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Bochner representable operators on variable exponent Lebesgue spaces

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Abstract. Let $L^{p(\cdot)}(\Omega)$ be a variable exponent Lebesgue space and X denote a Banach space. It is shown that a bounded linear operator $T : L^{p(\cdot)}(\Omega) \to X$ is Bochner representable if and only if $|||T|||_{p(\cdot)} < \infty$ (here $|||T|||_{p(\cdot)}$ denotes the norm of T in the Dinculeanu sense). As an application, we study the compactness property of these operators.

1. Introduction and preliminaries

Throughout the paper, assume that (Ω, Σ, μ) is a finite measure space and $(X, \|\cdot\|_X)$ is a real Banach space. By $\mathcal{S}(\Sigma)$, we denote the set of all Σ-simple functions on Ω. The problem of Bochner integral representation for linear operators $T : L^p(\Omega) \to X$ has been studied intensively (see [3, 9, 11, 20]). In particular, according to Diestel and Uhl [8, Theorem 8, page 110], we have the following result.

Theorem 1.1. *Assume that $1 < p < \infty$ and X has the Radon–Nikodym property with respect to the measure μ. Then, for a bounded linear operator $T : L^p(\Omega) \to X$, the following statements are equivalent:*

(i) $|||T|||_p := \sup\left\{\sum_{i=1}^n \|\alpha_i T(\mathbb{1}_{A_i})\|_X : s = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i} \in \mathcal{S}(\Sigma), \|s\|_p \leq 1\right\} < \infty$;

(ii) *T is Bochner representable, that is, there exists a unique function $g \in L^{p^*}(\Omega, X)$ ($\frac{1}{p} + \frac{1}{p^*} = 1$) such that*

$$T(u) = \int_\Omega u(t)g(t)d\mu \quad \text{for } u \in L^p(\Omega).$$

In this case, $|||T|||_p = \|g\|_{L^{p^*}(\Omega,X)}$.

Extensions of this result to the Orlicz spaces $L^\Phi(\Omega)$ were obtained by Uhl [19, Theorem 1], Nowak and Oelke [18, Theorem 3.1]. Compact operators on $L^\Phi(\Omega)$ have been studied by Uhl [19] and Nowak [17].

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The aim of this paper is to extend these results to the setting of operators defined on the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$.

The Lebesgue spaces can be generalized by replacing the constant exponent with a variable exponent function. In recent years, variable exponent Lebesgue spaces have been the object of much study (see [2,5,6,13,14,16]). In this paper, we consider the problem of Bochner integral representation for linear operators on variable exponent Lebesgue spaces. First, we establish terminology and basic facts concerning variable exponent Lebesgue spaces (see [4,7]).

Let $L^0(\Omega)$ be the space of μ-equivalence classes of all measurable real functions defined on Ω. We denote by $\mathcal{P}(\Omega)$ the set of all μ-measurable functions $p : \Omega \to (1,\infty)$, called exponent functions. Given an exponent function $p \in \mathcal{P}(\Omega)$, the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) := \{u \in L^0(\Omega) : \varrho_{p(\cdot)}(u/\lambda) < \infty \text{ for some } \lambda > 0\},$$

where

$$\varrho_{p(\cdot)}(u) := \int_\Omega |u(t)|^{p(t)}\, d\mu.$$

Then, $L^{p(\cdot)}(\Omega)$, equipped with the associated Luxemburg norm $\|\cdot\|_{p(\cdot)}$,

$$\|u\|_{p(\cdot)} := \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \le 1\} \quad \text{for } u \in L^{p(\cdot)}(\Omega),$$

is a Banach function space (with respect to the usual pointwise order). In the case of $p(\cdot) = p$ is constant, we have the Lebesgue space $L^p(\Omega)$.

