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Strongly elliptic equations with periodic coefficients in two-dimensional space

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Abstract. Regularity for strongly elliptic equations with highly oscillatory ε-periodic coefficients in two-dimensional space is addressed. In each ε-cell, the diffusion coefficients of the elliptic equations are $\omega^2 \in (1,\infty)$ in a small disk with radius $\frac{\varepsilon\mu}{4}$ ($< \frac{1}{4}$) and 1 outside the disk of the cell. Two cases are considered. Case one is that ε, μ, ω are independent in the elliptic equations. So, the diffusion coefficients of the elliptic equations are ε-periodic and discontinuous. L^p-gradient estimate uniformly in ε, μ, ω for the elliptic solutions is derived. However, the integrability p (> 2) of the solutions is not a large number. Case two is that ε, μ ($= \omega^{-1}$) are independent in the elliptic equations. The diffusion coefficients of the elliptic equations are ε-periodic, discontinuous, and L^1-bounded. Lipschitz estimate uniformly in ε, μ ($= \omega^{-1}$) for the elliptic solutions is obtained.

1. Introduction

Regularity for strongly elliptic equations with highly oscillatory ε-periodic coefficients in a domain $\Omega \subset \mathbb{R}^2$ is concerned. Let $\varepsilon, \mu \in (0,1)$, $\mathcal{Y} \equiv [\frac{-1}{2}, \frac{1}{2})^2$, $\mathcal{Y}_\mu$ be a disc centered at 0 with radius $\frac{\mu}{4}$, $\mathcal{Y}_f \equiv \mathcal{Y} \setminus \mathcal{Y}_\mu$, $\Omega(\varepsilon) \equiv \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) \geq \varepsilon\}$, $\mathcal{I}_\varepsilon \equiv \{\mathbf{j} \in \mathbb{Z}^2 \mid \varepsilon(\mathcal{Y} + \mathbf{j}) \subset \Omega(\varepsilon)\}$, $\Omega_\mu^\varepsilon \equiv \cup_{\mathbf{j} \in \mathcal{I}_\varepsilon} \varepsilon(\mathcal{Y}_\mu + \mathbf{j})$ be a disconnected subset of Ω, and Ω_f^ε ($\equiv \Omega \setminus \Omega_\mu^\varepsilon$) be a connected sub-region of Ω. Let $\mathbf{K}_{\delta,\mu}^\varepsilon$ for $\delta > 0$, $\varepsilon, \mu \in (0,1)$ represent the diffusion coefficients of the elliptic equations defined as

$$\mathbf{K}_{\delta,\mu}^\varepsilon(x) \equiv \begin{cases} 1 & \text{if } x \in \Omega_f^\varepsilon, \\ \delta & \text{if } x \in \Omega_\mu^\varepsilon. \end{cases}$$

The elliptic equations are

$$\begin{cases} -\nabla \cdot \big(\mathbf{K}_{\omega^2,\mu}^\varepsilon \nabla \Phi + Q\big) = G & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\omega \in (1,\infty)$, $\varepsilon, \mu \in (0,1)$, and Q, G are given functions. Problem (1.1) arises from flows in fractured media, contaminant flow problems, and the stress in composite materials (see [4, 5, 20, 21, 27] and references therein). Clearly, $H_0^1(\Omega)$ solutions of (1.1) exist and

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are bounded uniformly in ε, μ, ω if $Q, G \in L^2(\Omega)$. In this case, a sequence of the solutions of the strongly elliptic equations (1.1) converges weakly to a limit function in $H_0^1(\Omega)$ as ε closes to 0 [19]. If the diffusion coefficients of the strongly elliptic equations (1.1) are L^1-bounded, the sequence of the solutions converges uniformly to the limit function as ε closes to 0 [7,9,10]. Also, the limit function satisfies a constant conduction problem [7,21]. In three-dimensional space, uniform convergence of the solutions may not be true for the strongly elliptic equations (1.1) [10].

Let us recall the regularity for the elliptic equations with ε-periodic coefficients. For uniform elliptic equations (e.g., $\varepsilon \in (0, 1)$, $0 < \alpha_1 \leq \omega \leq \alpha_2$, and $\mu = 1$ in (1.1)) with Hölder ε-periodic coefficients, the Hölder estimate, $W^{1,p}$ estimate, and Lipschitz estimate uniformly in ε for the elliptic solutions are derived [2, 3, 11, 23, 28, 31]. For degenerate elliptic equations (e.g., $\varepsilon, \omega \in (0, 1)$ and $\mu = 1$ in (1.1)) with piecewise Hölder ε-periodic coefficients, the Hölder estimate, $W^{1,p}$ estimate, and Lipschitz estimate uniformly in ε, ω for the elliptic solutions are obtained [32, 33]. Regularity for strongly elliptic equations is different from the above two cases. Lamé system with partially infinite coefficients in a two-dimensional material is considered in [5]. The material contains two disjoint convex inclusions; Lamé constants are large inside the inclusions and bounded outside the inclusions. It shows that Lipschitz estimate for the solutions of the limit Lamé system (obtained as the Lamé constants inside the inclusions tend to ∞) blows up if the distance of the two inclusions goes to 0. The perfect and the insulated conductivity problems with finite multiple inclusions imbedded in a bounded domain are considered in [4]. It shows that the Lipschitz estimate for the solutions of the limit conductivity problems (as the conductivity inside the inclusions approaches ∞ or 0) blows up if any two inclusions close to each other. Regularity for strongly elliptic equations with wire-like stiff inclusions in $\mathbb{R}^3$ is considered in [8]. By separation of variables, the study of the three-dimensional strongly elliptic equations is reduced to the study of the two-dimensional Helmholtz-type strongly elliptic equations in ε-periodic composites Ω. The two-dimensional elliptic equations are similar to (1.1). For the elliptic equations (1.1), $W^{1,p}$ local estimate and $C^{1,\alpha}$ local convergence uniformly in $\varepsilon, \mu\ (= \omega^{-1})$ in the interior of the region Ω_f^ε are obtained. So, the $W^{1,p}$ estimate and $C^{1,\alpha}$ convergence uniformly in $\varepsilon, \mu\ (= \omega^{-1})$ for the solutions of the original three-dimensional elliptic equations are derived in a region away from the wire-like stiff inclusions. Elliptic system for elasticity with periodic grain-like stiff inclusions in $\mathbb{R}^n$ for $n \geq 3$ is considered in [28]. By the periodicity of the diffusion coefficients, local Lipschitz estimate uniformly in periodic size for the solutions of the elliptic system is proved. Recent works [25, 26] investigate the regularity of uniform elliptic equations with multiple oscillating scales in space (similar equations as the elliptic equations in [8] or the Darcy's equations in [20]). Without any separation condition on the oscillating scales, Hölder estimate and large-scale Lipschitz estimate uniformly in oscillating scales of the elliptic solutions are derived by a technique of reperiodization.

This work is interested in estimate uniformly in ε, μ, ω for the solutions of the problem (1.1). Suppose that ε, μ are independent and $\omega\mu^\alpha \leq 1$ for some $\alpha \in (0, 1)$ in problem (1.1) the solutions for (1.1) behave like the solutions for uniform elliptic equations in ε-periodic domains. In this case, the Lipschitz estimate for the solutions of (1.1) can be obtained by following the arguments in [2, 3]. Here, we study more general problems. Indeed, two cases are considered. Case one is that ε, μ, ω are independent in the strongly elliptic equations; case two is that ε, μ are independent and $\mu = \omega^{-1}$. For case one, the diffusion coefficients of the strongly elliptic equations (1.1) are ε-periodic and discontinuous. L^p-gradient estimate (for $p > 2$) uniformly in ε, μ, ω for the solutions of (1.1) is shown by employing the reverse Hölder inequality [18]. However, the integrability p of the solutions is not a large number. For case two, the diffusion coefficients of (1.1) are ε-periodic, discontinuous, and L^1-bounded. Lipschitz estimate uniformly in $\varepsilon, \mu\ (= \omega^{-1})$ for the elliptic solutions is obtained. This is done by applying the Lipschitz estimate for the Green's functions of (1.1). Lipschitz estimate for the Green's functions is proved by a three-step compactness argument in [2, 3].

The rest of this work is organized as follows: notations and main results are stated in Section 2. In Section 3, we study problem (1.1) for case one. (Here, ε, μ, ω are independent.) L^p-gradient estimate uniformly in ε, μ, ω for the solutions of (1.1) is derived (see Theorem 2.1). Next, we study problem (1.1) for case two. (Here, $\varepsilon, \mu\ (= \omega^{-1})$ are independent.) For this case, we resort to Green's functions and the corrector functions of the differential operators in (1.1). Local maximum norm for strongly elliptic equations (1.1) is shown in Section 4. Interior Lipschitz estimate for strongly elliptic equations (1.1) with L^1-bounded coefficients is considered in Section 5. Maximum norm of the corrector functions of the differential operators in (1.1) is derived in Section 6. Boundary Lipschitz estimate for strongly elliptic equations with L^1-bounded coefficients is obtained in Section 7. In Section 8, we prove the Lipschitz estimate uniformly in $\varepsilon, \mu\ (= \omega^{-1})$ for Green's functions as well as for the solutions of (1.1). Then, we get Theorem 2.2, which is the estimate result for case two. The proof of Lipschitz estimate for diffraction equations (i.e., Lemma 3.7) is given in Section 9.

2. Notation and main results

$C^{k,\alpha}$, L^p, $W^{k,p}$, and H^k are the Hölder space, Lebesgue space, Sobolev space, and Hilbert space, respectively, [30]. $C_0^\infty(\mathbb{R}^2)$ contains C^∞ functions with compact support; $H^1_{loc}(\mathbb{R}^2)$ contains local H^1 functions. $H^1_\#(\mathbb{R}^2) \subset H^1_{loc}(\mathbb{R}^2)$ contains periodic functions with period $\mathcal{Y}$; $L^p_\#(\mathbb{R}^2)$ and $W^{1,p}_\#(\mathbb{R}^2)$ are defined similarly. supp(ζ) is the support of ζ; $(\zeta)_B \equiv ⨍_B \zeta\, dx \equiv \frac{1}{|B|}\int_B \zeta\, dx$;

$$[\zeta]_{C^\alpha(B)} \equiv \sup_{x,y\in B} \frac{|\zeta(x) - \zeta(y)|}{|x-y|^\alpha}$$

is the α-th Hölder semi-norm of ζ in B. $B_r(x)$ is a disk centered at x with radius r; $|B|$ is the volume of B; $\overline{B}$ is the closure of a set B. If $r > 0$, then $\Omega/r \equiv \{x \mid rx \in \Omega\}$; $\partial\Omega/r$

is defined similarly. $B \Subset D$ means the compact closure of B is a subset of the interior of the set D; $\mathbf{K}^{\varepsilon,r}_{\omega,\mu}(x) \equiv \mathbf{K}^{\varepsilon}_{\omega,\mu}(rx)$; $\beta^x_r \equiv \text{dist}(x, \partial\Omega/r)$ denotes the distance from x to the boundary $\partial\Omega/r$; $\beta^x \equiv \beta^x_1$. Define $\mathbf{S}^{;r}_R(x) \equiv B_R(x) \cap \Omega/r$, $\mathbf{S}^{\varepsilon,r}_{f,R}(x) \equiv B_R(x) \cap \Omega^\varepsilon_f/r$, and $\mathbf{S}^{\varepsilon,r}_{\mu,R}(x) \equiv B_R(x) \cap \Omega^\varepsilon_\mu/r$. Let us make the following statements:

(A1) Ω is a bounded C^2 domain in $\mathbb{R}^2$;

(A2) $\varepsilon, \mu \in (0,1)$, $\omega\mu = 1$.

Our first result is the L^p-gradient estimate uniformly in ε, μ, ω for the strongly elliptic solutions (1.1). Here, ε, μ, ω are independent.

Theorem 2.1. *Suppose* (A1), $\omega \in (1,\infty)$*, and* $\varepsilon, \mu \in (0,1)$*; there is a number* $p_* > 2$ *with* $\frac{1}{p_*} + \frac{1}{p'_*} = 1$ *such that*

(I) *the solution of* (1.1) *for* $G = 0$ *satisfies*

$$\left\|\mathbf{K}^{\varepsilon}_{\omega^\tau,\mu}\nabla\Phi\right\|_{L^p(\Omega)} \le c\|\mathbf{K}^{\varepsilon}_{\omega^{\tau-2},\mu}Q\|_{L^p(\Omega)},$$

where $\tau \in [0,2]$, $p \in (p'_*, p_*)$*, and* c *is a constant independent of* ε, μ, ω, τ,

(II) *the solution of* (1.1) *for* $Q = 0$ *satisfies*

$$\left\|\mathbf{K}^{\varepsilon}_{\omega^2,\mu}\nabla\Phi\right\|_{L^p(\Omega)} \le c\|G\|_{W^{-1,p}(\Omega)},$$

where $p \in (p'_*, p_*)$ *and* c *is a constant independent of* ε, μ, ω,

(III) *the solution of* (1.1) *for* $Q = 0$ *satisfies*

$$\left\|\mathbf{K}^{\varepsilon}_{\omega^2,\mu}\nabla\Phi\right\|_{L^p(\Omega)} \le c\|G\|_{L^q(\Omega)},$$

where $p \in (2, p_*)$, $q \equiv \frac{2p}{2+p} \in (1,2)$*, and* c *is a constant independent of* ε, μ, ω.

Theorem 2.1 is proved in Section 3. For uniform elliptic equations with ε-periodic coefficients (e.g., $\varepsilon \in (0,1)$, $0 < \alpha_1 \le \omega \le \alpha_2$, and $\mu = 1$ in (1.1)), Lipschitz estimate uniformly in ε for the elliptic solutions is the best possible estimate (see [2, 22, 31]). Our second result is for problem (1.1) with ε-periodic, discontinuous, and L^1-bounded coefficients. We obtain Lipschitz estimate uniformly in ε, μ $(= \omega^{-1})$ for the solutions of strongly elliptic equations (1.1).

Theorem 2.2. *Under* (A1)–(A2) *and* $G \in L^p(\Omega)$ *for* $p > 2$*, the solution of*

$$\begin{cases} -\nabla\cdot(\mathbf{K}^{\varepsilon}_{\omega^2,\mu}\nabla\Phi) = G & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

satisfies

$$\left\|\mathbf{K}^{\varepsilon}_{\omega^2,\mu}\nabla\Phi\right\|_{L^\infty(\Omega)} \le c\|G\|_{L^p(\Omega)},$$

where c *is a constant independent of* ε, μ $(= \omega^{-1})$.

Theorem 2.2 is an extension of (III) of Theorem 2.1 and is proved in Section 8.

3. L^p-gradient estimate for strongly elliptic equations

This section is to derive L^p-gradient estimate uniformly in ε, μ, ω for the solutions of (1.1). It includes four subsections. $W^{k,p}$ estimate for interface problems with soft inclusions is derived in Section 3.1. $W^{k,p}$ estimate for interface problems with stiff inclusions (direct consequence of the results in Section 3.1) is in Section 3.2. Local L^p-gradient estimate for the solutions of strongly elliptic equations is in Section 3.3 by applying estimates in Section 3.2 and the reverse Hölder inequality [18]. Global L^p-gradient estimate for the strongly elliptic equations is derived in Section 3.4 by employing results in Section 3.3, partition of unity, and a duality argument.

