

Indefinite perturbations of an unbalanced growth eigenvalue problem

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Abstract. We consider indefinite perturbations of a double-phase eigenvalue problem. The perturbation is sublinear or superlinear, and it is in general sign-changing. Using the Nehari manifold, we prove the existence of two constant sign solutions for both cases (sublinear and superlinear), when the parameter $\lambda \in (0, \hat{\lambda}_1^\alpha)$ with $\hat{\lambda}_1^\alpha > 0$ being the principal eigenvalue of Dirichlet weighted p -Laplace operator $-\Delta_p^\alpha$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 -boundary $\partial\Omega$. In this paper, we study the following unbalanced growth Dirichlet problems:

$$\begin{cases} -\Delta_p^\alpha u(z) - \Delta_q u(z) = \lambda \alpha(z) |u(z)|^{p-2} u(z) + \beta(z) |u(z)|^{\tau-2} u(z) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

with $1 < \tau < q < p < N$, and

$$\begin{cases} -\Delta_p^\alpha u(z) - \Delta_q u(z) = \lambda \alpha(z) |u(z)|^{p-2} u(z) + \beta(z) |u(z)|^{r-2} u(z) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

with $1 < q < p < r < \min\{q^*, N\}$.

For $\alpha \in L^\infty(\Omega) \setminus \{0\}$ with $\alpha(z) \geq 0$ for a.a. $z \in \Omega$ and for $1 < s < \infty$, by Δ_s^α we denote the weighted s -Laplace differential operator defined by

$$\Delta_s^\alpha u = \operatorname{div}(\alpha(z) |Du|^{s-2} Du).$$

If $\alpha \equiv 1$, then we have the usual s -Laplace differential operator. In both problems (1.1) and (1.2), the equations are driven by the sum of two such operators and the differential operator is not homogeneous (since $q < p$). This differential operator is related to the so-called unbalanced growth integral functional

$$u \mapsto \int_\Omega [\alpha(z) |Du|^p + |Du|^q] dz.$$

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The density of this integral functional is

$$\eta(z, t) = \alpha(z)t^p + t^q \quad \text{for all } z \in \Omega, \text{ all } t \geq 0.$$

This is, in general, a Carathéodory function (that is, for all $t \geq 0$, $z \mapsto \eta(z, t)$ is measurable, and for a.a. $z \in \Omega$, $t \mapsto \eta(z, t)$ is continuous). Hence, $(z, t) \mapsto \eta(z, t)$ is jointly measurable. Since we do not assume that $0 < \text{essinf}_\Omega \alpha$, this density function exhibits an unbalanced growth, namely,

$$t^q \leq \eta(z, t) \leq c_0[1 + t^p] \quad \text{for a.a. } z \in \Omega, \text{ all } t \geq 0, \text{ for some } c_0 > 0.$$

Such functionals were first examined by Marcellini [20,21] and Zhikov [34,35] in the context of problems of the calculus of variations (including the Lavrentiev gap phenomenon) and of nonlinear elasticity. Double-phase operators are used to describe diffusion-type phenomena in a space where certain subdomains are distinguished from others (composite materials). The unbalanced growth implies that the classical Lebesgue and Sobolev spaces do not suffice to study these problems. We need to pass to the broader classes of Orlicz-Lebesgue and Orlicz-Sobolev spaces. Moreover, for such boundary value problems, there is no global regularity theory analogous to the one existing for balanced growth problems (see Lieberman [15]). There are only local regularity results. The papers of Marcellini [22, 23] and of Mingione–Rădulescu [24] provide an up-to-date presentation of the existing local regularity results. This lack of a global regularity theory removes from consideration some powerful tools and techniques which are readily available in the case of balanced growth problems. One way to overcome this obstacle is to use the Nehari method. This is done in the works of Liu–Dai [16], Deregowka–Gasiński–Papageorgiou [7], Gasiński–Papageorgiou [10], Gasiński–Winkert [12], Papageorgiou–Vetro–Vetro [31], Fiscella–Mishra–Tripathi [8], Mishra–Silva–Tripathi [25], Liu–Dai–Papageorgiou–Winkert [17], Liu–Papageorgiou [18], and Papageorgiou–Zhang [33]. More recently, Papageorgiou–Rădulescu–Zhang [29], Papageorgiou–Pudelko–Rădulescu [26], and Papageorgiou–Rădulescu–Wang [28] proved some spectral properties of the weighted Δ_p^α differential operator and used them to study resonant double-phase problems. Problems (1.1) and (1.2) can be viewed as perturbations of the eigenvalue problem considered in [26, 28, 29]. In problem (1.1), the perturbation is $(p - 1)$ -sublinear (since $\tau < q < p$), but in (1.2) the perturbation is $(p - 1)$ -superlinear (since $q < p < r$). The interesting feature in both problems is that this perturbation is in general indefinite (that is, the coefficient $\beta(\cdot)$ is sign-changing). Such indefinite perturbations of the eigenvalue problem for the Laplace differential operator were first studied using the Nehari method by Brown–Zhang [5] and Brown [4]. A possible alternative approach can be based on critical groups as it was done in the recent work of Liu–Papageorgiou [19]. Finally, we mention the work of Bai–Papageorgiou–Zeng [2], who studied a different class of parametric unbalanced growth problems by using critical point theory, and the recent interesting works of Aberqi–Benslimane–Knifda [1] and of Chen–Yang [6]. In [1], the authors studied systems with unbalanced growth on Riemannian manifolds using the Nehari method, while in [6] the authors dealt with a critical Choquard equation using variational methods.

2. Mathematical background

As we already mentioned in the Introduction, the unbalanced growth of $\eta(z, \cdot)$ leads to a functional framework based on generalized Orlicz spaces. A comprehensive presentation of the theory of these spaces can be found in the book of Harjulehto–Hästö [13].

By $C^{0,1}(\bar{\Omega})$ we denote the space of Lipschitz continuous functions on $\bar{\Omega}$. Also, by \mathcal{A}_p we denote the class of all p -Muckenhoupt weights (see Harjulehto–Hästö [13, p. 114]). Our hypotheses on the weight function $\alpha(\cdot)$, the coefficient $\beta(\cdot)$, and the exponents $1 < q < p$ are the following:

$$(H) : \alpha \in C^{0,1}(\bar{\Omega}) \cap \mathcal{A}_p \text{ with } \alpha(z) > 0 \text{ for all } z \in \Omega, \beta \in C(\Omega) \cap L^\infty(\Omega) \text{ with } \beta^+ \neq 0 \text{ and } \frac{p}{q} < 1 + \frac{1}{N}.$$

Let $L^0(\Omega)$ be the space of all measurable functions $u: \Omega \rightarrow \mathbb{R}$. We identify two such functions which differ only on a Lebesgue-null subset of Ω . Recall that

$$\eta(z, t) = \alpha(z)t^p + t^q \quad \text{for all } z \in \Omega, \text{ all } t \geq 0.$$

The generalized Lebesgue-Orlicz space $L^\eta(\Omega)$ is defined by

$$L^\eta(\Omega) = \left\{ u \in L^0(\Omega) \mid \rho_\eta(u) := \int_\Omega \eta(z, |u|) dz < \infty \right\}.$$

The function $\rho_\eta(\cdot)$ is known as the modular function corresponding to the density η . We equip this space with the so-called ‘‘Luxemburg norm’’ $\|\cdot\|_\eta$ defined by

$$\|u\|_\eta = \inf \left\{ \lambda > 0 \mid \rho_\eta\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

With this norm, $L^\eta(\Omega)$ becomes a Banach space which is separable and reflexive. In fact, $L^\eta(\Omega)$ is uniformly convex, since the density function $\eta(z, \cdot)$ is uniformly convex. Recall that a uniformly convex space is reflexive (by the Milman–Pettis theorem, see Papageorgiou–Winkert [32, p. 225]).

Using $L^\eta(\Omega)$, we can define the corresponding Orlicz-Sobolev space $W^{1,\eta}(\Omega)$ given by

$$W^{1,\eta}(\Omega) = \{u \in L^\eta(\Omega) \mid |Du| \in L^\eta(\Omega)\},$$

with Du being the weak gradient of u . The norm $\|\cdot\|_{1,\eta}$ of $W^{1,\eta}(\Omega)$ is defined by

$$\|u\|_{1,\eta} = \|u\|_\eta + \|Du\|_\eta \quad \text{for all } u \in W^{1,\eta}(\Omega).$$

Here, $\|Du\|_\eta = \||Du|\|_\eta$. Also, we set

$$W_0^{1,\eta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\eta}}.$$

The Poincaré inequality is valid on $W_0^{1,\eta}(\Omega)$. So, on $W_0^{1,\eta}(\Omega)$, we can consider the norm $\|\cdot\|$ defined by

$$\|u\| = \|Du\|_\eta \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Evidently, $W^{1,\eta}(\Omega)$ and $W_0^{1,\eta}(\Omega)$ are Banach spaces which are separable and reflexive (in fact, uniformly convex).

We have some useful embeddings involving these spaces.