Let us put $p_- := \operatorname{ess\,inf}\{p(t) : t \in \Omega\}$ and $p_+ := \operatorname{ess\,sup}\{p(t) : t \in \Omega\}$. By $p^*(\cdot)$, we denote the conjugate function of $p(\cdot)$, i.e., $p^*(\cdot)$ is the μ-measurable function such that $\frac{1}{p(t)} + \frac{1}{p^*(t)} = 1$ μ-a.e.

Proposition 1.2 ([4, 7], see also [1]). *If $p \in \mathcal{P}(\Omega)$ and $p_+ < \infty$, then the following statements hold:*

(1.1) *$(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is an order continuous Banach function space;*

(1.2) *the Banach dual $L^{p(\cdot)}(\Omega)^*$ of $L^{p(\cdot)}(\Omega)$ is given by*

$$L^{p(\cdot)}(\Omega)^* = \{\Phi_v : v \in L^{p^*(\cdot)}(\Omega)\},$$

where $\Phi_v(u) := \int_\Omega u(t)v(t)\,d\mu$ for $u \in L^{p(\cdot)}(\Omega)$;

(1.3) *$L^{p^*(\cdot)}(\Omega)$ coincides with the Köthe dual $L^{p(\cdot)}(\Omega)'$ of $L^{p(\cdot)}(\Omega)$, that is,*

$$L^{p^*(\cdot)}(\Omega) = \left\{v \in L^0(\Omega) : \int_\Omega |u(t)v(t)|\,d\mu < \infty \text{ for all } u \in L^{p(\cdot)}(\Omega)\right\},$$

and for $v \in L^{p^*(\cdot)}(\Omega)$, we have

$$\|v\|_{p^*(\cdot)} \le \|v\|'_{p^*(\cdot)} \le K_{p(\cdot)}\|v\|_{p^*(\cdot)},$$

where

$$\|v\|'_{p^*(\cdot)} := \sup\left\{\left|\int_\Omega u(t)v(t)d\mu\right| : u \in L^{p(\cdot)}(\Omega), \|u\|_{p(\cdot)} \le 1\right\}, \quad 1 \le K_{p(\cdot)} \le 2;$$

(1.4) *the set $\mathcal{S}(\Sigma)$ of all real simple functions on Ω is dense in $L^{p(\cdot)}(\Omega)$.*

The classical Hölder's inequality for $L^p(\Omega)$ can be extended to variable exponent spaces $L^{p(\cdot)}(\Omega)$ (see [4, Theorem 2.26 and Remark 2.27]).

Theorem 1.3. *Let $p \in \mathcal{P}(\Omega)$. Then, for $u \in L^{p(\cdot)}(\Omega)$, $v \in L^{p^*(\cdot)}(\Omega)$, we have $uv \in L^1(\Omega)$ and*

$$\int_\Omega |u(t)v(t)|\, d\mu \le K_{p(\cdot)}\|u\|_{p(\cdot)}\|v\|_{p^*(\cdot)},$$

where $1 \le K_{p(\cdot)} \le 2$.

Recall that a function $f : \Omega \to X$ is called *strongly measurable*, if there exists a sequence (h_n) of X-valued Σ-simple functions on Ω such that $\|h_n(t) - f(t)\|_X \to 0$ μ-a.e. (see [8, Definition 1, page 41]). Denote by $L^0(\Omega, X)$ the space of μ-equivalence classes of all X-valued strongly measurable functions defined on Ω. For $p \in \mathcal{P}(\Omega)$, let $L^{p(\cdot)}(\Omega, X)$ denote the *variable exponent Bochner–Lebesgue space*, that is,

$$L^{p(\cdot)}(\Omega, X) := \left\{f \in L^0(\Omega, X) : \|f(\cdot)\|_X \in L^{p(\cdot)}(\Omega)\right\}.$$

Then, $L^{p(\cdot)}(\Omega, X)$, equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\Omega,X)} := \big\|\|f(\cdot)\|_X\big\|_{p(\cdot)} \quad \text{for } f \in L^{p(\cdot)}(\Omega, X),$$

is a Banach space.

