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Symplectic cacti, virtualization and Berenstein–Kirillov groups

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Abstract. We explicitly realize an internal action of the *symplectic* cactus group, recently defined by Halacheva for any complex, reductive, finite-dimensional Lie algebra, on crystals of Kashiwara–Nakashima tableaux. Our methods include a symplectic version of jeu de taquin due to Sheats and Lecouvey, symplectic reversal, and virtualization due to Baker. As an application, we define and study a symplectic version of the Berenstein–Kirillov group and show that it is a quotient of the symplectic cactus group. In addition two relations for symplectic Berenstein–Kirillov group are given that do not follow from the defining relations of the symplectic cactus group.

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1. Introduction

The *cactus group* was originally defined by Henriques–Kamnitzer [24] in the context of coboundary categories defined by Drinfeld [16]. It has appeared in connection with the study of moduli spaces of rational curves with $n+1$ marked points [15,17,29,51]

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and has been generalized to other Coxeter types under the name *mock reflection group* in [13]. Coboundary categories are monoidal categories equipped with a *commutor*, that is, a collection of natural isomorphisms $\sigma_{A,B}\colon A \otimes B \to B \otimes A$ satisfying certain properties. The idea of studying the cactus group was originally due to A. Berenstein and was taken up by Henriques–Kamnitzer in [24], who defined it and further showed that it can be realized as the fundamental group of the moduli space of marked real genus zero stable curves. The original idea of Berenstein was to construct a commutor in the category of crystals of a complex, reductive, finite-dimensional Lie algebra, by first defining an involution $\xi_{\mathsf{B}}\colon \mathsf{B} \to \mathsf{B}$ for each crystal B which flips the crystal by exchanging highest weight elements with lowest weight elements. In the case of $\mathfrak{sl}(n,\mathbb{C})$ with the tableau model for the highest weight crystal $\mathsf{B}(\lambda)$ it was known that $\xi_{\mathsf{B}(\lambda)}$ coincides with the Schützenberger involution on semi-standard Young tableaux of shape λ (see [8]). See [11, Sections 4.3 and 14.3.3], and the references therein.

Let $\mathfrak{g}$ be a complex, semi-simple Lie algebra with Dynkin diagram X. There is a Dynkin diagram automorphism $\theta\colon X \to X$ defined by $\alpha_{\theta(i)} = -w_0\alpha_i$, where w_0 is the longest element of the Weyl group W of $\mathfrak{g}$. The *cactus group* $J_{\mathfrak{g}}$, defined by Halacheva in [21,22], is the group generated by σ_I, where I runs over all connected sub-Dynkin diagrams of X, subject to the following relations:

$$\begin{aligned} \sigma_I^2 &= 1, \\ \sigma_I\sigma_J &= \sigma_J\sigma_I \qquad \text{if } J \subseteq X,\ J \sqcup I \text{ is disconnected}, \\ \sigma_I\sigma_J &= \sigma_{\theta_I(J)}\sigma_I \quad \text{if } J \subset I, \end{aligned}$$

where θ_I is the automorphism on I defined by the longest element of the parabolic group W^I. Halacheva has defined an internal action of the cactus group $J_{\mathfrak{g}}$ on a normal $\mathfrak{g}$-crystal by partial Schützenberger–Lusztig involutions ξ_I. From this action we know that partial Schützenberger–Lusztig involutions satisfy the cactus group $J_{\mathfrak{g}}$ relations [23]. Halacheva [22] initiated a combinatorial study of the cactus group for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ by comparing the action of $J_n = J_{\mathfrak{sl}(n,\mathbb{C})}$ on a normal $\mathfrak{sl}(n,\mathbb{C})$-crystal with that of the Berenstein–Kirillov group on Gelfand–Tsetlin patterns (or semi-standard Young tableaux) [31]. Using a different approach, Chmutov, Glick and Pylyavskyy [12] have also found relationships between those two groups.

Our results compose a combinatorial study of the cactus group for the symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$. There are many combinatorial models for $\mathfrak{sp}(2n,\mathbb{C})$-crystals: De Concini tableaux [14], King tableaux [30], Lakshmibai–Seshadri [33] and Littelmann paths [40,41], the alcove path model of Gaussent–Littelmann [19] and the one of Lenart–Postnikov [39], but we work with Kashiwara–Nakashima tableaux, for which a rich combinatorial structure exists [25,28,36,37]. We review the basics in Sections 2 and 3. For each connected sub-Dynkin diagram I of X, we define the explicit action of ξ_I on a given Kashiwara–Nakashima tableau. The algorithmic procedure for that action is given by virtualization. In the case when I forms a Dynkin diagram of type C_{n-k}, it is also given by the *I-partial symplectic reversal*, a symplectic analogue of partial reversal on A_{n-1} semi-standard Young tableaux. Thereby we provide a combinatorial action of the cactus generators σ_I on the set of Kashiwara–Nakashima tableaux on the alphabet $\mathcal{C}_n$. This is addressed in Sections 7 and 8. The case of $I = X$ has already been developed by Santos in [47], where he defines an operation on straight shaped Kashiwara–Nakashima tableaux which is a *symplectic analogue* of the Schützenberger involution operation, also known as evacuation, on straight shaped A_{n-1} semi-standard Young tableaux. This procedure includes the symplectic jeu de taquin defined by Sheats in [49], and further developed by Lecouvey [36] using crystal isomorphisms. This is the content of Section 6.

For $I \subseteq X$ such that I forms a Dynkin diagram of type C_{n-k}, we define an algorithm for *I-partial symplectic reversal* which generalizes Santos' algorithm in the sense that, when $I = X$, our algorithm is exactly the same. The symplectic C_{n-k} reversal extends symplectic C_{n-k} evacuation to arbitrary semi-standard skew tableaux on the alphabet $\mathcal{C}_{n-k}$ whose shift of the entries by k are admissible on the alphabet $\mathcal{C}_n$. The C_{n-k} reversal of such a semi-standard skew tableau P on the alphabet $\mathcal{C}_{n-k}$, is characterized to be the unique skew tableau coplactic equivalent to P and plactic equivalent to the C_{n-k} evacuation of the symplectic rectification of P.

An important inspiration behind our generalization is the operation of *tableau-switching* of Benkart–Sottile–Stroomer [6] on A_{n-1} semi-standard Young tableaux. Given an admissible tableau on the alphabet $\mathcal{C}_n$, we start off by freezing the entries corresponding to nodes not appearing in I, creating at the same time a new Young tableau U with shape defined by the positive frozen entries as well as a skew tableau P consisting of the non-frozen entries. The tableau pair (U, P), sharing a common border, pass through each other via *symplectic jeu de taquin* (SJDT for short). After performing this procedure, a new pair (R, V) arises with R the symplectic rectification of P and V consisting of the entries of U as well as some new, *colored* letters. Each color records a precise instance of the symplectic rectification of P. Our *symplectic colorful tableau switching* is reversible since SJDT is reversible. It reduces to the A_{n-1} tableau switching on tableaux in the alphabet $[n]$. This work is carried out in detail in Section 8.2 of this paper, yielding formula (8.10), and illustrated in Section 8.3.

For the general case we use the virtualization map defined by Baker [3], that is, an injective map

$$E\colon \mathsf{KN}(\lambda, n) \to \mathsf{SSYT}(\lambda^A, n, \overline{n}),$$

which assigns to the $\mathfrak{sp}(2n, \mathbb{C})$-crystal $\mathsf{KN}(\lambda, n)$, a subset of the $\mathfrak{sl}(2n, \mathbb{C})$-crystal $\mathsf{SSYT}(\lambda^A, n, \overline{n})$ in a reversible way. This is discussed in Section 4. We show that one may apply the map E, then perform a certain partial Schützenberger–Lusztig involution in the type $\mathfrak{sl}(2n,\mathbb{C})$-crystal without leaving the image of E, reverse the virtualization map E and obtain our desired result. In Section 8.4.1 we show Theorem 8.13 and Theorem 8.14 using such algorithmic procedures. Additionally, in Definition 5.10, Section 5, we define the *virtual symplectic cactus group* $\widetilde{J}_{2n}$ and show that it is a subgroup of J_{2n} isomorphic to the symplectic cactus group $J_{\mathfrak{sp}(2n,\mathbb{C})}$. In Theorem 7.8, Section 7, an action of the virtual symplectic cactus group on the set $\mathsf{SSYT}(\lambda^A, n, \overline{n})$ is defined. The subset $E(\mathsf{KN}(\lambda, n))$ is preserved under this action as shown in Sections 8.4.1 and 8.5. In particular, in Section 8.5, we realize such an action of the virtual symplectic cactus group on the virtual images of Kashiwara–Nakashima tableaux and show that it *virtualizes* the action of the symplectic cactus group on Kashiwara–Nakashima tableaux. This work is illustrated in Section 8.6.

As an application, in Section 9, we define *symplectic* Bender–Knuth involutions combinatorially (Definition 9.11). We start off by defining the type C_n Berenstein–Kirillov group $\mathcal{BK}^{C_n}$ as the free group that is generated by the partial symplectic Schützenberger–Lusztig involutions with respect to connected sub-diagrams of the type C_n Dynkin diagram of the form $I = [n]$ modulo the relations they satisfy on Kashiwara–Nakashima tableaux of any straight shape. These generators of $\mathcal{BK}^{C_n}$ satisfy the relations of the symplectic cactus group (Theorem 9.10). We show that symplectic Bender–Knuth involutions are also generators of $\mathcal{BK}^{C_n}$.

We study relations for $\mathcal{BK}^{C_n}$ under the virtualization map E. More precisely, for each generator of the symplectic Berenstein–Kirillov group, one can associate a corresponding product of generators in the Berenstein–Kirillov group of type A_{2n-1} in a way analogous to the definition of the virtual symplectic cactus group. We call the group generated by them the *virtual symplectic Berenstein–Kirillov group* $\widetilde{\mathcal{BK}}_{2n}$ (see Definition 9.14). It is a subgroup of the type A_{2n-1} Berenstein–Kirillov group $\mathcal{BK}_{2n}$ satisfying, in particular, the relations of the virtual cactus group $\widetilde{J}_{2n}$ (Theorem 9.18). In Proposition 9.16 the virtual symplectic Bender–Knuth involutions are defined, in an analogous way to how the generators of $\widetilde{\mathcal{BK}}_{2n}$ are defined, as products of certain Bender–Knuth involutions in the type A_{2n-1} Berenstein–Kirillov group. These are shown to be themselves also generators of $\widetilde{\mathcal{BK}}_{2n}$. In Theorem 9.19, they are shown to be the virtualization of the symplectic Bender–Knuth involutions, that is, they commute with the virtualization map E. The virtual image of the group $\mathcal{BK}^{C_n}$ satisfies the relations of $\widetilde{\mathcal{BK}}_{2n}$. More precisely, it is shown that the corresponding map $\mathcal{BK}^{C_n} \to \widetilde{\mathcal{BK}}_{2n}$ is a group isomorphism. Some of the relations listed in Proposition 9.20 are obtained by applying the partial inverse to the virtualization map. Relations (9) and (10) in Proposition 9.20 are the only ones that do not follow from the defining relations of the symplectic cactus group $J_{\mathfrak{sp}(2n,\mathbb{C})}$. They are instead equivalent to the braid relations of the type C_n Weyl group.

2. Basics

Let $\mathfrak{g}$ be a finite-dimensional, complex, semi-simple Lie algebra. Let I be the Dynkin diagram associated to the root system of $\mathfrak{g}$, $\Delta = \{\alpha_i : i \in I\}$ the set of simple roots, W its Weyl group, generated by the simple reflections $\{r_i : i \in I\}$, and $w_0 \in W$ the longest Weyl group element. We will use the numbering of the vertices of I given by [10]. The Dynkin diagram has an automorphism, a permutation of its nodes which leaves the diagram invariant, $\theta\colon I \to I$ defined by $\alpha_{\theta(i)} = -w_0\alpha_i$ for any node $i \in I$, where w_0 is the longest element of W. We will also denote by Λ the integral weight lattice associated to the root system of $\mathfrak{g}$. It is generated by the fundamental weights $\omega_i, i \in I$. For a connected sub-diagram of I, $J \subseteq I$, denote by $\theta_J\colon J \to J$ the Dynkin diagram automorphism that satisfies $\alpha_{\theta_J(j)} = -w_0^J\alpha_j$ for any node $j \in J$, where w_0^J is the longest element of the parabolic subgroup $W^J \subseteq W$ (the Weyl group for $\mathfrak{g}$ restricted to J) [9]. When $J = I$ one has the original notation $\theta_I = \theta$. We focus on the cases where $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}), \mathfrak{sp}(2n, \mathbb{C})$. We will often abuse notation and write a Dynkin diagram I with n nodes as the interval $[n] = \{1 < \cdots < n\}$. The corresponding Weyl groups are the symmetric group $\mathfrak{S}_n$ on n letters and the hyperoctahedral group B_n respectively, where B_n is the free group generated by $r_1, \ldots, r_{n-1}, r_n$ subject to the relations

$$r_i^2 = 1, \quad 1 \le i \le n, \tag{2.1}$$
$$(r_i r_j)^2 = 1, \quad 1 \le i < j \le n, |i - j| > 1, \tag{2.2}$$
$$(r_i r_{i+1})^3 = 1, \quad 1 \le i \le n-2, \tag{2.3}$$
$$(r_{n-1} r_n)^4 = 1. \tag{2.4}$$

The free group generated by $r_1, \ldots, r_{n-1}$ subject to the relations above for $1 \le i$, $j < n$, is $\mathfrak{S}_n$ realized by the simple transpositions $r_i = (i, i+1)$ on the set $[n]$. The group B_n has $2^n n!$ elements and is realized by the signed transpositions

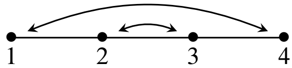
$$r_i = (i, i+1)(\overline{i}, \overline{i+1}), \quad i = 1, \ldots, n-1,$$

and $r_n = (n, \overline{n})$ on the set $\{1 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1}\}$. That is, we may see B_n embedded in $\mathfrak{S}_{2n}$ by folding $\{1 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1}\}$ through a central symmetry. The long element of B_n, $(r_n \cdots r_2 r_1)^n = (r_1 r_2 \cdots r_n)^n$, has length n^2, while the long element of $\mathfrak{S}_n$ has length $n(n-1)/2$ (see [9]).

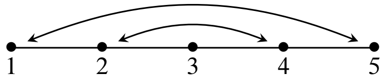
Occasionally, for the sake of clarity, we write w_0^A and w_0^C for the corresponding longest elements of $\mathfrak{S}_n$ and B_n respectively, or simply w_0 when there is no room for confusion. Given a vector $v \in \mathbb{Z}^n$, we have that r_i, with $i \in [n-1]$, acts on v by swapping the i-th and the $(i+1)$-st entries, and r_n acts on v by changing the sign of the last entry. Henceforth, w_0^A reverses v, $w_0^A(v_1, \ldots, v_n) = (v_n, \ldots, v_1)$, and w_0^C changes the sign of the entries of v, $w_0^C v = -v$.

Recall the $\mathfrak{sl}(n,\mathbb{C})$ simple roots $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, $i \in [n-1]$, and the $\mathfrak{sp}(2n,\mathbb{C})$ simple roots $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, $i \in [n-1]$, and $\alpha_n = 2\mathbf{e}_n$, where $\mathbf{e}_i$, $i \in [n]$, is the $\mathbb{R}^n$ standard basis. The A_{n-1} Dynkin diagram automorphism above is given by $\theta(i) = n-i$, since $-w_0\alpha_i = -(-\alpha_{n-i}) = \alpha_{n-i}$, with $i \in I = [n-1]$. For instance,

$n = 5 \quad A_4$



$n = 6 \quad A_5$



The C_n Dynkin diagram automorphism above is given by $\theta(i) = i$, since for $w_0 \in B_n$, $-w_0\alpha_i = -(-\alpha_i) = \alpha_i$, with $i \in I = [n]$. The weight lattices are $\Lambda = \mathbb{Z}^n$ for $\mathfrak{sp}(2n,\mathbb{C})$ and $\Lambda = \mathbb{Z}^n/\langle(1,\ldots,1)\rangle$ for $\mathfrak{sl}(n,\mathbb{C})$. We will often work with representatives in the case of $\mathfrak{sl}(n,\mathbb{C})$. The fundamental weights are

$$\omega_i = \sum_{j=1}^{i} e_i, \quad 1 \le i \le n,$$

and respectively have representatives ω_i, $1 \le i \le n-1$.

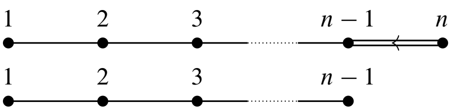
2.1. Levi sub-diagrams

Let I be a finite Dynkin diagram. A Levi sub-diagram J of I obtained by deleting from I a subset of its nodes is the Dynkin diagram of a semi-simple Lie algebra $\mathfrak{g}_J \subset \mathfrak{g}$ known as a *Levi sub-algebra* which is the Levi component of the parabolic Lie sub-algebra of $\mathfrak{g}$ generated by the Chevalley generators associated to the nodes of J.

Example 2.1. If we remove the last node (the one labeled by n) from the Dynkin diagram of type C_n, we obtain a Dynkin diagram of type A_{n-1} which corresponds to the Levi sub-algebra $\mathfrak{sl}(n,\mathbb{C})$ of $\mathfrak{sp}(2n,\mathbb{C})$:

C_n

A_{n-1}



Example 2.2. The semi-simple Lie algebra $\mathfrak{sl}(3,\mathbb{C}) \times \mathfrak{sp}(4,\mathbb{C})$ is a Levi sub-algebra of $\mathfrak{sp}(12,\mathbb{C})$. Note that the semi-simple Lie algebra $\mathfrak{sl}(n,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ is not a Levi sub-algebra of $\mathfrak{sp}(2n,\mathbb{C})$, as its Dynkin diagram of type $A_{n-1} \times A_1$ cannot be obtained from the type C_n diagram by deleting some of its vertices.

3. Normal crystals and Levi restrictions

Crystals corresponding to finite-dimensional (quantum group) $U_q(\mathfrak{g})$-representations belong to a family of crystals called *normal crystals* [11, 23]. In classical types, these crystals may be realized by a tableau model [28] and have nice combinatorial properties. These crystals decompose into connected components, one for each irreducible component to the representation at hand. The Levi restriction of a normal crystal is still a normal crystal, and the union of some connected components of a normal crystal is also a normal crystal [11, 23]. The crystals that we deal with are tableau crystals for finite-dimensional representations of $U_q(\mathfrak{sl}(n,\mathbb{C}))$ and $U_q(\mathfrak{sp}(2n,\mathbb{C}))$.

A $\mathfrak{g}$-*crystal* is a finite set B along with maps

$$\mathrm{wt}\colon \mathsf{B} \to \Lambda, \quad e_i, f_i\colon \mathsf{B} \to \mathsf{B} \sqcup \{0\}, \quad \varepsilon_i, \varphi_i\colon \mathsf{B} \to \mathbb{Z},$$

satisfying the following axioms for any $b, b' \in \mathsf{B}$ and $i \in I$:

(1) $b' = e_i(b)$ if and only if $b = f_i(b')$;

(2) if $f_i(b) \neq 0$, then $\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i$,
if $e_i(b) \neq 0$, then $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i$;

(3) $\varepsilon_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : e_i^a(b) \neq 0\}$ and $\varphi_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : f_i^a(b) \neq 0\}$;

(4) $\varphi_i(b) - \varepsilon_i(b) = \langle \mathrm{wt}(b), \alpha_i^\vee \rangle$,

where $\alpha_i^\vee = 2\alpha_i/\langle \alpha_i, \alpha_i \rangle$ are the simple coroots.

Remark 3.1. Our abstract $\mathfrak{g}$-crystals are defined with the additional condition that they are semi-normal [11].

The *crystal graph* of B is the directed graph with vertices in B and edges labeled by $i \in I$. If $f_i(b) = b'$ for $b, b' \in \mathsf{B}$, then we draw an edge $b \xrightarrow{i} b'$. See Example 3.5. Given an arbitrary subset $J \subseteq I$, B_J is defined to be the crystal B restricted to the sub-diagram J of I, the *Levi branched crystal*. The crystal graph of B_J has the same vertices as B, but the arrows are only those labeled in J; that is, we forget the maps e_i, f_i, φ_i, and ε_i for $i \notin J$ (see [11]). The weight map, which we denote by wt_J, is

$$\mathsf{B} \xrightarrow{\mathrm{wt}} \Lambda \xrightarrow{\mathrm{can}} \Lambda_J,$$

where wt is the weight map of B, Λ is the weight lattice of $\mathfrak{g}$, $\Lambda_J = \Lambda/\langle \omega_i : i \notin J \rangle$ is the weight lattice of $\mathfrak{g}_J$, and $\Lambda \xrightarrow{\mathrm{can}} \Lambda_J$ is the canonical projection. If $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$ and we restrict to $J = [n-1]$, then we obtain an $\mathfrak{sl}(n,\mathbb{C})$-crystal. Given $b \in \mathsf{B}$, $\mathsf{B}(b)$ denotes the connected component of B containing b.

A $\mathfrak{g}$-crystal is *normal* if it is isomorphic to a disjoint union of the crystals $\mathsf{B}(\lambda)$, where $\mathsf{B}(\lambda)$ is the crystal associated to an irreducible, finite-dimensional $\mathfrak{g}$-representation *of highest weight* λ, where $\lambda \in \Lambda$ is a *dominant weight*. In this work, where we focus on $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, respectively $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, dominant weights in $\mathbb{Z}^n$, respectively in $\mathbb{Z}^n/\langle(1, \dots, 1)\rangle$, correspond precisely to *partitions* with at most n parts, that is, weakly decreasing vectors in $\mathbb{Z}^n$ with non-negative entries, respectively to weakly decreasing vectors in $\mathbb{Z}^n$, and each such representative is equivalent to a unique partition in $\mathbb{Z}^{n-1} \hookrightarrow \mathbb{Z}^n$, where the last entry is fixed as zero. An important property of normal crystals B is the existence of a unique highest weight vertex for each connected component of B, that is, an element which is a source in the corresponding crystal graph, whose weight is dominant. Note that we work solely with *highest weight crystals*, namely, crystals B such that for each $b \in \mathsf{B}$, there exists a finite sequence $a_1, a_2, \dots, a_l \in I$ and a highest weight element $u_b \in \mathsf{B}(b)$ such that $b = f_{a_l} \cdots f_{a_2} f_{a_1}(u_b)$. For $b, b' \in \mathsf{B}$, we have $B(b) = B(b')$ if and only if $u_b = u_{b'}$. From now on, we will refer to $\mathfrak{sp}(2n, \mathbb{C})$-crystals by C_n-crystals, and $\mathfrak{sl}(n, \mathbb{C})$-crystals by A_{n-1}-crystals.

3.1. Kashiwara–Nakashima tableaux

Let $\mathsf{B}(\lambda)$ be the irreducible C_n-crystal with highest weight a partition λ of at most n parts. We realize $\mathsf{B}(\lambda)$ as the crystal $\mathsf{KN}(\lambda, n)$ of Kashiwara–Nakashima tableaux [28] of shape λ on the alphabet

$$\mathcal{C}_n = \{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\}.$$

The irreducible A_{n-1}-crystal with highest weight a partition λ of at most n parts is realized as the crystal $\mathsf{SSYT}(\lambda, n)$ of semi-standard tableaux of shape λ on the alphabet $[n]$. We also will refer to these tableaux as the A_{n-1} tableaux of shape λ. The crystal $\mathsf{SSYT}(\lambda, n)$ is a connected sub-crystal of $\mathsf{KN}(\lambda, n)$. The *weight* of an A_{n-1} tableau T, respectively a Kashiwara–Nakashima tableau U, is represented by, respectively is, the vector $(\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$, where μ_i denotes the number of i's in T, respectively the number of i's minus the number of $\overline{i}$'s in U.

Kashiwara–Nakashima tableaux (KN for short) are semi-standard Young tableaux in the alphabet $\mathcal{C}_n$ which satisfy some extra conditions. They are a variation of De Concini symplectic tableaux [14]. A semi-standard Young tableau of any shape (skew or straight) with entries in $\mathcal{C}_n$ is KN if and only if the following two conditions hold:

(1) each one of its columns is admissible;

(2) its splitting is a semi-standard Young tableau.

Definition 3.2. Let C be a semi-standard column in the alphabet $\mathcal{C}_n$ of length at most n. Let $Z = \{z_1 > \cdots > z_m\}$ be the set of non-barred letters z in $\mathcal{C}_n$ such that both z and $\overline{z}$ appear in C. We say that the column C is admissible if there exists a set

$$T = \{t_1 > \cdots > t_m\}$$

of unbarred letters t that satisfies the following:

(1) $t, \overline{t} \notin C$;

(2) $t_1 < z_1$ and is maximal with this property;

(3) $t_i < \min(t_{i-1}, z_i)$ and is maximal with this property.

The split of a column is the two-column tableau $lCrC$, where lC is the column obtained from C by replacing z_i by t_i and possibly re-ordering, and rC is obtained from C by replacing $\overline{z}_i$ by $\overline{t}_i$ and possibly re-ordering. The splitting of a tableau consisting of admissible columns is the concatenation of the splits of its columns.

Given $\mu \subseteq \lambda$ partitions with at most n parts, $\mathsf{KN}(\lambda/\mu, n)$ denotes the normal C_n-crystal of KN tableaux of skew shape λ/μ on the alphabet $\mathcal{C}_n$ [36, Lemma 6.1.3, Corollary 6.3.9].

Example 3.3. Let $n = 2$. The column $\begin{array}{|c|}\hline 2 \\ \hline \overline{2} \\ \hline\end{array}$ is admissible, however, $\begin{array}{|c|}\hline 1 \\ \hline \overline{1} \\ \hline\end{array}$ is not. Notice that although each of its columns is admissible, the tableau $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \overline{2} & \overline{2} \\ \hline\end{array}$ is not KN, because its split, $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 1 & 2 \\ \hline \overline{2} & \overline{1} & \overline{2} & \overline{1} \\ \hline\end{array}$ is not semi-standard.

We will mostly use the notation and definitions from [36, 37]. We also refer the reader to the references therein.

Remark 3.4 ([36, Remark 2.2.2]). The maximal height of an admissible column is n. Moreover, a column C is admissible if and only if, for any $m \in [n]$, the number $N(m)$ of letters x in C such that either $x \leq m$ or $x \geq \overline{m}$ satisfies $N(m) \leq m$. Moreover, if there exists in C a letter $m \leq n$ such that $N(m) > m$, then C contains a pair $(z, \overline{z})$ satisfying $N(z) > z$.

In [36], *coadmissible* columns are defined as well (see [36, p. 301]). We will not delve into the details here, however, we remark that there exists a bijection between admissible and coadmissible columns given by filling in the shape of the given admissible column C with the unbarred letters of lC from top to bottom in increasing order, followed by the barred letters of rC in the same fashion. We will denote this bijection by Φ and use it in Section 6.3.

The *reading word* of a KN tableau is the word in the alphabet $\mathcal{C}_n$ given by the Chinese/Japanese reading of its columns (from top to bottom and right to left). Recall the C_n *signature rule* [11, 28, 36] to compute the action of the Kashiwara crystal operators on a word in the alphabet $\mathcal{C}_n$. (We refer to Section 8.1 for more details.)

Example 3.5. The C_2 crystal $\mathsf{KN}(\lambda, 2)$ of shape $\lambda = (2, 1)$. Each node in the graph represents an element of the crystal. There is a blue, respectively red, arrow connecting an element a to an element b whenever $f_1(a) = b$, respectively $f_2(a) = b$, where b is computed by f_1, respectively f_2, according to the C_2 signature rule on the reading word of a. (See Figure 1.)

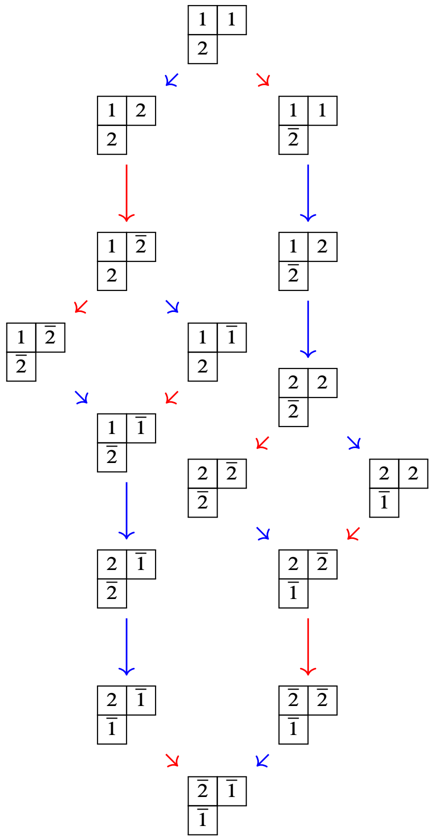


Figure 1. Example 3.5.

3.1.1. Levi branching of KN tableau crystals. For $J \subseteq I$, $\mathsf{KN}_J(\lambda, n)$ is the restriction of $\mathsf{KN}(\lambda, n)$ to the sub-diagram J of I: as a crystal graph it has the same set of vertices as $\mathsf{KN}(\lambda, n)$ but only contains the arrows labeled by J, and it is also a normal crystal. The highest weight elements of $\mathsf{KN}_J(\lambda, n)$ are those C_n tableaux in $\mathsf{KN}(\lambda, n)$ where the only incoming edges are colored in $[n] \setminus J$.

If $J = [p, q]$, $1 \leq p \leq q \leq n$, the crystal graph $\mathsf{KN}_J(\lambda, n)$ consists of the KN tableaux of $\mathsf{KN}(\lambda, n)$ with arrows colored in J.

If $q < n$, the Levi branched crystal $\mathsf{KN}_{[p,q]}(\lambda, n)$ is a type A_{q-p+1} normal crystal. The Weyl group is $W^J = \mathfrak{S}_{[p,q+1]}$, the symmetric group on the letters $\{p, \dots, q+1\}$ and generators $r_j = (j, j+1)(\overline{j}, \overline{j+1})$, $j \in J$. We say that the entries outside of

$$[\pm p, q+1] := \{p < \cdots < q+1\} \sqcup \{\overline{q+1} < \cdots < \overline{p}\}$$

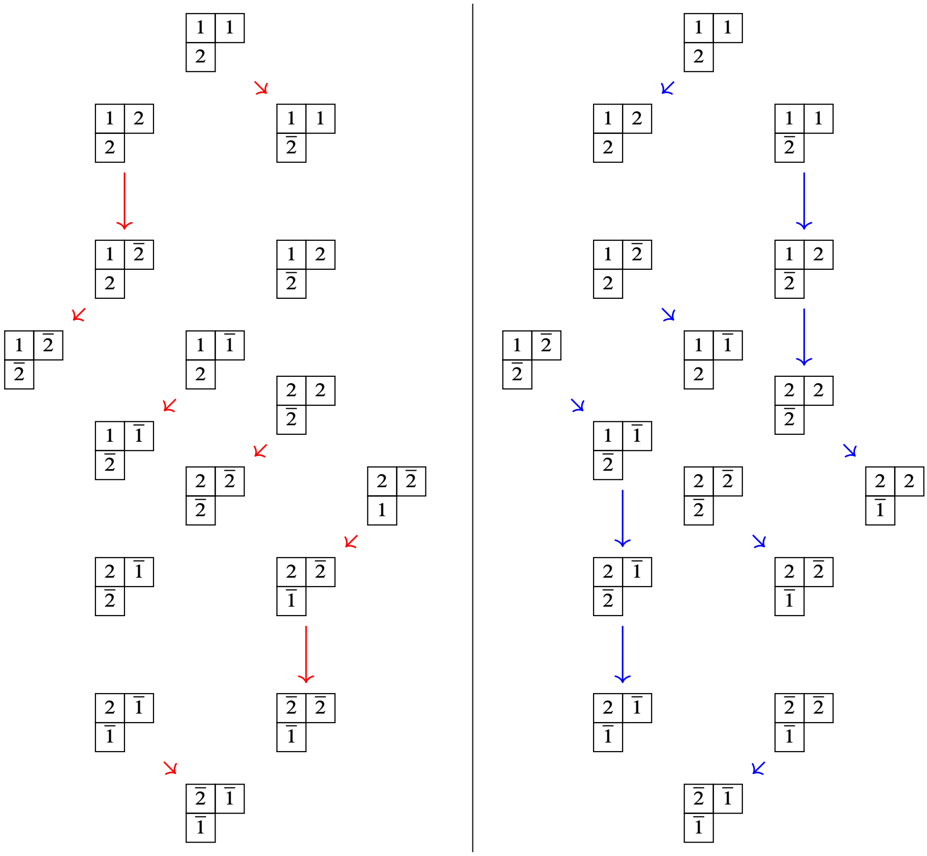


Figure 2. The Levi branched crystals $\mathsf{KN}_{\{2\}}(\lambda, 2)$ and $\mathsf{KN}_{\{1\}}(\lambda, 2)$ respectively from left to right for $\lambda = (2, 1)$.

are *frozen*. This is essentially the same as saying that the KN tableaux of the set $\mathsf{KN}(\lambda, n)$ in the same connected component of $\mathsf{KN}_{[p,q]}(\lambda, n)$ are stable in the entries over $\mathscr{C}_n \setminus [\pm p, q+1]$ under the action of the Kashiwara operators $f_i, e_i, i \in [p, q]$. That is, if $q < n$, in the same connected component of $\mathsf{KN}_{[p,q]}(\lambda, n)$, the subtableaux consisting of the letters

$$[1, p-1] = \{1 < \cdots < p-1\}, \quad [\overline{p-1}, \overline{1}] = \{\overline{p-1} < \cdots < \overline{1}\},$$

or

$$[\pm(q+2), n] = \{q+2 < \cdots < n < \overline{n} < \cdots < \overline{q+2}\}$$

are the same.

If $q = n$, the Levi branched crystal $\mathsf{KN}_{[p,n]}(\lambda, n)$ is isomorphic to a type C_{n-p+1} normal crystal. The Weyl group is $W^J = B_{[p,n]}$ generated by the signed permutations on the subset $\{p < \cdots < n < \overline{n} < \cdots < \overline{p}\}$. The entries outside of

$$[\pm p, n] = \{p < \cdots < n < \overline{n} < \cdots < \overline{p}\}$$

are *frozen*; within the same connected component of $\mathsf{KN}_J(\lambda, n)$, the subtableaux either consisting of the letters $\{1 < \cdots < p-1\}$ or $\{\overline{p-1} < \cdots < \overline{1}\}$ are the same. In Example 3.6, since $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{sp}(2, \mathbb{C})$, we get two crystals of types $A_1 = C_1$.

