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Hyperbolic convexity of holomorphic level sets

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Abstract. We prove that the sublevel set $\{z \in \mathbb{D} : k_{\mathbb{D}}(z, z_0) - k_{\mathbb{D}}(f(z), w_0) < \mu\}$, $\mu \in \mathbb{R}$, is geodesically convex with respect to the Poincaré distance $k_{\mathbb{D}}$ in the unit disk $\mathbb{D}$ for every $z_0, w_0 \in \mathbb{D}$ and every holomorphic $f\colon \mathbb{D} \to \mathbb{D}$ if and only if $\mu \leqslant 0$. An analogous result is established also for the set $\{z \in \mathbb{D} : 1 - |f(z)|^2 < \lambda(1 - |z|^2)\}$, $\lambda > 0$. This extends a result of Solynin (2007) and solves a problem posed by Arango, Mejía and Pommerenke (2019). We also propose several open questions aiming at possible extensions to more general settings.

Dedicated to the memory of Prof. Christian Pommerenke.

1. Introduction

In 2007, Solynin [22] proved a remarkable result, which in a slightly different form can be stated as follows.

Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map of the unit disk $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ equipped with the Poincaré distance $k_{\mathbb{D}}$, and let z_0 and w_0 be two points in $\mathbb{D}$. Then

$$\mathcal{D}(f) = \mathcal{D}(f; z_0, w_0) := \{z \in \mathbb{D} : k_{\mathbb{D}}(z, z_0) - k_{\mathbb{D}}(f(z), w_0) < 0\}$$

is either empty or it is a hyperbolically convex domain, which means that for every $z_1, z_2 \in \mathcal{D}(f)$, the hyperbolic geodesic segment $[z_1, z_2]_h$ joining these two points is entirely contained in $\mathcal{D}(f)$.

Similarly to the classical Schwarz–Pick lemma playing a fundamental role in complex analysis, both in one and higher dimensions, see e.g. [2], this result relates holomorphic mappings with the hyperbolic geometry in $\mathbb{D}$. To some extent, the interplay of hyperbolic convexity and holomorphicity has been previously exploited via Jørgensen's theorem [13], see e.g. [7, 8, 11, 16]. It is also worth mentioning that hyperbolically convex functions, i.e., conformal mappings onto hyperbolically convex domains, admit analytic characterization;

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their properties have been thoroughly studied, see e.g. [5,14,17,19] and references therein. Recently, using Solynin's result stated above, it has been found in [10] that non-linear resolvents of infinitesimal generators are hyperbolically convex functions.

In this paper, we give a different proof and extend Solynin's result to two families of nested subsets of $\mathbb{D}$:

$$\mathcal{D}_\mu(f) \subset \mathcal{D}_0(f) = \mathcal{D}(f) = \Omega_1(f) \subset \Omega_\lambda(f), \quad \mu < 0,\ \lambda > 1,$$

introduced as follows. First, the sets $\mathcal{D}_\mu(f)$, $\mu \in \mathbb{R}$, are defined in a very natural way, namely, as

$$\mathcal{D}_\mu(f) := \{z \in \mathbb{D} : k_{\mathbb{D}}(z, z_0) - k_{\mathbb{D}}(f(z), w_0) < \mu\}.$$

In Section 4, we prove that for every $\mu < 0$, this set is either empty or hyperbolically convex.

In order to define $\Omega_\lambda(f)$ in a less technical way, see (1.1) below, we observe that replacing f by the composition $T_2 \circ f \circ T_1$, where T_1 and T_2 are suitable conformal automorphisms of $\mathbb{D}$, we may suppose that $z_0 = w_0 = 0$ and accordingly rewrite the inequality $k_{\mathbb{D}}(f(z), w_0) > k_{\mathbb{D}}(z, z_0)$ in Euclidean terms as $|f(z)| > |z|$, or equivalently, as

$$\nu_f(z) := \frac{1 - |f(z)|^2}{1 - |z|^2} = \frac{\lambda_{\mathbb{D}}(z)}{\lambda_{\mathbb{D}}(f(z))} < 1,$$

where $\lambda_{\mathbb{D}}$ stands for the density of the Poincaré metric.

The result of Solynin stated above was conjectured (and proved for a special case) by Mejía and Pommerenke in their study [18] of the analytic fixed point function associated to f, which was shown to map $\mathbb{D}$ conformally onto

$$\mathcal{D}(f) = \Omega_1(f) := \{z \in \mathbb{D} : \nu_f(z) < 1\};$$

see also [21]. Much later, Arango, Mejía, and Pommerenke [4] considered the family

$$\Omega_\lambda(f) := \{z \in \mathbb{D} : \nu_f(z) < \lambda\}, \quad \lambda > 0, \tag{1.1}$$

and asked whether Ω_λ is necessarily hyperbolically convex for $\lambda > 1$. In Section 3, we give an affirmative answer to this question for all $\lambda \geqslant 1$. Surprisingly, this more general result is obtained by a simpler method as compared to the very elegant, but more involved ideas employed in [22].

It is also worth mentioning that $\mathcal{D}_\mu(f)$ fails to be hyperbolically convex if $\mu > 0$ already for automorphisms (see Example 4.2). Therefore, even though the interpretation of the level sets $\Omega_\lambda(f)$ in terms of the hyperbolic geometry seems to be less direct than that of $\mathcal{D}_\mu(f)$, these domains $\Omega_\lambda(f)$ appear to be a suitable way to extend the family $\mathcal{D}_\mu(f)$ beyond the "critical" level set $\mathcal{D}(f)$. Finally, note that the situation is symmetric: $\Omega_\lambda(f)$ are in general not hyperbolically convex for $\lambda \in (0, 1)$, see Example 3.2.

The plan of the paper is as follows. In the next section, we recall the basics of the hyperbolic geometry in $\mathbb{D}$ and prove some technical lemmas. We prove that the domains $\Omega_\lambda(f)$, $\lambda \geqslant 1$, and $\mathcal{D}_\mu(f)$, $\mu < 0$, are hyperbolically convex in Sections 3 and 4, respectively. In the concluding Section 5, we pose a few open questions indicating possible ways to develop similar results in various settings.

2. Preliminaries and auxiliary results

Recall that the hyperbolic (Poincaré) metric in $\mathbb{D}$ at a point $z \in \mathbb{D}$ is given by $(u, v) \mapsto \lambda_{\mathbb{D}}^2(z)\,\mathrm{Re}(u\bar{v})$, $u, v \in \mathbb{C}$, where

$$\lambda_{\mathbb{D}}(z) := \frac{2}{(1-|z|^2)}$$

is often referred to as the hyperbolic density in $\mathbb{D}$. This metric induces a distance given by

$$k_{\mathbb{D}}(z, w) := \log \frac{1+\rho_{\mathbb{D}}(z, w)}{1-\rho_{\mathbb{D}}(z, w)}, \quad \text{where } \rho_{\mathbb{D}}(z, w) := \left|\frac{z-w}{1-\bar{w}z}\right|.$$

The classical Schwarz–Pick lemma, see e.g. Theorem 3.2 in [6], states that holomorphic self-maps of $\mathbb{D}$ do not increase the Poincaré distance $k_{\mathbb{D}}$, while its isometries are exactly the conformal automorphisms of $\mathbb{D}$, the set of which we denote by $\mathsf{Aut}(\mathbb{D})$. In Euclidean terms, the infinitesimal version of this fact can be expressed precisely as follows:

$$|f'(z)| \leqslant \frac{\lambda_{\mathbb{D}}(z)}{\lambda_{\mathbb{D}}(f(z))} = \frac{1-|f(z)|^2}{1-|z|^2} =: \nu_f(z) \tag{2.1}$$

for all $z \in \mathbb{D}$ and every holomorphic function $f\colon \mathbb{D} \to \mathbb{D}$. The equality in (2.1) holds for some $z \in \mathbb{D}$ if and only if $f \in \mathsf{Aut}(\mathbb{D})$, in which case equality holds for all $z \in \mathbb{D}$. For further details, we refer interested readers to [6, 15].

