Z. Anal. Anwend. 45 (2026), 61–82
DOI 10.4171/ZAA/1814

Constraint minimizers for Choquard equations with different potentials

Min Liu and Rui Sun

Abstract. This paper is devoted to the normalized solutions of the Choquard equations and the magnetic Choquard equations involving different external potentials. Under the L^2-norm constraint, we apply the variational method and the Gagliardo–Nirenberg-type inequality with the Riesz potential to prove the existence of constraint minimizers of the energy functionals. Particularly, we obtain the compactness of minimizing sequences by establishing the relationship between minimal energies with respect to different mass. We extend and improve the research by Alves and Ji [J. Geom. Anal. 32 (2022), no. 5, article no. 165].

1. Introduction

In this paper, we are concerned with the normalized solutions of the following Choquard equation:

$$-\Delta v + (V(x) + \mu)v = (I_\theta * |v|^q)|v|^{q-2}v \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

under the constraint

$$\int_{\mathbb{R}^N} |v(x)|^2 \mathrm{d}x = a^2, \tag{1.2}$$

where the Riesz potential I_θ for $\theta \in (0, N)$ is defined as

$$I_\theta(x) = \frac{\Gamma(\frac{N-\theta}{2})}{2^\theta \pi^{\frac{N}{2}} \Gamma(\frac{\theta}{2})} |x|^{\theta-N}, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

The Choquard equations appeared in many physical models. When $N = 3$, $\theta = 2$, and $q = 2$, equation (1.1) is the well-known Pekar–Choquard equation [8], which was proposed by Pekar in 1954 to describe the quantum mechanics of a polaron at rest [20] and was also introduced by Choquard in 1976 in the modelling of a one-component plasma [15]. Moreover, the Choquard equations are often known as the Schrödinger–Newton equations in the models coupling Schrödinger equations of quantum physics with nonrelativistic Newtonian gravity [19].

Mathematics Subject Classification 2020: 35A15 (primary); 35J20, 35Q60 (secondary).
Keywords: Choquard equation, normalized solution, potential, magnetic field.

There are usually two perspectives to study equation (1.1). One is to fix $\mu \in \mathbb{R}$ as a constant, in which case equation (1.1) belongs to the fixed frequency problems and has been investigated widely. For instance, Moroz and Van Schaftingen [18] proved the existence, regularity, positivity, radial symmetry, and decay asymptotics at infinity of the ground state solutions for equation (1.1) when $V(x) \equiv 0$ and $\mu = 1$. The other is to treat $\mu \in \mathbb{R}$ as a Lagrange multiplier, in which case equation (1.1) is a fixed mass problem and its solutions are called the normalized solutions under the constraint (1.2). The pioneering result for equation (1.1) with (1.2) comes from [15], where Lieb proved the existence and uniqueness of the normalized solution by using symmetric decreasing rearrangement inequalities when $V(x) \equiv 0$, $N = 3$, $\theta = 2$, and $q = 2$. Later, Ye [29] studied the existence of normalized solutions of equation (1.1) by using an alternative method for $q = \frac{N+\theta+2}{N}$, $0 \le V(x) \in L^{\infty}_{loc}(\mathbb{R}^N)$, and $V(x) \to \infty$ as $|x| \to \infty$.

Over the years, normalized solutions of Choquard equations have received more and more attention. Cao, Jia, and Luo [7] studied the existence, multiplicity, and asymptotic behavior of normalized solutions for equation (1.1) with $V(x) \equiv 0$ and the nonlinearity replaced by $\nu(|x|^{-\alpha} * |v|^2)v + (|x|^{-\beta} * |v|^2)v$, and they also studied the stability of the corresponding standing waves for the related time-dependent problem. Jia and Luo [14] continued to analyze the case of $\beta = 4$. Cingolani and Tanaka [9] proved the existence of radially symmetric normalized solutions for

$$-\Delta v + \mu v = (I_\theta * G(v))G'(v) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $G \in C^1(\mathbb{R}, \mathbb{R})$ satisfies Berestycki–Lions-type conditions. Xia and Zhang [27] also considered equation (1.3) and established the existence of saddle-type normalized solutions under suitable assumptions on G by concentration compactness principle with a minimax procedure in the saddle-type symmetric subspace. Ao, Zhao, and Zou [4] proved the compactness of every minimizing sequence and the existence of normalized solutions for

$$-\Delta v + (V(x) + \mu)v = (I_\theta * G(v))G'(v) \quad \text{in } \mathbb{R}^N$$

by imposing appropriate conditions on V and G.

For other results about normalized solutions, we also refer to [3, 22, 23, 25] for nonlinear Schrödinger equations and [2, 5, 10, 13, 17, 28] for the elliptic equations with external potentials. Specifically, Alves and Ji [2] considered

$$-\Delta v + (V(x) + \mu)v = |v|^{q-2}v \quad \text{in } \mathbb{R}^N$$

and studied the existence of normalized solutions under different types of $V(x)$ when $q \in (2, \frac{2N+4}{N})$ and $N \ge 2$. They also proved some similar results for the nonlinear magnetic Schrödinger equations.

Driven by physical relevance and motivated by above works, we aim to study the existence of normalized solutions for equation (1.1) under the constraint (1.2). Different from the cases of Schrödinger equations, we need to develop new analytic techniques to overcome the lack of compactness due to the nonlocal convolution term. Towards this purpose, we assume that $V(x) \geq 0$ is continuous and characterized by the following four distinct features.

(V1) V is 1-periodic in $x_1, x_2, \ldots, x_N$.

(V2) V is asymptotically periodic: there is a 1-periodic function $V_P : \mathbb{R}^N \to \mathbb{R}$ such that $V(x) \leq V_P(x)$ for $x \in \mathbb{R}^N$ and

$$|V(x) - V_P(x)| \to 0 \quad \text{as } |x| \to \infty. \tag{1.4}$$

(V3) $V(x) = K(\varepsilon x)$, where $\varepsilon > 0$ is a parameter, $K \in L^\infty(\mathbb{R}^N)$ and

$$K_\infty := \liminf_{|x|\to\infty} K(x) > K_0 := \inf_{x\in\mathbb{R}^N} K(x) > 0. \tag{1.5}$$

(V4) $V(x) = \lambda W(x)$, where $\lambda > 0$ is a parameter, $W \in L^\infty(\mathbb{R}^N)$, and there is a constant $M_0 > 0$ such that

$$\left|\{x \in \mathbb{R}^N : W(x) < M_0\}\right| < \infty. \tag{1.6}$$

We would like to remark that the periodic potential (V1) was considered by Ackerman in [1] to address the non-coercive functionals. Stimulated by (V1), it is natural to consider the asymptotically periodic potential (V2). Rabinowitz [21] proposed (V3) in studying the existence of solutions for a class of nonlinear Schrödinger equations. The potential (V4) can be found in [2, 6], and it implies that $\lambda W(x)$ represents a potential well whose depth is controlled by λ. It is worth mentioning that (V4) is weaker than the usual coercivity assumption: $V(x) \to \infty$ as $|x| \to \infty$. The presence of $V(x)$ breaks translation invariance and makes the problem more interesting.

For simplicity, we introduce some notations. $B_r(x)$ is an open ball centered at x with radius $r > 0$ and $B_r^c(x) := \mathbb{R}^N \setminus B_r(x)$. The symbol C denotes any positive constant, whose values may change from line to line. $\|v\| := \|v\|_{H^1(\mathbb{R}^N)}$ and $\|v\|_p := \|v\|_{L^p(\mathbb{R}^N)}$ for $p \in [1, +\infty]$. $o_n(1)$ denotes real sequences tending to 0 as $n \to \infty$. $H^{-1}(\mathbb{R}^N)$ is the dual space of $H^1(\mathbb{R}^N)$.

Before stating our results, we give the definition of the normalized ground state solutions for equation (1.1) under the constraint (1.2).

Definition 1.1. $v \in H^1(\mathbb{R}^N)$ is called a normalized ground state solution of equation (1.1), provided that it solves equation (1.1) for some $\mu \in \mathbb{R}$ and has minimal energy among all the functions satisfying (1.2).

Now, we state our main results for equation (1.1).