We now give the definition of the Bochner representable operator on $L^{p(\cdot)}(\Omega)$.

Definition 1.4. A bounded linear operator $T : L^{p(\cdot)}(\Omega) \to X$ is called *Bochner representable* if there exists a unique function $g \in L^{p^*(\cdot)}(\Omega, X)$ such that

$$T(u) = \int_\Omega u(t)g(t)d\mu \quad \text{for all } u \in L^{p(\cdot)}(\Omega).$$

2. Bochner representable operators on $L^{p(\cdot)}(\Omega)$

Following [8, Definition 7, page 110] and [10, Section 13.3], for a bounded linear operator $T : L^{p(\cdot)}(\Omega) \to X$, define

$$|||T|||_{p(\cdot)} := \sup\left\{\sum_{i=1}^n \|\alpha_i T(\mathbb{1}_{A_i})\|_X : s = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i} \in \mathcal{S}(\Sigma), \|s\|_{p(\cdot)} \le 1\right\}.$$

Let us assume that $p_+ < \infty$. Since the set $\mathcal{S}(\Sigma)$ is dense in $L^{p(\cdot)}(\Omega)$, it follows that $\|T\| \le |||T|||_{p(\cdot)}$ (here $\|\cdot\|$ denotes the uniform operator norm).

Now, we can state our main result.

Theorem 2.1. *Assume that $p \in \mathcal{P}(\Omega)$ with $p_+ < \infty$ and X has the Radon–Nikodym property with respect to μ. For a bounded linear operator $T : L^{p(\cdot)}(\Omega) \to X$ the following statements are equivalent:*

(i) $|||T|||_{p(\cdot)} < \infty$;

(ii) *T is Bochner representable, that is, there exists a unique function $g \in L^{p^*(\cdot)}(\Omega, X)$ such that*

$$T(u) = \int_\Omega u(t)g(t)d\mu \quad \text{for all } u \in L^{p(\cdot)}(\Omega).$$

In this case,

$$\|g\|_{L^{p^*(\cdot)}(\Omega,X)} \le |||T|||_{p(\cdot)} \le K_{p(\cdot)}\|g\|_{L^{p^*(\cdot)}(\Omega,X)}.$$

Proof. (i)⇒(ii) Define a vector measure $m_T : \Sigma \to X$ by

$$m_T(A) := T(\mathbb{1}_A) \quad \text{for } A \in \Sigma.$$

Then, m_T is finitely additive. Moreover, since T is bounded, in view of [4, Corollary 2.23], we get

$$\|m_T(A)\|_X = \|T(\mathbb{1}_A)\|_X \le \|T\|\|\mathbb{1}_A\|_{p(\cdot)} \le \|T\| \max\left\{\mu(A)^{\frac{1}{p_+}}, \mu(A)^{\frac{1}{p_-}}\right\},$$

so m_T is μ-absolutely continuous and countably additive.

Now, set $\alpha := 1/\|\mathbb{1}_\Omega\|_{p(\cdot)}$. According to [4, Corollary 2.23], we obtain that

$$\min\left\{\mu(\Omega)^{\frac{1}{p_+}}, \mu(\Omega)^{\frac{1}{p_-}}\right\} \le \|\mathbb{1}_\Omega\|_{p(\cdot)},$$

and hence, we observe that $0 < \alpha < \infty$. For any finite partition $\{A_i \in \Sigma : 1 \le i \le n\}$ of Ω, we have $\alpha\mathbb{1}_\Omega = \sum_{i=1}^n \alpha\mathbb{1}_{A_i}$. Hence,

$$\alpha\sum_{i=1}^n \|m_T(A_i)\|_X = \sum_{i=1}^n \|\alpha T(\mathbb{1}_{A_i})\|_X \le |||T|||_{p(\cdot)} < \infty,$$

and this means that m_T has bounded variation. Since X has the Radon–Nikodym property with respect to μ, there exists a unique function $g \in L^1(\Omega, X)$ such that for all $A \in \Sigma$,

$$m_T(A) = \int_A g(t)d\mu \quad \text{and} \quad |m_T|(A) = \int_A \|g(t)\|_X\, d\mu.$$