Remark 2.1. As noticed in Section 2.2 of [4], the quantity $\nu_f(z)$ is also related to the boundary behavior of f and f'. Namely, let $\zeta \in \partial\mathbb{D}$. Then, according to the classical Julia–Wolff–Carathéodory theorem,

$$\alpha_f(\zeta) := \liminf_{\mathbb{D} \ni z \to \zeta} \nu_f(z) < +\infty$$

if and only if the angular limits

$$\angle \lim_{z \to \zeta} f(z) =: f(\zeta) \quad \text{and} \quad \angle \lim_{z \to \zeta} f'(z) =: f'(\zeta)$$

exist finitely and $f(\zeta) \in \partial\mathbb{D}$. Moreover, if these equivalent conditions hold, then we have $\nu_f(z) \to \alpha_f(\zeta)$ as $z \to \zeta$ non-tangentially,

$$\zeta\, \overline{f(\zeta)}\, f'(\zeta) = \alpha_f(\zeta) > 0,$$

and

$$|f(r\zeta) - f(\zeta)| < 2\alpha_f(\zeta)\,(1-r) \quad \text{for all } r \in [0, 1),$$

see, e.g., Sections 4.2-4.4 of [20], Section 1.2 of [1] or Chapter 2 of [3].

Returning to the hyperbolic geometry of $\mathbb{D}$, we recall that geodesics in this geometry are diameters of $\mathbb{D}$ and arcs of circles orthogonal to $\partial\mathbb{D}$; as a consequence, for each pair of points $z_1, z_2 \in \mathbb{D}$ there is a unique geodesic segment $[z_1, z_2]_h$ joining them. This gives a natural way to define an analogue of the classical Euclidean notion of convexity in the context of the hyperbolic geometry:

Definition 2.2. A set $\Omega \subset \mathbb{D}$ is said to be *hyperbolically convex*, or *h-convex* for short, if $[z_1, z_2]_h \subset \Omega$ whenever $z_1, z_2 \in \Omega$. If, in addition, every hyperbolic geodesic intersects $\partial\Omega$ at two points at most, then Ω is said to be *strictly h-convex*.

We will make use of a simple observation connecting hyperbolic convexity to starlikeness. Throughout the paper, we will use the symbol "$\subset$" to denote the inclusion in the wide sense.

Definition 2.3. A set $\Omega \subset \mathbb{C}$ is said to be *starlike* if $0 \in \Omega$ and for every straight line $L \subset \mathbb{C}$ passing through the origin, the intersection $L \cap \Omega$ is connected.

Lemma 2.4. *A set $\Omega \subset \mathbb{D}$ is h-convex if and only if for every $T \in \mathsf{Aut}(\mathbb{D})$ such that $0 \in T(\Omega)$, the set $T(\Omega)$ is starlike.*

Proof. The lemma follows easily from the observation that given two points $z_1, z_2 \in \Omega$, $z_1 \neq z_2$, the hyperbolic geodesic segment $[z_1, z_2]_h$ is contained in Ω if and only if the Euclidean segment $[0, T(z_2)]$ is contained in $T(\Omega)$ for some (and hence every) $T \in \mathsf{Aut}(\mathbb{D})$ with $T(z_1) = 0$. ■

We will also need the following technical lemma, that borrows an idea from Section 1.5 of [23].

Lemma 2.5. *Let $D \subset \mathbb{D}$ be a domain containing the origin. If D is not starlike, then there exists $a_0 \in \mathbb{D} \cap \partial D$ and $\delta > 0$ such that the Euclidean segment $[(1-\delta)a_0, (1+\delta)a_0]$ is contained in the relative closure $\mathrm{cl}_{\mathbb{D}}(D)$ of the domain D in $\mathbb{D}$.*

Proof. Consider the radius function

$$R(\omega) := \sup\{r > 0 : [0, r\omega] \subset D\}, \quad \omega \in \partial\mathbb{D}.$$

It is not difficult to see that R is lower semicontinuous. Indeed, if $\omega_0 \in \partial\mathbb{D}$ and $0 < r < R(\omega_0)$, then the straight line segment $[0, r\omega_0]$ is contained in D. Since this segment is compact, it follows that

$$\{t\omega_0 e^{i\theta} : t \in [0, r],\ \theta \in (-\varepsilon, \varepsilon)\} \subset D$$

for some $\varepsilon > 0$. Therefore, $R(\omega) > r$ for all $\omega \in \partial\mathbb{D}$ sufficiently close to ω_0, as desired.

Moreover, if $\omega_0 \in \partial\mathbb{D}$ and if every neighbourhood of $R(\omega_0)\omega_0$ contains some points of $\{r\omega_0 : r \geqslant 0\} \setminus \mathrm{cl}_{\mathbb{D}}(D)$, then the radius function is also upper semicontinuous at ω_0. To see this, we assume that $R(\omega_0) < 1$, since it is otherwise evident. Let $R(\omega_0) < r < 1$ and observe that, by assumption, there exists $\rho \in (R(\omega_0), r)$ for which $\rho\omega_0 \notin \mathrm{cl}_{\mathbb{D}}(D)$. Since $\mathbb{D} \setminus \mathrm{cl}_{\mathbb{D}}(D)$ is an open set, for all ω sufficiently close to ω_0 we have $\rho\omega \notin \mathrm{cl}_{\mathbb{D}}(D)$ and hence $R(\omega) < \rho < r$, as desired.

Finally, noting that $R(\omega)\omega \in \partial D$ for every $\omega \in \partial\mathbb{D}$, the above argument shows that if the conclusion of the lemma fails, then the radius function is continuous on $\partial\mathbb{D}$. Therefore, see, e.g., Lemma 1 in [23], the domain D is starlike, in contradiction to the hypothesis. ■

We complete this section with a lemma that allows us to reduce proving hyperbolic convexity to checking some local property at the boundary, namely, the existence of supporting hyperbolic half-planes. Denote

$$\mathbb{D}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

As usual, $\nabla u(z)$ will stand for the gradient of a real-valued function $u(x+iy) = u(z)$. In many cases, we identify $\nabla u(z)$ with the complex-valued function $\partial u/\partial x + i\,\partial u/\partial y$.