Theorem 1.2. *Let $q \in (\frac{N+\theta}{N}, \frac{N+\theta+2}{N})$ with $N \geq 2$ and $\theta \in (0, N)$.*

(1) *If V satisfies (V1), then, for each $a > 0$, there exists $\delta = \delta(a) > 0$ such that, for $|V|_\infty < \delta$, equation (1.1) possesses a positive normalized ground state solution for some $\mu > 0$.*

(2) *If V satisfies (V2), then, for each $a > 0$, there exists $\delta = \delta(a) > 0$, such that for $|V_P|_\infty < \delta$, equation (1.1) possesses a positive normalized ground state solution for some $\mu > 0$.*

(3) *If V satisfies (V3), then, for each $a > 0$, there exist $\delta = \delta(a) > 0$ and $\varepsilon^* = \varepsilon^*(a) > 0$, such that for $K_\infty < \delta$ and all $\varepsilon \in (0, \varepsilon^*)$, equation (1.1) possesses a positive normalized ground state solution for some $\mu > 0$.*

(4) *If V satisfies (V4), then, for each $a > 0$, there exist $\delta = \delta(a) > 0$ and $\lambda^* = \lambda^*(a) > 0$, such that for all $\lambda \in [\lambda^*, +\infty)$ and $\lambda|W|_\infty < \delta$, equation (1.1) possesses a positive normalized ground state solution for some $\mu > 0$.*

We mainly apply the constraint variational method to prove Theorem 1.2. The main difficulty is to analyze the compactness of minimizing sequences for the energy functionals. To achieve this, we establish the relationship between minimal energies with respect to different mass a. Particularly, we extend the results in [2] to the nonlocal case. Compared with [2], we relax the requirements of $V(x)$ in the third and fourth existence results and we improve some proof. The nonemptiness of the interior of $W^{-1}(0)$ in [2] is removed here.

The other part of this paper is dedicated to the following magnetic Choquard equation:

$$(-\mathrm{i}\nabla + A)^2 v + (V(x) + \mu)v = (I_\theta * |v|^q)|v|^{q-2}v \quad \text{in } \mathbb{R}^N, \tag{1.7}$$

where $\mu \in \mathbb{R}$ is unknown as the Lagrange multiplier, $V(x) \geq 0$ is continuous, and the magnetic potential $A : \mathbb{R}^N \to \mathbb{R}^N$ is continuous and bounded. Denote

$$\mathcal{A} := \sup_{x \in \mathbb{R}^N} |A(x) - A(0)|. \tag{1.8}$$

The definition of normalized ground state solutions of equation (1.7) is the same as that of equation (1.1). We have the following existence results about equation (1.7).

Theorem 1.3. *Let $q \in \left(\frac{N+\theta}{N}, \frac{N+\theta+2}{N}\right)$ with $N \geq 2$ and $\theta \in (0, N)$.*

(1) *If A is 1^N-periodic and V satisfies (V1), then, for each $a > 0$, there exists $\delta = \delta(a) > 0$ such that equation (1.7) possesses a normalized ground state solution for some $\mu > 0$, when $|V|_\infty < \delta$ and $\mathcal{A}^2 < \delta$.*

(2) *If A is 1^N-periodic and V satisfies (V2), then, for each $a > 0$, there exists $\delta = \delta(a) > 0$ such that equation (1.7) possesses a normalized ground state solution for some $\mu > 0$, when $|V_P|_\infty < \delta$ and $\mathcal{A}^2 < \delta$.*

(3) *If V satisfies (V3), $A(x) = \mathscr{A}(\varepsilon x)$ and there exists $\mathscr{A}_\infty \in \mathbb{R}^N$ satisfying $|\mathscr{A}(\varepsilon x) - \mathscr{A}_\infty| \to 0$ as $|x| \to \infty$, then, for each $a > 0$, there exist $\delta = \delta(a) > 0$ and $\varepsilon^\star = \varepsilon^\star(a) > 0$ such that, for all $\varepsilon \in (0, \varepsilon^\star)$, equation (1.7) possesses a normalized ground state solution for some $\mu > 0$, when $K_\infty < \delta$ and $\mathcal{A}^2 < \delta$.*

(4) *If V satisfies (V4), then, for each $a > 0$, there exist $\delta = \delta(a) > 0$ and $\lambda^\star = \lambda^\star(a) > 0$ such that, for all $\lambda \in [\lambda^\star, +\infty)$, equation (1.7) possesses a normalized ground state solution for some $\mu > 0$, when $\lambda|W|_\infty < \delta$ and $\mathcal{A}^2 < \delta$.*

The rest of this paper is organized as follows. Section 2 contains some preliminary results playing an important role in the arguments of main theorems. Sections 3 and 4 give the proof of the existence of normalized solutions for equations (1.1) and (1.7), respectively.

2. Preliminary results

In this section, we give some results which are necessary for the proof of our main theorems.

We first recall some useful facts about the Riesz potential. The Riesz potential with order $\theta \in (0, N)$ of a function $f \in L^1_{loc}(\mathbb{R}^N)$ is defined as

$$(I_\theta * f)(x) := \int_{\mathbb{R}^N} \frac{\Gamma(\frac{N-\theta}{2})}{2^\theta \pi^{\frac{N}{2}} \Gamma(\frac{\theta}{2})} \frac{f(y)}{|x-y|^{N-\theta}} \mathrm{d}y. \tag{2.1}$$

The integral in equation (2.1) converges in the classical Lebesgue sense for a.e. $x \in \mathbb{R}^N$ if and only if

$$f \in L^1\big(\mathbb{R}^N, (1+|x|)^{\theta-N}\big). \tag{2.2}$$

Moreover, if (2.2) does not hold, then (2.1) diverges to $+\infty$ everywhere in $\mathbb{R}^N$. The Riesz potential I_θ is well defined as an operator from $L^p(\mathbb{R}^N)$ to $L^\sigma(\mathbb{R}^N)$ if and only if $p \in [1, \frac{N}{\theta})$, where $\sigma := \frac{Np}{N-\theta p}$. Furthermore, if $p \in (1, \frac{N}{\theta})$, then $I_\theta : L^p(\mathbb{R}^N) \to L^\sigma(\mathbb{R}^N)$ is a bounded linear operator, which is a consequence of the following Hardy–Littlewood–Sobolev inequality.

Lemma 2.1 ([12, Theorem 382]). *Let $\theta \in (0, N)$ and $p \in (1, \frac{N}{\theta})$. Then, for any $f \in L^p(\mathbb{R}^N)$, we have $I_\theta * f \in L^{\frac{Np}{N-\theta p}}(\mathbb{R}^N)$ and there exists a constant $C_{N,\theta,p} > 0$ such that*

$$\Big(\int_{\mathbb{R}^N} |I_\theta * f|^{\frac{Np}{N-\theta p}} \mathrm{d}x\Big)^{\frac{1}{p}-\frac{\theta}{N}} \leq C_{N,\theta,p}\Big(\int_{\mathbb{R}^N} |f|^p \mathrm{d}x\Big)^{\frac{1}{p}}.$$

By Lemma 2.1 and the Hölder inequality, we have the following result.

Lemma 2.2 ([19, equation (3.3)]). *Let $\theta \in (0, N)$ and $p \in [1, +\infty)$. Then, for any $v \in L^{\frac{2Np}{N+\theta}}(\mathbb{R}^N)$, there exists a constant $C_{N,\theta} > 0$ such that*

$$\int_{\mathbb{R}^N} (I_\theta * |v|^p)|v|^p \mathrm{d}x \leq C_{N,\theta}\Big(\int_{\mathbb{R}^N} |v|^{\frac{2Np}{N+\theta}} \mathrm{d}x\Big)^{\frac{N+\theta}{N}}.$$

Lemma 2.1 also yields the Brézis–Lieb-type lemma for the convolution term.

Lemma 2.3 ([18, Lemma 2.4]). *Let $\theta \in (0, N)$, $p \in [1, \frac{2N}{N+\theta})$, and $\{v_n\}$ be a bounded sequence in $L^{\frac{2Np}{N+\theta}}(\mathbb{R}^N)$. If $v_n \to v$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$, then as $n \to \infty$,*

$$\int_{\mathbb{R}^N} (I_\theta * |v_n|^p)|v_n|^p \mathrm{d}x - \int_{\mathbb{R}^N} (I_\theta * |v_n - v|^p)|v_n - v|^p \mathrm{d}x \to \int_{\mathbb{R}^N} (I_\theta * |v|^p)|v|^p \mathrm{d}x.$$

By Lemma 2.2 and the classic Gagliardo–Nirenberg inequality [24, Lemma 2.4], we have the following result which is basic to show the boundedness from below of the energy functional.

Lemma 2.4 ([29, equation (1.6)]). *Let $\theta \in (0, N)$, $p \in (\frac{N+\theta}{N}, \frac{N+\theta}{N-2})$ with $N \geq 3$, and $p \in (\frac{N+\theta}{N}, +\infty)$ with $N = 1, 2$. Then, for any $v \in H^1(\mathbb{R}^N)$, there exists a constant $C_{N,\theta,p} > 0$ such that*

$$\int_{\mathbb{R}^N} (I_\theta * |v|^p)|v|^p \mathrm{d}x \leq C_{N,\theta,p}\Big(\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x\Big)^{\frac{Np-N-\theta}{2}} \Big(\int_{\mathbb{R}^N} |v|^2 \mathrm{d}x\Big)^{\frac{N+\theta-p(N-2)}{2}}.$$

Finally, we give the lemma from P. L. Lions in 1984. Let $2^* := \frac{2N}{N-2}$ for $N \geq 3$ and $2^* := +\infty$ for $N = 1, 2$.

Lemma 2.5 ([26, Lemma 1.21]). *Let $R > 0$ and $p \in [2, 2^*)$. If $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and*

$$\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |v_n(x)|^p \mathrm{d}x \to 0 \quad \text{as } n \to \infty,$$

then $v_n \to 0$ in $L^t(\mathbb{R}^N)$ for any $t \in (2, 2^*)$.