3.1. Interface problems with soft inclusions

Let $\delta, h > 0$; $B_h^+(0) \equiv B_h(0) \cap \{(z_1, z_2)|z_2 > 0\}$; $B_h^-(0) \equiv B_h(0) \cap \{(z_1, z_2)|z_2 < 0\}$; $\mathbb{I}_h(0) \equiv B_h(0) \cap \{(z_1, z_2)|z_2 = 0\}$; then,

$$\begin{cases} \mathbf{T}_\delta(x) \equiv \mathcal{X}_{\{(z_1,z_2)|z_2\geq 0\}}(x) + \delta \mathcal{X}_{\{(z_1,z_2)|z_2<0\}}(x), \\ \mathcal{P}_\delta(x) \equiv \mathbb{A}_1 \mathcal{X}_{\{(z_1,z_2)|z_2\geq 0\}}(x) + \delta \mathbb{A}_2 \mathcal{X}_{\{(z_1,z_2)|z_2<0\}}(x). \end{cases} \tag{3.1}$$

Lemma 3.1. *Assume $\mu, h \in (0,1)$ and $\tau \in [0,2]$, and assume $\mathbb{A}_1$, $\mathbb{A}_2$ in (3.1) are constant positive definite matrices. Any solution of*

$$\begin{cases} -\nabla \cdot (\mathcal{P}_{\mu^2} \nabla \phi + \mathbb{Q}) = \mathbb{G} & \text{in } B_h(0), \\ \phi = 0 & \text{on } \partial B_h(0) \end{cases}$$

satisfies

$$\|\mathbf{T}_{\mu^\tau} \nabla \phi\|_{L^2(B_h(0))} \leq c\big(\|\mathbf{T}_{\mu^{\tau-2}} \mathbb{Q}\|_{L^2(B_h(0))} + \|\mathbb{G}\|_{H^{-1}(B_h^+(0))} + \|\mu^{\tau-2}\mathbb{G}\|_{H^{-1}(B_h^-(0))}\big),$$

where c is a constant independent of μ, h, τ. See (3.1) for $\mathbf{T}_\delta$, $\mathcal{P}_\delta$.

Proof. Let c denote a constant independent of μ, h, τ; $\mathbf{D}$ is either $B_h^+(0)$ or $B_h^-(0)$; assume $\mathbb{Q} \in H_0^1(\mathbf{D})$. Find a $\widehat{\phi} \in H_0^1(\mathbf{D})$ satisfying

$$-\nabla \cdot (\mathcal{P}_{\mu^2} \nabla \widehat{\phi} + \mathbb{Q}) = \mathbb{G} \quad \text{in } \mathbf{D}.$$

By energy method,

$$\|\mathbf{T}_{\mu^\tau} \nabla \widehat{\phi}\|_{L^2(\mathbf{D})} \leq c\big(\|\mathbf{T}_{\mu^{\tau-2}} \mathbb{Q}\|_{L^2(\mathbf{D})} + \|\mathbf{T}_{\mu^{\tau-2}} \mathbb{G}\|_{H^{-1}(\mathbf{D})}\big). \tag{3.2}$$

Note $\widehat{\phi} \in H_0^1(B_h(0))$. Suppose $\psi = \phi - \widehat{\phi}$ in $B_h(0)$; then,

$$\begin{cases} -\nabla \cdot (\mathcal{P}_{\mu^2} \nabla \psi) = 0 & \text{in } B_h^+(0) \text{ and } B_h^-(0), \\ \psi = 0 & \text{on } \partial B_h(0), \\ \lfloor \psi \rfloor = 0 & \text{on } \mathbb{I}_h(0), \\ \lfloor \mathcal{P}_{\mu^2} \nabla \psi \cdot \vec{e}_2 \rfloor = -\lfloor \mathcal{P}_{\mu^2} \nabla \widehat{\phi} \cdot \vec{e}_2 \rfloor \equiv \zeta & \text{on } \mathbb{I}_h(0), \end{cases} \tag{3.3}$$

where $\vec{e}_2$ is the unit vector in the second coordinate direction in $\mathbb{R}^2$. Here, $\lfloor \psi(x_1,0) \rfloor \equiv \lim_{t\searrow 0}\psi(x_1,t) - \lim_{t\nearrow 0}\psi(x_1,t)$; $\lfloor \mathcal{P}_{\mu^2}\nabla\psi\cdot\vec{e}_2\rfloor$ is defined similarly. Let $\breve{\psi}$ denote the even extension function of $\psi|_{B_h^+(0)}$ with respect to $x_2=0$ in $B_h(0)$. Test (3.3)$_1$ against $\psi-\breve{\psi}$ to get $\|\nabla\psi\|_{L^2(B_h^-(0))}\le c\|\nabla\psi\|_{L^2(B_h^+(0))}$. So,

$$\|\nabla\psi\|_{L^2(B_h(0))}\le c\|\nabla\psi\|_{L^2(B_h^+(0))}. \tag{3.4}$$

Test (3.3)$_1$ against ψ to get, by trace theorem [19],

$$\|\nabla\psi\|_{L^2(B_h^+(0))}\le c\|\zeta\|_{H^{-1/2}(\mathbb{I}_h(0))}. \tag{3.5}$$

Equations (3.2)–(3.5) imply Lemma 3.1 for $\mathbb{Q}\in H_0^1(B_h^+(0))$ and $\mathbb{Q}\in H_0^1(B_h^-(0))$ case.

For general $\mathbb{Q}\in L^2(B_h(0))$, Lemma 3.1 is proved by a density argument. ■

Lemma 3.2. *Assume $\mu\in(0,1)$, $\tau\in[0,2]$, $0<\mathbf{d}<\mathbb{A}_1,\mathbb{A}_2\in C^2(B_2(0))$ in (3.1), and $q\ge 1$. Any solution of*

$$-\nabla\cdot(\mathcal{P}_{\mu^2}\nabla\psi+\mathbb{Q})=0 \quad \text{in } B_2(0) \tag{3.6}$$

satisfies

$$\|\mathbf{T}_{\mu^\tau}\nabla\psi\|_{L^2(B_{1/2}(0))}\le c\|\mathbf{T}_{\mu^{\tau-2}}\mathbb{Q}\|_{L^2(B_1(0))}+\begin{cases} c\|\mathbf{T}_{\mu^\tau}\psi\|_{L^2(B_1(0))} & \text{or}\\ c\|\mathbf{T}_{\mu^\tau}\nabla\psi\|_{L^q(B_1(0))}, \end{cases} \tag{3.7}$$

where c is a constant independent of μ, τ, q. See (3.1) for $\mathbf{T}_\delta$, $\mathcal{P}_\delta$.

Proof. Let $h\in(0,1)$ (determined later), and let $z\in B_1(0)$. The proof includes three steps.

Step I. If $B_h(z)\cap\mathbb{I}_1(0)=\emptyset$, take $\eta\in C_0^\infty(B_h(z))$ to be a non-negative function with $\eta=1$ in $B_{h/2}(z)$ and $|\nabla\eta|\le\frac{c}{h}$. Test (3.6) against $\psi\eta^2$ to see

$$\|\mathbf{T}_{\mu^\tau}\nabla\psi\|^2_{L^2(B_{h/2}(z))}\le c\big(h^{-2}\|\mathbf{T}_{\mu^\tau}\psi\|^2_{L^2(B_h(z))}+\|\mathbf{T}_{\mu^{\tau-2}}\mathbb{Q}\|^2_{L^2(B_h(z))}\big), \tag{3.8}$$

where c is a constant independent of μ, τ, z, h.

Step II. If $z=(z_1,0)\in\mathbb{I}_1(0)$, define a piecewise constant function as, for any $\delta,h>0$,

$$\mathcal{A}_\delta(x)\equiv\begin{cases}\lim_{t\searrow 0}\mathbb{A}_1(z_1,t) & \text{if } x\in B_h^+(z),\\ \lim_{t\nearrow 0}\delta\mathbb{A}_2(z_1,t) & \text{if } x\in B_h^-(z).\end{cases}$$

Let $\eta\in C_0^\infty(B_h(z))$ be a non-negative function with $\eta=1$ in $B_{h/2}(z)$ and $|\nabla\eta|\le\frac{c}{h}$. Multiply (3.6) by η to get

$$\begin{cases}-\nabla\cdot\big(\mathcal{A}_{\mu^2}\nabla(\psi\eta)+(\mathcal{P}_{\mu^2}-\mathcal{A}_{\mu^2})\nabla(\psi\eta)-\mathcal{P}_{\mu^2}\psi\nabla\eta+\mathbb{Q}\eta\big) & \\ \quad =-(\mathcal{P}_{\mu^2}\nabla\psi+\mathbb{Q})\nabla\eta & \text{in } B_h(z),\\ \psi\eta=0 & \text{on } \partial B_h(z).\end{cases}$$

By Lemma 3.1,

$$\|\mathbf{T}_{\mu^\tau}\nabla(\psi\eta)\|_{L^2(B_h(z))} \le c_0\big(\|\Delta^\dagger\mathcal{P}\|_{L^\infty(B_h(z))}\|\mathbf{T}_{\mu^\tau}\nabla(\psi\eta)\|_{L^2(B_h(z))} + h^{-1}\|\mathbf{T}_{\mu^\tau}\psi\|_{L^2(B_h(z))} + \|\mathbf{T}_{\mu^{\tau-2}}\mathbb{Q}\|_{L^2(B_h(z))}\big), \tag{3.9}$$

where $\Delta^\dagger\mathcal{P} \equiv \mathcal{A}_1 - \mathcal{P}_1$ and c_0 is independent of μ, τ, h, z. Since $\mathbb{A}_1, \mathbb{A}_2 \in C^2(B_2(0))$, there is a $h_\dagger \in (0,1)$ (independent of μ, τ, h, z) such that $c_0\|\Delta^\dagger\mathcal{P}\|_{L^\infty(B_{h_\dagger}(z))} \le \frac{1}{2}$. We choose h above as $h = h_\dagger$. Then, (3.9) implies

$$\|\mathbf{T}_{\mu^\tau}\nabla\psi\|^2_{L^2(B_{h_\dagger/2}(z))} \le c_1\big(h_\dagger^{-2}\|\mathbf{T}_{\mu^\tau}\psi\|^2_{L^2(B_{h_\dagger}(z))} + \|\mathbf{T}_{\mu^{\tau-2}}\mathbb{Q}\|^2_{L^2(B_{h_\dagger}(z))}\big), \tag{3.10}$$

where c_1 is independent of $\mu, \tau, z, h_\dagger$.

Step III. Find some points $\{\mathcal{P}_i\}_{i=1}^k$ so that (i) $B_{h_\dagger}(\mathcal{P}_i) \cap \mathbb{I}_1(0) = \emptyset$ or $\mathcal{P}_i \in \mathbb{I}_1(0)$ and (ii) $B_{1/2}(0) \subset \bigcup_{i=1}^k B_{h_\dagger/2}(\mathcal{P}_i) \subset \bigcup_{i=1}^k B_{2h_\dagger/3}(\mathcal{P}_i) \subset B_1(0)$. Apply (3.8) or (3.10) with $z = \mathcal{P}_i$ for all i and sum these inequalities to obtain $(3.7)_1$.

If ψ is a solution of (3.6), then $\psi - d$ for any $d \in \mathbb{R}$ is also a solution of (3.6). Equation $(3.7)_2$ follows by employing $(3.7)_1$ with $\psi - d$ for a proper $d \in \mathbb{R}$ and Sobolev embedding theorem [19]. ■

Lemma 3.3. *Assume* $\mu \in (0,1)$, $\tau \in [0,2]$, $0 < \mathbf{d} < \mathbb{A}_1, \mathbb{A}_2 \in C^2(B_2(0))$ *in* (3.1), *and* $q \ge 1$. *There is a constant* c *independent of* μ, τ, q *such that any solution of*

$$-\nabla\cdot(\mathcal{P}_{\mu^2}\nabla\psi) = 0 \quad \text{in } B_2(0) \tag{3.11}$$

satisfies

$$\|\nabla\psi\|_{L^\infty(B_1^+(0))} + \|\mathbf{T}_{\mu^\tau}\nabla\psi\|_{L^\infty(B_1^-(0))} \le \begin{cases} c\|\mathbf{T}_{\mu^\tau}\psi\|_{L^2(B_2(0))} & \textit{or} \\ c\|\mathbf{T}_{\mu^\tau}\nabla\psi\|_{L^q(B_2(0))}. \end{cases}$$

Proof. Let c denote a constant independent of μ, τ. We claim, for any $j = 1, 2, 3$,

$$\|\nabla^j\psi\|_{L^2(B_{1/2}^+(0))} + \|\mathbf{T}_{\mu^\tau}\nabla^j\psi\|_{L^2(B_{1/2}^-(0))} \le c\|\mathbf{T}_{\mu^\tau}\psi\|_{L^2(B_1(0))}. \tag{3.12}$$

If so, Lemma 3.3 follows from (3.12) and Sobolev embedding theorem [19].

Equation (3.12) for $j = 1$ is from $(3.7)_1$. Differentiate (3.11) with respect to x_1 and apply $(3.7)_1$ to get

$$\|\mathbf{T}_{\mu^\tau}\nabla\partial_{x_1}\psi\|_{L^2(B_{1/2}(0))} \le c\|\mathbf{T}_{\mu^\tau}\nabla\psi\|_{L^2(B_{2/3}(0))} \le c\|\mathbf{T}_{\mu^\tau}\psi\|_{L^2(B_1(0))}. \tag{3.13}$$

By (3.11) and (3.13),

$$\|\partial^2_{x_2}\psi\|_{L^2(B_{1/2}^+(0))} + \|\mathbf{T}_{\mu^\tau}\partial^2_{x_2}\psi\|_{L^2(B_{1/2}^-(0))} \le c\|\mathbf{T}_{\mu^\tau}\psi\|_{L^2(B_1(0))}.$$

So, (3.12) for $j = 2$ is true. Equation (3.12) for $j = 3$ is proved by an argument similar to that for (3.12) for $j = 2$ case. ■

3.2. Interface problems with stiff inclusions

By Lemmas 3.2–3.3 and change of variables, it is not difficult to see the following.

Corollary 3.4. *Assume* $\omega \in (1, \infty)$, $\tau \in [0, 2]$, $0 < \mathbf{d} < \mathbb{A}_1, \mathbb{A}_2 \in C^2(B_2(0))$ *in* (3.1), *and* $q \geq 1$. *Then,*

(I) *any solution of* $-\nabla \cdot (\mathcal{P}_{\omega^2} \nabla \psi + \mathbb{Q}) = 0$ *in* $B_2(0)$ *satisfies*

$$\|\mathbf{T}_{\omega^\tau} \nabla \psi\|_{L^2(B_1(0))} \leq c\|\mathbf{T}_{\omega^{\tau-2}} \mathbb{Q}\|_{L^2(B_2(0))} + \begin{cases} c\|\mathbf{T}_{\omega^\tau} \psi\|_{L^2(B_2(0))} & or \\ c\|\mathbf{T}_{\omega^\tau} \nabla \psi\|_{L^q(B_2(0))}, \end{cases}$$

where c is a constant independent of ω, τ, q;

(II) *any solution of* $-\nabla \cdot (\mathcal{P}_{\omega^2} \nabla \psi) = 0$ *in* $B_2(0)$ *satisfies*

$$\|\nabla \psi\|_{L^\infty(B_1^+(0))} + \|\mathbf{T}_{\omega^\tau} \nabla \psi\|_{L^\infty(B_1^-(0))} \leq \begin{cases} c\|\mathbf{T}_{\omega^\tau} \psi\|_{L^2(B_2(0))} & or \\ c\|\mathbf{T}_{\omega^\tau} \nabla \psi\|_{L^q(B_2(0))}, \end{cases}$$

where c is a constant independent of ω, τ, q. *See* (3.1) *for* $\mathbf{T}_\delta$, $\mathcal{P}_\delta$.