Then, for $s := \sum_{i=1}^n \alpha_i\mathbb{1}_{A_i} \in \mathcal{S}(\Sigma)$, we get

$$T(s) = \sum_{i=1}^n \alpha_i T(\mathbb{1}_{A_i}) = \sum_{i=1}^n \alpha_i m_T(A_i) = \sum_{i=1}^n \alpha_i \int_{A_i} g(t)d\mu = \int_\Omega s(t)g(t)\,d\mu$$

and

$$\sum_{i=1}^n |\alpha_i||m_T|(A_i) = \sum_{i=1}^n |\alpha_i| \int_{A_i} \|g(t)\|_X d\mu = \int_\Omega |s(t)|\|g(t)\|_X\, d\mu.$$

Let $\varepsilon > 0$ be given and assume that $\|s\|_{p(\cdot)} \leq 1$. Then, for $i \in \{1, \dots, n\}$, there exists a finite partition $\{A_{i,j} \in \Sigma : 1 \leq j \leq n_i\}$ of A_i such that

$$|m_T|(A_i) \leq \sum_{j=1}^{n_i} \|m_T(A_{i,j})\|_X + \frac{\varepsilon}{n|\alpha_i|} = \sum_{j=1}^{n_i} \|T(\mathbb{1}_{A_{i,j}})\|_X + \frac{\varepsilon}{n|\alpha_i|}.$$

Since $\sum_{i=1}^{n} \left(\sum_{j=1}^{n_i} \alpha_i \mathbb{1}_{A_{i,j}} \right) = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}$, it follows that

$$\begin{aligned} \int_\Omega |s(t)| \|g(t)\|_X \, d\mu &= \sum_{i=1}^{n} |\alpha_i| |m_T|(A_i) \leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n_i} \|\alpha_i T(\mathbb{1}_{A_{i,j}})\|_X \right) + \varepsilon \\ &\leq |||T|||_{p(\cdot)} + \varepsilon. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{i=1}^{n} \|\alpha_i T(\mathbb{1}_{A_i})\|_X &= \sum_{i=1}^{n} |\alpha_i| \|m_T(A_i)\|_X \leq \sum_{i=1}^{n} |\alpha_i| |m_T|(A_i) \\ &= \int_\Omega |s(t)| \|g(t)\|_X \, d\mu. \end{aligned}$$

Therefore, we conclude that

$$|||T|||_{p(\cdot)} = \sup \left\{ \int_\Omega |s(t)| \|g(t)\|_X \, d\mu : s \in \mathcal{S}(\Sigma), \|s\|_{p(\cdot)} \leq 1 \right\}.$$

Let $u \in L^{p(\cdot)}(\Omega) \subset L^0(\Omega)$. Then, there exists a sequence (s_n) in $\mathcal{S}(\Sigma)$ such that $0 \leq s_n(t) \uparrow |u(t)|$ for $t \in \Omega$. Since the norm $\| \cdot \|_{p(\cdot)}$ is order preserving (see [7, page 77]) and for every $n \in \mathbb{N}$, $s_n(t) \leq |u(t)|$ for $t \in \Omega$, it follows that $\|s_n\|_{p(\cdot)} \leq \|u\|_{p(\cdot)}$ for $n \in \mathbb{N}$. Assume that $u \neq 0$. Then, the norm of $s_n / \|u\|_{p(\cdot)}$ is less than or equal to 1. Using Fatou's lemma (see [15, Chapter 1, Section 6, Theorem 8]), we get

$$\int_\Omega \frac{|u(t)|}{\|u\|_{p(\cdot)}} \|g(t)\|_X \, d\mu \leq \sup_n \int_\Omega \frac{s_n(t)}{\|u\|_{p(\cdot)}} \|g(t)\|_X \, d\mu \leq |||T|||_{p(\cdot)} < \infty.$$