Example 3.6. Figure 2 shows the Levi branched crystals $\mathsf{KN}_{\{2\}}(\lambda, 2)$ and $\mathsf{KN}_{\{1\}}(\lambda, 2)$, respectively, from left to right for $\lambda = (2, 1)$. Both are A_1-crystals.

The highest weights, with multiplicity, in the left-hand side have representatives $(2,1), (1,2), (1,0), (0,1), (0,1), (-1,2), (-1,0), (-2,1)$. In the quotient of $\mathbb{Z}^2$ by the fundamental weight $\omega_1 = (1,0)$, these are equivalent to the vectors $(0,1), (0,2), (0,0)$, $(0, 1), (0, 1), (0, 2), (0, 0), (0, 1)$, respectively. In practice, this means that we have ignored the multiplicity of the letters $\{1, \overline{1}\}$ in the tableaux of the left-hand side to compute the highest weights. On the right-hand side, we consider another embedding of $\mathbb{Z} \hookrightarrow \mathbb{Z}^2$ given by the quotient $\mathbb{Z}^2/\langle(1,1)\rangle$, since $\omega_2 = (1,1)$. The computation of the highest weights on the right-hand side is similar to that of the left-hand side, and we thus leave it as an exercise for the reader.

4. Virtualization

In this section we closely follow Baker [3, Section 2] and adopt the notation used there. In Example 8.6, we present a detailed example of the content in this section. We include it later rather than earlier because it includes some more information which is not yet presented up to the end of this section.

4.1. Baker embedding and Baker recording tableau

Let $\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n \in \mathbb{Z}^n$ with $\omega_j = \sum_{i=1}^{j} e_i \in \mathbb{Z}^n$, $1 \le j \le n$, the fundamental weights of type C_n. Let

$$\omega_j^A = \sum_{i=1}^{j} e_i \in \mathbb{Z}^{2n}, \qquad 1 \le j \le n,$$

$$\omega_{\overline{j}}^A = \omega_{2n-j+1}^A = \sum_{i=1}^{2n-j+1} e_i \in \mathbb{Z}^{2n}, \quad 1 < j \le n,$$

be the A_{2n-1} fundamental weights, and consider as well the $\mathbb{Z}^{2n}$ partition

$$\lambda^A = 2\lambda_n\omega_n^A + \sum_{i=1}^{n-1}\lambda_i\big(\omega_i^A + \omega_{\overline{i+1}}^A\big).$$

Let $\mathsf{SSYT}(\lambda^A, n, \overline{n})$ be the A_{2n-1} crystal of semi-standard Young tableaux in the alphabet $\mathcal{C}_n$ of shape λ^A. We will denote the corresponding crystal operators by f_i^A for $i \in \mathcal{C}_n$ and consider, for $1 \le i \le n$, the operators $f_i^E = f_i^A f_{\overline{i+1}}^A$, $i < n$, and $f_n^E = (f_n^A)^2$. Let E denote the *virtualization map* defined on type C_n Kashiwara–Nakashima tableaux defined by Baker [3, Propositions 2.2 and 2.3]. More precisely, E is an injective map

$$E\colon \mathsf{KN}(\lambda, n) \hookrightarrow \mathsf{SSYT}(\lambda^A, n, \overline{n}) \tag{4.1}$$

such that $E(f_i(T)) = f_i^E(E(T))$ for $T \in \mathsf{KN}(\lambda, n)$, $1 \le i \le n$. We will denote by E^{-1} the restriction of any left inverse of E to the image of $\mathsf{KN}(\lambda, n)$ under E. Given an admissible column C in the alphabet $\mathcal{C}_n$ of shape ω_i, $1 \le i \le n$, denote by $\psi(C)$ its *Baker virtual split* [3, Proposition 2.2], a two column type A_{2n-1} tableau of shape $\omega_i^A + \omega_{2n-i}^A$. The map ψ is injective and embeds admissible columns of length i, in the alphabet $\mathcal{C}_n$, into $\mathsf{SSYT}(\omega_i^A + \omega_{2n-i}^A, n, \overline{n})$, $1 \le i \le n$. We define ψ^{-1} analogously to E^{-1}. From [3, Proposition 2.3] we know that, if we write T as a concatenation of its columns, that is, $T = C_k \cdots C_1$, then

$$E(T) = \big[\emptyset \leftarrow w(\psi(C_1)) \leftarrow \cdots \leftarrow w(\psi(C_k))\big],$$

where the word $w(\psi(C))$ of the type A_{2n-1} two-column tableau $\psi(C)$ is given by the Chinese/Japanese reading of its two columns (from top to bottom and right to left), and $P \leftarrow w$ is the Schensted column insertion of a word w into a type A_{2n-1} semi-standard Young tableau P in the alphabet $\mathcal{C}_n$ (see [18, 50]).

Let $T_\lambda \in \mathsf{KN}(\lambda, n)$ be the highest weight element; that is, T_λ is the Yamanouchi tableau of shape and weight λ on the alphabet $[n]$ (each row i is solely filled with the letter i). Then $E(T_\lambda) = T_{\lambda^A}$ is the highest weight element of $\mathsf{SSYT}(\lambda^A, n, \overline{n})$, that is, the A_{2n-1} Yamanouchi tableau of shape and weight λ^A in the alphabet $\mathcal{C}_n$. The image of $\mathsf{KN}(\lambda, n)$ under E in $\mathsf{SSYT}(\lambda^A, n, \overline{n})$ is the crystal generated by acting with the lowering operators f_i^E on the highest weight element T_λ^A of $\mathsf{SSYT}(\lambda^A, n, \overline{n})$. For $T \in \mathsf{KN}(\lambda, n)$, where $T = C_k \cdots C_1$, we write

$$w_T = w(\psi(C_1)) \cdots w(\psi(C_k)).$$

Then w_T is a word in $\mathcal{C}_n^*$, the monoid of words in the alphabet $\mathcal{C}_n$, and $E(T) = [\emptyset \leftarrow w_T]$. We will call the *recording tableau of the column insertion of* w_T, $Q(w_T)$, the *Baker recording tableau* associated to T.

Proposition 4.1. *For $T \in \mathsf{KN}(\lambda, n)$, the Baker recording tableau $Q(w_T)$ depends only on λ. From now on, we will denote by Q_λ the Baker recording tableau associated to λ.*

Proof. By abuse of notation, we denote by the same symbols the type A_{2n-1} crystal operators on the A_{2n-1} crystal $\mathcal{C}_n^*$ of words and those on semi-standard Young tableaux in the same alphabet. We know that there exists a sequence $1 \leq i_1, \dots, i_k \leq n$ such that $f_{i_k} \cdots f_{i_1}(T_\lambda) = T$. Therefore,

$$f_{i_k}^E \cdots f_{i_1}^E(E(T_\lambda)) = E(T),$$

where $E(T_\lambda) = T_{\lambda^A}$, the highest weight element of $\mathsf{SSYT}(\lambda^A, n, \overline{n})$, and so

$$f_{i_k}^E \cdots f_{i_1}^E(w_{T_\lambda}) = w_T,$$

since the connected components of the crystal $\mathcal{C}_n^*$ of words of type A_{2n-1} with highest weight elements w_{T_λ} and $w(E(T_\lambda)) = w(T_{\lambda^A})$ have the same highest weight λ^A and are hence isomorphic. In particular, both w_T and w_{T_λ} belong to the same connected component of the crystal $\mathcal{C}_n^*$ of words of type A_{2n-1}, namely, the connected component containing the Yamanouchi word w_{T_λ} of weight λ^A (recall that all words w_T have the same rectification shape λ^A and that all A_{2n-1} crystal operators commute with *jeu de taquin*). Now, we consider a version of the RSK correspondence [11, 18, 32, 50] which is a bijection

$$\mathcal{C}_n^* \overset{1:1}{\longleftrightarrow} \bigsqcup_{\substack{\mu \\ \ell(\mu)\leq 2n}} \mathsf{SSYT}(\mu, n, \overline{n}) \times \mathsf{SYT}(\mu), \quad w \overset{\mathrm{RSK}}{\longmapsto} (P(w), Q(w)),$$

where $\mathsf{SYT}(\mu)$ is the set of standard Young tableaux of shape μ, $P(w) = [\emptyset \leftarrow w]$ and $Q(w)$ is the corresponding recording tableau which encodes the sequence of shapes produced by the column insertion of w. In particular, for each standard Young tableau Q of shape μ, the pre-image $\mathrm{RSK}^{-1}(\mathsf{SSYT}(\mu, n, \overline{n}) \times \{Q\})$ is a crystal isomorphic to $\mathsf{SSYT}(\mu, n, \overline{n})$, and all of these pre-images are disjoint and cover $\mathcal{C}_n^*$. In particular, this means that all the words w_T for $T \in \mathsf{KN}(\lambda, n)$ are contained in the same connected component of $\mathcal{C}_n^*$ defined by:

$$\mathrm{RSK}^{-1}\big(\mathsf{SSYT}(\lambda^A, n, \overline{n}) \times \{Q(w_{T_\lambda})\}\big).$$

Thereby, $Q(w_T) = Q(w_{T_\lambda})$ for all $T \in \mathsf{KN}(\lambda, n)$. ■

Corollary 4.2. *Let $\lambda = \omega_{m_1} + \cdots + \omega_{m_k}$, $1 \leq m_1 \leq \cdots \leq m_k \leq n$, and let*

$$\lambda^A = \omega_{2n-m_1}^A + \omega_{m_1}^A + \cdots + \omega_{2n-m_k}^A + \omega_{m_k}^A \in \mathbb{Z}^{2n}.$$

Then Q_λ can be written out of the shape λ^A as a sequence of shapes by adding successively the columns $\omega^A_{m_1}, \omega^A_{2n-m_1}, \dots, \omega^A_{m_k}, \omega^A_{2n-m_k}$, whose boxes are filled along columns, top to bottom with 3 consecutive numbers from 1 to $|\lambda^A|$:

$$
\begin{aligned}
\emptyset &\subset \omega^A_{m_1}\\
&\subset \omega^A_{2n-m_1} + \omega^A_{m_1}\\
&\subset \omega^A_{m_2} + \omega^A_{2n-m_1} + \omega^A_{m_1}\\
&\subset \omega^A_{2n-m_2} + \omega^A_{m_2} + \omega^A_{2n-m_1} + \omega^A_{m_1}\\
&\vdots\\
&\subset \omega^A_{m_k} + \cdots + \omega^A_{2n-m_2} + \omega^A_{m_2} + \omega^A_{2n-m_1} + \omega^A_{m_1}\\
&\subset \omega^A_{2n-m_k} + \omega^A_{m_k} + \cdots + \omega^A_{2n-m_1} + \omega^A_{m_1} = \lambda^A.
\end{aligned}
$$

Given a partition λ with at most n parts, and $T = C_k \cdots C_1 \in \mathsf{KN}(\lambda, n)$, let $\Psi(T) = (w(\psi(C_1)), \dots, w(\psi(C_k))) \in \mathcal{C}^*_n$ (here the word is presented as a k-tuple) and $\Psi^{-1} = \underbrace{(\psi^{-1}, \dots, \psi^{-1})}_{k}$. Then $(E(T), Q_\lambda) = \mathrm{RSK}\,\Psi(T) = (P(w_T), Q_\lambda)$ and

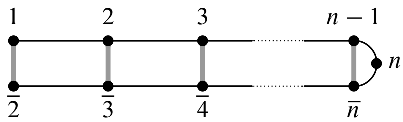
$$E^{-1} = \Psi^{-1}\,\mathrm{RSK}^{-1}_{|E(\mathsf{KN}(\lambda,n))\times\{Q_\lambda\}},$$

where $\mathrm{RSK}^{-1}_{|\mathsf{KN}(\lambda,n)\times\{Q_\lambda\}}$ denotes the inverse of RSK restricted to $E(\mathsf{KN}(\lambda,n))\times\{Q_\lambda\}$.

The computation of $\mathrm{RSK}^{-1}_{|\mathsf{KN}(\lambda,n)\times\{Q_\lambda\}}$ uses Q_λ to perform the inverse of column Schensted insertion. See the example in Section 8.6.

4.2. The Levi branched crystal and virtualization

Recall that a Levi branched crystal B_J, $J \subseteq I$, I a Dynkin diagram, is obtained by ignoring the maps $f_i, e_i, \varphi_i, \varepsilon_i$, for $i \notin J$. Let I be the A_{2n-1} Dynkin diagram with nodes $\{1, \dots, n, \overline{n}, \dots, \overline{2}\}$:



For each connected sub-diagram $J = [p, q]$ or $[k, n]$ with $1 \le p \le q < n$ and $k \le n$ of $[n]$, let $\overline{J} = [\overline{q+1}, \overline{p+1}]$ respectively $[\overline{n}, \overline{k+1}]$, if $k < n$, and $\overline{J} = \emptyset$ if $k = n$ be the corresponding connected sub-diagram of $[\overline{n}, \overline{2}]$.

Each connected component of the Levi branched crystal $\mathsf{KN}_{J \sqcup \overline{J}}(\lambda, n)$ with $J = [p, q]$, $[k, n]$, $1 \le p \le q < n$, $k \le n$, is embedded via E into a connected component of the Levi branched crystal $\mathsf{SSYT}_{J \sqcup \overline{J}}(\lambda, n)$ such that $J \sqcup \overline{J}$ is a disconnected diagram of $[1, \dots, n, \overline{n}, \dots, \overline{2}]$ if $q < n$, and otherwise, $J \sqcup \overline{J} = [k, \overline{k+1}]$ or $\overline{J} = J$ if $J = \{n\}$. Consider the Levi branching of the type C_n crystal $\mathsf{KN}(\lambda, n)$

to A_{q-p+1}, $1 \le p \le q < n$, and C_{n-k+1}, $k \le n$. The Levi type A_{q-p+1} Dynkin diagram is obtained via folding from the Levi subtype $A_{q-p+1} \times A_{q-p+1}$ of A_{2n-1}, which is obtained by removing the nodes $1, \dots, p-1, q+1, \dots, n, \overline{n}, \dots, \overline{q+2}, \overline{p+2}, \dots, \overline{2}$ from the A_{2n-1} Dynkin diagram. The Levi type C_{n-k+1}, $k \le n$, is obtained via folding from the Levi subtype $A_{2n-2k+1}$ of A_{2n-1} obtained by removing the nodes $1, \dots, k-1, \overline{k}, \dots, \overline{2}$ from the A_{2n-1} Dynkin diagram [11].

In [3, Proposition 2.3 (ii)], it is shown that given $b \in \mathsf{KN}(\lambda, n)$, the C_n crystal length functions $\varepsilon_i^C, \varphi_i^C$, $1 \le i \le n$, on b, and the A_{2n-1} crystal length functions ε_i^A, $\varepsilon_{\overline{i+1}}^A$, $1 \le i < n$, ε_n^A, and φ_i^A, $\varphi_{\overline{i+1}}^A$, $1 \le i < n$, φ_n^A on $E(b)$ are nicely related:

$$\varepsilon_i^C(b) = \varepsilon_i^A(E(b)) = \varepsilon_{\overline{i+1}}^A(E(b)), \quad 1 \le i < n, \quad \text{and} \quad \varepsilon_n^C(b) = \frac{1}{2}\varepsilon_n^A(E(b)),$$

and similarly for $\varphi_i^C(b)$, $1 \le i \le n$, where $\varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\ge 0} : e_i^k(b) \ne 0\}$ and $\varphi_i(b) = \max\{k \in \mathbb{Z}_{\ge 0} : f_i^k(b) \ne 0\}$. This means that b is the highest weight element of a connected component U of $\mathsf{KN}_J(\lambda, n)$ if and only if, for all $i \in J$,

$$\varepsilon_i^A(E(b)) = \varepsilon_{\overline{i+1}}^A(E(b)) = \varepsilon_i^C(b) = 0 \quad \text{for all } i \in J \setminus \{n\}$$

and

$$\varepsilon_n^A(E(b)) = \varepsilon_n^C(b) = 0 \quad \text{if } n \in J.$$

Similarly, in the case where b is the lowest weight element, by replacing appropriately, ε_i^C with φ_i^C and ε_i^A, $\varepsilon_{\overline{i+1}}^A$ with φ_i^A, respectively, $\varphi_{\overline{i+1}}^A$, and ε_n^A with φ_n^A. Henceforth,

$$\varepsilon_i^C(b) = 0, \ \ i \in J \quad \Leftrightarrow \quad \varepsilon_i^A(E(b)) = 0, \ \ i \in J \sqcup \overline{J}$$

and

$$\varphi_i^C(b) = 0, \ \ i \in J \quad \Leftrightarrow \quad \varphi_i^A(E(b)) = 0, \ \ i \in J \sqcup \overline{J}.$$

In other words, because our crystals are semi-normal, $E(b)$ is the highest weight element of the connected component V of $\mathsf{SSYT}_{J \sqcup \overline{J}}(\lambda, n)$ containing $E(b)$ and $E(U)$. It is therefore unique. A similar statement holds for the lowest weight element. The next proposition now easily follows.

Proposition 4.3. *Let $J \subseteq [n]$ be a connected sub-diagram of the type C_n Dynkin diagram. Let U be a connected component of the Levi branched crystal $\mathsf{KN}_J(\lambda, n)$ with highest and lowest weight elements u^{high} and u^{low} respectively. Then $E(U)$ is contained in a connected component of the Levi branched crystal $\mathsf{SSYT}_{J \sqcup \overline{J}}(\lambda, n)$ with highest and lowest weight elements $E(u^{\mathsf{high}})$ and $E(u^{\mathsf{low}})$ respectively.*

Remark 4.4. Given $T \in \mathsf{SSYT}(\mu, n, \overline{n})$, where μ is a partition with at most $2n$ parts, T may be decomposed into two disjoint semi-standard tableaux T^+ and T^- such that

$T = (T^+, T^-)$, where T^+ is the semi-standard tableau of shape $\mu+$ on the alphabet $[n]$ defined by the entries of T in $[n]$, that is, $T^+ \in \mathsf{SSYT}(\mu_+, n)$, called the positive part of T, and T^- is the semi-standard tableau of skew shape μ/μ_+ on the alphabet $[\overline{n}, \overline{1}]$ defined by the entries of T in $[\overline{n}, \overline{1}]$, that is, $T^- \in \mathsf{SSYT}(\mu/\mu_+, \overline{n})$, called the negative part of T. Provided that we only apply f_i^A, e_i^A with $i \in J \sqcup J'$ disconnected such that $J \subseteq [n-1]$ and $J' \subseteq [\overline{n}, \overline{2}]$, respectively, this shape decomposition is preserved. Those crystal operators preserve the shape decomposition above because, according to the type A_{2n-1} signature rule, they only change positive (resp. negative) letters into positive (resp. negative) letters. For $J \sqcup J'$ disconnected, $f_j^A f_{j'}^A = f_{j'}^A f_j^A$, $j \in J$, $j' \in J'$. We then write, for $\{j_1, \ldots, j_r\} \subseteq J$ and $\{j_1', \ldots, j_m'\} \subseteq J'$,

$$f_{j_r}^A \cdots f_{j_1}^A f_{j_m'}^A \cdots f_{j_1'}^A(T) = \big(f_{j_r}^A \cdots f_{j_1}^A(T^+), f_{j_m'}^A \cdots f_{j_1'}^A(T^-)\big). \tag{4.2}$$

5. The cactus group and virtualization

Halacheva [21, 23] has defined a more general version of the cactus group J_n originally defined by Henriques–Kamnitzer [24] in terms of generators and relations.

If I is the A_{n-1} Dynkin diagram, θ_J acts on J by reversing the connected interval of nodes J, whereas in the C_n type it depends on whether J contains the node with label n or not.

Definition 5.1 ([21, 23]). Let $\mathfrak{g}$ be a finite-dimensional, semi-simple Lie algebra with Dynkin diagram I. The *cactus group* $J_{\mathfrak{g}}$ has generators s_J where J runs over the connected sub-diagrams of the Dynkin diagram I of $\mathfrak{g}$, and relations:

(1$\mathfrak{g}$) $s_J^2 = 1$ for all $J \subseteq I$;

(2$\mathfrak{g}$) $s_J s_{J'} = s_{J'} s_J$ for all $J, J' \subseteq I$ such that $J \sqcup J'$ is disconnected;

(3$\mathfrak{g}$) $s_J s_{J'} = s_{\theta_J(J')} s_J$ for all $J' \subseteq J \subseteq I$.

Remark 5.2. Note that when $J' \subseteq J$, (3$\mathfrak{g}$) says that s_J commutes with $s_{J'}$ by reversing J' with respect to J. Recalling that W is the Weyl group of $\mathfrak{g}$, we also have a group epimorphism $J_{\mathfrak{g}} \to W$ taking s_J to w_0^J ([23], [21, Remark 10.0.1]). The kernel is called the *pure cactus group* (see [4] for examples), the fundamental group of the real locus of the Deligne–Mumford compactification $\overline{M_{0,n+1}}(\mathbb{R})$ of the moduli space of rational curves with $n+1$ marked points [15].

Lemma 5.3. *The cactus group $J_{\mathfrak{sl}(n,\mathbb{C})} = J_n$ is the group with generators s_J, where J runs over all connected sub-diagrams of $I = [n-1]$, the A_{n-1} Dynkin diagram, subject to the relations:*

(1A) $s_J^2 = 1$, $J \subseteq [n-1]$;

(2*A*) $s_J s_{J'} = s_{J'} s_J$ *for all* $J, J' \subseteq [n-1]$ *such that* $J \sqcup J'$ *is disconnected;*

(3*A*) $s_{[p,q]} s_{[k,l]} = s_{[p+q-l,p+q-k]} s_{[p,q]}$ *for* $[k,l] \subset [p,q] \subseteq [n-1]$.

Remark 5.4. The first and third relations ensure that the $n-1$ elements of the form

$$s_{[1,k]}, \quad 1 \leq k \leq n-1,$$

generate J_n, since any $s_{[i,j]}$ may be written as

$$s_{[i,j]} = s_{[1,j]} s_{[1,j-i+1]} s_{[1,j]}. \tag{5.1}$$

The elements $s_{[i,n-1]} = s_{[1,n-1]} s_{[1,n-i]} s_{[1,n-1]}$, $1 \leq i \leq n-1$, also form a set of generators.

Lemma 5.5. *The cactus group* $J_{\mathfrak{sp}(2n,\mathbb{C})}$ *is the group with generators* s_J*, where* J *runs over all connected sub-diagrams of the* C_n *Dynkin diagram* $I = [n]$*, subject to the relations:*

(1*C*) $s_J^2 = 1$, $J \subseteq [n]$;

(2*C*) $s_J s_{J'} = s_{J'} s_J$ *for all* $J, J' \subseteq [n]$ *such that* $J \sqcup J'$ *is disconnected;*

(3*C*) (i) $s_{[p,n]} s_{[k,l]} = s_{[k,l]} s_{[p,n]}$, $[k,l] \subseteq [p,n] \subseteq [n]$,

(ii) $s_{[p,q]} s_{[k,l]} = s_{[p+q-l,p+q-k]} s_{[p,q]}$, $[k,l] \subseteq [p,q] \subseteq [n-1]$.

Proof. Relations (1$\mathfrak{g}$) and (2$\mathfrak{g}$) translate directly to (1*A*) and (2*A*). Consider two nested intervals $[k,l] \subset [p,q]$. If $[p,q] \subset [n-1]$, we are in type A_{q-p}, hence (3*C* (ii)) holds, which is just relation (3*A*). If $q = n$, then we are in type C_{n-p+1}. The Weyl group $W^{[p,n]}$ is the restriction of the hyperoctahedral group B_n to the generators $r_p, \ldots, r_n$, as a group of signed permutations, it is the restriction to the set

$$[\pm p, n] = \{p < \cdots < n < \overline{n} < \cdots < \overline{p}\},$$

and $w_0^J(\alpha_j) = -\alpha_j$ for $j \in J$. Therefore, $\theta_{[p,n]}(d) = d$ for $d \in [k,l]$ and relation (3*C* (i)) follows directly from 3$\mathfrak{g}$. ∎

Remark 5.6. Note that the elements s_J, $J \subseteq [n-1]$, subject to the relations above, generate the cactus group J_n. As in (5.1), the following are alternative $2n-1$ generators of $J_{\mathfrak{sp}(2n,\mathbb{C})}$:

$$s_{[1,j]}, \quad 1 \leq j \leq n-1,$$
$$s_{[j,n]}, \quad 1 \leq j \leq n.$$

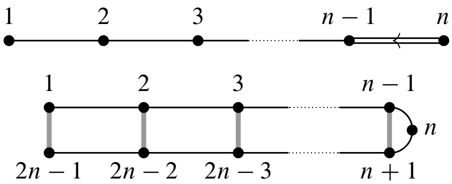
Remark 5.7. We may observe that J_n is a subgroup of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ defined by the subset of generators s_J, $J \subseteq [n-1]$, indexed by connected sub-diagrams of the A_{n-1} connected sub-diagram $[n-1]$ of the C_n Dynkin diagram $I = [n]$, subject to the relations above (1*C*), (2*C*) and (3*C* (ii)).

Proposition 5.8. *If $\mathfrak{g}$ is a finite-dimensional semi-simple Lie algebra, and $\mathfrak{l} \subset \mathfrak{g}$ is a Levi sub-algebra, then $J_{\mathfrak{l}}$ is a subgroup of $J_{\mathfrak{g}}$.*

Proof. Let I be the Dynkin diagram corresponding to $\mathfrak{g}$ and $J \subset I$ the sub-diagram corresponding to the Levi sub-algebra $\mathfrak{l}$. Any connected sub-diagram K of J is also a connected sub-diagram of I, hence one can define a map on generators by $s_K^J \mapsto s_K^I$. Here generators of $J_{\mathfrak{g}}$ are denoted by s_K^I, and generators of $J_{\mathfrak{l}}$ by s_K^J. Remark 5.2 implies that this map is a morphism of groups. The map is clearly injective because the generators of $J_{\mathfrak{g}}$ are all distinct. ∎

5.1. Embedding of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ into J_{2n}

We have observed that J_n is a subgroup of $J_{\mathfrak{sp}(2n,\mathbb{C})}$. We now show that there is a group embedding of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ into J_{2n} by folding the A_{2n-1} Dynkin diagram through the middle node n:



Why should such an embedding exist? Let us consider the following elements of J_{2n}:

$$s'_{[p,q]} := s_{[p,q]}s_{[2n-q,2n-p]} = s_{[2n-q,2n-p]}s_{[p,q]} \quad \text{for all } [p,q] \subseteq [n-1].$$

In Lemma 5.12 we show that these elements together with the generators $s_{[p,2n-p]}$ for $p \leq n$ generate a subgroup of J_{2n} isomorphic to $J_{\mathfrak{sp}(2n,\mathbb{C})}$. Notice the similarity between this and the construction of $\mathfrak{sp}(2n,\mathbb{C})$ as a sub-algebra of $\mathfrak{sl}(2n,\mathbb{C})$ by folding [26, Chapter 8, pp. 89–102]. Moreover, Lemma 5.9 below provides not only concrete combinatorial motivation for Lemma 5.12, but will also be the main ingredient in its proof.

Lemma 5.9. *The following relations hold in J_{2n}:*

$$s'^2_{[p,q]} = 1, \qquad 1 \leq p \leq q < n, \tag{5.2}$$

$$s^2_{[p,2n-p]} = 1, \qquad 1 \leq p < n, \tag{5.3}$$

$$s'_{[p,q]}s'_{[k,l]} = s'_{[k,l]}s'_{[p,q]}, \qquad [p,q] \sqcup [k,l] \subseteq [n-1] \text{ disconnected}, \tag{5.4}$$

$$s_{[p,2n-p]}s_{[k,2n-k]} = s_{[k,2n-k]}s_{[p,2n-p]}, \qquad 1 \le p < k < n, \tag{5.5}$$
$$s_{[p,2n-p]}s'_{[k,l]} = s'_{[k,l]}s_{[p,2n-p]}, \qquad 1 \le p \le k \le l < n, \tag{5.6}$$
$$s'_{[p,q]}s'_{[k,l]} = s'_{[p+q-l,p+q-k]}s'_{[p,q]}, \qquad 1 \le p \le k \le l \le q < n. \tag{5.7}$$

There are no more relations among the elements $s'_{[p,q]}$ *and* $s_{[k,2n-k]}$ *for all* $[p,q] \subseteq [n-1]$ *and* $[k,n] \subseteq [n]$.

Proof. We have the relations (5.2) and (5.3):

$$\begin{aligned} {s'_{[p,q]}}^2 &= \left(s_{[p,q]}s_{[2n-q,2n-p]}\right)^2 \\ &= s^2_{[p,q]}s^2_{[2n-q,2n-p]} \\ &\overset{(1A)}{=} 1, \\ s^2_{[p,2n-p]} &\overset{(1A)}{=} 1. \end{aligned}$$

For $1 \le p \le q < n$ and $1 \le k \le l < n$ such that $[p,q] \sqcup [k,l]$ is disconnected, the sub-diagrams $[p,q] \sqcup [2n-q, 2n-p]$ and $[k,l] \sqcup [2n-l, 2n-k]$ of $[2n-1]$ are disconnected, hence

$$s'_{[p,q]}s'_{[k,l]} \overset{(2A)}{=} s'_{[k,l]}s'_{[p,q]}.$$

This establishes (5.4). Additionally, if $q = n$, the sub-diagram $[k,l] \sqcup [p, 2n-p] \sqcup [2n-l, 2n-k]$ in $[2n-1]$ is disconnected, hence

$$s_{[p,2n-p]}s'_{[k,l]} \overset{(2A)}{=} s'_{[k,l]}s_{[p,2n-p]}.$$

Moreover, $s_{[p,2n-p]}s_{[k,2n-k]} \overset{(3A)}{=} s_{[2n-(2n-k),2n-k]}s_{[p,2n-p]} = s_{[k,2n-k]}s_{[p,2n-p]}$, for $1 \le p < k < n$, hence relation (5.5) holds. Now, for $1 \le p < k < l < n$, we have

$$\begin{aligned} s_{[p,2n-p]}s_{[k,l]}s_{[2n-l,2n-k]} &\overset{(3A)}{=} s_{[2n-l,2n-k]}s_{[p,2n-p]}s_{[2n-l,2n-k]} \\ &\overset{(3A)}{=} s_{[2n-l,2n-k]}s_{[2n-(2n-k),2n-(2n-l)]}s_{[p,2n-p]} \\ &= s_{[2n-l,2n-k]}s_{[k,l]}s_{[p,2n-p]}, \end{aligned}$$

which establishes relation (5.6). Finally, for $1 \le p < k < l < q < n$, the following holds:

$$\begin{aligned} s'_{[p,q]}s'_{[k,l]} &= s_{[p,q]}s_{[2n-q,2n-p]}s_{[k,l]}s_{[2n-l,2n-k]} \\ &\overset{(2A)}{=} s_{[p,q]}s_{[k,l]}s_{[2n-q,2n-p]}s_{[2n-l,2n-k]} \\ &\overset{(3A)}{=} s_{[p+q-l,p+q-k]}s_{[p,q]}s_{[2n-(p+q-k),2n-(p+q-l)]}s_{[2n-q,2n-p]} \\ &\overset{(2A)}{=} s_{[p+q-l,p+q-k]}s_{[2n-(p+q-k),2n-(p+q-l)]}s_{[p,q]}s_{[2n-q,2n-p]} \\ &= s'_{[p+q-l,p+q-k]}s'_{[p,q]}. \end{aligned}$$

This establishes relation (5.7). Any relation $R' = 1$ with the elements

$$s'_{[p,q]} = s_{[p,q]} s_{[2n-q,2n-p]} \quad \text{and} \quad s_{[k,2n-k]}$$

for some $[p,q] \subseteq [n-1]$ and $[k,n] \subseteq [n]$, translates to a relation $R = 1$ involving generators of J_{2n}, of the form $s_{[p,q]}$, $s_{[2n-q,2n-p]}$ in pairs, and $s_{[k,2n-k]}$, for some $[p,q] \subseteq [n-1]$ and $[k,n] \subseteq [n]$, which satisfy the same kind of relations as $s'_{[p,q]}$ and $s_{[k,2n-k]}$. Therefore, from $R = 1$ we do not get new relations $R' = 1$. ■

Definition 5.10. The *virtual symplectic cactus group* $\widetilde{J}_{2n}$ is the group with generators $\widetilde{s}_J$, where J runs over all sub-diagrams of $I = [2n-1]$, the A_{2n-1} Dynkin diagram, of the form $J = [p, 2n-p]$ for all $[p,n] \subseteq [n]$, or the form $J = [p,q] \sqcup [2n-q, 2n-p]$ for all $[p,q] \subseteq [n-1]$, subject to the relations:

(1$\widetilde{A}$) $\widetilde{s}_J^2 = 1$, $J \subseteq [2n-1]$;

(2$\widetilde{A}$) $\widetilde{s}_J \widetilde{s}_{J'} = \widetilde{s}_{J'} \widetilde{s}_J$ such that $J \sqcup J'$ is disconnected with respect to all $[p,q] \subseteq [n]$;

(3$\widetilde{A}$) (i) for $[q,l] \subseteq [p,n] \subseteq [n]$,

$$\widetilde{s}_{[p,2n-p]} \widetilde{s}_{[q,l] \sqcup [2n-l,2n-q]} = \widetilde{s}_{[q,l] \sqcup [2n-l,2n-q]} \widetilde{s}_{[p,2n-p]},$$

(ii) for $[k,l] \subseteq [p,q] \subseteq [n-1]$,

$$\begin{aligned} &\widetilde{s}_{[p,q] \sqcup [2n-q,2n-p]} \widetilde{s}_{[k,l] \sqcup [2n-l,2n-k]} \\ &\quad = \widetilde{s}_{[q+p-l,q+p-k] \sqcup [2n-p+2n-q-(2n-k),2n-p+2n-q-(2n-l)]} \\ &\qquad \cdot \widetilde{s}_{[p,q] \sqcup [2n-q,2n-p]} \\ &\quad = \widetilde{s}_{[q+p-l,q+p-k] \sqcup [2n-(p+q)+k,2n-(p+q)+l]} \widetilde{s}_{[p,q] \sqcup [2n-q,2n-p]}. \end{aligned}$$

The following are $2n-1$ alternative generators of $\widetilde{J}_{2n}$:

$$\widetilde{s}_{[1,j] \sqcup [2n-j,2n-1]}, \quad 1 \le j \le n-1, \tag{5.8}$$

$$\widetilde{s}_{[j,2n-j]}, \quad 1 \le j \le n. \tag{5.9}$$

Proposition 5.11. *There is an isomorphism* $\widetilde{J}_{2n} \simeq J_{\mathfrak{sp}(2n,\mathbb{C})}$.