3.3. Local L^p-gradient estimate

For any set $\mathbf{D}$, $H^1(\mathbf{D})/\mathbb{R} \equiv \{\zeta \in H^1(\mathbf{D}) \mid (\zeta)_{\mathbf{D}} = 0\}$. See Section 2 for $(\zeta)_{\mathbf{D}}$. Now, we give an extension result.

Lemma 3.5. *For any* $\mu \in (0, 1)$, *there is an operator* $\Pi_\mu : H^1(\mathcal{Y}_\mu)/\mathbb{R} \to H_0^1(2\mathcal{Y}_\mu)$ *such that if* $\zeta \in H^1(\mathcal{Y}_\mu)/\mathbb{R}$, *then*

$$\begin{cases} \Pi_\mu \zeta = \zeta & in\ \mathcal{Y}_\mu, \\ \|\Pi_\mu \zeta\|_{H^1(2\mathcal{Y}_\mu)} \leq c\|\nabla \zeta\|_{L^2(\mathcal{Y}_\mu)}, \end{cases}$$

where c is a constant independent of μ.

Proof. Let c denote a constant independent of μ. By Poincaré inequality [19, Theorem 7.25], there is an operator $\Pi : H^1(B_1(0))/\mathbb{R} \to H_0^1(B_2(0))$ such that, for any $\phi \in H^1(B_1(0))/\mathbb{R}$,

$$\begin{cases} \Pi\phi = \phi & \text{in } B_1(0), \\ \|\Pi\phi\|_{H^1(B_2(0))} \leq c\|\nabla \phi\|_{L^2(B_1(0))}. \end{cases}$$

Extend $\Pi\phi \in H_0^1(B_2(0))$ from $B_2(0)$ to $\mathbb{R}^2$ by 0 and regard $H_0^1(B_2(0)) \subset H^1(\mathbb{R}^2)$. Recall $\mathcal{Y}_\mu = B_{\mu/4}(0)$. Define an operator $\Pi_\mu : H^1(\mathcal{Y}_\mu)/\mathbb{R} \to H_0^1(\mathcal{Y})$ as follows: set $\phi(x) \equiv \zeta(\frac{\mu}{4}x)$ for any $\zeta \in H^1(\mathcal{Y}_\mu)/\mathbb{R}$ and $x \in B_1(0)$, and set $\Pi_\mu \zeta(y) \equiv \Pi\phi(\frac{4}{\mu}y)$ for $\Pi\phi \in H_0^1(B_2(0))$ and $y \in 2\mathcal{Y}_\mu$. Then,

$$\begin{cases} \Pi_\mu \zeta = \zeta & \text{in } \mathcal{Y}_\mu, \\ \|\Pi_\mu \zeta\|_{H^1(2\mathcal{Y}_\mu)} \leq c\|\Pi\phi\|_{H^1(B_2(0))} \leq c\|\nabla \phi\|_{L^2(B_1(0))} \leq c\|\nabla \zeta\|_{L^2(\mathcal{Y}_\mu)}. \end{cases}$$

So, we prove the lemma. ∎

If $\delta, \lambda > 0$ and $\mu \in (0,1)$, a function $\mathbb{K}_{\delta,\mu} \in L^\infty_\#(\mathbb{R}^2)$ (resp., $\mathbb{K}^\lambda_{\delta,\mu} \in L^\infty(\mathbb{R}^2)$) is defined as

$$\mathbb{K}_{\delta,\mu}(z) \equiv \begin{cases} 1 & \text{if } z \in \mathcal{Y}_f \\ \delta & \text{if } z \in \mathcal{Y}_\mu \end{cases} \quad \left(\text{resp., } \mathbb{K}^\lambda_{\delta,\mu}(z) \equiv \mathbb{K}_{\delta,\mu}\left(\frac{z}{\lambda}\right)\right). \tag{3.14}$$

Next is a local L^2-gradient estimate for elliptic solutions. The idea is from [28].

Lemma 3.6. *Suppose $\omega \in (1,\infty)$ and $\frac{\varepsilon}{r}, r, \mu \in (0,1)$; any weak solution of*

$$-\nabla \cdot \big(\mathbb{K}^{\varepsilon/r}_{\omega^2,\mu} \nabla \Psi + \mathbb{Q}\big) = 0 \quad \text{in } \frac{\varepsilon}{r}\mathcal{Y} \tag{3.15}$$

satisfies $\omega^2 \|\nabla\Psi\|_{L^2(\frac{\varepsilon}{r}\mathcal{Y}_\mu)} \le c(\|\nabla\Psi\|_{L^2(2\frac{\varepsilon}{r}\mathcal{Y}_\mu \setminus \frac{\varepsilon}{r}\mathcal{Y}_\mu)} + \|\mathbb{Q}\|_{L^2(2\frac{\varepsilon}{r}\mathcal{Y}_\mu)})$, where c is a constant independent of ε, μ, ω, r. See (3.14) for $\mathbb{K}^{\varepsilon/r}_{\omega^2,\mu}$.

Proof. Take a constant $\mathbf{h}$ so that the average $(\Psi - \mathbf{h})_{\frac{\varepsilon}{r}\mathcal{Y}_\mu} = 0$. By Lemma 3.5, there is a $V \in H^1_0(2\frac{\varepsilon}{r}\mathcal{Y}_\mu)$ satisfying

$$\begin{cases} V = \Psi - \mathbf{h} & \text{in } \frac{\varepsilon}{r}\mathcal{Y}_\mu, \\ \|V\|_{H^1(2\frac{\varepsilon}{r}\mathcal{Y}_\mu)} \le c\|\nabla\Psi\|_{L^2(\frac{\varepsilon}{r}\mathcal{Y}_\mu)}, \end{cases} \tag{3.16}$$

where c is a constant independent of ε, μ, ω, r. Test (3.15) against V to get

$$\int_{2\frac{\varepsilon}{r}\mathcal{Y}_\mu} \mathbb{K}^{\varepsilon/r}_{\omega^2,\mu} \nabla\Psi\nabla V \, dx + \int_{2\frac{\varepsilon}{r}\mathcal{Y}_\mu} \mathbb{Q}\nabla V \, dx = 0.$$

By (3.16),

$$\begin{aligned} \omega^2 \|\nabla\Psi\|^2_{L^2(\frac{\varepsilon}{r}\mathcal{Y}_\mu)} &\le \big(\|\nabla\Psi\|_{L^2(2\frac{\varepsilon}{r}\mathcal{Y}_\mu \setminus \frac{\varepsilon}{r}\mathcal{Y}_\mu)} + \|\mathbb{Q}\|_{L^2(2\frac{\varepsilon}{r}\mathcal{Y}_\mu)}\big)\|\nabla V\|_{L^2(2\frac{\varepsilon}{r}\mathcal{Y}_\mu)} \\ &\le c\big(\|\nabla\Psi\|_{L^2(2\frac{\varepsilon}{r}\mathcal{Y}_\mu \setminus \frac{\varepsilon}{r}\mathcal{Y}_\mu)} + \|\mathbb{Q}\|_{L^2(2\frac{\varepsilon}{r}\mathcal{Y}_\mu)}\big)\|\nabla\Psi\|_{L^2(\frac{\varepsilon}{r}\mathcal{Y}_\mu)}. \end{aligned}$$

So, the lemma follows from the above inequality. ∎

Lemma 3.7. *Assume $\omega \in (1,\infty)$, $\mu \in (0,1)$, and $\frac{|\ln\mu|}{\omega^2\mu} \le 1$. Any solution of*

$$-\nabla \cdot (\mathbb{K}_{\omega^2,\mu}\nabla\Psi) = 0 \quad \text{in } \mathcal{Y} \tag{3.17}$$

satisfies

$$\begin{cases} \|\nabla\Psi\|_{L^\infty(B_{1/2}(0))} \le c\|\Psi|_{L^2(\mathcal{Y}\setminus B_{1/4}(0))}, \\ \|\mathbb{K}_{\omega^2,\mu}\nabla\Psi\|_{L^\infty(2\mathcal{Y}_\mu)} \le c\|\Psi\|_{L^2(\mathcal{Y}\setminus B_{1/4}(0))}, \end{cases} \tag{3.18}$$

where c is a constant independent of μ, ω. See (3.14) for $\mathbb{K}^{\varepsilon/r}_{\omega^2,\mu}$.

Proof. Equation $(3.18)_1$ is proved in Section 9. Let c be independent of μ, ω. If $\phi(x) = \Psi(\mu x)$, then

$$-\nabla \cdot \big(\mathbb{K}^{1/\mu}_{\omega^2,\mu} \nabla \phi\big) = 0 \quad \text{in } \mathcal{Y}/\mu.$$

By partition of unity, argument [1, pp. 3964–3965], and (II) of Corollary 3.4 for $\tau = 2 = q$,

$$\big\|\mathbb{K}^{1/\mu}_{\omega^2,\mu} \nabla \phi\big\|_{L^\infty(2\mathcal{Y}_\mu/\mu)} \le c \big\|\mathbb{K}^{1/\mu}_{\omega^2,\mu} \nabla \phi\big\|_{L^2(4\mathcal{Y}_\mu/\mu)}. \tag{3.19}$$

By (3.19), Lemma 3.6, and $(3.18)_1$,

$$\|\mathbb{K}_{\omega^2,\mu} \nabla \Psi\|_{L^\infty(2\mathcal{Y}_\mu)} \le \frac{c}{\mu} \|\mathbb{K}_{\omega^2,\mu} \nabla \Psi\|_{L^2(4\mathcal{Y}_\mu)} \le \frac{c}{\mu} \|\nabla \Psi\|_{L^2(4\mathcal{Y}_\mu \setminus \mathcal{Y}_\mu)} \le c \|\Psi\|_{L^2(\mathcal{Y} \setminus B_{1/4}(0))}.$$

So, $(3.18)_2$ is proved. ■

Lemma 3.8. *Assume* (A1), $\omega \in (1,\infty)$, $\frac{\varepsilon}{r}, r, \mu \in (0,1)$, $\tau \in [0,2]$, $\mathcal{P} \in \Omega/r$, *and* $q \ge 1$.

(I) *If* $R \in (0, \frac{\varepsilon\mu}{r}]$, *any solution of*

$$\begin{cases} -\nabla \cdot \big(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu} \nabla \Phi + Q\big) = 0 & \text{in } \mathbf{S}^r_R(\mathcal{P}) \equiv B_R(\mathcal{P}) \cap \Omega/r, \\ \Phi = 0 & \text{on } B_R(\mathcal{P}) \cap \partial\Omega/r \end{cases} \tag{3.20}$$

satisfies

$$\begin{aligned} &\big\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu} \nabla \Phi\big\|_{L^2(\mathbf{S}^r_{R/2}(\mathcal{P}))} \\ &\quad \le c \big\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu} Q\big\|_{L^2(\mathbf{S}^r_R(\mathcal{P}))} + \begin{cases} \frac{c}{R} \|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu} \Phi\|_{L^2(\mathbf{S}^r_R(\mathcal{P}))} & \text{or} \\ cR\big|\big(|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu} \nabla \Phi|^q\big)_{\mathbf{S}^r_R(\mathcal{P})}\big|^{\frac{1}{q}}, \end{cases} \end{aligned}$$

where c is a constant independent of ε, μ, ω, r, τ, q, R, $\mathcal{P}$. See Section 2 for $(|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu} \nabla \Phi|^q)_{\mathbf{S}^r_R(\mathcal{P})}$.

(II) *If* $R \in (\frac{\varepsilon\mu}{r}, 32\frac{\varepsilon\mu}{r}]$, *any solution of* (3.20) *satisfies*

$$\big\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu} \nabla \Phi\big\|_{L^2(\mathbf{S}^r_{R/2}(\mathcal{P}))} \le cR\big|(|\nabla \Phi|^q)_{\mathbf{S}^r_R(\mathcal{P})}\big|^{1/q} + c\big\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu} Q\big\|_{L^2(\mathbf{S}^r_R(\mathcal{P}))},$$

where c is independent of ε, μ, ω, r, τ, q, R, $\mathcal{P}$. See Section 2 for $\mathbf{K}^{\varepsilon,r}_{\omega,\mu}$ *and* $(|\nabla \Phi|^q)_{\mathbf{S}^r_R(\mathcal{P})}$.

Proof.

For case (I). If $B_R(\mathcal{P}) \subset \Omega^\varepsilon_f/r$ or $B_R(\mathcal{P}) \subset \Omega^\varepsilon_\mu/r$ or $\mathcal{P} \in \partial\Omega/r$, then (I) of Lemma 3.8 is true by energy method. If $\mathcal{P} \in \frac{\varepsilon}{r}\partial(\mathcal{Y}_\mu + \mathbf{j})$ for $\mathbf{j} \in \mathcal{I}_{\varepsilon/r}$, we let $\mathcal{P} = 0$ by translation. Define $\psi(x) \equiv \Phi(Rx)$, $\mathbb{Q}(x) = RQ(Rx)$. Then, ψ, $\mathbb{Q}$ satisfy

$$-\nabla \cdot \big(\mathbf{K}^{\varepsilon,rR}_{\omega^2,\mu} \nabla \psi + \mathbb{Q}\big) = 0 \quad \text{in } B_1(0). \tag{3.21}$$

Any solution of (3.21) satisfies, by following the arguments [1, pp. 3964–3965] and (I) of Corollary 3.4,

$$\left\|\mathbf{K}^{\varepsilon,rR}_{\omega^\tau,\mu}\nabla\psi\right\|_{L^2(B_{\frac{1}{2}}(0))}\le c\left\|\mathbf{K}^{\varepsilon,rR}_{\omega^{\tau-2},\mu}\mathbb{Q}\right\|_{L^2(B_1(0))}+\begin{cases} c\left\|\mathbf{K}^{\varepsilon,rR}_{\omega^\tau,\mu}\psi\right\|_{L^2(B_1(0))} & \text{or}\\ c\left|\left(|\mathbf{K}^{\varepsilon,rR}_{\omega^\tau,\mu}\nabla\psi|^q\right)_{B_1(0)}\right|^{\frac{1}{q}}, \end{cases} \tag{3.22}$$

where $q\ge 1$ and c is independent of ε, μ, ω, r, τ, q, R. (I) of Lemma 3.8 is from (3.22) and change of variables.

