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Spherical Heinz transform of operator tuples

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Abstract. In this paper, we introduce the concept of the spherical Heinz transform for operator tuples, thereby extending the notion of the generalized mean transform in one-dimensional case to a multivariable operator setting. We study some of its properties and prove several inequalities involving the joint numerical radius and the joint operator norm of the spherical Aluthge and Heinz transforms. Moreover, several generalizations and refinements of some fundamental inequalities are also obtained.

1. Introduction

Let $\mathcal{H}$ denote a complex Hilbert space, and let $\mathfrak{B}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. An operator T is said to be *normal* if $T^*T = TT^*$, *quasinormal* if T commutes with T^*T, i.e., $TT^*T = T^*T^2$, *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$, and *hyponormal* if $T^*T \geq TT^*$. It is well known that

$$\text{normal} \Rightarrow \text{quasinormal} \Rightarrow \text{subnormal} \Rightarrow \text{hyponormal}.$$

The previous notions have been also transferred to a multivariable operator theory setting. First, for $\mathbf{T} = (T_1, \ldots, T_d) \in \mathfrak{B}(\mathcal{H})^d$, by $\mathbf{T}^*$ we denote the operator d-tuple $\mathbf{T}^* = (T_1^*, \ldots, T_d^*) \in \mathfrak{B}(\mathcal{H})^d$. For operators $S, T \in \mathfrak{B}(\mathcal{H})$, set $[S, T] := ST - TS$. We say that a d-tuple $\mathbf{T} = (T_1, \ldots, T_d)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{bmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_d^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_d^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_d] & [T_2^*, T_d] & \cdots & [T_d^*, T_d] \end{bmatrix}$$

is positive on the direct sum of n copies of $\mathcal{H}$ (cf. [3, 16, 19]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. For $i, j, k \in \{1, 2, \ldots, n\}$, $\mathbf{T}$ is called *matricially quasinormal* if each T_i commutes with each $T_j^* T_k$, $\mathbf{T}$ is (jointly) *quasinormal*

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if each T_i commutes with each $T_j^* T_j$, and *spherically quasinormal* if each T_i commutes with $\sum_{j=1}^n T_j^* T_j$ (see [28]). As shown in [4, 28], we have

$$\begin{aligned} \text{normal} &\Rightarrow \text{matricially quasinormal} \Rightarrow \text{(jointly) quasinormal} \\ &\Rightarrow \text{spherically quasinormal} \Rightarrow \text{subnormal.} \end{aligned}$$

Let $d \in \mathbb{N}$. For $T_1, \ldots, T_d \in \mathfrak{B}(\mathcal{H})$, consider a d-tuple $\mathbf{T} = \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix}$ as an operator from $\mathcal{H}$ into $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$, that is,

$$\mathbf{T} = \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix} : \mathcal{H} \to \begin{matrix} \mathcal{H} \\ \oplus \\ \vdots \\ \oplus \\ \mathcal{H} \end{matrix}.$$

We define (canonical) spherical polar decomposition of $\mathbf{T}$ (cf. [21, 22, 32]) as

$$\mathbf{T} = \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix} = \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix} P = \begin{pmatrix} V_1 P \\ \vdots \\ V_d P \end{pmatrix} = \mathbf{V} P,$$

where $P = \sqrt{T_1^* T_1 + \cdots + T_d^* T_d}$ is a positive operator on $\mathcal{H}$, and

$$\mathbf{V} = \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix} : \mathcal{H} \to \begin{matrix} \mathcal{H} \\ \oplus \\ \vdots \\ \oplus \\ \mathcal{H} \end{matrix}$$

is a spherical partial isometry from $\mathcal{H}$ into $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. In other words, $V_1^* V_1 + \cdots + V_d^* V_d$ is the (orthogonal) projection onto the initial space of the partial isometry $\mathbf{V}$ which is

$$\mathcal{N}(\mathbf{T})^\perp = \left(\bigcap_{i=1}^d \mathcal{N}(T_i) \right)^\perp = \mathcal{N}(P)^\perp = \left(\bigcap_{i=1}^d \mathcal{N}(V_i) \right)^\perp.$$

Using the spherical polar decomposition, we have the following useful characterizations of spherically quasinormal tuples.

Theorem 1.1 ([23, Lemma 2.1]). *Let* $\mathbf{T} = \mathbf{V} P = (V_1 P, \ldots, V_d P) \in \mathfrak{B}(\mathcal{H})^d$ *be the polar decomposition of a* d*-tuple* $\mathbf{T}$*. Then,* $\mathbf{T}$ *is spherically quasinormal if and only if* $V_i P = P V_i$*,* $i = 1, \ldots, d$*.*

Theorem 1.2. *Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$ be a d-tuple of commuting operators. The following conditions are equivalent:*

(i) $\mathbf{T}$ *is spherically quasinormal;*

(ii) $\mathbb{P}\mathbb{V} = \mathbb{V}\mathbb{P}$, *where*

$$\mathbb{V} = \begin{pmatrix} V_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ V_d & 0 & \cdots & 0 \end{pmatrix}$$

and

$$\mathbb{P} = \begin{pmatrix} P & & \\ & \ddots & \\ & & P \end{pmatrix}.$$

Proof. The proof follows from Theorem 1.1 and direct computation. ∎

For an account on various related notions of quasinormality in several variables, one may refer to [18, 41, 42].

Operator d-tuple $\mathbf{T} \in \mathfrak{B}(\mathcal{H})^d$ is said to be *Taylor invertible* if its associated Koszul complex $\mathcal{K}(\mathbf{T}, \mathcal{H})$ is exact. For $d = 2$, the Koszul complex $\mathcal{K}(\mathbf{T}, \mathcal{H})$ associated to $\mathbf{T} = (T_1, T_2)$ on $\mathcal{H}$ is given by

$$\mathcal{K}(\mathbf{T}, \mathcal{H}): \quad 0 \to \mathcal{H} \xrightarrow{\mathbf{T}} \mathcal{H} \oplus \mathcal{H} \xrightarrow{(-T_2\ T_1)} \mathcal{H} \to 0,$$

where $\mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$.

The Taylor spectrum of a commuting d-tuple $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$ is denoted by $\sigma_T(\mathbf{T})$ and it is defined as

$$\sigma_T(\mathbf{T}) = \{(\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d : \mathcal{K}((T_1 - \lambda_1, \dots, T_d - \lambda_d), \mathcal{H}) \text{ is not exact}\}.$$

Recall that $\sigma_T(\mathbf{T})$ is a nonempty compact subset of $\mathbb{C}^d$. The *joint spectral radius* of $\mathbf{T}$ is defined by

$$r(\mathbf{T}) = \max\{\|\lambda\|_2,\ \lambda = (\lambda_1, \dots, \lambda_d) \in \sigma_T(\mathbf{T})\},$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{C}^d$. For more information on the Taylor invertibility and Koszul complexes, we refer the reader to [33, 44, 45].

The *spherical norm* (or simply, *norm*) of $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$ is given by

$$\|\mathbf{T}\| := \sup\left\{\left(\sum_{k=1}^{d} \|T_k x\|^2\right)^{\frac{1}{2}} : x \in \mathcal{H},\ \|x\| = 1\right\},$$

while the *Euclidean norm* is defined as

$$\|\mathbf{T}\|_e := \sup_{(\lambda_1, \dots, \lambda_d) \in \overline{\mathbb{B}}_d} \|\lambda_1 T_1 + \cdots + \lambda_d T_d\|,$$

where $\mathbb{B}_d$ denotes the open unit ball of $\mathbb{C}^d$ with respect to the Euclidean norm $\|\cdot\|_2$. As shown recently, it turns out that $\|\mathbf{T}\|$ and $\|\mathbf{T}\|_e$ are always equivalent on $\mathfrak{B}(\mathcal{H})^d$ (see [38, Theorem 1.18] and [25, Proposition 2.1]):

$$\frac{1}{\sqrt{d}}\|\mathbf{T}\| \leq \|\mathbf{T}\|_e \leq \|\mathbf{T}\|, \quad \mathbf{T} \in \mathfrak{B}(\mathcal{H})^d.$$

For $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$, we define the *joint (or spherical) numerical radius* of $\mathbf{T}$ as

$$\omega(\mathbf{T}) = \sup\left\{\left(\sum_{k=1}^{d} |\langle T_k x, x\rangle|^2\right)^{\frac{1}{2}} : x \in \mathcal{H},\ \|x\| = 1\right\}.$$

For more details, see [15]. It was shown in [38, Proof of Theorem 1.19] that $\omega(\mathbf{T})$ coincides with the *Euclidean numerical radius* $\omega_e(\mathbf{T})$ given by

$$\omega_e(\mathbf{T}) := \sup_{(\lambda_1,\dots,\lambda_d)\in\mathbb{B}_d} \omega(\lambda_1 T_1 + \cdots + \lambda_d T_d).$$

