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Hardy–Littlewood maximal operator in a new vanishing Orlicz–Morrey space

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Abstract. We study mapping properties of maximal operator in a new vanishing subspace of Orlicz–Morrey spaces. We show that the vanishing property defining that subspace is preserved under the action of maximal operator. As an application, we also investigate the behavior of fractional operators.

1. Introduction

Morrey spaces $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ play an important role in the study of local behaviour and regularity properties of solutions to PDE. It is well known that the Morrey spaces are non-separable if $\lambda > 0$. The lack of approximation tools for the entire Morrey space has motivated the introduction of appropriate subspaces, like vanishing spaces. The definition of the vanishing Morrey spaces involves several vanishing conditions. Each condition generates a closed subspace of $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$. We use the notation of [3] and show these conditions as (V_0), (V_∞), and (V^*).

The space $V_0\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$, often called in the literature just by vanishing Morrey space, was already introduced in [5, 19, 20] motivated by regularity results of elliptic equations. The subspaces $V_\infty\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ and $V^{(*)}\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ were recently introduced in [3] to study the delicate problem in the approximation of Morrey functions by nice functions. A discussion on this approximation problem in generalized Morrey spaces can be found in [4]. Boundedness of many classical operators from harmonic analysis such as maximal operators, singular operators, potential operators, and Hardy operators, in these new vanishing Morrey and generalized Morrey spaces is investigated in the papers [1, 2].

A natural step in the theory of functions spaces was to study Orlicz–Morrey spaces

$$\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n),$$

where the "Morrey-type measuring" of regularity of functions is realized with respect to the Orlicz norm over balls instead of the Lebesgue one.

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We refer to [10,12–16,21] for the preservation of the vanishing property (V_0) and to [6, 7] for the preservation of the vanishing property (V_∞) of $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ by some classical operators, respectively.

In this paper, we focus on the condition (V^*). More precisely, the purpose of this paper is to introduce new vanishing Orlicz–Morrey space $V^{(*)}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ and to show that vanishing property (V^*) is preserved under the action of maximal operator.

We use the following notation: $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ and radius $r > 0$. The (Lebesgue) measure of a measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$ and χ_E denotes its characteristic function. We use C as a generic positive constant, i.e., a constant whose value may change with each appearance. The expression $A \lesssim B$ means that $A \leq CB$ for some independent constant $C > 0$, and $A \approx B$ means $A \lesssim B \lesssim A$.

2. Preliminaries

We recall the definition of Young functions.

Definition 2.1. A function $\Phi : [0,\infty] \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r\to+0}\Phi(r) = \Phi(0) = 0$, and $\lim_{r\to\infty}\Phi(r) = \Phi(\infty) = \infty$.

A Young function Φ is said to satisfy the Δ_2-condition, denoted also by $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq C\Phi(r), \quad r > 0$$

for some $C > 0$.

A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2C}\Phi(Cr), \quad r \geq 0$$

for some $C > 1$.

Next, we recall the generalized inverse of Young function Φ. For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\} \quad (\inf \emptyset = \infty).$$

We have

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r \quad \text{for } r \geq 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is the complementary function of Φ defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & r \in [0,\infty), \\ \infty, & r = \infty. \end{cases}$$

Definition 2.2 (Orlicz space). For a Young function Φ, the Orlicz space $L^\Phi(\mathbb{R}^n)$ is defined by

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}.$$

The space $L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n)$ is defined as the set of all measurable functions f such that $f\chi_B \in L^{\Phi}(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

$L^{\Phi}(\mathbb{R}^n)$ is a Banach space under the Luxemburg–Nakano norm

$$\|f\|_{L^{\Phi}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\right\}.$$

For $\Omega \subset \mathbb{R}^n$, let

$$\|f\|_{L^{\Phi}(\Omega)} := \|f\chi_{\Omega}\|_{L^{\Phi}}.$$

A tacit understanding is that f is defined to be zero outside Ω.

The following analog of the Hölder inequality is well known.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, and let f and g be measurable functions on Ω. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid:*

$$\int_{\Omega} |f(x)g(x)|dx \leq 2\|f\|_{L^{\Phi}(\Omega)}\|g\|_{L^{\tilde{\Phi}}(\Omega)}.$$

By elementary calculations, we have the following property.

Lemma 2.4. *Let Φ be a Young function, and let B be a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then,*

$$\|\chi_B\|_{L^{\Phi}} = \frac{1}{\Phi^{-1}(|B|^{-1})}.$$

By Theorem 2.3, Lemma 2.4, and (2.1), we get the following estimate.