Proposition 2.1. *Let $u \in W_0^{1,\eta}(\Omega)$. The following statements hold:*

(a) *for all $1 \leq s \leq q$, we have*

$$L^\eta(\Omega) \hookrightarrow L^s(\Omega) \quad \text{and} \quad W_0^{1,\eta}(\Omega) \hookrightarrow W_0^{1,s}(\Omega) \quad \text{continuously};$$

(b) *for all $1 \leq s \leq q^* = \frac{Nq}{N-q}$ (resp., all $1 \leq s < q^*$), we have*

$$W_0^{1,\eta}(\Omega) \hookrightarrow L^s(\Omega) \quad \text{continuously (resp., } W_0^{1,\eta}(\Omega) \hookrightarrow L^s(\Omega) \text{ compactly)}.$$

The modular function $\rho_\eta: W_0^{1,\eta}(\Omega) \rightarrow \mathbb{R}_+$ is clearly continuous and convex. Therefore, it is weakly lower semicontinuous (by using Mazur's lemma). We point out that the norm $\|\cdot\|$ of $W_0^{1,\eta}(\Omega)$ and the modular function $\rho_\eta(\cdot)$ are closely related.

Proposition 2.2. *The following statements hold:*

(a) *if $u \neq 0$, then $\|u\| = \lambda \Leftrightarrow \rho_\eta(\frac{Du}{\lambda}) = 1$.*

(b) *$\|u\| \leq 1$ (resp., $= 1, \geq 1$), $\Leftrightarrow \rho_\eta(Du) \leq 1$ (resp., $= 1, \geq 1$).*

(c) *$\|u\| \leq 1 \Rightarrow \|u\|^p \leq \rho_\eta(Du) \leq \|u\|^q$.*

(d) *$\|u\| \geq 1 \Rightarrow \|u\|^q \leq \rho_\eta(Du) \leq \|u\|^p$.*

(e) *$\|u\| \rightarrow \infty$ (resp., $\rightarrow 0$) $\Leftrightarrow \rho_\eta(Du) \rightarrow \infty$ (resp., $\rightarrow 0$).*

Let $\eta_0(z, t) = \alpha(z)t^p$ and consider the corresponding generalized Orlicz spaces $W_0^{1,\eta_0}(\Omega)$ and $L^{\eta_0}(\Omega)$. These are separable, reflexive Banach spaces and $W_0^{1,\eta_0}(\Omega) \hookrightarrow L^{\eta_0}(\Omega)$ compactly (see [26]). Let $\rho_{\eta_0}(u) := \int_\Omega \eta_0(z, |u|) dz$ be the modular function corresponding to the density $\eta_0(z, x)$. We consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p^\alpha u(z) = \lambda \alpha(z) |u(z)|^{p-2} u(z) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

From [26, 29], we know that this eigenvalue problem has the smallest eigenvalue $\hat{\lambda}_1^\alpha > 0$, which has the following variational characterization:

$$\hat{\lambda}_1^\alpha = \inf \left\{ \frac{\rho_{\eta_0}(Du)}{\rho_{\eta_0}(u)} \mid u \in W_0^{1,\eta_0}(\Omega), u \neq 0 \right\}. \quad (2.1)$$

This eigenvalue is simple; that is, if \hat{u}, \hat{v} are eigenfunctions corresponding to $\hat{\lambda}_1^\alpha$, then $\hat{u} = \theta \hat{v}$ for some $\theta \in \mathbb{R} \setminus \{0\}$. Also, $\hat{\lambda}_1^\alpha$ is isolated in the spectrum. The infimum in (2.1) is realized on the corresponding one-dimensional eigenspace, the elements of which have fixed sign. Let \hat{u}_1 denote the positive, $L^{\eta_0}(\Omega)$ -normalized eigenfunction (that is, $\|\hat{u}_1\|_{\eta_0} = 1$). We know that

$$\hat{u}_1 \in W_0^{1,\eta_0}(\Omega) \cap L^\infty(\Omega),$$

and if $K \subseteq \Omega$ is compact, then there exists $c_K > 0$ such that

$$c_K \leq \hat{u}_1(z) \quad \text{for a.a. } z \in K.$$

In what follows, for every $u \in L^0(\Omega)$ with the above property, we write $0 < u$. In the sequel, in order to simplify our notation, we write

$$\rho_{\eta_0}(u) := \int_{\Omega} \alpha(z)|u|^p dz \quad \text{for all } u \in L^{\eta_0}(\Omega),$$

and

$$\rho_{\eta_0}(Du) := \int_{\Omega} \alpha(z)|Du|^p dz \quad \text{for all } u \in W_0^{1,\eta_0}(\Omega).$$

These are continuous and convex functions on the corresponding spaces. Hence, they are also weakly lower semicontinuous.

If $u \in L^0(\Omega)$, then recall that

$$u^+ = \max\{u, 0\}, \quad u^- = \max\{-u, 0\}.$$

Then, $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,\eta}(\Omega)$, then $u^{\pm} \in W_0^{1,\eta}(\Omega)$. If X is a Banach space and $\varphi \in C^1(X)$, then $K_{\varphi} := \{u \in X \mid \varphi'(u) = 0\}$ (the critical set of φ).

For problem (1.1), the energy functional $\varphi_{\lambda}: W_0^{1,\eta}(\Omega) \rightarrow \mathbb{R}$ is defined by

$$\varphi_{\lambda}(u) := \frac{1}{p}\rho_{\eta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \frac{\lambda}{p}\rho_{\eta_0}(u) - \frac{1}{\tau}\int_{\Omega} \beta(z)|u|^{\tau} dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Given $u \in W_0^{1,\eta}(\Omega)$, the corresponding fibering function $\theta_u^{\lambda}: \mathbb{R}_+ \rightarrow \mathbb{R}$ is defined by

$$\theta_u^{\lambda}(t) = \frac{t^p}{p}\rho_{\eta_0}(Du) + \frac{t^q}{q}\|Du\|_q^q - \frac{\lambda t^p}{p}\rho_{\eta_0}(u) - \frac{t^{\tau}}{\tau}\int_{\Omega} \beta(z)|u|^{\tau} dz \quad \text{for all } t \geq 0.$$

Evidently, $\varphi_{\lambda} \in C^1(W_0^{1,\eta}(\Omega))$ and $\theta_u^{\lambda} \in C^2(\mathbb{R}_+)$. Similarly, for problem (1.2), the energy functional $\varphi_{\lambda}(\cdot)$ is given by

$$\varphi_{\lambda}(u) := \frac{1}{p}\rho_{\eta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \frac{\lambda}{p}\rho_{\eta_0}(u) - \frac{1}{r}\int_{\Omega} \beta(z)|u|^r dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Also, for $u \in W_0^{1,\eta}(\Omega)$, the corresponding fibering function $\theta_u^{\lambda}: \mathbb{R}_+ \rightarrow \mathbb{R}$ is given by

$$\theta_u^{\lambda}(t) := \frac{t^p}{p}\rho_{\eta_0}(Du) + \frac{t^q}{q}\|Du\|_q^q - \frac{\lambda t^p}{p}\rho_{\eta_0}(u) - \frac{t^r}{r}\int_{\Omega} \beta(z)|u|^r dz \quad \text{for all } t \geq 0.$$

Again, we have $\varphi_{\lambda} \in C^1(W_0^{1,\eta}(\Omega))$ and $\theta_u^{\lambda} \in C^2(\mathbb{R}_+)$. Then, for both problems, the Nehari manifold is defined by

$$N_{\lambda} := \{u \in W_0^{1,\eta}(\Omega) \mid \langle \varphi'_{\lambda}(u), u \rangle = 0 \text{ with } u \neq 0\}.$$

We decompose N_λ as follows:

$$N_\lambda = N_\lambda^+ \cup N_\lambda^- \cup N_\lambda^0,$$

where

$$\begin{aligned} N_\lambda^+ &:= \{u \in N_\lambda \mid (\theta_u^\lambda)''(1) > 0\} \quad (\text{local minima}), \\ N_\lambda^- &:= \{u \in N_\lambda \mid (\theta_u^\lambda)''(1) < 0\} \quad (\text{local maxima}), \\ N_\lambda^0 &:= \{u \in N_\lambda \mid (\theta_u^\lambda)''(1) = 0\} \quad (\text{inflection points}). \end{aligned}$$

3. Sublinear perturbation

In this section, we study problem (1.1). Indeed, we have

$$(\theta_u^\lambda)'(t) = t^{p-1}[\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u)] + t^{q-1}\|Du\|_q^q - t^{\tau-1} \int_\Omega \beta(z)|u|^\tau dz,$$

and

$$\begin{aligned} (\theta_u^\lambda)''(t) &= (p-1)t^{p-2}[\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u)] + (q-1)t^{q-2}\|Du\|_q^q \\ &\quad - (\tau-1)t^{\tau-2} \int_\Omega \beta(z)|u|^\tau dz, \end{aligned}$$

and

$$\langle \varphi'_\lambda(u), u \rangle = \rho_{\eta_0}(Du) + \|Du\|_q^q - \lambda\rho_{\eta_0}(u) - \int_\Omega \beta(z)|u|^\tau dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

This implies that

$$tu \in N_\lambda \Leftrightarrow (\theta_u^\lambda)'(t) = 0, \quad \text{hence } u \in N_\lambda \Leftrightarrow (\theta_u^\lambda)'(1) = 0. \quad (3.1)$$

Moreover, we observe the following:

“if $u \in W_0^{1,\eta}(\Omega) \setminus \{0\}$ satisfies $\int_\Omega \beta(z)|u|^\tau dz \leq 0$ and $0 < \lambda < \widehat{\lambda}_1^\alpha$, then $(\theta_u^\lambda)(\cdot)$ is strictly increasing and $(\theta_u^\lambda)(0) = 0$ ”.