If $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$ is a commuting d-tuple, then it was shown in [5, Theorem 2.2] and [6, Theorem 2.4] that

$$r(\mathbf{T}) \leq \max\left\{\frac{1}{2\sqrt{d}}\|\mathbf{T}\|,\ r(\mathbf{T})\right\} \leq \omega(\mathbf{T}) \leq \|\mathbf{T}\|_e \leq \|\mathbf{T}\|. \tag{1.1}$$

Next, recall the definition of the 2-variable weighted shifts. Let $\mathbb{Z}_+$ denote the set of all non-negative integers. Consider double-indexed non-negative bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in l^\infty(\mathbb{Z}_+^2)$, where $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}_+^2$, and let $l^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. We define the *2-variable weighted shift* $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ by

$$T_1 e_{(k_1,k_2)} = \alpha_{(k_1,k_2)} e_{(k_1+1,k_2)}$$

and

$$T_2 e_{(k_1,k_2)} = \beta_{(k_1,k_2)} e_{(k_1,k_2+1)}.$$

For all $(k_1, k_2) \in \mathbb{Z}_+^2$, it is easy to see that

$$T_1 T_2 = T_2 T_1 \iff \beta_{(k_1+1,k_2)}\alpha_{(k_1,k_2)} = \alpha_{(k_1,k_2+1)}\beta_{(k_1,k_2)}.$$

For the basic properties of the 2-variable weighted shift $\mathbf{W}_{(\alpha,\beta)}$, we refer the reader to [17, 20].

The *Aluthge transform* $\widetilde{T}$ of operator $T = U|T| \in \mathfrak{B}(\mathcal{H})$ is $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, the *Duggal transform* T^D of T is $T^D = |T|U$, and the *mean transform* $\widehat{T}$ of T is

$$\widehat{T} = \frac{1}{2}(U|T| + |T|U) = \frac{1}{2}(T + T^D).$$

Let $t \in [0,1]$. The *generalized Aluthge transform* of T is defined as $\widetilde{T}(t) = |T|^t U |T|^{1-t}$. Obviously, $\widetilde{T}(0) = T$, $\widetilde{T}(1) = T^D$, and $\widehat{T} = \frac{1}{2}(\widetilde{T}(0) + \widetilde{T}(1))$. The authors in [7] recently introduced the *generalized mean transform* of T as

$$\widehat{T}(t) = \frac{1}{2}(\widetilde{T}(t) + \widetilde{T}(1-t)).$$

For more details on the Aluthge and Duggal transform, see, for instance, [2, 14, 30]. In recent years, the mean transform also attracted considerable attention (see [10, 11, 35]).

In a multivariable case, for $\mathbf{T} = \mathbf{V}P = (V_1 P, \dots, V_d P) \in \mathfrak{B}(\mathcal{H})^d$, the following analogous concepts are introduced:

- *spherical Aluthge transform*:

$$\widetilde{\mathbf{T}} = (\widetilde{\mathbf{T}}_1, \dots, \widetilde{\mathbf{T}}_d) = (P^{\frac{1}{2}} V_1 P^{\frac{1}{2}}, \dots, P^{\frac{1}{2}} V_d P^{\frac{1}{2}});$$

- *spherical Duggal transform*:

$$\mathbf{T}^D = (\mathbf{T}_1^D, \dots, \mathbf{T}_d^D) = (P V_1, \dots, P V_d);$$

- *spherical mean transform*:

$$\widehat{\mathbf{T}} = \frac{1}{2}(\mathbf{T} + \mathbf{T}^D).$$

Recently, the authors in [8] introduced the *generalized spherical Aluthge transform* as

$$\widetilde{\mathbf{T}}(t) = (\widetilde{\mathbf{T}}_1(t), \dots, \widetilde{\mathbf{T}}_d(t)) = (P^t V_1 P^{1-t}, \dots, P^t V_d P^{1-t}),$$

where $t \in [0,1]$.

For more details on the mentioned concepts, we refer the reader to [21, 22, 26, 32, 43]. Naturally, we extend the notion of the generalized mean transform to a multivariable setting as well.

Definition 1.1. Let $\mathbf{T} = \mathbf{V}P = (V_1 P, \dots, V_d P) \in \mathfrak{B}(\mathcal{H})^d$ be the canonical spherical polar decomposition of a d-tuple $\mathbf{T}$, and let $t \in [0,1]$. The *spherical Heinz transform* (or *generalized spherical mean transform*) of $\mathbf{T}$ is defined as

$$\widehat{\mathbf{T}}(t) = \frac{1}{2}(\widetilde{\mathbf{T}}(t) + \widetilde{\mathbf{T}}(1-t)). \tag{1.2}$$

In other words, $\widehat{\mathbf{T}}(t) = (\widehat{\mathbf{T}}_1(t), \dots, \widehat{\mathbf{T}}_d(t))$, where

$$\widehat{\mathbf{T}}_i(t) = \frac{1}{2}\big(P^t V_i P^{1-t} + P^{1-t} V_i P^t\big), \quad i \in \{1, \dots, d\}.$$

Obviously, $\widehat{\mathbf{T}}(t) = \widehat{\mathbf{T}}(1-t)$, $\widehat{\mathbf{T}}(0) = \widehat{\mathbf{T}}(1) = \widehat{\mathbf{T}}$, and $\widehat{\mathbf{T}}(\frac{1}{2}) = \widetilde{\mathbf{T}}$.

Remark 1.1. Let $i \in \{1, \dots, d\}$. Observe that $\widehat{\mathbf{T}}_i(t)$ in the previous definition is not the generalized mean transform of T_i, as $T_i = V_i P$ is not the standard polar decomposition of T_i.

Remark 1.2. The motivation for the name "spherical Heinz transform" follows from the fact that the Heinz means for operators $A, B \in \mathfrak{B}(\mathcal{H})$ are defined by

$$\mathfrak{H}(A, B) = \frac{A^\alpha B^{1-\alpha} + A^{1-\alpha} B^\alpha}{2},$$

where $0 \leq \alpha \leq 1$ and $A, B \geq 0$ (see [31]).

In the paper, for brevity, we will use the following notation: for an operator d-tuple $\mathbf{T} = (T_1, \ldots, T_d)$, and $A, B \in \mathfrak{B}(\mathcal{H})$, $A\mathbf{T}B$ means

$$A\mathbf{T}B = (AT_1B, \ldots, AT_dB).$$

Also, for two operator d-tuples $\mathbf{A} = (A_1, \ldots, A_d)$ and $\mathbf{B} = (B_1, \ldots, B_d)$, we write

$$\mathbf{AB} = (A_1B_1, \ldots, A_dB_d).$$

Finally, $\mathbf{0}$ and $\mathbf{I}$ stand for operator tuples (of the appropriate dimension) $(0, \ldots, 0)$ and $(I, \ldots, I)$, respectively.

The paper is organized as follows. In Section 2, we give some properties of the spherical Heinz transform, which presents the basis for the further study on the topic. In Section 3, we describe how 2-variable weighted shifts behave under the transform. Finally, in Section 4, we give several norm and numerical radius inequalities related to the spherical Aluthge and Heinz transforms. We also generalize the known results in one-dimensional case to a multivariable operator theory setting.

2. General properties

In this section, we derive some general properties of the spherical Heinz transform. In many theorems, the case $t = 0$ (and $t = 1$) is excluded as it is already considered in [43].

We start with the following characterizations of injective spherically quasinormal tuples.

Theorem 2.1. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathfrak{B}(\mathcal{H})^d$ be a d-tuple of commuting operators such that $\mathcal{N}(\mathbf{T}) = \{0\}$, and let $t \in (0, 1) \setminus \{\frac{1}{2}\}$. The following conditions are equivalent:*

(i) $\mathbf{T}$ *is spherically quasinormal;*

(ii) $\widehat{\mathbf{T}}(t) = \widetilde{\mathbf{T}}(t)$*;*

(iii) $\widetilde{\mathbf{T}}(t) = \widetilde{\mathbf{T}}(1-t)$.

Proof. (i)⇒(ii): let $i \in \{1, \ldots, d\}$ be arbitrary. By Theorem 1.1, $V_iP = PV_i$, and so, $V_iP^\alpha = P^\alpha V_i$ for all $\alpha > 0$. Therefore,

$$\widehat{\mathbf{T}}_i(t) = \frac{1}{2}\left(P^tV_iP^{1-t} + P^{1-t}V_iP^t\right) = P^tV_iP^{1-t} = \widetilde{\mathbf{T}}_i(t).$$

Hence, $\widehat{\mathbf{T}}(t) = \widetilde{\mathbf{T}}(t)$.