Lemma 2.6. *Let* $u\colon \mathbb{D} \to \mathbb{R}$ *be a continuous function in* $\mathbb{D}$ *and let* Ω *be a connected component of* $\{z \in \mathbb{D} : u(z) > 0\}$. *Suppose that for every point* $\zeta \in \mathbb{D} \cap \partial\Omega$, *the function* $u(z)$ *is differentiable at* $z = \zeta$ *and satisfies the following two conditions*:

(a) $\nabla u(\zeta) \neq 0$, *and*

(b) *there exists* $\varepsilon > 0$ *such that* $\gamma_\zeta \cap \mathrm{cl}_{\mathbb{D}}(\Omega) \cap \mathbb{D}(\zeta,\varepsilon) = \{\zeta\}$, *where* γ_ζ *stands for the hyperbolic geodesic passing through the point* ζ *and orthogonal at* ζ *to* $\nabla u(\zeta)$.

Then Ω *is a strictly h-convex domain.*

Following the intuition from the Euclidean plane geometry, condition (b) can be regarded as a local version of the strict h-convexity: it means that the part of the closure of Ω contained in a sufficiently small neighbourhood of a boundary point ζ lies entirely to one side of a suitably chosen geodesic passing through ζ. Lemma 2.6 shows that similarly to the Euclidean setting, for a *connected* open set Ω, this local property satisfied at every boundary point implies the strict h-convexity in the global sense.

Proof of Lemma 2.6. Clearly, Ω is a domain, since it is a connected component of an open set. To show that Ω is h-convex, suppose on the contrary that it is not. Then by Lemma 2.4, there exists $T \in \mathrm{Aut}(\mathbb{D})$ such that the domain $D := T\big(\Omega\big)$ contains the origin but it is not starlike. By Lemma 2.5, it follows that $\partial D \cap \mathbb{D}$ contains a point a_0 such that

$$[(1-\delta)a_0, (1+\delta)a_0] \subset \mathrm{cl}_{\mathbb{D}}(D) \tag{2.2}$$

for a sufficiently small $\delta > 0$.

The preimage of $[(1-\delta)a_0, (1+\delta)a_0]$ with respect to T is a segment of the hyperbolic geodesic γ passing through $z := T^{-1}(0) \in \Omega$ and $\zeta := T^{-1}(a_0) \in \partial\Omega$. According to condition (b), there exists a punctured neighbourhood U of the point ζ such that the hyperbolic geodesic γ_ζ passing through ζ orthogonally to $\nabla u(\zeta)$ does not have common points with $\mathrm{cl}_{\mathbb{D}}(\Omega) \cap U$. Taking into account (2.2), it follows that $\gamma \neq \gamma_\zeta$. Being two distinct hyperbolic geodesics, γ and γ_ζ intersect at ζ transversally. Taking into account that $u(\zeta) = 0$ and $\nabla u(\zeta) \neq 0$, the latter implies that in every neighbourhood of ζ there are points of γ at which $u < 0$. This, however, contradicts (2.2) because $u(w) \geqslant 0$ for every $w \in \mathrm{cl}_{\mathbb{D}}(\Omega) = T^{-1}(\mathrm{cl}_{\mathbb{D}}(D))$.

To complete the proof, it remains to notice that if an h-convex domain is not strictly h-convex, then clearly its boundary contains a hyperbolic geodesic segment. However, for the domain Ω, this would contradict condition (b). ■

3. Level sets defined by the hyperbolic density

For $\lambda > 0$ and a holomorphic self-map $f\colon \mathbb{D} \to \mathbb{D}$, we consider the lower level set

$$\Omega_\lambda(f) := \left\{ z \in \mathbb{D} : \frac{1 - |f(z)|^2}{1 - |z|^2} < \lambda \right\}.$$

We state the main result of this section as follows.

Theorem 3.1. *Let* $\lambda \geqslant 1$ *and let* $f:\mathbb{D} \to \mathbb{D}$ *be holomorphic. In case* $\lambda = 1$, *we also suppose that* $f(0) \neq 0$ *and that* $f \notin \mathsf{Aut}(\mathbb{D})$. *Then the set* $\Omega_\lambda(f)$ *is a strictly hyperbolically convex domain containing the origin.*

Our assumptions in Theorem 3.1 are sharp. If $f(0) = 0$ then, by the Schwarz lemma, $\Omega_1(f) = \varnothing$. If $f(0) \neq 0$, then $0 \in \Omega_1(f)$ and, as Solynin [22] proved, $\Omega_1(f)$ is strictly h-convex unless f is an automorphism of $\mathbb{D}$. The situation for automorphisms is described in the example below.

Example 3.2. Let $f \in \mathsf{Aut}(\mathbb{D})$ and suppose that $f(0) \neq 0$. As a direct calculation shows,

$$\Omega_\lambda(f) = \{z: |z - z_0|^2 > (|z_0|^2 - 1)/\lambda\},$$

where $z_0 := 1/\bar{a}$, $a := f^{-1}(0)$. Consequently, $\Omega_\lambda(f)$ is h-convex if and only if $\lambda \geqslant 1$. Moreover, for $\lambda > 1$, $\Omega_\lambda(f)$ is strictly h-convex, but this is not the case for $\Omega_1(f)$. In fact, $\mathbb{D} \cap \partial\Omega_1(f)$ is precisely a geodesic.

Note that in the above example, $\Omega_\lambda(f)$ is the whole disk $\mathbb{D}$ if $\lambda \geqslant \frac{1+|f(0)|}{1-|f(0)|}$. At the end of this section, we will precisely characterise in which cases $\Omega_\lambda(f) = \mathbb{D}$; see Proposition 3.3.

Proof of Theorem 3.1. Observe that $\Omega_\lambda(f) = \{z \in \mathbb{D} : u(z) > 0\}$, where

$$u(z) := |f(z)|^2 - \lambda|z|^2 + \lambda - 1.$$

It immediately follows that $\Omega_\lambda(f)$ is an open set. Moreover, it is known (see [4], p. 415) that under the hypothesis of Theorem 3.1, the set $\Omega_\lambda(f)$ is starlike with respect to the origin, hence connected. Note also that the function u is real analytic throughout $\mathbb{D}$. Therefore, in order to apply Lemma 2.6, it is sufficient to check that for every point $\zeta \in \mathbb{D} \cap \partial\Omega_\lambda(f)$, the conditions (a) and (b) hold.

(a) Let $\zeta \in \mathbb{D} \cap \partial\Omega_\lambda(f)$. We compute

$$\nabla u(z) = 2\,\frac{\partial u}{\partial \bar{z}} = 2[f(z)\,\overline{f'(z)} - \lambda z], \quad z \in \mathbb{D},$$

so that $\nabla u(\zeta) = 0$ is equivalent to $f(\zeta)\overline{f'(\zeta)} = \lambda\zeta$. Moreover, $1 - |f(\zeta)|^2 = \lambda(1 - |\zeta|^2)$ because $\zeta \in \partial\Omega_\lambda(f)$. In particular, since $\lambda \geqslant 1$, we have that $|f(\zeta)| \leqslant |\zeta|$, with equality if and only if $\lambda = 1$. Therefore, if $\nabla u(\zeta)$ vanished, then using the Schwarz–Pick inequality (2.1) we would get

$$\lambda|\zeta| = |f(\zeta)f'(\zeta)| \leqslant |f(\zeta)|\,\frac{1 - |f(\zeta)|^2}{1 - |\zeta|^2} = \lambda|f(\zeta)| \leqslant \lambda|\zeta|,$$

where at least one of the inequalities would be strict unless $\lambda = 1$ and $f \in \mathsf{Aut}(\mathbb{D})$. However, the latter possibility is excluded by the hypotheses.