Proof. Clearly, $\widetilde{J}_{2n}$ and $J_{\mathfrak{sp}(2n,\mathbb{C})}$ satisfy the same relations corresponding to all connected sub-diagrams $[p,q] \subseteq [n]$. Furthermore, the maps

$$\begin{aligned} J_{\mathfrak{sp}(2n,\mathbb{C})} &\to \widetilde{J}_{2n}, \\ s_{[p,q]} &\mapsto \widetilde{s}_{[p,q] \sqcup [2n-q,2n-p]}, \\ s_{[p,n]} &\mapsto \widetilde{s}_{[p,2n-p]}, \end{aligned}$$

and

$$\begin{aligned}\widetilde{J}_{2n} &\to J_{\mathfrak{sp}(2n,\mathbb{C})},\\ \widetilde{s}_{[p,q]\sqcup[2n-q,2n-p]} &\mapsto s_{[p,q]},\\ \widetilde{s}_{[p,2n-p]} &\mapsto s_{[p,n]}\end{aligned}$$

are epimorphisms inverse to each other. This follows directly from the definitions of $\widetilde{J}_{2n}$ and $J_{\mathfrak{sp}(2n,\mathbb{C})}$ (see Definition 5.10 and Lemma 5.5, respectively). Therefore,

$$J_{\mathfrak{sp}(2n,\mathbb{C})} \simeq \widetilde{J}_{2n}.$$

■

Lemma 5.12. *The following assignment defines a group injection from $J_{\mathfrak{sp}(2n,\mathbb{C})}$ to J_{2n}:*

$$\begin{aligned}\Gamma\colon J_{\mathfrak{sp}(2n,\mathbb{C})} &\hookrightarrow J_{2n},\\ s_{[p,q]} &\mapsto s'_{[p,q]}, \qquad 1 \le p \le q < n,\\ s_{[p,n]} &\mapsto s_{[p,2n-p]}, \quad 1 \le p \le n.\end{aligned}$$

Proof. The map induced by Γ is indeed a group homomorphism. The relations ($1C$)–($3C$) from Lemma 5.5 follow from Lemma 5.9.

To show that it is injective, one needs to show that its left inverse defined by the assignment

$$\begin{aligned}\Gamma^{-1}_{\text{left}}\colon \operatorname{im}(\Gamma) \subset J_{2n} &\hookrightarrow J_{\mathfrak{sp}(2n,\mathbb{C})},\\ s'_{[p,q]} &\mapsto s_{[p,q]}, \qquad 1 \le p \le q < n,\\ s_{[p,2n-p]} &\mapsto s_{[p,n]}, \qquad 1 \le p \le n\end{aligned}$$

is also a group morphism. This however follows from Lemma 5.9 and the previous calculations: the generators of $\operatorname{im}(\Gamma)$ satisfy the relations from Lemma 5.5, and there are no more relations between them (all possible cases have been already covered above). ■

Proposition 5.13. *The group $\widetilde{J}_{2n}$ is isomorphic to a subgroup of J_{2n}.*

Proof. After composing the maps from Proposition 5.11 and Lemma 5.12, we obtain the group injection

$$\begin{aligned}\widetilde{J}_{2n} &\hookrightarrow J_{2n}\\ \widetilde{s}_{[p,q]\sqcup[2n-q,2n-p]} &\mapsto s'_{[p,q]}, \qquad 1 \le p \le q < n,\\ \widetilde{s}_{[p,2n-p]} &\mapsto s_{[p,2n-p]}, \quad 1 \le p \le n.\end{aligned}$$

■

6. Full Schützenberger–Lusztig involutions and algorithms

6.1. Full Schützenberger–Lusztig involution

Let $\mathsf{B}(\lambda)$ be the normal $\mathfrak{g}$-crystal with highest weight λ. Let u_λ and u_λ^{low} be the highest, respectively lowest, weight elements of $\mathsf{B}(\lambda)$. The *Schützenberger–Lusztig involution* $\xi\colon \mathsf{B}(\lambda) \to \mathsf{B}(\lambda)$ is the unique map of sets (hence, set involution) such that, for all $b \in \mathsf{B}(\lambda)$ and $i \in I$:

(1) $e_i\xi(b) = \xi f_{\theta(i)}(b)$;

(2) $f_i\xi(b) = \xi e_{\theta(i)}(b)$;

(3) $\mathsf{wt}(\xi(b)) = w_0\mathsf{wt}(\mathsf{b})$,

where w_0 is the long element of the Weyl group W. (For the existence and uniqueness of ξ, ξ^2 is a map of crystals, and hence $\xi^2 = 1$, see [11, 24].) The involution ξ acts by w_0 on the weights and interchanges the action of e_i and $f_{\theta(i)}$. It is an automorphism of the underlying unlabeled, non-oriented and non-weighted graph. For A_{n-1}, ξ acts by reversing the weight and interchanges the action of e_i and f_{n-i}; for C_n, ξ acts by changing the sign of the weight and interchanges the action of e_i and f_i.

If B is a normal $\mathfrak{g}$-crystal, B is the disjoint union of connected components, each of which is a crystal isomorphic to $\mathsf{B}(\lambda)$ for some dominant integral weight λ. We define ξ_{B} on B by applying ξ to each one of its connected components. Each element of $\mathsf{B}(\lambda)$ is generated by u_λ (resp. u_λ^{low}) by applying f_i's (resp. e_i's). Hence, the same sequence of f_i's (resp. e_i's) applies to the highest weight (resp. lowest weight) of any connected component of B isomorphic to $\mathsf{B}(\lambda)$.

The elements u_λ and u_λ^{low} are the unique elements of $\mathsf{B}(\lambda)$ of weight λ, respectively, $w_0\lambda$. Hence, since $\mathsf{wt}(\xi(u_\lambda)) = w_0\lambda$ and $\mathsf{wt}(\xi(u_\lambda^{\mathsf{low}})) = \lambda$, ξ interchanges highest and lowest weight elements of $B(\lambda)$, and so $u_\lambda^{\mathsf{low}} = \xi(u_\lambda)$, $\xi(u_\lambda^{\mathsf{low}}) = u_\lambda$. This implies that $u_\lambda = e_{j_r}\cdots e_{j_1}(u_\lambda^{\mathsf{low}})$ for some sequence $j_1,\dots,j_r \in I$, and

$$\begin{aligned} u_\lambda^{\mathsf{low}} = \xi(u_\lambda) &= \xi\big(e_{j_r}\cdots e_{j_1}(u_\lambda^{\mathsf{low}})\big) \\ &= f_{\theta(j_r)}\cdots f_{\theta(j_1)}\big(\xi(u_\lambda^{\mathsf{low}})\big) = f_{\theta(j_r)}\cdots f_{\theta(j_1)}(u_\lambda). \end{aligned}$$

Corollary 6.1. *Let $b \in \mathsf{B}(\lambda)$ and $b = f_{j_r}\cdots f_{j_1}(u_\lambda)$ for $j_r,\dots,j_1 \in I$. Then*

$$\xi(b) = e_{\theta(j_r)}\cdots e_{\theta(j_1)}(u_\lambda^{\mathsf{low}}), \quad \mathsf{wt}(\xi(b)) = w_0\mathsf{wt}(b).$$

In particular,

- *in type A_{n-1}, $\xi(b) = e_{n-j_r}\cdots e_{n-j_1}(u_\lambda^{\mathsf{low}})$ and* $\mathsf{wt}(\xi(b)) = \operatorname{rev}\mathsf{wt}(b)$, *where* rev *is the reverse permutation (long element) of $\mathfrak{S}_n$;*
- *in type C_n, $\xi(b) = e_{j_r}\cdots e_{j_1}(u_\lambda^{\mathsf{low}})$ and* $\mathsf{wt}(\xi(b)) = -\mathsf{wt}(b)$.

6.2. The full $\mathfrak{sl}(n, \mathbb{C})$ reversal

For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, ξ coincides with the Schützenberger involution [8, 38] also known as evacuation (evac for short) on $\mathsf{SSYT}(\lambda, n)$ (see [18, 50]), and as reversal on the set $\mathsf{SSYT}(\lambda/\mu, n)$ of skew semi-standard tableaux of shape λ/μ in the alphabet $[n]$ (see [6]).

Let $T \in \mathsf{B} = \mathsf{SSYT}(\lambda/\mu, n)$ and let $\mathsf{B}(T)$ be the connected component of the crystal $\mathsf{SSYT}(\lambda/\mu, n)$ containing T. Then $\mathsf{B}(T) \simeq \mathsf{B}(\nu)$ for some partition ν and $\mathsf{rect}(T) \in \mathsf{B}(\nu)$. Thereby, $\xi(T)$ is the unique tableau in $\mathsf{B}(T)$ such that

$$\begin{aligned} \mathsf{rect}\,\xi(T) &= \mathsf{evacuation}(\mathsf{rect}(T)), \\ \xi(T) &= \mathsf{arect}\big(\mathsf{evacuation}(\mathsf{rect}(T))\big), \end{aligned} \tag{6.1}$$

where arectification (arect for short) denotes the inverse process of rectification [1, 6]. More precisely, the rectification (rect for short) procedure is recorded by assigning a standard tableau S to the inner shape μ of T to form the tableau pair (S, T). The entries of S govern the *jeu de taquin* on T, as we slide out all letters in the filling of S, from largest to smallest, to get a new tableau pair $(\mathsf{rect}(T), S')$, where S' is the skew standard tableau consisting of the slid letters from S. The anti-rectification procedure, arect, is defined by the reverse *jeu de taquin* to $\mathsf{evacuation}(\mathsf{rect}(T)$ and is governed by the slid letters in S' in the tableau pair $(\mathsf{evacuation}(\mathsf{rect}(T)), S')$ from smallest to largest. Eventually one obtains the tableau pair $(S, \mathsf{reversal}(T))$, where

$$\mathsf{reversal}(T) := \mathsf{arect}\big(\mathsf{evacuation}(\mathsf{rect}(T))\big). \tag{6.2}$$

Next we will discuss $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$.

6.3. Lecouvey–Sheats symplectic jeu de taquin and symplectic Knuth equivalence

If T is a KN tableau, we consider its word $w(T) \in \mathcal{C}_n^*$ obtained by reading the columns of T in Chinese/Japanese order, from rightmost to leftmost, each column read from top to bottom. Recall Remark 3.4.

6.3.1. Lecouvey–Sheats symplectic jeu de taquin. In this section, which we include for the comfort of the reader, we recall material from [36, 49] which is relevant for our purposes. A punctured *skew* KN tableau is a skew KN tableau one of whose boxes, called its *puncture*, instead of being blank or containing a letter in $\mathcal{C}_n$, is distinguished by being filled in with an asterisk $*$ instead. The *splitting* $\mathrm{spl}(T)$ of a KN tableau T contains two punctures, in consecutive columns and in the same row, corresponding to the "splitting" of the puncture in T.

Let T be a punctured KN tableau with two columns C_1 and C_2 and split form $\mathrm{spl}(T) = lC_1 rC_1 lC_2 rC_2$, and let C_1 have the puncture $*$. Let α be the entry under the puncture of rC_1 and β the entry to the right of the puncture of rC_1:

$$\mathrm{spl}(T) = lC_1 rC_1 lC_2 rC_2 = \begin{array}{|c|c|c|c|}\hline \cdots & \cdots & \cdots & \cdots \\ \hline * & * & \beta & \cdots \\ \hline \cdots & \alpha & \cdots & \cdots \\ \hline \cdots & \cdots \\ \cline{1-2} \end{array},$$

where α or β may not necessarily exist. Following the wording in [47], the elementary steps of the symplectic jeu de taquin, or SJDT for short, are the following:

(A) If $\alpha \leq \beta$ or β does not exist, then the puncture of T will change its position with the cell beneath it. This is a vertical slide.

(B) If the slide is not vertical, then it is horizontal. We then have $\alpha > \beta$ or that α does not exist. Let C_1' and C_2' be the columns obtained after the slide. We have two subcases, depending on the sign of β:

(1) If β is barred, we are moving a barred letter, β, from lC_2 to the punctured box of rC_1, and the puncture will occupy β's place in lC_2. Note that lC_2 has the same barred part as C_2 and that rC_1 has the same barred part as $\Phi(C_1)$. Looking at T, we will have a horizontal slide of the puncture, getting

$$C_2' = C_2 \setminus \{\beta\} \sqcup \{*\} \quad \text{and} \quad C_1' = \Phi^{-1}\big(\Phi(C_1) \setminus \{*\} \sqcup \{\beta\}\big).$$

In a sense, β went from C_2 to $\Phi(C_1)$.

(2) If β is unbarred, the procedure is similar, but this time β will go from $\Phi(C_2)$ to C_1; hence,

$$C_1' = C_1 \setminus \{*\} \sqcup \{\beta\} \quad \text{and} \quad C_2' = \Phi^{-1}\big(\Phi(C_2) \setminus \{\beta\} \sqcup \{*\}\big).$$

However, in this case it may happen that C_1' is no longer admissible. In this situation, if i is the lowest entry such that $i, \overline{i}$ appear in C_1' and $N(i) > i$, we erase both i and $\overline{i}$ from the column, remove a cell from the bottom and from the top of the column and place all the remaining cells in order.

After applying elementary SJDT slides successively, the puncture will eventually reach a cell such that α and β do not exist. In this case we redefine the shape to not include this cell and the *jeu de taquin* ends. The SJDT when applied to semi-standard tableaux in the alphabet $[n]$ reduces to the ordinary *jeu de taquin*.

The SJDT is reversible, meaning that we can move $*$, the empty cell outside of μ, to the inner shape ν of a skew tableau T of shape μ/ν, simultaneously increasing both the inner and outer shapes of T by one cell. The slides work similarly to the previous case: the vertical slide means that an empty cell is going up, and a horizontal slide means that an entry goes from $\Phi(C_1)$ to C_2 or from C_1 to $\Phi(C_2)$, depending on whether the slid entry is barred or not, respectively. For an illustration of SJDT, we refer the reader to the first part of Example 8.7.

6.3.2. Symplectic Knuth equivalence. In this section, we gather the necessary tools from [34, 36]. For $w \in \mathcal{C}_n^*$, let $P(w)$ be the Kashiwara–Nakashima tableau obtained by performing the Baker–Lecouvey insertion algorithm on w. We do not need the algorithm in this paper, but refer the reader to [2, 36] for the original descriptions. A detailed account can also be found in [48]. Given $w_1, w_2 \in \mathcal{C}_n^*$, the relation

$$w_1 \sim w_2 \Leftrightarrow P(w_1) = P(w_2)$$

defines an equivalence relation on $\mathcal{C}_n^*$ known as *symplectic plactic equivalence*. It is the analogous relation defined by Knuth relations in the alphabet $[n]$ (see [18]). The *symplectic plactic monoid* is the quotient $\mathcal{C}_n^*/\sim$. Each *symplectic plactic class* is uniquely identified with a KN tableau.

The plactic monoid $\mathcal{C}_n^*/\sim$ can also be described as the quotient of $\mathcal{C}_n^*$ by the following *symplectic plactic relations* (we use the notation from [36]):

- (R1) $yzx \sim yxz$ for $x \leq y < z$ with $z \neq \overline{x}$ and $xzy \sim zxy$ for $x < y \leq z$ with $z \neq \overline{x}$;
- (R2) $y\overline{x-1}(x-1) \sim yx\overline{x}$ and $x\overline{x}y \sim \overline{x-1}(x-1)y$ for $1 < x \leq n$ and $x \leq y \leq \overline{x}$;
- (R3) (Symplectic contraction/dilation relation) $w \sim w \setminus \{z, \overline{z}\}$, where $w \in \mathcal{C}_n^*$ and $z \in [n]$ are such that w is a non-admissible column, z is the lowest non-barred letter in w such that $N(z) = z + 1$ and any proper factor of w is an admissible column.

For example, the words $23\overline{2}\overline{3}1$ and $\overline{1}113\overline{3}$ are symplectic Knuth related:

$$\overline{1}113\overline{3} \overset{(R1)}{\sim} \overline{1}131\overline{3} \overset{(R1)}{\sim} \overline{1}13\overline{3}1 \overset{(R2)}{\sim} 2\overline{2}3\overline{3}1 \overset{(R1)}{\sim} 23\overline{2}\overline{3}1.$$

Note that to apply the plactic relation (R3) to a non-admissible column word w, we need only check that all proper prefixes of w are admissible, as opposed to all proper factors [47]. For example,

$$234\overline{4}\overline{3} \overset{(R3)}{\sim} 23\overline{3} \quad \text{and} \quad 1234\overline{4}\overline{3} \overset{(R3)}{\sim} 123\overline{3} \overset{(R3)}{\sim} 12.$$

When Knuth relations are applied to factors of a word, the weight is preserved while the length may not be. Knuth relations can be seen as *jeu de taquin* moves on words or diagonally shaped tableaux, and each symplectic *jeu de taquin* slide preserves the Knuth class of the reading word of a tableau [36, Theorem 6.3.8].

6.4. Full symplectic reversal

6.4.1. Symplectic evacuation algorithm. In [47], Santos introduced a symplectic evacuation algorithm on tableaux in $\mathsf{KN}(\lambda, n)$ denoted by evac^{C_n} which he proved coincides with the full Lusztig–Schützenberger involution on a given $U_q(\mathfrak{sp}(2n, \mathbb{C}))$-crystal $\mathsf{B}(\lambda)$ associated to a representation of highest weight λ. Santos' evacuation algorithm mimics the Schützenberger evacuation on $\mathsf{SSYT}(\lambda, n)$. It replaces the action of the long element of $\mathfrak{S}_n$ with that of the long element of B_n and performs symplectic rectification or insertion using Lecouvey–Sheats symplectic jeu de taquin [36, 37, 49], or Baker–Lecouvey insertion [2, 36, 37], respectively. We refer the reader to [47, Section 5] for detailed examples of the algorithm.

6.4.2. Full symplectic reversal on KN skew tableaux. The set $\mathsf{KN}(\lambda/\mu, m)$ is a normal C_m crystal whose connected components are isomorphic to $\mathsf{KN}(\nu, m)$ for some partition ν whose number of boxes $|\nu|$ is less than or equal to $|\lambda| - |\mu|$. Let $n = m + j - 1$, where $1 \leq j - 1 < n$ is the number of parts of μ and $J = [j, n]$.

Let $\mathsf{B}(\lambda, \mu)$ be the subset of $\mathsf{KN}(\lambda, n)$ consisting of the tableaux in $\mathsf{KN}(\lambda, n)$ with entries exclusively in

$$1 < \cdots < j < j+1 < \cdots < j-1+m < \overline{j-1+m} < \cdots < \overline{j}$$

outside of the shape $\mu \subset \lambda$ and whose sub-tableau on the alphabet $\{1, \ldots, j-1\}$ is the fixed Yamanouchi tableau of shape μ. By [36, Lemma 6.1.3], $\mathsf{B}(\lambda, \mu)$ is a normal C_m crystal; in particular, it is stable under the action of $f_{i+j-1}, e_{i+j-1}, i = 1, \ldots, m$, so it is a sub-crystal of $\mathsf{KN}_J(\lambda, n)$. In particular, note that each one of its connected components is contained in a connected component of $\mathsf{KN}_J(\lambda, n)$. Shifting the entries of the KN skew tableaux in $\mathsf{KN}(\lambda/\mu, m)$ by $j - 1$, we may identify $\mathsf{KN}(\lambda/\mu, m)$ with $\mathsf{B}(\lambda, \mu) \subset \mathsf{KN}_J(\lambda, n)$. That is, the crystal operators, $f_i, e_i, i = 1, \ldots, m$, do not change the skew-shape of a KN tableau on the alphabet $\mathscr{C}_m$, and $\mathsf{KN}(\lambda/\mu, m)$ decomposes into connected components that can be identified with the connected components of $\mathsf{B}(\mu, \lambda)$.

In both type A_{n-1} and type C_n, Kashiwara operators e_i and f_i commute with SJDT slides. Let $T \in \mathsf{B} = \mathsf{KN}(\lambda/\mu, n)$. An inner corner in T is a box of μ such that the boxes below and to the right are not in μ; an outer corner in T is a box of λ such that the boxes below and to the right are not in λ. Let c be a fixed inner/outer corner of T. An SJDT *slide* or a *complete* SJDT *slide* to the inner corner c means a slide of the box c from an inner corner to an outer corner, or vice-versa. An SJDT slide to the inner/outer corner c of T gives a new KN skew tableau $\mathsf{SJDT}(T, c)$, possibly with fewer/more boxes. Applying an SJDT slide to the same inner corner c in all vertices of $\mathsf{B}(T)$ defines an isomorphic crystal $\mathsf{B}(\mathsf{SJDT}(T, c))$ [36, Theorem 6.3.8]. The images of the KN tableaux in the same connected component of $\mathsf{KN}(\lambda/\mu, m)$ under this crystal isomorphism have the same skew shape [36, Theorem 6.3.8]. Iterating the SJDT to all inner corners of T rectifies T, producing $\mathsf{rect}(T)$ ([49, Proposition 9.2], [36, Theorems 6.1.9 and 6.3.9]).

At the end of each SJDT slide, the inner corner (outer corner) where the slide started is filled, or the column where the slide started has two fewer (more) boxes ([49, Proposition 9.2], [36, Theorem 6.1.9]). The SJDT step where the tableau loses two boxes in a column has a previous step where this column is non-admissible but Knuth equivalent to the new column which is admissible. The step in reverse SJDT where the tableau gains two boxes in a column is (R3) Knuth equivalent to the previous one which is admissible. Therefore, in each step of SJDT we get crystals which are isomorphic. This allows, in the vein of reversal for A_{n-1} skew semi-standard tableaux, the definition of symplectic reversal, $\mathsf{reversal}^{C_n}$, on type C_n skew tableaux as a coplactic extension of $\mathsf{evacuation}^{C_n}$.

Lemma 6.2. *Let $T \in \mathsf{B} = \mathsf{KN}(\lambda/\mu, n)$. Then $\xi^{C_n}(T)$ is the unique KN tableau in $\mathsf{B}(T)$ that is symplectic Knuth equivalent to $\mathsf{evac}^{C_n}\ \mathsf{rect}(T)$, and*

$$\mathsf{rect}\ \xi^{C_n}(T) = \mathsf{evac}^{C_n}(\mathsf{rect}(T)). \tag{6.3}$$

Proof. The crystal $\mathsf{B}(T) \simeq \mathsf{B}(\nu)$ for some partition ν and $\mathsf{rect}(T) \in \mathsf{B}(\nu)$. The full Schützenberger–Lusztig involution on KN tableaux of straight shape satisfies

$$\xi^{C_n}(\mathsf{rect}(T)) = \mathsf{evac}^{C_n}(\mathsf{rect}(T)),$$

and crystal operators commute with SJDT when passing from $\mathsf{B}(T)$ to $\mathsf{B}(\nu)$. Therefore, (6.3) holds. ■

In Section 8.2.1, we will provide an algorithm for partial symplectic reversal on $\mathsf{KN}_J(\lambda, n)$ with $J = [j, n]$. An algorithm for full C_n reversal on $\mathsf{KN}(\lambda/\mu, n)$ will result as a special case by considering the normal sub-crystal $\mathsf{B}(\mu, \lambda)$ of $\mathsf{KN}_J(\lambda, n)$. See Remark 8.10.

7. Internal cactus group action on a normal crystal

7.1. Partial Schützenberger–Lusztig involutions

Partial Schützenberger involutions were first studied in the case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ by Berenstein and Kirillov [31] but have been defined by Halacheva in general: given $J \subseteq I$ any sub-diagram, the partial Schützenberger–Lusztig involution ξ_J is defined to be the Schützenberger–Lusztig involution ξ_{B_J} on the normal crystal B_J (see [21,22] and also [23]). The crystal B_J decomposes into connected components, and we apply the Schützenberger–Lusztig involution to each connected component. Let $b \in \mathsf{B}$, and let u^{high}, u^{low} be the highest and lowest weight elements of the connected component of B_J containing b. Let $b = f_{j_r} \cdots f_{j_1}(u^{\mathsf{high}})$, with $j_r \cdots j_1 \in J$. Then, for $j \in J$,

$$\xi_J e_j(b) = f_{\theta_J(j)} \xi_J(b), \tag{7.1}$$

$$\xi_J f_j(b) = e_{\theta_J(j)} \xi_J(b), \tag{7.2}$$

$$\mathsf{wt}_J(\xi_J(b)) = w_0^J \mathsf{wt}_J(b), \tag{7.3}$$

and $\xi_J(b) = e_{\theta_J(j_r)} \cdots e_{\theta_J(j_1)}(u^{\mathsf{low}})$.

Remark 7.1. If $J = K \sqcup K' \subseteq I$ is disconnected with K and K' connected sub-diagrams of I, we have the subtype Dynkin diagram $K \times K'$, and the Weyl group is $W^K \times W^{K'}$ with longest elements w_0^K and $w_0^{K'}$, respectively, such that

$$w_0^J = w_0^K w_0^{K'} = w_0^{K'} w_0^K.$$

The weight lattice of $\mathfrak{g}_K \oplus \mathfrak{g}_{K'}$ is $\Lambda_{K \sqcup K'} := \Lambda_K \oplus \Lambda_{K'}$ (see [11]). Then, if θ_K and $\theta_{K'}$ are the graph automorphisms defined by w_0^K and $w_0^{K'}$ in K and K', respectively, $\theta_J = \theta_K \theta_{K'} = \theta_{K'} \theta_K$ is a graph automorphism of the Dynkin graph $K \times K'$ and hence preserves the connected sub-diagrams K and K' of I as defined in Section 2. Thanks to ([21, Lemmas 10.1.3 and 10.1.4], [23, pp. 2368–2369]), the crystal operators act componentwise on the normal crystal $\mathsf{B}_{K \sqcup K'}$ (a normal $\mathfrak{g}_K \oplus \mathfrak{g}_{K'}$-crystal) and satisfy the following properties:

$$\begin{aligned}
f_k f_{k'} &= f_{k'} f_k, & f_k e_{k'} &= e_{k'} f_k, & & \\
e_k e_{k'} &= e_{k'} e_k, & e_k f_{k'} &= f_{k'} e_k & &\text{for } k \in K, k' \in K', && (7.4)\\
e_{k'} \xi_{\mathsf{B}_K} &= \xi_{\mathsf{B}_K} e_{k'}, & f_{k'} \xi_{\mathsf{B}_K} &= \xi_{\mathsf{B}_K} f_{k'} & &\text{for } k \in K, k' \in K', && (7.5)\\
e_k \xi_{\mathsf{B}_{K'}} &= \xi_{\mathsf{B}_{K'}} e_k, & f_k \xi_{\mathsf{B}_{K'}} &= \xi_{\mathsf{B}_{K'}} f_k & &\text{for } k \in K, k' \in K', && (7.6)
\end{aligned}$$

and ξ_K and $\xi_{K'}$ commute: $\xi_K \xi_{K'} = \xi_{K'} \xi_K$. This extends to a disconnected sub-diagram with more than two connected sub-diagrams.

Lemma 7.2. *Let $J = K \sqcup K' \subseteq I$ be a disconnected sub-diagram of I with K and K' connected. Then $\mathsf{B}_{K \sqcup K'}$ is a normal crystal, and the Schützenberger–Lusztig involution on $\mathsf{B}_{K \sqcup K'}$, $\xi_{K \sqcup K'}$, satisfies $\xi_{K \sqcup K'} = \xi_K \xi_{K'} = \xi_{K'} \xi_K$.*

Proof. The result follows from the previous remark: $\xi_K \xi_{K'} = \xi_{K'} \xi_K$ is an involution, and from (7.4), (7.5) and (7.6), it satisfies the conditions (7.1) and (7.2) above. In addition, the weight map

$$\mathsf{wt}_{K \sqcup K'} : \mathsf{B} \xrightarrow{\mathsf{wt}} \Lambda \xrightarrow{\text{can}} \Lambda_{K \sqcup K'} = \Lambda_K \oplus \Lambda_{K'},$$

and therefore

$$\mathsf{wt}_{K\sqcup K'}(\xi_K\xi_{K'}(b)) = \big(\mathsf{wt}_K(\xi_K(b)), \mathsf{wt}_{K'}(\xi_{K'}(b))\big) = w_0^K w_0^{K'}\big(\mathsf{wt}_K(b), \mathsf{wt}_{K'}(b)\big).$$

Since there is only one set involution on $\mathsf{B}_{K\sqcup K'}$ satisfying (7.1), (7.2), and (7.3), we have that $\xi_{K\sqcup K'} = \xi_K\xi_{K'} = \xi_{K'}\xi_K$. ∎

The partial Schützenberger–Lusztig involutions ξ_{J}, for any $J \subseteq I$ a *connected Dynkin sub-diagram* of I, satisfy the $J_{\mathfrak{g}}$ cactus relations.

Theorem 7.3 ([21]). *The map $s_J \mapsto \xi_{\mathsf{J}}$, for all $J \subseteq I$ connected Dynkin sub-diagrams of I, defines an action of the cactus group $J_{\mathfrak{g}}$ on the set B; that is, the following is a group homomorphism:*

$$\Phi_{\mathfrak{g}} \colon J_{\mathfrak{g}} \to \mathfrak{S}_{\mathsf{B}},$$
$$s_J \mapsto \xi_{\mathsf{J}}.$$

Moreover, $\mathsf{wt}_J(\xi_{\mathsf{J}}(b)) = w_0^J \mathsf{wt}_J(b)$*,* $b \in \mathsf{B}$.

The s_J act via ξ_{J} on each connected component of B_J as a graph automorphism of the underlying unlabeled, non-oriented and non-weighted connected graph by exchanging highest and lowest weight elements.

Remark 7.4. The Weyl group W of $\mathfrak{g}$ acts on the set B by $r_i.b = \xi_i(b)$, $i \in I$, where $\xi_i = \xi_{\{i\}}$; see Section 8.1 and Theorem 8.1. The action of $J_{\mathfrak{g}}$ factorizes then through the quotient by the braid relations of W.

The following corollary motivates what comes in the next section.

Corollary 7.5. *The following statements hold:*

(a) *For the $\mathfrak{sl}(n,\mathbb{C})$-crystal $\mathsf{SSYT}(\lambda,n)$, the map*

$$s_{[1,j]} \mapsto \xi_{[1,j]} = \mathsf{evac}_{j+1}, \quad 1 \le j \le n-1,$$

where evac_{j+1} denotes the evacuation on the sub-tableaux of straight shape obtained by restricting the entries to $\{1,\dots,j+1\}$ and fixing the remaining ones, defines an action of the cactus group J_n on the set $\mathsf{SSYT}(\lambda,n)$.

(b) *For the $\mathfrak{sp}(2n,\mathbb{C})$-crystal $\mathsf{KN}(\lambda,n)$, the map*

$$s_{[1,j]} \mapsto \xi^{C_n}_{[1,j]}, \quad 1 \le j \le n-1, \tag{7.7}$$
$$s_{[j,n]} \mapsto \xi^{C_n}_{[j,n]}, \quad 1 \le j \le n, \tag{7.8}$$

defines an action of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ on the set $\mathsf{KN}(\lambda,n)$, where $\xi^{C_n}_{[1,n]} = \xi^{C_n} = \mathsf{evac}^{C_n}$, $\xi^{C_n}_{[1,j]}$, $1 \le j \le n-1$, is given by the Baker embedding, Theorem 8.13, *and* $\xi^{C_n}_{[j,n]}$, $1 \le j \le n-1$ *is given either by the partial symplectic reversal in* (8.10)*, Section* 8.2.1*, or by the Baker embedding, Theorem* 8.14.

7.2. The virtual symplectic cactus group action on an $\mathfrak{sl}(2n,\mathbb{C})$-crystal and the virtualization of an $\mathfrak{sp}(2n,\mathbb{C})$-crystal

On A_{n-1} semi-standard tableaux, there is a straightforward algorithm to compute the action of a partial Schützenberger–Lusztig involution ξ_J with J a connected A_{n-1} Dynkin sub-diagram. Let $I=[n-1]$ and $J=[p,q]\subset I$, $1\le p\le q<n$, be a connected sub-diagram. The J*-partial reversal*, $\mathsf{reversal}_J$, is the reversal on $\mathsf{SSYT}_J(\lambda,n)$ which means the reversal or Schützeberger involution ξ applied to each connected component of $\mathsf{SSYT}_J(\lambda,n)$. Let $T\in\mathsf{SSYT}(\lambda,n)$, then, from (6.1) and (6.2):

$$\begin{aligned}\xi_J(T) &= \mathsf{reversal}_J(T)\\ &:= \big(T_{[1,p-1]}, \mathsf{reversal}(T_{[p,q+1]}), T_{[q+2,n]}\big)\\ &= \big(T_{[1,p-1]}, \mathsf{arect}\big(\mathsf{evacuation}(\mathsf{rect}(T_{[p,q+1]}))\big), T_{[q+2,n]}\big),\end{aligned} \tag{7.9}$$

where $T=(T_{[1,p-1]},T_{[p,q+1]},T_{[q+2,n]})$ is such that $T_{[1,p-1]}$ is the tableau obtained by restricting T to the alphabet $[1,p-1]$, $T_{[p,q+1]}$ is the skew tableau obtained by restricting to the alphabet $[p,q+1]$, and $T_{[q+2,n]}$ is obtained by restricting to the alphabet $[q+2,n]$. Indeed, if $J=[1,q]$, $\mathsf{reversal}_{[1,q]}(T)=\mathsf{evac}_{q+1}(T)$. The case where J is a disconnected sub-diagram of I will be a consequence of Lemma 7.2.

To define an internal action of the *virtual symplectic cactus group* $\widetilde{J}_{2n}$ on a crystal $\mathsf{SSYT}(\mu,n,\overline{n})$ with μ a partition with at most $2n$ parts, thanks to Lemma 7.2, we now explicitly characterize the partial Schützenberger–Lusztig involution on a disconnected sub-diagram $J\sqcup J'$ of the A_{2n-1} Dynkin diagram such that $J\subseteq[n-1]$ and $J'\subseteq[\overline{n},\overline{2}]$ are connected sub-diagrams. In the case of the A_{2n-1} Dynkin diagram, we label its nodes either in $[2n-1]$ or in $\{1,\dots,n,\overline{n},\dots,\overline{2}\}$.