Therefore, for $t > 0$, we have $tu \notin N_\lambda$ (see (3.1)). In addition, since $\theta_u^\lambda = \theta_{|u|}^\lambda$, we infer that for all $t < 0$ it holds $tu \notin N_\lambda$.

Proposition 3.1. *If hypotheses (H) hold, $0 < \lambda < \widehat{\lambda}_1^\alpha$, and $u \in W_0^{1,\eta}(\Omega)$ satisfies*

$$\int_\Omega \beta(z)|u|^\tau dz > 0,$$

then there exists unique $t_u > 0$ such that $t_u u \in N_\lambda$.

Proof. Since $\tau < q < p$ and $\lambda \in (0, \widehat{\lambda}_1^\alpha)$, we see that

$$\theta_u^\lambda(t) < 0 \quad \text{for } t \in (0, 1) \text{ small enough, and } \theta_u^\lambda(t) \rightarrow +\infty \quad \text{as } t \rightarrow +\infty.$$

On account of the continuity of $\theta_u^\lambda(\cdot)$, we can find $t_u > 0$ such that

$$\theta_u^\lambda(t_u) = \min_{t \geq 0} \theta_u^\lambda(t) < 0 \Rightarrow (\theta_u^\lambda)'(t_u) = 0 \Rightarrow t_u u \in N_\lambda \quad (\text{see (3.1)}).$$

Hence,

$$t_u^{p-\tau} [\rho_{\eta_0}(Du) - \lambda \rho_{\eta_0}(u)] + t_u^{q-\tau} \|Du\|_q^q = \int_\Omega \beta(z) |u|^\tau dz. \quad (3.2)$$

In equation (3.2), we see that the left-hand side as a function of $t \geq 0$ is strictly increasing, while the right-hand side is constant (does not depend on $t \geq 0$). Therefore, the solution $t_u > 0$ of (3.2) is unique. ■

Proposition 3.2. *If hypotheses (H) hold and $0 < \lambda < \widehat{\lambda}_1^\alpha$, then $N_\lambda \neq \emptyset$.*

Proof. Let $\Omega_+ = \{z \in \Omega \mid \beta(z) > 0\}$. Then, $\Omega_+ \subseteq \Omega$ is nonempty, open (see hypotheses (H)). Let $u \in C_0^1(\Omega_+) \setminus \{0\}$. We have $\int_{\Omega} \beta(z) |u|^\tau dz > 0$, and then using Proposition 3.1, we see that there exists unique $t_u > 0$ such that $t_u u \in N_\lambda$. Hence, $N_\lambda \neq \emptyset$. ■

Proposition 3.3. *If hypotheses (H) hold and $0 < \lambda < \widehat{\lambda}_1^\alpha$, then*

$$\emptyset \neq N_\lambda^+ = \left\{ u \in N_\lambda \mid \int_\Omega \beta(z) |u|^\tau dz > 0 \right\}.$$

Proof. Let $C_+ := \{u \in N_\lambda \mid \int_\Omega \beta(z) |u|^\tau dz > 0\}$. From Proposition 3.2 and its proof, we know that $C_+ \neq \emptyset$. Let $u \in N_\lambda^+$. By definition, we have

$$\begin{aligned} & (p-1)[\rho_{\eta_0}(Du) - \lambda \rho_{\eta_0}(u)] + (q-1)\|Du\|_q^q > (\tau-1) \int_\Omega \beta(z) |u|^\tau dz, \\ \Rightarrow & (p-1)[\rho_{\eta_0}(Du) - \lambda \rho_{\eta_0}(u) + \|Du\|_q^q] > (\tau-1) \int_\Omega \beta(z) |u|^\tau dz \quad (\text{since } q < p), \\ \Rightarrow & (p-\tau) \int_\Omega \beta(z) |u|^\tau dz > 0 \quad (\text{since } u \in N_\lambda^+ \subseteq N_\lambda), \\ \Rightarrow & \int_\Omega \beta(z) |u|^\tau dz > 0 \quad (\text{since } \tau < p), \\ \Rightarrow & N_\lambda^+ \subseteq C_+. \end{aligned}$$

Conversely, suppose $u \in C_+$. Then, since $u \in C_+ \subseteq N_\lambda$, and $\tau < q < p$, we have

$$\begin{aligned} & (p-1)[\rho_{\eta_0}(Du) - \lambda \rho_{\eta_0}(u)] + (q-1)\|Du\|_q^q \\ & > (\tau-1)[\rho_{\eta_0}(Du) - \lambda \rho_{\eta_0}(u) + \|Du\|_q^q] \\ & = (\tau-1) \int_\Omega \beta(z) |u|^\tau dz > 0, \\ \Rightarrow & (\theta_u^\lambda)''(1) > 0, \\ \Rightarrow & C_+ \subseteq N_\lambda^+. \end{aligned}$$

So, we have

$$\emptyset \neq N_\lambda^+ = C_+ = \left\{ u \in N_\lambda \mid \int_\Omega \beta(z)|u|^\tau dz > 0 \right\}. \quad \blacksquare$$

In the beginning of this section, we remarked that if $u \in W_0^{1,\eta}(\Omega) \setminus \{0\}$ satisfies

$$\int_\Omega \beta(z)|u|^\tau dz \leq 0,$$

then $u \notin N_\lambda$. This observation and Proposition 3.3 imply that we should focus on N_λ^+ .

Proposition 3.4. *If hypotheses (H) hold and $0 < \lambda < \hat{\lambda}_1^\alpha$, then $N_\lambda \subseteq W_0^{1,\eta}(\Omega)$ is bounded.*

Proof. We argue by contradiction. So, suppose that there is a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq N_\lambda$ such that

$$\|u_n\| \rightarrow \infty.$$

Let $y_n = \frac{u_n}{\|u_n\|}$ for all $n \in \mathbb{N}$. Then, $\|y_n\| = 1$ for all $n \in \mathbb{N}$. Hypotheses $\frac{p}{q} < 1 + \frac{1}{N}$ and (H) imply that $p < q^*$. So, using Proposition 2.1, we may assume that

$$y_n \rightarrow y \quad \text{weakly in } W_0^{1,\eta}(\Omega) \text{ and } y_n \rightarrow y \text{ in } L^p(\Omega). \quad (3.3)$$

Since $u_n \in N_\lambda$, we have

$$\begin{aligned} & \rho_{\eta_0}(Du_n) - \lambda \rho_{\eta_0}(y_n) + \|Du_n\|_q^q = \int_\Omega \beta(z)|u_n|^\tau dz, \\ \Rightarrow & \rho_{\eta_0}(Dy_n) - \lambda \rho_{\eta_0}(y_n) + \frac{1}{\|u_n\|^{p-q}} \|Dy_n\|_q^q \\ & = \frac{1}{\|u_n\|^{p-\tau}} \int_\Omega \beta(z)|y_n|^\tau dz \quad \text{for all } n \in \mathbb{N}. \end{aligned} \quad (3.4)$$

In (3.4), we pass to the limit as $n \rightarrow \infty$ and use (3.3) as well as the fact that $\tau < q < p$ to obtain

$$\rho_{\eta_0}(Dy) \leq \lambda \rho_{\eta_0}(y) \quad \Rightarrow \quad y = 0 \quad (\text{since } \lambda \in (0, \hat{\lambda}_1^\alpha), \text{ see (2.1)}). \quad (3.5)$$

We have the following claim.

Claim 1. $y_n \rightarrow y$ in $W_0^{1,\eta}(\Omega)$ as $n \rightarrow \infty$.

Suppose that the assertion of this claim is not true. On account of (3.3), at least one of the following strict inequalities holds:

$$\rho_{\eta_0}(Dy) < \liminf_{n \rightarrow \infty} \rho_{\eta_0}(Dy_n), \quad (3.6)$$

$$\|Dy\|_q^q < \liminf_{n \rightarrow \infty} \|Dy_n\|_q^q. \quad (3.7)$$

If (3.6) holds, then from (3.4) in the limit as $n \rightarrow \infty$ we obtain

$$\rho_{\eta_0}(Dy) < \lambda \rho_{\eta_0}(y) \quad (\text{see (3.3)}) \quad \Rightarrow \quad y \neq 0.$$

This contradicts with (3.5). Then, by Urysohn's criterion for the convergence of sequence (see Gasiński–Papageorgiou [9, p. 33]), we have

$$\rho_{\eta_0}(Dy_n) \rightarrow \rho_{\eta_0}(Dy). \quad (3.8)$$

Next, suppose that (3.7) holds. Since $\lambda \in (0, \hat{\lambda}_1^\alpha)$, from (3.4), we have

$$\|Dy_n\|_q^q \leq \frac{1}{\|u_n\|^{q-\tau}} \int_{\Omega} \beta(z)|y_n|^\tau dz \quad \Rightarrow \quad \|Dy\|_q^q < 0.$$

This is impossible. Therefore, for that sequence too, we have

$$\|Dy_n\|_q^q \rightarrow \|Dy\|_q^q. \quad (3.9)$$

From (3.8) and (3.9), we infer that

$$\rho_\eta(Dy_n) \rightarrow \rho_\eta(Dy).$$

Then, using the Brézis–Lieb lemma (see [3], and Papageorgiou–Winkert [32, p. 291]), we have

$$\rho_\eta(Dy_n - Dy) \rightarrow 0 \quad \Rightarrow \quad y_n \rightarrow y \quad \text{in } W_0^{1,\eta}(\Omega) \quad (\text{see Proposition 2.2}).$$

This proves the claim. The claim implies $\|y\| = 1$ and this contradicts (3.5). Therefore, we conclude that $N_\lambda \subseteq W_0^{1,\eta}(\Omega)$ is bounded. ■

Corollary 3.5. *If hypotheses (H) hold and $0 < \lambda < \hat{\lambda}_1^\alpha$, then $N_\lambda^+ \subseteq W_0^{1,\eta}(\Omega)$ is bounded.*

Now, we will use the boundedness of N_λ^+ to show that $\varphi_\lambda|_{N_\lambda^+}$ admits a minimizer.