For case (II). By translation, set $\mathcal{P}=0$. Define $h\equiv\frac{\varepsilon\mu}{r}$, $\psi(x)\equiv\Phi(hx)$, and $\mathbb{Q}(x)\equiv hQ(hx)$. Then, ψ, $\mathbb{Q}$ satisfy

$$\begin{cases} -\nabla\cdot\left(\mathbf{K}^{\varepsilon,rh}_{\omega^2,\mu}\nabla\psi+\mathbb{Q}\right)=0 & \text{in } \mathbf{S}^{,rh}_{R/h}(0)\equiv B_{R/h}(0)\cap\Omega/rh,\\ \psi=0 & \text{on } B_{R/h}(0)\cap\partial\Omega/rh. \end{cases} \tag{3.23}$$

Next, we cover the ball $B_{R/2h}(0)$ by some small regions. That is, find finite points $\{\mathcal{P}_i\}_{i=1}^k$ such that

(i) $B_{\frac{1}{2}}(\mathcal{P}_i)\cap\partial\frac{1}{\mu}(\mathcal{Y}_\mu+\mathbf{j})=\emptyset$ or $\mathcal{P}_i\in\partial\frac{1}{\mu}(\mathcal{Y}_\mu+\mathbf{j})$ for $\mathbf{j}\in\mathcal{I}_{\varepsilon/r}$ and

(ii) $B_{\frac{R}{2h}}(0)\subset\bigcup_{i=1}^k B_{\frac{1}{2}}(\mathcal{P}_i)\subset\bigcup_{i=1}^k B_{2/3}(\mathcal{P}_i)\subset B_{\frac{R}{h}}(0)$.

For each $\mathcal{P}_i$, we consider equation $(3.23)_1$ in $\mathbf{S}^{,rh}_{2/3}(\mathcal{P}_i)=B_{2/3}(\mathcal{P}_i)\cap\Omega/rh$. By following the argument [1, pp. 3964–3965] and (I) of Corollary 3.4 with $\tau=0$, any solution of $(3.23)_1$ satisfies

$$\|\nabla\psi\|_{L^2(\mathbf{S}^{,rh}_{1/2}(\mathcal{P}_i))}\le c\left(\|\psi\|_{L^2(\mathbf{S}^{,rh}_{2/3}(\mathcal{P}_i))}+\left\|\mathbf{K}^{\varepsilon,rh}_{\omega^{-2},\mu}\mathbb{Q}\right\|_{L^2(\mathbf{S}^{,rh}_{2/3}(\mathcal{P}_i))}\right). \tag{3.24}$$

Square both sides of (3.24) and sum i from 1 to k to get

$$\|\nabla\psi\|^2_{L^2(\mathbf{S}^{,rh}_{R/2h}(0))}\le c\left(\|\psi\|^2_{L^2(\mathbf{S}^{,rh}_{R/h}(0))}+\left\|\mathbf{K}^{\varepsilon,rh}_{\omega^{-2},\mu}\mathbb{Q}\right\|^2_{L^2(\mathbf{S}^{,rh}_{R/h}(0))}\right).$$

After change of variables, by $R\in(\frac{\varepsilon\mu}{r},32\frac{\varepsilon\mu}{r}]$,

$$\|\nabla\Phi\|^2_{L^2(\mathbf{S}^{,r}_{R/2}(0))}\le c\left(R^{-2}\|\Phi\|^2_{L^2(\mathbf{S}^{,r}_R(0))}+\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{-2},\mu}Q\right\|^2_{L^2(\mathbf{S}^{,r}_R(0))}\right). \tag{3.25}$$

By Lemma 3.6 and (3.25),

$$\begin{aligned}\left\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\right\|^2_{L^2(\mathbf{S}^{,r}_{R/4}(0))}&\le c\left(\|\nabla\Phi\|^2_{L^2(\mathbf{S}^{,r}_{R/2}(0))}+\|\omega^{\tau-2}Q\|^2_{L^2(\mathbf{S}^{,r}_{R/2}(0))}\right)\\&\le c\left(R^{-2}\|\Phi\|^2_{L^2(\mathbf{S}^{,r}_R(0))}+\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu}Q\right\|^2_{L^2(\mathbf{S}^{,r}_R(0))}\right)\\&\le c\left(R^2\left|(|\nabla\Phi|^q)_{\mathbf{S}^{,r}_R(0)}\right|^{\frac{2}{q}}+\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu}Q\right\|^2_{L^2(\mathbf{S}^{,r}_R(0))}\right),\end{aligned}$$

where $q\ge 1$. (II) of Lemma 3.8 follows from the above inequality. ∎

Lemma 3.9. *Assume* (A1), $\omega \in (1,\infty)$, $\frac{\varepsilon}{r}, r, \mu \in (0,1)$, $\tau \in [0,2]$, $\mathcal{P} \in \Omega/r$, *and assume that* Φ *satisfies* (3.20) *for* $R > 16\frac{\varepsilon\mu}{r}$. *If* $\frac{R}{2} \le \ell \le R - 8\frac{\varepsilon\mu}{r}$ *and* $\theta \in (0,1)$, *then*

$$\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\|^2_{L^2(\mathbf{S}^{,r}_\ell(\mathcal{P}))} \le \frac{c}{\theta(R-\ell)^2}\|\Phi\|^2_{L^2(\mathbf{S}^{,r}_R(\mathcal{P}))} + \left(\frac{c}{\omega^2}+\theta\right)\|\nabla\Phi\|^2_{L^2(\mathbf{S}^{,r}_R(\mathcal{P}))} + c\,\theta^{-1}\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu}Q\|^2_{L^2(\mathbf{S}^{,r}_R(\mathcal{P}))},$$

where c *is independent of* ε, μ, ω, r, τ, θ, ℓ, R, $\mathcal{P}$. *See Section* 2 *for* $\mathbf{K}^{\varepsilon,r}_{\omega,\mu}$.

Proof. Let c be independent of ε, μ, ω, r, τ, θ, R, ℓ, $\mathcal{P}$; recall $\mathbf{S}^{\varepsilon,r}_{f,R}(\mathcal{P})$, $\mathbf{S}^{\varepsilon,r}_{\mu,R}(\mathcal{P})$, $(\zeta)_B$ in Section 2. By Lemma 3.6 and summing up all $\frac{\varepsilon}{r}(\mathcal{Y}+\mathbf{j})$ with $\frac{\varepsilon}{r}(\mathcal{Y}+\mathbf{j}) \cap \mathbf{S}^{,r}_\ell(\mathcal{P}) \ne \emptyset$,

$$\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\|^2_{L^2(\mathbf{S}^{,r}_\ell(\mathcal{P}))} \le c\big(\|\nabla\Phi\|^2_{L^2(\mathbf{S}^{\varepsilon,r}_{f,\ell+\frac{\varepsilon\mu}{r}}(\mathcal{P}))} + \|\omega^{\tau-2}Q\|^2_{L^2(\mathbf{S}^{,r}_{\ell+\frac{\varepsilon\mu}{r}}(\mathcal{P}))}\big). \tag{3.26}$$

Let η be a function in $C^1_0(B_{R-3\frac{\varepsilon\mu}{r}}(\mathcal{P}))$ such that $\eta = 1$ in $B_{\ell+3\frac{\varepsilon\mu}{r}}(\mathcal{P})$ and

$$|\nabla\eta| \le c\left(R-\ell-6\frac{\varepsilon\mu}{r}\right)^{-1} \le c(R-\ell)^{-1},$$

where we used $R-\ell \ge 8\frac{\varepsilon\mu}{r}$. For each $\frac{\varepsilon}{r}(\mathcal{Y}_\mu+\mathbf{j})$ with $\mathbf{j} \in \mathcal{I}_{\varepsilon/r}$, we find $V_\mathbf{j} \in H^1_0(\frac{\varepsilon}{r}(\mathcal{Y}+\mathbf{j}))$ and $\mathbf{h_j} \in \mathbb{R}$ satisfying, by Lemma 3.5,

$$\begin{cases} V_\mathbf{j} = \Phi\eta^2 - \mathbf{h_j} & \text{in } \frac{\varepsilon}{r}(\mathcal{Y}_\mu+\mathbf{j}),\\ \|V_\mathbf{j}\|_{H^1(\frac{\varepsilon}{r}(\mathcal{Y}+\mathbf{j}))} \le c\|\nabla(\Phi\eta^2)\|_{L^2(\frac{\varepsilon}{r}(\mathcal{Y}_\mu+\mathbf{j}))}. \end{cases} \tag{3.27}$$

Note

$$\mathbf{h_j} = \begin{cases} 0 & \text{if } \overline{\frac{\varepsilon}{r}(\mathcal{Y}_\mu+\mathbf{j})} \setminus B_R(\mathcal{P}) \ne \emptyset\\ (\Phi\eta^2)_{\frac{\varepsilon}{r}(\mathcal{Y}_\mu+\mathbf{j})} & \text{otherwise.} \end{cases}$$

Extend $V_\mathbf{j}$ from $\frac{\varepsilon}{r}(\mathcal{Y}+\mathbf{j})$ to $\mathbb{R}^2$ by 0, and let $\zeta = \Phi\eta^2 - \sum_{\mathbf{j}\in\mathcal{I}_{\varepsilon/r}} V_\mathbf{j}$ in Ω/r. Note $\zeta \in H^1_0(B_R(\mathcal{P}))$ and $\nabla\zeta = 0$ in $\mathbf{S}^{\varepsilon,r}_{\mu,R}(\mathcal{P})$. Test (3.20) against ζ to get

$$\int_{\mathbf{S}^{\varepsilon,r}_{f,R}(\mathcal{P})} \nabla\Phi\nabla\zeta\,dx + \int_{\mathbf{S}^{\varepsilon,r}_{f,R}(\mathcal{P})} Q\nabla\zeta\,dx = 0. \tag{3.28}$$

By (3.28), (3.27), and Lemma 3.6,

$$\begin{aligned} &\left|\int_{\mathbf{S}^{\varepsilon,r}_{f,R}(\mathcal{P})} \nabla\Phi\nabla(\Phi\eta^2)dx\right| \\ &\le \sum_{\mathbf{j}\in\mathcal{I}_{\varepsilon/r}} \left|\int_{\frac{\varepsilon}{r}(\mathcal{Y}_f+\mathbf{j})} \nabla\Phi\nabla V_\mathbf{j}\,dx\right| + \|Q\nabla\zeta\|_{L^1(\mathbf{S}^{\varepsilon,r}_{f,R}(\mathcal{P}))} \\ &\le c\sum_{\mathbf{j}\in\mathcal{I}_{\varepsilon/r}} \|\nabla\Phi\|_{L^2(\frac{\varepsilon}{r}(\mathcal{Y}_f+\mathbf{j}))}\|\nabla(\Phi\eta^2)\|_{L^2(\frac{\varepsilon}{r}(\mathcal{Y}_\mu+\mathbf{j}))} + \|Q\nabla\zeta\|_{L^1(\mathbf{S}^{\varepsilon,r}_{f,R}(\mathcal{P}))} \end{aligned}$$

$$\leq c \sum_{\mathbf{j}\in\mathcal{I}_{\varepsilon/r}} \|\nabla\Phi\|_{L^2(\frac{\varepsilon}{r}(\mathcal{Y}_f+\mathbf{j}))}\big(\|\Phi\nabla\eta\|_{L^2(\frac{\varepsilon}{r}(\mathcal{Y}_\mu+\mathbf{j}))} + \omega^{-2}\|\nabla\Phi\|_{L^2(\frac{\varepsilon}{r}(\mathcal{Y}_f+\mathbf{j}))}$$
$$+ \omega^{-2}\|Q\|_{L^2(\frac{\varepsilon}{r}(\mathcal{Y}+\mathbf{j}))}\big) + \|Q\|_{L^2(\mathbf{S}^{\varepsilon,r}_{f,R}(\mathcal{P}))}\|\nabla\zeta\|_{L^2(\mathbf{S}^{\varepsilon,r}_{f,R}(\mathcal{P}))}.$$

For any $\theta \in (0,1)$,

$$\left|\int_{\mathbf{S}^{\varepsilon,r}_{f,R}(\mathcal{P})} \nabla\Phi\nabla(\Phi\eta^2)dx\right| \leq \frac{c}{\theta}\int_{\mathbf{S}^r_R(\mathcal{P})} |\Phi\nabla\eta|^2 dx + \left(\frac{c}{\omega^2}+\theta\right)\int_{\mathbf{S}^r_R(\mathcal{P})} |\nabla\Phi|^2 dx$$
$$+ c\,\theta^{-1}\left\|\mathbf{K}^{\varepsilon,r}_{1/\omega^2,\mu} Q\right\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))}. \tag{3.29}$$

The lemma follows from (3.26) and (3.29). ∎

Lemma 3.10. *Assume* (A1), $\omega \in (1,\infty)$, $\frac{\varepsilon}{r}, r, \mu \in (0,1)$, $\tau \in [0,2]$, $\mathcal{P} \in \Omega/r$, $q \geq 1$, *and assume that* Φ *is a weak solution of* (3.20) *for* $R \geq 32\frac{\varepsilon\mu}{r}$. *Then,*

$$\left\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\right\|_{L^2(\mathbf{S}^r_{R/2}(\mathcal{P}))} \leq cR|(|\nabla\Phi|^q)_{\mathbf{S}^r_R(\mathcal{P})}|^{\frac{1}{q}} + \frac{1}{2}\|\nabla\Phi\|_{L^2(\mathbf{S}^r_R(\mathcal{P}))}$$
$$+ c\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu} Q\right\|_{L^2(\mathbf{S}^r_R(\mathcal{P}))}, \tag{3.30}$$

where c *is independent of* ε, μ, ω, r, τ, q, R, $\mathcal{P}$.

Proof. Let k be the integer such that $\frac{R}{2^{k+2}} < 8\frac{\varepsilon\mu}{r} \leq \frac{R}{2^{k+1}}$. Let

$$\rho_i = R\left(1-\frac{1}{2^i}\right)$$

for $i \in \mathbb{N}$. Then, $\rho_{i+1} \geq 16\frac{\varepsilon\mu}{r}$ and $\frac{\rho_{i+1}}{2} \leq \rho_i \leq \rho_{i+1} - 8\frac{\varepsilon\mu}{r}$ for $1 \leq i \leq k$. By Lemma 3.9 with $R = \rho_{i+1}$ and $\ell = \rho_i$, for $\theta \in (0,1)$,

$$\left\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\right\|^2_{L^2(\mathbf{S}^r_{\rho_i}(\mathcal{P}))}$$
$$\leq \frac{c_0}{\theta(\rho_{i+1}-\rho_i)^2}\|\Phi\|^2_{L^2(\mathbf{S}^r_{\rho_{i+1}}(\mathcal{P}))}$$
$$+ \left(\frac{c_0}{\omega^2}+\theta\right)\|\nabla\Phi\|^2_{L^2(\mathbf{S}^r_{\rho_{i+1}}(\mathcal{P}))} + c_0\theta^{-1}\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu} Q\right\|^2_{L^2(\mathbf{S}^r_{\rho_{i+1}}(\mathcal{P}))},$$

where c_0 is independent of ε, μ, ω, r, τ, θ, ρ_i, ρ_{i+1}, $\mathcal{P}$. Thus, by an iteration argument,

$$\left\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\right\|^2_{L^2(\mathbf{S}^r_{\rho_1}(\mathcal{P}))}$$
$$\leq \frac{c_0}{\theta}\sum_{i=1}^{k} \frac{(\frac{c_0}{\omega^2}+\theta)^{i-1}}{(\rho_{i+1}-\rho_i)^2}\|\Phi\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))}$$
$$+ \left(\frac{c_0}{\omega^2}+\theta\right)^k\|\nabla\Phi\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))} + \frac{c_0}{\theta}\sum_{i=1}^{k}\left(\frac{c_0}{\omega^2}+\theta\right)^{i-1}\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu} Q\right\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))}.$$

Since $\rho_{i+1} - \rho_i = \frac{R}{2^{i+1}}$, we obtain

$$\begin{aligned}&\left\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\right\|^2_{L^2(\mathbf{S}^r_{R/2}(\mathcal{P}))}\\&\quad\le 4c_0\frac{\sum_{i=1}^{k}(4\frac{c_0}{\omega^2}+4\theta)^i}{\theta(\frac{c_0}{\omega^2}+\theta)R^2}\|\Phi\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))}\\&\qquad+\left(\frac{c_0}{\omega^2}+\theta\right)^k\|\nabla\Phi\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))}+\frac{c_0}{\theta}\sum_{i=1}^{k}\left(\frac{c_0}{\omega^2}+\theta\right)^{i-1}\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu}Q\right\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))}.\end{aligned}$$

Suppose $\frac{4c_0}{\omega^2} \le \frac{1}{2^2}$, then take $\theta = \frac{1}{2^4}$ and

$$\begin{aligned}&\left\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\right\|^2_{L^2(\mathbf{S}^r_{R/2}(\mathcal{P}))}\\&\quad\le\frac{c}{R^2}\|\Phi\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))}+\frac{1}{2^2}\|\nabla\Phi\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))}+c\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu}Q\right\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))}\\&\quad\le cR^2|(|\nabla\Phi|^q)_{\mathbf{S}^r_R(\mathcal{P})}|^{\frac{2}{q}}+\frac{1}{2^2}\|\nabla\Phi\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))}+c\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu}Q\right\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))}.\end{aligned}$$

This gives (3.30) for $\frac{4c_0}{\omega^2} \le \frac{1}{2^2}$ case.

