsilon \to 0$, we get

$$\mathsf{F}(U \cup V) \le \mathsf{F}(U) + \mathsf{F}(V),$$

as desired.

(ii) *Superadditivity.* We now show superadditivity holds for G_ε on disjoint sets. Let $\varepsilon > 0$ and let $U, V \in \mathcal{O}(\Omega)$ be open, disjoint sets. Pick two arbitrary families of cubes $\mathcal{H}_{\le\varepsilon(U)}$ and $\mathcal{H}_{\le\varepsilon(V)}$ as in the definition of G_ε. Then the union of these families is an admissible competitor for $\mathsf{G}_\varepsilon(f, U \cup V)$. This implies

$$\mathsf{G}_\varepsilon(f, U \cup V) \ge \sum_{Q\in\mathcal{H}_{\le\varepsilon(U)}} \ell(Q)^{n-1} \operatorname{Osc}(f,Q) + \sum_{Q'\in\mathcal{H}_{\le\varepsilon(V)}} \ell(Q)^{n-1} \operatorname{Osc}(f,Q).$$

Passing to the supremum in the right-hand side first over all families $\mathcal{H}_{\le\varepsilon(U)}$ and then over $\mathcal{H}_{\le\varepsilon(V)}$, we deduce the desired supperadditivity:

$$\mathsf{G}_\varepsilon(f, U \cup V) \ge \mathsf{G}_\varepsilon(f,U) + \mathsf{G}_\varepsilon(f,V).$$

Letting $\varepsilon \to 0$, we recover the superadditivity for F.

(iii) *Inner regularity.* We need to show $\mathsf{F}(U) = \sup\{\mathsf{F}(V) : V \in \mathcal{O}(\Omega),\ V \subset\subset U\}$ for every $U \in \mathcal{O}(\Omega)$. If $\mathsf{F}(U) = \infty$, then $|Df|(U) = \infty$ by (2.1) and therefore the claim follows from the fact that $|Df|$ is inner regular and another application of (2.1). If $\mathsf{F}(U) < \infty$, then $|Df|(U) < \infty$. Let $(A_k)_k$ be an exhaustion of relatively compact sets, with $\bar{A}_k \subset A_{k+1}$ for every $k \in \mathbb{N}$. Then, by subadditivity,

$$\mathsf{F}(U) \le \mathsf{F}(A_{k+1}) + \mathsf{F}(U \setminus \bar{A}_k) \le \mathsf{F}(A_{k+1}) + \frac{1}{2}|Df|(U \setminus \bar{A}_k).$$

Using the monotonicity of $|Df|$ along decreasing sequences, we get $|Df|(U \setminus \bar{A}_k) \to 0$ as $k \to \infty$. This yields the desired inner regularity property of F. ∎

Remark 2.2. By (1.3), (1.6), and the Poincaré inequality, the density g_f of the functional G satisfies

$$\frac{1}{4} \le g_f(x) \le \frac{1}{2} \quad \text{at } |Df|\text{-almost every } x. \tag{2.2}$$

3. Lower bound

In all that follows, we shall work with $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set.

Lemma 3.1. *Let $\tau \in [0, 1)$. For $|Df|$-almost every $x \in \Omega$, it holds*

$$g_f(x) \geq p_f^\tau(x).$$

If $n = 1$, then the same conclusion holds also for $\tau = 1$ (see Remark 3.2).

Proof. Let $\varepsilon, \delta > 0$. For each point $x \in \operatorname{spt}(|Df|)$, let $\mathcal{F}_\varepsilon(x)$ be the family of cubes $Q \subset \Omega$ satisfying the following properties:

(a) $x \in \tau Q$;

(b) $\ell(Q) \leq \varepsilon$;

(c) the following approximate continuity estimate holds:

$$\fint_Q |p_f^\tau(x) - p_f^\tau(y)| \, d|Df|(y) \leq \delta;$$

(d) the following almost maximizing property holds:

$$\frac{\operatorname{Osc}(f, Q)\, \ell(Q)^{n-1}}{|Df|(Q)} > p_f^\tau(x) - \delta;$$

(e) $|Df|(\partial Q) = 0$.

Notice that if a cube Q satisfies (a)–(b)–(c)–(d) and $|Df|(\partial Q) > 0$, we can slightly shrink Q so that it also satisfies (e). Then by the definition of $p_f^\tau(x)$ and by the Lebesgue differentiation theorem (see Theorem A.5), the family

$$\mathcal{F}_\varepsilon := \bigcup_{x \in \operatorname{spt}(|Df|)} \{\bar{Q} : Q \in \mathcal{F}_\varepsilon(x)\}$$

defines a fine cover of $\operatorname{spt}(|Df|)$. By property (a), we can invoke Theorem A.3, applied with the measure $\mu^* = |Df|$, to obtain a countable subcollection $\mathcal{F}'_\varepsilon \subset \mathcal{F}_\varepsilon$ of cubes with disjoint closures satisfying

$$|Df|\Big(\Omega \setminus \bigcup_{Q \in \mathcal{F}'_\varepsilon} \bar{Q}\Big) = 0. \tag{3.1}$$

We now use $\mathcal{F}'_\varepsilon$ as a competitor collection in the definition of G_ε to obtain a lower bound estimate:

$$\begin{aligned}
\mathsf{G}_\varepsilon(f, \Omega) &\geq \sum_{Q \in \mathcal{F}'_\varepsilon} \operatorname{Osc}(f, Q)\, \ell(Q)^{n-1} \overset{\text{(d)}}{\geq} \sum_{Q \in \mathcal{F}'_\varepsilon} (p_f^\tau(\bar{x}_Q) - \delta)|Df|(Q) \\
&\overset{\text{(c)}}{\geq} \sum_{Q \in \mathcal{F}'_\varepsilon} \Big(\fint_Q p_f^\tau(y)\, d|Df|(y) - 2\delta\Big)|Df|(Q) \\
&= \int_\Omega p_f^\tau(y)\, d|Df|(y) - 2\delta|Df|(\Omega).
\end{aligned}$$

In passing to the last equality, we used that $|Df|(\partial Q) = 0$ for every $Q \in \mathcal{F}_\varepsilon$, together with the Vitali property (3.1). Considering first the limit as $\delta \to 0$ and then the limit as $\varepsilon \to 0$, we deduce

$$\mathsf{G}(f, \Omega) \geq \int_\Omega p_f^\tau(y)\, d|Df|(y).$$

Since this holds for every Ω, an application of the Radon–Nikodym theorem implies that $g_f(x) \geq p_f^\tau(x)$ for $|Df|$-almost every x. This finishes the proof. ∎

Remark 3.2 (On the restriction $\tau < 1$). In our proof, the restriction $\tau < 1$ is essential for the lower bound, as it relies on the availability of a covering theorem. In the one-dimensional case, the proof above applies also to the case $\tau = 1$, as Theorem A.2 generalizes Theorem A.3 to this setting.

4. Good families of cubes

This section aims to establish the existence of families of cubes that exhibit near-optimality for G_ε and possess additional desirable properties. These families will play a pivotal role in the proof of the upper bound, presented in the subsequent section. The precise statement is the following.

Proposition 4.1 (Good families of cubes). *There exists a dimensional constant $\tau(n) \in (0, 1)$ with the following property: If $\tau \in [\tau(n), 1)$, then, for each $\varepsilon, \delta > 0$, there exists a family $\mathcal{F} \in \mathcal{H}_{\leq\varepsilon}(\Omega)$ satisfying the following properties:*

(i) *$\mathcal{F}$ is δ-almost maximizing for G_ε in the sense that*

$$\sum_{Q \in \mathcal{F}} \ell(Q)^{n-1} \operatorname{Osc}(f, Q) \geq (1 - \delta)\, \mathsf{G}_\varepsilon(f, \Omega).$$

(ii) *Every $Q \in \mathcal{F}$ satisfies*

$$\operatorname{Osc}(f, Q)\, \ell(Q)^{n-1} \geq \frac{1}{8}|Df|(Q).$$

(iii) *Every $Q \in \mathcal{F}$ satisfies*

$$|Df|(\tau Q) \geq \frac{1}{16}|Df|(Q).$$

In order to prove the proposition, we need a few preliminary results. First, we will need the following version of the Poincaré–Wirtinger inequality.

Lemma 4.2 (Modified Poincaré–Wirtinger). *Let $\tau \in (0, 1]$, and let Q be any given cube in $\mathbb{R}^n$. There exists a constant $C(n, \tau)$ such that*

$$\Big\| f - \fint_{\tau Q} f \Big\|_{L^{1^*}(Q)} \leq C(n, \tau)\, |Df|(Q)$$

for all $f \in \mathrm{BV}(Q)$. More precisely, we can choose $C(n, \tau) = C(n) + \frac{\sqrt{n}}{2} \frac{1-\tau}{\tau^{n+1}}$, where $C(n)$ is the classical Poincaré–Wirtinger constant for functions in L^{1^*} on the unit cube.

Proof. By scaling, we can assume that Q has volume one. By a standard approximation result for BV maps, it suffices to prove the statement for $f \in C^1(\overline{Q})$. For each $t \in (0,1)$, we will use the change of variables $y_t = tx + (1-t)\tau x$, which maps Q into $[\tau + t(1-\tau)]Q$. With this in mind, we apply the fundamental theorem of calculus to deduce the estimate

$$\begin{aligned}
\Big| \fint_Q f - \fint_{\tau Q} f \Big| &= \Big| \int_Q f(x) - f(\tau x)\, dx \Big| \\
&= \Big| \int_Q \int_0^1 \nabla f(tx + (1-t)\tau x) \cdot (1-\tau) x \, dt\, dx \Big| \\
&\le (1-\tau) \int_Q \int_0^1 |x| |\nabla f|(tx + (1-t)\tau x)\, dt\, dx \\
&\le (1-\tau) \int_0^1 \int_{[t+(1-t)\tau]Q} \frac{|y_t|}{|\tau + t(1-\tau)|^{n+1}} |\nabla f|(y_t)\, dy_t\, dt \\
&\le \frac{\sqrt{n}}{2}(1-\tau) \int_0^1 \frac{|Df|([t+(1-t)\tau]Q)}{|\tau + t(1-\tau)|^{n+1}}\, dt \\
&\le \frac{\sqrt{n}}{2} \frac{1-\tau}{\tau^{n+1}} |Df|(Q).
\end{aligned}$$

In particular, this allows us to estimate the L^1-difference between the averages on the large and small cubes as

$$\Big\| \fint_Q f - \fint_{\tau Q} f \Big\|_{L^{1^*}(Q)} \le \frac{\sqrt{n}}{2} \frac{1-\tau}{\tau^{n+1}} |Df|(Q).$$

In light of this estimate and the triangle inequality, we further get

$$\begin{aligned}
\Big\| f - \fint_{\tau Q} f \Big\|_{L^{1^*}(Q)} &\le \Big\| f - \fint_Q f \Big\|_{L^{1^*}(Q)} + \Big\| \fint_Q f - \fint_{\tau Q} f \Big\|_{L^{1^*}(Q)} \\
&\le \Big(C(n) + \frac{\sqrt{n}}{2} \frac{1-\tau}{\tau^{n+1}} \Big) |Df|(Q),
\end{aligned}$$

where $C(n)$ is the classical Poincaré–Wirtinger constant for functions in $L^{1^*}(Q)$. This finishes the proof. ∎

Next, we prove two technical lemmas that quantify the amount of mean oscillation on functions whose derivative concentrates near the boundary of a cube.

Lemma 4.3. *Let $\tau \in (0,1]$, $\eta > 0$ and let Q be a given cube in $\mathbb{R}^n$ such that*

$$|Df|(\tau Q) \le \eta |Df|(Q).$$

Then,

$$\frac{\operatorname{Osc}(f,Q)}{|Df|(Q)} \ell(Q)^{n-1} \le \frac{\eta}{2} + 3C(n,\tau)(1-\tau^n)^{1/n},$$

where $C(n,\tau)$ is the constant of Lemma 4.2.

Proof. We write the proof only for the case $n \geq 2$; if $n = 1$, only minor adaptations are needed to handle the fact that $1^* = \infty$. Since the left-hand side is invariant under translations, dilations, and multiplications by constants, there is no loss of generality in assuming that Q is the unit cube, $|Df|(Q) = 1$, and $\int_{\tau Q} f = 0$. By Lemma 4.2, we get

$$\Big(\int_Q |f|^{1^*}\Big)^{1/1^*} = \Big(\int_Q \Big|f - \fint_{\tau Q} f\Big|^{1^*}\Big)^{1/1^*} \leq C(n,\tau)|Df|(Q) = C(n,\tau).$$

From this estimate, we deduce the following auxiliary estimates. First, by Hölder's inequality, we get

$$\Big|\int_{Q\setminus\tau Q} f\Big| \leq \int_{Q\setminus\tau Q} |f| \leq \Big(\int_{Q\setminus\tau Q} |f|^{1^*}\Big)^{1/1^*} |Q \setminus \tau Q|^{1-1/1^*} \leq C(n,\tau)(1-\tau^n)^{1/n}.$$

On the other hand, the triangle inequality gives

$$\Big|\int_Q f\Big| \leq \Big|\int_{\tau Q} f\Big| + \Big|\int_{Q\setminus\tau Q} f\Big| \leq C(n,\tau)(1-\tau^n)^{1/n}.$$

Breaking $\mathrm{Osc}(f, Q)$ into smaller quantities that can be bounded by the quantities above, we obtain the sought estimate:

$$\begin{aligned}
\mathrm{Osc}(f,Q) &= \int_Q \Big|f - \int_Q f\Big| = \int_{\tau Q}\Big|f - \int_Q f\Big| + \int_{Q\setminus\tau Q}\Big|f - \int_Q f\Big| \\
&\leq \int_{\tau Q}\Big|f - \fint_{\tau Q} f\Big| + \int_{\tau Q}\Big|\fint_{\tau Q} f - \int_Q f\Big| + \int_{Q\setminus\tau Q}|f| + \int_{Q\setminus\tau Q}\Big|\int_Q f\Big| \\
&\leq \tau\frac{1}{2}|Df|(\tau Q) + |\tau Q|C(n,\tau)(1-\tau^n)^{1/n} \\
&\qquad + C(n,\tau)(1-\tau^n)^{1/n} + C(n,\tau)(1-\tau^n)^{1/n} \\
&\leq \frac{\eta}{2} + 3C(n,\tau)(1-\tau^n)^{1/n}.
\end{aligned}$$