Theorem 7.6. *Let* $J\sqcup J'$ *be a disconnected sub-diagram of the* A_{2n-1} *Dynkin diagram* $I=\{1,\dots,n,\overline{n},\dots,\overline{2}\}$ *such that* $J\subseteq[n-1]$ *and* $J'\subseteq[\overline{n},\overline{2}]$ *are connected sub-diagrams. Then* $\xi^{A_{2n-1}}_{J\sqcup J'}$*, the Schützenberger–Lusztig involution on* $\mathsf{SSYT}_{J\sqcup J'}(\mu,n,\overline{n})$*, with* μ *a partition with at most* $2n$ *parts, satisfies*

$$\begin{aligned}\xi^{A_{2n-1}}_{J\sqcup J'} &= \xi^{A_{2n-1}}_{J}\xi^{A_{2n-1}}_{J'} = \xi^{A_{2n-1}}_{J'}\xi^{A_{2n-1}}_{J}\\ &= \mathsf{reversal}^{A_{2n-1}}_{J}\mathsf{reversal}^{A_{2n-1}}_{J'} = \mathsf{reversal}^{A_{2n-1}}_{J'}\mathsf{reversal}^{A_{2n-1}}_{J},\end{aligned}$$

where $\xi^{A_{2n-1}}_{J}=\mathsf{reversal}^{A_{2n-1}}_{J}$ *and* $\xi^{A_{2n-1}}_{J'}=\mathsf{reversal}^{A_{2n-1}}_{J'}$ *are the Schützenberger–Lusztig involutions on* $\mathsf{SSYT}_J(\mu,n,\overline{n})$ *and* $\mathsf{SSYT}_{J'}(\mu,n,\overline{n})$*, respectively.*

Remark 7.7. This statement is indeed also valid for the Schützenberger–Lusztig involution on $\mathsf{SSYT}_{J\sqcup J'}(\mu, n)$, where $J \sqcup J'$ is a disconnected sub-diagram of the A_{n-1} Dynkin diagram with n odd.

The cactus group J_{2n} acts on an A_{2n-1}-crystal of semi-standard tableaux via partial reversals. We now conclude that the virtual symplectic cactus $\widetilde{J}_{2n}$ also does. In Section 8.5, we establish that this action preserves the subset $E(\mathsf{KN}(\lambda, n))$.

Theorem 7.8. *For the $\mathfrak{sl}(2n,\mathbb{C})$-crystal of tableaux $\mathsf{SSYT}(\mu, 2n)$, with μ a partition with at most $2n$ parts, the map*

$$\begin{aligned}\widetilde{s}_{[1,q]\sqcup[2n-q,2n-1]} \mapsto \xi^{A_{2n-1}}_{[1,q]\sqcup[2n-q,2n-1]} &= \xi^{A_{2n-1}}_{[1,q]}\,\xi^{A_{2n-1}}_{[2n-q,2n-1]} \\ &= \mathsf{evac}_{q+1}\,\mathsf{evac}_{2n}\,\mathsf{evac}_{q+1}\,\mathsf{evac}_{2n}, \quad 1 \le q \le n, \qquad (7.10)\\ \widetilde{s}_{[q,2n-q]} \mapsto \xi^{A_{2n-1}}_{[q,2n-q]} = \mathsf{reversal}^{A_{2n-1}}_{[q,2n-q]}, &\qquad 1 \le q \le n, \qquad (7.11)\end{aligned}$$

defines an action of the virtual symplectic cactus group $\widetilde{J}_{2n}$ on the set $\mathsf{SSYT}(\mu, 2n)$. That is, the following is a group homomorphism

$$\begin{aligned}\widetilde{\Phi}_{\mathfrak{sl}(2n,\mathbb{C})} \colon \widetilde{J}_{2n} &\to \mathfrak{S}_{\mathsf{B}},\\ \widetilde{s}_J &\mapsto \xi^{A_{2n-1}}_J,\end{aligned}$$

where $\mathsf{B} = \mathsf{SSYT}(\mu, 2n)$ and J as in (7.10) or (7.11). Moreover, the action of $\widetilde{J}_{2n}$ on $\mathsf{SSYT}(\lambda^A, n, \overline{n})$ preserves the subset $E(\mathsf{KN}(\lambda, n))$.

Proof. Since J_{2n} acts on $\mathsf{SSYT}(\mu, 2n)$, the partial Schützenberger involutions ξ_{J}, with J a connected sub-diagram of the A_{2n-1} Dynkin diagram $I = [2n-1]$, satisfy the J_{2n} cactus relations, namely, the ones in Lemma 5.9 which are the $\widetilde{J}_{2n}$ relations. We consider $\widetilde{J}_{2n}$ with generators (5.8), (5.9). In Sections 8.4.1 and 8.5, (8.13), we conclude that $\widetilde{J}_{2n}$ acts on the set $\mathsf{SSYT}(\lambda^A, n, \overline{n})$ permuting its elements in a way that the subset $E(\mathsf{KN}(\lambda, n))$ is preserved. ∎

Therefore, the partial Schützenberger involutions ξ_{J}, with J any connected sub-diagram of the A_{2n-1} Dynkin diagram of the form

$$J = [q, 2n-q], \quad [q, n] \subseteq [n],$$

or

$$J = [1, q] \sqcup [2n-q, 2n-1], \quad [1, q] \subseteq [n-1],$$

satisfy the virtual symplectic cactus $\widetilde{J}_{2n}$ relations.

8. Partial symplectic Schützenberger–Lusztig involutions and algorithms

For a connected sub-diagram J of the Dynkin diagram $I = [n-1]$ of type A_{n-1}, the partial Schützenberger involution ξ_J coincides with J*-partial reversal*, that is, $\mathsf{reversal}_J$ (see (7.9)). The case wherein J is a disconnected sub-diagram of I has been studied in Theorem 7.6 and Remark 7.7.

So far, there is no known form of tableau-switching for KN tableaux. The algorithm to compute J*-partial symplectic reversal*, $\mathsf{reversal}_J^{C_n}$, with $J = [p, n]$ a sub-diagram of the Dynkin diagram I of type C_n, presented in Section 8.2 and summarized in (8.10), is inspired by this problem and mimics the type A partial reversal algorithm on type A_{n-1} tableaux summarized in (7.9). The case $J = [p, q] \subseteq I$, $p < q < n$, is solved by virtualization in Section 8.4.1. In fact, all partial symplectic reversals can be virtualized as shown in Section 8.4.1.

8.1. Dynkin sub-diagram with a sole node and the Weyl group action

Let B be a normal crystal. If J has a sole node i of I, $\xi_i := \xi_{\{i\}}$, the Schützenberger–Lusztig involution on the i-strings (the connected components of $\mathsf{B}_{\{i\}}$), agrees with the Kashiwara $\mathfrak{g}$-crystal reflection operator S_i ([27, Section 7]) originally studied by Lascoux–Schützenberger [35] in the $\mathfrak{sl}(n, \mathbb{C})$ case, and rediscovered by Kashiwara for any Cartan type. Let $b \in \mathsf{B}$ and $k = \varphi_i(b) - \varepsilon_i(b)$. The *crystal reflection operator* S_i is defined by

$$S_i(b) = \begin{cases} f_i^k(b) & \text{if } k \geq 0, \\ e_i^{-k}(b) & \text{if } k \leq 0. \end{cases} \tag{8.1}$$

The following assertion combines the Schützenberger–Lusztig involution ξ_i on the i-strings and the Kashiwara crystal reflection operator S_i ([27, Section 7]).

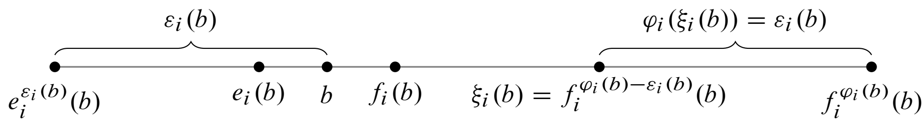
Theorem 8.1. *The operators* ξ_i, $i \in I$, *define an action of the Weyl group* W *on the underlying set of the normal crystal* B, $r_i.b = S_i(b) = \xi_i(b)$, $b \in \mathsf{B}$, *such that*

(1) $S_i^2 = 1$ *and* $e_i S_i = S_i f_i$, $S_i(0) := 0$;

(2) $r_i.\mathsf{wt}(b) = \mathsf{wt}(S_i(b))$;

(3) $u_\lambda^{\mathsf{low}} = w_0.u_\lambda^{\mathsf{high}}$ *if* $\mathsf{B} = \mathsf{B}(\lambda)$.

The action of ξ_i on the i-string of $b \in \mathsf{B}$: $\varphi_i(\xi_i(b)) = \varepsilon_i(b)$ or $\varepsilon_i(\xi_i(b)) = \varphi_i(b)$. An illustration with $k > 0$:



The propositions below follow from the action of the Weyl group W on i-strings, where useful information is gathered in the case of $\mathfrak{sp}(2n,\mathbb{C})$.

Notice that the action of $S_i = \xi_i$ (see (8.1)) on a KN tableau of type C_n is given by the *signature rule* on its reading word [28, 36] in the alphabet $\mathcal{C}_n$. Thus, the first point in the next proposition is just the translation of (8.1) on words in the alphabet $\mathcal{C}_n$. Let $i \in I$ and let w be a word in the alphabet $\mathcal{C}_n$ restricted to the alphabet

$$\{i < i+1 < \overline{i+1} < \overline{i}\}.$$

Substitute each letter in $\{i, \overline{i+1}\}$ by $+$ and by $-$ if in $\{i+1, \overline{i}\}$. Then bracket all factors $+-$ and erase all $(+-)$. The remaining $+, -$ form a subword $-^r +^s$ with $r = \varepsilon_i(w)$ and $s = \varphi_i(w)$. The Kashiwara operator e_i acts on the letter w_j of w associated to the rightmost unbracketed $-$ (i.e., not erased), whereas f_i acts on the letter w_j of w associated to the leftmost unbracketed $+$, and the other letters of w are unchanged:

$$e_i(w_j) = \begin{cases} i & \text{if } w_j = i+i \text{ and } i \neq n, \\ \overline{i+1} & \text{if } w_j = \overline{i} \text{ and } i \neq n, \\ n & \text{if } w_j = \overline{n} \text{ and } i = n, \end{cases} \qquad \text{and } f_i \text{ is the inverse map.}$$

If $s = 0$ then $f_i(w) = 0$, and if $r = 0$ then $e_i(w) = 0$.

Proposition 8.2. *Let $W = B_n$.*

(1) *For $U_q(\mathfrak{sp}(2n,\mathbb{C}))$ and the alphabet $\mathcal{C}_n$: given $i \in [n-1]$, let u^- be a word in the alphabet $\{\overline{i}, i+1\}$ with length $\ell(u^-) = r$, and let v^+ be a word in the alphabet $\{i, \overline{i+1}\}$ with length $\ell(v^+) = s$. Then, for all $r_i \in B_n$, $1 \leq i \leq n-1$,*

$$r_i.(u^- v^+) = \xi_i(u^- v^+) = \begin{cases} u_1^- e_i^{r-s}(u_2^-) v^+, & r > s, \\ u^- v^+, & r = s, \\ u^- f_i^{s-r}(v_1^+) v_2^+, & r < s, \end{cases} \tag{8.2}$$

such that when $r > s$, $u^- = u_1^- u_2^-$, with $\ell(u_2^-) = r - s$, and when $r < s$, $v = v_1^+ v_2^+$ with $\ell(v_1^+) = s - r$. When $i = n$,

$$r_n.(\overline{n}^r n^s) = \xi_n(\overline{n}^r n^s) = \overline{n}^s n^r. \tag{8.3}$$

(2) *For $U_q(\mathfrak{sp}(2n,\mathbb{C}))$: the crystal reflection operators ξ_i satisfy the relations of the Weyl group B_n:*

(i) $\xi_i^2 = 1$, $1 \leq i \leq n$, *and* $\xi_i \xi_j = \xi_j \xi_i$, $|i - j| > 1$, $1 \leq i, j \leq n$;

(ii) $(\xi_i \xi_{i+1})^3 = 1$, $1 \leq i \leq n-2$, *and* $(\xi_{n-1} \xi_n)^4 = 1$.

Example 8.3. Let $n = 4$ and

$$T = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & \bar{2} & \bar{1} \\ \hline 2 & \bar{4} & \bar{3} & \bar{3} & \bar{1} & \\ \cline{1-5} 4 & \bar{2} & \bar{1} & & & \\ \cline{1-3} \bar{4} & & & & & \\ \cline{1-1} \end{array}\ .$$

From (8.2) and (8.3), the action of ξ_i on the KN tableau T is given by the signature rule on its reading word as follows:

(1) For $i = 1$, the letters $\{1 < 2 < \bar{2} < \bar{1}\}$ in T are highlighted in blue. The word of T restricted to the alphabet $\{1 < 2 < \bar{2} < \bar{1}\}$ is $w = \bar{1}\,\bar{2}\bar{1}2\bar{1}2\bar{2}12$. The red color indicates the iterated action of e_1 on w:

$$\begin{aligned} w = \bar{1}\,\bar{2}\bar{1}2\bar{1}2\bar{2}12 &\to -(+-)---+(+-) \to -----+ = (-)^4(+)^1 \\ &\xrightarrow{(e_1)^3} -++++ = (-)^1(+)^4 \to -(+-)++++(+-) \\ &\to \bar{1}\bar{2}\bar{1}\,1\bar{2}1\,\bar{2}12 = \xi_1(w). \end{aligned}$$

Hence,

$$T = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & \bar{2} & \bar{1} \\ \hline 2 & \bar{4} & \bar{3} & \bar{3} & \bar{1} & \\ \cline{1-5} 4 & \bar{2} & \bar{1} & & & \\ \cline{1-3} \bar{4} & & & & & \\ \cline{1-1} \end{array} \to \begin{array}{|c|c|c|c|c|c|}\hline + & - & - & 3 & + & - \\ \hline - & \bar{4} & \bar{3} & \bar{3} & - & \\ \cline{1-5} 4 & + & - & & & \\ \cline{1-3} \bar{4} & & & & & \\ \cline{1-1} \end{array} \to \begin{array}{|c|c|c|c|c|c|}\hline + & + & + & 3 & + & - \\ \hline - & \bar{4} & \bar{3} & \bar{3} & - & \\ \cline{1-5} 4 & + & + & & & \\ \cline{1-3} \bar{4} & & & & & \\ \cline{1-1} \end{array}$$

$$\to \begin{array}{|c|c|c|c|c|c|}\hline + & 1 & 1 & 3 & + & - \\ \hline - & \bar{4} & \bar{3} & \bar{3} & - & \\ \cline{1-5} 4 & + & \bar{2} & & & \\ \cline{1-3} \bar{4} & & & & & \\ \cline{1-1} \end{array} \to \xi_1(T) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 3 & \bar{2} & \bar{1} \\ \hline 2 & \bar{4} & \bar{3} & \bar{3} & \bar{1} & \\ \cline{1-5} 4 & \bar{2} & \bar{2} & & & \\ \cline{1-3} \bar{4} & & & & & \\ \cline{1-1} \end{array}\ ,$$

where $\mathrm{wt}(\xi_1(T)) = r_1.\mathrm{wt}(T) = r_1(-2, 1, -1, -1) = (1, -2, -1, -1)$.

(2) For $i = 4$, the letters $\{4, \bar{4}\}$ in T are highlighted in blue. Now,

$$w = \bar{4}4\bar{4} \to -(+-) \xrightarrow{e_4} +(+-) \to 44\bar{4} = \xi_4(w)$$

and

$$\xi_4(T) = \xi_4 \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & \bar{2} & \bar{1} \\ \hline 2 & \bar{4} & \bar{3} & \bar{3} & \bar{1} & \\ \cline{1-5} 4 & \bar{2} & \bar{1} & & & \\ \cline{1-3} \bar{4} & & & & & \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & \bar{2} & \bar{1} \\ \hline 2 & 4 & \bar{3} & \bar{3} & \bar{1} & \\ \cline{1-5} 4 & \bar{2} & \bar{1} & & & \\ \cline{1-3} \bar{4} & & & & & \\ \cline{1-1} \end{array}\ ,$$

where $\mathrm{wt}(\xi_4(T)) = r_4.\mathrm{wt}(T) = r_4(-2, 1, -1, -1) = (-2, 1, -1, 1)$.

Proposition 8.4. *Let* $\mathsf{B}(\lambda)$ *be a* C_n *crystal and* $\mathsf{B}_J = \mathsf{B}_J(\lambda)$ *for* $J \subseteq I$*. Let* $b \in \mathsf{B}(\lambda)$*. The connected component of* B_J *containing* b *has highest weight element* b_J^{high} *and lowest weight element* b_J^{low}*. Then*

(1) $b_J^{\mathsf{low}} = r_a \cdots r_d . b_J^{\mathsf{high}} = \xi_a \cdots \xi_d (b_J^{\mathsf{high}})$*, where* $r_a \cdots r_d$ *is a reduced word for* $w_0^J \in W^J$ *with* $a, \dots, d \in J$*, and* $b = f_{j_r} \cdots f_{j_1}(b_J^{\mathsf{high}})$ *for some* $j_r, \dots, j_1 \in J$*;*

(2) *if* $J = [p,n]$*,* $\mathsf{B}_{[p,n]}$ *is a* C_{n-p+1} *crystal, then*

$$\xi_{\mathsf{J}}(b) = e_{j_r} \cdots e_{j_1}(r_a \cdots r_d . b_J^{\mathsf{high}}),$$
$$\mathsf{wt}_J(\xi_{\mathsf{J}}(b)) = -\mathsf{wt}_J(b);$$

(3) *if* $J = [p,q]$*,* $1 \le p \le q < n$*,* $\mathsf{B}_{[p,q]}$ *is an* A_{q-p+1}*, crystal, and*

$$\xi_{\mathsf{J}}(b) = e_{q-p-j_r+1} \cdots e_{q-p-j_1+1}(r_a \cdots r_d . b_J^{\mathsf{high}}),$$
$$\mathsf{wt}_J(\xi_{\mathsf{J}}(b)) = \mathrm{rev}(\mathsf{wt}_J(b)).$$

8.2. Dynkin sub-diagram $J = [j,n]$: J-symplectic reversal

Recall that the partial Schützenberger–Lusztig involution $\xi^{C_n}_{[j,n]}$ is the Schützenberger–Lusztig involution on each connected component of $\mathsf{KN}_{[j,n]}(\lambda, n)$. On the set $\mathsf{KN}(\lambda, n)$, $\xi^{C_n} = \xi^{C_n}_{[1,n]}$ coincides with Santos' symplectic evacuation evac^{C_n} (see Section 6.4.1 or [47, Section 5]).

8.2.1. The Knuth (R3) relation on a skew tableau. Given that $1 < j \le n$, the Levi branched crystal $\mathsf{KN}_{[j,n]}(\lambda, n)$ decomposes into connected components. We let $T \in \mathsf{KN}(\lambda, n)$, which belongs to some connected component of $\mathsf{KN}_{[j,n]}(\lambda, n)$, and let $T_{[\pm j,n]}$ denote the restriction of T to the alphabet $[\pm j, n]$. We claim that the connected component containing $T_{[\pm j,n]}$ is symplectic Knuth equivalent to a crystal connected component of admissible skew tableaux on the alphabet

$$[\pm j, n] = [\pm 1, n] \setminus \{1 < \cdots < j-1 < \overline{j-1} < \cdots < \overline{2} < \overline{1}\}$$

(of the same skew shape). The following remark gathers the missing observations we need to verify our claim.

Remark 8.5. (1) ([36, Proposition 2.3.3]) Let $C_1, \dots, C_k$ be admissible columns on the alphabet $\mathcal{C}_n$. Then $T = C_1 C_2 \cdots C_k$ is a KN tableau on the alphabet $\mathcal{C}_n$ if and only if $r(C_i) \le l(C_{i+1})$, that is, if $r(C_i) l(C_{i+1})$ is a type A_{2n-1} semi-standard tableau for $i = 1, \dots, k-1$.

(2) For $T \in \mathsf{KN}(\lambda, n)$, the restriction of T to the alphabet $[\pm j, n]$, $T_{[\pm j,n]}$, is a KN skew tableau on the alphabet $\mathcal{C}_n$, but $T_{[\pm j,n]}$ may have non-admissible columns with respect to the alphabet $[\pm j, n]$. In particular, we might produce a non-admissible skew tableau on the alphabet $\mathcal{C}_{n-j+1}$ (recall Definition 3.2 and Remark 3.4).

(3) Let C_1 and C_2 be two columns with entries on the alphabet $[\pm j, n]$ such that C_1C_2 is a KN skew tableau on the alphabet $\mathcal{C}_n$. Assume that C_1 and C_2 have exactly $m \geq 0$ and $t \geq 0$ pairs of symmetric entries $(x, \overline{x})$, respectively, with $N(x) > x$ with respect to the alphabet $[\pm j, n]$. Then C_1 has at least m boxes strictly below the row containing the last box of C_2, and C_2 has at least t boxes strictly above the row containing the top box of column C_1.

(4) Let $(\text{R3})^m$ denote the iteration of the Knuth relation (R3), $m \geq 0$ times, and let C_1 and C_2 be two columns with conditions as described in the previous point of this remark. Let $C_1 \overset{(\text{R3})^m}{\sim} X$, where X is an admissible column on $[\pm j, n]$, and $C_2 \overset{(\text{R3})^t}{\sim} Y$, where Y is an admissible column on $[\pm j, n]$. Then

$$C_1C_2 \overset{(\text{R3})^m}{\sim} XC_2 \overset{(\text{R3})^t}{\sim} XY$$

is a KN skew tableau on $[\pm j, n]$.

8.2.2. Reduced symplectic jeu de taquin. Given $T \in \mathsf{KN}(\lambda/\mu, n)$ and $j \in [n]$ such that T has all its entries in $[\pm j, n]$, the following is an algorithm to compute the *reduced symplectic jeu de taquin* on T on the interval $[\pm j, n]$, denoted SJDT_j. The skew tableau T might not be admissible on the alphabet $[\pm j, n]$. This means that we apply the SJDT after shifting all entries in T by $-(j-1)$ and iterating on T the contraction relation (R3) the needed number of times to get an admissible skew tableau on the alphabet $\mathcal{C}_{n-j+1}$. When $j = 1$, we recover the ordinary SJDT.

Definition 8.6. Reduced SJDT (SJDT_j).

- Let T_j be the tableau obtained by replacing each non-barred entry c and barred entry $\overline{c}$ in T by $c - j + 1$ and $\overline{c - j + 1}$, respectively.
- If T_j is not a KN tableau in $\mathsf{KN}(\lambda/\mu, n - j + 1)$, we have some columns containing pairs of the form $b, \overline{b}$ such that $b \in [n - j + 1]$ is lowest in the column and $N(b) > b$. Iteratively, we apply the Knuth contraction (R3) to T_j until we make all columns admissible. Define T_j to be the resulting tableau with all admissible columns.
- Compute SJDT on T_j as usual.
- Replace each non-barred entry m and $\overline{m}$ in $\mathsf{SJDT}(T_j)$ by $m+j-1$ and $\overline{m-j+1}$, respectively.

The *reduced rectification* to the alphabet $[\pm j, n]$, denoted $\mathsf{rectification}_j$ (rect_j for short), of T is the iteration of SJDT_j to all inner corners in T. Indeed, $\mathsf{rect}_j(T)$ is the shift by $j - 1$ of all entries of $\mathsf{rect}(T_j)$. When $j = 1$ we recover the ordinary rectification.

Example 8.7. For $T \in \mathsf{KN}((2,2,1)/(1),3)$, we compute a complete SJDT slide on the interval $[\pm 1, 3]$:

$$T = \begin{array}{|c|c|}\hline * & 2 \\ \hline 3 & \overline{2} \\ \hline \overline{3} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|}\hline 1 & \overline{1} \\ \hline 3 & * \\ \hline \overline{3} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

However, if we compute the complete SJDT_2 slide, that is, the complete SJDT slide reduced to the interval $[\pm 2, 3]$, we get:

$$\begin{array}{|c|c|}\hline * & 2 \\ \hline 3 & \overline{2} \\ \hline \overline{3} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \to T_2 = \begin{array}{|c|c|}\hline * & 1 \\ \hline 2 & \overline{1} \\ \hline \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \xrightarrow{(R3)} \begin{array}{|c|}\hline * \\ \hline 2 \\ \hline \overline{2} \\ \hline \end{array} \xrightarrow{\mathsf{SJDT}} \begin{array}{|c|}\hline 2 \\ \hline \overline{2} \\ \hline * \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|}\hline * & 2 \\ \hline 3 & \overline{2} \\ \hline \overline{3} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \xrightarrow{\mathsf{SJDT}_2} \begin{array}{|c|}\hline 3 \\ \hline \overline{3} \\ \hline * \\ \hline \end{array}.$$

8.2.3. Partial symplectic reversal: Colorful symplectic tableau switching. Next, we generalize the Benkart–Sottile–Stroomer reversal [6] on skew semi-standard tableaux to symplectic KN skew tableaux. The procedure consists of the *colorful symplectic tableau switching* governed by SJDT and, consequently, also by the Knuth (R3) relation due to SJDT case (B(2)); symplectic Santos evacuation; and reverse SJDT (RSJDT). More generally, we define $J = [j, n]$-*partial symplectic reversal* for KN skew tableaux, where J is a Dynkin sub-diagram of the ambient Dynkin diagram of type C_n containing the node n, $1 \le j \le n$. Supplementary ordered *colored letters* will be needed to record the use of the Knuth contractor (R3) relation when non-admissible columns occur: purple (8.4) for the possible non-admissible columns in $T_{[\pm j,n]}$, Figure 4, and red for the SJDT (B(2)) case, (8.5). For illustrations, we refer to Section 8.3.

Let $T \in \mathsf{KN}(\lambda, n)$ and let $j \ge 1$. Let B be the crystal connected component of $\mathsf{KN}_{[j,n]}(\lambda, n)$ containing T. We note that B is a highest weight crystal and all vertices of B are KN tableaux on the alphabet $\mathcal{C}_n$, with the letters in $[\pm 1, j-1]$ frozen, as the crystal operators in B are indexed by $[j, n]$ and do not act on the entries filled in $[\pm 1, j-1]$.

Let H be the highest weight element of B, and let $\mathrm{wt}(H_{[\pm j,n]}) \in \mathbb{Z}^{n-j+1}$ be its highest weight, where $H_{[\pm j,n]}$ is the restriction of H to the alphabet $[\pm j, n]$. The restriction of H to the alphabet $[\pm j, n]$ is a KN skew tableau on the alphabet $\mathcal{C}_n$. The entries of H in $[1, j-1]$ define a semi-standard tableau $T^{+}_{[1,j-1]}$ of shape, say μ, and the entries in $[\overline{j-1}, \overline{1}]$ define a skew semi-standard tableau $T^{-}_{[\overline{j-1},\overline{1}]}$ of shape λ/ν, where $\mu \subseteq \nu \subseteq \lambda$. Hence the cells of H filled in

$$[\pm j, n] = [\pm 1, n] \setminus \{1 < \cdots < j-1 < \overline{j-1} < \cdots < \overline{2} < \overline{1}\}$$

define the skew shape ν/μ, and because the crystal operators in B are indexed by $[j, n]$, they do not change the skew shape ν/μ either. Therefore, since all the vertices of B are connected to H through those crystal operators, the vertices of B restricted to the alphabet $[\pm j, n]$ have the same skew shape ν/μ and the same semi-standard tableaux $T^+_{[1,j-1]}$ and $T^-_{[\overline{j-1},\overline{1}]}$ (see [36, Lemma 6.1.3]).

Step I. The sequence of isomorphic crystals from $T_{[\pm j,n]}$ to its reduced rectification.

(I.1) *The C_{n-j+1} connected crystal* B^0 *containing* $T_{[\pm j,n]}$. Erase in the vertices of B the entries in $[\pm 1, j-1]$; i.e., erase the semi-standard tableaux $T^+_{[1,j-1]}$ and $T^-_{[\overline{j-1},\overline{1}]}$. We then obtain the connected C_{n-j+1} crystal B^0 of semi-standard skew tableaux of shape ν/μ with entries in the alphabet $[\pm j, n]$, possibly with some non-admissible columns, containing $T_{[\pm j,n]}$. (For short, we call to μ the inner shape of each vertex of B or B^0.) These KN skew tableaux over $\mathcal{C}_n$ might have non-admissible columns over $[\pm j, n]$. More precisely, B^0 is the connected crystal of the reading words of the aforesaid semi-standard skew tableaux on the alphabet $[\pm j, n]$, with highest weight element the word of $H_{[\pm j,n]}$. Hence, B^0 and B are isomorphic crystals, with a crystal isomorphism given by the reading word map on the alphabet $[\pm j, n]$.

(I.1.1) *The green inner standard tableau U_0 for any vertex of* B^0. Fix a standard tableau U_0 of shape μ filled in a completely ordered alphabet of *green letters*

$$\{g_1 < \cdots < g_{|\mu|}\},$$

where $|\mu|$ is the number of boxes of μ. Assign the inner standard tableau U_0 to the inner shape μ of each vertex of B^0. Recall that $T_{[\pm j,n]}$ is the image of T in B^0; see the tableau pair $(U_0, T_{[\pm j,n]})$ schematically depicted in Figure 3.

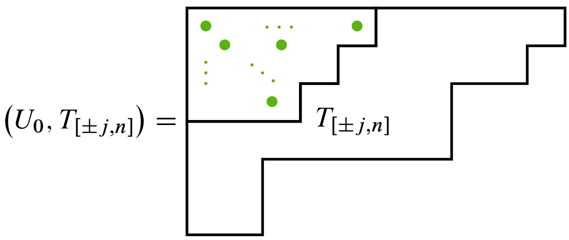


Figure 3. $T_{[\pm j,n]}$ in the crystal B^0 and the inner tableau U_0 in green.

(I.2) *The C_{n-j+1}-crystal* B^x *of KN skew tableaux* (R3) *isomorphic to* B^0. Let $H^0 := H_{[\pm j,n]}$ be the highest weight element of the C_{n-j+1} crystal B^0. The skew tableau H^0 of shape ν/μ may have non-admissible columns on the alphabet $[\pm j, n]$. Let $q < r < \cdots < s < t$ be the non-admissible columns of H^0. Then exactly the same columns in all vertices of B^0 are non-admissible. The Knuth contraction (R3) relation, in Section 6.3.2, defines a crystal isomorphism; it commutes with the crystal operators and preserves the weight. Moreover, each time (R3) is applied to a column of some vertex of B^0, it is also applied to the same column in every vertex of B^0 (see [36, Proposition 3.2.4 and Corollary 3.2.5]).

In each vertex of B^0, apply the contraction (R3) relation to column i, for $i = q, r, \dots, s, t$, until column i becomes admissible. For $i = q, r, \dots, s, t$, each time we apply (R3) to column i, a pair of entries $(k, \overline{k})$ is erased (whenever $k \in [n]$ is minimal for $N(k) > k$, k and $\overline{k}$ appear in the column and all prefixes are admissible). Then the cells from the top and the bottom of the current column i are emptied; the remaining entries are placed in order in the remaining cells between those erased. We obtain a new crystal of KN skew tableaux on the alphabet $[\pm j, n]$ isomorphic to the crystal B^0.

Let x be the total number of times (R3) has to be applied to H^0, from column r to column t as explained above, to get a KN skew tableau on alphabet $[\pm j, n]$. Denote the resulting KN skew tableau by H^x. Note that for each column of any vertex of B^0, the number of times (R3) is applied is the same. We then obtain the sequence of isomorphic crystals

$$\mathsf{B}^0 \overset{(R3)}{\simeq} \mathsf{B}^1 \overset{(R3)}{\simeq} \cdots \overset{(R3)}{\simeq} \mathsf{B}^{x_r} \overset{(R3)}{\simeq} \mathsf{B}^{x_r+1} \overset{(R3)}{\simeq} \cdots \overset{(R3)}{\simeq} \mathsf{B}^{x_r+x_s}$$
$$\overset{(R3)}{\simeq} \cdots \overset{(R3)}{\simeq} \mathsf{B}^{x_q+x_r+\cdots+x_s+x_t} = \mathsf{B}^x,$$

where $x = x_q + x_r + \cdots + x_s + x_t$ and x_i is the number of times we apply (R3) to column i of H^0 for $i = q, r, \dots, s, t$. The crystal B^x, isomorphic to B^0, is obtained by applying (R3) x times to each vertex of B^0, namely, x_i times to column i, for $i = q, \dots, s, t$, of each vertex of B^x. Equivalently, B^x is the crystal whose highest weight element is the KN skew tableau H^x of shape ν^x/μ^x, where $\nu^x \subseteq \nu$, $\mu \subseteq \mu^x$ and $|\mu^x| - |\mu| = |\nu| - |\nu^x| = x$ is the number of times (R3) has been applied to H^0 (or $T_{[\pm j,n]}$).

(I.2.1) *The pair* (U_x, V_x) *of green-purple inner and purple outer standard tableaux for any vertex of* B^x. Let

$$\{\mathbf{g}_1 < \cdots < \mathbf{g}_{|\mu|} < \mathbf{p}_1 < \mathbf{p}_2 < \cdots < \mathbf{p}_x < \mathbf{p}'_x < \cdots < \mathbf{p}'_2 < \mathbf{p}'_1\} \tag{8.4}$$

be a completely ordered alphabet of $|\mu| + 2x$ letters consisting of $|\mu|$ green letters and x unprimed and x primed *purple letters.*

Define the standard tableau U_x of shape μ^x, where $\mu \subseteq \mu^x$ and $|\mu^x| = |\mu| + x$, to be an extension of U_0 filled with the $|\mu|$ green letters by filling the extra x cells, the total number of cells made empty at the top of each non-admissible column in a vertex of B^0, with the unprimed purple letters $\{\mathbf{p}_1 < \cdots < \mathbf{p}_{x_r} < \cdots < \mathbf{p}_x\}$. Define the standard tableau V_x of shape ν/ν^x by filling the x cells made empty at the bottom of each non-admissible column in a vertex of B^0 with the primed purple letters $\mathbf{p}'_x < \cdots < \mathbf{p}'_{x_r} < \cdots < \mathbf{p}'_1$. The filling rule is as follows.