Suppose $\frac{4c_0}{\omega^2} > \frac{1}{2^2}$; then, $1 < \omega^2 < 2^4 c_0$. Equation (3.20) is a uniform elliptic equation. Let $\eta \in C_0^\infty(B_R(\mathcal{P}))$ be a non-negative bell-shaped function with $\eta = 1$ in $B_{R/2}(\mathcal{P})$. Test (3.20) against $\Phi\eta^2$ to get (3.30) for $\frac{4c_0}{\omega^2} > \frac{1}{2^2}$ case. ∎

Lemma 3.11. *Assume* (A1), $\omega \in (1,\infty)$, $\frac{\varepsilon}{r}, r, \mu \in (0,1)$, $\tau \in [0,2]$, $\mathcal{P} \in \Omega/r$, *and assume that* Φ *is a weak solution of* (3.20) *for* $R \in (0,1)$. *There is a number* $p_* > 2$ *such that, for any* $p \in (2, p_*)$,

$$\left\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\right\|_{L^p(\mathbf{S}^r_{R/2}(\mathcal{P}))}\le cR^{\frac{2}{p}-1}\left\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\right\|_{L^2(\mathbf{S}^r_R(\mathcal{P}))}+c\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu}Q\right\|_{L^p(\mathbf{S}^r_R(\mathcal{P}))},$$

where c *is independent of* ε, μ, ω, r, τ, q, R, $\mathcal{P}$.

Proof. By Lemma 3.8 (for $R \le 32\frac{\varepsilon\mu}{r}$) and Lemma 3.10 (for $R \ge 32\frac{\varepsilon\mu}{r}$),

$$\begin{aligned}\fint_{\mathbf{S}^r_{R/2}(\mathcal{P})}\left|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\right|^2 &\le c\left|\left(|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi|^q\right)_{\mathbf{S}^r_R(\mathcal{P})}\right|^{2/q}+\frac{1}{4}\fint_{\mathbf{S}^r_R(\mathcal{P})}|\nabla\Phi|^2\\&\quad+c\fint_{\mathbf{S}^r_R(\mathcal{P})}\left|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu}Q\right|^2,\end{aligned}$$

where $q \in (1,2)$ and c is a constant independent of ε, μ, ω, r, τ, q, R, $\mathcal{P}$. Suppose $s = \frac{2}{q} > 1$, $\psi = |\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi|^q$, and $\zeta = |\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu}Q|^q$; then,

$$\fint_{\mathbf{S}^r_{R/2}(\mathcal{P})}\psi^s \le c\left(\fint_{\mathbf{S}^r_R(\mathcal{P})}\psi\right)^s+\frac{1}{4}\fint_{\mathbf{S}^r_R(\mathcal{P})}\psi^s+c\fint_{\mathbf{S}^r_R(\mathcal{P})}\zeta^s. \tag{3.31}$$

Lemma 3.11 follows from (3.31) and [18, Proposition 1.1 in p. 122]. ∎

3.4. Global L^p-gradient estimate

By translation, we can move any point $\mathcal{P} \in \partial\Omega$ to 0. By (A1), there is a $\gamma_* > 0$ and a C^2 function $\Upsilon : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} \Upsilon(0) = 0, \\ \|\nabla\Upsilon\|_{L^\infty(\mathbb{R})} \le \mathbf{d}_1, \\ B_{\gamma_*}(0) \cap \Omega = B_{\gamma_*}(0) \cap \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \in \mathbb{R},\ z_2 > \Upsilon(z_1)\}, \\ \mathbf{d}_2\, z_2 \le \beta^z \le \mathbf{d}_3\, z_2 \quad \text{for } z = (0, z_2) \in B_{\gamma_*}(0) \cap \Omega, \end{cases} \tag{3.32}$$

where $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$ are constants independent of z. Constant γ_* is independent of $\mathcal{P} \in \partial\Omega$. See Section 2 for β^z. Let $\mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0) \equiv [-\mathbf{d}_4, \mathbf{d}_4] \times [-\mathbf{d}_5, \mathbf{d}_5]$ denote a rectangle for $\mathbf{d}_4, \mathbf{d}_5 \in [3, 4]$. If $0 < \varepsilon \le r \le 1$, $\mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0)$ is chosen such that the volume $|\mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0) \cap \frac{\varepsilon}{r}(\mathcal{Y} + \mathbf{j})|$ for $\mathbf{j} \in \mathcal{I}_{\varepsilon/r}$ is either 0 or $|\frac{\varepsilon}{r}|^2$. Note that $\mathbf{d}_4, \mathbf{d}_5$ depend on $\frac{\varepsilon}{r}$. Define

$$\begin{cases} \mathcal{Q}^{,r} \equiv \mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0) \cap \Omega/r, \\ \mathcal{Q}^{\varepsilon,r}_f \equiv \mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0) \cap \Omega^\varepsilon_f/r, \\ \mathcal{Q}^{\varepsilon,r}_\mu \equiv \mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0) \cap \Omega^\varepsilon_\mu/r, \\ \mathcal{Q}^{\varepsilon,r}_\dagger \equiv \bigcup_{\frac{\varepsilon}{r}(\mathcal{Y}+\mathbf{j}) \subset \mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0);\, \mathbf{j} \in \mathcal{I}_{\varepsilon/r}} \frac{\varepsilon}{r}(\mathcal{Y} + \mathbf{j}). \end{cases} \tag{3.33}$$

If $r = 0$, we define $\Omega/r \equiv \{(x_1, x_2) \mid x_2 > 0\}$. From $(3.33)_1$,

$$\begin{cases} \mathcal{Q}^{,r} \quad \text{are Lipschitz domains}, \\ \mathcal{Q}^{,r} \setminus \mathcal{Q}^{\varepsilon,r}_\dagger \subset \left\{x \in \Omega^\varepsilon_f/r \mid \beta^x_r \le \frac{3\varepsilon}{r}\right\}. \end{cases} \tag{3.34}$$

See Section 2 for β^x_r.

Lemma 3.12. *Assume* (A1), $\omega \in (1, \infty)$, $\frac{\varepsilon}{r}, r, \mu \in (0, 1)$, *and* $\tau \in [0, 2]$. *Any solution of*

$$\begin{cases} -\nabla \cdot \left(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu} \nabla\Phi + Q\right) = 0 & \text{in } \mathcal{Q}^{,r}, \\ \Phi = 0 & \text{on } \partial\mathcal{Q}^{,r} \end{cases} \tag{3.35}$$

satisfies $\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu} \nabla\Phi\|_{L^2(\mathcal{Q}^{,r})} \le c \|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu} Q\|_{L^2(\mathcal{Q}^{,r})}$, *where c is a constant independent of* ε, μ, ω, r, τ.

Proof. For each $\frac{\varepsilon}{r}(\mathcal{Y}_\mu + \mathbf{j})$ with $\mathbf{j} \in \mathcal{I}_{\varepsilon/r}$, we find $V_\mathbf{j} \in H^1_0(\frac{\varepsilon}{r}(\mathcal{Y} + \mathbf{j}))$ and $\mathbf{h}_\mathbf{j} \in \mathbb{R}$ satisfying, by Lemma 3.5,

$$\begin{cases} V_\mathbf{j} = \Phi - \mathbf{h}_\mathbf{j} \quad \text{in } \frac{\varepsilon}{r}(\mathcal{Y}_\mu + \mathbf{j}), \\ \|V_\mathbf{j}\|_{H^1(\frac{\varepsilon}{r}(\mathcal{Y}+\mathbf{j}))} \le c \|\nabla\Phi\|_{L^2(\frac{\varepsilon}{r}(\mathcal{Y}_\mu+\mathbf{j}))}. \end{cases}$$

Extend $V_\mathbf{j}$ from $\frac{\varepsilon}{r}(\mathcal{Y} + \mathbf{j})$ to $\mathcal{Q}^{,r}$ by 0, and let

$$\zeta = \Phi - \sum_{\mathbf{j} \in \mathcal{I}_{\varepsilon/r}} V_\mathbf{j} \quad \text{in } \mathcal{Q}^{,r}.$$

Note $\nabla\zeta = 0$ in $\Omega_\mu^{\varepsilon,r}$ and $\zeta \in H_0^1(\Omega^{,r})$. Test (3.35) against ζ to get

$$\int_{\Omega_f^{\varepsilon,r}} \nabla\Phi\nabla\zeta\, dx + \int_{\Omega_f^{\varepsilon,r}} Q\nabla\zeta\, dx = 0. \tag{3.36}$$

By (3.36) and Lemma 3.6, we obtain

$$\begin{aligned}\|\nabla\Phi\|^2_{L^2(\Omega_f^{\varepsilon,r})} &\le c\big(\|\nabla\Phi\|^2_{L^2(\Omega_\mu^{\varepsilon,r})} + \|Q\|^2_{L^2(\Omega_f^{\varepsilon,r})}\big)\\ &\le c\big(\omega^{-4}\|\nabla\Phi\|^2_{L^2(\Omega_f^{\varepsilon,r})} + c\|\mathbf{K}^{\varepsilon}_{1/\omega^2,\mu} Q\|^2_{L^2(\Omega^{,r})}\big).\end{aligned}$$

There is a $\omega_0 > 1$ so that if $\omega > \omega_0$, then $\|\nabla\Phi\|_{L^2(\Omega_f^{\varepsilon,r})} \le c\|\mathbf{K}^{\varepsilon}_{1/\omega^2,\mu} Q\|_{L^2(\Omega^{,r})}$. Applying Lemma 3.6 again, we see that Lemma 3.12 is true for $\omega > \omega_0$ case. Next, we consider $\omega \in (1, \omega_0)$. In this case, (3.35) is a uniform elliptic equation. Test (3.35) against Φ to obtain Lemma 3.12 for $\omega \in (1, \omega_0)$ case. ∎

Tracing the proofs of Lemmas 3.8–3.11, Lemma 3.11 is true if $\mathbf{S}_R^{,r}(\mathcal{P})$ in Lemma 3.11 is replaced by $B_R(\mathcal{P}) \cap \Omega^{,r}$. By Lemmas 3.11–3.12, we see the following.

Corollary 3.13. *Let p_* be a number same as Lemma 3.11. Assume (A1), $\omega \in (1, \infty)$, $\frac{\varepsilon}{r}, r, \mu \in (0, 1)$, $\tau \in [0, 2]$, $p \in (2, p_*)$, and $Q \in L^p(\Omega^{,r})$. The solution of (3.35) satisfies*

$$\big\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\big\|_{L^p(\Omega^{,r})} \le c\big\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu} Q\big\|_{L^p(\Omega^{,r})},$$

where c is a constant independent of ε, μ, ω, r, τ.

By a duality argument, we have the following lemma.

Lemma 3.14. *Let p_* be a number same as Lemma 3.11. Assume (A1), $\omega \in (1, \infty)$, $\frac{\varepsilon}{r}, r, \mu \in (0, 1)$, $\tau \in [0, 2]$, $p \in (2, p_*)$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $Q \in L^{p'}(\Omega^{,r})$. Then, a $W^{1,p'}(\Omega^{,r})$ solution of (3.35) exists uniquely and*

$$\big\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\big\|_{L^{p'}(\Omega^{,r})} \le c\big\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu} Q\big\|_{L^{p'}(\Omega^{,r})},$$

where c is a constant independent of ε, μ, ω, r, τ.

Proof. Let c be a constant independent of ε, μ, ω, r, τ. Suppose $Q, \zeta \in L^\infty(\Omega^{,r})$; let us find $\Phi \in H_0^1(\Omega^{,r})$ by solving

$$\begin{cases} -\nabla\cdot\big(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla\Phi + Q\big) = 0 & \text{in } \Omega^{,r},\\ \Phi = 0 & \text{on } \partial\Omega^{,r}\end{cases} \tag{3.37}$$

and find $\varphi \in H_0^1(\Omega^{,r})$ by solving

$$\begin{cases} -\nabla\cdot\big(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla\varphi + \zeta\big) = 0 & \text{in } \Omega^{,r},\\ \varphi = 0 & \text{on } \partial\Omega^{,r}.\end{cases} \tag{3.38}$$

The solutions of (3.37) and (3.38) exist uniquely by Lax–Milgram theorem [19]. By Corollary 3.13, the solution of (3.38) satisfies

$$\left\|\mathbf{K}^{\varepsilon,r}_{\omega^\sigma,\mu}\nabla\varphi\right\|_{L^p(\mathbb{Q}^{,r})} \le c\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{\sigma-2},\mu}\zeta\right\|_{L^p(\mathbb{Q}^{,r})}, \tag{3.39}$$

where $\sigma \in [0,2]$ and $p \in (2, p_*)$. Test (3.37) against the solution φ of (3.38), test (3.38) against the solution Φ of (3.37), and apply (3.39) with $\sigma = 2-\tau$ to see

$$\begin{aligned}\left|\int_{\mathbb{Q}^{,r}} \nabla\Phi\,\zeta\,dx\right| &= \left|\int_{\mathbb{Q}^{,r}} \mathbf{K}^{\varepsilon,r}_{\omega^{2-\tau},\mu}\nabla\varphi\,\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu} Q\,dx\right|\\ &\le c\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{-\tau},\mu}\zeta\right\|_{L^p(\mathbb{Q}^{,r})}\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu}Q\right\|_{L^{p'}(\mathbb{Q}^{,r})},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Since $L^\infty(\mathbb{Q}^{,r})$ is dense in $L^p(\mathbb{Q}^{,r})$ and in $L^{p'}(\mathbb{Q}^{,r})$, we obtain

$$\left\|\mathbf{K}^{\varepsilon,r}_{\omega^\tau,\mu}\nabla\Phi\right\|_{L^{p'}(\mathbb{Q}^{,r})} \le c\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{\tau-2},\mu}Q\right\|_{L^{p'}(\mathbb{Q}^{,r})}$$

by a density argument. ■

Lemma 3.15. *Let p_* be a number same as Lemma 3.11. Assume* (A1)*, $\omega \in (1,\infty)$, $\frac{\varepsilon}{r}, r, \mu \in (0,1)$, and $p \in (p'_*, p_*)$. Any solution of*

$$\begin{cases} -\nabla\cdot\left(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla\Phi\right) = G & \text{in } \mathbb{Q}^{,r},\\ \Phi = 0 & \text{on } \partial\mathbb{Q}^{,r}\end{cases} \tag{3.40}$$

satisfies $\|\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla\Phi\|_{L^p(\mathbb{Q}^{,r})} \le c\|G\|_{W^{-1,p}(\mathbb{Q}^{,r})}$, where c is independent of ε, μ, ω, r.