(ii)⇒(i): without loss of generality, we may assume that $0 < t < \frac{1}{2}$. Let $i \in \{1, \dots, d\}$ be arbitrary. From $\widehat{\mathbf{T}}(t) = \widetilde{\mathbf{T}}(t)$, it follows that $\widehat{\mathbf{T}}_i(t) = \widetilde{\mathbf{T}}_i(t)$, i.e.,

$$P^t V_i P^{1-t} + P^{1-t} V_i P^t = 2P^t V_i P^{1-t}.$$

From here,

$$P^t P^{1-2t} V_i P^t = P^t V_i P^{1-2t} P^t. \tag{2.1}$$

Since $\mathcal{N}(\mathbf{T}) = \{0\}$, it follows that $\mathcal{N}(P^t) = \mathcal{N}(P) = \{0\}$. Thus, P^t is one-to-one and has dense range. Equality (2.1) now yields

$$P^{1-2t} V_i = V_i P^{1-2t},$$

and so, $P V_i = V_i P$. Theorem 1.1 implies that $\mathbf{T}$ is spherically quasinormal.

(ii)⇔(iii): this is obviously true in general for any $\mathbf{T} \in \mathfrak{B}(\mathcal{H})^d$. ■

Theorem 2.2. *Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$ be a d-tuple of commuting operators such that $\mathcal{N}(\mathbf{T}) = \{0\}$. The following conditions are equivalent:*

(i) $\mathbf{T}$ *is spherically quasinormal;*

(ii) *there exist $t, s \in [0, 1]$, $t \neq s$, such that $\widetilde{\mathbf{T}}(t) = \widetilde{\mathbf{T}}(s)$.*

Proof. (i)⇒(ii): if $\mathbf{T}$ is spherically quasinormal, then Theorem 1.1 implies that $\widetilde{\mathbf{T}}(t) = \mathbf{T}$ for all $t \in [0, 1]$.

(ii)⇒(i): let $i \in \{1, \dots, d\}$ be arbitrary. Without loss of generality, we may assume that $t < s$. From $\widetilde{\mathbf{T}}(t) = \widetilde{\mathbf{T}}(s)$, we have

$$P^t V_i P^{1-t} = P^s V_i P^{1-s},$$

i.e.,

$$P^t V_i P^{s-t} P^{1-s} = P^t P^{s-t} V_i P^{1-s}.$$

Since $\mathcal{N}(P^\alpha) = \mathcal{N}(\mathbf{T}) = \{0\}$ for any $\alpha \in [0, 1]$, it follows that $V_i P^{s-t} = P^{s-t} V_i$. Since $s - t > 0$, we have that $V_i P = P V_i$, and thus, Theorem 1.1 yields that $\mathbf{T}$ is spherically quasinormal. ■

Our next results explore the equality of kernels of the original operator tuple and its spherical Heinz and Duggal transforms.

Theorem 2.3. *Let $\mathbf{T} = \mathbf{V}P = (V_1 P, \dots, V_d P) \in \mathfrak{B}(\mathcal{H})^d$ be a d-tuple of operators such that $\mathcal{N}(\mathbf{V}) \subseteq \mathcal{N}(\mathbf{V}^*)$, and let $t \in (0, 1)$. Then,*

$$\mathcal{N}(\widehat{\mathbf{T}}(t)) = \mathcal{N}(\mathbf{T}).$$

Proof. Let $x \in \mathcal{N}(\mathbf{T})$. Since $\mathcal{N}(\mathbf{T}) = \mathcal{N}(P) = \mathcal{N}(P^\alpha) = \mathcal{N}(\mathbf{V})$ (where $\alpha > 0$), we have that $P^\alpha x = V_i x = 0$ for all $\alpha > 0$ and all $i \in \{1, \dots, d\}$. Thus,

$$\widehat{\mathbf{T}}_i(t)x = \frac{1}{2}\big(P^t V_i P^{1-t} x + P^{1-t} V_i P^t x\big) = 0, \quad i = 1, \dots, d.$$

Hence, $x \in \bigcap_{i=1}^d \mathcal{N}(\widehat{\mathbf{T}}_i(t)) = \mathcal{N}(\widehat{\mathbf{T}}(t))$, and so, $\mathcal{N}(\mathbf{T}) \subseteq \mathcal{N}(\widehat{\mathbf{T}}(t))$.

Conversely, assume that $x \in \mathcal{N}(\widehat{\mathbf{T}}(t))$, and let $i \in \{1, \ldots, d\}$ be arbitrary. Without loss of generality, we may assume that $t \in (0, \frac{1}{2}]$. First, consider the case when $t \neq \frac{1}{2}$. We have that

$$\begin{aligned} 0 &= \widehat{\mathbf{T}}_i(t)\, x \\ &= \frac{1}{2}\big(P^t V_i P^{1-t} + P^{1-t} V_i P^t\big)x \\ &= \frac{1}{2} P^t \big(V_i P^{1-2t} + P^{1-2t} V_i\big) P^t x \\ &= \frac{1}{2} P^t \big(V_i P^{1-2t} + P^{1-2t} V_i\big) y, \end{aligned}$$

where $y = P^t x$. Since

$$\mathcal{N}(P^t) = \mathcal{N}(P) = \mathcal{N}(\mathbf{V}) \subseteq \mathcal{N}(\mathbf{V}^*) \subseteq \mathcal{N}(V_i^*), \tag{2.2}$$

it follows that

$$V_i^* V_i P^{1-2t} y + V_i^* P^{1-2t} V_i y = 0.$$

Summing over $i \in \{1, \ldots, d\}$, and using the fact that $\sum_{i=1}^{d} V_i^* V_i$ is the orthogonal projection onto $\overline{\mathcal{R}(P)} = \overline{\mathcal{R}(P^{1-2t})}$, we have

$$P^{1-2t} y + \sum_{i=1}^{d} V_i^* P^{1-2t} V_i y = 0.$$

Thus,

$$\langle P^{1-2t} y, y\rangle + \sum_{i=1}^{d} \langle P^{1-2t} V_i y, V_i y\rangle = 0.$$

Since operator P is positive, it follows that $y \in \mathcal{N}(P^{\frac{1-2t}{2}}) = \mathcal{N}(P^{1-t})$, and so,

$$Px = P^{1-t} y = 0.$$

Therefore, $x \in \mathcal{N}(P) = \mathcal{N}(\mathbf{T})$, and thus,

$$\mathcal{N}(\widehat{\mathbf{T}}(t)) \subseteq \mathcal{N}(\mathbf{T}).$$

The case $t = \frac{1}{2}$ can be proved in a similar manner. ∎

Directly from the previous theorem, we have the following corollary.

Corollary 2.4. *Let* $\mathbf{T} = \mathbf{V}P = (V_1 P, \ldots, V_d P) \in \mathfrak{B}(\mathcal{H})^d$ *be a* d*-tuple of operators, and let* $t \in (0, 1)$. *The following conditions are equivalent:*

(i) $\mathbf{T} = \mathbf{0}$;

(ii) $\widehat{\mathbf{T}}(t) = \mathbf{0}$ *and* $\mathcal{N}(\mathbf{V}) \subseteq \mathcal{N}(\mathbf{V}^*)$.

Theorem 2.5. *Let* $\mathbf{T} = \mathbf{V}P = (V_1 P, \dots, V_d P) \in \mathfrak{B}(\mathcal{H})^d$ *be a* d*-tuple of operators such that* $\mathcal{N}(\mathbf{V}) \subseteq \mathcal{N}(\mathbf{V}^*)$*. If* $\widehat{\mathbf{T}}(t) = \widetilde{\mathbf{T}}(t)$ *for some* $t \in (0,1) \setminus \{\frac{1}{2}\}$*, then*

$$\mathcal{N}(\mathbf{T}) = \mathcal{N}(\mathbf{T}^D).$$

Proof. Since $\widehat{\mathbf{T}}(t) = \widetilde{\mathbf{T}}(t)$, we have that (2.1) holds, i.e.,

$$P^t(P^{1-2t}V_i - V_i P^{1-2t})P^t = 0.$$

By (2.2), we have that

$$(V_i^* P^{1-2t} V_i - V_i^* V_i P^{1-2t})P^t = 0, \quad i = 1, \dots, d,$$

and so,

$$\left(\sum_{i=1}^{d} V_i^* P^{1-2t} V_i - P^{1-2t}\right) P^t = 0.$$

Thus,

$$P^{1-2t} = \sum_{i=1}^{d} V_i^* P^{1-2t} V_i \tag{2.3}$$

on $\overline{\mathcal{R}(P)}$. Obviously, (2.3) also holds on $\mathcal{N}(P)$, and so, it is true on whole

$$\mathcal{H} = \overline{\mathcal{R}(P)} \oplus \mathcal{N}(P).$$

Therefore,

$$\begin{aligned}
\mathcal{N}(\mathbf{T}) &= \mathcal{N}(P) = \mathcal{N}(P^{1-2t}) \\
&= \mathcal{N}\left(\sum_{i=1}^{d} V_i^* P^{1-2t} V_i\right) \\
&= \mathcal{N}\left(\sum_{i=1}^{d} (P^{\frac{1-2t}{2}} V_i)^* P^{\frac{1-2t}{2}} V_i\right) \\
&= \bigcap_{i=1}^{d} \mathcal{N}(P^{\frac{1-2t}{2}} V_i) = \bigcap_{i=1}^{d} \mathcal{N}(P V_i) \\
&= \mathcal{N}(\mathbf{T}^D).
\end{aligned}$$

■

The following theorem, similar in spirit to Corollary 2.4, is a generalization of [43, Theorem 2.4] to an arbitrary d-tuple of operators, and also to any $t \in [0,1]$. Note that the case $t = 0$ (and $t = 1$) proved in the mentioned theorem actually does not require the hyponormality condition imposed for $t \in (0,1)$.