This finishes the proof. ∎

Corollary 4.4. *There exists a dimensional constant* $\tau(n) \in (0,1)$ *with the following property*: *if* $\tau \in [\tau(n), 1)$ *and if* $f \in \mathrm{BV}(Q)$ *is such that*

$$|Df|(\tau Q) \leq \frac{1}{16}|Df|(Q),$$

then

$$\mathrm{Osc}(f,Q)\,\ell(Q)^{n-1} < \frac{1}{8}\,|Df|(Q).$$

Proof. Let

$$C(n,\tau) := C(n) + \frac{\sqrt{n}}{2}\,\frac{1-\tau}{\tau^{n+1}}$$

be the constant from Lemma 4.2. It suffices to apply Lemma 4.3 with $\eta = 1/16$ and notice that for τ sufficiently close to 1 it holds $3C(n,\tau)(1-\tau^n)^{1/n} < 1/16$. Observe that $C(n,\tau)$ is uniformly bounded if τ is chosen away from zero. ∎

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. Let $\eta \in (0, 1/2)$. By the definition of G_ε, we can find a disjoint collection $\mathcal{Q} \in \mathcal{H}_{\leq \varepsilon}(\Omega)$ satisfying

$$\sum_{Q \in \mathcal{Q}} \operatorname{Osc}(f, Q)\, \ell(Q)^{n-1} \geq (1-\eta)\, \mathsf{G}_\varepsilon(f, \Omega). \tag{4.1}$$

We then define a "good" subfamily of $\mathcal{Q}$ by setting

$$\mathcal{G} := \left\{ Q \in \mathcal{Q} : \operatorname{Osc}(f, Q)\, \ell(Q)^{n-1} \geq \tfrac{1}{8} |Df|(Q) \right\}.$$

Accordingly, we call $\mathcal{B} := \mathcal{Q} \setminus \mathcal{G}$ the "bad" subfamily of $\mathcal{Q}$. For each $Q \in \mathcal{B}$, we can select a collection $\mathcal{P}(Q)$ of pairwise disjoint cubes, contained in Q and satisfying

$$\sum_{Q' \in \mathcal{P}(Q)} \operatorname{Osc}(f, Q')\, \ell(Q')^{n-1} \geq (1-\eta)\, \mathsf{G}_\varepsilon(f, Q). \tag{4.2}$$

Note that the family $\widetilde{\mathcal{Q}} := \mathcal{G} \cup \bigcup_{Q \in \mathcal{B}} \mathcal{P}(Q)$ is still an admissible family of disjoint cubes and hence, by definition,

$$\sum_{Q' \in \widetilde{\mathcal{Q}}} \operatorname{Osc}(f, Q')\, \ell(Q')^{n-1} \leq \mathsf{G}_\varepsilon(f, \Omega). \tag{4.3}$$

From (4.1)–(4.3) and the definition of $\mathcal{Q}$, we infer that

$$\begin{aligned}
\eta\, \mathsf{G}_\varepsilon(f, \Omega) &\geq \sum_{Q \in \widetilde{\mathcal{Q}}} \operatorname{Osc}(f, Q)\, \ell(Q)^{n-1} - \sum_{Q \in \mathcal{Q}} \operatorname{Osc}(f, Q)\, \ell(Q)^{n-1} \\
&= \sum_{Q \in \mathcal{B}} \Big(\sum_{Q' \in \mathcal{P}(Q)} \operatorname{Osc}(f, Q')\, \ell(Q')^{n-1} - \operatorname{Osc}(f, Q)\, \ell(Q)^{n-1} \Big) \\
&\overset{(4.2)}{\geq} \sum_{Q \in \mathcal{B}} \Big((1-\eta)\, \mathsf{G}_\varepsilon(f, Q) - \frac{1}{8} |Df|(Q) \Big) =: A.
\end{aligned}$$

Using the lower bound

$$\frac{1}{4} |Df|(Q) \leq \mathsf{G}(f, Q) \leq \mathsf{G}_\varepsilon(f, Q)$$

(see (1.3)) and the Poincaré inequality

$$\operatorname{Osc}(f, Q)\, \ell(Q)^{n-1} \leq \frac{1}{2} |Df|(Q),$$

we further estimate

$$\begin{aligned}
\eta\, \mathsf{G}_\varepsilon(f, \Omega) \geq A &\geq \sum_{Q \in \mathcal{B}} \Big(\frac{1}{4}(1-\eta) - \frac{1}{8} \Big) |Df|(Q) \\
&= \frac{1}{8}(1-2\eta) \sum_{Q \in \mathcal{B}} |Df|(Q) \geq \frac{1}{4}(1-2\eta) \sum_{Q \in \mathcal{B}} \operatorname{Osc}(f, Q)\, \ell(Q)^{n-1}.
\end{aligned} \tag{4.4}$$

By construction, the collection $\mathcal{G}$ satisfies the following properties:

(a) $\ell(Q) \le \varepsilon$ for every $Q \in \mathcal{G}$;

(b) it satisfies

$$\begin{aligned}\sum_{Q\in\mathcal{G}} \operatorname{Osc}(f,Q)\,\ell(Q)^{n-1} &= \sum_{Q\in\mathcal{Q}} \operatorname{Osc}(f,Q)\,\ell(Q)^{n-1} - \sum_{Q\in\mathcal{B}} \operatorname{Osc}(f,Q)\,\ell(Q)^{n-1}\\ &\overset{(4.1)\text{–}(4.4)}{\ge} (1-\eta)\,\mathsf{G}_\varepsilon(f,\Omega) - \frac{4\eta}{1-2\eta}\,\mathsf{G}_\varepsilon(f,\Omega)\\ &= \Big(1-\eta\,\frac{5-2\eta}{1-2\eta}\Big)\,\mathsf{G}_\varepsilon(f,\Omega);\end{aligned}$$

(c) and $\operatorname{Osc}(f,Q)\,\ell(Q)^{n-1} \ge \frac{1}{8}|Df|(Q)$ for every $Q \in \mathcal{G}$ (by the definition of good subfamily).

Choosing η small enough so that $\eta\frac{5-2\eta}{1-2\eta} \le \delta$ we find out that $\mathcal{G}$ satisfies (i) and (ii). Finally, (iii) follows from (ii) by applying Corollary 4.4. ■

5. Upper bound

In this section, we prove that $p_f^\tau(x)$ bounds $g_f(x)$ from above for $|Df|$-almost every $x \in \mathbb{R}$.

Lemma 5.1. *Let $\tau(n) \le \tau \le 1$, where $\tau(n)$ is the constant defined in Corollary 4.4. Then, for $|Df|$-almost every $x \in \Omega$, it holds*

$$g_f(x) \le p_f^\tau(x). \tag{5.1}$$

Proof. We begin by recalling the definition

$$P_f^\tau(x,\rho) := \sup_{\substack{\tau Q \ni x,\\ \ell(Q)\le\rho}} \frac{\operatorname{Osc}(f,Q)}{|Df|(Q)}\,\ell(Q)^{n-1}.$$

Fix an open set $U \Subset \Omega$ so that $|Df|(U) < \infty$. Given $\delta, \rho > 0$, define the set

$$E_{\rho,\delta} := \big\{ y \in U : P_f^\tau(y,\rho) < p_f^\tau(y) + \delta \big\}. \tag{5.2}$$

Fix now $\delta > 0$. Then we can find $\rho > 0$ small enough such that

$$|Df|(U \setminus E_{\rho,\delta}) \le \delta.$$

Let us consider the set $A_{\rho,\delta} \subseteq E_{\rho,\delta}$ of $|Df|$-Lebesgue points of the function p_f^τ, namely the points x satisfying

$$\lim_{r\to 0} \fint_{Q_r} |p_f^\tau(y) - p_f^\tau(x)|\, d|Df|(y) = 0 \tag{5.3}$$

for every family of cubes Q_r such that $\ell(Q_r) = r$, $x \in \tau Q_r$. By Theorem A.5, $|Df|$-almost every point has this property, hence $|Df|(U \setminus A_{\rho,\delta}) \le \delta$. Moreover, by choosing some $\varepsilon \in (0, \rho)$ small enough and using (5.3), we can find a subset $C_{\rho,\delta} \subseteq A_{\rho,\delta}$ satisfying $|Df|(U \setminus C_{\rho,\delta}) \leq 2\delta$ and such that for every $x \in C_{\rho,\delta}$, it holds

$$p_f^\tau(x) \leq \fint_Q p_f^\tau(y)\, d|Df|(y) + \delta \tag{5.4}$$

for every Q with $\ell(Q) \leq \varepsilon$, $x \in \tau Q$.

If $\tau \in [\tau(n), 1)$, by Proposition 4.1, we can find a *good family* $\mathcal{F}_\varepsilon$ of disjoint cubes contained in U:

$$\ell(Q) \leq \varepsilon, \quad |Df|(\tau Q) \geq \frac{1}{16}|Df|(Q) \tag{5.5}$$

for every $Q \in \mathcal{F}_\varepsilon$, and

$$\mathsf{G}_\varepsilon(f, U) \leq \delta|Df|(U) + \sum_{Q\in\mathcal{F}_\varepsilon} \mathrm{Osc}(f, Q)\, \ell(Q)^{n-1}. \tag{5.6}$$

Notice that, if $\tau = 1$, we can enforce both (5.5) and (5.6) by considering an almost maximizing family, and thus we do not have to rely on Proposition 4.1. The next step is to split the last summand in (5.6) into two further sums. One, over cubes Q such that τQ intersects $C_{\rho,\delta}$, and the second, as the sum over the remaining cubes. To this end, let us introduce the family

$$\mathcal{A} := \{Q \in \mathcal{F}_\varepsilon : \tau Q \cap C_{\rho,\delta} \neq \emptyset\}.$$

Let us first consider cubes $Q \in \mathcal{F}_\varepsilon \setminus \mathcal{A}$. The Poincaré inequality yields the bound

$$\begin{aligned}\sum_{Q\in\mathcal{F}_\varepsilon\setminus\mathcal{A}} \mathrm{Osc}(f, Q)\, \ell(Q)^{n-1} &\leq \sum_{Q\in\mathcal{F}_\varepsilon\setminus\mathcal{A}} \frac{1}{2}|Df|(Q) \overset{(5.5)}{\leq} \sum_{Q\in\mathcal{F}_\varepsilon\setminus\mathcal{A}} 8|Df|(\tau Q)\\ &\leq 8|Df|(U \setminus C_{\rho,\delta}) \leq 16\delta.\end{aligned} \tag{5.7}$$

Let us now consider the case $Q \in \mathcal{A}$. By definition, for every such cube Q we can find a point contained in $\tau Q \cap C_{\rho,\delta}$, which we call $\bar{x}_Q$, and which satisfies (5.4). From this, we deduce the estimate

$$\begin{aligned}\sum_{Q\in\mathcal{A}} \mathrm{Osc}(f, Q)\, \ell(Q)^{n-1} &= \sum_{Q\in\mathcal{A}} \frac{\mathrm{Osc}(f, Q)\, \ell(Q)^{n-1}}{|Df|(Q)} |Df|(Q)\\ &\leq \sum_{Q\in\mathcal{A}} P_f^\tau(\bar{x}_Q, \rho)|Df|(Q) \overset{(5.2)}{\leq} \sum_{Q\in\mathcal{F}_\varepsilon} (p_f^\tau(\bar{x}_Q) + \delta)|Df|(Q)\\ &\overset{(5.4)}{\leq} \sum_{Q\in\mathcal{F}_\varepsilon} \Big(\fint_Q p_f^\tau(y)\, d|Df|(y) + 2\delta\Big)|Df|(Q)\\ &\leq \int_U p_f^\tau(y)\, d|Df|(y) + 2\delta|Df|(U).\end{aligned} \tag{5.8}$$

Plugging the estimates (5.7)–(5.8) into (5.6), we obtain

$$\mathsf{G}(f, U) \leq \mathsf{G}_\varepsilon(f, U) \leq 16\delta + 3\delta|Df|(U) + \int_U p_f^\tau(y)\, d|Df|(y). \tag{5.9}$$

As δ was arbitrary, we deduce

$$\mathsf{G}(f, U) \leq \int_U p_f^\tau(y)\, d|Df|(y).$$

Since this holds for every $U \Subset \Omega$, an application of the Radon–Nikodym theorem implies that $g_f(x) \leq p_f^\tau(x)$ for $|Df|$-almost every x. This finishes the proof. ∎

Remark 5.2 (On the need of largely uncentered cubes). The restriction $\tau \geq \tau(n)$ is an essential component in our proof of the upper bound. The argument builds on the existence of a good family of cubes (cf. Section 4), which requires us to work with large values of $\tau \leq 1$. In any case, we do not know the smallest τ for which (5.1) may hold, and it may happen that it is true also when $\tau = 0$, which would correspond to taking centered cubes.

Combining Lemma 3.1 and Lemma 5.1, we obtain the equality

$$g_f(x) = p_f^\tau(x) \quad \text{at } |Df|\text{-almost every point,}$$

provided that $\tau \in [\tau(n), 1)$, and thus Theorem 1.1 is proven. We record a very simple consequence of this equality and Remark 2.2.

Corollary 5.3. *Let $\tau \in [\tau(n), 1)$, and let $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. Then,*

$$\frac{1}{4} \leq p_f^\tau \leq p_f^1 \leq \frac{1}{2} \quad |Df|\text{-almost everywhere.} \tag{5.10}$$

6. Representation in terms of tangents

The goal of this section is to characterize p_f^τ in terms of the uncentered tangents of $f \in \mathrm{BV}$, which we introduce next.

6.1. Introducing oriented cubes

In this section, we treat cubes as objects with a specific orientation. An *orientation* of a cube Q is defined as an n-tuple $\mathbf{b} = \{b_1, \ldots, b_n\}$, where each b_i is the center of a face and $\{b_1 - x_Q, \ldots, b_n - x_Q\}$ (with x_Q being the cube's center) defines a frame of orthogonal vectors (see Figure 2). With a slight abuse of notation compared to the previous sections, we will use Q to denote an oriented cube $(Q, \mathbf{b}_Q)$. Now, for any oriented cube Q and its orientation $\mathbf{b}_Q$, there exists a unique angle-preserving affine map $T_Q\colon \mathbb{R}^n \to \mathbb{R}^n$ that satisfies two key properties:

- *Realizes the cube as the image of the unit cube:* $T(Q_0) = Q$.
- *Preserves orientation:* $T(\frac{1}{2}e_i) = b_i$ for each canonical basis vector e_i.

Furthermore, applying an angle-preserving affine map T to an oriented cube Q with orientation $\{b_1, \ldots, b_n\}$ naturally induces a new orientation $\{Tb_1, \ldots, Tb_n\}$ on the transformed cube $T(Q)$.