3. Proof of Theorem 1.2

In this section, we always assume that $q \in (\frac{N+\theta}{N}, \frac{N+\theta+2}{N})$ with $N \geq 2$ and $\theta \in (0, N)$. For equation (1.1), we define the energy functional $\mathcal{J} : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$\mathcal{J}(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|v|^2 \mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v|^q)|v|^q \mathrm{d}x,$$

and consider the existence of minimizers of the L^2-constraint minimization problem

$$\vartheta_a := \inf_{v \in S(a)} \mathcal{J}(v),$$

where

$$S(a) := \big\{v \in H^1(\mathbb{R}^N) : |v|_2 = a\big\}.$$

Lemma 3.1. *The energy functional $\mathcal{J}(v)$ is bounded from below on $S(a)$ for any $a > 0$.*

Proof. By Lemma 2.4, for $v \in S(a)$, we have

$$\mathcal{J}(v) \geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x - \frac{C_{N,\theta,q} a^{N+\theta-q(N-2)}}{2q}\Big(\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x\Big)^{\frac{Nq-N-\theta}{2}}. \tag{3.1}$$

Since $q \in (\frac{N+\theta}{N}, \frac{N+\theta+2}{N})$, we get $Nq - N - \theta < 2$ and then $\mathcal{J}$ is bounded from below on $S(a)$ for any $a > 0$. ■

Lemma 3.2. *If $V \in L^\infty(\mathbb{R}^N)$, then, for any $a > 0$, there exists a constant $\delta = \delta(a) > 0$ such that $\vartheta_a < 0$ when $|V|_\infty < \delta$.*

Proof. For any $a > 0$, we choose $w_0 \in S(a)$. Let $s \in \mathbb{R}$ and set $F(w_0, s)(x) = e^{\frac{Ns}{2}} w_0(e^s x)$ for $x \in \mathbb{R}^N$. By direct computations, we have

$$\int_{\mathbb{R}^N} |F(w_0, s)(x)|^2 \mathrm{d}x = a^2, \tag{3.2}$$

$$\int_{\mathbb{R}^N} |\nabla F(w_0, s)|^2 \mathrm{d}x = e^{2s} \int_{\mathbb{R}^N} |\nabla w_0|^2 \mathrm{d}x, \tag{3.3}$$

$$\int_{\mathbb{R}^N} \big(I_\theta * |F(w_0, s)(x)|^q\big)|F(w_0, s)(x)|^q \mathrm{d}x = e^{Nsq - Ns - s\theta} \int_{\mathbb{R}^N} (I_\theta * |w_0|^q)|w_0|^q \mathrm{d}x. \tag{3.4}$$

Combining (3.2), (3.3) with (3.4), we obtain $F(w_0, s) \in S(a)$ and

$$\mathcal{J}(F(w_0, s)) \leq \frac{e^{2s}}{2} \int_{\mathbb{R}^N} |\nabla w_0|^2 \mathrm{d}x + \frac{|V|_\infty a^2}{2} - \frac{e^{Nsq - Ns - s\theta}}{2q} \int_{\mathbb{R}^N} (I_\theta * |w_0|^q)|w_0|^q \mathrm{d}x.$$

Note that $q \in (\frac{N+\theta}{N}, \frac{N+\theta+2}{N})$, there exists a constant $s_1 < 0$ such that

$$\mathcal{K}_{s_1} := \frac{e^{2s_1}}{2} \int_{\mathbb{R}^N} |\nabla w_0|^2 \mathrm{d}x - \frac{e^{(Nq - N - \theta)s_1}}{2q} \int_{\mathbb{R}^N} (I_\theta * |w_0|^q)|w_0|^q \mathrm{d}x < 0.$$

Fix $\delta := \frac{-\mathcal{K}_{s_1}}{a^2}$, and for $|V|_\infty < \delta$, we get

$$\mathcal{J}(F(w_0, s_1)) < \mathcal{K}_{s_1} - \frac{\mathcal{K}_{s_1}}{2} = \frac{\mathcal{K}_{s_1}}{2} < 0.$$

Thus, $\vartheta_a < 0$. ■

Now, we establish the crucial relationship between minimal energies of $\mathcal{J}$ with respect to different parameters a.

Lemma 3.3. *If $0 < a_1 < a_2$ and $\vartheta_{a_1} < 0$, then $\vartheta_{a_2} < 0$ and $a_1^2 \vartheta_{a_2} < a_2^2 \vartheta_{a_1}$.*

Proof. Let $\kappa > 1$ satisfy $a_2 = \kappa a_1$. Choose $\{v_n\} \subset S(a_1)$ as a minimizing sequence of ϑ_{a_1}, then $\mathcal{J}(v_n) \to \vartheta_{a_1}$ as $n \to \infty$. Setting $z_n = \kappa v_n$, we get $z_n \in S(a_2)$ and

$$\vartheta_{a_2} \leq \mathcal{J}(z_n) = \kappa^2 \mathcal{J}(v_n) + \frac{\kappa^2 - \kappa^{2q}}{2q} \int_{\mathbb{R}^N} (I_\theta * |v_n|^q)|v_n|^q \mathrm{d}x.$$

Now, we show that there are $C > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^N} (I_\theta * |v_n|^q)|v_n|^q \mathrm{d}x \geq C \quad \forall n \geq n_0.$$

By contradiction method, going if necessary to a subsequence, we suppose $\int_{\mathbb{R}^N}(I_\theta * |v_n|^q)|v_n|^q \mathrm{d}x \to 0$ as $n \to \infty$. It follows from

$$\vartheta_{a_1} + o_n(1) = \mathcal{J}(v_n) \geq -\frac{1}{2q}\int_{\mathbb{R}^N}(I_\theta * |v_n|^q)|v_n|^q \mathrm{d}x$$

that $\vartheta_{a_1} \geq 0$ by letting $n \to \infty$, which contradicts with the assumption. Since $\kappa^2 - \kappa^{2q} < 0$, we have

$$\vartheta_{a_2} \leq \kappa^2 \vartheta_{a_1} + \frac{(\kappa^2 - \kappa^{2q})C}{2q} < \kappa^2 \vartheta_{a_1},$$

which implies $\vartheta_{a_2} < 0$ and $a_1^2 \vartheta_{a_2} < a_2^2 \vartheta_{a_1}$. ■

We next use Lemma 3.3 to prove the compactness of minimizing sequences.

Lemma 3.4. *Assume that $V \in L^\infty(\mathbb{R}^N)$. Let $\{v_n\} \subset S(a)$ be a minimizing sequence of $\vartheta_a < 0$ with $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$, $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, and $v \neq 0$. Then, $v_n \to v$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$, $v \in S(a)$, and $\mathcal{J}(v) = \vartheta_a$.*

Proof. If $|v|_2 := b \neq a$, by Fatou's lemma and the assumption $v \neq 0$, we get $b \in (0, a)$. Set $z_n = v_n - v$ and $w_n = |z_n|_2$, then we may assume $w_n \to w$ as $n \to \infty$. By the Brézis–Lieb lemma [26], we obtain $|v_n|_2^2 = |z_n|_2^2 + |v|_2^2 + o_n(1)$, $a^2 = b^2 + w^2$, and $w_n \in (0, a)$ for sufficiently large n. By Lemma 2.3, we have

$$\int_{\mathbb{R}^N}(I_\theta * |v_n|^q)|v_n|^q \mathrm{d}x = \int_{\mathbb{R}^N}(I_\theta * |z_n|^q)|z_n|^q \mathrm{d}x + \int_{\mathbb{R}^N}(I_\theta * |v|^q)|v|^q \mathrm{d}x + o_n(1).$$

Together with $|\nabla v_n|_2^2 = |\nabla z_n|_2^2 + |\nabla v|_2^2 + o_n(1)$, we conclude that

$$\vartheta_a + o_n(1) = \mathcal{J}(v_n) = \mathcal{J}(z_n) + \mathcal{J}(v) + o_n(1) \geq \vartheta_{w_n} + \vartheta_b + o_n(1). \tag{3.5}$$

If $\vartheta_{w_n} < 0$ for all sufficiently large n in (3.5), we obtain from Lemma 3.3 that

$$\vartheta_a \geq \frac{w_n^2}{a^2}\vartheta_a + \vartheta_b + o_n(1).$$

Letting $n \to \infty$, we get $b^2 \vartheta_a \geq a^2 \vartheta_b$. Then, it follows from $\vartheta_a < 0$ that $\vartheta_b < 0$, which implies $a^2 \vartheta_b > b^2 \vartheta_a$ and it is impossible. If $\vartheta_{w_n} \geq 0$ for some n large enough in (3.5), which together with $\vartheta_a < 0$ implies that $\vartheta_b < 0$. By Lemma 3.3 again, we have

$$\vartheta_a \geq \frac{b^2}{a^2}\vartheta_a + \vartheta_{w_n} + o_n(1),$$

then $w^2 \vartheta_a \geq a^2 \vartheta_{w_n} + o_n(1)$, which is impossible. Hence, $|v|_2 = a$ and $v \in S(a)$. Since $v_n \rightharpoonup v$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$ and $|v_n|_2 = |v|_2 = a$, we get $v_n \to v$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$. By Lemma 2.4, $\int_{\mathbb{R}^N}(I_\theta * |z_n|^q)|z_n|^q \mathrm{d}x \to 0$ as $n \to \infty$, which together with Lemma 2.3 implies that

$$\int_{\mathbb{R}^N}(I_\theta * |v_n|^q)|v_n|^q \mathrm{d}x \to \int_{\mathbb{R}^N}(I_\theta * |v|^q)|v|^q \mathrm{d}x \quad \text{as } n \to \infty. \tag{3.6}$$