(b) This part is more involved. We have to show that $u(z) < 0$ for all $z \in \gamma_\zeta \setminus \{\zeta\}$ close to ζ, where γ_ζ stands for the geodesic passing through ζ orthogonally to $\nabla u(\zeta)$.

Recall that $0 \in \Omega_\lambda(f)$. Taking this into account and replacing f by an appropriate rotation $e^{is} f(e^{it} z)$, we may assume without loss of generality that $\zeta > 0$ and $f(\zeta) \geqslant 0$.

To further simplify the setting, we apply an automorphism $\psi \in \mathsf{Aut}(\mathbb{D})$ that takes ζ to 0. Then the geodesic γ_ζ is mapped onto a diameter, i.e., $\psi(\gamma_\zeta) = (-\kappa, \kappa)$ for some $\kappa \in \partial\mathbb{D}$. The inverse automorphism $\varphi = \psi^{-1}$ is given by

$$\varphi(w) = \frac{\zeta + w}{1 + \zeta w} = \zeta + (1 - \zeta^2)(w - \zeta w^2) + O(w^3), \quad w \to 0. \tag{3.1}$$

Let $g := f \circ \varphi$ and write

$$g(w) = b + \sum_{n=1}^{\infty} b_n w^n, \quad \text{where} \quad b = f(\zeta) \in [0, 1).$$

Let

$$v(t) := u(\varphi(t\kappa)), \quad \text{for } t \in (-1, 1).$$

It is clear that $v(0) = 0$, since $\zeta \in \partial\Omega_\lambda(f)$. Also, we have that $v'(0) = 0$, because γ_ζ is orthogonal to $\nabla u(\zeta)$ at the point ζ. Our objective has now been reduced to showing that $v''(0) < 0$.

From the definition of u, we get

$$v(t) = |g(t\kappa)|^2 - \lambda|\varphi(t\kappa)|^2 + \lambda - 1,$$

and find the asymptotic expansion

$$\begin{aligned} v(t) &= |b + b_1 t\kappa + b_2 t^2\kappa^2|^2 - \lambda|\zeta + (1 - \zeta^2)(t\kappa - \zeta t^2\kappa^2)|^2 + \lambda - 1 + O(t^3) \\ &= b^2 - \lambda\zeta^2 + \lambda - 1 + 2\,\mathrm{Re}\{[bb_1 - \lambda\zeta(1 - \zeta^2)]\kappa\}t \\ &\quad + \left(|b_1|^2 - \lambda(1 - \zeta^2)^2 + 2\,\mathrm{Re}\{[bb_2 + \lambda\zeta^2(1 - \zeta^2)]\kappa^2\}\right)t^2 + O(t^3), \quad t \to 0. \end{aligned}$$

Note that $v(0) = 0$ is equivalent to

$$1 - b^2 = \lambda(1 - \zeta^2). \tag{3.2}$$

Let $A := bb_1 - \lambda\zeta(1 - \zeta^2)$. Then $v'(0) = 0$ is equivalent to

$$\mathrm{Re}(A\kappa) = 0. \tag{3.3}$$

Consider the holomorphic function $h\colon \mathbb{D} \to \overline{\mathbb{D}}$ given by

$$h(w) := \frac{b - g(w)}{w(1 - bg(w))} = \sum_{n=0}^{\infty} c_n w^n, \quad w \in \mathbb{D}.$$

Note that $h(\mathbb{D}) \subset \mathbb{D}$ if $f \notin \mathsf{Aut}(\mathbb{D})$, and $h \equiv \text{const} \in \partial\mathbb{D}$ if $f \in \mathsf{Aut}(\mathbb{D})$. Therefore,

$$|c_0| \leqslant 1 \quad \text{and} \quad |c_1| \leqslant 1 - |c_0|^2, \tag{3.4}$$

the latter by the Schwarz–Pick inequality (2.1), while the equality $|c_0| = 1$ occurs if and only if $f \in \mathsf{Aut}(\mathbb{D})$. Moreover, as we have already seen,

$$b = f(\zeta) \leqslant \zeta,$$

with the inequality being strict unless $\lambda = 1$.

Since in case $\lambda = 1$ we suppose that $f \notin \mathrm{Aut}(\mathbb{D})$, it follows that

$$b|c_0| < \zeta. \tag{3.5}$$

Elementary calculations show that

$$b_1 = -(1-b^2)c_0 \quad \text{and} \quad b_2 = -(1-b^2)(c_1 + bc_0^2). \tag{3.6}$$

Substituting in the expression for A and using (3.2), we obtain

$$A = -(1-b^2)(bc_0 + \zeta). \tag{3.7}$$

Combined with (3.5), this equality implies that $A \neq 0$. In view of (3.3), we may write $A\kappa = i\rho$ for some $\rho \in \mathbb{R} \setminus \{0\}$. Clearly, $\rho^2 = |A|^2$. Hence, using (3.7) we get

$$\kappa^2 = -\frac{\rho^2}{A^2} = -\frac{\overline{A}}{A} = -\frac{b\overline{c_0} + \zeta}{bc_0 + \zeta}. \tag{3.8}$$

Using (3.2) and (3.6), we get that the second coefficient in the asymptotic expansion for v is

$$\frac{v''(0)}{2} = \lambda(1-\zeta^2)^2(\lambda|c_0|^2 - 1) + 2\,\mathrm{Re}\{\lambda(1-\zeta^2)(\zeta^2 - bc_1 - b^2c_0^2)\kappa^2\} \leqslant \lambda(1-\zeta^2)\,\Phi,$$

where

$$\Phi := (1-\zeta^2)(\lambda|c_0|^2 - 1) + 2\,\mathrm{Re}\{(\zeta^2 - b^2c_0^2)\kappa^2\} + 2b|c_1|.$$

In view of (3.8), we have that

$$\mathrm{Re}\{(\zeta^2 - b^2c_0^2)\kappa^2\} = -\mathrm{Re}\{(\zeta - bc_0)(b\overline{c_0} + \zeta)\} = b^2|c_0|^2 - \zeta^2.$$

Now, using (3.4), we obtain

$$\begin{aligned}\Phi &= (1-\zeta^2)(\lambda|c_0|^2 - 1) + 2b^2|c_0|^2 - 2\zeta^2 + 2b|c_1| \\ &\leqslant (1-\zeta^2)(\lambda|c_0|^2 - 1) + 2b^2|c_0|^2 - 2\zeta^2 + 2b(1-|c_0|^2) \\ &= (1-b)^2|c_0|^2 + 2b - 1 - \zeta^2 \leqslant (1-b)^2 + 2b - 1 - \zeta^2 = b^2 - \zeta^2,\end{aligned}$$

with the last inequality being strict unless $f \in \mathrm{Aut}(\mathbb{D})$. Recalling that $\zeta > b$ if $\lambda > 1$, and $\zeta = b$ if $\lambda = 1$, we conclude that $\Phi < 0$. Thus also $v''(0) < 0$ as desired. ∎

Proposition 3.3. *Let $\lambda > 0$ and let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic. Then $\Omega_\lambda(f) = \mathbb{D}$ if and only if the following two conditions hold*:

(a) *f is a finite Blaschke product, i.e.,*

$$f(z) = e^{i\theta} \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k}z}, \qquad z \in \mathbb{D},$$

for some $\theta \in \mathbb{R}$, $n \in \mathbb{N}$, and some points $a_1, \ldots, a_n \in \mathbb{D}$.