It is worth noting that using oriented cubes primarily serves the purpose of defining a unique transformation map T_Q for each cube. Crucially, all the relevant quantities we consider below, such as $\mathrm{Osc}(f, Q)$ and $|Df|(Q)$, do not depend on the orientation of Q and are thus well defined.

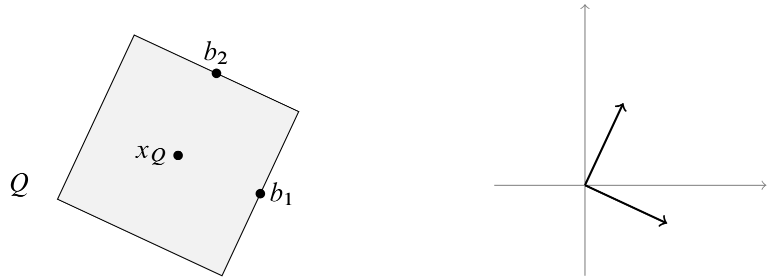

Figure 2. An oriented cube $Q \subset \mathbb{R}^2$. The vectors $b_1 - x_Q$ and $b_2 - x_Q$ are orthogonal.

6.2. Uncentered tangents

Given $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, and given an (oriented) cube Q such that $|Df|(Q) > 0$, let us define the rescaling $f_Q \colon Q_0 \to \mathbb{R}$ by setting

$$f_Q(y) := \frac{1}{|Df|(Q)} \Big(f(T_Q y) - \fint_Q f \Big) \, \ell(Q)^{n-1}. \tag{6.1}$$

Notice that f_Q has zero average and $|Df_Q|(Q_0) = 1$, and moreover,

$$Df_Q = \frac{(T_Q^{-1})_\# Df}{|Df|(Q)} \qquad \text{and} \qquad |Df_Q| = \frac{(T_Q^{-1})_\# |Df|}{|Df|(Q)}, \tag{6.2}$$

where

$$(T_Q^{-1})_\# : \mathcal{M}(Q; \mathbb{R}^n) \to \mathcal{M}(Q_0; \mathbb{R}^n)$$

is the usual push-forward of measures (cf. Definition 1.70 in [4]).

Definition 6.1 (Blow-up sequences and tangents). Fix $\tau \in [0, 1]$ and $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. Let $x \in \operatorname{spt}(|Df|)$ and let $(Q_j)_j$ be a sequence of oriented cubes contained in Ω. A sequence of rescaled functions $(f_{Q_j})_j \subset \mathrm{BV}(Q_0)$ is called a *τ-centered blow-up sequence of f at x* provided that

$$x \in \tau Q_j \quad \text{and} \quad \ell(Q_j) \to 0 \ \text{ as } j \to 0.$$

Any strong $L^1(Q_0)$-limit of a τ-centered blow-up sequence is called a *τ-centered tangent* (or simply *τ-tangent*) of f at x. We denote the set of all τ-tangents of f at x by $\mathrm{Tan}^\tau(f, x)$. If $x \notin \operatorname{spt}(|Df|)$, then we set $\mathrm{Tan}^\tau(f, x) = \emptyset$.

Remark 6.2. We observe that if $f_{Q_j} \to u$ in $L^1(Q_0)$, then also automatically

$$Df_{Q_j} \overset{*}{\rightharpoonup} Du \quad \text{locally weakly in the sense of measures,} \tag{6.3}$$

i.e., for every $\varphi \in C_c^0(Q_0)$,

$$\langle Df_{Q_j}, \varphi \rangle := \int_{Q_0} \varphi \, dDf_{Q_j} \to \int_{Q_0} \varphi \, dDu =: \langle Du, \varphi \rangle.$$

Indeed, given φ and $\varepsilon > 0$, there exists $\psi \in C_c^\infty(Q_0)$ such that $\|\varphi - \psi\|_{C^0} < \varepsilon$, hence

$$\begin{aligned}|\langle Df_{Q_j} - Du, \varphi\rangle| &\le |\langle Df_{Q_j} - Du, \varphi - \psi\rangle| + |\langle Df_{Q_j} - Du, \psi\rangle| \\ &\le 2\|\varphi - \psi\|_{C^0} + \|f_{Q_j} - u\|_{L^1}\|D\psi\|_{C^0},\end{aligned}$$

and passing to the limit as $j \to \infty$ and as $\varepsilon \to 0$, we reach the conclusion.

A standard result states that, at small scales around most points, BV functions are L^1-close to a one-directional and monotone function. The following result formalizes this in the context of τ-tangents.

Proposition 6.3. *Let $\tau \in [0,1)$. Then for $|Df|$-a.e. point $x \in \Omega$, every tangent $u \in \mathrm{Tan}^\tau(f,x)$ coincides with the restriction to the unit cube Q_0 of a function of the form $h(x \cdot e)$ for some monotone function $h\colon \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$, where $e = \frac{Df}{|Df|}(x)$ is the polar of Df at x.*

Proof. The proof follows the same lines of Lemma 4 in [7]. Indeed, by the uncentered differentiation Theorem A.4, one obtains that, at $|Df|$-almost all points x, all τ-tangents have constant polar. By approximation with smooth functions, one gets the conclusion. ∎

Combining Remark 6.2 with Proposition 6.3, one can improve (6.3) to an analogous convergence for the total variations, as the next lemma shows.

Lemma 6.4. *Let $\tau \in [0,1)$. For $|Df|$-almost every $x \in \Omega$, the following holds: if $(f_{Q_j}) \subset \mathrm{BV}(Q_0)$ is a τ-centered blow-up sequence at x, converging to $u \in \mathrm{Tan}^\tau(f,x)$, then*

$$|Df_{Q_j}| \overset{*}{\rightharpoonup} |Du| \quad \textit{locally weakly in the sense of measures in } Q_0.$$

Proof. We set $f_j := f_{Q_j}$ for simplicity and $r_j := \ell(Q_j)$ for each $j \in \mathbb{N}$. We write

$$Df = g|Df| \quad \text{and} \quad Df_j = g_j|Df_j|,$$

where g and g_j are the polars. Observe that for $|Df_j|$-a.e. z, by Radon–Nikodym we have the following identity:

$$\begin{aligned}g_j(z) &= \lim_{s\to 0} \frac{Df_j(B_s(z))}{|Df_j|(B_s(z))} \overset{(6.2)}{=} \lim_{s\to 0} \frac{Df(T_{Q_j}[B_s(z)])}{|Df|(T_{Q_j}[B_s(z)])} \\ &= \lim_{s\to 0} \frac{Df(B_{sr_j}(T_{Q_j}(z)))}{|Df|(B_{sr_j}(T_{Q_j}(z)))} = g(T_{Q_j}(z)).\end{aligned} \tag{6.4}$$

Let $\varphi \in C_c(Q_0)$ and observe

$$\begin{aligned}\int_{Q_0} \varphi(z)\, d|Df_j|(z) &= \int_{Q_0} \varphi(z)\, g_j(z) \cdot g_j(z)\, d|Df_j|(z) \\ &= \int_{Q_0} \varphi(z)\, g(x) \cdot g_j(z)\, d|Df_j|(z) + R_j \\ &= \int_{Q_0} \varphi(z)\, g(x) \cdot dDf_j(z) + R_j \to \int_{Q_0} \varphi(z)\, g(x) \cdot dDu(z),\end{aligned}$$

where we have used that $Df_{Q_j} \overset{*}{\rightharpoonup} Du$ by Remark 6.2 and the fact that R_j goes to zero, since

$$
\begin{aligned}
|R_j| &:= \Big| \int_{Q_0} \varphi(z)\,(g(x) - g_j(z)) \cdot g_j(z)\, d|Df_j|(z) \Big| \\
&\le \|\varphi\|_{C^0} \int_{Q_0} |g(x) - g_j(z)|\, d|Df_j|(z) \\
&\overset{(6.4)}{=} \|\varphi\|_{C^0} \int_{Q_0} |g(x) - g(T_{Q_j}(z))|\, d|Df_j|(z) \\
&= \|\varphi\|_{C^0} \int_{Q_0} |g(x) - g(y)|\, d(T_{Q_j})_{\#}|Df_j|(y) \\
&\overset{(6.2)}{=} \|\varphi\|_{C^0} \fint_{Q_j} |g(x) - g(y)|\, d|Df|(y) \to 0
\end{aligned}
$$

for $|Df|$-a.e. x by Theorem A.5. Since φ was chosen arbitrarily, this shows that

$$|Df_j| \overset{*}{\rightharpoonup} g(x) \cdot Du \quad \text{locally weakly in the sense of measures in } Q_0.$$

By Proposition 6.3, for $|Df|$-a.e. x, the polar of Du is constant and equal to $e = \frac{Df}{|Df|}(x) = g(x)$, therefore we obtain $g(x) \cdot Du = |Du|$, i.e.,

$$|Df_j| \overset{*}{\rightharpoonup} |Du| \quad \text{locally weakly in the sense of measures in } Q_0.$$

Hence the sought conclusion holds true at $|Df|$-a.e. x, and this finishes the proof. ■

Remark 6.5. By standard BV-compactness, $\mathrm{Tan}^\tau(f, x)$ is non-empty for all $x \in \mathrm{spt}(|Df|)$. We shall see later by means of Proposition 6.9 that, additionally, $\mathrm{Tan}^\tau(f, x) \neq \{0\}$ for $|Df|$-almost every x (cf. Corollary 6.10).

In the following, we will reserve the letter u to denote tangents. By the lower semi-continuity of the total variation, every tangent $u\colon Q_0 \to \mathbb{R}$ is a BV function on Q_0 with $|Du|(Q_0) \le 1$. It will be convenient to introduce a special subclass of tangents, the set of *normalized* τ-tangents,

$$\mathrm{Tan}_1^\tau(f, x) := \{u \in \mathrm{Tan}^\tau(f, x) : |Du|(Q_0) = 1\}.$$

Next, we record a couple of simple observations concerning blow-up sequences and τ-tangents.

Lemma 6.6 (Composition rule). *Let Q be any cube in $\mathbb{R}^n$, and let S be a cube contained in Q_0. Let $T_Q\colon Q_0 \to Q$ be the affine map associated with Q. Then*

$$(f_Q)_S = f_{T_Q(S)}. \tag{6.5}$$

Proof. If $T_S\colon Q_0 \to S$ is the affine map associated with S, then $T_{T_Q(S)} = T_Q \circ T_S$. Indeed, by definition,

$$T_Q \circ T_S(Q_0) = T_Q(S) = T_{T_Q(S)}(Q_0).$$

Moreover, let $\mathbf{b}_Q = \{q_1, \ldots, q_n\}$ and $\mathbf{b}_S = \{s_1, \ldots, s_n\}$ be the orientations of Q and S. Then

$$T_Q \circ T_S(\tfrac{1}{2}e_i) = T_Q(s_i) = T_{T_Q(S)}(\tfrac{1}{2}e_i).$$

It follows that the maps $T_{T_Q(S)}$ and $T_Q \circ T_S$ coincide on the basis $\{e_i\}$, so they must coincide. Therefore, $(f_Q)_S$ and $f_{T_Q(S)}$ have the same affine rescaling in the inner variable, have average zero and total variation one. This uniquely determines the function, so they must coincide. ∎

Lemma 6.7 (Tangents are compact). *Fix $\tau \in [0, 1]$ and let $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. Then, for every $x \in \Omega$, the set $\mathrm{Tan}^\tau(f, x)$ is compact (and thus closed) in the $L^1(Q_0)$ topology.*

Proof. We can clearly assume that $x \in \mathrm{spt}(|Df|)$. Take a sequence $u_j \in \mathrm{Tan}^\tau(f, x)$. By definition, $|Du_j|(Q_0) \leq 1$ and $\fint_{Q_0} u_j = 0$. By compactness in BV, up to subsequences, u_j converge in $L^1(Q_0)$ to some $v \in \mathrm{BV}(Q_0)$ with $|Dv|(Q_0) \leq 1$. For every j, let $(Q_j^i)_{i\in\mathbb{N}}$ be a sequence of cubes with $x \in \tau Q_j^i$ such that

$$f_{Q_j^i} \to u_j \quad \text{in } L^1(Q_0) \text{ as } i \to \infty.$$

Then, by a diagonal argument we can find a sequence $Q_j^{i(j)}$ such that $\ell(Q_j^{i(j)}) \to 0$ and

$$f_{Q_j^{i(j)}} \to v \quad \text{in } L^1(Q_0) \text{ as } j \to \infty.$$

This shows that $v \in \mathrm{Tan}^\tau(f, x)$ and concludes the proof. ∎

6.3. Local Poincaré constants and tangents

Let us now turn to the main result of this section, which states that $p_f^\tau(x)$ can be recovered by maximizing the oscillation of (normalized) τ-tangents at x. First, we define the quantities

$$m^\tau(x) := \sup\left\{ \frac{\mathrm{Osc}(u, Q_0)}{|Du|(Q_0)} : u \in \mathrm{Tan}^\tau(f, x),\ |Du|(Q_0) > 0 \right\}$$
$$m_1^\tau(x) := \sup\{\mathrm{Osc}(u, Q_0) : u \in \mathrm{Tan}_1^\tau(f, x)\}.$$

We convene that the supremum over the emptyset is zero.

Lemma 6.8. *Let $\tau \in [0, 1)$ and let $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. For $|Df|$-a.e. $x \in \Omega$ the following holds:*

(i) *If $u \in \mathrm{Tan}^\tau(f, x)$ is not the zero function, then the function $|Du|(Q_0)^{-1}u$ also belongs to $\mathrm{Tan}^\tau(f, x)$, and hence to $\mathrm{Tan}_1^\tau(f, x)$.*

(ii) *We have that $m^\tau(x) = m_1^\tau(x)$. Moreover, if $\mathrm{Tan}^\tau(f, x) \neq \{0\}$, then both suprema are attained. In addition, they also coincide with*

$$\sup\{\mathrm{Osc}(u, Q_0) : u \in \mathrm{Tan}^\tau(f, x)\}.$$

Proof. (i) Let Q_j be such that $x \in \tau Q_j$, $\ell(Q_j) \to 0$ and $f_{Q_j} \to u$ in $L^1(Q_0)$.