Since $\int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)|v|^2)\mathrm{d}x$ is continuous and convex in $H^1(\mathbb{R}^N)$, by the weakly lower semicontinuous, we obtain

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla v_n|^2 + V(x)|v_n|^2)\mathrm{d}x \geq \int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)|v|^2)\mathrm{d}x.$$

Noticing

$$\vartheta_a = \lim_{n\to\infty}\mathcal{J}(v_n) \geq \mathcal{J}(v), \quad v \in S(a),$$

we get $\mathcal{J}(v) = \vartheta_a$. Thus, $\mathcal{J}(v_n) \to \mathcal{J}(v)$ as $n \to \infty$, which jointly with (3.6), ensures that $v_n \to v$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. ∎

3.1. The case of (V1)

We assume (V1) holds in this subsection. The periodicity of V is essential to find the nontrivial minimizer of ϑ_a.

Lemma 3.5. *Assume that $\vartheta_a < 0$ for $a > 0$. Then, any minimizing sequence of ϑ_a is bounded and the minimizing sequence $\{v_n\} \subset S(a)$ of ϑ_a can be chosen such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ with $v \neq 0$.*

Proof. Let $\{u_n\} \subset S(a)$ be a minimizing sequence of ϑ_a, then $\mathcal{J}(u_n) \to \vartheta_a$ as $n \to \infty$. Noticing $q \in (\frac{N+\theta}{N}, \frac{N+\theta+2}{N})$ and (3.1), we get that $|\nabla u_n|_2$ is bounded, from which it follows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Similar to the proof of Lemma 3.3, there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^N}(I_\theta * |u_n|^q)|u_n|^q\mathrm{d}x \geq C_1 \tag{3.7}$$

for sufficiently large n, since $\vartheta_a < 0$. Additionally, there exist $R > 0, \sigma > 0$ and $t_n \in \mathbb{R}^N$ such that

$$\int_{B_R(t_n)}|u_n|^2\mathrm{d}x \geq \sigma. \tag{3.8}$$

Otherwise, by Lemma 2.5, $u_n \to 0$ in $L^t(\mathbb{R}^N)$ as $n \to \infty$ for any $t \in (2, 2^*)$, which together with Lemma 2.2 implies that

$$\int_{\mathbb{R}^N}(I_\theta * |u_n|^q)|u_n|^q\mathrm{d}x \to 0 \quad \text{as } n \to \infty$$

which is a contradiction with (3.7). We may choose $t_n \in \mathbb{Z}^N$ and increase R if necessary in (3.8). Setting $v_n(x) = u_n(x + t_n)$, we get that $\{v_n\} \subset S(a)$ is also a bounded minimizing sequence of ϑ_a in view of (V1). Hence, there exists $v \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ up to a subsequence. By (3.8), we have $\int_{B_R(0)}|v_n|^2\mathrm{d}x \geq \sigma$, which implies $v \neq 0$. ∎

Proposition 3.6. *For each $a > 0$, there exists a constant $\delta = \delta(a) > 0$ such that if $|V|_\infty < \delta$, then $\vartheta_a < 0$ and ϑ_a is attained by a positive function.*

Proof. By Lemma 3.2, for any $a > 0$, there is $\delta = \delta(a) > 0$ such that $\vartheta_a < 0$ when $|V|_\infty < \delta$. By Lemma 3.5, there exists a minimizing sequence $\{v_n\} \subset S(a)$ of ϑ_a such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ with $v \neq 0$. By Lemma 3.4, $v \in S(a)$, $\mathcal{J}(v) = \vartheta_a$ and $v_n \to v$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Applying the Lagrange multiplier rule, there exists $\mu_a \in \mathbb{R}$ such that

$$\mathcal{J}'(v) + \mu_a \Phi'(v) = 0 \quad \text{in } H^{-1}(\mathbb{R}^N), \tag{3.9}$$

where $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$ is given by $\Phi(v) = \int_{\mathbb{R}^N} |v|^2 \mathrm{d}x$ for $v \in H^1(\mathbb{R}^N)$. Following (3.9), we conclude that (v, μ_a) is a couple of solution to the following equation:

$$-\Delta v + (V(x) + \mu_a)v = (I_\theta * |v|^q)|v|^{q-2}v \quad \text{in } \mathbb{R}^N.$$

Multiplying the above equation by v and integrating, we obtain

$$\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \int_{\mathbb{R}^N} (V(x) + \mu_a)v^2 \mathrm{d}x = \int_{\mathbb{R}^N} (I_\theta * |v|^q)|v|^q \mathrm{d}x.$$

Since $\vartheta_a < 0$ and $q \in (\frac{N+\theta}{N}, \frac{N+\theta+2}{N})$, we get

$$\mu_a a^2 = -2\vartheta_a + \frac{q-1}{q} \int_{\mathbb{R}^N} (I_\theta * |v|^q)|v|^q \mathrm{d}x > 0,$$

which shows that $\mu_a > 0$.

Next, we will prove that v can be chosen to be positive. Obviously, if $v \in H^1(\mathbb{R}^N)$, then $|v| \in H^1(\mathbb{R}^N)$ with $|\nabla |v||_2^2 = |\nabla v|_2^2$. Moreover, $v \in S(a)$ implies that $|v| \in S(a)$. Then, we get that $\vartheta_a = \mathcal{J}(v) = \mathcal{J}(|v|)$. Thus, we can replace v by $|v|$. By the standard regularity theory [19], $v \in C^2(\mathbb{R}^N)$. Assume by contradition that there exists $x_1 \in \mathbb{R}^N$ such that $v(x_1) = 0$. Since $v \neq 0$, there is $x_2 \in \mathbb{R}^N$ such that $v(x_2) > 0$. Then, fix $R > 0$ large enough such that $x_1, x_2 \in B_R(0)$. By [11, Theorem 8.20], there exists a constant $C > 0$ such that

$$\sup_{z \in B_R(0)} v(z) \le C \inf_{z \in B_R(0)} v(z),$$

which is impossible, since

$$\sup_{z \in B_R(0)} v(z) > 0 \quad \text{and} \quad \inf_{z \in B_R(0)} v(z) = 0. \qquad \blacksquare$$

3.2. The case of (V2)

We assume (V2) holds in this subsection and $V \not\equiv V_P$. Then, there exists a measurable set $\mathcal{D} \subset \mathbb{R}^N$ with $|\mathcal{D}| > 0$ such that $V(x) < V_P(x)$ for all $x \in \mathcal{D}$. We consider $\mathcal{J}_P$: $H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\mathcal{J}_P(v) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^N} V_P(x)|v|^2 \mathrm{d}x - \frac{1}{2q} \int_{\mathbb{R}^N} (I_\theta * |v|^q)|v|^q \mathrm{d}x$$

and define

$$\vartheta_{a,P} := \inf_{v \in S(a)} \mathcal{J}_P(v).$$

According to Theorem 1.2 (1) there exists a positive function $v_P \in S(a)$ such that $\mathcal{J}_P(v_P) = \vartheta_{a,P} < 0$ when $|V_P|_\infty < \delta(a)$, where $\delta(a)$ is the existing constant in Theorem 1.2 (1). Since $V(x) < V_P(x)$ for $x \in \mathcal{D}$ with $|\mathcal{D}| > 0$, we get that, for $|V_P|_\infty < \delta(a)$,

$$\vartheta_a = \inf_{v \in S(a)} \mathcal{J}(v) \leq \mathcal{J}(v_P) < \mathcal{J}_P(v_P) = \vartheta_{a,P} < 0. \tag{3.10}$$

In view of Lemma 3.1, $\mathcal{J}(v)$ is bounded from below on $S(a)$ for any $a > 0$.