Proof. Let c denote a constant independent of ε, μ, ω, r. Suppose $\zeta \in L^\infty(\mathbb{Q}^{,r})$; find $\varphi \in H^1_0(\mathbb{Q}^{,r})$ by solving

$$\begin{cases} -\nabla\cdot\left(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla\varphi + \zeta\right) = 0 & \text{in } \mathbb{Q}^{,r},\\ \varphi = 0 & \text{on } \partial\mathbb{Q}^{,r}.\end{cases} \tag{3.41}$$

By Corollary 3.13 and Lemma 3.14,

$$\|\nabla\varphi\|_{L^{p'}(\mathbb{Q}^{,r})} \le c\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{-2},\mu}\zeta\right\|_{L^{p'}(\mathbb{Q}^{,r})}, \tag{3.42}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Test (3.40) against the solution φ of (3.41), test (3.41) against the solution Φ of (3.40), and apply (3.42) to see

$$\begin{aligned}\left|\int_{\mathbb{Q}^{,r}} \nabla\Phi\,\zeta\right| = \left|\int_{\mathbb{Q}^{,r}} \varphi\,G\right| &\le c\|\nabla\varphi\|_{L^{p'}(\mathbb{Q}^{,r})}\|G\|_{W^{-1,p}(\mathbb{Q}^{,r})}\\ &\le c\left\|\mathbf{K}^{\varepsilon,r}_{\omega^{-2},\mu}\zeta\right\|_{L^{p'}(\mathbb{Q}^{,r})}\|G\|_{W^{-1,p}(\mathbb{Q}^{,r})}.\end{aligned}$$

Since $L^\infty(\mathbb{Q}^{,r})$ is dense in $L^{p'}(\mathbb{Q}^r)$, the lemma follows by a density argument. ■

Corollary 3.16. *Let p_* be a number same as Lemma 3.11. Assume* (A1)*,* $\omega \in (1,\infty)$*,* $\frac{\varepsilon}{r}, r, \mu \in (0,1)$*, and* $q = \frac{2p}{2+p} \in (1,2)$ *for* $p \in (2, p_*)$*. Any solution of* (3.40) *satisfies*

$$\left\| \mathbf{K}^{\varepsilon,r}_{\omega^2,\mu} \nabla \Phi \right\|_{L^p(\mathcal{Q}^{,r})} \leq c \|G\|_{L^q(\mathcal{Q}^{,r})},$$

where c *is a constant independent of* ε*,* μ*,* ω*,* r*.*

Proof. By Lemma 3.15, we should show $\|G\|_{W^{-1,p}(\mathcal{Q}^{,r})} \leq c\|G\|_{L^q(\mathcal{Q}^{,r})}$. If $q' \equiv \frac{2p'}{2-p'}$, then

$$\frac{1}{q'} + \frac{1}{q} = 1.$$

By Sobolev embedding theorem [19], $W^{1,p'}(\mathcal{Q}^{,r}) \subset L^{\frac{2p'}{2-p'}}(\mathcal{Q}^{,r}) = L^{q'}(\mathcal{Q}^{,r})$. Therefore, if $\phi \in W^{1,p'}(\mathcal{Q}^{,r})$,

$$\left| \int_{\mathcal{Q}^{,r}} G\, \phi\, dx \right| \leq \|G\|_{L^q(\mathcal{Q}^{,r})} \|\phi\|_{L^{q'}(\mathcal{Q}^{,r})} \leq \|G\|_{L^q(\mathcal{Q}^{,r})} \|\phi\|_{W^{1,p'}(\mathcal{Q}^{,r})}.$$

So, we prove the claim. ∎

Proof of Theorem 2.1. From the arguments in this subsection (i.e., Section 3.4), we see if $r = 1$ in (3.33) and if $\mathbf{d}_4, \mathbf{d}_5$ in (3.33) are large (e.g., $\mathbf{d}_4, \mathbf{d}_5 > 5$) so that $\Omega \subset \mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0)$, Corollaries 3.13–3.16 and Lemmas 3.14–3.15 are still true. Therefore,

- (I) of Theorem 2.1 is a direct consequence of Corollary 3.13 and Lemma 3.14,
- (II) of Theorem 2.1 is from Lemma 3.15,
- (III) of Theorem 2.1 is from Corollary 3.16. ∎

4. Local maximum norm for strongly elliptic equations

First, we recall a local L^∞ estimate for strongly elliptic equations. By energy method or [19, Theorem 8.1], we see the following.

Remark 4.1. Assume $\omega \in (1,\infty)$, $\frac{\varepsilon}{r}, r, \mu \in (0,1)$, and $\mathbf{D}$ is a Lipschitz domain. Any weak solution $\Psi \in H^1(\mathbf{D})$ of $-\nabla \cdot (\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu} \nabla \Psi) = 0$ in $\mathbf{D}$ satisfies

$$\sup_{\mathbf{D}} \Psi(-\Psi) \leq \sup_{\partial \mathbf{D}} \Psi^+(\Psi^-),$$

where

$$\Psi^+ \equiv \max\{0, \Psi\}, \quad \Psi^- \equiv \max\{0, -\Psi\},$$

and c is a constant independent of ε, μ, ω, r. By (II) of Corollary 3.4, the weak solution Ψ is a Lipschitz function in $\mathbf{D}$.

Next is a convergence result.

Lemma 4.2. *Consider*

$$\begin{cases} -\nabla\cdot\big(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla\Psi\big)=0 & \text{in } \mathbf{S}^{;r}_1(0)\equiv B_1(0)\cap\Omega/r,\\ \Psi=0 & \text{on } B_1(0)\cap\partial\Omega/r. \end{cases} \tag{4.1}$$

If $\{\varepsilon_k,\mu_k,\omega_k,r_k,\Psi_k,\mathbf{S}^{;r_k}_1(0)\}$ *satisfies* (A1), $\omega_k\in(1,\infty)$, $\frac{\varepsilon_k}{r_k},r_k,\mu_k\in(0,1)$, $\mathbf{d}\in(1,2)$, (3.32), (4.1), *and*

(i) $\|\Psi_k\|_{W^{1,\mathbf{d}}(\mathbf{S}^{;r_k}_1(0))}$ *are bounded independent of* k,

(ii) $|\mathbf{S}^{;r_k}_1(0)\setminus\mathbf{S}|+|\mathbf{S}\setminus\mathbf{S}^{;r_k}_1(0)|\to 0$ *as* $k\to\infty$,

(iii) $\Psi_k\in W^{1,\mathbf{d}}(\mathbf{S}^{;r_k}_1(0))\cap C(\mathbf{S}^{;r_k}_1(0))\to\Psi\in W^{1,\mathbf{d}}(\mathbf{S})\cap C(\mathbf{S})$ *weakly in* $W^{1,\mathbf{d}}(\mathbf{S})$ *as* $k\to\infty$,

then $\|\Psi_k-\Psi\|_{L^\infty(\mathbf{S}^{;r_k}_1(0)\cap\mathbf{S})}\to 0$ *as* $k\to\infty$.

Proof. Step I. Define $\phi_k\equiv\Psi_k\mathcal{X}_{\mathbf{S}^{;r_k}_1(0)}$. By (i), $\|\phi_k\|_{W^{1,\mathbf{d}}(B_1(0))}\le c$, where c is independent of k. There are $\widehat{\Psi}\in C(B_1(0))$ and a subsequence of $\{\phi_k\}$ (same notation for subsequence) so that, by (ii), (iii), and [17, Theorem 7 on p. 8],

$$\begin{cases} \phi_k & \text{converges quasi-uniformly to } \widehat{\Psi}\in C(B_1(0)) \text{ and } \widehat{\Psi}|_{\mathbf{S}}=\Psi,\\ \widehat{\Psi}(x)=0 & \text{if } x\in B_1(0)\setminus\bigcup_k\mathbf{S}^{;r_k}_1(0). \end{cases} \tag{4.2}$$

Step II. Claim that the subsequence of ϕ_k converges to $\widehat{\Psi}$ in $C(B_{\frac12}(0))$. If so, the whole sequence ϕ_k converges to $\widehat{\Psi}$ in $C(B_{\frac12}(0))$ by a contradiction argument. ■

Proof of the claim. For any $m\ (\in\mathbb{N})>6$, by $\widehat{\Psi}\in C(B_1(0))$, there is a $\delta_m\in(0,\frac{1}{m})$ satisfying

$$|\widehat{\Psi}(x)-\widehat{\Psi}(y)|<|2m|^{-1}\quad \text{if } x,y\in\overline{B_{2/3}(0)},\ |x-y|\le\delta_m. \tag{4.3}$$

Let $\mathfrak{C}_s(\mathfrak{L})$ for $s\in(1,\mathbf{d})$ denote the s-capacity of a unit segment $\mathfrak{L}$ in $\mathbb{R}^2$ (see [9, p. 458]). For any $s\in(1,\mathbf{d})$, by $(4.2)_1$ and [17, Theorem 7 on p. 8], there is a relatively open subset $\mathcal{O}_{s,m}$ of $B_{2/3}(0)$ and a positive number $\mathcal{N}_{s,m}\in\mathbb{N}$ satisfying

$$\begin{cases} \mathfrak{C}_s(\mathcal{O}_{s,m})<\mathfrak{C}_s(\mathfrak{L})|\delta_m|^{2-s},\\ |\phi_k(x)-\widehat{\Psi}(x)|<|2m|^{-1} & \text{if } k\ge\mathcal{N}_{s,m},\ x\in B_{2/3}(0)\setminus\mathcal{O}_{s,m}, \end{cases} \tag{4.4}$$

where $\mathfrak{C}_s(\mathcal{O}_{s,m})$ is the s-capacity of $\mathcal{O}_{s,m}$. Moreover, by $(4.2)_2$, (4.3), and (A1),

$$|\widehat{\Psi}(x)|<|2m|^{-1}\quad \text{if } k\ge\mathcal{N}_{s,m},\ x\in B_{2/3}(0)\setminus\mathbf{S}^{;r_k}_1(0). \tag{4.5}$$

Then, (4.5) and the definition of ϕ_k imply

$$|\phi_k(x)-\widehat{\Psi}(x)|<|2m|^{-1}\quad \text{if } k\ge\mathcal{N}_{s,m},\ x\in B_{2/3}(0)\setminus\mathbf{S}^{;r_k}_1(0). \tag{4.6}$$

Find a connected component $\mathcal{O}$ of $\mathcal{O}_{s,m}$ satisfying

$$\overline{\mathcal{O}} \cap \overline{B_{1/2}(0)} \neq \emptyset.$$

Since $\mathcal{O}$ is connected, for any $y, z \in \mathcal{O}$, there is a curve $\mathbb{L}$ ($\subset \mathcal{O} \subset \mathcal{O}_{s,m}$) connecting y and z. By [9, Lemma 2.8] and $(4.4)_1$,

$$\mathfrak{C}_s(\mathfrak{L})|y-z|^{2-s} \leq \mathfrak{C}_s(\mathbb{L}) \leq \mathfrak{C}_s(\mathcal{O}_{s,m}) \leq \mathfrak{C}_s(\mathfrak{L})|\delta_m|^{2-s}. \tag{4.7}$$

By (4.7), we see $\mathrm{diam}(\mathcal{O}) \leq \delta_m$. (Here, $\mathrm{diam}(\mathcal{O})$ means the diameter of $\mathcal{O}$.) Therefore,

$$|\mathcal{O}| \leq \pi|\delta_m|^2, \quad \overline{\mathcal{O}} \subset \overline{B_{1/2+\delta_m}(0)}, \quad \partial\mathcal{O} \subset \overline{B_{2/3}(0)} \setminus \mathcal{O}_{s,m}. \tag{4.8}$$

If $\{\varepsilon_k, \mu_k, \omega_k, r_k, \phi_k, \mathbf{S}_1^{;r_k}(0)\}$ satisfy (4.1), Remark 4.1 implies

$$\min_{\partial(\mathcal{O}\cap\Omega/r_k)} \phi_k \leq \phi_k(x) \leq \max_{\partial(\mathcal{O}\cap\Omega/r_k)} \phi_k \quad \text{for } x \in \mathcal{O} \cap \Omega/r_k. \tag{4.9}$$

By $(4.4)_2$, (4.6), (4.8), and (4.3),

$$\begin{cases} \min_{\partial(\mathcal{O}\cap\Omega/r_k)} \phi_k \geq \min_{\partial(\mathcal{O}\cap\Omega/r_k)} \widehat{\Psi} - \frac{1}{2m} \geq \widehat{\Psi}(x) - \frac{1}{m} \\ \max_{\partial(\mathcal{O}\cap\Omega/r_k)} \phi_k \leq \max_{\partial(\mathcal{O}\cap\Omega/r_k)} \widehat{\Psi} + \frac{1}{2m} \leq \widehat{\Psi}(x) + \frac{1}{m} \end{cases} \quad \text{for } k \geq \mathcal{N}_{s,m},\ x \in \overline{\mathcal{O}}. \tag{4.10}$$

Hence, (4.9)–(4.10) imply $\|\phi_k - \widehat{\Psi}\|_{L^\infty(\mathcal{O}\cap\Omega/r_k)} < \frac{1}{m}$ for $m \in \mathbb{N}$ and $k \geq \mathcal{N}_{s,m}$. Since $\mathcal{O}_{s,m}$ is the union of its connected components, $(4.4)_2$ implies

$$\|\phi_k - \widehat{\Psi}\|_{L^\infty(B_{1/2}(0)\cap\Omega/r_k)} < \frac{1}{m}$$

for $m \in \mathbb{N}, k \geq \mathcal{N}_{s,m}$. So, we prove the claim. ∎

Lemma 4.3. *Assume* (A1), $\omega \in (1,\infty)$, $\frac{\varepsilon}{rR}, r, R, \mu \in (0,1)$, *and* $\mathbf{d} \in (1,2)$, *and assume that* $\Psi \in W^{1,\mathbf{d}}(\mathbf{S}_R^{;r}(0))$ *is a weak solution of*

$$\begin{cases} -\nabla \cdot \left(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu} \nabla\Psi\right) = 0 & \text{in } \mathbf{S}_R^{;r}(0) \equiv B_R(0) \cap \Omega/r, \\ \Psi = 0 & \text{on } B_R(0) \cap \partial\Omega/r. \end{cases} \tag{4.11}$$

Then,

$$\|\Psi\|_{L^\infty(\mathbf{S}_{R/2}^{;r}(0))} \leq c\left((|\Psi|)_{\mathbf{S}_R^{;r}(0)} + R\left|(|\nabla\Psi|^{\mathbf{d}})_{\mathbf{S}_R^{;r}(0)}\right|^{1/\mathbf{d}}\right), \tag{4.12}$$

where c is a constant independent of ε, μ, ω, r, R.