Lemma 2.5. *For a Young function Φ and a ball B, the following inequality is valid:*

$$\int_B |f(y)|dy \leq 2|B|\Phi^{-1}(|B|^{-1})\|f\|_{L^{\Phi}(B)}.$$

We have the following scaling law.

Lemma 2.6 ([9, Lemma 10]). *Let $\beta > 0$, Φ a Young function, $\Phi_\beta(t) = \Phi(t^{1/\beta})$, and B a ball. Then,*

$$\big\||f|^{\beta}\big\|_{L^{\Phi_\beta}(B)} = \|f\|^{\beta}_{L^{\Phi}(B)}$$

for all measurable functions f.

Now, we define operators investigated in this paper.

The Hardy–Littlewood maximal operator is one of the most central operators in modern harmonic analysis and theory of partial differential equations. It is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

It is well known that the maximal operator controls various other important operators of harmonic analysis. This is the case of the sharp maximal function

$$M^{\sharp} f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy,$$

where $f_B = \frac{1}{|B|} \int_B f(z) dz$. By straightforward calculations, we have

$$(M^{\sharp} f)(x) \leq 2(Mf)(x), \quad x \in \mathbb{R}^n. \tag{2.2}$$

The multidimensional Hardy operators H and $\mathcal{H}$ are defined by

$$Hf(x) := \frac{1}{|x|^n} \int_{|y|<|x|} f(y) dy \quad \text{and} \quad \mathcal{H} f(x) := \int_{|y|>|x|} \frac{f(y)}{|y|^n} dy.$$

Using that $|x - y| < 2|x|$ in the integral defining H, we get the pointwise estimate

$$H(|f|)(x) \leq 2^n v_n Mf(x), \quad x \in \mathbb{R}^n. \tag{2.3}$$

We will consider more general Hardy-type operator H_α and $\mathcal{H}_\alpha$, $0 \leq \alpha < n$, defined for appropriate functions f by

$$H_\alpha f(x) := |x|^{\alpha-n} \int_{|y|<|x|} f(y) dy \quad \text{and} \quad \mathcal{H}_\alpha f(x) := |x|^\alpha \int_{|y|>|x|} \frac{f(y)}{|y|^n} dy.$$

It can be easily shown that operator H_α is now dominated by the fractional maximal operator M_α and the Riesz potential operator I_α for $x \in \mathbb{R}^n$:

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |f(y)| dy, \quad I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

More precisely, for $0 < \alpha < n$, there holds

$$|H_\alpha f(x)| \leq v_n 2^{n-\alpha} (M_\alpha f)(x) \leq 2^{n-\alpha} I_\alpha(|f|)(x), \quad x \in \mathbb{R}^n. \tag{2.4}$$

The Hardy operator $\mathcal{H}_\alpha$ is also dominated by the Riesz potential operator (cf. [1, Lemma 2.3]) as follows:

$$|\mathcal{H}_\alpha f(x)| \leq 2^{n-\alpha} I_\alpha(|f|)(x), \quad x \in \mathbb{R}^n. \tag{2.5}$$

In [11], the Orlicz–Morrey space $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ was introduced to unify Orlicz spaces and generalized Morrey spaces. The definition of $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is as follows.

Definition 2.7. Let φ be a positive measurable function on $(0,\infty)$ and Φ any Young function. The Orlicz–Morrey space $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is the space of functions $f \in L^{\Phi}_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{\|f\|_{L^\Phi(B(x,r))}}{\varphi(r)} < \infty.$$

For a Young function Φ, we denote by $\mathcal{G}_\Phi$ the set of all almost increasing $\varphi : (0,\infty) \to (0,\infty)$ functions such that $t \in (0,\infty) \mapsto \varphi(t)\Phi^{-1}(t^{-n})$ is almost decreasing.

It will be assumed that the functions φ are of the class $\mathcal{G}_\Phi$ in the sequel. We refer to [9, Section 5] for more information about these spaces.

3. Auxiliary estimates

In the next section where we prove our main estimates, we take into account the next results.

Theorem 3.1 ([18]). *If $\Phi \in \nabla_2$, then the operator M is bounded on $L^{\Phi}(\mathbb{R}^n)$.*

Theorem 3.2 ([11, Theorem 4.6]). *Let Φ be a Young function with $\Phi \in \nabla_2$ and $\varphi \in \mathcal{G}_{\Phi}$. Then, the maximal operator M is bounded on $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.*

The following pointwise estimate also plays an essential role in the proof of our results.