Theorem 2.6. *Let* $\mathbf{T} = \mathbf{V}P = (V_1 P, \ldots, V_d P) \in \mathfrak{B}(\mathcal{H})^d$ *be a* d*-tuple of operators, and let* $t \in (0,1)$. *The following conditions are equivalent:*

(i) $\mathbf{T} = \mathbf{I}$;

(ii) $\widehat{\mathbf{T}}(t) = \mathbf{I}$, V_i *are hyponormal,* $i \in \{1, \ldots, d\}$, *and* $V_i V_j^* = V_j^* V_i = \frac{1}{d} I$ *for* $i \neq j$, $i, j \in \{1, \ldots, d\}$;

(iii) $\widehat{\mathbf{T}}(t) = \mathbf{I}$, $\sum_{i=1}^d V_i$ *is hyponormal and* $\sum_{i\neq j} V_i^* V_j = (d-1)I$.

Proof. (i)⇒(ii): if $\mathbf{T} = \mathbf{I}$, then, obviously, $P = \sqrt{d}\, I$ and $V_i = \frac{1}{\sqrt{d}} I$ for all $i \in \{1, \ldots, d\}$. This immediately yields (ii).

(ii)⇒(iii): since V_1 and V_2 are hyponormal and V_1 commutes with V_2^*, it is easy to see that $V_1 + V_2$ must also be hyponormal. Thus, V_3 and $V_1 + V_2$ are hyponormal and V_3 commutes with $(V_1 + V_2)^*$, which implies that $V_1 + V_2 + V_3$ is hyponormal. By continuing this process, we conclude that $\sum_{i=1}^d V_i$ is hyponormal. Also,

$$\sum_{i\neq j} V_i^* V_j = d(d-1) \cdot \frac{1}{d} I = (d-1)I.$$

(iii)⇒(i): now, assume that (iii) holds, and without loss of generality, we may also assume that $t \in (0, \frac{1}{2}]$. For each $i \in \{1, \ldots, d\}$, $\widehat{\mathbf{T}}(t) = \mathbf{I}$ implies that

$$P^t V_i P^{1-t} + P^{1-t} V_i P^t = 2I, \tag{2.4}$$

i.e.,

$$P^t (V_i P^{1-2t} + P^{1-2t} V_i) P^t = 2I.$$

From here, it follows that P^t is invertible, and so,

$$V_i P^{1-2t} + P^{1-2t} V_i = 2P^{-2t}.$$

By multiplying from the left-hand side by V_i^* and summing over all $i \in \{1, \ldots, d\}$, we obtain

$$P^{1-2t} + \sum_{i=1}^d V_i^* P^{1-2t} V_i = 2\left(\sum_{i=1}^d V_i^*\right) P^{-2t}. \tag{2.5}$$

It follows that

$$P^{-2t}\left(\sum_{i=1}^d V_i\right) = \left(\sum_{i=1}^d V_i^*\right) P^{-2t} \geq 0, \tag{2.6}$$

and so,

$$P^{-2t}\left(\sum_{i=1}^d V_i\right) P^{2t} = \left(\sum_{i=1}^d V_i\right)^*. \tag{2.7}$$

Since operator P^{2t} is positive and invertible, we have that $0 \notin \overline{\mathcal{W}(P^{2t})}$, where $\mathcal{W}(P^{2t})$ denotes the numerical range of P^{2t}. Using [46, Theorem 1] (also cf. [40]), we conclude that $\sum_{i=1}^{d} V_i$ is self-adjoint. Now, (2.7) implies that $\sum_{i=1}^{d} V_i$ commutes with P^{2t}, and so, it also commutes with P^{α} for any $\alpha > 0$. Inequality (2.6) now yields

$$P^{-t}\left(\sum_{i=1}^{d} V_i\right)P^{-t} \geq 0,$$

and by multiplying from both sides by P^t, we get that $\sum_{i=1}^{d} V_i$ is positive. Also, observe that since $\sum_{i=1}^{d} V_i^* V_i$ is an orthogonal projection onto $\overline{\mathcal{R}(P)} = \mathcal{H}$, we have that

$$\sum_{i=1}^{d} V_i^* V_i = I,$$

and so,

$$\begin{aligned}\left(\sum_{i=1}^{d} V_i\right)^2 &= \left(\sum_{i=1}^{d} V_i^*\right)\left(\sum_{i=1}^{d} V_i\right)\\ &= \sum_{i=1}^{d} V_i^* V_i + \sum_{i\neq j} V_i^* V_j\\ &= I + (d-1)I\\ &= dI.\end{aligned}$$

From the uniqueness of a positive square root, it follows that $\sum_{i=1}^{d} V_i = \sqrt{d}I$. Now, summing over $i \in \{1, \ldots, d\}$ in (2.4), we get

$$2\sqrt{d}P = 2dI,$$

and thus, $P = \sqrt{d}I$. Furthermore, for each $i \in \{1, \ldots, d\}$, equation (2.4) now implies that $2\sqrt{d}V_i = 2I$, and so,

$$V_i = \frac{1}{\sqrt{d}}I, \quad i \in \{1, \ldots, d\}.$$

Finally, we conclude that $\mathbf{T} = \mathbf{I}$. ■

Remark 2.1. Observe that the condition (iii) of the previous theorem can also be replaced with the following one:

(iii′) $\widehat{\mathbf{T}}(t) = \mathbf{I}$, $\sum_{i=1}^{d} V_i$ is hyponormal and

$$\left|\sum_{i=1}^{d} V_i\right|^{1-2t} \leq P^{1-2t} \leq \sum_{i=1}^{d} V_i^* P^{1-2t} V_i. \tag{2.8}$$

Indeed, as in the proof of (iii)⇒(i), we may conclude that $\sum_{i=1}^{d} V_i$ is positive and commutes with P. Therefore, (2.8) implies that $\sum_{i=1}^{d} V_i \leq P$. From (2.5) and (2.8), we then have

$$\begin{aligned}
2P^{1-2t} &= 2P^{-t}PP^{-t} \\
&\geq 2P^{-t}\left(\sum_{i=1}^{d} V_i\right)P^{-t} \\
&= 2\left(\sum_{i=1}^{d} V_i\right)P^{-2t} \\
&= P^{1-2t} + \sum_{i=1}^{d} V_i^* P^{1-2t} V_i \\
&\geq 2P^{1-2t}.
\end{aligned}$$

Thus, $(\sum_{i=1}^{d} V_i)P^{-2t} = P^{1-2t}$, and so, $\sum_{i=1}^{d} V_i = P$. Summing over $i \in \{1, \dots, d\}$ in (2.4), we get

$$2P^2 = 2dI,$$

and thus, $P = \sqrt{d}I$. We now deduce that $\mathbf{T} = \mathbf{I}$ in the same way as in Theorem 2.6. This proves (iii′)⇒(i). Implication (i)⇒(iii′) follows from the fact that, under the assumption that $\mathbf{T} = \mathbf{I}$, the inequalities in (2.8) are actually equalities.

Let us now introduce the following notation: for $\mathbf{T} = (T_1, \dots, T_m) \in \mathfrak{B}(\mathcal{H})^m$ and $\mathbf{S} = (S_1, \dots, S_n) \in \mathfrak{B}(\mathcal{H})^n$, we write

$$\mathbf{T} \circ \mathbf{S} := (T_1S_1, \dots, T_1S_n, \dots, T_mS_1, \dots, T_mS_n).$$

Theorem 2.7. *Let* $\mathbf{T} = \mathbf{V}P = (V_1P, \dots, V_dP) \in \mathfrak{B}(\mathcal{H})^d$ *be a* d*-tuple of operators and let* $t \in (0, 1]$. *The following conditions are equivalent:*

(i) $\widehat{\mathbf{T}}(t) = \frac{1}{2}\widetilde{\mathbf{T}}(1-t)$;

(ii) $\widetilde{\mathbf{T}}(t) = \mathbf{0}$;

(iii) $\mathbf{T} \circ \mathbf{T} = \mathbf{0}$;

(iv) $\mathbf{V} \circ \mathbf{V} = \mathbf{0}$.