Successively fill the pair of cells made empty each time (R3) is applied with one unprimed purple letter and one primed purple letter,

$$\mathbf{p}_1 < \mathbf{p}'_1, \ldots, \mathbf{p}_{x_r} < \mathbf{p}'_{x_r}, \mathbf{p}_{x_r+1} < \mathbf{p}'_{x_r+1}, \ldots, \mathbf{p}_{x_r+x_s} < \mathbf{p}'_{x_r+x_s}, \ldots, \mathbf{p}_x < \mathbf{p}'_x,$$

with the unprimed letter at the top of the column and the primed letter at the bottom of the column. We impose the order

$$\begin{aligned}\mathbf{g}_1 < \cdots < \mathbf{g}_{|\mu|} < \mathbf{p}_1 < \cdots < \mathbf{p}_{x_r} < \mathbf{p}_{x_r+1} < \cdots < \mathbf{p}_{x_r+x_s} < \cdots < \mathbf{p}_x \\ < \mathbf{p}'_x < \cdots < \mathbf{p}'_{x_r+x_s} < \cdots < \mathbf{p}'_{x_r+1} < \mathbf{p}'_{x_r} < \cdots < \mathbf{p}'_1.\end{aligned}$$

That is, each time an unprimed purple letter and a primed purple letter are added to U_x and V_x, respectively, the unprimed letter is strictly larger than any green letter and any unprimed purple letter already added to U_x, and simultaneously, the primed purple letter is strictly smaller than any primed purple letter already added to V_x.

By construction, the pair (U_x, V_x) of inner and outer standard tableaux is the same for any vertex of B^x. More precisely, U_x of shape μ^x is the extension of U_0 filled with the alphabet $\{\mathbf{g}_1 < \cdots < \mathbf{g}_{|\mu|} < \mathbf{p}_1 < \mathbf{p}_2 < \cdots < \mathbf{p}_x\}$; V_x of skew shape ν/ν^x is filled with the alphabet of primed purple letters

$$\mathbf{p}'_x < \cdots < \mathbf{p}'_{x_r+\cdots+x_q} < \cdots < \mathbf{p}'_{x_r+x_s} < \cdots < \mathbf{p}'_{x_r+1} < \mathbf{p}'_{x_r} < \cdots < \mathbf{p}'_1.$$

Regarding U_x, extend the column r of U_0 with the x_r unprimed purple letters $\mathbf{p}_1 < \cdots < \mathbf{p}_{x_r}$, the column s with the x_s unprimed purple letters $\mathbf{p}_{x_r+1} < \cdots < \mathbf{p}_{x_r+x_s}$, and finally the column t with the x_t unprimed purple letters

$$\mathbf{p}_{x_r+\cdots+x_q+1} < \cdots < \mathbf{p}_{x_r+\cdots+x_q+x_t} = \mathbf{p}_x;$$

regarding V_x of skew shape ν/ν^x, start with the skew shape ν/μ^x, and fill the bottom x_r boxes of column r with the alphabet of primed purple letters $\mathbf{p}'_{x_r} < \cdots < \mathbf{p}'_1$, the bottom x_s of column s with the alphabet $\mathbf{p}'_{x_r+x_s} < \cdots < \mathbf{p}'_{x_r+1}$, and, finally, the bottom x_t boxes of column t with the alphabet $\mathbf{p}'_x < \cdots < \mathbf{p}'_{x_r+x_s+\cdots+x_q+1}$. See the triple (U_x, H^x, V_x) in Figure 4.

(I.3) *Rectification of the C_{n-j+1} crystal B^x and reduced rectification of $T_{[\pm j,n]}$.* Consider the triple of tableaux (U_x, H^x, V_x) previously defined. Apply complete SJDT_j slides successively to the cells of U_x, from the largest entry to the smallest one, to rectify H^x. At the end of each complete SJDT_j slide, we get an outer cell filled with the letter where the slide started in U_x. While H^x is being rectified, the cells of U_x are slid to end up as outer corners and added to the skew standard tableau V_x.

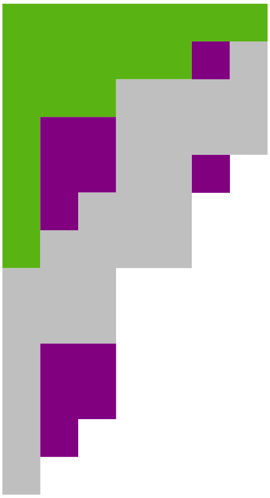


Figure 4. The triple (U_x, H^x, V_x) with H^x in gray, V_x in purple, and $U_0(\subseteq U_x)$ in green. U_x consists of the green region together with the purple inner regions of V_x.

The rectification of H^x does not depend on the choice of the inner corner made in each step during the rectification process [36, Corollary 6.3.9]. Applying SJDT_j to any corner of U_x in an element of B^x (recall that for all elements of B^x, U_x is the same) gives a crystal isomorphism. This observation is equivalent to the fact that the rectification does not depend on the filling of U_x: U_x is a choice to keep track of the rectification of H^x (or of any other vertex of B^x). If we may apply a complete SJDT_j slide to an inner corner of H^x, then we may apply a complete SJDT_j slide to the same inner corner in every vertex of the crystal B^x, and this slide will create the same outer corner filled with the same letter.

However, if the number of boxes of H^x, $|H^x|$, exceeds the minimal number of boxes of its Knuth class, it will be necessary to apply SJDT_j more than $|U_x| = |\mu| + x$ times to rectify H^x. When H^x has the minimal number of boxes of its Knuth class, only x unprimed purple letters and $|\mu|$ green letters will slide outwards and join the outer tableau V_x.

Let $2y \geq 0$ be the number of boxes of H^x that exceeds the minimal number of boxes of its Knuth class, that is,

$$2y = |H^x| - |\mathsf{rect}_j(H^x)|.$$

Necessarily, $2y$ boxes of H^x will be lost in the SJDT_j rectification process. Henceforth, the SJDT_j (B(2)) case will be applied y times, each of which will create a non-admissible column followed by the application of an (R3) contraction relation, each of which will result in the loss of two boxes.

Remark 8.8. As stated in [36, Theorem 6.1.9], if the (B(2)) case appears with the creation of a non-admissible column when applying complete SJDT to an inner corner of a KN skew tableau, it has to be at the initial column where the inner corner was originally contained.

This observation implies that each of the y mentioned non-admissible columns will only occur in the columns containing the inner corners where the slide started.

The complete SJDT_j slides applied successively to the entries of U_x, as mentioned, will transform the crystal B^x into an isomorphic crystal of KN skew tableaux, as long as the SJDT_j (B(2)) case does not create a non-admissible column. Otherwise, one has an isomorphic crystal where each vertex has a non-admissible column. In this case, we apply the contraction relation (R3) to that column in each vertex, erasing a pair $(k, \overline{k})$ if $k \in [\pm j, n]$ is the lowest entry such that $N(k) > k$. Then, as in (I.2) above, the cells from the top and the bottom of the current column are emptied and the remaining entries are placed in order. We get a new isomorphic crystal of KN skew tableaux where each vertex has two fewer boxes. As observed above, this may only happen in the y columns where SJDT_j was applied, specifically, those containing the inner corners where the slides started; no other boxes are deleted in the rectification process of B^x.

Eventually, H^x is rectified to $\mathsf{rect}(H^x)$, as are all vertices of B^x, and we get

$$\mathsf{B}^0 \simeq \mathsf{B}^x \simeq \mathsf{R},$$

where the crystal R is of straight KN tableaux with highest weight element $\mathsf{rect}(H^x)$.

(I.3.1) *The green-purple-red standard tableau V of every vertex in the C_{n-j+1} crystal R containing $\mathsf{rect}_j(T_{[\pm j,n]})$.* Let

$$\mathsf{B}^{x,1}, \mathsf{B}^{x,1,\text{-}}, \mathsf{B}^{x,2}, \mathsf{B}^{x,2,\text{-}} \dots, \mathsf{B}^{x,y}, \mathsf{B}^{x,y,\text{-}}$$

be the sequence of $2y$ isomorphic crystals appearing in the rectification process from B^x to R, tracking each complete SJDT_j slide which triggers a (B(2)) case, and the subsequent application of an (R3) contraction relation to that non-admissible column. Namely, $\mathsf{B}^{x,i}$ correspond to the (B(2)) case and $\mathsf{B}^{x,i,\text{-}}$ to the (R3) contraction relation.

For $i = 1, \dots, y$, let $H^{x,i}$ and $H^{x,i,\text{-}}$ be the pair of highest weight elements of the crystal pair $\mathsf{B}^{x,i}$ and $\mathsf{B}^{x,i,\text{-}}$, respectively. Each $H^{x,i}$ has exactly one non-admissible column, and $H^{x,i,\text{-}}$ has no non-admissible columns.

We have to store $2y$ new auxiliary letters to record the $2y$ empty cells created by the y applications of (R3) as a consequence of the creation of y non-admissible columns by the complete SJDT_j slide where the (B(2)) case appeared and created a non-admissible column.

Consider the triple of tableaux (U_x, H^x, V_x) corresponding to the crystal B^x. Let $(U_{x,1}, H^{x,1}, V_{x,1})$ be the triple of tableaux obtained from (U_x, H^x, V_x) by applying complete SJDT_j slides to the entries of U_x and transforming the KN skew tableau H^x into $H^{x,1}$, where for the first time in the complete SJDT_j slide, the (B(2)) case appears and creates a non-admissible column. After the said complete SJDT_j slides to U_x, $U_{x,1}$ is the inner standard tableau of $H^{x,1}$, and $V_{x,1}$ is obtained from V_x by adding the slid entries from U_x to V_x. $V_{x,1}$ is indeed a standard tableau because by construction the green entries of U_x are strictly smaller than the primed purple entries of V_x. The pair $(U_{x,1}, V_{x,1})$ of inner and outer standard tableaux is the same for every vertex of $\mathsf{B}^{x,1}$: $U_{x,1} \subseteq U_x$, $V_{x,1} \supseteq V_x$.

We have to apply an (R3) contraction relation to $H^{x,1}$ (and to every vertex of $\mathsf{B}^{x,1}$) to transform the non-admissible column into an admissible one: a pair of symmetric entries in each vertex of $\mathsf{B}^{x,1}$ will be deleted, the top and bottom cells of that column will be emptied and the remaining entries will be placed in order. Fill the empty entries with *red letters* $\mathbf{r}_1 < \mathbf{r}_1'$, with $\mathbf{r}_1$ on the top and $\mathbf{r}_1'$ on the bottom, where in the complete SJDT_j slide, the B.2 case appears and has created a non-admissible column such that $\mathbf{r}_1$ is strictly larger than any entry of $U_{x,1}$, and $\mathbf{r}_1'$ is strictly smaller than any entry of $V_{x,1}$. $V_{x,1}$ is filled with the entries of U_x already slid and with all primed purple letters. The cell with the red letter $\mathbf{r}_1$ was the cell of U_x, where the complete SJDT_j slide started and the (B(2)) case appeared with the creation of a non-admissible column.

Let $U_{x,1,+}$ be the standard tableau obtained by adding the red letter $\mathbf{r}_1$ to $U_{x,1}$, and let $V_{x,1,+}$ be the standard tableau obtained by adding the primed red letter $\mathbf{r}_1'$ to $V_{x,1}$ in the manner described,

$$U_{x,1} \subset U_{x,1,+} \subseteq U_x, \quad V_{x,1,+} \supset V_{x,1} \supseteq V_x.$$

We keep applying complete SJDT_j slides to entries of $U_{x,1,+}$, from the largest to the smallest, to rectify $H^{x,1,-}$, so the cell $\mathbf{r}_1$ will be the first to slide outwards and become an outer corner.

Let $(U_{x,2}, H^{x,2}, V_{x,2})$ be the triple of tableaux which is obtained from the triple $(U_{x,1,+}, H^{x,1,-}, V_{x,1,+})$ by applying complete SJDT_j slides to the entries of $U_{x,1,+}$ and transforming the KN skew tableau $H^{x,1,-}$ into $H^{x,2}$, where for the second time in the complete SJDT_j slide, the (B(2)) case appears with the creation of a non-admissible column; that is, $H^{x,2}$ has a non-admissible column, and the highest weight elements of all previous crystals obtained from $\mathsf{B}^{x,1,-}$ had all columns admissible. After these complete SJDT_j slides to $U_{x,1,+}$, $U_{x,2}$ is the inner standard tableau of $H^{x,2}$; $V_{x,2}$ is obtained from $V_{x,1,+}$ by adding the slid entries from $U_{x,1,+}$ to $V_{x,1,+}$. $V_{x,2}$ is indeed a standard tableau because by construction the entries of $U_{x,1,+}$ are strictly smaller than the entries of $V_{x,1,+}$. At this point, the red letter $\mathbf{r}_1$ has already slid from

$U_{x,1,+}$ to $V_{x,2}$; that is, $\mathbf{r}_1$ is no longer in $U_{x,2}$ and instead belongs to $V_{x,2}$,

$$U_{x,2} \subseteq U_{x,1} \subset U_{x,1,+} \subseteq U_x, \quad V_{x,2} \supset V_{x,1,+} \supset V_{x,1} \supseteq V_x.$$

Continuing in this fashion for $i = 2, \dots, y-1$, we eventually reach a point wherein the $2(y-1)$ red letters have the following relative ordering,

$$\mathbf{r}_{y-1} < \mathbf{r}'_{y-1} < \cdots < \mathbf{r}_2 < \mathbf{r}'_2 < \cdots < \mathbf{r}_1 < \mathbf{r}'_1$$

and the rectification storing tableaux are such that

$$U_{x,y} \subset U_{x,y-1} \subset U_{x,y-1+} \subset \cdots \subset U_{x,2} \subset U_{x,2,+} \subseteq U_{x,1} \subset U_{x,1,+} \subseteq U_x,$$
$$V_{x,y} \supset V_{x,y-1,+} \supset V_{x,y-1} \supset \cdots \supset V_{x,2,+} \supset V_{x,2} \supset V_{x,1,+} \supset V_{x,1} \supseteq V_x.$$

Let $\mathsf{B}^{x,y}$ be the crystal with highest weight element $H^{x,y}$. We have to apply the (R3) contractor operator to $H^{x,y}$ (and to every vertex of $\mathsf{B}^{x,y}$) to transform the non-admissible column into an admissible one: a pair of symmetric entries in each vertex of $\mathsf{B}^{x,y}$ will be deleted, the top and bottom cells of that column will be emptied and the remaining entries will be placed in order. Let $\mathsf{B}^{x,y,-}$ be the new crystal of KN skew tableaux isomorphic to $\mathsf{B}^{x,y}$, and let $H^{x,y,-}$ be its highest weight element (it has two fewer boxes than $H^{x,y}$). Fill the empty entries with red letters $\mathbf{r}_y < \mathbf{r}'_y$, $\mathbf{r}_y$ on the top and $\mathbf{r}'_y$ on the bottom of the column, where in the complete SJDT_j slide, the (B(2)) case appears and creates a non-admissible column such that $\mathbf{r}_y$ is strictly larger than any entry of $U_{x,y}$, and $\mathbf{r}'_y$ is strictly smaller than any entry of $U_{x,y}$ already slid. The primed letters are considered to be slid because by the time they are created, they are outer corners.

The cell with the red letter $\mathbf{r}_y$ was the cell of $U_{x,y-1,+}$ where the complete SJDT_j slide started and the (B(2)) case appeared with the creation of a non-admissible column. Let $U_{x,y,+}$ be the standard tableau obtained by adding the red letter $\mathbf{r}_y$ to $U_{x,y}$, and let $V_{x,y,+}$ be the standard tableau obtained by adding the primed red letter $\mathbf{r}'_y$ to $V_{x,y}$.

We keep applying complete SJDT_j slides to the entries of $U_{x,y,+}$ from the largest to the smallest, and eventually, we rectify $H^{x,y,-}$ without further use of the contractor (R3). We reach the crystal R, where every vertex is rectified. The crystal R is called the rectification of B^0. At this point, one has the following relative ordering among the $2y$ red letters:

$$\mathbf{r}_{y} < \mathbf{r}'_{y} < \cdots < \mathbf{r}_2 < \mathbf{r}'_2 < \cdots < \mathbf{r}_1 < \mathbf{r}'_1$$

and the rectification storing tableaux

$$\emptyset \subset U_{x,y} \subset U_{x,y,+} \subseteq U_{x,y-1} \subset \cdots \subset U_{x,2} \subset U_{x,2,+} \subseteq U_{x,1} \subset U_{x,1,+} \subseteq U_x,$$
$$V \supset V_{x,y,+} \supset V_{x,y} \supset V_{x,y-1,+} \supset V_{x,y-1} \supset \cdots \supset V_{x,2,+} \supset V_{x,2} \supset V_{x,1,+}$$
$$\supset V_{x,1} \supseteq V_x,$$

where V is the standard tableau obtained by adding to $V_{x,y,+}$ via sliding the letters from $U_{x,y,+}$. We have the following ordering of all colored letters, green, purple (primed and unprimed), and red (primed and unprimed) in the skew standard tableau V:

$$\begin{aligned} \mathbf{g}_1 < \cdots < \mathbf{r}_y < \mathbf{r}'_y < \mathbf{g}_l < \cdots < \mathbf{r}_d < \mathbf{r}'_d < \cdots < \mathbf{g}_{|\mu|} \\ < \mathbf{p}_1 < \mathbf{p}_2 < \cdots < \mathbf{r}_k < \mathbf{r}'_k < \cdots < \mathbf{p}_i < \cdots < \mathbf{r}_1 \\ < \mathbf{r}'_1 < \cdots < \mathbf{p}_x < \mathbf{p}'_x < \cdots < \mathbf{p}'_1. \end{aligned} \tag{8.5}$$

We have constructed the following sequence of isomorphic crystals, stored in V via the slid colorful letters:

$$\mathsf{B}^0 \overset{(R3)}{\simeq} \cdots \mathsf{B}^{x_r} \overset{(R3)}{\simeq} \cdots \mathsf{B}^{x_r+x_s} \overset{(R3)}{\simeq} \cdots \mathsf{B}^{x_r+x_s+\cdots+x_t} = \mathsf{B}^x \tag{8.6}$$

$$\begin{aligned} \mathsf{B}^x \overset{\mathsf{SJDT}_j}{\simeq} \cdots \mathsf{B}^{x,1} \overset{(R3)}{\simeq} \mathsf{B}^{x,1,-} \overset{\mathsf{SJDT}_j}{\simeq} \cdots \mathsf{B}^{x,2} \overset{R3}{\simeq} \mathsf{B}^{x,2,-} \\ \overset{\mathsf{SJDT}_j}{\simeq} \cdots \mathsf{B}^{x,y} \overset{(R3)}{\simeq} \mathsf{B}^{x,y,-} \end{aligned} \tag{8.7}$$

$$\mathsf{B}^{x,y,-} \overset{\mathsf{SJDT}_j}{\simeq} \cdots \overset{\mathsf{SJDT}_j}{\simeq} \mathsf{R}. \tag{8.8}$$

Remark 8.9. In our construction, purple letters are larger than all green ones (8.4). However, for the red ones together with the two other colors, we just write (8.5).

(I.4) *The Schützenberger–Lusztig involution on the C_{n-j+1} crystal B^0, its rectification and the reversal.* Let L^0 be the lowest weight element of the C_{n-j+1} connected normal crystal B^0. The crystal R with highest weight element $\mathsf{rect}_j(H^0)$ is the rectification of the crystal B^0 and contains $\mathsf{rect}_j(T_{[\pm j,n]})$. Let F be the composition of the sequence of lowering operators connecting H^0 to $T_{[\pm j,n]}$ in B^0, $F(H^0) = T_{[\pm j,n]}$. The Schützenberger–Lusztig involution ξ in B^0 gives $\xi(T_{[\pm j,n]}) = F^{-1}(L^0)$, where F^{-1} means the sequence obtained by replacing each lowering operator f_i in F with the corresponding raising operator e_i. In each crystal of the sequence (8.6)–(8.8) above, the same sequence F (F^{-1}) connects the corresponding highest (lowest) weight element to the corresponding coplactic image of $T_{[\pm j,n]}$ ($\xi(T_{[\pm j,n]})$). In particular, F connects $\mathsf{rect}_j(H^0)$ to $\mathsf{rect}_j(T_{[\pm j,n]})$:

$$F(\mathsf{rect}_j\ H^0) = \mathsf{rect}_j(T_{[\pm j,n]}).$$

By Santos [47], the Schützenberger–Lusztig involution in R guarantees that

$$\mathsf{evac}^{C_{n-j+1}}\big(\mathsf{rect}_j(T_{[\pm j,n]})\big) = F^{-1}(\mathsf{rect}_j(L^0))$$

is in R. Thanks to the crystal isomorphisms (8.6)–(8.8) and Lemma 6.2,

$$\mathsf{reversal}^{C_{n-j+1}}(T_{[\pm j,n]}) = F^{-1}(L^0) = \mathsf{arect}_j\ \mathsf{evac}^{C_{n-j+1}}\big(\mathsf{rect}_j(T_{[\pm j,n]})\big). \tag{8.9}$$

To compute the reversal of $T_{[\pm j,n]}$ in B^0 without using the sequence F of crystal operators and the highest/lowest weight elements H^0, L_0 of B^0, we use Santos' evacuation on $\mathsf{rect}_j(T_{[\pm j,n]})$ and the rectification sequence of crystals backwards in (8.6)–(8.8) stored in the standard skew tableau V.

Step II. Computation of symplectic evacuation of $\mathsf{rect}_j(T_{[\pm j,n]})$ in the C_{n-j+1} crystal R. The tableau $\mathsf{rect}_j(T_{[\pm j,n]})$ is admissible in the alphabet $[\pm j, n]$. We use Santos' algorithm as follows: take π-rotation and change the sign of $\mathsf{rect}_j(T_{[\pm j,n]})$; then, apply SJDT_j to obtain $\mathsf{evac}^{C_{n-j+1}}(\mathsf{rect}_j(T_{[\pm j,n]}))$ in the crystal R. Replace the tableau pair $(\mathsf{rect}_j(T_{[\pm j,n]}), V)$ with $(\mathsf{evac}^{C_{n-j+1}}(\mathsf{rect}_j(T_{[\pm j,n]})), V)$.

Step III. Symplectic reversal of $T_{[\pm j,n]}$ in the C_{n-j+1} crystal B^0. Consider the pair of tableaux $(\mathsf{evac}^{C_{n-j+1}}(\mathsf{rect}_j(T_{[\pm j,n]})), V)$, where V is the standard tableau consisting of all the slid letters in the rectification sequence (8.6)–(8.8) on the alphabet of green, purple and red letters.

Apply the *reverse* SJDT_j, $R\,\mathsf{SJDT}_j$, to the entries of V from the smallest to the largest to send $\mathsf{evac}^{C_{n-j+1}}(\mathsf{rect}_j(T_{[\pm j,n]}))$ to $\mathsf{reversal}(T_{[\pm j,n]}) = F^{-1}(L_0)$ in the C_{n-j+1} crystal B^0.

When the SJDT_j is applied to an unprimed red letter $\mathbf{r}_i$, $i \in \{1, \dots, y\}$, in V, the letter $\mathbf{r}_i$ slides to the top of a column with the cell $\mathbf{r}_i'$ on the bottom. At this point, we have reached the crystal $\mathsf{B}^{x,i}$. Then we apply the operator (R3) to the column containing the pair $(\mathbf{r}_i, \mathbf{r}_i')$ by erasing those entries and adding a pair of symmetric entries $(k, \overline{k})$ so that we get a non-admissible column on the alphabet $[\pm j, n]$. The SJDT_j applies now to the next letter bigger than $\mathbf{r}_i'$. In this complete reverse slide, the SJDT_j (B(2)) case occurs.

When the reverse SJDT_j slides have been applied to all non-primed purple letters, we have reached the crystal B^x, where columns $r < s < \cdots < t$ have x_i non-primed purple letters $\mathbf{p}_{x_1+\cdots+x_{i-1}+1} < \cdots < \mathbf{p}_{x_1+\cdots+x_i}$ on the top and the corresponding primed letters on the bottom for $i = r, s, \dots, t$. Then, for $i = r, \dots, s, t$, we apply (R3) x_i times to each such column i, and we reach the crystal B^0, where each vertex has non-admissible columns $r, s, \dots, t$. In particular, we obtain $\mathsf{reversal}^{C_{n-j+1}}(T_{[\pm j,n]})$.

Step IV. Partial symplectic reversal of T computes $\xi^{C_n}_{[j,n]}(T)$. We let $\xi^{C_n}_{[j,n]}(T)$ be the Schützenberger–Lusztig involution of $T = (T^+_{[j-1]}, T_{[\pm j,n]}, T^-_{[\overline{j-1},\overline{1}]})$ in the crystal connected component $\mathsf{B} \simeq \mathsf{B}^0$ of $\mathsf{KN}_{[j,n]}(\lambda, n)$. We then replace $T_{[\pm j,n]}$ with $\mathsf{reversal}^{C_{n-j+1}}(T_{[\pm j,n]})$ (see (8.9)) in T, to give the formula

$$\begin{aligned}\xi^{C_n}_{[j,n]}(T) &= \mathsf{reversal}^{C_n}_{[j,n]}(T)\\ &= \big(T^+_{[j-1]}, \mathsf{arect}_j\ \mathsf{evac}^{C_{n-j+1}}\big(\mathsf{rect}_j(T_{[\pm j,n]})\big), T^-_{[\overline{j-1},\overline{1}]}\big).\end{aligned} \tag{8.10}$$

Remark 8.10. Some consequences of our colorful algorithm.

(1) If we put $j = 1$ in the colorful algorithm, we reduce to Step II of C_n evacuation.

(2) The algorithm for the full C_m reversal of a KN skew tableau $T \in \mathsf{KN}(\lambda/\mu, m)$ results from our colorful tableau switching algorithm by considering the image of T, $(T_\mu, \widehat{T})$, in the sub-crystal $\mathsf{B}(\mu, \lambda) \subseteq \mathsf{KN}_{[j,n]}(\lambda, n)$, where $n = m + j - 1$ (Section 6.4.2). Let B be the crystal connected component of $\mathsf{B}(\mu, \lambda)$ containing $(T_\mu, \widehat{T})$, where T_μ is the Yamanouchi tableau of shape μ and $\widehat{T}$ is obtained by increasing each of the entries of T by $j - 1$. Then, restricting $(T_\mu, \widehat{T})$ to the alphabet $[\pm j, n]$, $\widehat{T}$ is an admissible C_{n-j+1} skew tableau in the C_{n-j+1} crystal B^0. Our algorithm reduces to Step I with just green and red, Step II and Step III. Finally, we subtract $j - 1$ from the entries of $\mathsf{reversal}^{C_{n-j+1}}(\widehat{T})$ to get $\mathsf{reversal}^{C_m}(T)$. However, subtraction by $j - 1$ cancels the last step in the reduced SJDT_j (Definition 8.6), and therefore it is enough to apply SJDT.

This means that the algorithm for the full C_m reversal of the KN skew tableau T results from our algorithm with $\mathsf{B}^0 = \mathsf{B}(T)$ a type C_m crystal, $x = 0$ and applying SJDT to U_0 to get $(\mathsf{rect}(T), V)$, where V is a skew standard tableau without purple letters. Then RSJDT applied to V gives

$$\mathsf{reversal}^{C_m}(T) = \mathsf{arect}\ \mathsf{evac}^{C_m}\big(\mathsf{rect}(T)\big).$$

8.3. Examples of full and partial symplectic reversal

Example 8.11. Full reversal of a skew tableau, $J = I$. Let

$$T = \begin{array}{|c|c|c|c|}\hline \ & 2 & \bar{2} & \bar{1} \\ \hline \bar{2} & 2 & \bar{1} \\ \cline{1-3} \bar{1} \\ \cline{1-1} \end{array} \in \mathsf{KN}\big((4,3,2)/(1), 3\big).$$

We compute $\xi^{C_3}(T)$ as follows. First, we fill in the empty box in T with a green letter (it defines the one box standard tableau U_0), to which we perform symplectic jeu de taquin until it becomes an outer corner:

$$(U_0, T) = \begin{array}{|c|c|c|c|}\hline g & 2 & \bar{2} & \bar{1} \\ \hline \bar{2} & 2 & \bar{1} \\ \cline{1-3} \bar{1} \\ \cline{1-1} \end{array} \xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|c|}\hline 1 & g & \bar{2} & \bar{1} \\ \hline \bar{2} & \bar{1} & \bar{1} \\ \cline{1-3} \bar{1} & \ \\ \cline{1-2} \end{array} \xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|c|}\hline r & g & \bar{2} & \bar{1} \\ \hline \bar{2} & \bar{1} & \bar{1} \\ \cline{1-3} r' \\ \cline{1-1} \end{array}$$

$$\xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|c|}\hline r & \bar{2} & g & \bar{1} \\ \hline \bar{2} & \bar{1} & \bar{1} \\ \cline{1-3} r' \\ \cline{1-1} \end{array} \xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|c|}\hline r & \bar{2} & \bar{1} & \bar{1} \\ \hline \bar{2} & \bar{1} & g \\ \cline{1-3} r' \\ \cline{1-1} \end{array}$$

$$\xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|c|}\hline \bar2&\bar2&\bar1&\bar1\\ \hline r&\bar1&g& \\ \cline{1-3} r'&&&\\ \cline{1-1}\end{array} \xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|c|}\hline \bar2&\bar2&\bar1&\bar1\\ \hline \bar1&r&g& \\ \cline{1-3} r'&&&\\ \cline{1-1}\end{array}$$

$$\Rightarrow \mathsf{rect}(T) = \begin{array}{|c|c|c|c|}\hline \bar2&\bar2&\bar1&\bar1\\ \hline \bar1&&&\\ \cline{1-1}\end{array} \qquad V = \begin{array}{|c|c|c|c|}\hline \ &\ &\ &\ \\ \hline \ &r&g&\\ \cline{1-3} r'&&&\\ \cline{1-1}\end{array}, \quad r < r' < g.$$

Taking π-rotation and changing the signs of $\mathsf{rect}(T)$, we again apply SJDT to compute $\mathsf{evac}^{C_3}(\mathsf{rect}(T))$:

$$\begin{array}{|c|c|c|c|}\hline *&*&*&1\\ \hline 1&1&2&2\\ \hline\end{array} \xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|c|}\hline *&*&1&*\\ \hline 1&1&2&2\\ \hline\end{array} \xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|c|}\hline *&*&1&2\\ \hline 1&1&2&*\\ \hline\end{array} \xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|c|}\hline *&1&1&2\\ \hline 1&*&2&*\\ \hline\end{array}$$

$$\xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|c|}\hline *&1&1&1\\ \hline 1&2&*&*\\ \hline\end{array} \xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline *&2&*&*\\ \hline\end{array}$$

$$\xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&*&*&*\\ \hline\end{array} = \mathsf{evac}^{C_3}(\mathsf{rect}(T)).$$

We replace $\mathsf{rect}(T)$ with $\mathsf{evac}^{C_3}(\mathsf{rect}(T))$ in $(\mathsf{rect}(T), V)$ and apply reverse SJDT to V to compute $\xi^{C_3}(T) = \mathsf{reversal}^{C_3}(T)$:

$$\left(\mathsf{evac}^{C_3}(\mathsf{rect}(T)), V\right) = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&r&g&\\ \cline{1-3} r'&&&\\ \cline{1-1}\end{array} \xrightarrow{\mathsf{RSJDT}} \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline r&2&g&\\ \cline{1-3} r'&&&\\ \cline{1-1}\end{array}$$

$$\xrightarrow{\mathsf{RSJDT}} \begin{array}{|c|c|c|c|}\hline r&1&1&2\\ \hline 1&2&g&\\ \cline{1-3} r'&&&\\ \cline{1-1}\end{array} \overset{(\text{R3})}{\sim} \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&2&g&\\ \cline{1-3} \bar2&&&\\ \cline{1-1}\end{array}$$

$$\xrightarrow{\mathsf{RSJDT}} \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&g&2&\\ \cline{1-3} \bar2&&&\\ \cline{1-1}\end{array} \xrightarrow{\mathsf{RSJDT}} \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline g&2&2&\\ \cline{1-3} \bar2&&&\\ \cline{1-1}\end{array}$$

$$\xrightarrow{\mathsf{RSJDT}} \begin{array}{|c|c|c|c|}\hline g&1&1&2\\ \hline 1&2&2&\\ \cline{1-3} \bar2&&&\\ \cline{1-1}\end{array} = \left(U_0, \xi^{C_3}(T)\right)$$

$$\Rightarrow \xi^{C_3}(T) = \begin{array}{|c|c|c|c|}\hline \ &1&1&2\\ \hline 1&2&2&\\ \cline{1-3} \bar2&&&\\ \cline{1-1}\end{array}.$$

Example 8.12. Let

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & \overline{1} \\ \hline 4 & 4 & \overline{3} \\ \cline{1-3} \overline{4} & \overline{2} & \overline{1} \\ \cline{1-3} \overline{3} \\ \cline{1-1} \end{array} \in \mathsf{KN}\big((4,3,3,1),4\big).$$

We have $\mathsf{wt}(P) = (-1, 1, -2, 1)$. To compute

$$\xi^{C_4}_{[2,4]}(P) = \mathsf{reversal}^{C_4}_{[2,4]}(P),$$

we freeze the letters $1, \overline{1}$ in P and consider $P_{[\pm 2,4]}$, which is not an admissible C_3 tableau in the alphabet $[\pm 2, 4]$: the second column $24\overline{2}$ is not an admissible C_3 column. The column reading of $P_{[\pm 2,4]}$ is

$$2\overline{3}24\overline{2}4\overline{4}\overline{3} \overset{(R3)}{\sim} 2\overline{3}44\overline{4}\overline{3}.$$

In order to rectify $P_{[\pm 2,4]}$, we include this non-admissible second column in the SJDT_2 sequence.