Lemma 3.3 ([8]). *Let Φ be a Young function, $\varphi \in \mathcal{G}_{\Phi}$, and $\beta \in (0,1)$. If the condition*

$$t^{\alpha}\varphi(t) + \int_r^{\infty} t^{\alpha}\varphi(t)\Phi^{-1}(t^{-n})\frac{dt}{t} \lesssim \varphi(t)^{\beta} \tag{3.1}$$

holds, then there exists a positive constant C such that, for all $f \in \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ and for every $x \in \mathbb{R}^n$,

$$I_{\alpha} f(x) \leq C(Mf(x))^{\beta}\|f\|^{1-\beta}_{\mathcal{M}^{\Phi,\varphi}}. \tag{3.2}$$

4. Main result

In this section, we show that the subspace $V^{(*)}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is invariant with respect to maximal and fractional operators.

We consider the following subspace of $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.

Definition 4.1. The vanishing Orlicz–Morrey space $V^{(*)}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{N\to\infty} \mathcal{A}_{N,\Phi}(f) := \lim_{N\to\infty} \sup_{x\in\mathbb{R}^n} \|f\chi_N\|_{L^{\Phi}(B(x,1))} = 0,$$

with the notation $\chi_N := \chi_{\mathbb{R}^n\setminus B(0,N)}$, $N \in \mathbb{N}$.

The idea of the proof of the following theorem comes from [1].

Theorem 4.2. *Let Φ be a Young function with $\Phi \in \nabla_2$ and $\varphi \in \mathcal{G}_{\Phi}$. Let also*

$$\lim_{t\to\infty} \varphi(t)\Phi^{-1}(t^{-n}) = 0. \tag{4.1}$$

Then, M is bounded on $V^{(*)}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.

Proof. Since M is bounded on $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ (cf. Theorem 3.2), we only have to show that it preserves the vanishing property (V^*), that is,

$$\lim_{N\to\infty} \mathcal{A}_{N,\Phi}(f) = 0 \;\Rightarrow\; \lim_{N\to\infty} \mathcal{A}_{N,\Phi}(Mf) = 0.$$

Given $x \in \mathbb{R}^n$ and $N \in \mathbb{N}$, we split f into

$$f = f_1 + f_2 \quad \text{with } f_1 := f\chi_{\Omega_{x,N/2}},\ f_2 = f\chi_{\mathbb{R}^n \setminus \Omega_{x,N/2}},$$

where, for short, we use the notation

$$\Omega_{x,N} := B(x,2) \cap (\mathbb{R}^n \setminus B(0,N)).$$

Since M is sublinear, we have

$$\mathcal{A}_{N,\Phi}(Mf) \leq \mathcal{A}_{N,\Phi}(Mf_1) + \mathcal{A}_{N,\Phi}(Mf_2). \tag{4.2}$$

We show next that both quantities on the right-hand side of (4.2) tend to zero as $N \to \infty$. From the boundedness of M in $L^\Phi(\mathbb{R}^n)$ (cf. Theorem 3.1), we get

$$\|Mf_1 \cdot \chi_N\|_{L^\Phi(B(x,1))} \leq \|Mf_1\|_{L^\Phi(\mathbb{R}^n)} \lesssim \|f_1\|_{L^\Phi(\mathbb{R}^n)} = \|f\|_{L^\Phi(\Omega_{x,N/2})}$$

with the implicit constant independent of x, N, and f. Since $f \in V^{(*)}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$, the right-hand side above tends to zero uniformly on x as $N \to \infty$. (Note that the property (V^*) does not depend on the particular value of the radius taken in the balls centered at x, cf. [3, Lemma 3.4].) Therefore,

$$\lim_{N \to \infty} \mathcal{A}_{N,\Phi}(Mf_1) = 0.$$

Now, we deal with the second term of the sum in (4.2). Let $\varepsilon > 0$ be arbitrary. By condition (4.1), there exists $t_1 > 1$ such that $\varphi(t)\Phi^{-1}(t^{-n}) < \varepsilon$ for all $t \geq t_1$. For such fixed t_1, we have

$$\|Mf_2 \cdot \chi_N\|_{L^\Phi(B(x,1))} \leq I_1(x,N) + I_2(x,N),$$

where

$$I_1(x,N) := \|\chi_N(\cdot) \sup_{0<t<t_1} \frac{1}{|B(\cdot,t)|} \int_{B(\cdot,t)} |f(z)|\chi_{\mathbb{R}^n \setminus \Omega_{x,N/2}}(z)dz\|_{L^\Phi(B(x,1))}$$

and

$$I_2(x,N) := \|\chi_N(\cdot) \sup_{t \geq t_1} \frac{1}{|B(\cdot,t)|} \int_{B(\cdot,t)} |f(z)|\chi_{\mathbb{R}^n \setminus \Omega_{x,N/2}}(z)dz\|_{L^\Phi(B(x,1))}.$$