Proof. (i)⇔(ii): this is obvious.

(ii)⇒(iii): assume that $\widetilde{\mathbf{T}}(t) = \mathbf{0}$. Then, for all $j \in \{1, \dots, d\}$, we have $P^tV_jP^{1-t} = 0$. Therefore,

$$T_iT_j = V_iPV_jP = V_iP^{1-t}P^tV_jP^{1-t}P^t = 0$$

for all $i, j \in \{1, \dots, d\}$. This implies that $\mathbf{T} \circ \mathbf{T} = \mathbf{0}$.

(iii)⇒(ii): now, assume that $\mathbf{T} \circ \mathbf{T} = \mathbf{0}$, and let $i, j \in \{1, \ldots, d\}$ be arbitrary. Then,

$$V_i P V_j P = T_i T_j = 0.$$

By multiplying on the left-hand side by V_i^* and summing over $i \in \{1, \ldots, d\}$, we have that $P V_j P = 0$. Thus, $P V_j = 0$ on $\overline{\mathcal{R}(P)}$, and obviously, $P V_j = 0$ on $\mathcal{N}(P)$. Thus, $P V_j = 0$, and using the fact that $\mathcal{N}(P) = \mathcal{N}(P^t)$, we immediately get that

$$P^t V_j P^{1-t} = 0$$

for all $j \in \{1, \ldots, d\}$. Therefore, (ii) holds.

(iii)⇒(iv): let $i, j \in \{1, \ldots, d\}$ be arbitrary. As in the proof of the previous implication, we have that $P V_j P = 0$ for all $j \in \{1, \ldots, d\}$. Since $\mathcal{N}(P) \subseteq \mathcal{N}(V_i)$, it follows that $V_i V_j P = 0$. Thus, $V_i V_j = 0$ on $\overline{\mathcal{R}(P)}$, and obviously, $V_i V_j = 0$ on $\mathcal{N}(P)$. Thus, $V_i V_j = 0$, i.e., $\mathbf{V} \circ \mathbf{V} = \mathbf{0}$.

(iv)⇒(iii): let $i, j \in \{1, \ldots, d\}$ be arbitrary and assume that $\mathbf{V} \circ \mathbf{V} = \mathbf{0}$. Then, $V_i V_j = 0$, and so, $\mathcal{R}(V_j) \subseteq \mathcal{N}(V_i)$ for all $i \in \{1, \ldots, d\}$. Thus,

$$\mathcal{R}(V_j) \subseteq \mathcal{N}(P) = \bigcap_{i=1}^{d} \mathcal{N}(V_i),$$

and so, $P V_j = 0$. From here, we get

$$T_i T_j = V_i P V_j P = 0,$$

which implies that $\mathbf{T} \circ \mathbf{T} = \mathbf{0}$. ■

As the immediate corollaries of the previous theorem, we obtain the following known results in one-dimensional case.

Corollary 2.8 ([10, Lemma 2.4]). *Let $t \in (0, 1]$ and $T \in \mathfrak{B}(\mathcal{H})$. Then,*

$$\widetilde{T}(t) = 0 \iff T^2 = 0.$$

Corollary 2.9 ([27, p. 73]). *Let $T = U|T|$ be the polar decomposition of an operator T. Then, $T^2 = 0$ if and only if $U^2 = 0$.*

The following theorem shows that the spherical Heinz transform respects unitary equivalence.

Theorem 2.10. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathfrak{B}(\mathcal{H})^d$, and let $U \in \mathfrak{B}(\mathcal{H})$ be a unitary operator. Then,*

$$\widehat{U\mathbf{T}U^*}(t) = U\widehat{\mathbf{T}}(t)U^*$$

for all $t \in [0, 1]$.

Proof. Let $\mathbf{S} = U\mathbf{T}U^* = (UT_1U^*, \dots, UT_dU^*)$. Consider the spherical polar decompositions $\mathbf{T} = \mathbf{V}P$ and $\mathbf{S} = \mathbf{W}P_S$. As in the proof of [43, Theorem 2.5], we can show that $P_S = UPU^*$ and $\mathbf{W} = U\mathbf{V}U^*$. Now, for an arbitrary $t \in [0,1]$ and $i \in \{1, \dots, d\}$,

$$\begin{aligned}\widehat{\mathbf{S}}_i(t) &= \frac{1}{2}(P_S^t W_i P_S^{1-t} + P_S^{1-t} W_i P_S^t) \\ &= \frac{1}{2}(UP^tU^*UV_iU^*UP^{1-t}U^* + UP^{1-t}U^*UV_iU^*UP^tU^*) \\ &= U\frac{P^tV_iP^{1-t} + P^{1-t}V_iP^t}{2}U^* \\ &= U\widehat{\mathbf{T}}_i(t)U^*\end{aligned}$$

for $i = 1, \dots, d$, which yields the wanted result. ■

Our next result considers the behaviour of the spherical partial isometry under the spherical Heinz transform. As we will see from Theorem 2.12 and Remark 2.2 below, the cases $t \in (0,1)$ and $t = 0$ $(t = 1)$ differ. First, we need to recall the following definition and theorem. Observe that we use a slightly different approach than the one used in [43].

Definition 2.1. Let $\mathbf{A} = (A_1, \dots, A_n)$ and $\mathbf{B} = (B_1, \dots, B_n)$ be two n-tuples of operators on $\mathcal{H}$. We say that $\mathbf{A}$ and $\mathbf{B}$ *criss-cross commute* (or that $\mathbf{A}$ *criss-cross commutes* with $\mathbf{B}$) if $A_iB_jA_k = A_kB_jA_i$ and $B_iA_jB_k = B_kA_jB_i$ for all $i, j, k = 1, \dots, n$.

Theorem 2.11 ([13, Corollary 3.4]). *Let $\mathbf{A} = (A, \dots, A)$ and $\mathbf{B} = (B_1, \dots, B_n)$ be criss-cross commuting. If A is normal, then $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$.*

Theorem 2.12. *Let $\mathbf{V} = (V_1, \dots, V_d) \in \mathfrak{B}(\mathcal{H})^d$ be a spherical partial isometry, and let $t \in (0,1)$. Then,*

$$\widehat{\mathbf{V}}(t) = \mathbf{V}^D$$

and

$$\sigma_T(\mathbf{V}) = \sigma_T(\widehat{\mathbf{V}}(t)).$$

Proof. The spherical polar decomposition of $\mathbf{V}$ is given by $V_i = V_iP$, $i = 1, \dots, d$, where $P = (\sum_{i=1}^d V_i^*V_i)^{\frac{1}{2}}$. Since P is an orthogonal projection, it is easy to see that $P^\alpha = P$ for each $\alpha > 0$. Now, for $i \in \{1, \dots, d\}$, we have

$$\begin{aligned}\widehat{\mathbf{V}}_i(t) &= \frac{1}{2}\left(P^tV_iP^{1-t} + P^{1-t}V_iP^t\right) \\ &= \frac{1}{2}(PV_iP + PV_iP) \\ &= \frac{1}{2}(PV_i + PV_i) \\ &= PV_i,\end{aligned}$$

which proves the first part of the claim.

Next, we will show that $\mathbf{V} = (V_1, \dots, V_d)$ criss-cross commutes with $\mathbf{P} = (P, \dots, P)$. Using the fact that $V_i P = V_i$, $i = 1, \dots, d$ and that $\mathbf{V}$ is commuting, we have that

$$V_i P V_j = V_i V_j = V_j V_i = V_j P V_i, \quad i, j \in \{1, \dots, d\}.$$

By definition, $\mathbf{V}$ criss-cross commutes with $\mathbf{P}$ as the other condition in cross-commutativity trivially holds. Also, observe that we can write $\widehat{\mathbf{V}}(t) = \mathbf{PV}$. Theorem 2.11 now yields

$$\sigma_T(\widehat{\mathbf{V}}(t)) = \sigma_T(\mathbf{PV}) = \sigma_T(\mathbf{VP}) = \sigma_T(\mathbf{V}). \qquad \blacksquare$$

Remark 2.2. The case $t = 0$ (and $t = 1$) in the analogue of the previous theorem was considered in [43, Theorem 2.8], where it was shown that

$$\widehat{\mathbf{V}} = \frac{I + P}{2}\mathbf{V}.$$

We now deal with the continuity property of the map $\mathbf{T} \mapsto \widehat{\mathbf{T}(t)}$, where $t \in (0, 1)$. Although the next theorem was originally stated for an operator pair, it also works for any d-tuple.

Theorem 2.13 (Proposition 4.2 [24]). *Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$ be a d-tuple of commuting operators, and let $t \in (0, 1)$. Then, the generalized spherical Aluthge transform map $\mathbf{T} \mapsto \widetilde{\mathbf{T}}(t)$ is $(\|\cdot\|, \|\cdot\|)$-continuous.*

Directly from (1.2) and the previous theorem, we have the following corollary.