Define the points $y_j := T_{Q_j}^{-1}(x) \in \tau Q_0$. Up to a (not relabeled) subsequence, we can assume that $y_j \to y_\infty \in \tau \bar{Q}_0$. For $\eta > 0$, we also define the similarity

$$D_\eta(y) := y_\infty + \eta(y - y_\infty).$$

For the sake of simplicity, we write $Q_0^\eta := D_\eta(Q_0)$ and $Q_j^\eta := T_{Q_j}(Q_0^\eta) \subseteq Q_j$. Observe that for sufficiently large j, we still have $y_j \in \tau Q_0^\eta$, or equivalently, $x \in \tau Q_j^\eta$. Therefore $(Q_j^\eta)_j$ is an admissible sequence for $\operatorname{Tan}^\tau(f, x)$.

For all $\delta > 0$, we can always choose η such that $1 - \delta < \eta < 1$ and $|Du|(\partial Q_0^\eta) = 0$. By Lemma 6.4, for $|Df|$-a.e. x, the measures $|Df_{Q_j}|$ converge locally weakly-star to $|Du|$, hence in particular,

$$|Du|(Q_0^\eta) = \lim_{j\to\infty} |Df_{Q_j}|(Q_0^\eta).$$

By definition and the composition rule (6.5),

$$f_{Q_j^\eta} = (f_{Q_j})_{Q_0^\eta} = \frac{1}{|Df_{Q_j}|(Q_0^\eta)}\Big(f_{Q_j} \circ D_\eta - \fint_{Q_0^\eta} f_{Q_j}\Big)\ell(Q_0^\eta)^{n-1}.$$

Moreover, this sequence L^1-converges to

$$\frac{1}{|Du|(Q_0^\eta)}\Big(u \circ D_\eta - \fint_{Q_0^\eta} u\Big)\eta^{n-1}.$$

Observe that, as $\eta \to 1$, this family L^1-converges to $|Du|(Q_0)^{-1}u$. Now if we fix a sequence $\eta_j \to 1$, we can extract a diagonal subsequence so that $f_{Q_j^{\eta_j}}$ converge to $|Du|(Q_0)^{-1}u$. This concludes the proof of (i).

(ii) Since $\operatorname{Tan}^\tau(f, x) \neq \{0\}$, then $m^\tau(x) > 0$. Indeed, the oscillation is zero if and only if the function is constant, and thus identically zero (because every tangent has zero average). Let now $u_j \in \operatorname{Tan}^\tau(f, x)$ be a maximizing sequence, i.e.,

$$\frac{\operatorname{Osc}(u_j, Q_0)}{|Du_j|(Q_0)} \to m^\tau(x) > 0.$$

Then, by (i), each $\tilde{u}_j := |Du_j|^{-1}(Q_0)u_j$ belongs to $\operatorname{Tan}_1^\tau(f, x)$. By Lemma 6.7, up to a (not relabeled) subsequence, $\tilde{u}_j \to v$ for some $v \in \operatorname{Tan}^\tau(f, x)$. First, we will show that $v \neq 0$. By Lemma 4.3, there exist constants $\eta \in (0, 1)$ and $\delta > 0$ such that $|D\tilde{u}_j|(\eta Q_0) \geq \delta|D\tilde{u}_j|(Q_0) = \delta$ for every j. This implies, taking again into account Lemma 6.4, that $|Dv|(\overline{\eta Q_0}) \geq \limsup_j |D\tilde{u}_j|(\overline{\eta Q_0}) \geq \delta$. Now we use the continuity of the oscillation and the lower semicontinuity of the total variation on open sets to infer that

$$\frac{\operatorname{Osc}(v, Q_0)}{|Dv|(Q_0)} \geq \limsup_j \frac{\operatorname{Osc}(\tilde{u}_j, Q_0)}{|D\tilde{u}_j|(Q_0)} = m^\tau(x).$$

This shows that v is a non-zero maximizer in $\operatorname{Tan}^\tau(f, x)$, with $|Dv|(Q_0) = 1$; for otherwise there would be a strict inequality above, contradicting that $m^\tau(x)$ is the supremum. In particular, v is also a maximizer in $\operatorname{Tan}_1^\tau(f, x)$. Since $\operatorname{Tan}_1^\tau(f, x) \subseteq \operatorname{Tan}^\tau(f, x)$, we conclude that $m^\tau(x) = m_1^\tau(x)$. Finally, the last claim in (ii) follows from the inequalities

$$m_1^\tau(x) \leq \sup\{\operatorname{Osc}(u, Q_0) : u \in \operatorname{Tan}^\tau(f, x)\} \leq m^\tau(x).$$

This finishes the proof. ∎

We now prove the following result, which is the last step towards the proof of Theorem 1.3.

Proposition 6.9. *Let* $\tau \in [0,1)$ *and let* $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. *Then, for* $|Df|$*-a.e.* $x \in \Omega$, *it holds*

$$p_f^\tau(x) = m^\tau(x) = m_1^\tau(x). \tag{6.6}$$

Proof. We divide the proof into steps consisting of different inequalities.

Step 1. $m^\tau(x) \geq p_f^\tau(x)$.

We may assume that $p_f^\tau(x) > 0$. Let Q_j be a sequence of cubes such that $x \in \tau Q_j$, $\ell(Q_j) \to 0$ and

$$\mathrm{Osc}(f_{Q_j}, Q_0) = \frac{\mathrm{Osc}(f, Q_j)}{|Df|(Q_j)}\,\ell(Q_j)^{n-1} \to p_f^\tau(x), \quad \text{as } j \to \infty.$$

Since $|Df_{Q_j}|(Q_0) = 1$, f_{Q_j} is a pre-compact sequence in $L^1(Q_0)$ and converges (up to not relabeled subsequences) to some $u \in \mathrm{BV}(Q_0)$ with $|Du|(Q_0) \leq 1$. We claim that $|Du|(Q_0) > 0$. Indeed,

$$\mathrm{Osc}(u, Q_0) = \lim_{j\to\infty} \mathrm{Osc}(f_{Q_j}, Q_0) = p_f^\tau(x) > 0.$$

Therefore, u cannot be constant and hence $|Du|(Q_0) > 0$. This proves that $\mathrm{Tan}^\tau(f,x) \neq \{0\}$. Moreover, this also shows that

$$\frac{\mathrm{Osc}(u, Q_0)}{|Du|(Q_0)} \geq \mathrm{Osc}(u, Q_0) \geq p_f^\tau(x),$$

whence $m^\tau(x) \geq p_f^\tau(x)$.

Step 2. $p_f^\tau(x) \geq m_1^\tau(x)$.

As before, we may assume that $m_1^\tau(x) > 0$, and thus that $\mathrm{Tan}_1^\tau(f,x) \neq \emptyset$. Take any $v \in \mathrm{Tan}_1^\tau(f,x)$, and a generating blow-up sequence $f_{Q_j} \to v$. Then

$$\begin{aligned}\mathrm{Osc}(v, Q_0) &= \lim_j \mathrm{Osc}(f_{Q_j}, Q_0) = \lim_j \frac{\mathrm{Osc}(f, Q_j)}{|Df|(Q_j)}\,\ell(Q_j)^{n-1} \\ &\leq \lim_j P^\tau(x, \ell(Q_j)) = p_f^\tau(x).\end{aligned}$$

Taking the supremum among all $v \in \mathrm{Tan}_1^\tau(f,x)$, we obtain the sought assertion.

Step 3. *Conclusion.*

By Lemma 6.8 and by the previous points, we have $p_f^\tau(x) \leq m^\tau(x) = m_1^\tau(x) \leq p_f^\tau(x)$, and therefore all inequalities are equalities. This completes the proof. ∎

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. If $\mathrm{Tan}^\tau(f,x) \neq \{0\}$, we can invoke Lemma 6.8 (ii), together with Proposition 6.9, to conclude that

$$p_f^\tau(x) = \sup_{u \in \mathrm{Tan}^\tau(f,x)} \mathrm{Osc}(u, Q_0)$$

and that the supremum is attained. If, on the contrary, $\mathrm{Tan}^\tau(f,x) = \{0\}$, then the right-hand side is trivially zero. Moreover, by definition $m^\tau(x) = m_1^\tau(x) = 0$ (because we convene that the sup on the emptyset is zero) and we conclude that also $p_f^\tau(x) = 0$ applying again Proposition 6.9. ■

From the assertion of Proposition 6.9 and Theorem 1.1, we deduce the following.

Corollary 6.10. *Let $\tau \in [\tau(n), 1)$ and $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. Then, for $|Df|$-almost every $x \in \Omega$, the tangent set $\mathrm{Tan}^\tau(f,x)$ contains some non-zero function.*

Proof. By Theorem 1.1, we know that, for $|Df|$-almost every x, $p_f^\tau(x) = g_f(x)$. Moreover, $g_f(x) \in [1/4, 1/2]$ for $|Df|$-almost every x. Proposition 6.9 shows that $p_f^\tau(x)$ also coincides with the supremum of $\mathrm{Osc}(u, Q_0)$ among all tangents $u \in \mathrm{Tan}^\tau(f,x)$. It follows that there must exist at least one τ-tangent for which $\mathrm{Osc}(u, Q_0) \neq 0$, and which therefore cannot be the zero function. ■

6.4. Tangents to tangents are tangents

We now demonstrate that repeatedly applying the blow-up procedure is equivalent to a single blow-up operation (at least almost everywhere, in a sense to be precisely defined later). This type of stability result was first established by Preiss [26] for *tangent measures*. A few key distinctions from Preiss' original setting are worth noting before delving into technical details. First, in our case the tangent is only defined in the cube Q_0, while Preiss' tangents are defined in the whole space. Additionally, our definition of tangents employs uncentered rescalings, while Preiss used only centered rescalings. Nevertheless, the proof follows closely the original one, or rather the one that can be found in Theorem 14.16 of [23].

Proposition 6.11. *Let $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and let $\tau \in [0,1)$. At $|Df|$-almost every point $a \in \Omega$, every $u \in \mathrm{Tan}^\tau(f,a)$ has the following properties*:

(i) $u_Q \in \mathrm{Tan}^\tau(f,a)$ *for all subcubes $Q \subset Q_0$ with $\tau Q \cap \mathrm{spt}(|Du|) \neq \emptyset$*;

(ii) $\mathrm{Tan}^\tau(u,y) \subseteq \mathrm{Tan}^\tau(f,a)$ *for every $y \in \mathrm{spt}(|Du|)$*.

Proof. (i) By Lemma 6.8 (i), we can assume that $u \in \mathrm{Tan}_1^\tau(f,x)$, namely, that no mass is lost at the boundary along the sequence that generates u. Indeed, the rescaling u_Q is invariant with respect to scalar multiplication of u.

We first prove the claim under the additional assumption that $|Du|(\partial Q) = 0$. For positive integers k and m, let $A_{k,m}$ be the set of all points $a \in \Omega$ for which there exists $u_a \in \mathrm{Tan}^\tau(f,a)$ and a cube $Q_a \subset Q_0$ satisfying $|Du_a|(\tau Q_a) > 0$, $|Du_a|(\partial Q_a) = 0$ and moreover

$$\|(u_a)_{Q_a} - f_Q\|_{L^1(Q_0)} > \frac{1}{k} \tag{6.7}$$

for all cubes Q with $a \in \tau Q$ and $0 < \ell(Q) < 1/m$.

In order to reach the conclusion, it suffices to show that $|Df|(A_{k,m}) = 0$ for all k and m.

Suppose by contradiction that $|Df|(A_{k,m}) > 0$ for some k and m. By the separability of $L^1(Q_0)$, we find a set $A \subset A_{k,m}$ with $|Df|(A) > 0$ and such that

$$\|(u_a)_{Q_a} - (u_b)_{Q_b}\|_{L^1(Q_0)} < \frac{1}{2k} \tag{6.8}$$

for every $a, b \in A$. Now, let a be a density point of A for $|Df|$, i.e.,

$$1 = \lim_{r \to 0} \frac{|Df|(A \cap Q_r)}{|Df|(Q_r)} \tag{6.9}$$

for every family of cubes $\{Q_r\}_r$ with $\ell(Q_r) = r$ and $a \in \tau Q_r$. In light of Theorem A.4 (which can be applied to arbitrary sets), $|Df|$-almost every point $a \in A$ is a density point of A; we now show that the conclusion holds at all such points.

Let $(Q_j)_j$ be a sequence of cubes such that $a \in \tau Q_j$, $\ell(Q_j) \to 0$ and

$$u_a = \lim_{j \to \infty} f_{Q_j}.$$

Call $T_j := T_{Q_j}$ the affine maps sending Q_0 to Q_j. We claim that

$$T_j(\tau Q_a) \cap A \neq \emptyset \quad \text{for } j \text{ large enough.} \tag{6.10}$$

Suppose by contradiction that this is not the case. Then, up to extracting a (not relabeled) subsequence, it holds

$$T_j(\tau Q_a) \cap A = \emptyset \quad \text{for every } j. \tag{6.11}$$

Since $\tau Q_a \cap \operatorname{spt}(|Du_a|) \neq \emptyset$ by assumption, we further get

$$0 < |Du_a|(\tau Q_a) \leq \liminf_{j \to \infty} |Df_{Q_j}|(\tau Q_a) \overset{(6.2)}{=} \liminf_{j \to \infty} \frac{|Df|(T_j(\tau Q_a))}{|Df|(Q_j)}.$$

From this, we deduce that

$$T_j(\tau Q_a) \cap \operatorname{spt}(|Df|) = T_j(\tau Q_a \cap \operatorname{spt}(|Df_{Q_j}|)) \neq \emptyset$$

whenever j is sufficiently large. It follows from (6.11) that

$$\begin{aligned}
\frac{|Df|(Q_j \cap A)}{|Df|(Q_j)} &= \frac{|Df|([Q_j \cap (T_j(\tau Q_a))^c \cap A)}{|Df|(Q_j)} \\
&\leq \frac{|Df|([Q_j \cap (T_j(\tau Q_a))^c)}{|Df|(Q_j)} = 1 - \frac{|Df|(T_j(\tau Q_a))}{|Df|(Q_j)} \\
&\overset{(6.2)}{=} 1 - \frac{|Df_{Q_j}|(\tau Q_a)}{|Df_{Q_j}|(Q_0)} = 1 - |Df_{Q_j}|(\tau Q_a).
\end{aligned}$$

Therefore, by the lower semicontinuity of the total variation,

$$1 \overset{(6.9)}{=} \lim_{j \to \infty} \frac{|Df|(Q_j \cap A)}{|Df|(Q_j)} \leq 1 - \liminf_{j \to \infty} |Df_{Q_j}|(\tau Q_a) \leq 1 - |Du_a|(\tau Q_a) < 1,$$

where we used the assumption $|Du_a|(\tau Q_a) > 0$. We have reached a contradiction, thus proving the claim (6.10).