Proof. If $\phi(y) \equiv \Psi(Ry)$, then $\phi \in W^{1,\mathbf{d}}(\mathbf{S}_1^{;rR}(0))$ and (4.11) can be written as

$$\begin{cases} -\nabla \cdot \left(\mathbf{K}^{\varepsilon,rR}_{\omega^2,\mu} \nabla\phi\right) = 0 & \text{in } \mathbf{S}_1^{;rR}(0) = B_1(0) \cap \Omega/rR, \\ \phi = 0 & \text{on } B_1(0) \cap \partial\Omega/rR. \end{cases} \tag{4.13}$$

Step I. For $B_1(0) \Subset \Omega/rR$ case. We claim, for any $\alpha > 0$, there is a constant c_α so that the solution of $(4.13)_1$ satisfies

$$\|\phi - (\phi)_{B_1(0)}\|_{L^\infty(B_{1/2}(0))} \leq c_\alpha \|\phi - (\phi)_{B_1(0)}\|_{L^1(B_1(0))} + \alpha\|\nabla\phi\|_{L^{\mathbf{d}}(B_1(0))}. \tag{4.14}$$

■

Proof of the claim. If not, there are a $\alpha > 0$ and a sequence $\{\varepsilon_k, \mu_k, \omega_k, r_k, R_k, \phi_k\}$ satisfying $r_k, R_k \to r, R \in [0, 1]$, $\omega_k \in (1, \infty)$, $\mu_k, \frac{\varepsilon_k}{r_k R_k} \in (0, 1)$, $(4.13)_1$, and

$$\|\phi_k - (\phi_k)_{B_1(0)}\|_{L^\infty(B_{\frac{1}{2}}(0))} > k\|\phi_k - (\phi_k)_{B_1(0)}\|_{L^1(B_1(0))} + \alpha\|\nabla\phi_k\|_{L^{\mathbf{d}}(B_1(0))} \tag{4.15}$$

for any $k \in \mathbb{N}$. Define

$$\mathbb{V}_k \equiv \frac{\phi_k - (\phi_k)_{B_1(0)}}{\|\phi_k - (\phi_k)_{B_1(0)}\|_{L^\infty(B_{1/2}(0))}} \quad \text{on } B_1(0).$$

Then, $\mathbb{V}_k$ satisfies $(4.13)_1$, $\mathbb{V}_k \in W^{1,\mathbf{d}}(B_1(0))$, and $\|\mathbb{V}_k\|_{L^\infty(B_{1/2}(0))} = 1$. By (4.15),

$$1 > k\|\mathbb{V}_k\|_{L^1(B_1(0))} + \alpha\|\nabla\mathbb{V}_k\|_{L^{\mathbf{d}}(B_1(0))}.$$

Note $\|\mathbb{V}_k\|_{L^1(B_1(0))} \to 0$ as $k \to \infty$. By Lemma 4.2, the sequence $\mathbb{V}_k$ converges to 0 in $C(B_{1/2}(0))$, which contradicts $\|\mathbb{V}_k\|_{L^\infty(B_{1/2}(0))} = 1$. So, (4.14) is true. Therefore,

$$\|\phi\|_{L^\infty(B_{1/2}(0))} \leq c\big(|(\phi)_{B_1(0)}| + \|\nabla\phi\|_{L^{\mathbf{d}}(B_1(0))}\big).$$

After change of variables, we obtain (4.12).

Step II. For $0 \in \partial\Omega/rR$ case. We claim, for any $\alpha > 0$, there is a constant c_α so that the solution of (4.13) satisfies

$$\|\phi\|_{L^\infty(\mathbf{S}^{;rR}_{1/2}(0))} \leq c_\alpha\|\phi\|_{L^1(\mathbf{S}^{;rR}_1(0))} + \alpha\|\nabla\phi\|_{L^{\mathbf{d}}(\mathbf{S}^{;rR}_1(0))}. \tag{4.16}$$

■

Proof of claim. If not, there is a $\alpha > 0$ and a sequence $\{\varepsilon_k, \mu_k, \omega_k, r_k, R_k, \phi_k\}$ so that $r_k, R_k \to r, R \in [0, 1]$, $\omega_k \in (1, \infty)$, $\mu_k, \frac{\varepsilon_k}{r_k R_k} \in (0, 1)$, and (4.13) is true and, for $k \in \mathbb{N}$,

$$\|\phi_k\|_{L^\infty(\mathbf{S}^{;r_kR_k}_{1/2}(0))} > k\|\phi_k\|_{L^1(\mathbf{S}^{;r_kR_k}_1(0))} + \alpha\|\nabla\phi_k\|_{L^{\mathbf{d}}(\mathbf{S}^{;r_kR_k}_1(0))}. \tag{4.17}$$

Define

$$\mathbb{V}_k \equiv \frac{\phi_k}{\|\phi_k\|_{L^\infty(\mathbf{S}^{;r_kR_k}_{1/2}(0))}} \quad \text{on } \mathbf{S}^{;r_kR_k}_1(0).$$

Then, $\mathbb{V}_k$ satisfies (4.13), $\mathbb{V}_k \in W^{1,\mathbf{d}}(\mathbf{S}^{;r_kR_k}_1(0))$, and $\|\mathbb{V}_k\|_{L^\infty(\mathbf{S}^{;r_kR_k}_{1/2}(0))} = 1$. By (4.17),

$$1 > k\|\mathbb{V}_k\|_{L^1(\mathbf{S}^{;r_kR_k}_1(0))} + \alpha\|\nabla\mathbb{V}_k\|_{L^{\mathbf{d}}(\mathbf{S}^{;r_kR_k}_1(0))}.$$

Note $\|\mathbb{V}_k\|_{L^1(\mathbf{S}^{;r_kR_k}_1(0))} \to 0$ as $k \to \infty$. By Lemma 4.2, the sequence $\mathbb{V}_k$ converges to 0 in $C(\mathbf{S}^{;r_kR_k}_{1/2}(0))$, which contradicts $\|\mathbb{V}_k\|_{L^\infty(\mathbf{S}^{;r_kR_k}_{1/2}(0))} = 1$. So, (4.16) is true. After change of variables, we obtain (4.12). So, the lemma is proved. ■

5. Interior Lipschitz estimate for strongly elliptic equations

We consider the interior Lipschitz estimate for strongly elliptic equations in Ω. For any $\omega \in (1,\infty)$, $\mu \in (0,1)$, and $i \in \{1,2\}$, find $\mathbb{X}_{\omega,\mu,i} \in H^1_\#(\mathbb{R}^2)$ by solving

$$\begin{cases} -\nabla \cdot \big(\mathbb{K}_{\omega^2,\mu}(\nabla \mathbb{X}_{\omega,\mu,i} + \vec{e}_i)\big) = 0 & \text{in } \mathcal{Y}, \\ (\mathbb{X}_{\omega,\mu,i})_{\mathcal{Y}_\mu} = 0, \end{cases} \tag{5.1}$$

where $\vec{e}_i$ is the unit vector in the i-th coordinate direction. See (3.14) for $\mathbb{K}_{\omega^2,\mu}$ and Section 2 for $H^1_\#(\mathbb{R}^2)$ and $(\mathbb{X}_{\omega,\mu,i})_{\mathcal{Y}}$.

Remark 5.1. A solution of (5.1) exists uniquely by Lax–Milgram theorem. Also,

$$\|\nabla \mathbb{X}_{\omega,\mu,i}\|_{L^2(\mathcal{Y})} \le c, \tag{5.2}$$

where c is independent of μ, ω. Here, μ, ω are independent. Equation (5.2) is proved by tracing the argument of Lemma 3.12. Indeed, by Lemma 3.5, we find $V \in H^1_0(\mathcal{Y})$ so that

$$\begin{cases} V = \mathbb{X}_{\omega,\mu,i} & \text{in } \mathcal{Y}_\mu, \\ \|V\|_{H^1(\mathcal{Y})} \le c\|\nabla \mathbb{X}_{\omega,\mu,i}\|_{L^2(\mathcal{Y}_\mu)}. \end{cases}$$

Set $\zeta \equiv \mathbb{X}_{\omega,\mu,i} - V$. Then, $\zeta = 0$ in $\mathcal{Y}_\mu$ and $\zeta \in H^1_\#(\mathcal{Y})$. Test (5.1) against ζ to get

$$\|\nabla \mathbb{X}_{\omega,\mu,i}\|_{L^2(\mathcal{Y}_f)} \le c\|\nabla \mathbb{X}_{\omega,\mu,i}\|_{L^2(\mathcal{Y}_\mu)} + c,$$

where c is independent of μ, ω. By Lemma 3.6, any solution of (5.1) satisfies

$$\|\nabla \mathbb{X}_{\omega,\mu,i}\|_{L^2(\mathcal{Y}_\mu)} \le c\|\omega^{-2}\nabla \mathbb{X}_{\omega,\mu,i}\|_{L^2(\mathcal{Y}_f)} + c,$$

where c is independent of μ, ω. So, there is a $\omega_0 > 1$ such that (5.2) holds if $\omega \ge \omega_0$. If $\omega \in (1,\omega_0)$, then (5.1) is a uniform elliptic equation. In this case, (5.2) follows by testing (5.1) against $\mathbb{X}_{\omega,\mu,i}$.

If $y = (y_1, y_2)$ and $\varphi_i(y) = \mathbb{X}_{\omega,\mu,i}(y) + y_i$ for $i = 1, 2$, then $\nabla \cdot (\mathbb{K}_{\omega^2,\mu}\nabla\varphi_i) = 0$ in $\mathcal{Y}$. By Lemma 3.6 and (5.2),

$$\big\|\mathbb{K}_{\omega^2,\mu}(\nabla \mathbb{X}_{\omega,\mu,i} + \vec{e}_i)\big\|_{L^2(\mathcal{Y})} = \big\|\mathbb{K}_{\omega^2,\mu}\nabla\varphi_i\big\|_{L^2(\mathcal{Y})} \le \|\nabla\varphi_i\|_{L^2(\mathcal{Y}_f)} \le c, \tag{5.3}$$

where c is independent of μ, ω. Define a 2×2 matrix $\mathcal{K}_{\omega,\mu}$, whose (i,j)-entry is

$$\int_{\mathcal{Y}} \mathbb{K}_{\omega^2,\mu}\big(\delta_{i,j} + \partial_j \mathbb{X}_{\omega,\mu,i}(z)\big)dz, \quad \text{where } \delta_{i,j} \equiv \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases} \tag{5.4}$$

By remark [21, pp. 43–44] and (5.2)–(5.3), $\mathcal{K}_{\omega,\mu}$ is a positive function depending only on μ, ω and satisfies

$$\begin{cases} 0 < \mathbf{d}_6 I \le \mathcal{K}_{\omega,\mu} \le \mathbf{d}_7 I, \\ \mathcal{K}_{\omega,\mu} \quad \text{is continuous in } \{(\omega,\mu) \mid \omega \in (1,\infty), \mu \in (0,1)\}, \end{cases} \tag{5.5}$$

where I is the identity matrix and $\mathbf{d}_6$, $\mathbf{d}_7$ are constants independent of μ, ω. Let us remark that (5.2)–(5.3) and (5.5) are true even though μ, ω are independent.

By (A2), Lemma 3.7, (5.2), and $(5.1)_2$,

$$\begin{cases} \left\|\mathbb{K}_{\omega^2,\mu}(\nabla\mathbb{X}_{\omega,\mu,i}+\vec{e}_i)\right\|_{L^\infty(\mathcal{Y})} = \left\|\mathbb{K}_{\omega^2,\mu}\nabla\varphi_i\right\|_{L^\infty(\mathcal{Y})} \le c\|\nabla\varphi_i\|_{L^2(\mathcal{Y})} \le c, \\ \|\mathbb{X}_{\omega,\mu,i}\|_{L^\infty(\mathcal{Y})} \le c, \end{cases} \tag{5.6}$$

where c is independent of $i, \mu\ (=\omega^{-1})$. If $\nu > 0$ and $i = 1, 2$, define

$$\mathbb{X}^\nu_{\omega,\mu,i}(x) \equiv \nu\mathbb{X}_{\omega,\mu,i}(x/\nu), \quad \mathbb{X}^\nu_{\omega,\mu}(x) \equiv \left(\mathbb{X}^\nu_{\omega,\mu,1}(x), \mathbb{X}^\nu_{\omega,\mu,2}(x)\right). \tag{5.7}$$

Assume $B_1(0) \Subset \Omega$ in this section.

Lemma 5.2. *Under* (A2) *and* $\alpha \in (0,1)$, *there are constants* $\theta, \varepsilon_0 \in (0,1)$ *depending on* α *such that if*

$$\begin{cases} -\nabla\cdot\left(\mathbf{K}^\nu_{\omega^2,\mu}\nabla\mathbb{V}\right) = 0 \quad \text{in } B_1(0), \\ \nu, \mu \in (0, \varepsilon_0), \\ \|\mathbb{V}\|_{L^\infty(B_1(0))} \le 1, \end{cases} \tag{5.8}$$

then

$$\sup_{z\in B_\theta(0)} \left|\mathbb{V}(z) - \mathbb{V}(0) - \left(z + \mathbb{X}^\nu_{\omega,\mu}(z)\right)\mathbf{b}_{\omega,\mu,\nu}\right| \le \theta^{1+\alpha}, \tag{5.9}$$

where $\mathbf{b}_{\omega,\mu,\nu} \equiv \mathcal{K}^{-1}_{\omega,\mu}(\mathbf{K}^\nu_{\omega^2,\mu}\nabla\mathbb{V})_{B_\theta(0)}$ *and* $\mathcal{K}^{-1}_{\omega,\mu}$ *is the inverse matrix of* $\mathcal{K}_{\omega,\mu}$. *See Section* 2 *for* $(\mathbf{K}^\nu_{\omega^2,\mu}\nabla\mathbb{V})_{B_\theta(0)}$.