Lemma 3.7. *For each $a > 0$, there exists a constant $\delta = \delta(a) > 0$ such that if $|V_P|_\infty < \delta$, then any minimizing sequence of ϑ_a is bounded and the minimizing sequence $\{v_n\} \subset S(a)$ of ϑ_a can be chosen such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ with $v \neq 0$.*

Proof. It follows from $q \in (\frac{N+\theta}{N}, \frac{N+\theta+2}{N})$ and (3.1) that any minimizing sequence of ϑ_a is bounded in $H^1(\mathbb{R}^N)$. Hence, there exist $v \in H^1(\mathbb{R}^N)$ and a subsequence of $\{v_n\}$, still denoted by itself such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Suppose $v = 0$, then $v_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$ as $n \to \infty$, which together with the assumption (1.4) implies that

$$\int_{\mathbb{R}^N} (V(x) - V_P(x))|v_n|^2 \mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Observe that

$$\vartheta_a + o_n(1) = \mathcal{J}(v_n) \geq \vartheta_{a,P} + \frac{1}{2}\int_{\mathbb{R}^N} (V(x) - V_P(x))|v_n|^2 \mathrm{d}x. \tag{3.11}$$

Letting $n \to \infty$ in (3.11), we get $\vartheta_a \geq \vartheta_{a,P}$, which contradicts with (3.10). Thus, $v \neq 0$. ■

Proposition 3.8. *For each $a > 0$, there exists a constant $\delta = \delta(a) > 0$ such that if $|V_P|_\infty < \delta$, then $\vartheta_a < 0$ and ϑ_a is attained by a positive function.*

Proof. By Lemma 3.2, for any $a > 0$, there is $\delta = \delta(a) > 0$ such that $\vartheta_{a,P} < 0$ when $|V_P|_\infty < \delta$. Due to (3.10), $\vartheta_a < 0$. By Lemma 3.7, there exists a minimizing sequence $\{v_n\} \subset S(a)$ of ϑ_a such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ with $v \neq 0$. By Lemma 3.4, we obtain $v \in S(a)$, $\mathcal{J}(v) = \vartheta_a$ and $v_n \to v$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. The following arguments are similar to that of Proposition 3.6 and the details are omitted. ■

3.3. The case of (V3)

We assume (V3) holds in this subsection. The third result is associated with the energy functional $\mathcal{J}_\varepsilon : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{J}_\varepsilon(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} K(\varepsilon x)|v|^2 \mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v|^q)|v|^q \mathrm{d}x$$

and the minimization problem

$$\vartheta_{a,\varepsilon} := \inf_{v\in S(a)} \mathcal{J}_\varepsilon(v).$$

By Lemma 3.1, $\mathcal{J}_\varepsilon(v)$ is bounded from below on $S(a)$ for any $a > 0$.

Denote by $\mathcal{J}_0, \mathcal{J}_\infty : H^1(\mathbb{R}^N) \to \mathbb{R}$ the following functionals:

$$\mathcal{J}_0(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} K_0|v|^2 \mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v|^q)|v|^q \mathrm{d}x,$$
$$\mathcal{J}_\infty(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} K_\infty|v|^2 \mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v|^q)|v|^q \mathrm{d}x.$$

Define

$$\vartheta_{a,0} := \inf_{v\in S(a)} \mathcal{J}_0(v),$$
$$\vartheta_{a,\infty} := \inf_{v\in S(a)} \mathcal{J}_\infty(v).$$

Applying Theorem 1.2 (1) and (1.5), we get that, for each $a > 0$, there exists $\delta = \delta(a) > 0$ such that if $K_\infty < \delta$, then there exist positive functions $v_0, v_\infty \in S(a)$ such that $\mathcal{J}_0(v_0) = \vartheta_{a,0}$ and $\mathcal{J}_\infty(v_\infty) = \vartheta_{a,\infty}$. Moreover, when $K_\infty < \delta$, we have

$$\vartheta_{a,0} < \vartheta_{a,\infty} < 0. \tag{3.12}$$

Lemma 3.9. *We have* $\limsup_{\varepsilon\to 0^+} \vartheta_{a,\varepsilon} \le \vartheta_{a,0}$.

Proof. Let $\{x_k\} \subset \mathbb{R}^N$ satisfy $K(x_k) \to K_0$ as $k \to \infty$. If $|x_k| \to \infty$ as $k \to \infty$, then $K_0 \ge K_\infty$, which contradicts with (1.5). Thus, there is $x_0 \in \mathbb{R}^N$ such that $K(x_0) = K_0$. Set $u_\varepsilon(x) = v_0(x - \frac{x_0}{\varepsilon})$. Then, $u_\varepsilon(x) \in S(a)$ and

$$\vartheta_{a,\varepsilon} \le \mathcal{J}_\varepsilon(u_\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_0|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} K(\varepsilon x + x_0)|v_0|^2 \mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v_0|^q)|v_0|^q \mathrm{d}x.$$

Noticing $K \in L^\infty(\mathbb{R}^N)$, we have

$$\limsup_{\varepsilon\to 0^+} \vartheta_{a,\varepsilon} \le \lim_{\varepsilon\to 0^+} \mathcal{J}_\varepsilon(u_\varepsilon) = \mathcal{J}_0(v_0) = \vartheta_{a,0}. \qquad \blacksquare$$

Remark 3.10. By Lemma 3.9 and (3.12), there exists a constant $\varepsilon^* = \varepsilon^*(a) > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$, $\vartheta_{a,\varepsilon} < \vartheta_{a,\infty} < 0$, when $K_\infty < \delta$.

Lemma 3.11. *If $K_\infty < \delta$, then for all $\varepsilon \in (0, \varepsilon^*)$, any minimizing sequence of $\vartheta_{a,\varepsilon}$ is uniformly bounded with respect to ε in $H^1(\mathbb{R}^N)$ and the minimizing sequence $\{v_{n,\varepsilon}\} \subset S(a)$ of $\vartheta_{a,\varepsilon}$ can be chosen such that $v_{n,\varepsilon} \rightharpoonup v_\varepsilon$ in $H^1(\mathbb{R}^N)$ and $v_{n,\varepsilon}(x) \to v_\varepsilon(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ with $v_\varepsilon \ne 0$, where ε^* is given in Remark 3.10.*

Proof. Due to Remark 3.10, $\vartheta_{a,\varepsilon} < 0$, which together with $q \in (\frac{N+\theta}{N}, \frac{N+\theta+2}{N})$ and (3.1), implies that any minimizing sequence of $\vartheta_{a,\varepsilon}$ is uniformly bounded with respect to ε in $H^1(\mathbb{R}^N)$. Thus, there are $v_\varepsilon \in H^1(\mathbb{R}^N)$ and a subsequence of $\{v_{n,\varepsilon}\}$, still denoted by itself, such that $v_{n,\varepsilon} \rightharpoonup v_\varepsilon$ in $H^1(\mathbb{R}^N)$ and $v_{n,\varepsilon}(x) \to v_\varepsilon(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. By (V3), for any given $\varsigma > 0$, there is $R > 0$ such that $K(x) \geq K_\infty - \varsigma$ for all $|x| \geq R$. If $v_\varepsilon = 0$, then $v_{n,\varepsilon} \to 0$ in $L^2(B_{\frac{R}{\varepsilon}}(0))$ as $n \to \infty$. Noting that

$$\begin{aligned}\vartheta_{a,\varepsilon} + o_n(1) = \mathcal{J}_\varepsilon(v_{n,\varepsilon}) &= \mathcal{J}_\infty(v_{n,\varepsilon}) + \frac{1}{2}\int_{\mathbb{R}^N} (K(\varepsilon x) - K_\infty)|v_{n,\varepsilon}|^2 \mathrm{d}x \\ &\geq \mathcal{J}_\infty(v_{n,\varepsilon}) + \frac{1}{2}\int_{B_{\frac{R}{\varepsilon}}(0)} (K(\varepsilon x) - K_\infty)|v_{n,\varepsilon}|^2 \mathrm{d}x - \frac{1}{2}\varsigma \int_{B^c_{\frac{R}{\varepsilon}}(0)} |v_{n,\varepsilon}|^2 \mathrm{d}x,\end{aligned}$$

we obtain $\vartheta_{a,\varepsilon} \geq \vartheta_{a,\infty} - \varsigma C$. Since $\varsigma > 0$ is arbitrary, we get $\vartheta_{a,\varepsilon} \geq \vartheta_{a,\infty}$, which contradicts with Remark 3.10. Hence, $v_\varepsilon \neq 0$. ∎

Proposition 3.12. *For each $a > 0$, there exist constants $\delta = \delta(a) > 0$ and $\varepsilon^* = \varepsilon^*(a) > 0$ such that if $K_\infty < \delta$, then $\vartheta_{a,\varepsilon} < 0$ and $\vartheta_{a,\varepsilon}$ is attained by a positive function for all $\varepsilon \in (0, \varepsilon^*)$.*

Proof. By Lemma 3.2 and Remark 3.10, for any $a > 0$, there are $\delta = \delta(a) > 0$ and $\varepsilon^* = \varepsilon^*(a) > 0$ such that $\vartheta_{a,\varepsilon} < 0$ for all $\varepsilon \in (0, \varepsilon^*)$ when $K_\infty < \delta$. By Lemma 3.11, there exists a minimizing sequence $\{v_{n,\varepsilon}\} \subset S(a)$ of $\vartheta_{a,\varepsilon}$ such that $v_{n,\varepsilon} \rightharpoonup v_\varepsilon$ in $H^1(\mathbb{R}^N)$ and $v_{n,\varepsilon}(x) \to v_\varepsilon(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ with $v_\varepsilon \neq 0$. By Lemma 3.4, we get that $v_\varepsilon \in S(a)$, $\mathcal{J}_\varepsilon(v_\varepsilon) = \vartheta_{a,\varepsilon}$ and $v_{n,\varepsilon} \to v_\varepsilon$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. The following arguments are similar to the proof of Proposition 3.6. ∎

3.4. The case of (V4)

We assume (V4) holds in this subsection. For $v \in H^1(\mathbb{R}^N)$, we choose the equivalent norm

$$\|v\|_1 = \left[\int_{\mathbb{R}^N} \left(|\nabla v|^2 + (\lambda W(x) + 1)|v|^2\right)\mathrm{d}x\right]^{\frac{1}{2}}.$$

We study the energy functional $\mathcal{J}_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\mathcal{J}_\lambda(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \lambda W(x)|v|^2 \mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v|^q)|v|^q \mathrm{d}x$$

and the minimization problem

$$\vartheta_{a,\lambda} := \inf_{v \in S(a)} \mathcal{J}_\lambda(v),$$

where $\mathcal{J}_\lambda(v)$ is bounded from below on $S(a)$ for any $a > 0$ by Lemma 3.1.