In light of (6.10), we can choose j such that $\ell(Q_j) < 1/m$ and such that there exists a point $b \in A \cap T_j(\tau Q_a)$. By (6.8) we must have

$$\|(u_a)_{Q_a} - (u_b)_{Q_b}\|_{L^1(Q_0)} < \frac{1}{2k}. \tag{6.12}$$

By the composition rule (6.5), we obtain the identity

$$(u_a)_{Q_a} = \lim_{j\to\infty} (f_{Q_j})_{Q_a} = \lim_{j\to\infty} f_{T_j(Q_a)},$$

Here, to deduce the first equality, we have used that $|Du_a|(\partial Q_a) = 0$, the equality $|Df_{Q_j}|(Q_0) = |Du_a|(Q_0) = 1$, and also Lemma A.7 to infer that $|Du_a|(Q_a) = \lim_{j\to\infty} |Df_{Q_j}|(Q_a)$. This says that we can choose j such that $\ell(Q_j) < 1/m$ and

$$\|(u_a)_{Q_a} - f_{T_j(Q_a)}\|_{L^1(Q_0)} < \frac{1}{2k}.$$

Applying the triangle inequality with this estimate and (6.12) gives

$$\|(u_b)_{Q_b} - f_{T_j(Q_a)}\|_{L^1} < \frac{1}{k}.$$

However, given that $b \in \tau T_j(Q_a)$ and $\ell(T_j(Q_a)) \le \ell(Q_j) < 1/m$, this contradicts (6.7) for the point b using $Q = T_j(Q_a)$. We have thus reached a contradiction, and whence $|Df|(A_{k,m}) = 0$. This finishes the proof of (i) under the assumption that $|Du|(\partial Q) = 0$.

To prove the general case we argue by approximation with inner cubes: given any $Q \subset Q_0$, let $Q_k \subset Q$, $k \in \mathbb{N}$, be a sequence of cubes such that $Q_k \nearrow Q$ and $|Du|(\partial Q_k) = 0$. Applying the first part of the proposition to each u_{Q_k}, we discover that $u_{Q_k} \in \operatorname{Tan}^\tau(f, x)$. Since $u_{Q_k} \to u_Q$ in $L^1(Q_0)$ as $k \to \infty$, the closedness of the tangents (cf. Lemma 6.7) yields $u_Q \in \operatorname{Tan}^\tau(f, x)$. This proves (i).

(ii) This part follows directly from (i) and the closedness property of $\operatorname{Tan}^\tau(f, x)$. This finishes the proof. ■

Remark 6.12. The previous proposition holds also for $\tau = 1$ when $n = 1$. It is sufficient to apply the Lebesgue differentiation theorem with uncentered intervals, which is a consequence of Theorem A.2.

Corollary 6.13. *Let $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and let $\tau \in [0, 1)$. For $|Df|$-almost every point $x \in \mathbb{R}^n$ the following holds: for all tangents $u \in \operatorname{Tan}^\tau(f, x)$ and all cubes $Q \subseteq Q_0$ satisfying $\tau Q \cap \operatorname{spt}(|Du|) \neq \emptyset$, it holds*

$$\operatorname{Osc}(u_Q, Q_0) = \frac{\operatorname{Osc}(u, Q)}{|Du|(Q)}\,\ell(Q)^{n-1} \le p_f^\tau(x). \tag{6.13}$$

In particular, for $|Df|$-almost every $x \in \Omega$,

$$p_u^\tau(y) \le p_f^\tau(x) \quad \textit{for every } y \in \operatorname{spt}(|Du|)$$

for all $u \in \operatorname{Tan}^\tau(f, x)$.

Proof. The proof follows directly from Proposition 6.11 and 6.9. ■

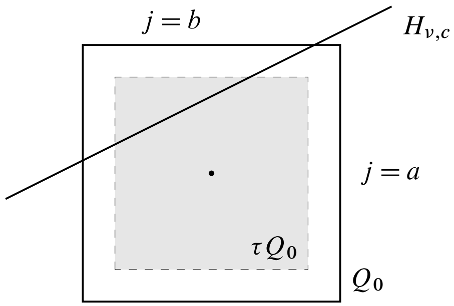

Figure 3. The function $j = j_{a,b,\nu,c}$ of Lemma 6.14 (in 2d). The function j takes constant values a, b and jumps from value a to b across the line $H_{\nu,c}$.

Corollary 1.5 is a direct consequence of this result.

Proof of Corollary 1.5. Thanks to Corollary 6.13, at $|Df|$-a.e. x, for every τ-tangent $u \in \operatorname{Tan}^\tau(f, x)$ and every interval $I \subseteq (-1/2, 1/2)$ with $\operatorname{spt}(|Du|) \cap I \neq \emptyset$, it holds

$$\operatorname{Osc}(u, I) \leq \frac{\operatorname{Osc}(u, I)}{|Du|(I)} \leq p_f^\tau(x).$$

Taking the supremum over all intervals, we obtain $\|u\|_{\mathrm{BMO}((-1/2,1/2))} \leq p_f^\tau(x)$ for every τ-tangent u. On the other hand, by Lemma 6.8 and Proposition 6.9, we can find a tangent u realizing $p_f^\tau(x) = \operatorname{Osc}(u, (-1/2, 1/2)) \leq \|u\|_{\mathrm{BMO}((-1/2,1/2))}$, and the claimed equality follows. ∎

6.5. Tangents of SBV functions

In this section, we compute explicitly the tangent sets $\operatorname{Tan}^\tau(f, x)$ associated with an SBV function f. Given a hyperplane $H_{\nu,c} := \{x \in \mathbb{R}^n : x \cdot \nu = c\}$, we define $j_{a,b,\nu,c}$ as the function that jumps from the value a to the value b when crossing the hyperplane $H_{\nu,c}$ in direction ν (see Figure 3).

The following lemma is a direct consequence of the fine properties of BV functions (see [4]), and its proof is omitted.

Lemma 6.14. *Let* $f \in \mathrm{SBV}_{\mathrm{loc}}(\Omega)$, *and* $\tau \in [0, 1)$. *Then*

(i) *at* $|D^a f|$*-almost every point,* $\operatorname{Tan}^\tau(f, x)$ *consists of all linear maps* $u\colon Q_0 \to \mathbb{R}$ *with gradient of modulus* 1;

(ii) *at* $|D^j f|$*-almost every point,* $\operatorname{Tan}^\tau(f, x)$ *consists of all the maps of the form* $j_{a,b,\nu,c}$ *among all* $a, b \in \mathbb{R}$, $\nu \in \mathbb{S}^1$ *and* $c \in \mathbb{R}$ *such that* $H_{\nu,c} \cap \tau\overline{Q_0} \neq \emptyset$ *and*

$$\int_{Q_0} j_{a,b,\nu,c}(x)\, dx = 0, \quad |Dj_{a,b,\nu,c}|(Q_0) = 1.$$

This result allows us to give a proof of Corollary 1.4, i.e.,

$$\mathsf{K}_0(f) = \mathsf{G}(f, \Omega), \quad f \in \mathrm{SBV}_{\mathrm{loc}}(\Omega).$$

Proof of Corollary 1.4. By (1.4), we know that $\mathsf{K}_0(f) = \frac{1}{4}|D^a f|(\Omega) + \frac{1}{2}|D^j f|(\Omega)$. Let us fix $\tau \in (0,1)$ sufficiently close to 1 so that Theorem 1.1 applies. We only need to verify that $p_f^\tau(x)$ is $1/4$ at $|D^a f|$-almost every point, and $1/2$ at $|D^j f|$-almost every point. Recall from Theorem 1.3 that $p_f^\tau(x)$ coincides with the largest oscillation among the τ-tangents. Let us analyze absolutely continuous and jump points separately.

For $|D^a f|$-almost every point, tangents are linear with modulus 1, and by Lemma 3.1 in [19], the supremum of $\operatorname{Osc}(u, Q_0)$ in this class is precisely $1/4$. In fact, it is attained by functions whose gradient is aligned with one of the coordinate axes.

For $|D^j f|$-almost every point, the jump function $u = j(-1/2, 1/2, H_{e_1,0})$ (which is in $\operatorname{Tan}^\tau(f, x)$ by Lemma 6.14) satisfies $\operatorname{Osc}(u, Q_0) = 1/2$. Since $p_f^\tau(x) \le 1/2$ always, we conclude $p_f^\tau(x) = 1/2$.

The proof is complete. ∎

7. Rigidity associated with extremal Poincaré constants

Motivated by the analysis in the SBV case, one might hope to characterize the jump points of $f \in \mathrm{BV}$ as those where $p_f^\tau(x) = 1/2$, and the absolutely continuous points as those where $p_f^\tau(x) = 1/4$. Unfortunately, this conclusion is not always true, as there exist Cantor functions on $[0,1]$ that reach value $1/4$ or $1/2$ at $|Df|$-almost every point. Nevertheless, when p_f^τ attains an extreme value, it is possible to infer certain rigidity properties on the set of tangents. More specifically, we establish (see Propositions 7.2 and 7.3) the following rigidity properties of $p_f^\tau(x)$:

- the value $p_f^\tau(x) = 1/2$ is attained if and only if there exists a jump tangent at x;
- on the other hand, $p_f^\tau(x) = 1/4$ if and only if every tangent $u \in \operatorname{Tan}^\tau(f, x)$ satisfies $p_u^\tau = 1/4$ at $|Du|$-almost every point.

We begin by recording the following generalization of Hadwiger's result [21], about the rigidity of functions that attain the maximal Poincaré constant on the unit cube.

Lemma 7.1 (Maximizers of Poincaré inequality on the unit cube). *Let $u \in \mathrm{BV}(Q_0)$ attain the maximal Poincaré constant, namely,*

$$\frac{\operatorname{Osc}(u, Q_0)}{|Du|(Q_0)} = \frac{1}{2}.$$

Then, up to addition and multiplication by a constant, u is the characteristic function of a half-cube[1] *of Q_0.*

Proof. We know that $\operatorname{Osc}(v, Q_0) \le \frac{1}{2}|Dv|(Q_0)$ for every $v \in L^1(Q_0)$. Therefore, the functions that attain equality are precisely, up to multiplication by a constant, the maximizers of

$$\max\{\operatorname{Osc}(v, Q_0) : |Dv|(Q_0) \le 1\}.$$

[1] By a half-cube of Q_0, we refer to a set of the form $\{y \in Q_0 : y \cdot e_i \ge 0\}$ or $\{y \in Q_0 : y \cdot e_i \le 0\}$, where $e_1, \dots, e_n$ is the canonical basis of $\mathbb{R}^n$.

Observe that the map $v \mapsto \operatorname{Osc}(v, Q_0)$ is convex. Moreover, the set of zero-average functions $v \in L^1(Q_0)$ with $|Dv|(Q_0) \le 1$ is convex and compact in $L^1(Q_0)$. Let thus u be a maximizer, that we can suppose satisfies $|Du|(Q_0) = 1$.

Using the coarea formula, we write u as a convex combination of rescaled characteristic functions of its super- and sub-level sets. More precisely, setting

$$m := \mathscr{L}^n\text{-essinf}_{Q_0} u \quad \text{and} \quad M := \mathscr{L}^n\text{-esssup}_{Q_0} u,$$

we observe that the zero-average condition entails $m \le 0$ and $M \ge 0$, so that

$$u(x) = \int_0^M \mathbb{1}_{\{u>t\}}(x)\, dt - \int_m^0 \mathbb{1}_{\{u<t\}}(x)\, dt$$

and

$$|Du|(Q_0) = \int_0^M |D\mathbb{1}_{\{u>t\}}|(Q_0)\, dt + \int_m^0 |D\mathbb{1}_{\{u<t\}}|(Q_0)\, dt.$$

Define

$$u_t(x) := \begin{cases} \frac{1}{|D\mathbb{1}_{\{u>t\}}|(Q_0)}\, \mathbb{1}_{\{u>t\}}(x) & \text{if } t \in (0, M), \\ -\frac{1}{|D\mathbb{1}_{\{u<t\}}|(Q_0)}\, \mathbb{1}_{\{u<t\}}(x) & \text{if } t \in (m, 0), \end{cases}$$

and

$$\lambda(t) := \begin{cases} |D\mathbb{1}_{\{u>t\}}|(Q_0) & \text{if } t \in (0, M), \\ |D\mathbb{1}_{\{u<t\}}|(Q_0) & \text{if } t \in (m, 0). \end{cases}$$

Notice that $\lambda(t) > 0$ for $t \in (m, M)$, because in this range the sub- and super-level sets are non-trivial. Hence, the functions u_t are well defined, and moreover,

$$\int_m^M \lambda(t)\, dt = |Du|(Q_0) = 1.$$

By construction, $|Du_t|(Q_0) = 1$ for every $t \in (m, M)$, and u can be written as a convex combination of the u_t's:

$$u(x) = \int_m^M \lambda(t)\, u_t(x)\, dt.$$

Since u maximizes the oscillation, and by convexity of the latter, we get

$$\frac{1}{2} = \operatorname{Osc}(u, Q_0) \le \int_m^M \lambda(t) \operatorname{Osc}(u_t, Q_0)\, dt \le \int_m^M \lambda(t)\, \frac{1}{2}\, |Du_t|(Q_0)\, dt = \frac{1}{2}.$$

This means that all inequalities are equalities. In particular,

$$\operatorname{Osc}(u_t, Q_0) = \frac{1}{2} \quad \text{for } \mathscr{L}^1\text{-almost every } t \in (m, M).$$

In turn, this implies that u_t maximizes the oscillation in Q_0 for $\mathscr{L}^1$-almost every $t \in (m, M)$. Hadwiger [21] proved that, among characteristic functions $v = \mathbb{1}_E$, those satisfying $\operatorname{Osc}(v, Q_0) = \frac{1}{2}|Dv|(Q_0)$ correspond to the case when E is a half-cube. This shows that $\mathscr{L}^1$-almost every super- or sub-level set (in the range (m, M)) of u must be a half-cube. Now, by monotonicity of super- and sub-level sets, and by the zero-average condition, there exists a half-cube H such that

$$\mathbb{1}_{\{u>t\}} = \mathbb{1}_H \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0, M),$$
$$\mathbb{1}_{\{u<t\}} = \mathbb{1}_{H^c} \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (m, 0).$$

The conclusion now follows. ■

As a direct consequence of this result, combined with Theorem 1.3, we obtain the following rigidity for tangents at points in the set $\{p_f^\tau = 1/2\}$.