(b) $\lambda \geqslant \sup_{z \in \mathbb{D}} |f'(z)|$ *and* $\lambda > 1$.

Proof. (Necessity). Assume that $\Omega_\lambda(f) = \mathbb{D}$, i.e.,

$$\nu_f(z) := \frac{1-|f(z)|^2}{1-|z|^2} < \lambda \quad \text{for all } z \in \mathbb{D}. \tag{3.9}$$

Then in view of the Schwarz–Pick lemma, $|f'| < \lambda$ in $\mathbb{D}$. In particular, f extends continuously to the closure of $\mathbb{D}$. Appealing again to (3.9), we have that $|f(\zeta)| = 1$ for every $\zeta \in \partial\mathbb{D}$. Thus, f must be a finite Blaschke product by a theorem of Fatou; see, e.g., Theorem 3.5.2 in [12]. To see that $\lambda > 1$ in this case, notice that

$$2\pi n = \int_0^{2\pi} |f'(e^{it})|\,\mathrm{d}t \leqslant 2\pi\lambda,$$

where n is the degree of the Blaschke product f. It follows that $\lambda \geqslant n \geqslant 1$. If $\lambda = 1$, then we get that $f \in \mathsf{Aut}(\mathbb{D})$ and $|f'| < 1$ in $\mathbb{D}$, which is impossible.

(Sufficiency). Suppose conditions (a) and (b) hold. To show that the domain $\Omega_\lambda(f)$ coincides with $\mathbb{D}$, assume on the contrary that $\Omega_\lambda(f) \neq \mathbb{D}$. Recall that $\Omega_\lambda(f)$ is not empty because $0 \in \Omega_\lambda(f)$ when $\lambda > 1$. Take any geodesic γ tangent to $\partial\Omega_\lambda(f)$. Since $\Omega_\lambda(f)$ is h-convex by Theorem 3.1, we have that $\Omega_\lambda(f)$ is contained in one of the two connected components of $\mathbb{D}\setminus\gamma$. Let U stand for the other connected component. Fix an open arc $\Gamma \subset \partial\mathbb{D}$ contained in ∂U. Since f maps $\partial\mathbb{D}$ into $\partial\mathbb{D}$, it is easy to see (using, e.g., Remark 2.1) that

$$\zeta f'(\zeta)/f(\zeta) > 0 \quad \text{and that} \quad |f'(\zeta)| = \lim_{r\to 1^-} \nu_f(r\zeta) \quad \text{for all } \zeta \in \partial\mathbb{D}.$$

Therefore, $|f'(\zeta)| \geqslant \lambda$ for all $\zeta \in \Gamma$. At the same time, $|f'(\zeta)| \leqslant \lambda$ for all $\zeta \in \partial\mathbb{D}$ by condition (b). As a result, $\zeta f'(\zeta)/f(\zeta) = \lambda$ for all $\zeta \in \Gamma$. Thus, the rational function $z \mapsto zf'(z)/f(z)$ is constant and equal to λ, which is only possible if $f(z) = e^{i\theta}z^n$ and $\lambda = n$ for some $n \in \mathbb{N}$ and some $\theta \in \mathbb{R}$. Recalling that $\lambda > 1$ by the hypothesis, it is easy to see that in this case (3.9) holds, contradicting our assumption. ■

Remark 3.4. Note that if at every point ζ of some open arc $\Gamma \subset \partial\mathbb{D}$ a holomorphic self-map $f : \mathbb{D} \to \mathbb{D}$ has radial limit $\lim_{r\to 1^-} f(r\zeta)$ belonging to $\partial\mathbb{D}$ and attained uniformly with respect to $\zeta \in \Gamma$, then f extends holomorphically to Γ by the Schwarz reflection principle. Combining this fact with the Julia–Wolff–Carathéodory theorem (see Remark 2.1), it is possible to establish a sort of local version of the above Proposition 3.3:

An open arc $\Gamma \subset \partial\mathbb{D}$ is contained in the boundary of $\Omega_\lambda(f)$, $\lambda > 1$, if and only if the function f admits a holomorphic extension to Γ satisfying $f(\zeta) \in \partial\mathbb{D}$ and $|f'(\zeta)| \leqslant \lambda$ for all $\zeta \in \Gamma$.

Furthermore, since

$$\liminf_{z\to\zeta} \nu_f(z) = \exp\left\{\liminf_{z\to\zeta} k_{\mathbb{D}}(z,0) - k_{\mathbb{D}}(f(z),0)\right\},$$

see, e.g., Propositions 2.1.15 and 2.1.21 in [3], a similar statement holds for the level sets $\mathcal{D}_\mu(f)$ studied in the next section.

4. Level sets defined by the hyperbolic distance

In this section, we consider the sublevel sets

$$\mathcal{D}_\mu(f) = \mathcal{D}_\mu(f; z_0, w_0) := \{z \in \mathbb{D} \colon k_{\mathbb{D}}\big(z, z_0\big) - k_{\mathbb{D}}\big(f(z), w_0\big) < \mu\}, \quad \mu \in \mathbb{R},$$

where $f \colon \mathbb{D} \to \mathbb{D}$ is a holomorphic function, and z_0 and w_0 are points in $\mathbb{D}$. Setting

$$g := T_2^{-1} \circ f \circ T_1,$$

for $T_1, T_2 \in \mathsf{Aut}(\mathbb{D})$ that map the origin to z_0 and w_0, respectively, we see that

$$\mathcal{D}_\mu(f; z_0, w_0) = T_1\big(\mathcal{D}_\mu(g; 0, 0)\big).$$

Recalling that automorphisms preserve h-convexity, in what follows we assume that

$$z_0 = w_0 = 0.$$

Moreover, taking into account that $\mathcal{D}_0(f) = \Omega_1(f)$ for this choice of z_0 and w_0, we may exclude $\mu = 0$ from consideration.

Our objective is to prove the following.

Theorem 4.1. *If $\mu < 0$ and $f \colon \mathbb{D} \to \mathbb{D}$ is holomorphic, then the set $\mathcal{D}_\mu(f)$ is either empty or a strictly hyperbolically convex domain.*

This is sharp since for $\mu > 0$, the following example shows that h-convexity may fail.

Example 4.2. Let $\mu > 0$ and set $b := \tanh(\mu/2) \in (0, 1)$. Consider the disk-automorphism $f(z) := (z - b)/(1 - bz)$, $z \in \mathbb{D}$. Note that

$$z \in \mathcal{D}_\mu(f) \quad \Longleftrightarrow \quad b - |z| + (1 - b|z|)|f(z)| > 0.$$

Since $|f(z)| \geqslant f(|z|)$ for all $z \in \mathbb{D}$, with equality if and only if $z \in [b, 1)$, it is easy to see that in this case, $\mathcal{D}_\mu(f) = \mathbb{D} \setminus [b, 1)$, which is not a hyperbolically convex domain.

Observe that we may rewrite the defining inequality of the set $\mathcal{D}_\mu(f)$ as

$$\mathcal{D}_\mu(f) = \left\{z \in \mathbb{D} : \frac{|z| - |f(z)|}{1 - |zf(z)|} < \tanh(\mu/2)\right\} = \{z \in \mathbb{D} : u(z) > 0\}, \tag{4.1}$$

where

$$u(z) := (1 + a|z|)|f(z)| - |z| - a \quad \text{and} \quad a := -\tanh(\mu/2). \tag{4.2}$$

We begin with some basic considerations in the following proposition.