Theorem 2.14. *Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$ be a d-tuple of commuting operators, and let $t \in (0, 1)$. Then, the generalized spherical mean transform map $\mathbf{T} \mapsto \widehat{\mathbf{T}}(t)$ is $(\|\cdot\|, \|\cdot\|)$-continuous.*

3. Spherical Heinz transform of 2-variable weighted shifts

In this brief section, our main goal is the derivation of the formula of the spherical Heinz transform of an arbitrary 2-variable weighted shift.

Theorem 3.1. *Let $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ be a 2-variable weighted shift, and let $t \in [0, 1]$. Then, $\widehat{\mathbf{W}_{(\alpha,\beta)}}(t) = (\widehat{\mathbf{W}}_1(t), \widehat{\mathbf{W}}_2(t))$ is given by*

$$\widehat{\mathbf{W}}_1(t)\, e_{\mathbf{k}} = \gamma_{\mathbf{k}} \frac{(\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2)^{\frac{1-t}{2}} (\alpha_{\mathbf{k}+\varepsilon_1}^2 + \beta_{\mathbf{k}+\varepsilon_1}^2)^{\frac{t}{2}} + (\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2)^{\frac{t}{2}} (\alpha_{\mathbf{k}+\varepsilon_1}^2 + \beta_{\mathbf{k}+\varepsilon_1}^2)^{\frac{1-t}{2}}}{2} e_{\mathbf{k}+\varepsilon_1},$$

$$\widehat{\mathbf{W}}_2(t)\, e_{\mathbf{k}} = \delta_{\mathbf{k}} \frac{(\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2)^{\frac{1-t}{2}} (\alpha_{\mathbf{k}+\varepsilon_2}^2 + \beta_{\mathbf{k}+\varepsilon_2}^2)^{\frac{t}{2}} + (\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2)^{\frac{t}{2}} (\alpha_{\mathbf{k}+\varepsilon_2}^2 + \beta_{\mathbf{k}+\varepsilon_2}^2)^{\frac{1-t}{2}}}{2} e_{\mathbf{k}+\varepsilon_2}$$

for all $\mathbf{k} \in \mathbb{Z}_+^2$, where $\varepsilon_1 = (1, 0)$, $\varepsilon_2 = (0, 1)$,

$$\gamma_{\mathbf{k}} = \begin{cases} \frac{\alpha_{\mathbf{k}}}{\sqrt{\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2}} & \text{if } \alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 \neq 0, \\ 0 & \text{if } \alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 0, \end{cases} \tag{3.1}$$

and

$$\delta_{\mathbf{k}} = \begin{cases} \frac{\beta_{\mathbf{k}}}{\sqrt{\alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2}} & \text{if } \alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 \neq 0, \\ 0 & \text{if } \alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 0. \end{cases} \tag{3.2}$$

Proof. Let $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}_+^2$ be arbitrary. First, note that

$$(T_1^*T_1 + T_2^*T_2)e_{(k_1,k_2)} = (\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2)e_{(k_1,k_2)}.$$

and thus,

$$Pe_{(k_1,k_2)} = \sqrt{\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2}\,e_{(k_1,k_2)}.$$

It is now easy to see that

$$V_1 e_{(k_1,k_2)} = \gamma_{(k_1,k_2)} e_{(k_1+1,k_2)} \quad \text{and} \quad V_2 e_{(k_1,k_2)} = \delta_{(k_1,k_2)} e_{(k_1,k_2+1)},$$

where $\gamma_{(k_1,k_2)}$ and $\delta_{(k_1,k_2)}$ are given by (3.1) and (3.2), respectively.

It now follows that

$$\begin{aligned} P^t V_1 P^{1-t} e_{(k_1,k_2)} &= \left(\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2\right)^{\frac{1-t}{2}} P^t V_1 e_{(k_1,k_2)} \\ &= \gamma_{(k_1,k_2)}\left(\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2\right)^{\frac{1-t}{2}} P^t e_{(k_1+1,k_2)} \\ &= \gamma_{(k_1,k_2)}\left(\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2\right)^{\frac{1-t}{2}} \left(\alpha_{(k_1+1,k_2)}^2 + \beta_{(k_1+1,k_2)}^2\right)^{\frac{t}{2}}, \end{aligned}$$

and similarly,

$$P^{1-t} V_1 P^t e_{(k_1,k_2)} = \gamma_{(k_1,k_2)}\left(\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2\right)^{\frac{t}{2}} \left(\alpha_{(k_1+1,k_2)}^2 + \beta_{(k_1+1,k_2)}^2\right)^{\frac{1-t}{2}}.$$

Analogously,

$$\begin{aligned} P^t V_2 P^{1-t} e_{(k_1,k_2)} &= \left(\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2\right)^{\frac{1-t}{2}} P^t V_2 e_{(k_1,k_2)} \\ &= \delta_{(k_1,k_2)}\left(\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2\right)^{\frac{1-t}{2}} \left(\alpha_{(k_1,k_2+1)}^2 + \beta_{(k_1,k_2+1)}^2\right)^{\frac{t}{2}}, \end{aligned}$$

and

$$P^{1-t} V_2 P^t e_{(k_1,k_2)} = \delta_{(k_1,k_2)}\left(\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2\right)^{\frac{t}{2}} \left(\alpha_{(k_1,k_2+1)}^2 + \beta_{(k_1,k_2+1)}^2\right)^{\frac{1-t}{2}}.$$

Using the notation $\mathbf{k} = (k_1, k_2)$, $\varepsilon_1 = (1, 0)$, $\varepsilon_2 = (0, 1)$, we immediately obtain that

$$\begin{aligned} \widehat{\mathbf{W}}_1(t)\, e_{\mathbf{k}} &= \frac{1}{2}(P^t V_1 P^{1-t} + P^{1-t} V_1 P^t)e_{\mathbf{k}} \\ &= \gamma_{\mathbf{k}} \frac{(\alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2)^{\frac{1-t}{2}}(\alpha_{\mathbf{k}+\varepsilon_1}^2+\beta_{\mathbf{k}+\varepsilon_1}^2)^{\frac{t}{2}} + (\alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2)^{\frac{t}{2}}(\alpha_{\mathbf{k}+\varepsilon_1}^2+\beta_{\mathbf{k}+\varepsilon_1}^2)^{\frac{1-t}{2}}}{2} e_{\mathbf{k}+\varepsilon_1}, \end{aligned}$$

and

$$\begin{aligned} \widehat{\mathbf{W}}_2(t)\, e_{\mathbf{k}} &= \frac{1}{2}(P^t V_2 P^{1-t} + P^{1-t} V_2 P^t)e_{\mathbf{k}} \\ &= \delta_{\mathbf{k}} \frac{(\alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2)^{\frac{1-t}{2}}(\alpha_{\mathbf{k}+\varepsilon_2}^2+\beta_{\mathbf{k}+\varepsilon_2}^2)^{\frac{t}{2}} + (\alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2)^{\frac{t}{2}}(\alpha_{\mathbf{k}+\varepsilon_2}^2+\beta_{\mathbf{k}+\varepsilon_2}^2)^{\frac{1-t}{2}}}{2} e_{\mathbf{k}+\varepsilon_2}. \end{aligned}$$

This completes the proof. ∎

Remark 3.1. Let $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ be a 2-variable weighted shift, and let $t \in [0,1]$. From the previous theorem, we have that $\widehat{\mathbf{W}_{(\alpha,\beta)}}(t) = (\widehat{\mathbf{W}}_1(t), \widehat{\mathbf{W}}_2(t))$ is a 2-variable weighted shift as well.

Corollary 3.2. *Let $\mathbf{W}_{(\alpha,\beta)} = (T_1, T_2)$ be a 2-variable weighted shift such that $\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 \neq 0$ for each $\mathbf{k} \in \mathbb{Z}_+^2$, and let $t \in [0,1]$. Then, $\widehat{\mathbf{W}_{(\alpha,\beta)}}(t) = (\widehat{\mathbf{W}}_1(t), \widehat{\mathbf{W}}_2(t))$ is given by*

$$\widehat{\mathbf{W}}_1(t)\, e_{\mathbf{k}} = \frac{\alpha_{\mathbf{k}}}{2}\left(\left(\frac{\alpha_{\mathbf{k}+\varepsilon_1}^2 + \beta_{\mathbf{k}+\varepsilon_1}^2}{\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2}\right)^{\frac{t}{2}} + \left(\frac{\alpha_{\mathbf{k}+\varepsilon_1}^2 + \beta_{\mathbf{k}+\varepsilon_1}^2}{\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2}\right)^{\frac{1-t}{2}}\right) e_{\mathbf{k}+\varepsilon_1},$$
$$\widehat{\mathbf{W}}_2(t)\, e_{\mathbf{k}} = \frac{\beta_{\mathbf{k}}}{2}\left(\left(\frac{\alpha_{\mathbf{k}+\varepsilon_2}^2 + \beta_{\mathbf{k}+\varepsilon_2}^2}{\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2}\right)^{\frac{t}{2}} + \left(\frac{\alpha_{\mathbf{k}+\varepsilon_2}^2 + \beta_{\mathbf{k}+\varepsilon_2}^2}{\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2}\right)^{\frac{1-t}{2}}\right) e_{\mathbf{k}+\varepsilon_2}$$

for all $\mathbf{k} \in \mathbb{Z}_+^2$, where $\varepsilon_1 = (1,0)$, $\varepsilon_2 = (0,1)$.