(1) Rectification of $P_{[\pm 2,4]}$:

$$\big(U_0, P_{[\pm 2,4]}\big) = \begin{array}{|c|c|c|c|}\hline g & 2 & 2 & \\ \hline 4 & 4 & \overline{3} \\ \cline{1-3} \overline{4} & \overline{2} & \\ \cline{1-3} \overline{3} \\ \cline{1-1} \end{array} \xrightarrow{\mathsf{SJDT}_2} \begin{array}{|c|c|c|c|}\hline g & p & 2 & \\ \hline 4 & 4 & \overline{3} \\ \cline{1-3} \overline{4} & p' & \\ \cline{1-3} \overline{3} \\ \cline{1-1} \end{array} \xrightarrow{\mathsf{SJDT}_2} \begin{array}{|c|c|c|c|}\hline g & 2 & \overline{3} & \\ \hline 4 & 4 & p \\ \cline{1-3} \overline{4} & p' & \\ \cline{1-3} \overline{3} \\ \cline{1-1} \end{array}$$

$$\xrightarrow{\mathsf{SJDT}_2} \begin{array}{|c|c|c|c|}\hline 2 & 4 & \overline{3} & \\ \hline 4 & g & p \\ \cline{1-3} \overline{4} & p' & \\ \cline{1-3} \overline{3} \\ \cline{1-1} \end{array} \xrightarrow{\mathsf{SJDT}_2} \begin{array}{|c|c|c|c|}\hline r & 4 & \overline{3} & \\ \hline 2 & g & p \\ \cline{1-3} \overline{3} & p' & \\ \cline{1-3} r' \\ \cline{1-1} \end{array} \xrightarrow{\mathsf{SJDT}_2} \begin{array}{|c|c|c|c|}\hline 2 & 4 & \overline{3} & \\ \hline \overline{3} & g & p \\ \cline{1-3} r & p' & \\ \cline{1-3} r' \\ \cline{1-1} \end{array}$$

$$= \big(\mathsf{rect}_2\, P_{[\pm 2,4]}, V\big)$$

$$\Rightarrow \mathsf{rect}_2\, P_{[\pm 2,4]} = \begin{array}{|c|c|c|}\hline 2 & 4 & \overline{3} \\ \hline \overline{3} \\ \cline{1-1} \end{array},$$

and

$$V = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & g & p \\ \cline{1-3} r & p' \\ \cline{1-2} r' \\ \cline{1-1} \end{array}, \quad r < r' < g < p < p'.$$

(2) Computation of $\mathsf{evac}^{C_3}\ \mathsf{rect}_2(P_{[\pm 2,4]})$. Taking π-rotation and changing the signs of $\mathsf{rect}_2(P_{[\pm 2,4]})$, we again apply SJDT_2:

$$\begin{array}{|c|c|c|}\hline & & 3 \\ \hline 3 & \overline{4} & \overline{2} \\ \hline \end{array} \xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|}\hline & 3 & \overline{2} \\ \hline 3 & \overline{4} \\ \cline{1-2} \end{array} \xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|}\hline 3 & 3 & \overline{2} \\ \hline & \overline{4} \\ \cline{1-2} \end{array}$$

$$\xrightarrow{\mathsf{SJDT}} \begin{array}{|c|c|c|}\hline 3 & 3 & \overline{2} \\ \hline \overline{4} \\ \cline{1-1} \end{array} = \mathsf{evac}^{C_3}\ \mathsf{rect}_2\ P_{[\pm 2,4]}.$$

(3) Reversal of $P_{[\pm 2,4]}$. Replace $\mathsf{rect}_2(P_{[\pm 2,4]})$ with $\mathsf{evac}^{C_3}(\mathsf{rect}_2(P_{[\pm 2,4]}))$ in $(\mathsf{rect}_2(P_{[\pm 2,4]}), V)$ and apply $R\,\mathsf{SJDT}_2$ to V:

$$\left(\mathsf{evac}^{C_3}\ (\mathsf{rect}_2\ P_{[\pm 2,4]}), V\right) = \begin{array}{|c|c|c|c|}\hline 3 & 3 & \overline{2} & \\ \hline \overline{4} & g & p \\ \cline{1-3} r & p' & \\ \cline{1-3} r' \\ \cline{1-1} \end{array} \xrightarrow{\mathsf{RSJDT}_2} \begin{array}{|c|c|c|c|}\hline r & 3 & \overline{2} & \\ \hline 3 & g & p \\ \cline{1-3} \overline{4} & p' & \\ \cline{1-3} r' \\ \cline{1-1} \end{array}$$

$$\xrightarrow{\mathsf{RSJDT}_2} \begin{array}{|c|c|c|c|}\hline 2 & 3 & \overline{2} & \\ \hline 3 & g & p \\ \cline{1-3} \overline{4} & p' & \\ \cline{1-3} \overline{2} \\ \cline{1-1} \end{array} \xrightarrow{\mathsf{RSJDT}_2} \begin{array}{|c|c|c|c|}\hline g & 2 & \overline{2} & \\ \hline 3 & 3 & p \\ \cline{1-3} \overline{4} & p' & \\ \cline{1-3} \overline{2} \\ \cline{1-1} \end{array}$$

$$\xrightarrow{\mathsf{RSJDT}_2} \begin{array}{|c|c|c|c|}\hline g & p & 3 & \\ \hline 3 & 3 & \overline{3} & \\ \hline \overline{4} & p' & \\ \cline{1-3} \overline{2} \\ \cline{1-1} \end{array} \xrightarrow{\mathsf{RSJDT}_2} \begin{array}{|c|c|c|c|}\hline g & 2 & 3 & \\ \hline 3 & 3 & \overline{3} & \\ \hline \overline{4} & \overline{2} & \\ \cline{1-3} \overline{2} \\ \cline{1-1} \end{array}$$

$$= \left(U_0, \mathsf{reversal}^{C_3}(P_{[\pm 2,4]})\right)$$

$$\Rightarrow \mathsf{reversal}^{C_3}\left(P_{[\pm 2,4]}\right) = \begin{array}{|c|c|c|c|}\hline & 2 & 3 & \\ \hline 3 & 3 & \overline{3} & \\ \hline \overline{4} & \overline{2} & \\ \cline{1-3} \overline{2} \\ \cline{1-1} \end{array}.$$

(4) Replace $P_{[\pm 2,4]}$ with $\mathsf{reversal}^{C_3}(P_{[\pm 2,4]})$ in P to obtain

$$\mathsf{reversal}^{C_4}_{[2,4]}(P) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & \overline{1} \\ \hline 3 & 3 & \overline{3} \\ \cline{1-3} \overline{4} & \overline{2} & \overline{1} \\ \cline{1-3} \overline{2} \\ \cline{1-1} \end{array}, \quad \begin{aligned} \mathsf{wt}_{[2,4]}\left(\xi^{C_4}_{[2,4]}(P)\right) &= -\mathsf{wt}_{[2,4]}(P) \\ &= (-1, 2, -1). \end{aligned}$$

8.4. General Dynkin sub-diagram and virtualization

Let $\xi^{C_n}_{[1,j]}$, $1 \leq j < n$, be the Schützenberger–Lusztig involution on $\mathsf{KN}_{[1,j]}(\lambda, n)$. Notice that the unique crystal operators which change the signs of the entries are f_n and e_n, which are forgotten. Next we give a computation of $\xi^{C_n}_{[p,q]}$ for any $1 \leq p \leq q \leq n$, via virtualization E and bring it back to $\mathsf{KN}_{[p,q]}(\lambda, n)$ by applying E^{-1}. See Theorem 8.13 below for $1 \leq q < n$, and Theorem 8.14 otherwise.

8.4.1. Embedding of a partial symplectic Schützenberger–Lusztig involution and back. Let $J \subseteq [n]$ be a sub-Dynkin diagram of the type C_n Dynkin diagram $I = [n]$. Let U be a connected component of the Levi branched crystal $\mathsf{KN}_J(\lambda, n)$ with $J \subseteq [n]$ and highest and lowest weight elements u^{high} and u^{low}, respectively. Recall from Section 4.2, Proposition 4.3, that each connected component U of the Levi branched crystal $\mathsf{KN}_J(\lambda, n)$ is embedded via E into a connected component of the Levi branched crystal $\mathsf{SSYT}_{J\sqcup\overline{J}}(\lambda, n)$ with highest and lowest weight elements $E(u^{\mathsf{high}})$ and $E(u^{\mathsf{low}})$, respectively.

Let $P = (P^+, P^-) \in \mathsf{SSYT}(\lambda^A, n, \overline{n})$. The crystals

$$\mathsf{SSYT}_{[p,q]}(\lambda^A, n, \overline{n}) \quad \text{and} \quad \mathsf{SSYT}_{[\overline{q+1},\overline{p+1}]}(\lambda^A, n, \overline{n})$$

are isomorphic to $\mathsf{SSYT}_{[p,q]}(\lambda^A_+, n)$ respectively to $\mathsf{SSYT}_{[\overline{q+1},\overline{p+1}]}(\lambda^A/\lambda^A_+, \overline{n})$ (recall Remark 4.4). The corresponding pair of isomorphic crystals has the same multiset of highest weight vectors in $\mathbb{Z}^{q-p+1}$ respectively, regarding the sub-Dynkin diagram $[p, q]$. We may then write

$$\xi^{A_{2n-1}}_{[p,q]} \xi^{A_{2n-1}}_{[\overline{q+1},\overline{p+1}]}(P) = \left(\xi^{A_{n-1}}_{[p,q]}(P^+), \xi^{A_{n-1}}_{[\overline{q+1},\overline{p+1}]}(P^-)\right).$$

Theorem 8.13. *Let $T \in \mathsf{KN}_{[p,q]}(\lambda, n)$ of type A_{p-q+1}, $1 \leq p \leq q < n$. Then*

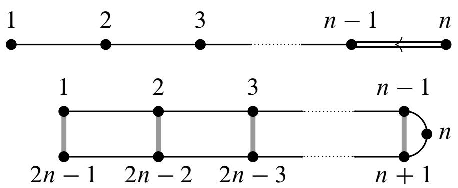
$$\xi^{A_{2n-1}}_{[p,q]\sqcup[\overline{q+1},\overline{p+1}]}(E(T)) = \xi^{A_{2n-1}}_{[p,q]} \xi^{A_{2n-1}}_{[\overline{q+1},\overline{p+1}]}(E(T)) = E\left(\xi^{C_n}_{[p,q]}(T)\right).$$

Moreover,

$$\xi^{C_n}_{[p,q]}(T) = E^{-1}\mathsf{reversal}^{A_{2n-1}}_{[p,q]}\mathsf{reversal}^{A_{2n-1}}_{[\overline{q+1},\overline{p+1}]}E(T).$$

Proof. Recall Proposition 4.3, Remark 4.4 and (4.2). Then the result follows from Theorem 7.6. ∎

It is now convenient to change the labeling of the A_{2n-1} Dynkin diagram. Instead of $[k, \overline{k+1}]$, we write $[k, 2n-k]$, and $\mathsf{SSYT}_{[k,2n-k]}(\lambda, n, \overline{n})$. This relabeling is illustrated in the picture below:



Theorem 8.14. *Let* $T \in \mathsf{KN}_{[k,n]}(\lambda, n)$ *of type* C_{n-k+1} *for some* $1 \le k \le n$*. Then*

$$\xi^{A_{2n-1}}_{[k,2n-k]}(E(T)) = E\big(\xi^{C_n}_{[k,n]}(T)\big).$$

Moreover, on $\mathsf{SSYT}(\lambda^A, n, \overline{n})$, $\xi^{A_{2n-1}}_{[k,2n-k]} = \mathsf{reversal}^{A_{2n-1}}_{[k,2n-k]}$*, and*

$$\xi^{C_n}_{[k,n]} = E^{-1}\mathsf{reversal}^{A_{2n-1}}_{[k,2n-k]}E. \tag{8.11}$$

Proof. Recall Corollary 6.1 and the fact that, in the case of the branched crystal $\mathsf{SSYT}_{[k,2n-k]}(\lambda, n, \overline{n})$, we have $\theta(i) = 2n - i$ for $i \in [k, 2n-k]$. Let U be the connected component of $\mathsf{KN}_{[k,n]}(\lambda, n)$ containing T, and let the highest and lowest weight elements of U be u^{high} and u^{low}, respectively.

$$\begin{aligned} T &= f_{i_r}\dots f_{i_1}(u^{\mathsf{high}}), & i_1,\dots,i_r &\in [k,n],\\ E(T) &= f^A_{i_r} f^A_{2n-i_r}\dots f^A_{i_1} f^A_{2n-i_1}\big(E(u^{\mathsf{high}})\big), & i_1,\dots,i_r &\in [k,n], \end{aligned} \tag{8.12}$$

and

$$\xi^{C_n}_{[k,n]}(T) = e_{i_r}\dots e_{i_1}(u^{\mathsf{low}}).$$

Then, from Section 4.1 and (4.1),

$$E\big(\xi^{C_n}_{[k,n]}(T)\big) = E\big(e_{i_r}\dots e_{i_1}(u^{\mathsf{low}})\big) = e^A_{i_r}e^A_{2n-i_r}\dots e^A_{i_1}e^A_{2n-i_1}E(u^{\mathsf{low}}),$$

and from (8.12),

$$\begin{aligned} \xi^A_{[k,2n-k]}(E(T)) &= e^A_{\theta(i_r)}e^A_{\theta(2n-i_r)}\dots e^A_{\theta(i_1)}e^A_{\theta(2n-i_1)}\big(E(u^{\mathsf{low}})\big)\\ &= e^A_{2n-i_r}e^A_{i_r}\dots e^A_{2n-i_1}e^A_{i_1}\big(E(u^{\mathsf{low}})\big)\\ &= e^A_{i_r}e^A_{2n-i_r}\dots e^A_{i_1}e^A_{2n-i_1}E(u^{\mathsf{low}}) = E\big(\xi^C_{[k,n]}(T)\big). \end{aligned}$$

Finally, (8.11) follows from (7.9). ∎

Using a generalized form of Lemma 7.2, the following corollary is a generalization of the two theorems above.

Corollary 8.15. *Let* $T \in \mathsf{KN}_{[p,q]\sqcup[k,n]}(\lambda, n)$ *of subtype* $A_{p-q+1} \times C_{n-k+1}$ *for some* $1 \le p \le q < k-1 < n$*. Then*

$$\begin{aligned} \xi^{A_{2n-1}}_{[p,q]\sqcup[2n-q,2n-p]\sqcup[k,2n-k]}(E(T)) &= \xi^{A_{2n-1}}_{[p,q]}\xi^{A_{2n-1}}_{[2n-q,2n-p]}\xi^{A_{2n-1}}_{[k,2n-k]}(E(T))\\ &= E\big(\xi^{C_n}_{[p,q]\sqcup[k,n]}(T)\big). \end{aligned}$$

Moreover, on $\mathsf{SSYT}(\lambda^A, n, \overline{n})$,

$$\begin{aligned} \xi^{A_{2n-1}}_{[p,q]} &= \mathsf{reversal}^{A_{2n-1}}_{[p,q]},\\ \xi^{A_{2n-1}}_{[k,2n-k]} &= \mathsf{reversal}^{A_{2n-1}}_{[k,2n-k]},\\ \xi^{A_{2n-1}}_{[2n-q,2n-p]} &= \mathsf{reversal}^{A_{2n-1}}_{[2n-q,2n-p]}. \end{aligned}$$

Remark 8.16. Both $\xi^{A_{2n-1}}_{[p,q]\sqcup[2n-q,2n-p]}$ and $\xi^{A_{2n-1}}_{[k,2n-k]}$ act on the set $\mathsf{SSYT}(\lambda^A, n, \overline{n})$ to define a graph automorphism of the underlying graph such that the subset $E(\mathsf{KN}(\lambda, n))$ is preserved. In other words, each of these involutions defines a graph automorphism of the underlying graph of $E(\mathsf{KN}(\lambda, n))$ when its action is restricted to this subset.

Corollary 8.17. *Let* $\mathsf{SSYT}(\mu, 2n)$ *with* μ *a partition with at most* $2n$ *parts, and let* B_n *be the Weyl group realized as*

$$\langle r_i = (i\ i+1)(2n-i\ 2n-i+1), r_n = (n\ n+1) : 1 \le i \le n-1 \rangle.$$

Then $\xi^{A_{2n-1}}_i \xi^{A_{2n-1}}_{2n-i}$, $\xi^{A_{2n-1}}_n$, $1 \le i \le n-1$, *define an action of* B_n *on* $\mathsf{SSYT}(\mu, 2n)$ *by* $r_i.b = \xi^{A_{2n-1}}_i \xi^{A_{2n-1}}_{2n-i}(b)$, $1 \le i \le n-1$, *and* $r_n.b = \xi^{A_{2n-1}}_n(b)$ *for* $b \in \mathsf{SSYT}(\mu, 2n)$ *such that*

1. $e^{A_{2n-1}}_i e^{A_{2n-1}}_{2n-i} \xi^{A_{2n-1}}_i \xi^{A_{2n-1}}_{2n-i} = \xi^{A_{2n-1}}_i \xi^{A_{2n-1}}_{2n-i} f^{A_{2n-1}}_i f^{A_{2n-1}}_{2n-i}$, $1 \le i < n$;
2. $e^{A_{2n-1}}_n e^{A_{2n-1}}_n \xi^{A_{2n-1}}_n = \xi^{A_{2n-1}}_n f^{A_{2n-1}}_n f^{A_{2n-1}}_n$;
3. $\mathsf{wt}(r_i.b) = r_i.\mathsf{wt}(b)$, $1 \le i \le n$;
4. $w_0.T_\mu = T^{\mathsf{low}}_\mu$, w_0 *the long element of* B_n;
5. *if* $\mu = \lambda^A$ *for some* λ, *it preserves the action of* $r_i \in B_n$ *on the* i*-strings,* $1 \le i \le n$, *of the crystal* $\mathsf{KN}(\lambda, n)$ *embedded in the crystal* $\mathsf{SSYT}(\lambda^A, 2n)$.

Proof. Recall that ξ_i, $1 \le i \le 2n-1$, define an action of $\mathfrak{S}_{2n}$ on $\mathsf{SSYT}(\mu, 2n)$. Thus the involutions $\xi^{A_{2n-1}}_i \xi^{A_{2n-1}}_{2n-i}$ and $\xi^{A_{2n-1}}_n$, $1 \le i < n-1$, satisfy the braid relations of B_n. The connected components of the crystal $\mathsf{B}_{\{i\}\sqcup\{2n-i\}}$, the restriction of $\mathsf{B} = \mathsf{SSYT}(\mu, 2n)$ to the Dynkin diagram $\{i\} \sqcup \{2n-i\}$ of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, are grids of rectangles where we have (1). It follows from Theorem 8.13 and Theorem 8.14 with $p = q$ and $k = n$, respectively, that

$$\xi^{A_{2n-1}}_p \xi^{A_{2n-1}}_{p+1}(E(T)) = \xi^{A_{2n-1}}_p \xi^{A_{2n-1}}_{2n-p}(E(T)) = E\big(\xi^{C_n}_p(T)\big)$$

and

$$\xi^{A_{2n-1}}_n(E(T)) = E\big(\xi^{C_n}_n(T)\big).$$

From Proposition 8.2 (2), the involutions $\xi^{C_n}_i$, $1 \le i \le n$, define an action of B_n on $\mathsf{KN}(\lambda, n)$. Therefore, $\xi^{A_{2n-1}}_i \xi^{A_{2n-1}}_{2n-i}, \xi^{A_{2n-1}}_n$, $1 \le i \le n-1$, are the translations of this action to the embedded crystal $E(\mathsf{KN}(\lambda, n))$ in $\mathsf{SSYT}(\lambda^A, 2n)$. This proves (5). ∎

8.5. Virtualization of the action of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ on the crystal $\mathsf{KN}(\lambda, n)$

We have the following commutative diagram corresponding to the crystal embedding E and the partial C_n and A_{2n-1} Schützenberger–Lusztig involutions, where

$[p,q] \subseteq [n-1]$ and $[p,n] \subseteq [n]$ are connected subintervals of the Dynkin diagram of C_n:

$$\begin{array}{ccc} \mathsf{KN}(\lambda,n) & \xrightarrow{E} & \mathsf{SSYT}(\lambda^A,n,\overline{n}) \\ \xi^{C_n}_{[p,n]} \Big\downarrow \xi^{C_n}_{[p,q]} & & \xi^{A_{2n-1}}_{[p,\overline{p+1}]} \Big\downarrow \xi^{A_{2n-1}}_{[p,q]\sqcup[\overline{q+1},\overline{p+1}]} \\ \mathsf{KN}(\lambda,n) & \xrightarrow{E} & \mathsf{SSYT}(\lambda^A,n,\overline{n}). \end{array}$$

Theorem 7.8 and Remark 8.16 imply that the action of $\widetilde{J}_{2n}$ on $\mathsf{SSYT}(\lambda^A,n,\overline{n})$ preserves the subset $E(\mathsf{KN}(\lambda,n))$, and thus, we then have an action of $\widetilde{J}_{2n}$ on the set $E(\mathsf{KN}(\lambda,n))$ defined by

$$\begin{aligned} \widetilde{\Phi}^E_{\mathfrak{sl}(2n,\mathbb{C})}\colon \widetilde{J}_{2n} &\to \mathfrak{S}_{E(\mathsf{KN}(\lambda,n))}, \\ \widetilde{s}_{[p,q]\sqcup[\overline{q+1},\overline{p+1}]} &\mapsto \xi^{A_{2n-1}}_{[p,q]\sqcup[\overline{q+1},\overline{p+1}]} = \xi^{A_{2n-1}}_{[p,q]}\xi^{A_{2n-1}}_{[\overline{q+1},\overline{p+1}]}, \\ \widetilde{s}_{[p,\overline{p+1}]} &\mapsto \xi^{A_{2n-1}}_{[p,\overline{p+1}]} \end{aligned} \tag{8.13}$$

such that

$$\widetilde{\Phi}^E_{\mathfrak{sl}(2n,\mathbb{C})}(\widetilde{s}_J) = \widetilde{\Phi}_{\mathfrak{sl}(2n,\mathbb{C})}(\widetilde{s}_J)_{|E(\mathsf{KN}(\lambda,n))} \in \mathfrak{S}_{E(\mathsf{KN}(\lambda,n))}.$$

Let $\widetilde{\iota}\colon J_{\mathfrak{sp}(2n,\mathbb{C})} \to \widetilde{J}_{2n}$ be the group isomorphism defined by $s_{[1,j]} \mapsto \widetilde{s}_{[1,j]\sqcup[\overline{j+1},\overline{2}]}$, $1 \le j < n$, and $s_{[j,n]} \mapsto \widetilde{s}_{[j,\overline{j+1}]}$, $1 \le j < n$, (see Proposition 5.11), and $\iota\colon \mathfrak{S}_{\mathsf{KN}(\lambda,n)} \to \mathfrak{S}_{E(\mathsf{KN}(\lambda,n))}$ the group isomorphism defined by $\iota(\sigma) = E\sigma E^{-1}$. The virtualization of the action of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ on the crystal $\mathsf{KN}(\lambda,n)$ is then realized from the following commutative diagram:

$$\begin{array}{ccc} J_{\mathfrak{sp}(2n,\mathbb{C})} & \xrightarrow{\Phi_{\mathfrak{sp}(2n,\mathbb{C})}} & \mathfrak{S}_{\mathsf{KN}(\lambda,n)} \\ \widetilde{\iota}\Big\downarrow & & \iota\Big\downarrow \widetilde{\Phi}^E_{\mathfrak{sl}(2n,\mathbb{C})}\widetilde{\iota}=\iota\Phi_{\mathfrak{sp}(2n,\mathbb{C})} \\ \widetilde{J}_{2n} & \xrightarrow{\widetilde{\Phi}^E_{\mathfrak{sl}(2n,\mathbb{C})}} & \mathfrak{S}_{E(\mathsf{KN}(\lambda,n))}. \end{array} \tag{8.14}$$

From (8.14),

$$\begin{aligned} \widetilde{\Phi}^E_{\mathfrak{sl}(2n,\mathbb{C})}\widetilde{\iota}(s_{[1,j]}) &= \widetilde{\Phi}^E_{\mathfrak{sl}(2n,\mathbb{C})}(\widetilde{s}_{[1,j]\sqcup[\overline{j+1},\overline{2}]}) = \xi^{A_{2n-1}}_{[1,j]\sqcup[\overline{j+1},\overline{2}]} \\ &= \iota\Phi_{\mathfrak{sp}(2n,\mathbb{C})}(s_{[1,j]}) = \iota\xi^{C_n}_{[1,j]} \\ &= E\xi^{C_n}_{[1,j]}E^{-1} = \xi^{A_{2n-1}}_{[1,j]\sqcup[\overline{j+1},\overline{2}]}, \end{aligned}$$

and

$$\begin{aligned} \widetilde{\Phi}^E_{\mathfrak{sl}(2n,\mathbb{C})}\widetilde{\iota}(s_{[j,n]}) &= \widetilde{\Phi}^E_{\mathfrak{sl}(2n,\mathbb{C})}(s_{[j,\overline{j+1}]}) = \xi^{A_{2n-1}}_{[j,\overline{j+1}]} \\ &= \iota\Phi_{\mathfrak{sp}(2n,\mathbb{C})}(s_{[j,n]}) = \iota\xi^{C_n}_{[j,n]} \\ &= E\xi^{C_n}_{[j,n]}E^{-1} = \xi^{A_{2n-1}}_{[j,\overline{j+1}]}. \end{aligned}$$

8.6. Virtualization example

Consider $n = 5$, $J = [1, 4]$ and the KN tableau T of shape $\lambda = \omega_4 + \omega_3$:

$$T = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \bar{5} \\ \hline \bar{4} & \bar{3} \\ \hline \bar{3} \\ \cline{1-1} \end{array}, \quad \mathrm{wt}(T) = (2, 0, -1, -1, -1).$$

Following the conventions in Section 4, and labeling the columns of T from right to left as C_1 and C_2, we obtain $E(T)$ with shape $\lambda^A = \omega_7 + \omega_3 + \omega_6 + \omega_4$:

$$\psi(C_1) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \bar{5} \\ \hline 4 & \bar{3} \\ \hline \bar{5} \\ \cline{1-1} \bar{4} \\ \cline{1-1} \bar{3} \\ \cline{1-1} \bar{2} \\ \cline{1-1} \end{array}, \psi(C_2) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 5 & \bar{4} \\ \hline \bar{5} & \bar{2} \\ \hline \bar{4} \\ \cline{1-1} \bar{3} \\ \cline{1-1} \end{array}$$

$$\Rightarrow E(T) = \left[\emptyset \leftarrow w(\psi(C_1)) \leftarrow w(\psi(C_2))\right] = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 4 & \bar{5} \\ \hline 3 & \bar{5} & \bar{4} & \bar{3} \\ \hline 5 & \bar{4} & \bar{2} \\ \cline{1-3} \bar{5} & \bar{3} \\ \cline{1-2} \bar{4} & \bar{2} \\ \cline{1-2} \bar{3} \\ \cline{1-1} \end{array}.$$

Considering the barred and unbarred parts of $E(T)$ separately, we compute the evacuation, evac, of the unbarred part and the reversal, reversal, of the barred part, yielding

$$\mathrm{evac} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 4 \\ \cline{1-3} 3 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 5 & 5 \\ \cline{1-3} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}, \quad \mathrm{reversal} \begin{array}{|c|c|c|c|}\hline * & * & * & * \\ \hline * & * & * & \bar{5} \\ \hline * & \bar{5} & \bar{4} & \bar{3} \\ \hline * & \bar{4} & \bar{2} \\ \cline{1-3} \bar{5} & \bar{3} \\ \cline{1-2} \bar{4} & \bar{2} \\ \cline{1-2} \bar{3} \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|}\hline * & * & * & * \\ \hline * & * & * & \bar{3} \\ \hline * & \bar{4} & \bar{2} & \bar{2} \\ \hline * & \bar{3} & \bar{1} \\ \cline{1-3} \bar{4} & \bar{2} \\ \cline{1-2} \bar{3} & \bar{1} \\ \cline{1-2} \bar{1} \\ \cline{1-1} \end{array}.$$

Putting these tableaux together, one obtains the A_9 tableau

$$\xi^{A_9}_{[1,4]\sqcup[\bar{5},\bar{2}]}(E(T)) = \left(\mathrm{evac}(E(T)^+), \mathrm{reversal}(E(T)^-)\right).$$

Using Q_λ to perform the reverse column Schensted insertion on $\xi^{A_9}_{[1,4]\sqcup[\bar{5},\bar{2}]}(E(T))$ provides the image under ψ of two KN columns C_1' and C_2'. Applying ψ^{-1} to each column results in

$$Q_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 11 & 15 \\ \hline 2 & 5 & 12 & 16 \\ \hline 3 & 6 & 13 & 17 \\ \hline 7 & 14 & 18 \\ \cline{1-3} 8 & 19 \\ \cline{1-2} 9 & 20 \\ \cline{1-2} 10 \\ \cline{1-1}\end{array} \Rightarrow \psi(C_1') = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 4 & \bar{3} \\ \hline 5 & \bar{2} \\ \hline \bar{4} \\ \cline{1-1} \bar{3} \\ \cline{1-1} \bar{2} \\ \cline{1-1} \bar{1} \\ \cline{1-1}\end{array}, \psi(C_2') = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline 5 & \bar{2} \\ \hline \bar{4} & \bar{1} \\ \hline \bar{3} \\ \cline{1-1} \bar{1} \\ \cline{1-1}\end{array}$$

$$\Rightarrow C_2'C_1' = \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 5 & \bar{3} \\ \hline \bar{3} & \bar{2} \\ \hline \bar{1} \\ \cline{1-1}\end{array} = \xi^C_{[1,4]}(T).$$

We note that $\mathrm{wt}(\xi^C_{[1,4]}(T)) = \mathrm{rev}(\mathrm{wt}(T)) = (-1,-1,-1,0,2)$.

9. The type C_n Berenstein–Kirillov group

9.1. The type A Berenstein–Kirillov group

The type *A Berenstein–Kirillov group* $\mathcal{BK}$ (or *Gelfand–Tsetlin group*) [31] is the free group generated by the Bender–Knuth involutions [5] t_i, $i > 0$, modulo the relations they satisfy on semi-standard Young tableaux of any (straight) shape.

Definition 9.1. The Bender–Knuth involution t_i, $i \geq 1$, is an operation that acts on a semi-standard tableau T of any shape (skew or straight) as follows:

- pairs $(i, i+1)$ within each column of T are considered fixed, and other occurrences of i's or $i+1$'s are considered free;
- if a row within T has k free i's followed by l free $i+1$'s, then we replace these letters by l free i's followed by k free $i+1$'s.

The t_i's have many known relations in $\mathcal{BK}$ [12, 31]:

$$t_i^2 = 1, \quad i \geq 1, \quad ([31, \text{Corollary 1.1}]), \tag{9.1}$$

$$t_i t_j = t_j t_i, \quad |i-j| > 1, \quad ([31, \text{Corollary 1.1}]), \tag{9.2}$$

$$\left(t_1 q_{[1,i]}\right)^4 = 1, \quad i > 2, \quad ([31, \text{Corollary 1.1}]), \tag{9.3}$$

$$(t_1 t_2)^6 = 1, \quad ([31, \text{Corollary 1.1}]), \tag{9.4}$$

$$\left(t_i q_{[j,k-1]}\right)^2 = 1, \quad i+1 < j < k, \quad ([12]), \tag{9.5}$$

where

$$q_{[1,i]} := t_1(t_2t_1)\cdots(t_it_{i-1}\cdots t_1), \quad i \geq 1, \tag{9.6}$$

$$q_{[j,k-1]} := q_{[1,k-1]}q_{[1,k-j]}q_{[1,k-1]}, \quad j < k. \tag{9.7}$$

Remark 9.2. (1) It is not known whether the latter set forms a complete set of relations [7].

(2) ([31, Section 2]) On straight-shaped semi-standard Young tableaux,

$$\begin{aligned} q_{[1,i]} &= \xi_{[1,i]}, \qquad i \geq 1, \\ q_{[j,k-1]} &= \xi_{[j,k-1]}, \quad j < k, \end{aligned} \tag{9.8}$$

and $q_{[j,j]} = q_{[1,j]}q_{[1,1]}q_{[1,j]}$ computes the crystal reflection operator $\xi_j = \xi_{[j,j]}$, where $q_{[1,1]} = \xi_{[1,1]} = t_1$ for $j \geq 1$. In particular, $q_{[1,i]} = \xi_{[1,i]} = \mathsf{evac}_{i+1}$, the evacuation restricted to the alphabet $[1, i+1]$, and $q_{[j,k-1]}$ computes the Schützenberger evacuation restricted to the alphabet $[j,k]$,

$$\xi_{[j,k-1]} = \mathsf{evac}_k\ \mathsf{evac}_{k-j+1}\ \mathsf{evac}_k, \quad j < k.$$

(3) Relation (9.5) implies that in particular, $(t_i\xi_j)^2 = 1$, $j > i+1$, which generalizes the relation $(t_1q_{[1,i]})^4 = 1$.

(4) For a generic (straight or skew) shaped semi-standard Young tableau T,

$$\mathsf{wt}(t_i(T)) = \mathsf{wt}(\xi_i(T)) = r_i.\mathsf{wt}(T), \quad r_i \in \mathfrak{S}_n \text{ for all } n \geq 1.$$

However, $t_i \neq \xi_i$ for $i > 1$; t_1 and ξ_1 need only coincide on straight shaped semi-standard Young tableaux, whereas t_i and ξ_i for $i > 1$ do not. Moreover, t_i, $1 \leq i < n$, do not need to satisfy the braid relations of $\mathfrak{S}_n$, however, they do on key tableaux, that is, straight shaped tableaux whose weight is a permutation of its shape [18].

Let $\mathcal{BK}_n$ be the subgroup of $\mathcal{BK}$ generated by $t_1, \ldots, t_{n-1}$.

Proposition 9.3 ([31, Remark 1.3]). *As elements of $\mathcal{BK}$,*

$$t_i = q_{[1,i-1]}q_{[1,i]}q_{[1,i-1]}q_{[1,i-2]}, \quad i \geq 1, q_{[1,0]} = q_{[1,-1]} := 1.$$

The elements $q_{[1,1]}, \ldots, q_{[1,n-1]}$ are generators of $\mathcal{BK}_n$.

The following result is both a consequence of the combinatorial action of the cactus group J_n via partial Schützenberger involutions $\xi_{[1,i]}$ on the straight-shape tableau crystal $\mathsf{SSYT}(\lambda, n)$, as defined by Halacheva [21], and the cactus group J_n relations satisfied by the generators $q_{[i,j]} = \xi_{[i,j]}$ of $\mathcal{BK}_n$ when acting on $\mathsf{SSYT}(\lambda, n)$, as studied by Chmutov, Glick and Pylyavskyy via the growth diagram approach [12].