Hence, by (1.3), we get $\|g(\cdot)\|_X \in L^{p^*(\cdot)}(\Omega)$ and $ug \in L^1(\Omega, X)$. Moreover, we derive that

$$\|g\|_{L^{p^*(\cdot)}(\Omega, X)} = \big\| \|g(\cdot)\|_X \big\|_{p^*(\cdot)} \leq |||T|||_{p(\cdot)}. \tag{2.1}$$

Now, define $T_g : L^{p(\cdot)}(\Omega) \to X$ by

$$T_g(u) := \int_\Omega u(t) g(t) d\mu \quad \text{for } u \in L^{p(\cdot)}(\Omega).$$

We will show that

$$T_g(u) = T(u) \quad \text{for all } u \in L^{p(\cdot)}(\Omega).$$

Since $\mathcal{S}(\Sigma)$ is dense in $L^{p(\cdot)}(\Omega)$ (see (1.4)), for $u \in L^{p(\cdot)}(\Omega)$, there exists a sequence (s_n) in $\mathcal{S}(\Sigma)$ such that $\|s_n - u\|_{p(\cdot)} \to 0$. Then, $\|T(s_n) - T(u)\|_X \to 0$. On the other hand, using Hölder's inequality (see Theorem 1.3), we get

$$\begin{aligned}\|T(s_n) - T_g(u)\|_X &= \left\| \int_\Omega s_n(t)g(t)d\mu - \int_\Omega u(t)g(t)\,d\mu \right\|_X \\ &\leq \int_\Omega |s_n(t) - u(t)|\|g(t)\|_X\,d\mu \\ &\leq K_{p(\cdot)}\|s_n - u\|_{p(\cdot)} \big\| \|g(\cdot)\|_X \big\|_{p^*(\cdot)} \to 0.\end{aligned}$$

Then,

$$T(u) = T_g(u) = \int_\Omega u(t)g(t)d\mu,$$

as desired.

(ii)⇒(i) Assume that there exists $g \in L^{p^*(\cdot)}(\Omega, X)$ such that

$$T(u) = \int_\Omega u(t)g(t)d\mu \quad \text{for } u \in L^{p(\cdot)}(\Omega).$$

It follows that

$$\|T(u)\|_X \leq \int_\Omega |u(t)|\|g(t)\|_X\,d\mu \quad \text{for all } u \in L^{p(\cdot)}(\Omega).$$

Then, using Hölder's inequality, for $s = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i} \in \mathcal{S}(\Sigma)$ with $\|s\|_{p(\cdot)} \leq 1$, we get

$$\begin{aligned}\sum_{i=1}^n \|\alpha_i T(\mathbb{1}_{A_i})\|_X &\leq \sum_{i=1}^n |\alpha_i| \int_\Omega \mathbb{1}_{A_i}(t)\|g(t)\|_X\,d\mu = \int_\Omega |s(t)|\|g(t)\|_X\,d\mu \\ &\leq K_{p(\cdot)}\|s\|_{p(\cdot)} \big\| \|g(\cdot)\|_X \big\|_{p^*(\cdot)} \leq K_{p(\cdot)}\|g\|_{L^{p^*(\cdot)}(\Omega,X)}.\end{aligned}$$

Hence,

$$|||T|||_{p(\cdot)} \leq K_{p(\cdot)}\|g\|_{L^{p^*(\cdot)}(\Omega,X)}. \tag{2.2}$$