Lemma 3.13. *For any $a > 0$, there exists a constant $\delta = \delta(a) > 0$ such that if $\lambda|W|_\infty < \delta$, then $\vartheta_{a,\lambda} < 0$ and*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} (I_\theta * |v_{n,\lambda}|^q)|v_{n,\lambda}|^q \mathrm{d}x \geq \gamma,$$

where $\{v_{n,\lambda}\} \subset S(a)$ is a minimizing sequence of $\vartheta_{a,\lambda}$ and $\gamma = \gamma(a) > 0$ is a constant.

Proof. Similar to the proof of Lemma 3.2, we can find $\delta = \delta(a) > 0$ and $\gamma = \gamma(a) > 0$ such that $\vartheta_{a,\lambda} < -\frac{\gamma}{2q}$ when $\lambda|W|_\infty < \delta$. Since

$$\begin{aligned}\vartheta_{a,\lambda} + o_n(1) = \mathcal{J}_\lambda(v_{n,\lambda}) = &\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_{n,\lambda}|^2 \mathrm{d}x + \frac{\lambda}{2}\int_{\mathbb{R}^N} W(x)|v_{n,\lambda}|^2 \mathrm{d}x \\ &- \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v_{n,\lambda}|^q)|v_{n,\lambda}|^q \mathrm{d}x,\end{aligned}$$

we have

$$-\frac{\gamma}{2q} + o_n(1) \geq -\frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v_{n,\lambda}|^q)|v_{n,\lambda}|^q \mathrm{d}x,$$

which implies that this lemma holds. ■

Lemma 3.14. *If $\lambda|W|_\infty < \delta$, then any minimizing sequence $\{v_{n,\lambda}\}$ of $\vartheta_{a,\lambda}$ is uniformly bounded with respect to λ in $H^1(\mathbb{R}^N)$, where δ is defined in Lemma 3.13.*

Proof. Due to Lemma 3.13, we have $\vartheta_{a,\lambda} < 0$, which combines with $q \in (\frac{N+\theta}{N}, \frac{N+\theta+2}{N})$ and (3.1) ensures that $|\nabla v_{n,\lambda}|_2$ is uniformly bounded with respect to λ. Observe that $\{\int_{\mathbb{R}^N} \lambda W(x)|v_{n,\lambda}|^2 \mathrm{d}x\}$ is also uniformly bounded with respect to λ when $\lambda|W|_\infty < \delta$. Hence, $\{v_{n,\lambda}\}$ is uniformly bounded with respect to λ in $H^1(\mathbb{R}^N)$. ■

Lemma 3.15. *There exist $R > 0$ and $\lambda^* = \lambda^*(a) > 0$ such that if $\lambda \in [\lambda^*, +\infty)$ and $\lambda|W|_\infty < \delta$; then,*

$$\limsup_{n\to\infty} \int_{B_R^c(0)} (I_\theta * |v_{n,\lambda}|^q)|v_{n,\lambda}|^q \mathrm{d}x \leq \frac{\gamma}{2},$$

where $\{v_{n,\lambda}\} \subset S(a)$ is a minimizing sequence of $\vartheta_{a,\lambda}$ and δ, γ are given in Lemma 3.13.

Proof. Similar to [6], for any $r > 0$, we consider

$$A(r) := \{x \in \mathbb{R}^N : |x| > r, W(x) \geq M_0\}, \quad B(r) := \{x \in \mathbb{R}^N : |x| > r, W(x) < M_0\},$$

where M_0 is given in (1.6). By Lemma 3.14, there is $M > 0$ independent of λ such that $\|v_{n,\lambda}\|_1^2 \leq M$. Thus, we obtain

$$\begin{aligned}\int_{A(r)} |v_{n,\lambda}|^2 \mathrm{d}x &\leq \frac{1}{\lambda M_0 + 1}\int_{\mathbb{R}^N} (\lambda W(x) + 1)|v_{n,\lambda}|^2 \mathrm{d}x \\ &\leq \frac{1}{\lambda M_0 + 1}\int_{\mathbb{R}^N} (|\nabla v_{n,\lambda}|^2 + (\lambda W(x) + 1)|v_{n,\lambda}|^2)\mathrm{d}x \\ &= \frac{1}{\lambda M_0 + 1}\|v_{n,\lambda}\|_1^2 \leq \frac{M}{\lambda M_0 + 1}.\end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned}\int_{B(r)}|v_{n,\lambda}|^2\mathrm{d}x &\le \left(\int_{B(r)}|v_{n,\lambda}|^{2p}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B(r)}\mathrm{d}x\right)^{\frac{1}{p'}}\\ &\le \left(\int_{\mathbb{R}^N}|v_{n,\lambda}|^{2p}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B(r)}\mathrm{d}x\right)^{\frac{1}{p'}}\\ &\le C\|v_{n,\lambda}\|_1^2|B(r)|^{\frac{1}{p'}} \le CM|B(r)|^{\frac{1}{p'}},\end{aligned}$$

where $p \in (1, \frac{N}{N-2})$ with $\frac{1}{p} + \frac{1}{p'} = 1$. By Lemma 2.4, we get

$$\begin{aligned}&\int_{B_r^c(0)}(I_\theta * |v_{n,\lambda}|^q)|v_{n,\lambda}|^q\mathrm{d}x\\ &\le C\left(\int_{B_r^c(0)}|\nabla v_{n,\lambda}|^2\mathrm{d}x\right)^{\frac{Nq-N-\theta}{2}}\left(\int_{B_r^c(0)}|v_{n,\lambda}|^2\mathrm{d}x\right)^{\frac{N+\theta-q(N-2)}{2}}\\ &\le C\|v_{n,\lambda}\|_1^{Nq-N-\theta}\left(\int_{A(r)}|v_{n,\lambda}|^2\mathrm{d}x + \int_{B(r)}|v_{n,\lambda}|^2\mathrm{d}x\right)^{\frac{N+\theta-q(N-2)}{2}}\\ &\le C\left(\int_{A(r)}|v_{n,\lambda}|^2\mathrm{d}x + \int_{B(r)}|v_{n,\lambda}|^2\mathrm{d}x\right)^{\frac{N+\theta-q(N-2)}{2}}\\ &\le C\left(\frac{1}{\lambda M_0+1} + |B(r)|^{\frac{1}{p'}}\right)^{\frac{N+\theta-q(N-2)}{2}}.\end{aligned}$$

The first term on the right-hand side of the above inequality can be arbitrarily small for λ large enough. The second term on the right-hand side of the above inequality can be also arbitrarily small if r is large enough, since $|B(r)| \to 0$ as $r \to \infty$. ∎

Lemma 3.16. *For all $\lambda \in [\lambda^*, +\infty)$ and $\lambda|W|_\infty < \delta$, the minimizing sequence $\{v_{n,\lambda}\} \subset S(a)$ of $\vartheta_{a,\lambda}$ can be chosen such that $v_{n,\lambda} \rightharpoonup v_\lambda$ in $H^1(\mathbb{R}^N)$ and $v_{n,\lambda}(x) \to v_\lambda(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ with $v_\lambda \neq 0$, where λ^* and δ are given in Lemmas 3.15 and 3.13, respectively.*

Proof. By Lemma 3.14, there is $v_\lambda \in H^1(\mathbb{R}^N)$ such that $v_{n,\lambda} \rightharpoonup v_\lambda$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$ along a subsequence. Assume by contradiction that $v_\lambda = 0$. Note that $v_{n,\lambda} \to 0$ in $L^2(B_R(0))$ as $n \to \infty$ for $R > 0$ given in Lemma 3.15, it follows from Lemmas 2.4, 3.13, and 3.15 that

$$\begin{aligned}\gamma \le \liminf_{n\to\infty}\int_{\mathbb{R}^N}(I_\theta * |v_{n,\lambda}|^q)|v_{n,\lambda}|^q\mathrm{d}x &= \liminf_{n\to\infty}\int_{B_R^c(0)}(I_\theta * |v_{n,\lambda}|^q)|v_{n,\lambda}|^q\mathrm{d}x\\ &\le \limsup_{n\to\infty}\int_{B_R^c(0)}(I_\theta * |v_{n,\lambda}|^q)|v_{n,\lambda}|^q\mathrm{d}x \le \frac{\gamma}{2},\end{aligned}$$

which is impossible. Thus, $v_\lambda \neq 0$. ∎

Lemma 3.17. *Let $\{v_{n,\lambda}\} \subset S(a)$ be a minimizing sequence of $\vartheta_{a,\lambda} < 0$ with $v_{n,\lambda} \rightharpoonup v_\lambda$ in $H^1(\mathbb{R}^N)$, $v_{n,\lambda}(x) \to v_\lambda(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, and $v_\lambda \neq 0$. Then, $v_{n,\lambda} \to v_\lambda$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$, $v_\lambda \in S(a)$ and $\mathcal{J}(v_\lambda) = \vartheta_{a,\lambda}$.*

The proof of Lemma 3.17 is similar to that of Lemma 3.4 and so is omitted.