Proposition 4.3. *Let $\mu < 0$ and let f be a holomorphic self-map of $\mathbb{D}$. Then $\mathcal{D}_\mu(f) \neq \varnothing$ if and only if $|f(0)| > -\tanh(\mu/2)$, in which case $\mathcal{D}_\mu(f)$ is a proper subset of $\mathbb{D}$ containing the origin.*

Proof. If $|f(0)| > a = -\tanh(\mu/2)$, then by (4.1) it is immediate that $0 \in \mathcal{D}_\mu(f)$. The reverse inequality $|f(0)| \leqslant -\tanh(\mu/2)$ is equivalent to $k_{\mathbb{D}}(f(0), 0) \leqslant -\mu$. In such a case, applying the triangle inequality and the Schwarz–Pick lemma (Theorem 3.2 in [6]), for all $z \in \mathbb{D}$ we obtain

$$k_{\mathbb{D}}(f(z), 0) \leqslant k_{\mathbb{D}}(f(z), f(0)) + k_{\mathbb{D}}(f(0), 0) \leqslant k_{\mathbb{D}}(z, 0) - \mu,$$

which shows that $\mathcal{D}_\mu(f) = \varnothing$.

To see that $\mathcal{D}_\mu(f)$ is not the whole unit disk when $|f(0)| > a$, it remains to recall that $\mathcal{D}_\mu(f) \subset \mathcal{D}_0(f) = \Omega_1(f)$ and to use Proposition 3.3. ■

In the proof of Theorem 4.1, we will use the following lemma.

Lemma 4.4. *Let $\mu < 0$ and let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic with $|f(0)| > -\tanh(\mu/2)$. Then for every $\zeta \in \partial\mathcal{D}_\mu(f) \cap \mathbb{D}$, we have $|f(\zeta)| > |\zeta| > 0$ and*

$$\operatorname{Re}\{\bar{\zeta}\, \nabla u(\zeta)\} < 0, \tag{4.3}$$

where u is defined by (4.2).

In particular, $\mathcal{D}_\mu(f)$ is a domain that is starlike with respect to the origin.

Proof. Since $\mu < 0$, we have that $a = -\tanh(\mu/2) \in (0, 1)$. Let $\zeta \in \partial\mathcal{D}_\mu(f) \cap \mathbb{D}$. Note that $\zeta \neq 0$ by Proposition 4.3. Using the fact that $u(\zeta) = 0$, we obtain

$$|f(\zeta)| = \frac{|\zeta| + a}{1 + a|\zeta|} > |\zeta|. \tag{4.4}$$

In particular, $f(\zeta) \neq 0$.

Note that $u(z)$ is real-analytic whenever $z \neq 0$ and $f(z) \neq 0$. Employing the formula $\nabla u(z) = 2\,\partial u(z)/\partial\bar{z}$, we compute

$$\operatorname{Re}\{\bar{z}\, \nabla u(z)\} = |z|(a|f(z)| - 1) + (1 + a|z|)\, \operatorname{Re}\left\{\frac{f(z)}{|f(z)|}\, \overline{zf'(z)}\right\}.$$

Taking into account the equality in (4.4) and using (2.1), we therefore obtain

$$\begin{aligned} \operatorname{Re}\left\{\bar{\zeta}\, \nabla u(\zeta)\right\} &= \frac{|\zeta|\,(a^2 - 1)}{1 + a|\zeta|} + (1 + a|\zeta|)\operatorname{Re}\left\{\frac{f(\zeta)}{|f(\zeta)|}\, \overline{\zeta f'(\zeta)}\right\} \\ &\leqslant \frac{|\zeta|\,(a^2 - 1)}{1 + a|\zeta|} + (1 + a|\zeta|)\,|\zeta f'(\zeta)| \leqslant\ 0, \end{aligned}$$

with equalities occurring in both inequalities if and only if $f \in \mathrm{Aut}(\mathbb{D})$ and $\overline{\zeta f'(\zeta)}\, f(\zeta) > 0$. The only automorphisms satisfying this last inequality and (4.4) are unimodular multiples of the function

$$f_0(z) := \frac{z + z_0}{1 + \bar{z}_0\, z}, \quad z_0 := a\,\frac{\zeta}{|\zeta|}.$$

However, $|f_0(0)| = a$, which contradicts the hypothesis of the lemma. Thus, the desired strict inequality holds.

The set $\mathcal{D}_\mu(f)$ is open because u is continuous in $\mathbb{D}$. To prove that $\mathcal{D}_\mu(f)$ is a starlike domain, we fix an arbitrary $\theta \in \mathbb{R}$. The function

$$q(t) := u(te^{i\theta})$$

is continuous in $[0, 1)$ and satisfies $q(0) > 0$ because $0 \in \mathcal{D}_\mu(f)$ by Proposition 4.3. It is enough to show that

$$E := \{t \in [0, 1) : q(t) > 0\}$$

is connected. Suppose that it is not the case. Then E has a connected component not containing 0. Since E is relatively open in $[0,1)$, this connected component is an interval of the form (t_1,t_2) with $t_1 \in (0,1)$. We have $q(t) > 0$ for all $t \in (t_1,t_2)$ and $q(t_1) = 0$. It follows that

$$\zeta := t_1 e^{i\theta} \in \partial\mathcal{D}_\mu(f) \cap \mathbb{D}$$

and, at the same time,

$$\operatorname{Re}\{\bar{\zeta}\,\nabla u(\zeta)\} = t_1 q'(t_1) \geqslant 0,$$

which contradicts (4.3). ■

Proof of Theorem 4.1. Let $\mu < 0$ and let f be a holomorphic self-map of $\mathbb{D}$ such that $\mathcal{D}_\mu(f) \neq \varnothing$. Fix an arbitrary $\zeta \in \mathbb{D} \cap \partial\mathcal{D}_\mu(f)$. Using rotations, we may assume without loss of generality that $\zeta > 0$ and $b := f(\zeta) > 0$.

In view of (4.1), Lemmas 2.6 and 4.4, in order to show that $\mathcal{D}_\mu(f)$ is strictly h-convex, working exactly as in the proof of Theorem 3.1, it suffices to prove the following claim.

Claim. *Let* $v(t) := u\big(\varphi(t\kappa)\big)$, $t \in (-1,1)$, *where* $\varphi(z) := (\zeta + z)/(1+\zeta z)$ *and* $\kappa \in \partial\mathbb{D}$ *is chosen in such a way that* $v'(0) = 0$. *Then* $v''(0) < 0$.