Proposition 7.2 (Rigidity for $p_f^\tau = 1/2$). *Let $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and let $\tau \in [0, 1)$. Then for every $x \in \Omega$, the following conditions are equivalent*:

(i) $\mathrm{Tan}^\tau(f, x)$ *contains a (non-trivial) jump function across a half-cube.*

(ii) $p_f^\tau(x) = 1/2$.

We now turn to the analysis of rigidity of tangents corresponding to points in the set $\{p_f^\tau = 1/4\}$:

Proposition 7.3 (Rigidity for $p_f^\tau = 1/4$). *Let $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and let $\tau \in [0, 1)$. Then at $|Df|$-almost every $x \in \{p_f^\tau = 1/4\}$, every $u \in \mathrm{Tan}^\tau(f, x)$ satisfies*

$$\mathrm{Osc}(u, Q)\, \ell(Q)^{n-1} \leq \frac{1}{4}|Du|(Q)$$

for every cube $Q \subset Q_0$ such that $\tau Q \cap \mathrm{spt}\,|Du| \neq \emptyset$. In particular,

$$p_u^\tau = \frac{1}{4} \quad |Du|\text{-almost everywhere.}$$

Proof. Without loss of generality, we can restrict to the set of points $x \in \Omega$ where tangents to tangents are tangents, because this set has full measure by Proposition 6.11. Fix such an $x \in \{p_f^\tau = 1/4\}$. Then, Corollary 6.13 tells us that every $u \in \mathrm{Tan}^\tau(f, x)$ satisfies the first assertion of the proposition, and also

$$p_u^\tau \leq \frac{1}{4} \quad |Du|\text{-almost everywhere.}$$

Since it always holds $p_u^\tau \geq 1/4$ $|Du|$-almost everywhere, the proof is complete. ■

8. A few interesting questions

Let us now delve into a few intriguing questions about the structure of a function, its Poincaré constant and its τ-tangents. For the rest of this section, we fix $\tau \in [0, 1)$ and let $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$.

8.1. Mono-directionality

Recall that by Proposition 6.3, at almost every point the τ-tangents of a BV function are mono-directional. It is natural to ask if there exist optimal tangents (realizing the Poincaré constant in the cell-formula maximization problem appearing in Theorem 1.3), whose mono-directionality is aligned with one of the sides of Q_0.

Question 1 (Alignment of optimal tangents). *Is it true that for all (or some) tangents $u \in \operatorname{Tan}^\tau(f,x)$ realizing $p_f^\tau(x)$, we have $u(x) = h(x \cdot e)$ for some monotone function $h\colon \mathbb{R} \to \mathbb{R}$ and where e is one of the coordinate directions?*

A positive answer to this question would indicate the possibility of reducing the proofs presented in this work to the one-dimensional case. This would create a viable pathway for proving Theorem 1.1 for $\tau = 1$ in all dimensions $n > 1$.

8.2. Poincaré constants of tangents

We have observed that $\operatorname{Tan}^\tau(f,x)$ is naturally a compact set in $L^1(Q_0)$. We have also discussed that tangents to tangents qualify as tangents themselves. Additionally, Poincaré constants are always confined to take values in the compact interval $[1/4, 1/2]$. Furthermore, the inequality $p_u^\tau \le p_f^\tau(x)$ holds for all tangents $u \in \operatorname{Tan}^\tau(f,x)$, suggesting the possibility of minimizing p_u^τ within $\operatorname{Tan}(f,x)$. This raises the prospect of finding at least one tangent with minimal and constant local Poincaré constants everywhere:

Question 2. *For $|Df|$-almost every $x \in \Omega$, does $\operatorname{Tan}^\tau(f,x)$ contain a tangent $\bar u$ attaining the minimum value*

$$\alpha(x) := \inf_{u \in \operatorname{Tan}^\tau(f,x)} \int_{Q_0} p_u^\tau \, d|Du| = \inf_{u \in \operatorname{Tan}^\tau(f,x)} \mathsf{G}(u, Q_0),$$

and, if so, is $p_{\bar u}^\tau$ constant $|D\bar u|$-almost everywhere?

The key challenge in proving this claim lies in the lack of lower semicontinuity of the functional $\mathsf{G}(\bullet, Q_0)$ with respect to L^1-convergence (or weak*-convergence in the BV space for that matter).

The $|Df|$-approximate continuity of p_f^τ implies that, at $|Df|$-almost every point x, $p_f^\tau(x)$ is very close to being constant on small neighborhoods. Since tangents look at the infinitesimal behavior of f, this may suggest the existence of a tangent at x whose local Poincaré constant is constantly equal to $p_f^\tau(x)$. With this in mind, we formulate the following question:

Question 3. *Is it true that, at $|Df|$-almost every point $x \in \Omega$, there exists $u \in \operatorname{Tan}^\tau(f,x)$ such that $p_u^\tau = p_f^\tau(x)$ almost everywhere with respect to $|Du|$?*

The answer to this question is affirmative for all points x such that $p_f^\tau(x) \in \{1/4, 1/2\}$ (cf. Propositions 7.2 and 7.3).

In addition, we observe that if one were able to prove that

$$p_f^\tau(x) = \max_{u \in \operatorname{Tan}^\tau(f,x)} \mathsf{G}(u, Q_0), \tag{8.1}$$

then Question 3 would have a positive resolution. Indeed, let $\bar{u} \in \operatorname{Tan}^\tau(f,x)$ be a maximizer, namely assume that

$$p_f^\tau(x) = \max_{u \in \operatorname{Tan}^\tau(f,x)} \mathsf{G}(u, Q_0) = \mathsf{G}(\bar{u}, Q_0).$$

Then, by Corollary 6.13,

$$\begin{aligned}\int_{Q_0} p_{\bar{u}}^\tau(y)\, d|D\bar{u}|(y) &\le \int_{Q_0} p_f^\tau(x)\, d|D\bar{u}|(y) \le p_f^\tau(x) = \mathsf{G}(\bar{u}, Q_0) \\ &= \int_{Q_0} p_{\bar{u}}^\tau(y)\, d|D\bar{u}|(y).\end{aligned}$$

It follows that all inequalities are equalities, and thus $p_{\bar{u}}^\tau(y) = p_f^\tau(x)$ for $|D\bar{u}|$-almost every point.

At the moment, we are not able to prove (8.1). However, we can establish the following weaker result.

Proposition 8.1. *Let $\tau \in [\tau(n), 1)$. For $|Df|$-almost every x, it holds*

$$p_f^\tau(x) = \sup_{u \in \operatorname{Tan}^\tau(f,x)} \mathsf{G}_1(u, Q_0).$$

Proof. Fix $u \in \operatorname{Tan}^\tau(f,x)$. By the definition of G_1, we have

$$\mathsf{G}_1(u, Q_0) \ge \operatorname{Osc}(u, Q_0)$$

because the unit cube belongs to $\mathcal{H}_{\le 1}(Q_0)$. Taking the supremum over all tangents and applying Theorem 1.3, we deduce the inequality

$$\sup_{u \in \operatorname{Tan}^\tau(f,x)} \mathsf{G}_1(u, Q_0) \ge p_f^\tau(x).$$

The other inequality follows from Proposition 4.1 and Corollary 6.13. Let $u \in \operatorname{Tan}^\tau(f,x)$ and let $\delta > 0$. We apply Proposition 4.1 to find a good family $\mathcal{F} \in \mathcal{H}_{\le 1}(Q_0)$ for $\mathsf{G}(u, Q_0)$. Notice that, for any cube Q, if $|Du|(Q) = 0$ then $\operatorname{Osc}(u, Q) = 0$. Therefore we can assume without loss of generality that $|Du|(Q) > 0$ for all cubes $Q \in \mathcal{F}$. Further, by the definition of good family, for every $Q \in \mathcal{F}$, it holds

$$|Du|(\tau Q) \ge \frac{1}{16} |Du|(Q) > 0.$$

This means that $\operatorname{spt}(|Du|) \cap \tau Q \ne \emptyset$ for all $Q \in \mathcal{F}$, which allows us to apply Corollary 6.13 for each cube of $\mathcal{F}$:

$$\begin{aligned}(1-\delta)\,\mathsf{G}_1(u, Q_0) &\le \sum_{Q \in \mathcal{F}} \operatorname{Osc}(u, Q)\, \ell(Q)^{n-1} \\ &\overset{(6.13)}{\le} \sum_{Q \in \mathcal{F}} p_f^\tau(x) |Du|(Q) \le p_f^\tau(x) |Du|(Q_0) \le p_f^\tau(x).\end{aligned}$$

Letting $\delta \downarrow 0$ first, and then taking the supremum over all tangents, we obtain the other inequality and we conclude the proof. ∎

Appealing to the same line of thought of the previous proof, one can show that the following inequalities hold at $|Df|$-almost every point: for every $\varepsilon \in (0,1)$ and every $u \in \mathrm{Tan}^\tau(f,x)$,

$$\mathsf{G}(u, Q_0) \leq \mathsf{G}_\varepsilon(u, Q_0) \leq \mathsf{G}_1(u, Q_0) \leq p_f^\tau(x).$$

Motivated by Question 2 and Question 3, one may also wonder whether the following weaker conclusion holds.

Question 4. *Is it true that, at $|Df|$-almost every point $x \in \Omega$, there exists $u \in \mathrm{Tan}^\tau(f,x)$ such that p_u^τ is constant almost everywhere with respect to $|Du|$?*

While a conclusive answer remains elusive at present, we are able to establish the following result, which gives a partial answer to Question 4.

Proposition 8.2 (Almost minimizing tangents). *Let $\tau \in [\tau(n), 1)$. For $x \in \Omega$, let*

$$\beta(x) := \inf\{|Du|\text{-essinf } p_u^\tau : u \in \mathrm{Tan}^\tau(f,x)\}. \tag{8.2}$$

For $|Df|$-almost every point $x \in \Omega$, it holds

$$\frac{1}{4} \leq \beta(x) \leq \frac{1}{2}.$$

Moreover, at such points, for each $\delta > 0$ there exists a non-trivial tangent $v \in \mathrm{Tan}^\tau(f,x)$ satisfying

$$\beta(x) \leq p_v^\tau(y) \leq \beta(x) + \delta$$

for $|Dv|$-almost every $y \in Q_0$.

Proof. Let $A \subset \mathrm{spt}(|Df|)$ be the set of points x where tangents to tangents are tangents, Corollary 6.13 applies, and $\mathrm{Tan}^\tau(f,x) \neq \{0\}$, which has full $|Df|$ measure (cf. Proposition 6.11 and Corollary 6.10). Observe that, thanks to Corollary 5.3, we deduce that $\beta(x) \in [1/4, 1/2]$ for every $x \in A$. This proves the first assertion.

For $x \in A$ and $\delta > 0$, we may hence find a non-trivial tangent $u \in \mathrm{Tan}^\tau(f,x)$ satisfying

$$|Du|\text{-essinf } p_u^\tau < \beta(x) + \delta.$$

Choose $y \in \mathrm{spt}(|Du|)$ such that $\mathrm{Tan}^\tau(u,y) \neq \{0\}$ and $p_u^\tau(y) \leq \beta(x) + \delta$. Now, let $v \in \mathrm{Tan}^\tau(u,y) \subset \mathrm{Tan}^\tau(f,x)$ be a non-trivial tangent. Then, by Corollary 6.13 we have $p_v^\tau \leq p_u^\tau(y) \leq \beta(x) + \delta$ at $|Dv|$-almost all points. On the other hand, by the definition of $\beta(x)$, we have $p_v^\tau \geq \beta(x)$ at $|Dv|$-almost all points. This proves the second assertion. ∎

Remark 8.3. If the minimization problem posed in Question 2 has a minimizer $\bar{u}$ in $\mathrm{Tan}_1^\tau(f,x)$, then $\alpha(x) = \beta(x)$, where $\beta(x)$ is defined in (8.2), at least if $\tau \in [\tau(n), 1)$. Indeed, let $\bar{u} \in \mathrm{Tan}_1^\tau(f,x)$ be the minimizer for $\alpha(x)$ and let $\tilde{u}$ be a tangent given by Proposition 8.2. Then, by minimality,

$$\alpha(x) = \mathsf{G}(\bar{u}, Q_0) \leq \mathsf{G}(\tilde{u}, Q_0) = \int_{Q_0} p_{\tilde{u}}^\tau \, d|D\tilde{u}| \leq \beta(x) + \delta.$$

On the other hand, by the definition of $\beta(x)$, it holds $\beta(x) \leq p_{\bar{u}}^{\tau}$ almost everywhere with respect to $|D\bar{u}|$, hence by integration,

$$\beta(x) \leq \int_{Q_0} p_{\bar{u}}^{\tau}\, d|D\bar{u}| = \mathsf{G}(\bar{u}, Q_0) = \alpha(x).$$

Sending $\delta \to 0$ we conclude that $\alpha(x) = \beta(x)$, as we wanted. Moreover, in this case it follows that $p_{\bar{u}}^{\tau}$ is constant and equal to $\alpha(x)$ almost everywhere with respect to $|D\bar{u}|$, thus answering Question 4.

8.3. Rigidity at points with minimal Poincaré constant

As we have already seen in Propositions 7.2 and 7.3, there are certain additional properties of $\mathrm{Tan}^{\tau}(f, x)$, when $p_f^{\tau}(x)$ attains an extremal value. We also know that affine functions are a prototype of a mono-directional monotone function with local Poincaré constant equal to $1/4$ almost everywhere. In fact, if $u\colon Q_0 \to \mathbb{R}$ is linear, then

$$\frac{\mathrm{Osc}(u, Q)}{|Du|(Q)}\, \ell(Q)^{n-1} = \frac{1}{4}$$

for all subcubes $Q \subset Q_0$ with one face orthogonal to the gradient of u. In this context, the following question probes the validity of a converse statement, specifically exploring whether the upper bound $1/4$ on such quotients enforces affinity:

Question 5. *Let $u\colon Q_0 \to \mathbb{R}$ be a mono-directional (in direction e) and monotone BV function. Does the inequality*

$$\mathrm{Osc}(u, Q)\, \ell(Q)^{n-1} \leq \frac{1}{4}|Du|(Q),$$

for all cubes $Q \subset Q_0$, imply that u is affine? If not, could we reach this conclusion by imposing the equality condition

$$\mathrm{Osc}(u, Q)\, \ell(Q)^{n-1} = \frac{1}{4}|Du|(Q) \tag{8.3}$$

for all cubes $Q \subset Q_0$ with one face orthogonal to e?