Proposition 3.6. *If hypotheses (H) hold and $0 < \lambda < \hat{\lambda}_1^\alpha$, then there exists $\hat{u} \in N_\lambda^+$ such that*

$$\varphi_\lambda(\hat{u}) = \inf_{N_\lambda^+} \varphi_\lambda =: m_\lambda^+ < 0.$$

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subseteq N_\lambda^+$ be a minimizing sequence, that is,

$$\varphi_\lambda(u_n) \downarrow m_\lambda^+ \quad \text{as } n \rightarrow \infty.$$

Since $\tau < q < p$, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \varphi_\lambda(u_n) &\leq \frac{1}{q} [\rho_{\eta_0}(Du_n) - \lambda \rho_{\eta_0}(u_n) + \|Du_n\|_q^q] - \frac{1}{\tau} \int_{\Omega} \beta(z)|u_n|^\tau dz \\ &= \left(\frac{1}{q} - \frac{1}{\tau} \right) \int_{\Omega} \beta(z)|u_n|^\tau dz \\ &< 0, \end{aligned}$$

where we have used the fact that $u_n \in N_\lambda^+ \subseteq N_\lambda$. This means that $m_\lambda^+ < 0$ owing to $\tau < q$ and Proposition 3.3. From Corollary 3.5, we know that $\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$ is bounded. So, we may assume that

$$u_n \rightarrow \hat{u} \text{ weakly in } W_0^{1,\eta}(\Omega) \quad \text{and} \quad u_n \rightarrow \hat{u} \text{ in } L^p(\Omega) \text{ as } n \rightarrow \infty. \quad (3.10)$$

Since $u_n \in N_\lambda^+$, we have

$$0 < \int_{\Omega} \beta(z) |u_n|^\tau dz \quad (\text{see Proposition 3.3}) \quad \Rightarrow \quad 0 \leq \int_{\Omega} \beta(z) |\hat{u}|^\tau dz,$$

where we used the facts (3.10) and $\tau < q < p$. If $\int_{\Omega} \beta(z) |\hat{u}|^\tau dz = 0$, then since $u_n \in N_\lambda^+$, we have

$$\begin{aligned} \rho_{\eta_0}(Du_n) - \lambda \rho_{\eta_0}(u_n) + \|Du_n\|_q^q &= \int_{\Omega} \beta(z) |u_n|^\tau dz \\ \Rightarrow \rho_{\eta_0}(D\hat{u}) - \lambda \rho_{\eta_0}(\hat{u}) + \|D\hat{u}\|_q^q &\leq 0, \end{aligned}$$

(see (3.10)). This together with $\lambda \in (0, \hat{\lambda}_1^q)$ and Proposition 2.2 implies

$$\hat{u} = 0. \quad (3.11)$$

Note that

$$\rho_\eta(Du_n) = \lambda \rho_{\eta_0}(u_n) + \int_{\Omega} \beta(z) |u_n|^\tau dz \quad \text{for all } n \in \mathbb{N} \quad (\text{since } u_n \in N_\lambda^+),$$

so, we have

$$\begin{aligned} \rho_\eta(Du_n) &\rightarrow 0 \quad (\text{see (3.10) and (3.11)}) \\ \Rightarrow u_n &\rightarrow 0 \quad \text{in } W_0^{1,\eta}(\Omega) \quad (\text{see Proposition 2.2}) \\ \Rightarrow \varphi_\lambda(u_n) &\rightarrow 0. \end{aligned}$$

But, $\varphi_\lambda(u_n) \rightarrow m_\lambda^+ < 0$ leads to a contradiction. Therefore, we obtain

$$0 < \int_{\Omega} \beta(z) |\hat{u}|^\tau dz. \quad (3.12)$$

Moreover, we need the following claim.

Claim 2. $u_n \rightarrow \hat{u}$ in $W_0^{1,\eta}(\Omega)$ as $n \rightarrow \infty$.

We argue by contradiction. So, suppose that this claim is not true. On account of (3.10) and the weak lower semicontinuity of the modular function $\rho_\eta(\cdot)$, we have

$$\rho_\eta(D\hat{u}) < \liminf_{n \rightarrow \infty} \rho_\eta(Du_n). \quad (3.13)$$

Since $u_n \in N_\lambda^+$, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \rho_\eta(Du_n) &= \lambda \rho_{\eta_0}(u_n) + \int_\Omega \beta(z)|u_n|^\tau dz \quad \text{for all } n \in \mathbb{N} \\ \Rightarrow \rho_\eta(D\hat{u}) &< \lambda \rho_{\eta_0}(\hat{u}) + \int_\Omega \beta(z)|\hat{u}|^\tau dz, \end{aligned}$$

where we have used (3.10) and (3.13). Hence,

$$(\theta_{\hat{u}}^\lambda)'(1) < 0. \tag{3.14}$$

From (3.12) and Proposition 3.1, we know that there exists unique $t_{\hat{u}} > 0$ such that $t_{\hat{u}}\hat{u} \in N_\lambda$. On account of (3.14) and the properties of the fibering function (recall $\tau < q < p$), we have $t_{\hat{u}} > 1$. It follows that

$$\varphi_\lambda(t_{\hat{u}}\hat{u}) \leq \varphi_\lambda(\hat{u}) < \liminf_{n \rightarrow \infty} \varphi_\lambda(u_n) = m_\lambda^+. \tag{3.15}$$

But $t_{\hat{u}}\hat{u} \in N_\lambda$ and (3.12) imply $t_{\hat{u}}\hat{u} \in N_\lambda^+$ (see Proposition 3.3) and so, $m_\lambda^+ \leq \varphi_\lambda(t_{\hat{u}}\hat{u})$, which contradicts (3.15). This proves the claim.

Using Claim 2, we have

$$\varphi_\lambda(u_n) \rightarrow \varphi_\lambda(\hat{u}).$$

Consequently, from (3.12), we have $\varphi_\lambda(\hat{u}) = m_\lambda^+$ and $\hat{u} \in N_\lambda^+$. ■

In the next proposition, we show that \hat{u} is a critical point of φ_λ (that is, $\hat{u} \in K_{\varphi_\lambda}$). This means that N_λ^+ is a natural constraint for $\varphi_\lambda(\cdot)$ (see Papageorgiou–Rădulescu–Repovš [27, p. 425]).

Proposition 3.7. *If hypotheses (H) hold, $0 < \lambda < \hat{\lambda}_1^\alpha$, and $\hat{u} \in N_\lambda^+$ is the minimizer of $\varphi_\lambda|_{N_\lambda^+}$ (see Proposition 3.6), then $\hat{u} \in K_{\varphi_\lambda}$.*

Proof. Consider the C^1 -functionals $\psi_\lambda, k: W_0^{1,\eta}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\psi_\lambda(u) := \langle \varphi'_\lambda(u), u \rangle \quad \text{and} \quad k(u) := \int_\Omega \beta(z)|u|^\tau dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

From Propositions 3.3 and 3.6, we have

$$\varphi_\lambda(\hat{u}) = \inf\{\varphi_\lambda(u) \mid u \in W_0^{1,\eta}(\Omega) \setminus \{0\}, \psi_\lambda(u) = 0 \text{ and } k(u) > 0\}.$$

Invoking the Lagrange multiplier rule of Ioffe–Tihomirov [14, Theorem 4, p. 74], we can find $\mu_1 \in \mathbb{R}$ and $\mu_2 \leq 0$ such that

$$\varphi'_\lambda(\hat{u}) = \mu_1 \psi'_\lambda(\hat{u}) + \mu_2 k'(\hat{u}) \quad \text{in } W_0^{1,\eta}(\Omega)^* \text{ and } \mu_2 k(\hat{u}) = 0. \tag{3.16}$$

Recall that $k(\hat{u}) > 0$ (since $\hat{u} \in N_\lambda^+$). So, from the second equality in (3.16), we have $\mu_2 = 0$. Then, the first equality in (3.16) becomes

$$\varphi'_\lambda(\hat{u}) = \mu_1 \psi'_\lambda(\hat{u}) \quad \text{in } W_0^{1,\eta}(\Omega)^*. \tag{3.17}$$

On (3.17), we act with $\hat{u} \in W_0^{1,\eta}(\Omega)$. Since $\hat{u} \in N_\lambda^+$, we obtain

$$0 = \mu_1 \langle \psi'_\lambda(\hat{u}), \hat{u} \rangle = \mu_1 (\theta_\mu^\lambda)''(1). \quad (3.18)$$

Since $\hat{u} \in N_\lambda^+$, we have $(\theta_\mu^\lambda)''(1) > 0$, and so, from (3.18), it follows that $\mu_1 = 0$. Then, we have

$$\varphi'_\lambda(\hat{u}) = 0 \quad (\text{see (3.17)}).$$

Therefore, it holds $\hat{u} \in K_{\varphi_\lambda}$. ■

Remark 3.8. We can have an alternative proof of Proposition 3.7. The continuity of $k(\cdot)$ implies that the set $D_+ := \{u \in W_0^{1,\eta}(\Omega) \mid k(u) > 0\}$ is open. Proposition 3.3 implies that

$$N_\lambda^+ = N_\lambda \cap D_+ \quad \Rightarrow \quad N_\lambda^+ \subseteq N_\lambda \quad \text{is open (for the subspace topology)}.$$

Then, on account of Proposition 3.6, we have that

$$\hat{u} \text{ is a local minimizer of } \varphi_\lambda|_{N_\lambda}.$$

Invoking the classical Lagrange multiplier rule (see Papageorgiou–Rădulescu–Repovš [27, Theorem 5.5.9, p. 422]), we can find $\mu \in \mathbb{R}$ such that

$$\varphi'_\lambda(\hat{u}) = \mu \psi'_\lambda(\hat{u}) \quad \text{in } W_0^{1,\eta}(\Omega)^*. \quad (3.19)$$

On (3.19), we act with $\hat{u} \in N_\lambda^+$. Then, since $\hat{u} \in N_\lambda^+$, we have

$$0 = \mu \langle \psi'_\lambda(\hat{u}), \hat{u} \rangle \quad \Rightarrow \quad \mu (\theta_\mu^\lambda)''(1) = 0 \quad \Rightarrow \quad \mu = 0 \quad (\text{since } \hat{u} \in N_\lambda^+).$$

From (3.19), we conclude that $\hat{u} \in K_{\varphi_\lambda}$.