Following the proof of Theorem 3.1, we set $g := f \circ \varphi$ and express $v'(0)$ and $v''(0)$ in terms of b, ζ and the first two Taylor coefficients of

$$h(z) := \frac{b - g(z)}{z(1 - bg(z))} = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \mathbb{D},$$

which satisfy

$$|c_0| \leqslant 1 \quad \text{and} \quad |c_1| \leqslant 1 - |c_0|^2. \tag{4.5}$$

Furthermore, the relation $0 = v(0) = (1 + a\zeta)b - \zeta - a$ allows us to eliminate the parameter a:

$$a = \frac{b - \zeta}{1 - b\zeta}. \tag{4.6}$$

With an asymptotic expansion for $t \to 0$, we see that

$$\mathrm{d}[u \circ \varphi(t\tilde{\kappa})]/\mathrm{d}t\,\big|_{t=0} = \operatorname{Re}(A\tilde{\kappa}),$$

for arbitrary $\tilde{\kappa} \in \partial\mathbb{D}$, where

$$A := -\frac{(1-\zeta^2)(1-b^2)(c_0+1)}{1 - b\zeta}. \tag{4.7}$$

On the other hand, we get

$$\mathrm{d}[u \circ \varphi(t\tilde{\kappa})]/\mathrm{d}t\,\big|_{t=0} = \operatorname{Re}[\overline{\nabla u(\zeta)}\,\varphi'(0)\tilde{\kappa}]$$

by the chain rule. Since $\nabla u(\zeta) \neq 0$ by Lemma 4.4, it follows that

$$A = \overline{\nabla u(\zeta)}\,\varphi'(0) \neq 0,$$

thus $c_0 \neq -1$.

To obtain an expression for $v''(0)$ suitable for our purpose, notice first that if a function $q\colon \mathbb{D} \to \mathbb{C}$ admits at the origin an expansion of the form

$$q(w) = q_0 + q_1 w + q_2 w^2 + O(w^3), \quad \text{with } q_0 > 0,$$

then

$$|q(w)| = q_0 + \operatorname{Re}(q_1 w + q_2 w^2) + \frac{(\operatorname{Im}(q_1 w))^2}{2q_0} + O(w^3), \quad w \to 0.$$

In particular, with the help of (3.1) and (3.6), we get

$$|\varphi(w)| = \zeta + (1-\zeta^2)\Big[\operatorname{Re} w - \zeta \operatorname{Re}(w^2) + \frac{1-\zeta^2}{2\zeta}(\operatorname{Im} w)^2\Big] + O(w^3) \quad \text{and}$$
$$|g(w)| = b - (1-b^2)\Big[\operatorname{Re}(c_0 w) + \operatorname{Re}\{(c_1 + bc_0^2)w^2\} - \frac{1-b^2}{2b}(\operatorname{Im}(c_0 w))^2\Big] + O(w^3)$$

as $w \to 0$. Observe that

$$v(t) = U\big(|\varphi(t\kappa)|, |g(t\kappa)|\big), \quad \text{where } U(X,Y) := -a + Y - X + aXY.$$

With (4.6) and $\operatorname{Re}\{(t\kappa)^2\} = t^2 - 2(\operatorname{Im}(t\kappa))^2$ taken into account, a laborious but elementary calculation leads to

$$\text{(4.8)} \quad \frac{v''(0)}{2} = \frac{(1-\zeta^2)(1-b^2)}{1-b\zeta}\Big[\frac{(1-b^2)(\operatorname{Im}(c_0\kappa))^2}{2b} - \operatorname{Re}\{(c_1 + bc_0^2)\kappa^2\}$$
$$- (b-\zeta)\operatorname{Re}(\kappa)\operatorname{Re}(c_0\kappa) + \zeta - \frac{(1+3\zeta^2)(\operatorname{Im}\kappa)^2}{2\zeta}\Big].$$

Since $\operatorname{Re}(A\kappa) = v'(0) = 0$, we may write $A\kappa = i\rho$ for a suitable $\rho > 0$ (replacing, if necessary, κ by $-\kappa$). We set

$$c_0 + 1 =: re^{i\theta}, \quad \text{with} \quad |\theta| < \frac{\pi}{2}, \quad 0 < r \leqslant 2\cos\theta.$$

Taking into account (4.7), in this notation we have that

$$\kappa = \frac{i\rho}{A} = -i\,\frac{|c_0+1|}{c_0+1} = -ie^{-i\theta}.$$

With the help of the equalities

$$\kappa = -\sin\theta + i(-\cos\theta), \quad |c_0|^2 = r^2 - 2r\cos\theta + 1,$$
$$c_0\kappa = \sin\theta + i(\cos\theta - r), \quad \text{and} \quad \operatorname{Re}(c_0^2\kappa^2) = 1 - 2\cos^2\theta + 2r\cos\theta - r^2,$$

formula (4.8) can be rewritten as

$$\Phi := \frac{1-b\zeta}{(1-\zeta^2)(1-b^2)}\,\frac{v''(0)}{2} = \alpha r^2 - 2\alpha r\cos\theta + \beta\cos^2\theta - \operatorname{Re}(c_1\kappa^2),$$

where

$$\alpha := \frac{1+b^2}{2b} \quad \text{and} \quad \beta := \frac{1+b^2}{2b} - \frac{1+\zeta^2}{2\zeta}.$$

According to (4.5), we have

$$-\operatorname{Re}(c_1\kappa^2) \leqslant |c_1| \leqslant 1-|c_0|^2 = 2r\cos\theta - r^2.$$

Hence,

$$\begin{aligned}\Phi \leqslant p(r) &:= (\alpha-1)r^2 - 2(\alpha-1)r\cos\theta + \beta\cos^2\theta\\ &= (\alpha-1)r(r-2\cos\theta) + \beta\cos^2\theta.\end{aligned}$$

Since

$$\alpha - 1 = (1-b)^2/(2b) > 0,$$

the quadratic polynomial p is convex with respect to the variable r. Hence, it suffices to verify that p is negative at the endpoints $r=0$ and $r=2\cos\theta$. We have

$$p(0) = p(2\cos\theta) = \beta\cos^2\theta,$$

with $\beta < 0$, since $0<\zeta<b<1$ in view of Lemma 4.4. Thus, $v''(0)<0$, which proves the claim. ■

Remark 4.5. Recall that the hyperbolic geometry can be transferred from the unit disk to any simply connected hyperbolic domain via any of its Riemann mappings, carrying with it the notion of h-convexity. Hence, Theorem 4.1 proved above can be rewritten as follows.

If $\mu<0$ and $f\colon D_1 \to D_2$ is a holomorphic map between two simply connected hyperbolic domains $D_j \subset \mathbb{C}$, $j=1,2$, then for every $z_1 \in D_1$ and every $z_2 \in D_2$, the set

$$\mathcal{D}_\mu(f;D_1,D_2) := \{z \in D_1 : k_{D_1}(z,z_1) - k_{D_2}(f(z),z_2) < \mu\},$$

where k_{D_j} stands for the hyperbolic distance in D_j, is either empty or a strictly h-convex domain in D_1. For $\mu = 0$, the same statement holds[1] *unless f maps D_1 conformally onto D_2, in which case $\mathcal{D}_\mu(f;D_1,D_2)$ is merely h-convex (not strictly).*

5. Concluding remarks and open questions

Theorem 4.1, asserting the h-convexity of the sets $\mathcal{D}_\mu(f)$, $\mu \leqslant 0$ (where the case $\mu=0$ is due to Solynin [22]), is stated in terms of hyperbolic geometry. However, our proof is entirely in Euclidean terms.

Problem 5.1. *Give an intrinsic proof, i.e., a proof purely in terms of hyperbolic geometry, of the h-convexity of $\mathcal{D}_\mu(f)$, $\mu \leqslant 0$.*

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[1] As for the case $\mu = 0$, recall that it is covered by Theorem 1 in [22]; see also Theorem 3.1 and Example 3.2.