Theorem 9.4. *The following are group epimorphisms from J_n to $\mathcal{BK}_n$:*

(1) $s_{[i,j]} \mapsto q_{[i,j]}$ ([12, Theorem 1.4])*;*

(2) $s_{[1,j]} \mapsto q_{[1,j]}$ *(*[31, Remark 1.3]*,* [21, Section 10.2]*,* [22, Remark 3.9]*).*

The group $\mathcal{BK}_n$ is isomorphic to a quotient of J_n. The generators $q_{[1,1]}, \dots, q_{[1,n-1]}$ of $\mathcal{BK}_n$ (and therefore $q_{[i,j]}$) satisfy the relations of J_n.

Remark 9.5. It follows from [12] that (9.4) is the only known relation which does not follow from the cactus group J_n relations. It is in fact equivalent to the braid relations satisfied by the crystal reflection operators $\xi_i = \xi_{[1,i]} t_1 \xi_{[1,i]}$, $1 \le i < n$, on a $U_q(\mathfrak{sl}(n, \mathbb{C}))$ crystal [31, Proposition 1.4], [43].

Remark 9.6. We may define the two dual sets of generators of $\mathcal{BK}_n$:

$$\widetilde{t}_{n-i} := q_{[1,n-1]} t_i q_{[1,n-1]}, \quad 1 \le i < n, \tag{9.9}$$

called dual Bender–Knuth involutions, and

$$\widetilde{q}_{[1,i]} := q_{[1,n-1]} q_{[1,i]} q_{[1,n-1]} = q_{[n-i,n-1]}, \quad 1 \le i < n, \tag{9.10}$$

for $\mathcal{BK}_n$. Indeed, from Proposition 9.3 and Theorem 9.4, one has

$$\widetilde{t}_{n-i} = q_{[n-i+1,n-1]} q_{[n-i,n-1]} q_{[n-i+1,n-1]} q_{[n-i+2,n-1]}, \quad 1 \le i < n, \tag{9.11}$$

with $q_{[n,n-1]} = q_{[n+1,n-1]} := 1$, and $\mathsf{wt}(\widetilde{t}_{n-i}(T)) = r_{n-i}.\mathsf{wt}(T)$ for $T \in \mathsf{SSYT}(\lambda, n)$ and $r_i \in \mathfrak{S}_n$, $i < n$.

The dual generators satisfy a list of relations similar to (9.1)–(9.5):

$$\widetilde{t}_{n-i}^2 = 1, \qquad i \ge 1, \tag{9.12}$$

$$\widetilde{t}_{n-i}\widetilde{t}_{n-j} = \widetilde{t}_{n-j}\widetilde{t}_{n-i}, \qquad |i-j| > 1, \tag{9.13}$$

$$(\widetilde{t}_{n-1}\widetilde{t}_{n-2})^6 = 1, \tag{9.14}$$

$$\begin{aligned}\left(\widetilde{t}_{n-i}\widetilde{q}_{[j,k-1]}\right)^2 &= \left(\widetilde{t}_{n-i} q_{[n-k+1,n-j]}\right)^2 \\ &= 1, \qquad n-k < n-j < n-i-1,\end{aligned} \tag{9.15}$$

where

$$\widetilde{q}_{[1,i]} = \widetilde{t}_{n-1}(\widetilde{t}_{n-2}\widetilde{t}_{n-1}) \cdots (\widetilde{t}_{n-i}\widetilde{t}_{n-i+1} \cdots \widetilde{t}_{n-1}), \quad i \ge 1, \tag{9.16}$$

$$\begin{aligned}\widetilde{q}_{[j,k-1]} &:= \widetilde{q}_{[1,k-1]}\widetilde{q}_{[1,k-j]}\widetilde{q}_{[1,k-1]} \\ &= q_{[n-k+1,n-j]} \\ &= q_{[n-k+1,n-1]} q_{[n-k+j,n-1]} q_{[n-k+1,n-1]}, \quad j < k.\end{aligned} \tag{9.17}$$

On $\mathsf{SSYT}(\lambda, n)$, $\widetilde{q}_{[j,k-1]} = \xi_{[n-k+1,n-j]}$.

Remark 9.7. We note some features of the operators (9.9) when acting on straight shaped semi-standard tableaux. Set $\mathsf{evac} := \mathsf{evac}_n$. Let $1 \le i < n$ and $T = (A, B) \in \mathsf{SSYT}(\lambda, n)$, where A of straight shape is the restriction of T to the alphabet $[1, n-i-1]$ and B, an extension of A, is the restriction of T to the alphabet $[n-i, n]$. One has

$$\mathsf{evac}(A, B) = \big(\mathsf{evac}\ \mathsf{rect}(B), X\big)$$

with X such that $\mathsf{rect}\,(X) = \mathsf{evac}\,(A)$. Therefore,

$$\begin{aligned}
\tilde{t}_{n-i}(T) = \tilde{t}_{n-i}(A, B) &= \mathsf{evac}\, t_i\, \mathsf{evac}(A, B) && \text{(by (9.9))}\\
&= \mathsf{evac}\, t_i\, (\mathsf{evac}\,\mathsf{rect}(B), X) && \text{(such that } \mathsf{rect}(X) = \mathsf{evac}(A))\\
&= \mathsf{evac}(t_i\, \mathsf{evac}\,\mathsf{rect}(B), X) &&\\
&= (\mathsf{evac}\,\mathsf{rect}(X), Z) &&\\
&= (A, Z) && \text{(such that}\\
& && \mathsf{rect}(Z) = \mathsf{evac}\, t_i\, \mathsf{evac}\,\mathsf{rect}(B)).
\end{aligned}$$

9.2. The type C_n Berenstein–Kirillov group and virtualization

Symplectic Bender–Knuth involutions $t_i^{C_n}$ are not known for KN tableaux. Motivated by the fact that, for $n \ge 1$, $q_{[1,1]}, \dots, q_{[1,n-1]}$ are generators for the Berenstein–Kirillov group $\mathcal{BK}_n$ in type A, and that on straight shaped semi-standard tableaux, they coincide with the action of the partial Schützenberger–Lusztig involutions $\xi_{[1,i]}$, $1 \le i < n$, we use the action of the partial Schützenberger–Lusztig involutions $\xi^{C_n}_{[1,i]}$, $1 \le i \le n-1$, and $\xi^{C_n}_{[i,n]}$, $1 \le i \le n$, on KN tableaux of any straight shape on the alphabet $\mathcal{C}_n$ to define the type C_n *Berenstein–Kirillov group* $\mathcal{BK}^{C_n}$.

Definition 9.8. Given $n \ge 1$, the *symplectic Berenstein–Kirillov group* $\mathcal{BK}^{C_n}$ is the free group generated by the $2n-1$ partial Schützenberger–Lusztig involutions

$$\begin{aligned}
q^{C_n}_{[1,i]} &:= \xi^{C_n}_{[1,i]}, \quad 1 \le i < n,\\
q^{C_n}_{[i,n]} &:= \xi^{C_n}_{[i,n]}, \quad 1 \le i \le n,
\end{aligned}$$

on straight shaped KN tableaux on the alphabet $\mathcal{C}_n$ modulo the relations they satisfy on those tableaux. We also define

$$q^{C_n}_{[1,-1]} = q^{C_n}_{[1,0]} = q^{C_n}_{[0,n]} = q^{C_n}_{[n+1,n]} := 1$$

and

$$q^{C_n}_{[j,k-1]} := q^{C_n}_{[1,k-1]} q^{C_n}_{[1,k-j]} q^{C_n}_{[1,k-1]}, \quad 1 \le j < k \le n.$$

Remark 9.9. Likewise in type A, Remark 9.2, (9.8), one also has in type C_n, thanks to Theorem 8.13,

$$q^{C_n}_{[j,k-1]} = \xi^{C_n}_{[j,k-1]} = E^{-1}\xi_{[j,k-1]}\xi_{[2n-k+1,2n-j]}E, \quad 1 \le j < k \le n.$$

For $1 \le j < k \le n$,

$$\begin{aligned}
q^{C_n}_{[j,k-1]} &= q^{C_n}_{[1,k-1]}q^{C_n}_{[1,k-j]}q^{C_n}_{[1,k-1]} = \xi^{C_n}_{[1,k-1]}\xi^{C_n}_{[1,k-j]}\xi^{C_n}_{[1,k-1]} && \text{(by Definition 9.8)}\\
&= E^{-1}\xi^{A_{2n-1}}_{[1,k-1]}\xi^{A_{2n-1}}_{[2n-k+1,2n-1]}\xi^{A_{2n-1}}_{[1,k-j]}\\
&\quad\cdot \xi^{A_{2n-1}}_{[2n-k+j,2n-1]}\xi^{A_{2n-1}}_{[1,k-1]}\xi^{A_{2n-1}}_{[2n-k+1,2n-1]}E && \text{(by Theorem 8.13)}\\
&= E^{-1}\xi^{A_{2n-1}}_{[1,k-1]}\xi^{A_{2n-1}}_{[1,k-j]}\xi^{A_{2n-1}}_{[1,k-1]}\\
&\quad\cdot \xi^{A_{2n-1}}_{[2n-k+1,2n-1]}\xi^{A_{2n-1}}_{[2n-k+j,2n-1]}\xi^{A_{2n-1}}_{[2n-k+1,2n-1]}E\\
&= E^{-1}\xi^{A_{2n-1}}_{[j,k-1]}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]}E\\
&\quad \text{(by Remark 9.2, (9.8), and Remark 9.6, (9.17), in type } A_{2n-1})\\
&= E^{-1}q_{[j,k-1]}q_{[2n-k+1,2n-j]}E\\
&\quad \text{(by Remark 9.2, (9.8), in type } A_{2n-1})\\
&= E^{-1}q_{[j,k-1]}\tilde{q}_{[j,k-1]}E && \text{(by (9.17) in type } A_{2n-1})\\
&= \xi^{C_n}_{[j,k-1]} && \text{(by Theorem 8.13).}
\end{aligned}$$

Thanks to Theorem 7.3, (7.7) and (7.8), one has therefore that $\mathcal{BK}^{C_n}$ is a quotient of $J_{\mathfrak{sp}(2n,\mathbb{C})}$. The generators of $\mathcal{BK}^{C_n}$ satisfy the cactus group $J_{\mathfrak{sp}(2n,\mathbb{C})}$ relations.

Theorem 9.10. *The following is a group epimorphism from $J_{\mathfrak{sp}(2n,\mathbb{C})}$ to $\mathcal{BK}^{C_n}$:*

$$s_{[1,j]} \mapsto q^{C_n}_{[1,j]}, \quad 1 \le j < n, \qquad s_{[j,n]} \mapsto q^{C_n}_{[j,n]}, \quad 1 \le j \le n.$$

Therefore, $\mathcal{BK}^{C_n}$ is isomorphic to a quotient of $J_{\mathfrak{sp}(2n,\mathbb{C})}$.

We next define symplectic Bender–Knuth involutions $t_i^{C_n}$, $1 \le i \le 2n-1$, on straight shaped KN tableaux that in turn generate $\mathcal{BK}^{C_n}$.

Definition 9.11. The $2n-1$ symplectic Bender–Knuth involutions $t_i^{C_n}$ on KN tableaux of straight shape on the alphabet $\mathcal{C}_n$ are defined by

$$\begin{aligned}
t_i^{C_n} &:= q^{C_n}_{[1,i-1]}q^{C_n}_{[1,i]}q^{C_n}_{[1,i-1]}q^{C_n}_{[1,i-2]}, && 1 \le i \le n-1,\\
t_{n-1+i}^{C_n} &:= q^{C_n}_{[n-i+1,n]}q^{C_n}_{[n-i+2,n]}, && 1 \le i \le n.
\end{aligned}$$

Thanks to the $J_{\mathfrak{sp}(2n,\mathbb{C})}$ relations satisfied by the generators of $\mathcal{BK}^{C_n}$,

$$q^{C_n}_{[j,j]} = q^{C_n}_{[1,j]}q^{C_n}_{[1,1]}q^{C_n}_{[1,j]} = q^{C_n}_{[1,j]}t_1^{C_n}q^{C_n}_{[1,j]} \tag{9.19}$$

(by Definition 9.8 with $j = k-1$) computes the symplectic crystal reflection operator $\xi_j^{C_n}$, $1 \le j \le n$, on KN tableaux (see Proposition 8.2 (2)).

Remark 9.12. The symplectic Bender–Knuth involutions $t_i^{C_n}$, $1 \le i \le n$, act on the elements of the set $\mathsf{KN}(\lambda, n)$ such that $\mathsf{wt}(t_i^{C_n}(T)) = \mathsf{wt}(\xi_i^{C_n}(T)) = r_i.\mathsf{wt}(T)$, $r_i \in B_n$, $1 \le i \le n$. This induces an action of the Weyl group B_n on the weights in $\mathbb{Z}^n$, although, as we shall see in Section 9.3, they do not define an action of the Weyl group B_n on the set $\mathsf{KN}(\lambda, n)$. Let $T \in \mathsf{KN}(\lambda, n)$ and $\mathsf{wt}(T) = (v_1, \dots, v_n) \in \mathbb{Z}^n$, then

$$\begin{aligned}
\mathsf{wt}\big(t_i^{C_n}(T)\big) &= r_i \mathsf{wt}(T), && 1 \le i < n,\\
\mathsf{wt}\big(t_n^{C_n}(T)\big) &= (v_1, \dots, -v_n) = r_n \mathsf{wt}(T), &&\\
\mathsf{wt}\big(t_{2n-i}^{C_n}(T)\big) &= (v_1, \dots, -v_i, \dots, v_n) &&\\
&= r_{n-1} \cdots r_{n-i} r_n r_{n-i} \cdots r_{n-1}(v_1, \dots, v_n) &&\\
&= \mathsf{wt}\big(t_{n-1}^{C_n} \cdots t_{n-i}^{C_n} t_n^{C_n} t_{n-i}^{C_n} \cdots t_{n-1}^{C_n}(T)\big), && 1 \le i < n.
\end{aligned}$$

Proposition 9.13. *The symplectic Bender–Knuth involutions $t_i^{C_n}$, $1 \le i \le 2n-1$, generate $\mathcal{BK}^{C_n}$. In particular,*

(1) $q_{[1,i]}^{C_n} = p_1^{C_n} p_2^{C_n} \cdots p_i^{C_n}$, $1 \le i < n$; *and*

(2) $q_{[n-i,n]}^{C_n} = t_{n+i}^{C_n} \cdots t_n^{C_n}$, $0 \le i \le n-1$,

where $p_i^{C_n} := t_i^{C_n} \cdots t_2^{C_n} t_1^{C_n}$, $1 \le i \le 2n-1$, is the symplectic promotion.

Proof. (1) We show by induction on i that $q_{[1,i]}^{C_n} = q_{[1,i-1]}^{C_n} p_i^{C_n}$. Note that

$$q_{[1,1]}^{C_n} = p_1^{C_n} = t_1^{C_n}.$$

Furthermore, for $i > 1$, we have

$$q_{[1,i]}^{C_n} = q_{[1,i-1]}^{C_n} t_i^{C_n} q_{[1,i-2]}^{C_n} q_{[1,i-1]}^{C_n}.$$

Assuming that for some fixed positive integer k, we have $q_{[1,j]}^{C_n} = q_{[1,j-1]}^{C_n} p_i^{C_n}$ for all $j \in [1, k-1]$, our inductive hypothesis implies

$$\begin{aligned}
q_{[1,k]}^{C_n} &= q_{[1,k-1]}^{C_n} t_k^{C_n} q_{[1,k-2]}^{C_n} q_{[1,k-1]}^{C_n}\\
&= q_{[1,k-1]}^{C_n} t_k^{C_n} q_{[1,k-2]}^{C_n} q_{[1,k-2]}^{C_n} p_{k-1}^{C_n}\\
&= q_{[1,k-1]}^{C_n} t_k^{C_n} p_{k-1}^{C_n}\\
&= q_{[1,k-1]}^{C_n} p_k^{C_n}.
\end{aligned}$$

(2) We proceed by induction on i. As a base case, when $i = 0$, we have $t_n^{C_n} = q_{[n,n]}^{C_n}$. As an inductive step, we assume the statement is true for all $j \in [0, k-1]$ for some

fixed positive integer $k < n-1$, so

$$t^{C_n}_{n+k} = q^{C_n}_{[n-k,n]} q^{C_n}_{[n-(k-1),n]} \Rightarrow t^{C_n}_{n+k} q^{C_n}_{[n-(k-1),n]} = q^{C_n}_{[n-k,n]}$$
$$\Rightarrow q^{C_n}_{[n-k,n]} = t^{C_n}_{n+k} t^{C_n}_{n+k-1} \cdots t^{C_n}_{n}. \qquad \blacksquare$$

Henceforth, t_i and $q_{[i,j]}$ will be denoted in $\mathcal{BK}_n$ by $t_i^{A_{n-1}}$ and $q_{[i,j]}^{A_{n-1}}$ to distinguish from the corresponding symplectic involutions. By Theorem 9.4, the involutions $q^{A_{2n-1}}_{[i,j]} \in \mathcal{BK}_{2n}$, $1 \le i \le j < 2n$, satisfy the cactus J_{2n} relations. Consider in $\mathcal{BK}_{2n}$ the involution $q^{A_{2n-1}}_{[1,i]}$ with its dual $\tilde{q}^{A_{2n-1}}_{[1,i]} := q^{A_{2n-1}}_{[2n-i,2n-1]}$, $1 \le i < n$ (Remark 9.6, (9.10)), and $q^{A_{2n-1}}_{[i,2n-i]}$, $1 \le i \le n$. Indeed,

$$q^{A_{2n-1}}_{[1,i]} \tilde{q}^{A_{2n-1}}_{[1,j]} = \tilde{q}^{A_{2n-1}}_{[1,j]} q^{A_{2n-1}}_{[1,i]}, \quad 1 \le i, j < n.$$

Definition 9.14. The *virtual symplectic Berenstein–Kirillov group* $\widetilde{\mathcal{BK}}_{2n}$ is the subgroup of $\mathcal{BK}_{2n}$ generated by the $2n-1$ involutions

$$\begin{aligned} q^{A_{2n-1}}_{[1,i]\sqcup[2n-i,2n-1]} &:= q^{A_{2n-1}}_{[1,i]} \tilde{q}^{A_{2n-1}}_{[1,i]} = \tilde{q}^{A_{2n-1}}_{[1,i]} q^{A_{2n-1}}_{[1,i]} \\ &= q^{A_{2n-1}}_{[1,i]} q^{A_{2n-1}}_{[2n-i,2n-1]} \\ &= q^{A_{2n-1}}_{[2n-i,2n-1]} q^{A_{2n-1}}_{[1,i]}, \qquad 1 \le i < n, \qquad (9.20a) \\ q^{A_{2n-1}}_{[i,2n-i]}, & \qquad 1 \le i \le n, \qquad (9.20b) \end{aligned}$$

modulo the relations they satisfy when acting upon semi-standard tableaux of any straight shape.

By Theorem 7.6, $q^{A_{2n-1}}_{[1,i]\sqcup[2n-i,2n-1]}$ coincides with $\xi^{A_{2n-1}}_{[1,i]\sqcup[2n-i,2n-1]}$ on semi-standard tableaux of any straight shape for $1 \le i < n$. (In particular, in $E(\mathsf{KN}(\lambda, n))$, for any partition λ with at most n parts.)

Bender–Knuth involutions $t_i^{A_{2n-1}}$ and dual Bender–Knuth involutions $\tilde{t}^{A_{2n-1}}_{2n-j}$, $1 \le i, j < n$, in $\mathcal{BK}_{2n}$ commute as the next lemma shows.

Lemma 9.15. *For* $1 \le i, j < n$, *we have* $(t_i^{A_{2n-1}} \tilde{t}^{A_{2n-1}}_{2n-j})^2 = 1$.

Proof. By Proposition 9.3, Remark 9.6, (9.11), in $\mathcal{BK}_{2n}$, Lemma 5.3 (2*A*), and Theorem 9.4, it follows that

$$\begin{aligned} t_i^{A_{2n-1}} \tilde{t}^{A_{2n-1}}_{2n-j} &= q^{A_{2n-1}}_{[1,i-1]} q^{A_{2n-1}}_{[1,i]} q^{A_{2n-1}}_{[1,i-1]} q^{A_{2n-1}}_{[1,i-2]} q^{A_{2n-1}}_{[2n-j+1,2n-1]} \\ &\quad \cdot q^{A_{2n-1}}_{[2n-j,2n-1]} q^{A_{2n-1}}_{[2n-j+1,2n-1]} q^{A_{2n-1}}_{[2n-j+2,2n-1]} \\ &= q^{A_{2n-1}}_{[2n-j+1,2n-1]} q^{A_{2n-1}}_{[2n-j,2n-1]} \\ &\quad \cdot q^{A_{2n-1}}_{[2n-j+1,2n-1]} q^{A_{2n-1}}_{[1,i-1]} q^{A_{2n-1}}_{[1,i]} q^{A_{2n-1}}_{[1,i-1]} q^{A_{2n-1}}_{[1,i-2]} \\ &= \tilde{t}^{A_{2n-1}}_{2n-j} t_i^{A_{2n-1}}, \quad 1 \le i, j < n. \end{aligned} \qquad \blacksquare$$

Proposition 9.16. *For $1 \leq i < n$, consider the Bender–Knuth involution $t_i^{A_{2n-1}}$ with its dual $\tilde{t}_{2n-i}^{A_{2n-1}}$ in $\mathcal{BK}_{2n}$. The group $\widetilde{\mathcal{BK}}_{2n}$ also has the $2n-1$ involution generators:*

$$t_{\{i\}\sqcup\{2n-i\}}^{A_{2n-1}} := t_i^{A_{2n-1}}\tilde{t}_{2n-i}^{A_{2n-1}} = \tilde{t}_{2n-i}^{A_{2n-1}} t_i^{A_{2n-1}}, \quad 1 \leq i < n, \tag{9.21}$$

$$\begin{aligned} t_{[n-i+1,n+i]}^{A_{2n-1}} &:= q_{[n-i+1,n+i-1]}^{A_{2n-1}} q_{[n-i+2,n+i-2]}^{A_{2n-1}} \\ &= q_{[n-i+2,n+i-2]}^{A_{2n-1}} q_{[n-i+1,n+i-1]}^{A_{2n-1}}, \quad 1 \leq i \leq n, \end{aligned} \tag{9.22}$$

where $q_{[n+1,n-1]}^{A_{2n-1}} := 1$. We call them the virtual symplectic Bender–Knuth involutions.

Proof. The group $\mathcal{BK}_{2n}$ satisfies the J_{2n} relations and $\widetilde{\mathcal{BK}}_{2n} \subseteq \mathcal{BK}_{2n}$. Hence, by Lemma 5.3 (3$A$) and Theorem 9.4, one finds

$$q_{[n-i+1,n+i-1]}^{A_{2n-1}} q_{[n-i+2,n+i-2]}^{A_{2n-1}} = q_{[n-i+2,n+i-2]}^{A_{2n-1}} q_{[n-i+1,n+i-1]}^{A_{2n-1}}, \quad 1 \leq i \leq n.$$

The identity $t_i^{A_{2n-1}}\tilde{t}_{2n-i}^{A_{2n-1}} = \tilde{t}_{2n-i}^{A_{2n-1}} t_i^{A_{2n-1}}$, $1 \leq i < n$, follows from Lemma 9.15 with $i = j$. Considering Definition 9.14, (9.20a), for $1 \leq i < n$, we have

$$\begin{aligned} q_{[1,i]\sqcup[2n-i,2n-1]}^{A_{2n-1}} &= q_{[1,i]}^{A_{2n-1}} q_{[2n-i,2n-1]}^{A_{2n-1}} \\ &= p_1^{A_{2n-1}} \cdots p_i^{A_{2n-1}} q_{[1,2n-1]}^{A_{2n-1}} p_1^{A_{2n-1}} \cdots \\ &\qquad \cdots p_i^{A_{2n-1}} q_{[1,2n-1]}^{A_{2n-1}} \qquad \text{(by (9.6) and (9.7))} \\ &= p_1^{A_{2n-1}} \cdots p_i^{A_{2n-1}} \tilde{p}_{2n-1}^{A_{2n-1}} \cdots \tilde{p}_{2n-i}^{A_{2n-1}} \\ &\quad \text{(since } q_{[1,2n-1]}^{A_{2n-1}} \text{ is an involution)} \\ &= t_1^{A_{2n-1}}\tilde{t}_{2n-1}^{A_{2n-1}} \big(t_2^{A_{2n-1}}\tilde{t}_{2n-2}^{A_{2n-1}} t_1^{A_{2n-1}}\tilde{t}_{2n-1}^{A_{2n-1}}\big) \cdots \\ &\qquad \cdots \big(t_i^{A_{2n-1}}\tilde{t}_{2n-i}^{A_{2n-1}} \cdots t_2^{A_{2n-1}}\tilde{t}_{2n-2}^{A_{2n-1}} t_1^{A_{2n-1}}\tilde{t}_{2n-1}^{A_{2n-1}}\big) \\ &\quad \text{(by Lemma 9.15)} \\ &= t_{\{1\}\sqcup\{2n-1\}}^{A_{2n-1}} \big(t_{\{2\}\sqcup\{2n-2\}}^{A_{2n-1}} t_{\{1\}\sqcup\{2n-1\}}^{A_{2n-1}}\big) \cdots \\ &\qquad \cdots \big(t_{\{i\}\sqcup\{2n-i\}}^{A_{2n-1}} \cdots t_{\{2\}\sqcup\{2n-2\}}^{A_{2n-1}} t_{\{1\}\sqcup\{2n-1\}}^{A_{2n-1}}\big) \ \text{(by (9.21))}, \end{aligned}$$

where $q_{[1,i]}^{A_{2n-1}} = p_1^{A_{2n-1}} \cdots p_i^{A_{2n-1}}$ with

$$p_i^{A_{2n-1}} := t_i^{A_{2n-1}} \cdots t_2^{A_{2n-1}} t_1^{A_{2n-1}}$$

and

$$\begin{aligned} \tilde{p}_{2n-i}^{A_{2n-1}} &:= q_{[1,2n-1]}^{A_{2n-1}} p_i^{A_{2n-1}} q_{[1,2n-1]}^{A_{2n-1}} \\ &= \tilde{t}_{2n-i}^{A_{2n-1}} \cdots \tilde{t}_{2n-2}^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}}, \quad 1 \leq i < n. \end{aligned}$$

On the other hand, considering Definition 9.14, (9.20b), for $1 \leq i \leq n$, we have

$$\begin{aligned} q^{A_{2n-1}}_{[n-i+1,n+i-1]} &= q^{A_{2n-1}}_{[n,n]}\big(q^{A_{2n-1}}_{[n,n]} q^{A_{2n-1}}_{[n-1,n+1]}\big)\big(q^{A_{2n-1}}_{[n-1,n+1]} q^{A_{2n-1}}_{[n-2,n+2]}\big)\cdots \\ &\quad \cdots \big(q^{A_{2n-1}}_{[n-(i-2),n+i-2]} q^{A_{2n-1}}_{[n-i+1,n+i-1]}\big) \\ &\quad \text{(since } q^{A_{2n-1}}_{[n-i,n+i]} \text{ is an involution)} \\ &= t^{A_{2n-1}}_{[n,n+1]} t^{A_{2n-1}}_{[n-1,n+2]} t^{A_{2n-1}}_{[n-2,n+3]} \cdots \\ &\quad \cdots t^{A_{2n-1}}_{[n-i+2,n+i-1]} t^{A_{2n-1}}_{[n-i+1,n+i]} \qquad \text{(by (9.22))}, \end{aligned}$$

concluding the proof. ∎

Remark 9.17. If T is an A_{2n-1} semi-standard tableau,

$$\mathsf{wt}\big(t^{A_{2n-1}}_{\{i\}\sqcup\{2n-i\}}(T)\big) = r_i r_{2n-i}.\mathsf{wt}(T),$$

where $r_i = (i, i+1)$ and $r_{2n-i} = (2n-i, 2n-i+1)$ are simple transpositions in $\mathfrak{S}_{2n}$ for $1 \leq i < n$, and

$$\mathsf{wt}\big(t_{[n-i+1,n+i]}(T)\big) = (n-i+1, n+i)\mathsf{wt}(T),$$

where $(n-i+1, n+i)$ is the transposition of $\mathfrak{S}_{2n}$ that swaps $n-i+1$ and $n+i$ for $1 \leq i \leq n$.The virtual symplectic Bender–Knuth involutions $t^{A_{2n-1}}_{\{i\}\sqcup\{2n-i\}}$, $1 \leq i < n$, and $t^{A_{2n-1}}_{[n,n+1]}$ act on the elements in $\mathsf{SSYT}(\lambda, 2n)$, inducing an action of the Weyl group B_n, realized as $\langle (i, i+1)(2n-i, 2n-i+1), (n, n+1) : 1 \leq i < n \rangle$, on the weights in $\mathbb{Z}^{2n}$.

Thanks to Theorem 7.8, we have that $\widetilde{\mathcal{BK}}_{2n}$ is a quotient of the virtual symplectic cactus $\widetilde{J}_{2n}$. The generators (9.20a) and (9.20b) of the group $\widetilde{\mathcal{BK}}_{2n}$ satisfy the relations of the cactus group $\widetilde{J}_{2n}$ or equivalently those of the cactus group $J_{\mathfrak{sp}(2n,\mathbb{C})}$.

Theorem 9.18. *The following is a group epimorphism from $\widetilde{J}_{2n}$ to $\widetilde{\mathcal{BK}}_{2n}$:*

$$\begin{aligned} \widetilde{s}_{[1,j]\sqcup[2n-j,2n-1]} &\mapsto q^{A_{2n-1}}_{[1,j]\sqcup[2n-j,2n-1]}, & 1 \leq j < n, \\ \widetilde{s}_{[j,2n-j]} &\mapsto q^{A_{2n-1}}_{[j,2n-j]}, & 1 \leq j \leq n. \end{aligned}$$

The group $\widetilde{\mathcal{BK}}_{2n}$ is isomorphic to a quotient of $\widetilde{J}_{2n}$, and via the isomorphism between $J_{\mathfrak{sp}(2n,\mathbb{C})}$ and $\widetilde{J}_{2n}$ defined via

$$\begin{aligned} s_{[1,j]} &\mapsto \widetilde{s}_{[1,j]\sqcup[2n-j,2n-1]}, & 1 \leq j < n, \\ s_{[j,n]} &\mapsto \widetilde{s}_{[j,2n-j]}, & 1 \leq j \leq n, \end{aligned}$$

is also isomorphic to a quotient of $J_{\mathfrak{sp}(2n,\mathbb{C})}$.

Since the action of $\widetilde{J}_{2n}$ on the set $\mathsf{SSYT}(\lambda^A, n, \overline{n})$ preserves the subset $E(\mathsf{KN}(\lambda, n))$ (see Remark 8.16), we now relate the virtual symplectic and symplectic Bender–Knuth involutions by embedding the crystal $\mathsf{KN}(\lambda, n)$ into the crystal $\mathsf{SSYT}(\lambda^A, n, \overline{n})$.

Theorem 9.19. *The symplectic Bender–Knuth involutions $t_i^{C_n}$, $1 \le i \le 2n-1$, in the symplectic Berenstein–Kirillov group $\mathcal{BK}^{C_n}$ can be realized by the virtual symplectic Bender–Knuth involutions $t_i^{A_{2n-i}} \tilde{t}_{2n-i}^{A_{2n-i}}$, $1 \le i < n$, and $t_{[n-i+1,n+i]}^{A_{2n-1}}$, $1 \le i \le n$, in $\widetilde{\mathcal{BK}}_{2n}$, and vice-versa,*

$$t_i^{C_n} = E^{-1} t_i^{A_{2n-1}} \tilde{t}_{\overline{i+1}}^{A_{2n-1}} E = E^{-1} t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} E, \quad 1 \le i < n, \tag{9.23}$$

$$t_{n+i-1}^{C_n} = E^{-1} t_{[n-i+1,n+i]}^{A_{2n-1}} E, \quad 1 \le i \le n. \tag{9.24}$$

In particular, the map $\mathcal{BK}^{C_n} \to \widetilde{\mathcal{BK}}_{2n}$ defined on generators by

$$t_i^{C_n} \mapsto t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-i}}, \quad 1 \le i < n, \tag{9.25}$$

$$t_{n+i-1}^{C_n} \mapsto t_{[n-i+1,n+i]}^{A_{2n-1}}, \quad 1 \le i \le n, \tag{9.26}$$

is an isomorphism of groups.

Proof. For $1 \le i < n$, we have

$$\begin{aligned} t_i^{C_n} &= q_{[1,i-1]}^{C_n} q_{[1,i]}^{C_n} q_{[1,i-1]}^{C_n} q_{[1,i-2]}^{C_n} \qquad \text{(by Definition 9.11)} \\ &= E^{-1}\big(\xi_{[1,i-1]}^{A_{2n-1}} \xi_{[\overline{i},\overline{2}]}^{A_{2n-1}}\big) E E^{-1}\big(\xi_{[1,i]}^{A_{2n-1}} \xi_{[\overline{i+1},\overline{2}]}^{A_{2n-1}}\big) E \\ &\quad \cdot E^{-1}\big(\xi_{[1,i-1]}^{A_{2n-1}} \xi_{[\overline{i},\overline{2}]}^{A_{2n-1}}\big) E E^{-1}\big(\xi_{[1,i-2]}^{A_{2n-1}} \xi_{[\overline{i-1},\overline{2}]}^{A_{2n-1}}\big) E \\ &\quad \text{(by Definition 9.8 and Theorem 8.13)} \\ &= E^{-1}\big(\xi_{[1,i-1]}^{A_{2n-1}} \xi_{[1,i]}^{A_{2n-1}} \xi_{[1,i-1]}^{A_{2n-1}} \xi_{[1,i-2]}^{A_{2n-1}} \xi_{[\overline{i},\overline{2}]}^{A_{2n-1}} \xi_{[\overline{i+1},\overline{2}]}^{A_{2n-1}} \xi_{[\overline{i},\overline{2}]}^{A_{2n-1}} \xi_{[\overline{i-1},\overline{2}]}^{A_{2n-1}}\big) E \\ &= E^{-1}\big(t_i^{A_{2n-1}} \tilde{t}_{\overline{i+1}}^{A_{2n-1}}\big) E. \end{aligned}$$

By Definition 9.11, we have

$$t_{n+i-1}^{C_n} = q_{[n-i+1,n]}^{C_n} q_{[n-i+2,n]}^{C_n}, \quad 2 \le i \le n,$$

and

$$t_n^{C_n} = q_{[n,n]}^{C_n} = \xi_n^{C_n}.$$

Then the result follows from Theorem 8.14 and Proposition 9.16.