First, we estimate $I_2(x,N)$. By Lemma 2.5, we have

$$\frac{1}{|B(y,t)|} \int_{B(y,t)} |f(z)|dz \lesssim \Phi^{-1}(t^{-n})\|f\|_{L^\Phi(B(y,t))}.$$

Hence, we have

$$\begin{aligned} I_2(x,N) &\leq \|\sup_{t \geq t_1} \Phi^{-1}(t^{-n})\|f\|_{L^\Phi(B(\cdot,t))}\|_{L^\Phi(B(x,1))} \\ &\leq \|\sup_{t \geq t_1} \Phi^{-1}(t^{-n})\varphi(t)\|f\|_{\mathcal{M}^{\Phi,\varphi}}\|_{L^\Phi(B(x,1))} \lesssim \varepsilon\|f\|_{\mathcal{M}^{\Phi,\varphi}}. \end{aligned}$$

Now, we estimate $I_1(x,N)$. There are two different cases to be analyzed for $z \in B(y,t)$ and $z \notin \Omega_{x,N/2}$. If $z \in B(0,\frac{N}{2})$, then $t > |z-y| \geq |y| - |z| > \frac{N}{2}$. Thus, there is no contribution to the supremum on $t \in (0,t_1)$ for $N \geq 2t_1$. If $z \notin B(x,2)$, then $t > |z-y| \geq |z-x| - |y-x| \geq 1$. Hence, it remains to handle $I_1(x,N)$ when the supremum is taken over all $t \in (1,t_1)$. For such values of t, we have

$$t^{-n}\int_{B(y,t)} |f(z)|\chi_{\mathbb{R}^n\setminus\Omega_{x,N/2}}(z)dz \leq \int_{B(y,t_1)} |f(z)|dz = \int_{B(0,t_1)} |f(y-z)|dz.$$

Using Minkowski's inequality for L^Φ, which is a special case of [17, Theorem 3.1], and making a simple change of variables yield

$$\begin{aligned}
I_1(x,N) &\leq \left\| \chi_N(\cdot)\int_{B(0,t_1)} |f(\cdot - z)|dz \right\|_{L^\Phi(B(x,1))} \\
&\leq \left\| \int_{\mathbb{R}^n} \chi_{B(x,1)\cap(\mathbb{R}^n\setminus B(0,N))}(\cdot)\chi_{B(0,t_1)}(z)|f(\cdot - z)|dz \right\|_{L^\Phi(\mathbb{R}^n)} \\
&\leq \int_{B(0,t_1)} \|\chi_{B(x-z,1)\cap(\mathbb{R}^n\setminus B(-z,N))} f\|_{L^\Phi(\mathbb{R}^n)} dz \\
&\leq \int_{B(0,t_1)} \mathcal{A}_{N-|z|,\Phi}(f)dz.
\end{aligned}$$

There is an $M > 0$ such that for any $N > M$

$$I_1(x,N) \lesssim t_1^n \mathcal{A}_{N-M,\Phi}(f),$$

which implies that

$$I_1(x,N) \to 0 \text{ uniformly on } x, \quad \text{as } N \to \infty$$

since $f \in V^{(*)}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$. ∎

In view of the inequalities (2.2) and (2.3), we get the following corollary.

Corollary 4.3. *Under the same assumptions of Theorem 4.2, the operators $M^\sharp$ and H are bounded in $V^{(*)}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.*

We end this section by showing that the vanishing property (V^*) is also preserved by the operators T_α, where T_α stands for any of the operators I_α, M_α, H_α, and $\mathcal{H}_\alpha$.

Theorem 4.4. *Let Φ be a Young function with $\Phi \in \nabla_2$, and let $\varphi \in \mathcal{G}_\Phi$. Let $\beta \in (0,1)$ and define $\eta(t) \equiv \varphi(t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$. If conditions (3.1) and (4.1) hold, then T_α is bounded from $V^{(*)}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $V^{(*)}\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*

Proof. Since the operators M_α, H_α, and $\mathcal{H}_\alpha$ are pointwise dominated by the Riesz potential operator I_α (cf. (2.4) and (2.5)), it suffices to show the result for the latter operator. From (3.2) and Lemma 2.6, we get

$$\mathcal{A}_{N,\Psi}(I_\alpha f) \lesssim \|f\|^{1-\beta}_{\mathcal{M}^{\Phi,\varphi}} \mathcal{A}^\beta_{N,\Phi}(Mf),$$

with the implicit constant not depending on f and $N \in \mathbb{N}$. If $f \in V^{(*)}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$, then $Mf \in V^{(*)}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ by Theorem 4.2. Consequently, we also have $I_\alpha f \in V^{(*)}\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$ by the previous estimate. ∎

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