Now, we are ready for the multiplicity result to the “sublinear” case.

Theorem 3.9. *If hypotheses (H) hold and $0 < \lambda < \hat{\lambda}_1^\alpha$, then problem (1.1) has a solution $\hat{u} \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega) \setminus \{0\}$ such that*

$$0 \leq \hat{u} \quad \text{and} \quad \varphi_\lambda(\hat{u}) < 0.$$

Moreover, since $\varphi_\lambda(u) = \varphi_\lambda(|u|)$ for all $u \in W_0^{1,\eta}(\Omega)$, we have that $\hat{v} = -\hat{u} \leq 0$ is also another solution of problem (1.1).

Proof. Let $\hat{u} \in N_\lambda^+$ be the minimizer of $\varphi_\lambda|_{N_\lambda^+}$ produced in Proposition 3.6. We may assume that $\hat{u} \geq 0$. From Proposition 3.7, we know that $\hat{u} \in K_{\varphi_\lambda}$. Hence,

$$\begin{cases} -\Delta_p^\alpha \hat{u}(z) - \Delta_q \hat{u}(z) = \lambda \alpha(z) \hat{u}(z)^{p-1} + \beta(z) u(z)^{\tau-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The condition on the exponents p, q (see hypotheses (H)) and Gasiński–Winkert [11, Theorem 3.1] imply that $\hat{u} \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$. Moreover, we have

$$\varphi_\lambda(\hat{u}) = m_\lambda^+ < 0.$$

Evidently, $\hat{v} = -\hat{u} \leq 0$ is also a solution of problem (1.1). \blacksquare

We conclude our study of the “sublinear” case by examining what happens as $\lambda \rightarrow (\hat{\lambda}_1^\alpha)^-$.

Proposition 3.10. *If hypotheses (H) hold and $\int_\Omega \beta(z) \hat{u}_1^\tau dz > 0$, then $m_\lambda^+ \rightarrow -\infty$ as $\lambda \rightarrow (\hat{\lambda}_1^\alpha)^-$.*

Proof. For $\mu \in (0, 1)$, we set $\tilde{u}_1 := \mu \hat{u}_1$. By hypotheses, we have $\int_\Omega \beta(z) \tilde{u}_1^\tau dz > 0$. Then, Propositions 3.1 and 3.3 imply that there is unique $t_0 = t_0(\mu) > 0$ such that

$$t_0 \tilde{u}_1 \in N_\lambda^+. \quad (3.20)$$

From (3.2), we see that, by taking $\mu \in (0, 1)$ small, it holds

$$t_0 > 1 \quad \text{and} \quad \int_\Omega \beta(z) \tilde{u}_1^\tau dz - \|D\tilde{u}_1\|_q^q - (\hat{\lambda}_1^\alpha - \lambda) \rho_{\eta_0}(\tilde{u}_1) > 0. \quad (3.21)$$

From (3.2) and (3.20), we have

$$\begin{aligned} t_0^{p-\tau} (\hat{\lambda}_1^\alpha - \lambda) \rho_{\eta_0}(\tilde{u}_1) + t_0^{q-\tau} \|D\tilde{u}_1\|_q^q &= \int_\Omega \beta(z) \tilde{u}_1^\tau dz \\ \Rightarrow t_0^{p-\tau} [(\hat{\lambda}_1^\alpha - \lambda) \rho_{\eta_0}(\tilde{u}_1) + \|D\tilde{u}_1\|_q^q] &\geq \int_\Omega \beta(z) \tilde{u}_1^\tau dz, \end{aligned}$$

where we have used the facts (3.21) and $\mu < q < p$. This indicates that

$$t_0 \geq \left[\frac{\int_\Omega \beta(z) \tilde{u}_1^\tau dz}{(\hat{\lambda}_1^\alpha - \lambda) \rho_{\eta_0}(\tilde{u}_1) + \|D\tilde{u}_1\|_q^q} \right]^{\frac{1}{p-\tau}}.$$

Using (3.21) and since $\tau < q < p$, we obtain

$$\begin{aligned} \varphi_\lambda(t_0 \tilde{u}_1) &\leq \frac{t_0^\tau}{p} \left[(\hat{\lambda}_1^\alpha - \lambda) \rho_{\eta_0}(\tilde{u}_1) + \|D\tilde{u}_1\|_q^q - \int_\Omega \beta(z) \tilde{u}_1^\tau dz \right] \\ &\leq \frac{1}{p} \left[\frac{\int_\Omega \beta(z) \tilde{u}_1^\tau dz}{(\hat{\lambda}_1^\alpha - \lambda) \rho_{\eta_0}(\tilde{u}_1) + \|D\tilde{u}_1\|_q^q} \right]^{\frac{\tau}{p-\tau}} \xi \end{aligned}$$

with $\xi = (\hat{\lambda}_1^\alpha - \lambda) \rho_{\eta_0}(\tilde{u}_1) + \|D\tilde{u}_1\|_q^q - \int_\Omega \beta(z) \tilde{u}_1^\tau dz < 0$. Hence,

$$\limsup_{\lambda \rightarrow (\hat{\lambda}_1^\alpha)^-} \varphi_\lambda(t_0 \tilde{u}_1) \leq \frac{\xi'}{p\mu^{\frac{(q-\tau)\tau}{p-\tau}}},$$

with $\xi' = \left[\frac{\int_\Omega \beta(z) \hat{u}_1^\tau dz}{\|D\hat{u}_1\|_q^q} \right]^{\frac{\tau}{p-\tau}} (\|D\tilde{u}_1\|_q^q - \int_\Omega \beta(z) \tilde{u}_1^\tau dz) < 0$. Since $\mu \in (0, 1)$ can be arbitrarily small, we conclude that

$$m_\lambda^+ \rightarrow -\infty \quad \text{as } \lambda \rightarrow (\hat{\lambda}_1^\alpha)^-. \quad \blacksquare$$

4. Superlinear perturbation

In this section, we study problem (1.2). Suppose that $u \in W_0^{1,\eta}(\Omega) \setminus \{0\}$ with

$$\int_{\Omega} \beta(z)|u|^r dz \leq 0.$$

We have

$$\begin{aligned} (\theta_u^\lambda)'(t) &= t^{p-1}[\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u)] + t^{q-1}\|Du\|_q^q \\ &\quad - t^{r-1} \int_{\Omega} \beta(z)|u|^r dz > 0 \quad \text{for all } t > 0 \text{ (see (2.1))} \\ \Rightarrow (\theta_u^\lambda)(\cdot) &\text{ is strictly increasing on } \mathbb{R}_+ \quad (\text{with } \theta_u^\lambda(0) = 0). \end{aligned}$$

Then, from (1.2), it follows that $N_\lambda = \emptyset$. This suggests that we should focus on those $u \in W_0^{1,\eta}(\Omega) \setminus \{0\}$ with $\int_{\Omega} \beta(z)|u|^r dz > 0$ (see Proposition 4.3 below).

We start with a result similar to Proposition 3.1, but now the behavior of the fibering function is different.

Proposition 4.1. *If hypotheses (H) hold, $0 < \lambda < \hat{\lambda}_1^\alpha$, and $u \in W_0^{1,\eta}(\Omega)$ satisfies*

$$\int_{\Omega} \beta(z)|u|^r dz > 0,$$

then there exists unique $t_u > 0$ such that $t_u u \in N_\lambda$.

Proof. In this case, since $q < p < r$, we see that

$$\theta_u^\lambda(t) > 0, \quad \text{for } t \in (0, 1) \text{ small,} \quad \theta_u^\lambda(t) < 0, \quad \text{for } t > 1 \text{ large.}$$

So, we can find $t_u > 0$ such that

$$\theta_u^\lambda(t_u) = \max_{t \geq 0} \theta_u^\lambda(t) \Rightarrow (\theta_u^\lambda)'(t_u) = 0 \Rightarrow t_u u \in N_\lambda \quad (\text{see (3.1)}).$$

We have

$$\frac{1}{t_u^{r-p}}[\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u)] + \frac{1}{t_u^{r-q}}\|Du\|_q^q = \int_{\Omega} \beta(z)|u|^r dz. \quad (4.1)$$

In this equation, the left-hand side as a function of $t > 0$ is strictly decreasing, while the right-hand side is a constant (independent of $t > 0$). So, the solution $t_u > 0$ of (4.1) is unique. \blacksquare

Reasoning as in the proof of Proposition 3.2, we obtain the following result.