Proposition 3.18. *For each $a > 0$, there exist constants $\delta = \delta(a) > 0$ and $\lambda^* = \lambda^*(a) > 0$ such that if $\lambda \in [\lambda^*, +\infty)$ and $\lambda|W|_\infty < \delta$, then $\vartheta_{a,\lambda} < 0$ and $\vartheta_{a,\lambda}$ is attained by a positive function.*

Proof. By Lemma 3.13, for any $a > 0$, there is $\delta = \delta(a) > 0$ such that $\vartheta_{a,\lambda} < 0$ when $\lambda|W|_\infty < \delta$. By Lemmas 3.14, 3.15, and 3.16, there exists a minimizing sequence $\{v_{n,\lambda}\} \subset S(a)$ of $\vartheta_{a,\lambda}$ such that $v_{n,\lambda} \rightharpoonup v_\lambda$ in $H^1(\mathbb{R}^N)$, $v_{n,\lambda}(x) \to v_\lambda(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ with $v_\lambda \neq 0$. By Lemma 3.17, we have $v_\lambda \in S(a)$, $\mathcal{J}_\lambda(v_\lambda) = \vartheta_{a,\lambda}$ and $v_{n,\lambda} \to v_\lambda$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. The following proof is similar to that of Proposition 3.6. ■

4. Proof of Theorem 1.3

In this section, we set $\nabla_A := -i\nabla + A$ and assume that the magnetic potential $A : \mathbb{R}^N \to \mathbb{R}^N$ is continuous with $A \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$. We consider the Hilbert space $H^1_A(\mathbb{R}^N, \mathbb{C})$ with the inner product

$$\langle u, v\rangle := \Re e\left(\int_{\mathbb{R}^N} (\nabla_A u\overline{\nabla_A v} + u\bar{v})\mathrm{d}x\right)$$

and the norm

$$\|v\|_A^2 := \int_{\mathbb{R}^N} (|\nabla_A v|^2 + |v|^2)\mathrm{d}x.$$

Lemma 4.1 ([16] Diamagnetic inequality). *If $v \in H^1_A(\mathbb{R}^N, \mathbb{C})$, then $|v| \in H^1(\mathbb{R}^N)$ and*

$$|\nabla|v|(x)| \le |\nabla_A v(x)| \quad \forall x \in \mathbb{R}^N.$$

As a consequence of Lemma 4.1, the embeddings $H^1_A(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^p(\mathbb{R}^N, \mathbb{C})$ is continuous for $p \in [2, 2^*]$ and $H^1_A(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^p_{loc}(\mathbb{R}^N, \mathbb{C})$ is compact for $p \in [2, 2^*)$.

Define the energy functional $\mathcal{J}_A : H^1_A(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ by

$$\mathcal{J}_A(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A v|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|v|^2\mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v|^q)|v|^q\mathrm{d}x,$$

and define the minimization problem

$$\vartheta_{a,A} := \inf_{v\in S_A(a)} \mathcal{J}_A(v),$$

where

$$S_A(a) := \left\{v \in H^1_A(\mathbb{R}^N, \mathbb{C}) : |v|_2 = a\right\}.$$

By Lemmas 4.1 and 2.4, we have

$$\mathcal{J}_A(v) \geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla |v||^2 \mathrm{d}x - \frac{C_{N,\theta,q} a^{N+\theta-q(N-2)}}{2q}\Big(\int_{\mathbb{R}^N} |\nabla |v||^2 \mathrm{d}x\Big)^{\frac{Nq-N-\theta}{2}}. \tag{4.1}$$

Since $q \in (\frac{N+\theta}{N}, \frac{N+\theta+2}{N})$, we infer that $Nq - N - \theta < 2$ and $\mathcal{J}_A$ is bounded from below on $S_A(a)$ for any $a > 0$.

Lemma 4.2. *Assume that $V \in L^\infty(\mathbb{R}^N)$. For each $a > 0$, there exists a constant $\delta = \delta(a) > 0$ such that if $|V|_\infty < \delta$ and $\mathcal{A}^2 < \delta$, then $\vartheta_{a,A} < 0$, where $\mathcal{A}$ is given by (1.8).*

Proof. For any $a > 0$, we choose $w_0 \in S_A(a)$. Let $s \in \mathbb{R}$ and set $F(w_0, s)(x) = e^{\frac{Ns}{2}} e^{iA(0)\cdot x} w_0(e^s x)$ for $x \in \mathbb{R}^N$. By direct computations, we have

$$\int_{\mathbb{R}^N} |F(w_0, s)(x)|^2 \mathrm{d}x = a^2, \tag{4.2}$$

$$\int_{\mathbb{R}^N} |\nabla_A F(w_0, s)|^2 \mathrm{d}x = e^{2s} \int_{\mathbb{R}^N} |\nabla w_0|^2 \mathrm{d}x + \int_{\mathbb{R}^N} |A(0) - A(e^{-s}x)|^2 |w_0|^2 \mathrm{d}x, \tag{4.3}$$

$$\int_{\mathbb{R}^N} (I_\theta * |F(w_0, s)(x)|^q)|F(w_0, s)(x)|^q \mathrm{d}x = e^{Nsq-Ns-s\theta} \int_{\mathbb{R}^N} (I_\theta * |w_0|^q)|w_0|^q \mathrm{d}x. \tag{4.4}$$

Combining (4.2), (4.3) with (4.4), we obtain

$$\begin{aligned}
&\mathcal{J}_A(F(w_0, s)) \\
&\leq \frac{e^{2s}}{2} \int_{\mathbb{R}^N} |\nabla w_0|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^N} |A(0) - A(e^{-s}x)|^2 |w_0|^2 \mathrm{d}x \\
&\quad + \frac{|V|_\infty a^2}{2} - \frac{e^{Nsq-Ns-s\theta}}{2q} \int_{\mathbb{R}^N} (I_\theta * |w_0|^q)|w_0|^q \mathrm{d}x \\
&\leq \frac{e^{2s}}{2} \int_{\mathbb{R}^N} |\nabla w_0|^2 \mathrm{d}x + \frac{(\mathcal{A}^2 + |V|_\infty) a^2}{2} - \frac{e^{Nsq-Ns-s\theta}}{2q} \int_{\mathbb{R}^N} (I_\theta * |w_0|^q)|w_0|^q \mathrm{d}x.
\end{aligned}$$

Since $q \in (\frac{N+\theta}{N}, \frac{N+\theta+2}{N})$, there exists a constant $s_2 < 0$ such that

$$\mathcal{F}_{s_2} := \frac{e^{2s_2}}{2} \int_{\mathbb{R}^N} |\nabla w_0|^2 \mathrm{d}x - \frac{e^{(Nq-N-\theta)s_2}}{2q} \int_{\mathbb{R}^N} (I_\theta * |w_0|^q)|w_0|^q \mathrm{d}x < 0.$$

Fix $\delta := \frac{-\mathcal{F}_{s_2}}{2a^2}$. If $|V|_\infty < \delta$ and $\mathcal{A}^2 < \delta$, then

$$\mathcal{J}_A(F(w_0, s_2)) < \mathcal{F}_{s_2} - \frac{\mathcal{F}_{s_2}}{2} = \frac{\mathcal{F}_{s_2}}{2} < 0,$$

which implies $\vartheta_{a,A} < 0$. ∎

Similar to Lemmas 3.3–3.4, we can also establish the relationship between minimal energies of $\mathcal{J}_A$ with respect to different parameters a, and prove the compactness of minimizing sequences of $\vartheta_{a,A}$. The periodicity assumption of A in Theorem 1.3 (1) and Theorem 1.3 (2) is used to find the nontrivial normalized solution for equation (1.7). When V satisfies (V1) or (V2), the discussion for equation (1.7) is similar to that of equation (1.1), so we omit the details.