Remark 8.4. Without the monotonicity assumption, the answer to the first part of Question 5 is negative: a direct computation shows that the function $u(x) = |x|$, defined on $(-1/2, 1/2)$, satisfies $\mathrm{Osc}(u, I) \leq \frac{1}{4}|Du|(I)$ for every interval $I \subseteq (-1/2, 1/2)$.

Remark 8.5. At the time this paper was accepted for publication, Conti and the first author established in [8] that any locally integrable $u\colon (a, b) \to \mathbb{R}$ satisfying (8.3) must indeed be affine. In particular, this answers the second part of Question 5 for the one-dimensional case.

Proposition 7.3 underscores the significance of this question: A positive resolution of either claim could lead to a positive answer to the following questions.

Question 6. *Is it true that, for $|Df|$-almost every $x \in \{p_f^{\tau} = 1/4\}$, there* exists *a linear tangent in* $\mathrm{Tan}^{\tau}(f, x)$*?*

Question 7. *Is it true that, for $|Df|$-almost every $x \in \{p_f^\tau = 1/4\}$,* every *tangent in* $\mathrm{Tan}^\tau(f, x)$ *is linear?*

Lastly, we explore a potential converse implication in Proposition 7.3.

Question 8. *Does $|Df|$-almost every point $x \in \Omega$ have the following property*: *if*

$$p_u^\tau = \frac{1}{4} \quad |Du|\text{-almost everywhere}$$

for every tangent $u \in \mathrm{Tan}^\tau(f, x)$, then $p_f^\tau(x) = 1/4$?

8.4. Size of points with minimal Poincaré constant

By standard measure-theoretic arguments concerning the *invariance directions* of tangent measures (see [6]; see also [5, 18]), the following dimensional estimate holds: the set of points where all tangents of f are linear has full dimension (provided it has positive $|Df|$-measure). This suggests the following question, which would have a positive answer in case Question 7 is solved affirmatively.

Question 9. *Provided that it has positive $|Df|$-measure, does the set $\{p_f^\tau = 1/4\}$ have Hausdorff dimension n?*

A. Some measure theoretic results

A.1. A covering theorem in $\mathbb{R}$

We recall some definitions. Let $A \subset \mathbb{R}$. A *cover* of A is any family of sets $\mathcal{F}$ such that for every $x \in A$ there exists $U \in \mathcal{F}$ with $x \in U$. We call $\mathcal{F}(x)$ the collection of sets from $\mathcal{F}$ that contain x. We say that $\mathcal{F}$ is a *fine cover* of A if for every $x \in A$ and for every $\delta > 0$ there exists $U \in \mathcal{F}(x)$ with $\mathrm{diam}(U) < \delta$.

Definition A.1 (Vitali property). Given a cover $\mathcal{F}$ of A, and a Radon measure μ, we say that $\mathcal{F}$ has the μ-*Vitali property* if there exists a disjoint collection $\mathcal{F}' \subset \mathcal{F}$ such that

$$\mu\Big(A \setminus \bigcup_{U \in \mathcal{F}'} U\Big) = 0.$$

It is well known (see, e.g., Theorem 2.2 in [23]) that any fine cover $\mathcal{F}$ of A with closed balls (also uncentered ones) has the $\mathcal{L}^n$-Vitali property. In the one-dimensional case, relying on the natural order relation on the real line, one can actually show the same for every measure μ. This observation can be found in Remark 5, Chapter 1 of [14], along with some hints for its proof. For completeness, we report here a detailed argument.

Theorem A.2 (Vitali with closed intervals in $\mathbb{R}$). *Let μ be any Radon measure on the real line $\mathbb{R}$. Then every fine cover $\mathcal{F}$ of a set E with closed intervals has the μ-Vitali property.*

Proof. We can assume without loss of generality that $E \subseteq (0, 1)$, and that every element of the cover is contained in $(0, 1)$. Moreover, we can assume that $E \subseteq \mathrm{spt}(\mu)$, and therefore that $\mu(I) > 0$ for every $I \in \mathcal{F}$.

We start by selecting $S_1 \in \mathcal{F}$ such that

$$\mu(S_1) \geq \frac{1}{2} \sup\{\mu(I) : I \in \mathcal{F}\}.$$

Then we inductively select S_k, $k \geq 2$, so that

$$S_k \cap \bigcup_{i=1}^{k-1} S_i = \emptyset \quad \text{and} \quad \mu(S_k) \geq \frac{1}{2} \sup \Big\{\mu(I) : I \in \mathcal{F},\ I \cap \bigcup_{i=1}^{k-1} S_i = \emptyset\Big\}.$$

If the process terminates in a finite number of steps then we are done, as μ-almost all the set E must be covered. Suppose then that the process goes on, giving an infinite sequence S_k, $k \geq 1$. Since the S_k's are disjoint and $(0, 1)$ has finite μ-measure, we must have

$$\mu(S_k) \to 0 \quad \text{as } k \to \infty. \tag{A.1}$$

We now claim that

$$I \cap \bigcup_{k=1}^{\infty} S_k \neq \emptyset \quad \text{for every } I \in \mathcal{F}. \tag{A.2}$$

Indeed, otherwise there would exist $I \in \mathcal{F}$ disjoint from all the S_k, but then it must have been selected at some point in the process because of (A.1) (recall that $\mu(I) > 0$ for every $I \in \mathcal{F}$). Therefore (A.2) is proven.

Now we prove that

$$\mu\Big(E \setminus \bigcup_{k=1}^{\infty} S_k\Big) = 0.$$

For every $k \geq 1$, let us define the *enlarged* set

$$\widehat{S}_k := \bigcup \{S \in \mathcal{F} : \mu(S) \leq 2\mu(S_k),\ S \cap S_k \neq \emptyset\}.$$

We have the following chain of inclusions:

$$\begin{aligned} E \setminus \bigcup_{k=1}^{h} S_k &\subseteq \bigcup \Big\{S \in \mathcal{F} : S \cap \bigcup_{k=1}^{h} S_k = \emptyset\Big\} \\ &= \bigcup \Big\{S \in \mathcal{F} : S \cap \bigcup_{k=1}^{h} S_k = \emptyset,\ S \cap \bigcup_{k=h+1}^{\infty} S_k \neq \emptyset\Big\} \\ &= \bigcup_{j=h}^{\infty} \Big\{S \in \mathcal{F} : \bigcup_{k=1}^{j} S_k = \emptyset,\ S \cap S_{j+1} \neq \emptyset\Big\} \subseteq \bigcup_{j=h}^{\infty} \widehat{S}_{j+1}. \end{aligned} \tag{A.3}$$

The final step is to show that

$$\mu(\widehat{S}_k) \leq 5\mu(S_k). \tag{A.4}$$

This is sufficient to conclude in view of (A.3), and sending $h \to \infty$.

Now (A.4) follows from the following inequality:

$$\mu(\widehat{I}) \leq \mu(I) + \sup\{\mu(I_L) + \mu(I_R) : I_L, I_R \cap I \neq \emptyset,\ \mu(I_L), \mu(I_R) \leq 2\mu(I)\}.$$

Indeed, it is sufficient to choose I_L and I_R as the biggest intervals extending to the left and to the right of I (and in case there is no biggest, one can choose a "maximizing" sequence and use inner regularity of μ). ■

Theorem A.3. *Let $E \subset \Omega \subset \mathbb{R}^n$, with Ω open set, let $\mu^*: \mathcal{P}(\mathbb{R}^n) \to [0,\infty]$ be an outer Radon measure on $\mathbb{R}^n$ and let $\tau \in [0,1)$. Moreover, let $\mathcal{F}$ be a cover of E with closed cubes such that for every $x \in E$ there exist arbitrarily small cubes $Q \in \mathcal{F}$ with $x \in \tau Q$. Then $\mathcal{F}$ has the μ^*-Vitali property, i.e., there exists a countable family $\mathcal{F}_0 \subset \mathcal{F}$ of cubes with pairwise disjoint closures such that*

$$\mu^*\Big(E \setminus \bigcup_{Q \in \mathcal{F}_0} \bar{Q}\Big) = 0.$$

Proof. We can apply Theorem 1.147 in [17]. Indeed, the cover $\mathcal{F}$ is a fine γ-Morse cover (see Definition 1.137 in [17]), with $\gamma = 2\sqrt{n}/(1-\tau)$. Indeed, if $x \in \tau Q$ then by elementary considerations $\overline{B}(x, \frac{1-\tau}{2}\ell(Q)) \subset \overline{Q} \subset \overline{B}(x, \sqrt{n}\ell(Q))$. ■

We now recall the following Lebesgue-differentiation type results. Notice the set A in the next statement need not be measurable (see also Corollary 2.14(1) and Remark 2.15(2) in [23]).

Theorem A.4. *Let $\Omega \subset \mathbb{R}^n$, and μ a Radon non-negative measure on Ω. Fix $\tau \in [0,1)$ and a set $A \subset \Omega$. Then for μ-almost every $x \in A$, the following happens: for every family of cubes $(Q_r)_{r>0}$, with $x \in \tau Q_r$ and $\ell(Q_r) \to 0$ as $r \to 0$,*

$$\lim_{r\to 0} \frac{\mu(A \cap Q_r)}{\mu(Q_r)} = 1.$$

Proof. We start observing that in the case when A is Borel we can just apply Remark 1.160 in [17], after noticing that

$$B\Big(x, \frac{1-\tau}{2}\,\ell(Q_r)\Big) \subset Q_r \subset B(x, \sqrt{n}\ell(Q_r)).$$

In the general case, we assume that μ is (extended to be) a Borel outer measure. Let

$$A_{\mathrm{bad}} := \Big\{x \in A : \liminf_{r\to 0} \frac{\mu(A \cap Q_r^x)}{\mu(Q_r^x)} < 1 \quad \text{for some } Q_r^x, \ell(Q_r^x) \to 0, x \in \tau Q_r^x\Big\}.$$

Then $A_{\mathrm{bad}} = \bigcup_{\varepsilon>0} A_\varepsilon$, where

$$A_\varepsilon := \{x \in A : \exists Q_j^x,\ \ell(Q_j^x) \to 0, x \in \tau Q_j,\ \mu(A \cap Q_j^x) < (1-\varepsilon)\mu(Q_j^x)\}.$$

We now fix $\varepsilon > 0$ and show that $\mu(A_\varepsilon) = 0$. Since each cube Q_j^x in the definition of A_ε is open, by inner approximation it is possible to find another sequence of open cubes $\tilde{Q}_j^x \subset Q_j^x$ such that

$$x \in \tau\tilde{Q}_j^x, \quad \mu(A \cap \tilde{Q}_j^x) < (1-\varepsilon)\mu(\tilde{Q}_j^x) \quad \text{and} \quad \mu(\partial\tilde{Q}_j^x) = 0.$$

We may henceforth assume that for each $x \in A_\varepsilon$, the corresponding cubes Q_j^x satisfy $\mu(\partial Q_j^x) = 0$. By definition,

$$\mu(A_\varepsilon) = \inf\{\mu(U) : U \supset A_\varepsilon,\ U \text{ open}\}.$$

Fix $\delta > 0$ and take $U_\delta \supset A_\varepsilon$ open with $\mu(U_\delta) \le \mu(A_\varepsilon) + \delta$. The cubes $\{Q_j^x\}_{x,j}$ that are contained in U_δ provide a fine cover of A_ε. By Theorem A.3, and since $A_\varepsilon \subseteq A$, there is a disjoint subcollection $\mathcal{F}$ with

$$\mu\Big(A_\varepsilon \setminus \bigcup_{Q\in\mathcal{F}} (Q \cap A)\Big) = \mu\Big(A_\varepsilon \setminus \bigcup_{Q\in\mathcal{F}} Q\Big) = \mu\Big(A_\varepsilon \setminus \bigcup_{Q\in\mathcal{F}} \bar{Q}\Big) = 0.$$

Then

$$\begin{aligned}\mu(A_\varepsilon) &\le \mu\Big(A_\varepsilon \setminus \bigcup_{Q\in\mathcal{F}} (Q \cap A)\Big) + \mu\Big(\bigcup_{Q\in\mathcal{F}} Q \cap A\Big)\\ &\le \sum_{Q\in\mathcal{F}} \mu(Q \cap A) < (1-\varepsilon) \sum_{Q\in\mathcal{F}} \mu(Q) \le (1-\varepsilon)\mu(U_\delta) \le (1-\varepsilon)(\mu(A_\varepsilon)+\delta).\end{aligned}$$

Sending $\delta \to 0$, we discover that $\mu(A_\varepsilon) \le (1-\varepsilon)\mu(A_\varepsilon)$, thus $\mu(A_\varepsilon) = 0$. ∎

We need the following version of Lebesgue's theorem for essentially bounded functions (notice that we are only applying this result to either p^τ in Lemmas 5.1 and 3.1, or to the polar g in Lemma 6.4, which are bounded functions). Its validity remains true for locally μ-integrable functions, but the proof is slightly more complicated.

Theorem A.5. *Let $\Omega \subset \mathbb{R}^n$ and let μ be a Radon non-negative measure on Ω. Let u be a Borel, μ-essentially bounded function. Then for μ-a.e. $x \in \Omega$ the following happens: for every family of cubes $(Q_r)_{r>0}$, with $x \in \tau Q_r$ and $\ell(Q_r) \to 0$ as $r \to 0$,*

$$\lim_{r\to 0} \fint_{Q_r} |u(y) - u(x)|\, d\mu(y) = 0.$$

Proof. Fix $\varepsilon > 0$, and let

$$A_j^\varepsilon := \{x \in \Omega :\ j\varepsilon < u(x) \le (j+1)\varepsilon\}.$$

By Theorem A.4, for every j and for μ-a.e. $x \in A_j^\varepsilon$,

$$\lim_{r\to 0} \frac{\mu(Q_r \setminus A_j^\varepsilon)}{\mu(Q_r)} = 0.$$

For μ-a.e. $x \in A_j^\varepsilon$,

$$\begin{aligned}\int_{Q_r} |u(x)-u(y)|\, d\mu(y) &= \int_{Q_r\cap A_j^\varepsilon} |u(x)-u(y)|\, d\mu(y) + \int_{Q_r\setminus A_j^\varepsilon} |u(x)-u(y)|\, d\mu(y)\\ &\le \varepsilon\mu(Q_r \cap A_j^\varepsilon) + 2\|u\|_{L^\infty_\mu}\, \mu(Q_r \setminus A_j^\varepsilon),\end{aligned}$$

and since this holds independently of j, from the limit above we conclude that, for μ-a.e. $x \in \Omega$,

$$\limsup_{r\to 0} \fint_{Q_r} |u(x) - u(y)|\, d\mu(y) \le \varepsilon.$$

Considering a sequence $\varepsilon_k \to 0$ and using σ-subadditivity, we reach the conclusion. ∎

A.2. A criterion for set-functionals

Let Ω be an open set in $\mathbb{R}^n$, and let $\mathcal{O}(\Omega)$ be the family of all open subsets of Ω.