We expect that such a proof could reveal some new interesting relationship between the hyperbolic metric and holomorphic mappings. Moreover, having a hyperbolic-geometric proof, one may hope that Theorem 4.1 can be extended to a more general setting. To be more concrete, we need to recall some definitions.

Recall that a *geodesic* in a metric space (X,k) is an isometry $\gamma\colon I \to X$ from an interval $I \subset \mathbb{R}$, endowed with the Euclidean distance $k_{\mathbb{R}}(s,t) := |t-s|$, to (X,k). As usual, the image $\gamma(I)$ of such an isometry is also referred to as a geodesic. The metric space (X,k) is said to be *geodesic* if every pair of distinct points in X can be joined by a geodesic. Note that, in general, such a geodesic does not have to be unique. Moreover, in addition to these geodesics, there can exist also *local geodesics* joining the same two points. Recall that $\gamma\colon I \to X$ is called a local geodesic[2] if every $t_0 \in I$ is contained in a subinterval $J \subset I$ such that the restriction of γ to J is a geodesic. As a result, in a geodesic metric space there are four different conditions that can be considered as analogues of convexity in $\mathbb{R}^n$:

(i) for every $x_1, x_2 \in A \subset X$, $x_1 \neq x_2$, the set A contains a local geodesic joining x_1 and x_2;

(ii) for every $x_1, x_2 \in A \subset X$, $x_1 \neq x_2$, the set A contains a geodesic joining x_1 and x_2;

(iii) every geodesic with end-points in A is contained in A;

(iv) every local geodesic with end-points in A is contained in A.

Clearly, (i) $\Leftarrow$ (ii) $\Leftarrow$ (iii) $\Leftarrow$ (iv). Moreover, if a local geodesic joining two distinct points is unique, as we have, e.g., for a simply connected domain endowed with the hyperbolic distance, then the four conditions are equivalent. However, this is not true for multiply connected hyperbolic domains in $\mathbb{C}$, with an annulus giving a simplest example. It is therefore natural to ask the following question.

Problem 5.2. *Fix $\mu \in \mathbb{R}$ and two hyperbolic (multiply connected) domains $D_j \subset \mathbb{C}$, $j = 1,2$, equipped with hyperbolic distance functions k_{D_j}. Further, let $z_j \in D_j$, $j = 1,2$. In this setting, which of the conditions (i)–(iv) hold for $A := \mathcal{D}_\mu(f; D_1, D_2)$ and for every holomorphic map $f\colon D_1 \to D_2$?*

Remark. Note that the above problem makes sense also in several complex variables. For example, one can ask the same question with D_1 and D_2 replaced by Kobayashi hyperbolic complex manifolds provided that the first manifold is a geodesic metric space with respect to the Kobayashi distance.[3]

In contrast to the level sets $\mathcal{D}_\mu(f)$, the hyperbolic-density family

$$\Omega_\lambda(f) := \left\{ z \in \mathbb{D} : \frac{\lambda_{\mathbb{D}}(z)}{\lambda_{\mathbb{D}}(f(z))} < \lambda \right\}$$

is not conformally invariant. Therefore, replacing $\mathbb{D}$ with another hyperbolic domain, even simply connected, may change the situation.

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[2] It is worth mentioning that in Riemannian geometry the terminology is a bit different: geodesics are called *minimizers* or *length minimizing geodesics*, and local geodesics are referred to simply as geodesics.

[3] This is the case, e.g., for bounded convex domains in $\mathbb{C}^n$, see Theorem 2.6.19 in [1]; and for complete Kobayashi hyperbolic complex manifolds on which the Kobayashi metric is Finsler, as follows from the Hopf–Rinow theorem, see, e.g., Corollary 1 in Section 7, Chapter 8, of [9].

Problem 5.3. *For which pairs of hyperbolic domains* $D_j \subset \mathbb{C}$, $j = 1, 2$, *the sets*

$$\Omega_\lambda(f; D_1, D_2) := \left\{ z \in \mathbb{D} : \frac{\lambda_{D_1}(z)}{\lambda_{D_2}(f(z))} < \lambda \right\}$$

satisfy, for every holomorphic map $f\colon D_1 \to D_2$ *and every* $\lambda \geqslant 1$, *at least one of the conditions* (i)–(iv)? *Here* λ_{D_j}, $j = 1, 2$, *stands for the density of the hyperbolic metric in* D_j.

Returning to the original setting of the paper, we have another question, which seems to be of interest too.

Problem 5.4. *Fix some* $\mu \in \mathbb{R}$. *For which strictly increasing functions* $\Phi\colon [0, +\infty) \to \mathbb{R}$ *the sets* $\mathcal{D}_{\Phi,\mu}(f)$ *consisting of all* $z \in \mathbb{D}$ *satisfying*

$$\Phi(k_\mathbb{D}(z, 0)) - \Phi(k_\mathbb{D}(f(z), 0)) < \mu$$

are hyperbolically convex whenever $f\colon \mathbb{D} \to \mathbb{D}$ *is holomorphic?*

Clearly, for $\mu = 0$, every strictly increasing function Φ defines the same hyperbolically convex set $\mathcal{D}_{\Phi,0}(f) = \mathcal{D}(f)$. Furthermore, our results show that the functions $\Phi_-(x) := x$ if $\mu < 0$ and $\Phi_+(x) := \log\cosh(x/2)$ if $\mu > 0$ are suitable choices. Given some fixed $\mu \neq 0$, does there exist any other Φ for which $\mathcal{D}_{\Phi,\mu}(f)$ is hyperbolically convex? Are there any choices valid for all μ in some open interval rather than for a single fixed value, aside from multiples of Φ_- and Φ_+?

We conclude this section with a question related to the notion of geodesic convexity of a *real-valued function*. Recall that a function $u\colon X \to \mathbb{R}$ in a geodesic metric space (X, k) is said to be *geodesically* (or *geodetically*) *convex* if for every geodesic $\gamma \subset X$, the function $u \circ \gamma$ of a real variable is convex.

It is easy to see that the sublevel sets of a geodesically convex function u are geodesically convex sets, actually in the strongest sense (iv). In this respect, it is worth noticing the following: our argument in the proof of Theorem 3.1 shows that for every $\lambda > 1$, every holomorphic self-map $f : \mathbb{D} \to \mathbb{D}$ and every point $\zeta \in \partial\Omega_\lambda(f)$, the function

$$u_1(z) := \lambda|z|^2 - |f(z)|^2,$$

restricted to the geodesic γ_ζ tangent at ζ to $\partial\Omega_\lambda(f)$, is convex in a neighbourhood of ζ because $(u_1 \circ \gamma_\zeta)''$ is positive at the point $\gamma_\zeta^{-1}(\zeta)$. A similar observation applies to the function $u_2(z) := |z| - (1 + a|z|)|f(z)|$, $a \in (0, 1)$, near boundary points of $\mathcal{D}_\mu(f)$ with $\mu := \log\big((1 - a)/(1 + a)\big)$.

Problem 5.5. *Is it possible to establish Theorems* 3.1 *and* 4.1 *by proving geodesic convexity of* $u(z) := F(z, f(z))$ *in* $(\mathbb{D}, k_\mathbb{D})$ *for a suitable choice of* $F\colon \mathbb{D}^2 \to \mathbb{R}$?

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