By taking $t = 0$ ($t = 1$), we obtain the already known formula disclosed in [43, Corollary 3.3].

4. Joint norm and joint numerical radius of the spherical Heinz transform

In this section, we derive several inequalities involving the joint numerical radius and the joint operator norm of the spherical Aluthge and Heinz transforms. In order to prove our results, we need the following two lemmas.

Lemma 4.1 ([1, Lemma 2.1]). *Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$. Then,*

$$\|\mathbf{T}\| = \|\mathbb{T}\|,$$

where $\mathbb{T}$ denotes the following operator matrix on $\mathcal{H}_d = \bigoplus_{i=1}^d \mathcal{H}$:

$$\mathbb{T} = \begin{pmatrix} T_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ T_d & 0 & \cdots & 0 \end{pmatrix}.$$

Lemma 4.2 ([26, Lemma 2.1]). *Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$. Then,*

$$\|\mathbf{T}\| = \left\| \sum_{k=1}^d T_k^* T_k \right\|^{\frac{1}{2}}.$$

Note that the previous formula can be viewed as a natural definition of the norm of operator tuples in the following way: by considering $\mathfrak{B}(\mathcal{H})^d$ as a Hilbert C^*-module over

C^*-algebra $\mathfrak{B}(\mathcal{H})$, we directly have that

$$\|\mathbf{T}\|_{\mathfrak{B}(\mathcal{H})^d} = \|\langle \mathbf{T}, \mathbf{T}\rangle\|^{\frac{1}{2}}_{\mathfrak{B}(\mathcal{H})} = \left\|\sum_{k=1}^{d} T_k^* T_k\right\|^{\frac{1}{2}},$$

where $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$.

For more details on the Hilbert C^*-modules, we refer the reader to [34,36]. We believe that utilizing the Hilbert C^*-module approach may lead to further developments. For example, using the mentioned approach, we are able to give another proof of the following lemma which appears in [26].

Lemma 4.3 ([26, Lemma 2.2]). *Let $A, X_k \in \mathfrak{B}(\mathcal{H})$ for $k = 1, 2, \dots, d$. Then,*

$$\left\|\sum_{k=1}^{d} X_k^* A X_k\right\| \le \left\|\sum_{k=1}^{d} X_k^* X_k\right\| \|A\|.$$

Proof. Let $\mathbf{X} = (X_1, \dots, X_d)$ and $\mathbf{Y} = (AX_1, \dots, AX_n)$. Using the Cauchy–Schwarz inequality for Hilbert C^*-modules and [37, Proposition 2.3], we have

$$\begin{aligned}
\left\|\sum_{k=1}^{d} X_k^* A X_k\right\| &= \|\langle \mathbf{X}, \mathbf{Y}\rangle\|_{\mathfrak{B}(\mathcal{H})} \\
&\le \|\mathbf{X}\|_{\mathfrak{B}(\mathcal{H})^d} \|\mathbf{Y}\|_{\mathfrak{B}(\mathcal{H})^d} \\
&= \left\|\sum_{k=1}^{d} X_k^* X_k\right\|^{\frac{1}{2}} \left\|\sum_{k=1}^{d} X_k^* A^* A X_k\right\|^{\frac{1}{2}} \\
&= \left\|\sum_{k=1}^{d} X_k^* X_k\right\|^{\frac{1}{2}} \left\|\sum_{k=1}^{d} \langle X_k, A^*\rangle\langle A^*, X_k\rangle\right\|^{\frac{1}{2}} \\
&\le \left\|\sum_{k=1}^{d} X_k^* X_k\right\|^{\frac{1}{2}} \left\|\sum_{k=1}^{d} \|A^*\|^2 \langle X_k, X_k\rangle\right\|^{\frac{1}{2}} \\
&= \|A\| \left\|\sum_{k=1}^{d} X_k^* X_k\right\|^{\frac{1}{2}} \left\|\sum_{k=1}^{d} X_k^* X_k\right\|^{\frac{1}{2}} \\
&= \|A\| \left\|\sum_{k=1}^{d} X_k^* X_k\right\|.
\end{aligned}$$

■

In the following theorem, we present a refinement of the inequality obtained in [26, Theorem 1].

Theorem 4.4. *Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$. Then,*

$$\|\widehat{\mathbf{T}}(t)\| \le \|\widehat{\mathbf{T}}\| \le \|\mathbf{T}\| \tag{4.1}$$

for any $t \in [0,1]$. *In particular,*

$$\|\widetilde{\mathbf{T}}\| \leq \|\widehat{\mathbf{T}}\| \leq \|\mathbf{T}\|. \tag{4.2}$$

Proof. Let $t \in [0,1]$ be arbitrary, and let $\mathbf{T} = \mathbf{V}P = (V_1 P, \dots, V_d P)$ be the spherical polar decomposition of $\mathbf{T}$. Let $\mathbb{V}$ and $\mathbb{P}$ be as in Theorem 1.2. Observe that the operator $\mathbb{P}$ is positive. Using Lemma 4.1, we have

$$\begin{aligned} \|\widehat{\mathbf{T}}(t)\| &= \frac{1}{2}\left\| \begin{pmatrix} P^t V_1 P^{1-t} + P^{1-t} V_1 P^t & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ P^t V_d P^{1-t} + P^{1-t} V_d P^t & 0 & \cdots & 0 \end{pmatrix} \right\| \\ &= \frac{1}{2}\|\mathbb{P}^t \mathbb{V} \mathbb{P}^{1-t} + \mathbb{P}^{1-t} \mathbb{V} \mathbb{P}^t\|. \end{aligned}$$

Now, recall the following Heinz inequality [29]: for positive operators $A, B \in \mathfrak{B}(\mathcal{H})$,

$$\|A^{1-t} T B^t + A^t T B^{1-t}\| \leq \|AT + TB\| \tag{4.3}$$

for any $T \in \mathfrak{B}(\mathcal{H})$ and any $0 \leq t \leq 1$. It follows now that

$$\|\widehat{\mathbf{T}}(t)\| \leq \frac{1}{2}\|\mathbb{P}\mathbb{V} + \mathbb{V}\mathbb{P}\| = \|\widehat{\mathbf{T}}\|.$$

Also, by triangle inequality and Lemma 4.1,

$$\begin{aligned} \|\widehat{\mathbf{T}}\| &= \frac{1}{2}\|\mathbb{P}\mathbb{V} + \mathbb{V}\mathbb{P}\| \\ &\leq \frac{1}{2}(\|\mathbb{P}\mathbb{V}\| + \|\mathbb{V}\mathbb{P}\|) \\ &\leq \|\mathbb{P}\|\|\mathbb{V}\| \\ &= \|P\|\|\mathbf{V}\| \\ &= \|\mathbf{T}\|\left\|\sum_{i=1}^d V_i^* V_i\right\|^{\frac{1}{2}} \\ &\leq \|\mathbf{T}\|. \end{aligned}$$

Here, we used the facts that $\sum_{i=1}^d V_i^* V_i$ is the orthogonal projection onto $\overline{\mathcal{R}(P)}$ and $\|\mathbf{T}\| = \|P\|$, by Lemma 4.2. This proves inequality (4.1). Inequality (4.2) follows directly from (4.1) by taking $t = \frac{1}{2}$. ■

Theorem 4.5. *Let* $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$. *Then,*

$$\|\widehat{\mathbf{T}}(t)\|_e \leq \|\widehat{\mathbf{T}}\|_e \leq \|\mathbf{T}\|_e \tag{4.4}$$

for any $t \in [0,1]$. *In particular,*

$$\|\widetilde{\mathbf{T}}\|_e \leq \|\widehat{\mathbf{T}}\|_e \leq \|\mathbf{T}\|_e. \tag{4.5}$$

Proof. Let $t \in [0, 1]$ be arbitrary, and let

$$\mathbf{T} = \mathbf{V}P = (V_1 P, \dots, V_d P)$$

be the spherical polar decomposition of $\mathbf{T}$. First, observe that, by definition,

$$\|\widehat{\mathbf{T}}(t)\|_e = \sup_{(\lambda_1,\dots,\lambda_d)\in\overline{\mathbb{B}}_d} \left\| \frac{P^t U_\lambda P^{1-t} + P^{1-t} U_\lambda P^t}{2} \right\|, \tag{4.6}$$

where $U_\lambda = \sum_{i=1}^d \lambda_i V_i$.