Theorem A.6 (De Giorgi–Letta, Theorem 1.53 in [4]). *Let* $\mathsf{F}\colon \mathcal{O}(\Omega) \to [0,\infty]$ *be an increasing set function with* $\mathsf{F}(\emptyset) = 0$. *Assume that the following properties hold*:

(i) *If* $U, V \in \mathcal{O}(\Omega)$, *then* $\mathsf{F}(U \cup V) \leq \mathsf{F}(U) + \mathsf{F}(V)$ (subadditivity);

(ii) *If* $U, V \in \mathcal{O}(\Omega)$ *and* $U \cap V = \emptyset$, *then* $\mathsf{F}(U \cup V) \geq \mathsf{F}(U) + \mathsf{F}(V)$ (superadditivity);

(iii) $\mathsf{F}(U) = \sup\{\mathsf{F}(V) : V \in \mathcal{O}(\Omega),\ V \subset\subset U\}$ (inner regularity).

Then there is a uniquely determined measure μ *that extends* F *to all Borel subsets, given by*

$$\mu(B) := \inf\{\mathsf{F}(U) : U \in \mathcal{O}(\Omega),\ U \supset B\}, \qquad B \text{ Borel}.$$

A.3. An elementary convergence lemma

We record here the following observation (its proof is a modification of the standard upper semicontinuity property of weak convergence on compact sets, see Example 1.63 in [4]).

Lemma A.7. *Let* $Q \subset \mathbb{R}^n$ *be a cube. Let* $(\mu_j)_j$ *be a sequence of non-negative measures on* Q *such that* $\mu_j \overset{*}{\rightharpoonup} \mu$ *and* $\mu_j(Q) \to \mu(Q)$ *as* $j \to \infty$ *for some finite measure* μ *on* Q. *Then for every relatively closed set* $F \subset Q$, *it holds*

$$\mu(F) \geq \limsup_{j \to +\infty} \mu_j(F).$$

In particular, for every Borel set $B \subset Q$ *such that* $\mu(\partial B \cap Q) = 0$, *it holds*

$$\lim_{j \to +\infty} \mu_j(B) = \mu(B).$$

B. An example of non-existence of K_0 for a BV function

Lemma B.1. *There exists a purely cantorian* $u \in \mathrm{BV}(0,1)$, *with* $|Du|((0,1)) = 1$, *satisfying*

$$\liminf_{\varepsilon \to 0} \mathsf{K}_\varepsilon(u) = \frac{1}{4} \quad \text{and} \quad \limsup_{\varepsilon \to 0} \mathsf{K}_\varepsilon(u) = \frac{1}{2}.$$

Proof. The construction consists of several steps.

Step 1. Let $I = [a,b] \subset [0,1]$ be a given interval and let k be a given positive integer. We define $I^k \subset [0,1]$ as the (disjoint) union of the k closed intervals $\{[a_j, a_j + |I|k^{-2}]\}_{j=1}^k$ where the $\{a_j\}_{j=1}^k$ are equi-distributed (i.e., $a_j - a_{j-1} = |I|k^{-1}$) and $a_1 = a$ is the left extreme of I. Notice that

$$|I^k| = k \times |I|k^{-2} = \frac{|I|}{k}.$$

Step 2. Let $\{k_i\}_{i=1}^\infty$ be a sequence of positive integers that are growing sufficiently fast, for instance such that

$$\frac{k_i^2}{k_{i+1}} \leq \frac{1}{2^i}.$$

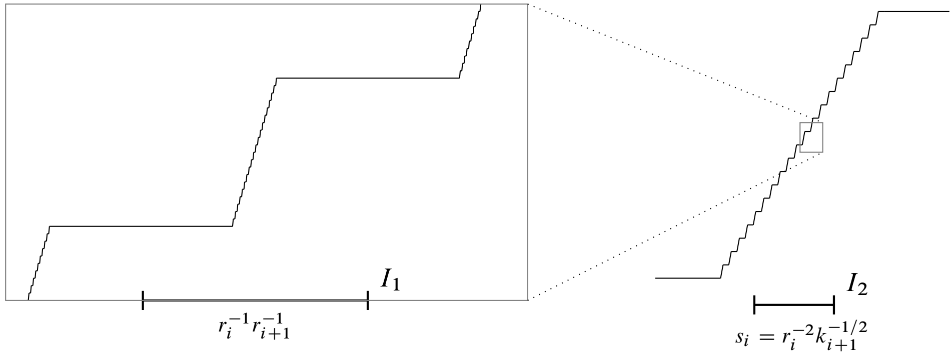

Figure 4. On the left: on the interval I_1 of scale $r_i^{-1} k_i^{-1}$ the function is close to a jump. On the right: on the interval I_2 of intermediate scale r_i^{-1} the function looks affine.

We define $J_0 := [0, 1]$ and we define the J_i inductively. Assuming that J_i is a union of $r_i = k_0 \times \cdots \times k_i$ disjoint intervals (with $k_0 = 1$) of the same length, we define $J_{i+1} \subset J_i$ as follows: denoting by $J_i(r)$, with $r = 1, \ldots, r_i$, the disjoint sub-intervals making up J_i, we set

$$J_{i+1} = \bigcup_{r=1}^{r_i} J_i(r)^{k_{i+1}}.$$

The new set J_{i+1} consists of $r_i \times k_{i+1} = r_{i+1}$ disjoint intervals. Since all $J_i(r)$ have the same length it follows from the definition in Step 1 that all $J_{i+1}(r)$ are also of the same length. Moreover, since $|J_i(r)|^{k_{i+1}} = k_{i+1}^{-1} |J_i(r)|$, we have

$$|J_{i+1}| = \frac{|J_i|}{k_{i+1}} = \frac{|J_{i-1}|}{k_{i+1} k_i} \cdots = \frac{|J_1|}{k_{i+1} \times \cdots \times k_2} = \frac{1}{r_{i+1}}.$$

Step 3. For each $i \in \mathbb{N}$, we define a probability density $h_i : [0, 1] \to \mathbb{R}$ by setting $h_i = r_i \mathbb{1}_{J_i}$. We then consider its primitive $u_i : [0, 1] \to \mathbb{R}$ defined by

$$u_i(t) := \int_0^t h_i(x)\, dx,$$

which defines a continuous and non-decreasing function. By construction $u_i(0) = 0$, $u_i(1) = 1$ and u_i has bounded variation $|Du_i|(0, 1) = \|h_i\|_{L^1} = 1$. It is not hard to see that $u_{i+1} \geq u_i$ and

$$u_{i+1}(t) - u_i(t) \leq \frac{1}{k_i k_{i+1}} \leq \frac{1}{k_{i+1}}.$$

This, in turn, gives (assuming that $m \geq n$)

$$\|u_m - u_n\|_\infty \leq \sum_{i=n+1}^{m} \frac{1}{k_i} \leq \sum_{i=n+1}^{m} \frac{1}{2^{i-1}} \leq \frac{1}{2^{n-1}}.$$

It follows that the sequence $\{u_i\}_{i=1}^{\infty}$ is Cauchy in C^0, and thus, it converges to some continuous u with $u(0) = 0$ and $u(1) = 1$, which is also non-decreasing. Hence, u is BV and $D^j u = 0$. Moreover,

$$\operatorname{spt}(|Du|) = J_\infty := \bigcap_{i=1}^{\infty} J_i.$$

Since $|J_\infty| \le |J_i| \to 0$, we further deduce that $D^a u = 0$. This proves that u is purely cantorian.

Step 4. By construction, at scale $r_i^{-1} r_{i+1}^{-1}$, u is closer and closer to a function that jumps (see interval I_1 in Figure 4), and therefore

$$\lim_{i\to\infty} \mathsf{K}_{r_i^{-1} r_{i+1}^{-1}}(u) = \frac{1}{2}.$$

On the other side, at an intermediate scale s_i with $r_i^{-1} r_{i+1}^{-1} \ll s_i \ll r_i^{-2}$, say $s_i = r_i^{-2} k_{i+1}^{-1/2}$, u looks pretty much like a piecewise linear map oscillating between constants and affine maps of the same slope (see interval I_2 in Figure 4). Observe that the total contribution of the intervals that contain both the almost-affine part and the constant part is negligible in the limit $i \to \infty$. It follows that

$$\lim_{i\to\infty} \mathsf{K}_{r_i^{-2} k_{i+1}^{-1/2}}(u) = \frac{1}{4}. \qquad \blacksquare$$

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References

[1] Ambrosio, L., Bourgain, J., Brezis, H. and Figalli, A.: Perimeter of sets and BMO-type norms. *C. R. Math. Acad. Sci. Paris* **352** (2014), no. 9, 697–698. Zbl 1316.46026 MR 3258259

[2] Ambrosio, L., Bourgain, J., Brezis, H. and Figalli, A.: BMO-type norms related to the perimeter of sets. *Comm. Pure Appl. Math.* **69** (2016), no. 6, 1062–1086. Zbl 1352.46026 MR 3493625

[3] Ambrosio, L. and Comi, G. E.: Anisotropic surface measures as limits of volume fractions. In *Current research in nonlinear analysis*, pp. 1–32. Springer Optim. Appl. 135, Springer, Cham, 2018. Zbl 1408.49008 MR 3793260

[4] Ambrosio, L., Fusco, N. and Pallara, D.: *Functions of bounded variation and free discontinuity problems*. Oxford Math. Monogr., The Clarendon Press, Oxford University Press, New York, 2000. Zbl 0957.49001 MR 1857292

[5] Ambrosio, L. and Soner, H. M.: A measure-theoretic approach to higher codimension mean curvature flows. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **25** (1997), no. 1-2, 27–49. Zbl 1043.35136 MR 1655508

[6] Arroyo-Rabasa, A.: An elementary approach to the dimension of measures satisfying a first-order linear PDE constraint. *Proc. Amer. Math. Soc.* **148** (2020), no. 1, 273–282. Zbl 1447.28003 MR 4042849

[7] Arroyo-Rabasa, A., Bonicatto, P. and Del Nin, G.: Representation of the total variation as a Γ-limit of BMO-type seminorms. *Indiana Univ. Math. J.* **73** (2024), no. 1, 341–365. Zbl 07834170 MR 4721319

[8] Arroyo-Rabasa, A. and Conti, S.: Affine rigidity of functions with additive oscillation. Preprint 2025, arXiv:2510.27360v1.

[9] Bourgain, J., Brezis, H. and Mironescu, P.: Another look at Sobolev spaces. In *Optimal control and partial differential equations*, pp. 439–455. IOS, Amsterdam, 2001. Zbl 1103.46310 MR 3586796

[10] Bourgain, J., Brezis, H. and Mironescu, P.: A new function space and applications. *J. Eur. Math. Soc. (JEMS)* **17** (2015), no. 9, 2083–2101. Zbl 1339.46028 MR 3420502

[11] Brezis, H., Seeger, A., Van Schaftingen, J. and Yung, P.-L.: Families of functionals representing Sobolev norms. *Anal. PDE* **17** (2024), no. 3, 943–979. Zbl 1551.46026 MR 4736522

[12] Brezis, K.: How to recognize constant functions. A connection with Sobolev spaces. *Russian Math. Surv.* **57** (2002), no. 4, 693–708; translated from *Uspekhi Mat. Nauk* **57** (2002), no. 4 (346), 59–74. Zbl 1072.46020 MR 1942116

[13] Dávila, J.: On an open question about functions of bounded variation. *Calc. Var. Partial Differential Equations* **15** (2002), no. 4, 519–527. Zbl 1047.46025 MR 1942130

[14] de Guzmán, M.: *Differentiation of integrals in* $\mathbb{R}^n$. Lecture Notes Math. 481, Springer, Berlin-New York, 1975. Zbl 0327.26010 MR 0457661

[15] de Philippis, G., Fusco, N. and Pratelli, A.: On the approximation of SBV functions. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **28** (2017), no. 2, 369–413. Zbl 1366.26020 MR 3649354

[16] Farroni, F., Fusco, N., Guarino Lo Bianco, S. and Schiattarella, R.: A formula for the anisotropic total variation of SBV functions. *J. Funct. Anal.* **278** (2020), no. 9, article no. 108451, 32 pp. Zbl 1434.26029 MR 4062330

[17] Fonseca, I. and Leoni, G.: *Modern methods in the calculus of variations: L^p spaces*. Springer Monogr. Math., Springer, New York, 2007. Zbl 1153.49001 MR 2341508

[18] Fragalà, I. and Mantegazza, C.: On some notions of tangent space to a measure. *Proc. Roy. Soc. Edinburgh Sect. A* **129** (1999), no. 2, 331–342. Zbl 0937.58009 MR 1686704

[19] Fusco, N., Moscariello, G. and Sbordone, C.: A formula for the total variation of SBV functions. *J. Funct. Anal.* **270** (2016), no. 1, 419–446. Zbl 1332.26018 MR 3419767

[20] Fusco, N., Moscariello, G. and Sbordone, C.: BMO-type seminorms and Sobolev functions. *ESAIM Control Optim. Calc. Var.* **24** (2018), no. 2, 835–847. Zbl 1410.46021 MR 3816417

[21] Hadwiger, H.: Gitterperiodische Punktmengen und Isoperimetrie. *Monatsh. Math.* **76** (1972), 410–418. Zbl 0248.52012 MR 0324550

[22] Lahti, P. and Nguyen, Q.-H.: BMO-type functionals, total variation, and Γ-convergence. *Proc. Amer. Math. Soc.* **152** (2024), no. 9, 3817–3830. Zbl 07892864 MR 4781976

[23] Mattila, P.: *Geometry of sets and measures in Euclidean spaces*. Cambridge Stud. Adv. Math. 44, Cambridge University Press, Cambridge, 1995. Zbl 0819.28004 MR 1333890

[24] Nguyen, H.-M.: Further characterizations of Sobolev spaces. *J. Eur. Math. Soc. (JEMS)* **10** (2008), no. 1, 191–229. Zbl 1228.46033 MR 2349901

[25] Ponce, A. C. and Spector, D.: On formulae decoupling the total variation of BV functions. *Nonlinear Anal.* **154** (2017), 241–257. Zbl 1359.26011 MR 3614653

[26] Preiss, D.: Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities. *Ann. of Math. (2)* **125** (1987), no. 3, 537–643. Zbl 0627.28008 MR 0890162

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