Proposition 4.2. *If hypotheses (H) hold and $0 < \lambda < \hat{\lambda}_1^\alpha$, then $N_\lambda \neq \emptyset$.*

In contrast to the sublinear case, the relevant component of N_λ for the superlinear case is N_λ^- . The other two components N_λ^+ and N_λ^0 are empty.

Proposition 4.3. *If hypotheses (H) hold and $0 < \lambda < \hat{\lambda}_1^\alpha$, then*

$$\emptyset \neq N_\lambda^- = \left\{ u \in N_\lambda \mid \int_\Omega \beta(z)|u|^r dz > 0 \right\} \quad \text{and} \quad N_\lambda^+, N_\lambda^0 = \emptyset.$$

Proof. Let $\hat{C}_+ := \{u \in N_\lambda \mid \int_\Omega \beta(z)|u|^r dz > 0\}$. If $u \in N_\lambda^-$, then by definition we have

$$\begin{aligned} & (\theta_u^\lambda)''(1) < 0 \\ \Rightarrow & (p-1)[\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u)] + (q-1)\|Du\|_q^q - (r-1)\int_\Omega \beta(z)|u|^r dz < 0 \\ \Rightarrow & (q-1)[\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u) + \|Du\|_q^q] < (r-1)\int_\Omega \beta(z)|u|^r dz \\ & \text{(since } q < p \text{ and } \lambda \in (0, \hat{\lambda}_1^\alpha)) \\ \Rightarrow & (r-q)\int_\Omega \beta(z)|u|^r dz > 0 \quad \text{(since } u \in N_\lambda^- \subset N_\lambda) \\ \Rightarrow & \int_\Omega \beta(z)|u|^r dz > 0 \quad \text{(since } q < r) \\ \Rightarrow & N_\lambda^- \subseteq \hat{C}_+. \end{aligned}$$

On the other hand, if $u \in \hat{C}_+$, then

$$\begin{aligned} (\theta_u^\lambda)''(1) &= (p-1)[\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u)] + (q-1)\|Du\|_q^q - (r-1)\int_\Omega \beta(z)|u|^r dz \\ &\leq (p-1)[\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u) + \|Du\|_q^q] - (r-1)\int_\Omega \beta(z)|u|^r dz \\ &\quad \text{(since } q < p \text{ and } \lambda \in (0, \hat{\lambda}_1^\alpha)) \\ &= (p-r)\int_\Omega \beta(z)|u|^r dz < 0 \quad \text{(since } u \in N_\lambda \text{ and } p < r). \end{aligned}$$

Then, we have $\hat{C}_+ \subseteq N_\lambda^-$. So, we conclude that $N_\lambda^- = \hat{C}_+$.

If $u \in N_\lambda$ and $\lambda \in (0, \hat{\lambda}_1^\alpha)$, then

$$\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u) + \|Du\|_q^q > 0 \quad \text{(see (2.1)).} \quad (4.2)$$

Whereas, from $u \in N_\lambda$, (4.2) and $p < r$, we have

$$\begin{aligned} (\theta_u^\lambda)''(1) &= (p-1)[\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u)] + (q-1)\|Du\|_q^q - (r-1)\int_\Omega \beta(z)|u|^r dz \\ &\leq (p-1)[\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u) + \|Du\|_q^q] - (r-1)\int_\Omega \beta(z)|u|^r dz \\ &= (p-r)[\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u) + \|Du\|_q^q] \quad \text{(since } u \in N_\lambda) \\ &< 0 \quad \text{(see (4.2) and recall } p < r). \end{aligned}$$

This implies that $\emptyset \neq N_\lambda^- = N_\lambda$ and $N_\lambda^+, N_\lambda^0 = \emptyset$. ■

Proposition 4.4. *If hypotheses (H) hold and $0 < \lambda < \widehat{\lambda}_1^\alpha$, then there exists $\widehat{c}_\lambda > 0$ such that $\widehat{c}_\lambda \leq \|u\|$ for all $u \in N_\lambda = N_\lambda^-$.*

Proof. Let $u \in N_\lambda$ and without any loss of generality, we assume that $\|u\| \leq 1$. Then, we have

$$\begin{aligned}
 & \rho_{\eta_0}(Du) - \lambda \rho_{\eta_0}(u) + \|Du\|_q^q = \int_{\Omega} \beta(z)|u|^r dz, \\
 \Rightarrow & \left(1 - \frac{\lambda}{\widehat{\lambda}_1^\alpha}\right) \rho_{\eta_0}(Du) + \|Du\|_q^q \leq c_1 \|u\|^r \\
 & \text{(see (2.1), hypotheses (H) and Proposition 2.1)} \\
 \Rightarrow & \rho_{\eta}(Du) \leq c_2 \|u\|^r \quad \text{for some } c_2 = c_2(\lambda) > 0 \quad \text{(since } \lambda \in (0, \widehat{\lambda}_1^\alpha)) \\
 \Rightarrow & \|u\|^p \leq c_2 \|u\|^r \quad \text{(since } \|u\| \leq 1, \text{ see Proposition 2.2)} \\
 \Rightarrow & 0 < \left(\frac{1}{c_2}\right)^{\frac{1}{r-p}} \leq \|u\| \quad \text{for all } u \in N_\lambda
 \end{aligned}$$

for some $c_1 > 0$. This completes the proof of the proposition. \blacksquare

So, according to the above proposition, the elements of the Nehari manifold $N_\lambda = N_\lambda^-$ are bounded away from zero. We will use this fact to show that $\varphi_\lambda|_{N_\lambda}$ is coercive.

Proposition 4.5. *If hypotheses (H) hold and $0 < \lambda < \widehat{\lambda}_1^\alpha$, then $\varphi_\lambda|_{N_\lambda}$ is coercive and $0 < c_\lambda^* \leq \varphi_\lambda|_{N_\lambda}$.*

Proof. Let $u \in N_\lambda = N_\lambda^-$. We have

$$\begin{aligned}
 \varphi_\lambda(u) & \geq \frac{1}{p} [\rho_{\eta_0}(Du) - \lambda \rho_{\eta_0}(u) + \|Du\|_q^q] - \frac{1}{r} \int_{\Omega} \beta(z)|u|^r dz \\
 & \hspace{15em} \text{(since } q < p \text{ and } \lambda \in (0, \widehat{\lambda}_1^\alpha)) \\
 & = \left(\frac{1}{p} - \frac{1}{r}\right) [\rho_{\eta_0}(Du) - \lambda \rho_{\eta_0}(u) + \|Du\|_q^q] \quad \text{(since } u \in N_\lambda, \text{ see (3.2))} \\
 & \geq \frac{r-p}{rp} \left(1 - \frac{\lambda}{\widehat{\lambda}_1^\alpha}\right) \rho_{\eta_0}(Du) \quad \text{(see (3.2))} \\
 & \geq \frac{r-p}{rp} \left(1 - \frac{\lambda}{\widehat{\lambda}_1^\alpha}\right) \min\{\|u\|^p, \|u\|^q\} \quad \text{(see Proposition 2.2)}.
 \end{aligned}$$

From the estimates above and Proposition 3.3 as well as $p < r$ and $\lambda \in (0, \widehat{\lambda}_1^\alpha)$, we see that $\varphi_\lambda|_{N_\lambda}$ is coercive. \blacksquare

Let $m_\lambda^- = \inf_{N_\lambda^-} \varphi_\lambda$. From Proposition 4.5, we see that $m_\lambda^- > 0$. We show that this minimization problem has a solution.

Proposition 4.6. *If hypotheses (H) hold and $0 < \lambda < \widehat{\lambda}_1^\alpha$, then there exists $\widehat{u}_0 \in N_\lambda^-$ such that $\varphi_\lambda(\widehat{u}_0) = m_\lambda^-$.*

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subseteq N_\lambda^-$ be a minimizing sequence, that is,

$$\varphi_\lambda(u_n) \downarrow m_\lambda^- \quad \text{as } n \rightarrow \infty.$$

From Proposition 4.5, we know that $\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$ is bounded. So, we may assume that

$$u_n \rightarrow \hat{u}_0 \text{ weakly in } W_0^{1,\eta}(\Omega) \quad \text{and} \quad u_n \rightarrow \hat{u}_0 \text{ in } L^r(\Omega) \text{ as } n \rightarrow \infty \quad (4.3)$$

thanks to $r < q^*$ and Proposition 2.1. Since $u_n \in N_\lambda^- = N_\lambda$, for all $n \in \mathbb{N}$ and $q < p$, we have

$$\begin{aligned} \varphi_\lambda(u_n) &\leq \left(\frac{1}{q} - \frac{1}{r}\right) \int_\Omega \beta(z) |u_n|^r dz \\ \Rightarrow 0 < m_\lambda^- &\leq \left(\frac{1}{q} - \frac{1}{r}\right) \int_\Omega \beta(z) |\hat{u}_0|^r dz \quad (\text{see (4.3)}). \end{aligned}$$

This reveals that $\hat{u}_0 \neq 0$.

In addition, we have the following claim.

Claim 3. $u_n \rightarrow \hat{u}_0$ in $W_0^{1,\eta}(\Omega)$ as $n \rightarrow \infty$.