Let $(\lambda_1, \dots, \lambda_n) \in \overline{\mathbb{B}}_d$ be arbitrary. Using inequality (4.3), we have

$$\left\| \frac{P^t U_\lambda P^{1-t} + P^{1-t} U_\lambda P^t}{2} \right\| \le \left\| \frac{U_\lambda P + P U_\lambda}{2} \right\|.$$

By taking supremum in the last inequality over all $(\lambda_1, \dots, \lambda_d) \in \overline{\mathbb{B}}_d$, (4.6) directly implies (4.4) (and (4.5) by taking $t = \frac{1}{2}$). ∎

In order to prove our next theorem, we need the following result which was first proved in [39].

Lemma 4.6 ([9, Lemma 3.2]). *Let $A, X \in \mathfrak{B}(\mathcal{H})$ be such that A is positive. Then,*

$$\omega(A^{\frac{1}{2}} X A^{\frac{1}{2}}) \le \omega\left(\frac{A^\alpha X A^{1-\alpha} + A^{1-\alpha} X A^\alpha}{2} \right)$$

for all $0 \le \alpha \le 1$.

Theorem 4.7. *Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathfrak{B}(\mathcal{H})^d$. Then,*

$$\omega(\widetilde{\mathbf{T}}) \le \omega(\widehat{\mathbf{T}}(t)) \le \frac{\omega(\mathbf{T}) + \|\mathbf{T}\|}{2} \tag{4.7}$$

for any $t \in [0, 1]$.

Proof. Let $t \in [0, 1]$ be arbitrary, and let $\mathbf{T} = \mathbf{V}P = (V_1 P, \dots, V_d P)$ be the spherical polar decomposition of $\mathbf{T}$. Since the joint numerical radius and Euclidean numerical radius coincide for any d-tuple of operators, we have that

$$\omega(\widetilde{\mathbf{T}}(t)) = \sup_{(\lambda_1,\dots,\lambda_d)\in\overline{\mathbb{B}}_d} \omega(P^t U_\lambda P^{1-t}), \tag{4.8}$$

and

$$\omega(\widehat{\mathbf{T}}(t)) = \sup_{(\lambda_1,\dots,\lambda_d)\in\overline{\mathbb{B}}_d} \omega\left(\frac{P^t U_\lambda P^{1-t} + P^{1-t} U_\lambda P^t}{2} \right), \tag{4.9}$$

where $U_\lambda := \sum_{i=1}^d \lambda_i V_i$.

Now, let $(\lambda_1, \ldots, \lambda_n) \in \overline{\mathbb{B}}_d$ be arbitrary. Using Lemma 4.6, it follows that

$$\begin{aligned}
\omega(P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}}) &\le \omega\left(\frac{P^t U_\lambda P^{1-t} + P^{1-t} U_\lambda P^t}{2}\right) \\
&\le \frac{\omega(P^t U_\lambda P^{1-t}) + \omega(P^{1-t} U_\lambda P^t)}{2} \\
&\le \frac{\omega(\widetilde{\mathbf{T}}(t)) + \omega(\widetilde{\mathbf{T}}(1-t))}{2} \\
&\le \frac{\omega(\mathbf{T}) + \|\mathbf{T}\|}{2},
\end{aligned}$$

where the last inequality is obtained using [1, Theorem 2.3]. By taking supremum over all $(\lambda_1, \ldots, \lambda_d) \in \overline{\mathbb{B}}_d$, and using (4.8) and (4.9), we promptly obtain (4.7). ■

By inspecting the proof of the previous theorem, we also get a generalization of [26, Theorem 2.2].

Corollary 4.8. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathfrak{B}(\mathcal{H})^d$. *Then,*

$$\omega(\widetilde{\mathbf{T}}) \le \frac{1}{2}\omega(\widetilde{\mathbf{T}}(t)) + \frac{1}{2}\omega(\widetilde{\mathbf{T}}(1-t))$$

for each $t \in [0, 1]$.

In a case of commuting tuples, Theorem 4.7 takes the following, more compact, form.

Theorem 4.9. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathfrak{B}(\mathcal{H})^d$ *be a commuting* d*-tuple of operators. Then,*

$$\omega(\widetilde{\mathbf{T}}) \le \omega(\widehat{\mathbf{T}}) \le \omega(\mathbf{T}). \tag{4.10}$$

Proof. The first inequality follows from (4.7) by taking $t = 0$. Considering the second inequality, first, observe that

$$\omega(\widehat{\mathbf{T}}) = \frac{1}{2}\omega(\mathbf{T} + \mathbf{T}^D) \le \frac{1}{2}\omega(\mathbf{T}) + \frac{1}{2}\omega(\mathbf{T}^D). \tag{4.11}$$

By careful examination of the proof of [26, Theorem 2.3], we note that $\omega(\mathbf{T}^D) \le \omega(\mathbf{T})$. Therefore, (4.11) now implies that $\omega(\widehat{\mathbf{T}}) \le \omega(\mathbf{T})$, which completes the proof. ■

In a single variable case, we obtain as a corollary the following result which was proved in [9].

Corollary 4.10 ([9, Corollary 3.3]). *Let* $T \in \mathfrak{B}(\mathcal{H})$. *Then,*

$$\omega(\widetilde{T}) \le \omega(\widehat{T}) \le \omega(T).$$

In the case of jointly hyponormal tuples, we can obtain an even stronger inequality than (4.10).

Theorem 4.11. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathfrak{B}(\mathcal{H})^d$ *be a jointly hyponormal* d*-tuple of operators. Then,*

$$\omega(\widetilde{\mathbf{T}}) \le \omega(\widehat{\mathbf{T}}(t)) \le \omega(\mathbf{T})$$

for any $t \in [0, 1]$.

Proof. Using [12, Lemma 3.10], and bearing in mind Lemma 4.2, it follows that $r(\mathbf{T}) = \|\mathbf{T}\|$. Since $r(\mathbf{T}) \le \omega(\mathbf{T}) \le \|\mathbf{T}\|$ always holds for the commuting tuples (see (1.1)), the conclusion directly follows from Theorem 4.7. ■

We finish the section by providing a simpler proof of the inequality (4.12) which first appeared in [1] and which is closely related to the previously established inequalities.

Theorem 4.12 ([1, Theorem 2.6]). *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathfrak{B}(\mathcal{H})^d$. *Then,*

$$\omega(\mathbf{T}) \le \frac{1}{4}\left\| \left(\sum_{i=1}^{d} |T_i|^2\right)^t + \left(\sum_{i=1}^{d} |T_i|^2\right)^{1-t} \right\| + \frac{1}{2}\omega(\widetilde{\mathbf{T}}(t)) \tag{4.12}$$

for each $t \in [0, 1]$.

Proof. Let $t \in [0, 1]$ be arbitrary, and let $\mathbf{T} = \mathbf{V}P = (V_1 P, \ldots, V_d P)$ be the spherical polar decomposition of T. From [1, Theorem 2.2] and Lemma 4.2, we have that

$$\omega(\mathbf{T}) \le \frac{1}{2}\|\mathbf{T}\| + \frac{1}{2}\omega(\widetilde{\mathbf{T}}(t)) = \frac{1}{2}\|P\| + \frac{1}{2}\omega(\widetilde{\mathbf{T}}(t)).$$

Using the fact that for any commuting positive operators $A, B \in \mathfrak{B}(\mathcal{H})$,

$$\sqrt{AB} \le \frac{A + B}{2},$$

we have that

$$P = \sqrt{P^{2t} P^{2(1-t)}} \le \frac{P^{2t} + P^{2(1-t)}}{2} = \frac{1}{2}\left(\left(\sum_{i=1}^{d} |T_i|^2\right)^t + \left(\sum_{i=1}^{d} |T_i|^2\right)^{1-t}\right),$$

and thus,

$$\begin{aligned}\omega(\mathbf{T}) &\le \frac{1}{2}\|P\| + \frac{1}{2}\omega(\widetilde{\mathbf{T}}(t)) \\ &\le \frac{1}{4}\left\| \left(\sum_{i=1}^{d} |T_i|^2\right)^t + \left(\sum_{i=1}^{d} |T_i|^2\right)^{1-t} \right\| + \frac{1}{2}\omega(\widetilde{\mathbf{T}}(t)).\end{aligned}$$ ■

Specially, from the proof of the previous theorem, we observe that the following norm equality holds:

$$\|\mathbf{T}\| = \frac{1}{2}\min_{t\in[0,1]}\left\| \left(\sum_{i=1}^{d} |T_i|^2\right)^t + \left(\sum_{i=1}^{d} |T_i|^2\right)^{1-t} \right\|.$$

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