Since (i)⇔(ii) holds, in view of (2.1) and (2.2), we obtain

$$\|g\|_{L^{p^*(\cdot)}(\Omega,X)} \leq |||T|||_{p(\cdot)} \leq K_{p(\cdot)}\|g\|_{L^{p^*(\cdot)}(\Omega,X)}.$$ ■

Remark 2.2. It should be noted that in the proof of Theorem 2.1, we use deep results concerning the topological and order structure of the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$. In particular, we use the fact that the set of simple functions $\mathcal{S}(\Sigma)$ is dense in $L^{p(\cdot)}(\Omega)$ and the Banach dual $L^{p(\cdot)}(\Omega)^*$ of $L^{p(\cdot)}(\Omega)$ coincides with the Köthe dual $L^{p(\cdot)}(\Omega)'$ when $p_+ < \infty$. Moreover, the Hölder's inequality for the space $L^{p(\cdot)}(\Omega)$ is of importance.

As a consequence of Theorem 2.1, we have the following corollary.

Corollary 2.3. *Suppose that* $1 < p_- \le p_+ < \infty$ *and* X *has the Radon–Nikodym property with respect to* μ*. Then, every linear operator* $T : L^{p(\cdot)}(\Omega) \to X$ *with* $|||T|||_{p(\cdot)} < \infty$ *is compact.*

Proof. In view of Theorem 2.1, there exists a function $g \in L^{p^*(\cdot)}(\Omega, X)$ such that

$$T(u) = \int_\Omega u(t)g(t)d\mu \quad \text{for all } u \in L^{p(\cdot)}(\Omega),$$

where $\frac{1}{p(t)} + \frac{1}{p^*(t)} = 1$ μ-a.e. Since $p_- > 1$, then we have $p^*_+ < \infty$. In view of [12, Section 1.C, Theorem 6], there exists a sequence (h_n) of X-valued Σ-simple functions on Ω such that $\|h_n(t) - g(t)\|_X \to 0$ μ-a.e. and $\|h_n(t)\|_X \le \|g(t)\|_X$ μ-a.e. for all $n \in \mathbb{N}$. Hence, $\|h_n(t) - g(t)\|_X \le 2\|g(t)\|_X$ μ-a.e. for all $n \in \mathbb{N}$. This means that the sequence $(\|(h_n - g)(\cdot)\|_X)$ is order convergent to 0 in $L^{p^*(\cdot)}(\Omega)$. Since $p^*_+ < \infty$, we know that $L^{p^*(\cdot)}(\Omega)$ is an order continuous Banach function space, and it follows that

$$\|h_n - g\|_{L^{p^*(\cdot)}(\Omega,X)} = \big\| \|h_n(\cdot) - g(\cdot)\|_X \big\|_{p^*(\cdot)} \xrightarrow{n} 0.$$

Now, for $n \in \mathbb{N}$, we define the linear operator $T_n : L^{p(\cdot)}(\Omega) \to X$ by

$$T_n(u) := \int_\Omega u(t)h_n(t)d\mu \quad \text{for } u \in L^{p(\cdot)}(\Omega).$$

Then, T_n are bounded operators of finite dimensional range, so T_n are compact. Note that, in view of Hölder's inequality (see Theorem 1.3), we have

$$\begin{aligned}
\|T - T_n\| &= \sup_{\|u\|_{p(\cdot)} \le 1} \left\| \int_\Omega u(t)(g - h_n)(t)\, d\mu \right\|_X \\
&\le \sup_{\|u\|_{p(\cdot)} \le 1} \int_\Omega |u(t)| \|(g - h_n)(t)\|_X \, d\mu \\
&\le K_{p(\cdot)} \big\| \|(g - h_n)(\cdot)\|_X \big\|_{p^*(\cdot)} \\
&= K_{p(\cdot)} \|g - h_n\|_{L^{p^*(\cdot)}(\Omega,X)} \xrightarrow{n} 0.
\end{aligned}$$

Thus, T is a compact operator as the limit of compact operators. ■

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