As before, we proceed indirectly. So, we assume that the assertion of Claim 3 is not true. Then, on account of (4.3), the following strict inequality holds:

$$\rho_\eta(D\hat{u}_0) < \liminf_{n \rightarrow \infty} \rho_\eta(Du_n). \quad (4.4)$$

Since $u_n \in N_\lambda^- = N_\lambda$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \rho_\eta(Du_n) - \lambda \rho_{\eta_0}(u_n) - \int_\Omega \beta(z) |u_n|^r dz &= 0, \\ \Rightarrow \rho_\eta(D\hat{u}_0) - \lambda \rho_{\eta_0}(\hat{u}_0) - \int_\Omega \beta(z) |\hat{u}_0|^r dz &< 0 \quad (\text{see (4.3) and (4.4)}), \quad (4.5) \end{aligned}$$

$$\Rightarrow \int_\Omega \beta(z) |\hat{u}_0|^r dz > 0. \quad (4.6)$$

From Proposition 4.1 and its proof, we know that there exists unique $\hat{t}_0 = \hat{t}_{\hat{u}_0} > 0$ such that

$$\varphi_\lambda(\hat{t}_0 \hat{u}_0) = \theta_{\hat{u}_0}^\lambda(t_0) = \max_{t \geq 0} \theta_{\hat{u}_0}^\lambda(t) \Rightarrow \hat{t}_0 \hat{u}_0 \in N_\lambda \quad (\text{see (3.1)}). \quad (4.7)$$

From the properties of the fibering function $\theta_{\hat{u}_0}^\lambda$ and (4.5), it follows that $\hat{t}_0 < 1$. Since $u_n \in N_\lambda^- = N_\lambda$, we have

$$\varphi_\lambda(\hat{t}_0 u_n) \leq \varphi_\lambda(u_n) \quad \text{for all } n \in \mathbb{N} \Rightarrow \varphi_\lambda(\hat{t}_0 \hat{u}_0) < \hat{m}_\lambda^- \quad (\text{see (4.4)}).$$

But this contradicts (4.6) (see (4.7) and Proposition 4.3). This proves the claim.

Notice that

$$\begin{aligned} & \rho_\eta(Du_n) - \lambda\rho_{\eta_0}(u_n) - \int_\Omega \beta(z)|u_n|^r dz = 0 \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow & \rho_\eta(D\hat{u}_0) - \lambda\rho_{\eta_0}(\hat{u}_0) - \int_\Omega \beta(z)|\hat{u}_0|^r dz = 0 \quad (\text{see Claim 3}) \\ \Rightarrow & \hat{u}_0 \in N_\lambda = N_\lambda^-. \end{aligned}$$

Finally, we have $m_\lambda^- = \varphi_\lambda(\hat{u}_0) \in N_\lambda^-$. \blacksquare

Reasoning as in the proof of Proposition 3.7 (again we have two proofs), we show that $N_\lambda^- = N_\lambda$ is a natural constraint for $\varphi_\lambda(\cdot)$.

Proposition 4.7. *If hypotheses (H) hold, $0 < \lambda < \hat{\lambda}_1^\alpha$, and $\hat{u}_0 \in N_\lambda^-$ is the minimizer of $\varphi_\lambda|_{N_\lambda^-}$, then $\hat{u}_0 \in K_{\varphi_\lambda}$.*

This leads to the following multiplicity theorem for problem (1.2). Recall that if $h \in L^\infty(\Omega)$, then we write $0 \prec h$ if for all $K \subseteq \Omega$ compact we have $0 < c_K \leq h(z)$ for a.a. $z \in K$.

Theorem 4.8. *If hypotheses (H) hold and $0 < \lambda < \hat{\lambda}_1^\alpha$, then problem (1.2) has a solution*

$$\hat{u}_0 \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega) \setminus \{0\}, \quad 0 \prec \hat{u}_0,$$

with $\varphi_\lambda(\hat{u}_0) > 0$ (positive energy). Also, $\hat{v}_0 = -\hat{u}_0 \prec 0$ is another solution of problem (1.2) with positive energy.

Proof. From Propositions 4.6 and 4.7, we can find $\hat{u}_0 \in N_\lambda^-$ such that

$$0 < m_\lambda^- = \varphi_\lambda(\hat{u}_0) \quad \text{and} \quad \hat{u}_0 \in K_{\varphi_\lambda}.$$

Since $\varphi_\lambda(u) = \varphi_\lambda(|u|)$ for all $u \in W_0^{1,\eta}(\Omega)$, we may assume that $\hat{u}_0 \geq 0$. Then,

$$\begin{cases} -\Delta_p^\alpha \hat{u}_0 - \Delta_q \hat{u}_0 = \lambda\alpha(z)\hat{u}_0^{p-1} + \beta(z)\hat{u}_0^{r-1} & \text{in } \Omega, \\ \hat{u}_0 = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.8)$$

This combined with [11] implies $\hat{u}_0 \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$. Let $\varrho = \|\hat{u}_0\|_\infty$. Since $\beta \in L^\infty(\Omega)$ (see hypotheses (H)) and $r > p$, we can find $\hat{\xi}_\varrho > 0$ such that

$$\begin{aligned} & (\lambda + \hat{\xi}_\varrho)x^{p-1} + \beta(z)x^{r-1} \geq 0 \quad \text{for a.a. } z \in \Omega, \quad \text{all } 0 \leq x \leq \varrho, \\ \Rightarrow & -\Delta_p^\alpha \hat{u}_0 - \Delta_q \hat{u}_0 + \hat{\xi}_\varrho \hat{u}_0^{p-1} \geq 0 \quad \text{in } \Omega \quad (\text{see (4.8)}), \\ \Rightarrow & 0 \prec \hat{u}_0 \quad (\text{see [30, Proposition 4]}). \end{aligned}$$

Clearly, $\hat{v}_0 = -\hat{u}_0 \prec 0$ is also a solution of problem (1.2) and $\varphi_\lambda(\hat{v}_0) > 0$. \blacksquare

Finally, as we did in the sublinear case, we examine what happens to the ground state energy as $\lambda \rightarrow (\hat{\lambda}_1^\alpha)^-$.

Proposition 4.9. *If hypotheses (H) hold and $\int_{\Omega} \beta(z) \hat{u}_0^r dz > 0$, then $m_{\lambda}^{-} \rightarrow 0^+$ as $\lambda \rightarrow (\hat{\lambda}_1^{\alpha})^{-}$.*

Proof. The reasoning is similar to that in the proof of Proposition 3.10. Let $\mu > 1$ and set $\tilde{u}_1 = \mu \hat{u}_0$. Then, using the fact $\int_{\Omega} \beta(z) \tilde{u}_1^r dz > 0$ and Proposition 4.1, we can find unique $\hat{t} > 0$ such that

$$\hat{t} \tilde{u}_1 \in N_{\lambda} = N_{\lambda}^{-}. \quad (4.9)$$

On account of (3.2), if $\mu > 1$ is large, then

$$\hat{t} < 1 \quad \text{and} \quad \int_{\Omega} \beta(z) \tilde{u}_1^r dz - \|D\tilde{u}_1\|_q^q > 0 \quad (\text{since } q < p < r).$$

From (4.9), we have

$$\begin{aligned} & \hat{t}^p (\hat{\lambda}_1^{\alpha} - \lambda) \rho_{\eta_0}(\tilde{u}_1) + \hat{t}^q \|D\tilde{u}_1\|_q^q = \hat{t}^r \int_{\Omega} \beta(z) \tilde{u}_1^r dz \\ \Rightarrow & \hat{t}^p (\hat{\lambda}_1^{\alpha} - \lambda) \rho_{\eta_0}(\tilde{u}_1) \leq \hat{t}^r \left[\int_{\Omega} \beta(z) \tilde{u}_1^r dz - \|D\tilde{u}_1\|_q^q \right] \quad (\text{since } \hat{t} < 1 \text{ and } q < p < r) \\ \Rightarrow & \hat{t} \geq \left[\frac{(\hat{\lambda}_1^{\alpha} - \lambda) \rho_{\eta_0}(\tilde{u}_1)}{\int_{\Omega} \beta(z) \tilde{u}_1^r dz - \|D\tilde{u}_1\|_q^q} \right]^{\frac{1}{r-p}}. \end{aligned} \quad (4.10)$$

Also, we have

$$\hat{t} \leq \left[\frac{(\hat{\lambda}_1^{\alpha} - \lambda) \rho_{\eta_0}(\tilde{u}_1) + \|D\tilde{u}_1\|_q^q}{\int_{\Omega} \beta(z) \tilde{u}_1^r dz} \right]^{\frac{1}{r-q}}. \quad (4.11)$$

Then, using (4.10) and (4.11), we have

$$\begin{aligned} \varphi_{\lambda}(\hat{t} \tilde{u}_1) & \leq \frac{1}{q} \left[\frac{(\hat{\lambda}_1^{\alpha} - \lambda) \rho_{\eta_0}(D\tilde{u}_1) + \|D\tilde{u}_1\|_q^q}{\int_{\Omega} \beta(z) \tilde{u}_1^r dz} \right]^{\frac{q}{r-q}} \\ & \quad - \frac{1}{r} \left[\frac{(\hat{\lambda}_1^{\alpha} - \lambda) \rho_{\eta_0}(\tilde{u}_1)}{\int_{\Omega} \beta(z) \tilde{u}_1^r dz - \|D\tilde{u}_1\|_q^q} \right]^{\frac{r}{r-p}} \\ & \quad (\text{since } q < p < r) \\ \Rightarrow \limsup_{\lambda \rightarrow (\hat{\lambda}_1^{\alpha})^{-}} \varphi_{\lambda}(\hat{t} \tilde{u}_1) & \leq \frac{1}{q} \left(\frac{\|D\hat{u}_1\|_q^q}{\mu^{r-q} \int_{\Omega} \beta(z) \hat{u}_1^r dz} \right)^{\frac{q}{r-q}}. \end{aligned}$$