Note that the map defined by (9.25) and (9.26) is an isomorphism as a consequence, since the description (9.23) and (9.24) of the generators of $\mathcal{BK}^{C_n}$ in terms of those of $\widetilde{\mathcal{BK}}_{2n}$ implies that both sets of generators will satisfy precisely the same relations. ∎

9.3. Symplectic Bender–Knuth involutions and the character of a KN tableau crystal

In the C_2 Weyl group $B_2 = \langle r_1, r_2 : r_i^2 = 1, (r_1 r_2)^4 = 1\rangle$ with long element $r_2 r_1 r_2 r_1$, the C_2 symplectic Bender–Knuth involutions are

$$t_1^{C_2} = \xi_1^{C_2}, \quad t_2^{C_2} = \xi_2^{C_2}, \quad t_3^{C_2} = \xi_{[1,2]}^{C_2}\xi_2^{C_2} = \xi_2^{C_2}\xi_{[1,2]}^{C_2};$$

therefore, in this case, $t_1^{C_2}$ and $t_2^{C_2}$ define an action of the Weyl group B_2 on $\mathsf{KN}(\lambda, 2)$. However, in general, for $n \geq 3$, the symplectic Bender–Knuth involutions $t_1^{C_n}, \dots, t_n^{C_n}$ do not define an action of the Weyl group B_n on the set $\mathsf{KN}(\lambda, n)$. Recall that the first $n-1$ generators of the Weyl group B_n satisfy the braid relations (2.3) of $\mathfrak{S}_n$, but we claim that, in general, $t_i^{C_n} t_{i+1}^{C_n} t_i^{C_n} \neq t_{i+1}^{C_n} t_i^{C_n} t_{i+1}^{C_n}$, i.e., $(t_i^{C_n} t_{i+1}^{C_n})^3 \neq 1$ for $1 \leq i < n$. To show this inequality, note that by Theorem 9.19, it is enough to consider the virtual symplectic Bender–Knuth involutions and the corresponding virtual inequality

$$\begin{aligned} &t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} t_{i+1}^{A_{2n-1}} \tilde{t}_{2n-(i+1)}^{A_{2n-1}} t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} \\ &\qquad \neq t_{i+1}^{A_{2n-1}} \tilde{t}_{2n-(i+1)}^{A_{2n-1}} t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} t_{i+1}^{A_{2n-1}} \tilde{t}_{2n-(i+1)}^{A_{2n-1}}. \end{aligned} \tag{9.27}$$

From Proposition 9.3 and Remark 9.6, $t_i^{A_{2n-1}} \tilde{t}_{2n-(i+1)}^{A_{2n-1}} = \tilde{t}_{2n-(i+1)}^{A_{2n-1}} t_i^{A_{2n-1}}$, $1 \leq i < n$. If we had equality in (9.27), then

$$\begin{aligned} &\tilde{t}_{2n-i}^{A_{2n-1}} \tilde{t}_{2n-(i+1)}^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} t_i^{A_{2n-1}} t_{i+1}^{A_{2n-1}} t_i^{A_{2n-1}} \\ &\qquad = \tilde{t}_{2n-(i+1)}^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} \tilde{t}_{2n-(i+1)}^{A_{2n-1}} t_{i+1}^{A_{2n-1}} t_i^{A_{2n-1}} t_{i+1}^{A_{2n-1}} \\ &\Leftrightarrow \left(\tilde{t}_{2n-(i+1)}^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}}\right)^3 = \left(t_i^{A_{2n-1}} t_{i+1}^{A_{2n-1}}\right)^3. \end{aligned}$$

Applying this identity to the A_9 tableau $E(T) = (P^+, P^-)$ in the virtualization Example 8.6 would imply that

$$\begin{aligned} &(\tilde{t}_6^{A_{11}} \tilde{t}_5^{A_{11}})^3(E(T)) = (t_4^{A_{11}} t_5^{A_{11}})^3(E(T)) \\ &\quad \Leftrightarrow \left(P^+, (\tilde{t}_6^{A_{11}} \tilde{t}_5^{A_{11}})^3(P^-)\right) = \left((t_4^{A_{11}} t_5^{A_{11}})^3(P^+), P^-\right), \end{aligned} \tag{9.28}$$

but this is impossible as $(t_2^{A_9} t_3^{A_9})^3(P^+) \neq P^+$. Note that the left-hand side of (9.28) follows from $\tilde{t}_{2n-i}(P^+, P^-) = \mathsf{evac}\ t_i\ \mathsf{evac}(P^+, P^-)$ and Remark 9.7.

Though the symplectic Bender–Knuth involutions do not define an action of B_n on $\mathsf{KN}(\lambda, n)$, they can be used to show that the character of the crystal $\mathsf{KN}(\lambda, n)$ is a symmetric Laurent polynomial with respect to the action of B_n. Let $\mathcal{E} := \mathbb{Z}[x_1^{\pm}, \dots, x_n^{\pm}]$ be the ring of Laurent polynomials in n variables over $\mathbb{Z}$, and let

$$\mathcal{E}^{B_n} = \{f \in \mathcal{E} : r_i.f = f, r_i \in B_n, 1 \leq i \leq n\},$$

where $r_i.x^\alpha := x^{r_i.\alpha}$, for $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha \in \mathbb{Z}^n$ and $r_i \in B_n$, be the subring of symmetric Laurent polynomials.

The character of $\mathsf{KN}(\lambda, n)$ is the symplectic Schur function $\mathrm{sp}_\lambda(x)$ in the sequence of variables $x = (x_1, \ldots, x_n)$. Thanks to Remark 9.12, $\mathsf{wt}(t_i^{C_n}.b) = r_i.\mathsf{wt}(b)$ for any $b \in \mathsf{KN}(\lambda, n)$ and $1 \le i \le n$. Therefore, since $t_i^{C_n}$, $1 \le i \le n$, is an involution on the set $\mathsf{KN}(\lambda, n)$, we obtain a proof that $\mathrm{sp}_\lambda(x)$ is a symmetric Laurent polynomial:

$$\begin{aligned}\mathrm{sp}_\lambda(x) &= \sum_{b \in \mathsf{KN}(\lambda,n)} x^{\mathsf{wt}(b)} = \sum_{b \in \mathsf{KN}(\lambda,n)} x^{\mathsf{wt}(t_i^{C_n} b)} \\ &= \sum_{b \in \mathsf{KN}(\lambda,n)} x^{r_i.\mathsf{wt}(b)} = \mathrm{sp}_\lambda(r_i.x), \quad 1 \le i \le n.\end{aligned}$$

9.4. Relations for the symplectic Berenstein–Kirillov group

Thanks to Theorems 9.10 and 9.18, we now provide the following relations for $\mathcal{BK}^{C_n}$, equivalently $\widetilde{\mathcal{BK}}_{2n}$. The relations (9) and (10) below are the only ones known for $\mathcal{BK}^{C_n}$, equivalently $\widetilde{\mathcal{BK}}_{2n}$, which do not follow from the cactus group $J_{\mathfrak{sp}(2n,\mathbb{C})}$ relations equivalently the virtual cactus group $\widetilde{J}_{2n}$ relations (see also Remark 9.21).

Proposition 9.20. *The symplectic Bender–Knuth involutions $t_i^{C_n}$, $i = 1, \ldots, 2n-1$, satisfy the following relations:*

(1) $(t_i^{C_n})^2 = 1$, $i = 1, \ldots, 2n-1$;
(2) $(t_{n+i-1}^{C_n} t_{n+j-1}^{C_n})^2 = 1$, $1 \le i, j \le n$;
(3) $(t_i^{C_n} t_j^{C_n})^2 = 1$, $|i-j| > 1$, $1 \le i, j < n$;
(4) $(t_i^{C_n} t_{n+j-1}^{C_n})^2 = 1$, $i < n-j$;
(5) $(t_i^{C_n} q_{[j,k-1]}^{C_n})^2 = 1$, $i+1 < j < k \le n$;
(6) $(t_i^{C_n} q_{[j,n]}^{C_n})^2 = 1$, $i+1 < j \le n$;
(7) $(t_{n+i-1}^{C_n} q_{[j,n]}^{C_n})^2 = 1$, $1 \le i, j \le n$;
(8) $(t_{n+i-1}^{C_n} q_{[j,k-1]}^{C_n})^2 = 1$, $n-i+1 < j < k \le n$;
(9) $(t_1^{C_n} t_2^{C_n})^6 = 1$, $n \ge 3$;
(10) $(t_{n-1}^{C_n} \cdots t_2^{C_n} t_1^{C_n} t_2^{C_n} \cdots t_{n-1}^{C_n} t_n^{C_n})^4 = 1$.

The virtual symplectic Bender–Knuth involutions

$$\begin{aligned} t_i^{A_{2n-i}} \tilde{t}_{2n-i}^{A_{2n-i}} &= E t_i^{C_n} E^{-1}, && 1 \le i < n, \\ t_{[n-i+1,n+i]}^{A_{2n-1}} &= E t_{n-i+1}^{C_n} E^{-1}, && 1 \le i \le n, \quad \text{in } \widetilde{\mathcal{BK}}_{2n}, \end{aligned}$$

satisfy the same relations as those of $\mathcal{BK}^{C_n}$ by replacing $t_i^{C_n}$ by $t_i^{A_{2n-i}} \tilde{t}_{2n-i}^{A_{2n-i}}$, $1 \le i < n$, and $t_{n+i-1}^{C_n}$ by $t_{[n-i+1,n+i]}^{A_{2n-1}}$, $1 \le i \le n$.

Proof. Recall Definitions 9.8 and 9.11, and Theorems 9.10, 9.18 and 9.19.

(1) $(t_1^{C_n})^2 = (q_{[1,1]}^{C_n})^2 = 1$, $(t_n^{C_n})^2 = (q_{[1,n]}^{C_n})^2 = 1$. For $2 \le i \le n$, we have that

$$\big(t_{n-1+i}^{C_n}\big)^2 = \big(q_{[n-i+1,n]}^{C_n} q_{[n-i+2,n]}^{C_n}\big)^2 = 1$$

is equivalent to the $J_{\mathfrak{sp}(2n,\mathbb{C})}$ relation (3C.i) of Lemma 5.5.

For $2 \le i \le n-1$, we observe that

$$(t_i^{C_n})^2 = q_{[1,i-1]}^{C_n} q_{[1,i]}^{C_n} q_{[1,i-1]}^{C_n} q_{[1,i-2]}^{C_n} q_{[1,i-1]}^{C_n} q_{[1,i]}^{C_n} q_{[1,i-1]}^{C_n} q_{[1,i-2]}^{C_n} = 1$$

follows from the cactus relation $J_{\mathfrak{sp}(2n,\mathbb{C})}$ of Lemma 5.5 (3C (ii)), and the observations that

$$\begin{aligned} q_{[1,i-1]}^{C_n} q_{[1,i-2]}^{C_n} &= q_{[2,i-1]}^{C_n} q_{[1,i-1]}^{C_n}, \\ q_{[1,i-1]}^{C_n} q_{[2,i-1]}^{C_n} &= q_{[1,i-2]}^{C_n} q_{[1,i-1]}^{C_n}, \\ q_{[1,i]}^{C_n} q_{[2,i-1]}^{C_n} &= q_{[2,i-1]}^{C_n} q_{[1,i]}^{C_n}. \end{aligned}$$

(2) Let $i \neq j$. From Lemma 5.5 (3C (i)),

$$\begin{aligned} t_{n+i-1}^{C_n} t_{n+j-1}^{C_n} &= q_{[n-i+1,n]}^{C_n} q_{[n-i+2,n]}^{C_n} q_{[n-j+1,n]}^{C_n} q_{[n-j+2,n]}^{C_n} \\ &= q_{[n-j+1,n]}^{C_n} q_{[n-j+2,n]}^{C_n} q_{[n-i+1,n]}^{C_n} q_{[n-i+2,n]}^{C_n} \\ &= t_{n+j-1}^{C_n} t_{n+i-1}^{C_n}. \end{aligned}$$

(3) Recall Theorem 9.19, (9.2), Remark 9.6, (9.13), and Lemma 9.15. Then

$$\begin{aligned} \big(t_i^{C_n} t_j^{C_n}\big)^2 &= \big(E^{-1} t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} E E^{-1} t_j^{A_{2n-1}} \tilde{t}_{2n-j}^{A_{2n-1}} E\big)^2 \\ &= E^{-1} \big(t_i^{A_{2n-1}} t_j^{A_{2n-1}}\big)^2 \big(\tilde{t}_{2n-i}^{A_{2n-1}} \tilde{t}_{2n-j}^{A_{2n-1}}\big)^2 E \\ &= 1, \quad |i-j| > 1,\ 1 \le i, j \le n-1. \end{aligned}$$

(4) For $i < n-j$, due to the $J_{\mathfrak{sp}(2n,\mathbb{C})}$ relation (2C), we have

$$\begin{aligned} t_i^{C_n} t_{n+j-1}^{C_n} &= q_{[1,i-1]}^{C_n} q_{[1,i]}^{C_n} q_{[1,i-1]}^{C_n} q_{[1,i-2]}^{C_n} q_{[n-j+1,n]}^{C_n} q_{[n-j+2,n]}^{C_n} \\ &= q_{[n-j+1,n]}^{C_n} q_{[n-j+2,n]}^{C_n} q_{[1,i-1]}^{C_n} q_{[1,i]}^{C_n} q_{[1,i-1]}^{C_n} q_{[1,i-2]}^{C_n} \\ &= t_{n+j-1}^{C_n} t_i^{C_n}. \end{aligned}$$

(5) Recall Theorems 8.13 and 9.19, Definition 9.8, (9.5), and Remark 9.6, (9.15). We first observe the following forms of (9.5), respectively (9.15), in type A_{2n-1}. For $i+1 < j < k \le n$, we have

$$\big(t_i^{A_{2n-1}} q_{[2n-k+1,2n-j]}^{A_{2n-1}}\big)^2 = 1, \tag{9.29}$$

$$\big(\tilde{t}_{2n-i}^{A_{2n-1}} q_{[j,k-1]}^{A_{2n-1}}\big)^2 = 1. \tag{9.30}$$

Since $i+1<j<n<2n-k+1<2n-j+1<2n-i<2n$, we obtain

$$t_i^{A_{2n-1}} q_{[2n-k+1,2n-j]}^{A_{2n-1}} = q_{[2n-k+1,2n-j]}^{A_{2n-1}} t_i^{A_{2n-1}},$$

by (9.5) with $k:=2n-j+1$ and $j:=2n-k+1$ for $i+1<j:=2n-k+1<k:=2n-j+1$, which proves (9.29). Thus, the identity (9.30) is the dual version of (9.29) with $k:=2n-j+1$ and $j:=2n-k+1$.

Henceforth, for $i+1<j<k\leq n$, we have

$$\begin{aligned}
\big(t_i^{C_n} q_{[j,k-1]}^{C_n}\big)^2 &= \big(E^{-1} t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} E E^{-1} \xi_{[j,k-1]}^{A_{2n-1}} \xi_{[2n-k+1,2n-j]}^{A_{2n-1}} E\big)^2 \\
&\quad \text{(by Theorem 9.19 and Remark 9.9)} \\
&= E^{-1}\big(t_i^{A_{2n-1}} \xi_{[j,k-1]}^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} \xi_{[2n-k+1,2n-j]}^{A_{2n-1}}\big)^2 E \qquad \text{(by (9.30))} \\
&= E^{-1} t_i^{A_{2n-1}} \xi_{[j,k-1]}^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} \xi_{[2n-k+1,2n-j]}^{A_{2n-1}} t_i^{A_{2n-1}} \xi_{[j,k-1]}^{A_{2n-1}} \\
&\qquad \cdot \tilde{t}_{2n-i}^{A_{2n-1}} \xi_{[2n-k+1,2n-j]}^{A_{2n-1}} E \\
&= E^{-1} t_i^{A_{2n-1}} \xi_{[j,k-1]}^{A_{2n-1}} t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} \xi_{[2n-k+1,2n-j]}^{A_{2n-1}} \xi_{[j,k-1]}^{A_{2n-1}} \\
&\qquad \cdot \tilde{t}_{2n-i}^{A_{2n-1}} \xi_{[2n-k+1,2n-j]}^{A_{2n-1}} E \\
&\quad \text{(by (9.29) and Lemma 9.15)} \\
&= E^{-1} t_i^{A_{2n-1}} \xi_{[j,k-1]}^{A_{2n-1}} t_i^{A_{2n-1}} \xi_{[j,k-1]}^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} \xi_{[2n-k+1,2n-j]}^{A_{2n-1}} \\
&\qquad \cdot \tilde{t}_{2n-i}^{A_{2n-1}} \xi_{[2n-k+1,2n-j]}^{A_{2n-1}} E \\
&\quad \text{(by Lemma 5.3 (2A), } j\leq k-1<n<, 2n-k+1, \text{ and (9.30))} \\
&= E^{-1}\big(t_i^{A_{2n-1}} \xi_{[j,k-1]}^{A_{2n-1}}\big)^2 \big(\tilde{t}_{2n-i}^{A_{2n-1}} \xi_{[2n-k+1,2n-j]}^{A_{2n-1}}\big)^2 E \\
&= 1 \qquad \text{(by (9.5) and (9.15))}.
\end{aligned}$$

(6) For $i+1<j\leq n$, equivalently $2n-i-1>2n-j\geq n>j-1$, we have

$$\begin{aligned}
t_i^{C_n} q_{[j,n]}^{C_n} &= E^{-1} t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} \xi_{[j,2n-j]}^{A_{2n-1}} E \qquad \text{(by Theorem 8.14)} \\
&= E^{-1} t_i^{A_{2n-1}} \xi_{[j,2n-j]}^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} E \\
&\quad \text{(by (9.15) with } k:=2n-j+1 \text{ and } 2n-i-1>2n-j>j-1) \\
&= E^{-1} \xi_{[j,2n-j]}^{A_{2n-1}} t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} E \\
&\quad \text{(by (9.5) with } k:=2n-j+1 \text{ and } 2n-j+1>j>i+1) \\
&= q_{[j,n]}^{C_n} t_i^{C_n}.
\end{aligned}$$

(7) We will prove $(t_{n+i-1}^{C_n} q_{[j,n]}^{C_n})^2=1$, $1\leq i,j\leq n$. Recalling Definition 9.11, Theorem 8.14, and Theorem 9.19, (9.24), we have

$$\begin{aligned}
t_{n+i-1}^{C_n} q_{[j,n]}^{C_n} &= E^{-1} t_{[n-i+1,n+i]}^{A_{2n-1}} \xi_{[j,2n-j]}^{A_{2n-1}} E \\
&= E^{-1} \xi_{[n-(i-1),n+(i-1)]}^{A_{2n-1}} \xi_{[n-(i-2),n+(i-2)]}^{A_{2n-1}} \xi_{[j,2n-j]}^{A_{2n-1}} E \quad \text{(by (9.22))}
\end{aligned}$$

$$= E^{-1}\xi^{A_{2n-1}}_{[j,2n-j]}\xi^{A_{2n-1}}_{[n-(i-1),n+(i-1)]}\xi^{A_{2n-1}}_{[n-(i-2),n+(i-2)]}E$$

(by Theorem 9.18 and Lemma 5.9, (5.5))

$$= q^{C_n}_{[j,n]}t^{C_n}_{n+i-1}.$$

(8) We now prove $(t^{C_n}_{n+i-1}q^{C_n}_{[j,k-1]})^2 = 1$, for $n-i+1 < j < k \le n$. It follows from Theorems 8.13 and 9.19, and Remark 9.9 that

$$\begin{aligned} t^{C_n}_{n+i-1}q^{C_n}_{[j,k-1]} &= E^{-1}\xi^{A_{2n-1}}_{[n-(i-1),n+(i-1)]}\xi^{A_{2n-1}}_{[n-(i-2),n+(i-2)]}\xi^{A_{2n-1}}_{[j,k-1]}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]}E \\ &= E^{-1}\xi^{A_{2n-1}}_{[n-(i-1),n+(i-1)]}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]} \\ &\quad \cdot \xi^{A_{2n-1}}_{[n-(i-2),n+(i-2)]}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]}E \\ &\quad \text{(by Lemma 5.3 (3}A\text{) with } n-i+1 < j < k \le n) \\ &= E^{-1}\xi^{A_{2n-1}}_{[jk-1]}\xi^{A_{2n-1}}_{[n-(i-1),n+(i-1)]}\xi^{A_{2n-1}}_{[j,k-1]}\xi^{A_{2n-1}}_{[n-(i-2),n+(i-2)]}E \\ &\quad \text{(by Lemma 5.3 (3}A\text{) with } n+i-1 > 2n-j > 2n-k \ge n) \\ &= E^{-1}\xi^{A_{2n-1}}_{[j,k-1]}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]}\xi^{A_{2n-1}}_{[n-(i-1),n+(i-1)]}\xi^{A_{2n-1}}_{[n-(i-2),n+(i-2)]}E \\ &\quad \text{(by Lemma 5.3 (3}A\text{))} \\ &= q^{C_n}_{[j,k-1]}t^{C_n}_{n+i-1}. \end{aligned}$$

(9) Recall Theorem 9.19, (9.23) and Remark 9.6. Then, for $n \ge 3$, we have

$$\begin{aligned} \big(t^{C_n}_1 t^{C_n}_2\big)^6 &= E^{-1}\big(t^{A_{2n-1}}_1\tilde{t}^{A_{2n-1}}_2 t^{A_{2n-1}}_2\tilde{t}^{A_{2n-1}}_3\big)^6 E \\ &= E^{-1}\big(t^{A_{2n-1}}_1 t^{A_{2n-1}}_2\tilde{t}^{A_{2n-1}}_2\tilde{t}^{A_{2n-1}}_3\big)^6 E && \text{(by Lemma 9.15)} \\ &= E^{-1}\big(t^{A_{2n-1}}_1 t^{A_{2n-1}}_2\big)^6\big(\tilde{t}^{A_{2n-1}}_2\tilde{t}^{A_{2n-1}}_3\big)^6 E && \text{(by Lemma 9.15)} \\ &= E^{-1}\big(\tilde{t}^{A_{2n-1}}_2\tilde{t}^{A_{2n-1}}_3\big)^6 E && \text{(by (9.4))} \\ &= E^{-1}\big(\tilde{t}^{A_{2n-1}}_{2n-1}\tilde{t}^{A_{2n-1}}_{2n-2}\big)^6 E = 1 && \text{(by (9.14)).} \end{aligned}$$

(10) From Proposition 8.2 (2), we know that $(\xi^{C_n}_{n-1}\xi^{C_n}_n)^4 = 1$. We prove that

$$\big(t^{C_n}_{n-1}\cdots t^{C_n}_2 t^{C_n}_1 t^{C_n}_2\cdots t^{C_n}_{n-1}t^{C_n}_n\big)^4 = 1$$

is equivalent to $(\xi^{C_n}_{n-1}\xi^{C_n}_n)^4 = 1$. Recall Proposition 9.13. Then, from Proposition 9.13, one has the following:

$$\begin{aligned} \big(\xi^{C_n}_{n-1}\xi^{C_n}_n\big)^4 &= \big(q^{C_n}_{[1,n-1]}t^{C_n}_1 q^{C_n}_{[1,n-1]}t^{C_n}_n\big)^4 && \text{(by (9.19))} \\ &= \big(q^{C_n}_{[1,n-2]}p^{C_n}_{n-1}t^{C_n}_1 q^{C_n}_{[1,n-2]}p^{C_n}_{n-1}t^{C_n}_n\big)^4. \end{aligned} \tag{9.31}$$

From [31, Proposition 1.4 (a)] and mimicking its proof in conjunction with relation (5), we may write

$$\begin{aligned} q^{C_n}_{[1,n-1]} &= q^{C_n}_{[1,n-2]} p^{C_n}_{n-1} \\ &= \left(p^{C_n}_{n-1}\right)^{-1} q^{C_n}_{[1,n-2]}. \end{aligned} \tag{9.32}$$

Therefore, using (9.32),

$$\begin{aligned} \left(\xi^{C_n}_{n-1}\xi^{C_n}_{n}\right)^4 &= \left(q^{C_n}_{[1,n-2]} p^{C_n}_{n-1} t^{C_n}_1 (p^{C_n}_{n-1})^{-1} q^{C_n}_{[1,n-2]} t^{C_n}_n\right)^4 && \text{(by (9.31))} \\ &= \left(q^{C_n}_{[1,n-2]} p^{C_n}_{n-1} t^{C_n}_1 (p^{C_n}_{n-1})^{-1} t^{C_n}_n q^{C_n}_{[1,n-2]}\right)^4 && \text{(by relation (7))} \\ &= q^{C_n}_{[1,n-2]} \left(p^{C_n}_{n-1} t^{C_n}_1 (p^{C_n}_{n-1})^{-1} t^{C_n}_n\right)^4 q^{C_n}_{[1,n-2]} \\ &= q^{C_n}_{[1,n-2]} \left(p^{C_n}_{n-1} t^{C_n}_1 (p^{C_n}_{n})^{-1}\right)^4 q^{C_n}_{[1,n-2]}. \end{aligned}$$

Since $(q^{C_n}_{[1,n-2]})^2 = 1$, we get

$$\begin{aligned} &\left(\xi^{C_n}_{n-1}\xi^{C_n}_{n}\right)^4 = 1 \\ &\quad \Leftrightarrow \left(p^{C_n}_{n-1} t^{C_n}_1 (p^{C_n}_{n})^{-1}\right)^4 = \left(t^{C_n}_{n-1}\cdots t^{C_n}_2 t^{C_n}_1 t^{C_n}_2 \cdots t^{C_n}_n\right)^4 = 1, \end{aligned} \tag{9.33}$$

where $p^{C_n}_0 := 1$. In particular, for $n = 2$, $(t^{C_n}_1 t^{C_n}_2)^4 = 1$. ■

Remark 9.21. (1) Proposition 9.20 (9) in $\mathcal{BK}^{C_n}$, respectively,

$$\left(t^{A_{2n-1}}_1 t^{A_{2n-1}}_2\right)^6 \left(\tilde{t}^{A_{2n-1}}_{2n-1} \tilde{t}^{A_{2n-1}}_{2n-2}\right)^6 = 1 \quad \text{in } \widetilde{\mathcal{BK}}_{2n},$$

is equivalent to the braid relations of B_n, $(\xi^{C_n}_i \xi^{C_n}_{i+1})^3 = 1$ for $1 \le i < n-1$, respectively,

$$\left(\xi^{A_{2n-1}}_i \xi^{A_{2n-1}}_{2n-i} \xi^{A_{2n-1}}_{i+1} \xi^{A_{2n-1}}_{2n-i-1}\right)^3 = \left(\xi^{A_{2n-1}}_i \xi^{A_{2n-1}}_{i+1}\right)^3 \left(\xi^{A_{2n-1}}_{2n-i} \xi^{A_{2n-1}}_{2n-i-1}\right)^3 = 1,$$

the braid relations of $\mathfrak{S}_{2n}$ for $1 \le i < n-1$ ([31, Proposition 1.4 (d)]).

(2) The identity $(\xi^{C_n}_{n-1}\xi^{C_n}_n)^4 = 1$ in the group $\mathcal{BK}^{C_n}$ translates to the isomorphic group $\widetilde{\mathcal{BK}}_{2n}$ as

$$\left(\xi^{A_{2n-1}}_{n-1} \xi^{A_{2n-1}}_{n+1} \xi^{A_{2n-1}}_{n}\right)^4 = 1. \tag{9.34}$$

Thus Proposition 9.20 (10) in $\mathcal{BK}^{C_n}$ translates to $\widetilde{\mathcal{BK}}_{2n}$ as

$$\left(t^{A_{2n-1}}_{n-1} \cdots t^{A_{2n-1}}_1 \tilde{t}^{A_{2n-1}}_{n+1} \cdots \tilde{t}^{A_{2n-1}}_{2n-1} \xi^{A_{2n-1}}_n\right)^4 = 1. \tag{9.35}$$

Recall in Corollary 8.17, we saw that involutions $\xi^{A_{2n-1}}_i \xi^{A_{2n-1}}_{2n-i}$, $\xi^{A_{2n-1}}_n$, $1 \le i \le n-1$, define an action of B_n on the embedded crystal $E(\mathsf{KN}(\lambda, n))$ in $\mathsf{SSYT}(\lambda^A, 2n)$. Hence, as (9.33) shows, the relation (10) in $\mathcal{BK}^{C_n}$, respectively, (9.35) in $\widetilde{\mathcal{BK}}_{2n}$, is equivalent to the braid relation $(\xi^{C_n}_{n-1}\xi^{C_n}_n)^4 = 1$ of B_n, respectively, the braid relation (9.34) of B_n, realized in $\mathfrak{S}_{2n}$.

10. Open questions and final remarks

Similarly to $\mathcal{BK}$, it remains to establish whether or not $\mathcal{BK}^{C_n}$ satisfies additional relations besides those listed in Proposition 9.20. Chmutov, Glick and Pylyavskyy [12] have determined relationships between subsets of relations in the groups $\mathcal{BK}_n$ and J_n which yield a presentation for the cactus group J_n in terms of Bender–Knuth generators. Rodrigues [42–45] has also introduced a shifted Berenstein–Kirillov group with many parallels with the original $\mathcal{BK}$ group. Following Halacheva, Rodrigues has defined a cactus group action of J_n via partial shifted Schützenberger–Lusztig involutions (partial shifted reversal) on the Gillespie–Levinson–Purbhoo shifted tableau crystal [20]. On the other hand, with the shifted tableau switching she has defined shifted Bender–Knuth involutions, and following Chmutov, Glick and Pylyavskyy she has obtained a presentation for the cactus group J_n in terms of shifted Bender–Knuth generators. In the same vein, it is natural to seek precise relationships between subsets of relations in the two groups $\widetilde{\mathcal{BK}}_{2n}$ and the virtual symplectic cactus group $\widetilde{J}_n$. It is also natural to seek a presentation of the virtual symplectic cactus group $\widetilde{J}_{2n}$ in terms of the virtual symplectic Bender–Knuth generators.

Glossary

B_n	The hyperoctahedral group, i.e., the free group generated by $r_1, \dots, r_{n-1}, r_n$ subject to the relations (2.1)–(2.4).
B	A normal crystal.
B_J	The Levi branched normal crystal B_J, the restriction of B to the subdiagram J of I.
$\mathsf{B}(\lambda)$	The normal $\mathfrak{g}$-crystal with highest weight λ.
$\mathcal{BK}$	The Berenstein–Kirillov group (or Gelfand–Tsetlin group) [31].
$\mathcal{BK}^{C_n}$	The type C_n symplectic Berenstein–Kirillov group.
$\mathcal{BK}_n$	The subgroup of $\mathcal{BK}$ generated by the first $n-1$ Bender–Knuth involutions $t_1, \dots, t_{n-1}$.
$\widetilde{\mathcal{BK}}_{2n}$	The virtual symplectic Berenstein–Kirillov group, a subgroup of $\mathcal{BK}_{2n}$ satisfying the relations of the virtual symplectic cactus group $\widetilde{J}_{2n}$.
$\mathcal{C}_n$	$\{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$.
$\mathcal{C}_n^*$	The monoid of words in the alphabet $\mathcal{C}_n$.
E	The virtualization map defined by Baker [3, Propositions 2.2 and 2.3] on type C_n Kashiwara–Nakashima tableaux.

$\mathfrak{g}$ — Finite-dimensional, complex, semi-simple Lie algebra.

J_n — The cactus group $J_{\mathfrak{sl}(n,\mathbb{C})}$.

$\widetilde{J}_{2n}$ — The virtual symplectic cactus group.

$J_{\mathfrak{sp}(2n,\mathbb{C})}$ — The symplectic cactus group with generators s_J for J any connected sub-diagram of the C_n Dynkin diagram subject to the relations in Lemma 5.5.

$\mathsf{KN}(\lambda, n)$ — The $U_q(\mathfrak{sp}(2n,\mathbb{C}))$ crystal of Kashiwara–Nakashima tableaux of shape λ in the alphabet $\mathcal{C}_n$.

$\mathsf{KN}_J(\lambda, n)$ — The Levi branched crystal of Kashiwara–Nakashima tableaux obtained by deleting in $\mathsf{KN}(\lambda, n)$ all the arrows not labeled in $J \subset I$.

reversal — Combinatorial procedure to compute the Schützenberger involution ξ on $\mathsf{SSYT}(\lambda, n)$.

$\mathsf{reversal}^{C_n}$ — Combinatorial procedure to compute the Schützenberger involution ξ on $\mathsf{KN}(\lambda, n)$.

$\mathsf{reversal}_J$ — J-partial reversal, the reversal on $\mathsf{SSYT}_J(\lambda, n)$ with $J \subseteq [n-1]$.

$\mathsf{reversal}_J^{C_n}$ — J-partial symplectic reversal, the symplectic reversal $\mathsf{KN}_J(\lambda, n)$ with $J \subseteq [n]$ a connected sub-diagram containing the node n.

R3 — The symplectic contraction/dilation relation in the symplectic plactic monoid $\mathcal{C}_n^*/\sim$.

$\mathsf{SSYT}(\lambda, n)$ — The $U_q(\mathfrak{sl}(n,\mathbb{C}))$-crystal of semi-standard Young tableaux of shape λ and entries in $[n]$.

$\mathsf{SSYT}(\lambda^A, n, \overline{n})$ — The $U_q(\mathfrak{sl}(2n,\mathbb{C}))$-crystal of semi-standard Young tableaux of shape λ^A and entries in $\mathcal{C}_n$.

$\mathsf{SSYT}_J(\lambda, n)$ — The Levi branched crystal, the restriction of $\mathsf{SSYT}(\lambda, n)$ to $J \subseteq [n-1]$.

ξ — The Schützenberger–Lusztig involution on $\mathsf{B}(\lambda)$.

ξ_{B} — The Schützenberger–Lusztig involution on the normal crystal B.

ξ_{J} — The partial Schützenberger–Lusztig involution to the sub-diagram $J \subseteq I$ is the Schützenberger–Lusztig involution ξ_{B_J} on the normal crystal B_J.

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