Now, let us assume that V satisfies (V3). The associated energy functional is $\mathcal{J}_{A,\varepsilon} : H^1_A(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ given by

$$\mathcal{J}_{A,\varepsilon}(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{\mathscr{A}(\varepsilon x)} v|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} K(\varepsilon x)|v|^2 \mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v|^q)|v|^q \mathrm{d}x$$

and we define

$$\vartheta_{a,A,\varepsilon} := \inf_{v \in S_A(a)} \mathcal{J}_{A,\varepsilon}(v).$$

In addition, we assume that there exists $\mathscr{A}_\infty \in \mathbb{R}^N$ such that

$$\lim_{|x|\to\infty} |\mathscr{A}(\varepsilon x) - \mathscr{A}_\infty| = 0. \tag{4.5}$$

Denote by $\mathcal{J}_{A,0}, \mathcal{J}_{A,\infty} : H^1_A(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ the following functionals:

$$\mathcal{J}_{A,0}(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{\mathscr{A}(0)} v|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} K_0|v|^2 \mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v|^q)|v|^q \mathrm{d}x,$$

$$\mathcal{J}_{A,\infty}(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{\mathscr{A}_\infty} v|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} K_\infty|v|^2 \mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v|^q)|v|^q \mathrm{d}x$$

and define

$$\vartheta_{a,A,0} := \inf_{v \in S_A(a)} \mathcal{J}_{A,0}(v), \quad \vartheta_{a,A,\infty} := \inf_{v \in S_A(a)} \mathcal{J}_{A,\infty}(v).$$

Due to Theorem 1.3 (1), for each $a > 0$, there exists $\delta = \delta(a) > 0$ such that if $K_\infty < \delta$ and $\mathcal{A}^2 < \delta$, then there exist $v_{A,0}, v_{A,\infty} \in S_A(a)$ such that $\mathcal{J}_{A,0}(v_{A,0}) = \vartheta_{a,A,0}$ and $\mathcal{J}_{A,\infty}(v_{A,\infty}) = \vartheta_{a,A,\infty}$.

The following result plays a key role in the proof of Theorem 1.3 (3).

Lemma 4.3. *There holds $\vartheta_{a,A,0} = \vartheta_{a,0}$ and $\vartheta_{a,A,\infty} = \vartheta_{a,\infty}$.*

Proof. We prove $\vartheta_{a,A,0} = \vartheta_{a,0}$. By Lemma 4.1, we get $\vartheta_{a,A,0} \geq \vartheta_{a,0}$. Consider $\hat{v}_0(x) = e^{i\mathscr{A}(0)\cdot x} v_0(x)$ for $x \in \mathbb{R}^N$, where $v_0 \in S(a)$ is defined in Section 3.3 satisfying $\mathcal{J}_0(v_0) = \vartheta_{a,0}$. By direct computations, we get $\hat{v}_0 \in S_A(a)$ and

$$\vartheta_{a,A,0} \leq \mathcal{J}_{A,0}(\hat{v}_0) = \mathcal{J}_0(v_0) = \vartheta_{a,0}.$$

Similarly, $\vartheta_{a,A,\infty} = \vartheta_{a,\infty}$. ∎

Remark 4.4. It follows from Lemmas 4.2, 4.3 and (3.12) that $\vartheta_{a,A,0} < \vartheta_{a,A,\infty} < 0$ when $K_\infty < \delta$ and $\mathcal{A}^2 < \delta$.

Lemma 4.5. *The following holds:* $\limsup_{\varepsilon\to 0^+} \vartheta_{a,A,\varepsilon} \le \vartheta_{a,A,0}$.

Proof. Similar to the proof of Lemma 3.9, there is $x_0 \in \mathbb{R}^N$ satisfying $K(x_0) = K_0$. Define $u_\varepsilon(x) = e^{i\mathscr{A}(x_0)\cdot x} v_0(x - \frac{x_0}{\varepsilon})$, where $v_0 \in S(a)$ is defined in Subsection 3.3. Then, $u_\varepsilon(x) \in S_A(a)$ and

$$\begin{aligned}
\vartheta_{a,A,\varepsilon} &\le \mathcal{J}_{A,\varepsilon}(u_\varepsilon)\\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{\mathscr{A}(\varepsilon x)} u_\varepsilon|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} K(\varepsilon x)|u_\varepsilon|^2 \mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |u_\varepsilon|^q)|u_\varepsilon|^q \mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_0|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} |\mathscr{A}(x_0) - \mathscr{A}(\varepsilon x + x_0)|^2 |v_0|^2 \mathrm{d}x\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} K(\varepsilon x + x_0)|v_0|^2 \mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v_0|^q)|v_0|^q \mathrm{d}x.
\end{aligned}$$

Letting $\varepsilon \to 0^+$ and by Lemma 4.3, we have

$$\limsup_{\varepsilon\to 0^+} \vartheta_{a,A,\varepsilon} \le \lim_{\varepsilon\to 0^+} \mathcal{J}_{A,\varepsilon}(u_\varepsilon) = \mathcal{J}_0(v_0) = \vartheta_{a,0} = \vartheta_{a,A,0}. \qquad \blacksquare$$

Remark 4.6. By Lemma 4.5 and Remark 4.4, there exists a constant $\varepsilon^\star = \varepsilon^\star(a) > 0$ such that $\vartheta_{a,A,\varepsilon} < \vartheta_{a,A,\infty} < 0$ for all $\varepsilon \in (0, \varepsilon^\star)$, when $K_\infty < \delta$ and $\mathcal{A}^2 < \delta$.

Lemma 4.7. *If $K_\infty < \delta$ and $\mathcal{A}^2 < \delta$, then for all $\varepsilon \in (0, \varepsilon^\star)$, any minimizing sequence of $\vartheta_{a,A,\varepsilon}$ is uniformly bounded with respect to ε in $H^1_A(\mathbb{R}^N, \mathbb{C})$ and the minimizing sequence $\{v_{n,A,\varepsilon}\} \subset S_A(a)$ of $\vartheta_{a,A,\varepsilon}$ can be chosen such that its weak limit $v_{A,\varepsilon}$ is nontrivial, where $\varepsilon^\star$ is given in Remark 4.6.*

Proof. In view of $q \in (\frac{N+\theta}{N}, \frac{N+\theta+2}{N})$ and Remark 4.6, we get that any minimizing sequence of $\vartheta_{a,A,\varepsilon}$ is uniformly bounded by (4.1). Suppose by contradiction that $v_{A,\varepsilon} = 0$. By Lemma 4.1 and (4.5), we obtain

$$\vartheta_{a,A,\varepsilon} + o_n(1) = \mathcal{J}_{A,\varepsilon}(v_{n,A,\varepsilon}) \ge \mathcal{J}_\infty(|v_{n,A,\varepsilon}|) + \int_{\mathbb{R}^N} (K(\varepsilon x) - K_\infty)|v_{n,A,\varepsilon}|^2 \mathrm{d}x.$$

Similar to the proof of Lemma 3.11 and applying Lemma 4.3, we have $\vartheta_{a,A,\varepsilon} \ge \vartheta_{a,A,\infty}$, which contradicts Remark 4.6. ■

Finally, we assume V satisfies (V4) and study the energy functional

$$\mathcal{J}_{A,\lambda}(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A v|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \lambda W(x)|v|^2 \mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\theta * |v|^q)|v|^q \mathrm{d}x$$

and the minimization problem

$$\vartheta_{a,A,\lambda} := \inf_{v\in S_A(a)} \mathcal{J}_{A,\lambda}(v).$$

By Lemmas 3.1 and 4.1, $\mathcal{J}_{A,\lambda}(v)$ is bounded from below on $S_A(a)$ for any $a > 0$. As the proof of Lemma 4.2, we have the following lemma.

Lemma 4.8. *For each $a > 0$, there exists a constant $\delta = \delta(a) > 0$ such that if $\lambda|W|_\infty < \delta$ and $\mathcal{A}^2 < \delta$, then $\vartheta_{a,A,\lambda} < 0$.*

Furthermore, we can find $\lambda^\star = \lambda^\star(a) > 0$ such that for all $\lambda \in [\lambda^\star, +\infty)$, any minimizing sequence of $\vartheta_{a,A,\lambda}$ is bounded uniformly in λ with nontrivial weak limit and $\vartheta_{a,A,\lambda}$ can be attained, when $\lambda|W|_\infty < \delta$ and $\mathcal{A}^2 < \delta$. Similar to the proof in Section 3, we can complete the proof of Theorem 1.3 by combining Lemmas 4.7 and 4.8.

Acknowledgments. The authors thank the editor and reviewer for their helpful suggestions and valuable comments, which have greatly improved the paper.

Funding. This work was partially supported by the NSF of China (Grant 12326306), NSF of Liaoning Province (Grant 2025-MSLH-432) and Youth Fund Project of Liaoning Provincial Department of Education (Grant LJ212510165001)

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Received 14 April 2024; revised 21 December 2025.

Min Liu
School of Mathematics, Liaoning Normal University, 116029 Dalian, P. R. China;
minliu@lnnu.edu.cn

Rui Sun
School of Mathematics and Statistics, Beijing Jiaotong University, 100044 Beijing, P. R. China;
sunrui@bjtu.edu.cn