Since $\mu > 1$ can be arbitrarily large, we infer that

$$\limsup_{\lambda \rightarrow (\hat{\lambda}_1^{\alpha})^{-}} \varphi_{\lambda}(\hat{t} \tilde{u}_1) \leq 0.$$

Therefore, we conclude that $m_{\lambda}^{-} \rightarrow 0^+$ as $\lambda \rightarrow (\hat{\lambda}_1^{\alpha})^{-}$ (see (4.9) and Proposition 4.5). ■

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References

- [1] A. Aberqi, O. Benslimane, and M. Knifda, [Double phase systems with convex-concave non-linearity on complete manifold](#). *Rend. Circ. Mat. Palermo (2)* **73** (2024), no. 3, 1155–1176 Zbl 1540.35171 MR 4722476
- [2] Y. Bai, N. S. Papageorgiou, and S. Zeng, [Positive solutions for a double phase eigenvalue problem](#). *Positivity* **26** (2022), no. 5, article no. 81 Zbl 1500.35180 MR 4499505
- [3] H. Brézis and E. Lieb, [A relation between pointwise convergence of functions and convergence of functionals](#). *Proc. Amer. Math. Soc.* **88** (1983), no. 3, 486–490 Zbl 0526.46037 MR 0699419
- [4] K. J. Brown, [The Nehari manifold for a semilinear elliptic equation involving a sublinear term](#). *Calc. Var. Partial Differential Equations* **22** (2005), no. 4, 483–494 Zbl 1130.35049 MR 2124630
- [5] K. J. Brown and Y. Zhang, [The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function](#). *J. Differential Equations* **193** (2003), no. 2, 481–499 Zbl 1074.35032 MR 1998965
- [6] Y. Chen and Z. Yang, [The existence of multiple solutions for a class of upper critical Choquard equation in a bounded domain](#). *Demonstr. Math.* **57** (2024), no. 1, article no. 20230152 Zbl 1541.35213 MR 4736175
- [7] B. Deręgowska, L. Gasiński, and N. S. Papageorgiou, [A multiplicity theorem for superlinear double phase problems](#). *Symmetry* **13** (2021), no. 9, article no. 1556

- [8] A. Fiscella, P. K. Mishra, and V. M. Tripathi, [Nehari manifold method for singular double phase problem with optimal control on parameter](#). *J. Math. Phys.* **64** (2023), no. 11, article no. 111504 Zbl [1531.35146](#) MR [4662254](#)
- [9] L. Gasiński and N. S. Papageorgiou, *Exercises in analysis. Part 1*. Probl. Books in Math., Springer, Cham, 2014 Zbl [1298.00007](#) MR [3307732](#)
- [10] L. Gasiński and N. S. Papageorgiou, [Constant sign and nodal solutions for superlinear double phase problems](#). *Adv. Calc. Var.* **14** (2021), no. 4, 613–626 Zbl [1478.35118](#) MR [4319046](#)
- [11] L. Gasiński and P. Winkert, [Constant sign solutions for double phase problems with superlinear nonlinearity](#). *Nonlinear Anal.* **195** (2020), article no. 111739 Zbl [1437.35233](#) MR [4050785](#)
- [12] L. Gasiński and P. Winkert, [Sign changing solution for a double phase problem with nonlinear boundary condition via the Nehari manifold](#). *J. Differential Equations* **274** (2021), 1037–1066 Zbl [1458.35149](#) MR [4189002](#)
- [13] P. Harjulehto and P. Hästö, *Orlicz spaces and generalized Orlicz spaces*. Lecture Notes in Math. 2236, Springer, Cham, 2019 Zbl [1436.46002](#) MR [3931352](#)
- [14] A. D. Ioffe and V. M. Tihomirov, *Theory of extremal problems*. Stud. Math. Appl. 6, North-Holland, Amsterdam, 1979 Zbl [0407.90051](#) MR [528295](#)
- [15] G. M. Lieberman, [The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations](#). *Comm. Partial Differential Equations* **16** (1991), no. 2-3, 311–361 Zbl [0742.35028](#) MR [1104103](#)
- [16] W. Liu and G. Dai, [Existence and multiplicity results for double phase problem](#). *J. Differential Equations* **265** (2018), no. 9, 4311–4334 Zbl [1401.35103](#) MR [3843302](#)
- [17] W. Liu, G. Dai, N. S. Papageorgiou, and P. Winkert, [Existence of solutions for singular double phase problems via the Nehari manifold method](#). *Anal. Math. Phys.* **12** (2022), no. 3, article no. 75 Zbl [1496.35185](#) MR [4411687](#)
- [18] Z. Liu and N. S. Papageorgiou, [Positive solutions for double phase problems with combined nonlinearities](#). *Positivity* **26** (2022), no. 1, article no. 24 Zbl [1485.35249](#) MR [4385399](#)
- [19] Z. Liu and N. S. Papageorgiou, [Nonautonomous \$\(p, q\)\$ -equations with unbalanced growth and competing nonlinearities](#). *J. Math. Pures Appl. (9)* **182** (2024), 164–194 Zbl [1536.35165](#) MR [4677156](#)
- [20] P. Marcellini, [Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions](#). *Arch. Rational Mech. Anal.* **105** (1989), no. 3, 267–284 Zbl [0667.49032](#) MR [0969900](#)
- [21] P. Marcellini, [Regularity and existence of solutions of elliptic equations with \$p, q\$ -growth conditions](#). *J. Differential Equations* **90** (1991), no. 1, 1–30 Zbl [0724.35043](#) MR [1094446](#)
- [22] P. Marcellini, [Growth conditions and regularity for weak solutions to nonlinear elliptic pdes](#). *J. Math. Anal. Appl.* **501** (2021), no. 1, article no. 124408 Zbl [1512.35301](#) MR [4258802](#)
- [23] P. Marcellini, [Local Lipschitz continuity for \$p, q\$ -PDEs with explicit \$u\$ -dependence](#). *Nonlinear Anal.* **226** (2023), article no. 113066 Zbl [1501.35188](#) MR [4502254](#)
- [24] G. Mingione and V. Rădulescu, [Recent developments in problems with nonstandard growth and nonuniform ellipticity](#). *J. Math. Anal. Appl.* **501** (2021), no. 1, article no. 125197 Zbl [1467.49003](#) MR [4258810](#)
- [25] P. K. Mishra, K. Silva, and V. M. Tripathi, [Extremal parameter for double phase problem with concave-convex nonlinearity](#). *Commun. Nonlinear Sci. Numer. Simul.* **126** (2023), article no. 107463 Zbl [1528.35067](#) MR [4626392](#)
- [26] N. S. Papageorgiou, A. Pudelko, and V. D. Rădulescu, [Non-autonomous \$\(p, q\)\$ -equations with unbalanced growth](#). *Math. Ann.* **385** (2023), no. 3-4, 1707–1745 Zbl [1511.35197](#) MR [4566704](#)

- [27] N. S. Papageorgiou, V. D. Rădulescu, and D. D. Repovš, *Nonlinear analysis—theory and methods*. Springer Monogr. Math., Springer, Cham, 2019 Zbl 1414.46003 MR 3890060
- [28] N. S. Papageorgiou, V. D. Rădulescu, and Y. Wang, [Constant sign and nodal solutions for resonant double phase problems](#). *Ann. Fenn. Math.* **48** (2023), no. 2, 757–777 Zbl 1532.35245 MR 4673800
- [29] N. S. Papageorgiou, V. D. Rădulescu, and Y. Zhang, [Resonant double phase equations](#). *Nonlinear Anal. Real World Appl.* **64** (2022), article no. 103454 Zbl 1479.35291 MR 4339038
- [30] N. S. Papageorgiou, C. Vetro, and F. Vetro, [Multiple solutions for parametric double phase Dirichlet problems](#). *Commun. Contemp. Math.* **23** (2021), no. 4, article no. 2050006 Zbl 1467.35181 MR 4237564
- [31] N. S. Papageorgiou, C. Vetro, and F. Vetro, [Least energy solutions with sign information for parametric double phase problems](#). *Results Math.* **77** (2022), no. 1, article no. 35 Zbl 1482.35102 MR 4354713
- [32] N. S. Papageorgiou and P. Winkert, *Applied nonlinear functional analysis. An introduction*. De Gruyter Grad., De Gruyter, Berlin, 2018 Zbl 1404.46001 MR 3823796
- [33] N. S. Papageorgiou and C. Zhang, [Multiple ground-state solutions with sign information for double-phase Robin problems](#). *Israel J. Math.* **253** (2023), no. 1, 419–443 Zbl 1518.35410 MR 4575414
- [34] V. V. Zhikov, [Averaging of functionals of the calculus of variations and elasticity theory](#). *Math. USSR Izv.* **29** (1987), no. 1, 33–66 Zbl 0599.49031
- [35] V. V. Zhikov, [On Lavrentiev’s phenomenon](#). *Russian J. Math. Phys.* **3** (1995), no. 2, 249–269 Zbl 0910.49020 MR 1350506

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