Proof. Consider $-\Delta\mathbb{V} = 0$ in $B_{4/5}(0)$. By [19, Theorem 4.6], there is a small $\theta \in (0,1)$ such that, for $\breve{\alpha} \in (\alpha, 1)$,

$$\sup_{B_\theta(0)} \left|\mathbb{V}(z) - \mathbb{V}(0) - z(\nabla\mathbb{V})_{B_\theta(0)}\right| \le \theta^{1+\breve{\alpha}}\|\mathbb{V}\|_{L^\infty(B_{4/5}(0))}. \tag{5.10}$$

We claim (5.9). If not, there is a sequence $\{\nu, \mu_\nu, \omega_\nu, \mathbb{V}_\nu, \mathcal{K}_{\omega_\nu,\mu_\nu}\}$ satisfying (5.8) and

$$\begin{cases} \nu, \mu_\nu \to 0, \\ \widetilde{\mathcal{K}} = \lim_{\nu\to 0}\mathcal{K}_{\omega_\nu,\mu_\nu}, \\ \sup_{z\in B_\theta(0)}\left|\mathbb{V}_\nu(z) - \mathbb{V}_\nu(0) - \left(z + \mathbb{X}^\nu_{\omega_\nu,\mu_\nu}(z)\right)\mathbf{b}_{\omega_\nu,\mu_\nu,\nu}\right| > \theta^{1+\alpha}. \end{cases} \tag{5.11}$$

See (5.4) for $\mathcal{K}_{\omega_\nu,\mu_\nu}$. Equation $(5.11)_2$ is due to (5.5). By Lemma 3.6, (A2), and $(5.8)_3$,

$$\left\|\mathbf{K}^\nu_{\omega_\nu^2,\mu_\nu}\nabla\mathbb{V}_\nu\right\|_{L^2(B_{4/5}(0))} \le c\|\nabla\mathbb{V}_\nu\|_{L^2(B_{5/6}(0))} \le c, \tag{5.12}$$

where c is independent of ν, $\mu_\nu (= \omega_\nu^{-1})$. By (5.12), Lemma 3.11, Sobolev embedding theorem [19], $(5.8)_3$, [7, Theorem 1], and remark [21, pp. 43–44], there is a subsequence (same notation for subsequence) of $\{\mathbb{V}_\nu\}$ such that, as $\nu \to 0$,

$$\begin{cases} \mathbb{V}_\nu \to \mathbb{V} & \text{in } C(B_{4/5}(0)), \\ \|\mathbb{V}\|_{L^\infty(B_1(0))} \leq 1, & \\ \mathbf{K}^\nu_{\omega_\nu^2,\mu_\nu} \nabla \mathbb{V}_\nu \to \widetilde{\mathcal{K}} \nabla \mathbb{V} & \text{in } L^2(B_{4/5}(0)) \text{ weakly}, \\ -\nabla \cdot (\widetilde{\mathcal{K}} \nabla \mathbb{V}) = 0 & \text{in } B_{4/5}(0), \end{cases} \tag{5.13}$$

where $\widetilde{\mathcal{K}}$ is a positive constant satisfying $(5.5)_1$. By $(5.11)_2$ and $(5.13)_3$, we see

$$\lim_{\nu \to 0} \mathbf{b}_{\omega_\nu,\mu_\nu,\nu} = \lim_{\nu \to 0} \mathcal{K}^{-1}_{\omega_\nu,\mu_\nu} ⨍_{B_\theta(0)} \mathbf{K}^\nu_{\omega_\nu^2,\mu_\nu} \nabla \mathbb{V}_\nu \, dx = (\nabla \mathbb{V})_{B_\theta(0)}. \tag{5.14}$$

By (5.11) (5.13), (5.6)–(5.7), (5.14), and (5.10),

$$\begin{aligned} \theta^{1+\alpha} &\leq \lim_{\nu \to 0} \sup_{z \in B_\theta(0)} \left| \mathbb{V}_\nu(z) - \mathbb{V}_\nu(0) - \big(z + \mathbb{X}^\nu_{\omega_\nu,\mu_\nu}(z)\big) \mathbf{b}_{\omega_\nu,\mu_\nu,\nu} \right| \\ &= \sup_{z \in B_\theta(0)} \left| \mathbb{V}(z) - \mathbb{V}(0) - z(\nabla \mathbb{V})_{B_\theta(0)} \right| \leq \theta^{1+\breve{\alpha}} \|\mathbb{V}\|_{L^\infty(B_{4/5}(0))}. \end{aligned}$$

We get contradiction. So, (5.9) holds. ■

Lemma 5.3. *Under* (A2) *and* $\alpha \in (0,1)$*, there are constants* $\theta, \varepsilon_0 \in (0,1)$ *depending on* α *such that if* $\varepsilon, \mu \in (0, \varepsilon_0)$ *and*

$$-\nabla \cdot \big(\mathbf{K}^\varepsilon_{\omega^2,\mu} \nabla V\big) = 0 \quad \text{in } B_1(0), \tag{5.15}$$

then, for any $k \in \mathbb{N}$ *with* $\frac{\varepsilon}{\varepsilon_0} < \theta^k$*, there are constants* $\mathbf{a}^{\varepsilon,k}_{\omega,\mu}$*,* $\mathbf{b}^{\varepsilon,k}_{\omega,\mu}$ *satisfying*

$$\begin{cases} |\mathbf{a}^{\varepsilon,k}_{\omega,\mu}| + |\mathbf{b}^{\varepsilon,k}_{\omega,\mu}| \leq cJ, \\ \sup_{z \in B_{\theta^k}(0)} \left| V(z) - V(0) - \varepsilon \mathbf{a}^{\varepsilon,k}_{\omega,\mu} - \big(z + \mathbb{X}^\varepsilon_{\omega,\mu}(z)\big) \mathbf{b}^{\varepsilon,k}_{\omega,\mu} \right| \leq \theta^{k(1+\alpha)} J. \end{cases} \tag{5.16}$$

Here, $J \equiv \|V\|_{L^\infty(B_1(0))}$ *and* c *is a constant independent of* $\varepsilon, \mu\ (= \omega^{-1})$.

Proof. For $k = 1$, (5.16) is from Lemma 5.2 with $\nu = \varepsilon$ and $\mathbb{V} = \frac{V}{J}$. In this case, $\mathbf{a}^{\varepsilon,1}_{\omega,\mu} = 0$, $\mathbf{b}^{\varepsilon,1}_{\omega,\mu} = \mathcal{K}^{-1}_{\omega,\mu}(\mathbf{K}^\varepsilon_{\omega^2,\mu} \nabla V)_{B_\theta(0)}$. By Lemma 3.6 and (A2), we see $|\mathbf{b}^{\varepsilon,1}_{\omega,\mu}| \leq cJ$, where c is independent of $\varepsilon, \mu\ (= \omega^{-1})$. If (5.16) holds for some $k \in \mathbb{N}$ with $\frac{\varepsilon}{\varepsilon_0} < \theta^k$, define

$$\mathbb{V}(z) \equiv \frac{V(\theta^k z) - V(0) - \varepsilon \mathbf{a}^{\varepsilon,k}_{\omega,\mu} - (\theta^k z + \mathbb{X}^\varepsilon_{\omega,\mu}(\theta^k z)) \mathbf{b}^{\varepsilon,k}_{\omega,\mu}}{\theta^{k(1+\alpha)} J} \quad \text{in } B_1(0).$$

By induction, $\mathbb{V}$ satisfies (5.8) with $\nu = \varepsilon/\theta^k$. Apply Lemma 5.2 to obtain

$$\sup_{z \in B_\theta(0)} \left| \mathbb{V}(z) - \mathbb{V}(0) - \big(z + \mathbb{X}^{\varepsilon/\theta^k}_{\omega,\mu}(z)\big) \mathbf{b}_{\omega,\mu,\varepsilon/\theta^k} \right| \leq \theta^{1+\alpha}, \tag{5.17}$$

where $\mathbf{b}_{\omega,\mu,\varepsilon/\theta^k} \equiv \mathcal{K}^{-1}_{\omega,\mu}(\mathbf{K}^{\varepsilon/\theta^k}_{\omega^2,\mu}\nabla\mathbb{V})_{B_\theta(0)}$. Define

$$\mathbf{a}^{\varepsilon,k+1}_{\omega,\mu} \equiv -\mathbb{X}^1_{\omega,\mu}(0)\mathbf{b}^{\varepsilon,k}_{\omega,\mu} \quad \text{and} \quad \mathbf{b}^{\varepsilon,k+1}_{\omega,\mu} \equiv \mathbf{b}^{\varepsilon,k}_{\omega,\mu} + J\theta^{k\alpha}\mathbf{b}_{\omega,\mu,\varepsilon/\theta^k}. \tag{5.18}$$

By Lemma 3.6, (A2), and $\|\mathbb{V}\|_{L^\infty(B_1(0))} \leq 1$, we see that $|\mathbf{b}_{\omega,\mu,\varepsilon/\theta^k}|$ are bounded uniformly in $\varepsilon, \mu\ (= \omega^{-1}), k$. By (5.6) and (5.18), we obtain $(5.16)_1$. Rewrite (5.17) in terms of V in $B_{\theta^{k+1}}(0)$ and apply (5.18) to obtain $(5.16)_2$. ■

Lemma 5.4. *Under* (A2)*, there is a number* $\varepsilon_0 \in (0,1)$ *such that if* $\varepsilon, \mu \in (0, \varepsilon_0)$*, any solution of* (5.15) *satisfies*

$$\left\|\mathbf{K}^{\varepsilon}_{\omega^2,\mu}\nabla V\right\|_{L^\infty(B_{1/2}(0))} \leq c\|V\|_{L^\infty(B_1(0))}, \tag{5.19}$$

where c is a constant independent of $\varepsilon, \mu\ (= \omega^{-1})$.

Proof. Let α, θ, ε_0, J be the same as Lemma 5.3 and c be a constant independent of $\varepsilon, \mu\ (= \omega^{-1})$. Suppose $k \in \mathbb{N}$ satisfying $\theta^{k+1} \leq \frac{\varepsilon}{\varepsilon_0} < \theta^k$, by Lemma 5.3,

$$\sup_{z\in B_{\varepsilon/\varepsilon_0}(0)} \left|V(z) - V(0) - \varepsilon\mathbf{a}^{\varepsilon,k}_{\omega,\mu} - \big(z + \mathbb{X}^{\varepsilon}_{\omega,\mu}(z)\big)\mathbf{b}^{\varepsilon,k}_{\omega,\mu}\right| \leq c\left|\frac{\varepsilon}{\varepsilon_0}\right|^{1+\alpha} J. \tag{5.20}$$

Define

$$\mathbb{V}(z) \equiv \frac{V(\varepsilon z) - V(0) - \varepsilon\mathbf{a}^{\varepsilon,k}_{\omega,\mu} - (\varepsilon z + \varepsilon\mathbb{X}^1_{\omega,\mu}(z))\mathbf{b}^{\varepsilon,k}_{\omega,\mu}}{\varepsilon^{1+\alpha}J} \quad \text{in } B_{1/\varepsilon_0(0)}.$$

Equation (5.20) implies that $\mathbb{V}$ satisfies (5.8) with $\nu = 1$. Lemma 3.7 implies

$$\left\|\mathbf{K}^{\varepsilon,\varepsilon}_{\omega^2,\mu}\nabla\mathbb{V}\right\|_{L^\infty(B_{1/2\varepsilon_0}(0))} \leq c. \tag{5.21}$$

Equations (5.21), $(5.16)_1$, (5.6) imply (5.19). ■

Remark 5.5. Let ε_0 be same as in Lemma 5.4. If $\mu \in [\varepsilon_0, 1]$, equation (5.15) is a uniform elliptic equation by (A2). By [32], we know the following.

> *Under* (A2) *and* $\mu \in [\varepsilon_0, 1]$, *any solution of* (5.15) *satisfies* (5.19).

By Lemma 3.7, we see the following.

> *Under* (A2) *and* $\varepsilon \in [\varepsilon_0, 1]$, *any solution of* (5.15) *satisfies* (5.19).

Combining with Lemma 5.4, we conclude the following.

> *Under* (A2)*, any solution of* (5.15) *satisfies* (5.19).

6. Maximum norm of corrector functions

We now study Green's functions and the corrector functions of the strongly elliptic operators $-\nabla\cdot(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla)$ in $\mathcal{Q}^{\varepsilon,r}$. We first derive the pointwise estimates of Green's functions; then, consider the maximum norm of the corrector functions.

6.1. Estimates for Green's functions

Recall a weak Lebesgue L^2 space [6, 15]:

$$L^{2,\infty}(\mathbf{D}) \equiv \big\{\zeta : \mathbf{D}(\subset \mathbb{R}^2) \to \mathbb{R} \mid \|\zeta\|_{L^{2,\infty}(\mathbf{D})} < \infty\big\},$$

where

$$\|\zeta\|_{L^{2,\infty}(\mathbf{D})} \equiv \sup_{t\geq 0} t\big|\{x \in \mathbf{D} \mid |\zeta(x)| \geq t\}\big|^{1/2}.$$

Lemma 6.1. *Assume* (A1)*,* $\omega \in (1,\infty)$*,* $\frac{\varepsilon}{r}, r, \mu \in (0,1)$*, and* $x \in \mathcal{Q}^r$*. A solution of*

$$\begin{cases} -\nabla_y \cdot \big(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu} \nabla_y \mathbf{G}^{\varepsilon,r}(x,y) + \frac{x-y}{2\pi|x-y|^2}\big) = 0 & \text{in } \mathcal{Q}^r, \\ \mathbf{G}^{\varepsilon,r}(x,\cdot) = 0 & \text{on } \partial\mathcal{Q}^r \end{cases} \tag{6.1}$$

exists uniquely in $L^{2,\infty}(\mathcal{Q}^r)$ *and satisfies, for any relatively open set* $\mathcal{O} \subset \mathcal{Q}^r$*,*

$$\big\|\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu} \nabla \mathbf{G}^{\varepsilon,r}(x,\cdot)\big\|_{L^{\mathbf{d}}(\mathcal{O})} < c|\mathcal{O}|^{\frac{1}{\mathbf{d}}-\frac{1}{2}}, \tag{6.2}$$

where $\mathbf{d} \in [1,2)$ *and c is independent of* ε*,* μ*,* ω*,* r*,* x*,* $\mathcal{O}$*. See* (3.33) *for* $\mathcal{Q}^r$*.*

Proof. Uniqueness of the solution $\mathbf{G}^{\varepsilon,r}(x,y)$ of (6.1) is obvious.

Step I. Consider the following problem:

$$\begin{cases} -\nabla \cdot \big(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu} \nabla \Psi + \mathbb{Q}\big) = 0 & \text{in } \mathcal{Q}^r, \\ \Psi = 0 & \text{on } \partial\mathcal{Q}^r. \end{cases} \tag{6.3}$$

Let p_* be the positive number in Lemma 3.11. If $\frac{1}{p_*} + \frac{1}{p'_*} = 1$ and $p \in (p'_*, p_*)$, Corollary 3.13 and Lemma 3.14 imply

$$\big\|\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu} \nabla \Psi\big\|_{L^p(\mathcal{Q}^r)} \leq c\|\mathbb{Q}\|_{L^p(\mathcal{Q}^r)}, \tag{6.4}$$

where c is independent of ε, μ, ω, r. Define a map $\mathbb{T}(\mathbb{Q}) = \mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla\Psi$, where Ψ is the solution of (6.3). By (6.4), $\mathbb{T} : L^p(\mathcal{Q}^r) \to L^p(\mathcal{Q}^r)$ for $p \in (p'_*, p_*)$ are bounded linear operators. By [15, Lemma 1], the map $\mathbb{T} : L^{2,\infty}(\mathcal{Q}^r) \to L^{2,\infty}(\mathcal{Q}^r)$ is a bounded linear operator.

Step II. Let us take $\mathbb{Q} = \frac{x-y}{2\pi|x-y|^2} \in L^{2,\infty}(\mathcal{Q}^r)$. The solution Ψ of (6.3) satisfies $\Psi = \mathbf{G}^{\varepsilon,r}(x,\cdot)$ and, by (6.4),

$$\big\|\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu} \nabla \mathbf{G}^{\varepsilon,r}(x,\cdot)\big\|_{L^{2,\infty}(\mathcal{Q}^r)} < c, \tag{6.5}$$

where c is independent of $\varepsilon, \mu, \omega, r, x$. Equation (6.2) follows from [6, (2.2)] and (6.5). ■

Lemma 6.2. *Assume* (A1)*,* $\omega \in (1,\infty)$*, and* $\frac{\varepsilon}{r}, r, \mu \in (0,1)$*. The solution of* (6.1) *satisfies*

$$\big|\mathbf{G}^{\varepsilon,r}(x,y)\big| \leq c\big(1 + |\ln|x-y||\big) \quad \text{for } x, y \in \mathcal{Q}^r, \tag{6.6}$$

where c is independent of ε*,* μ*,* ω*,* r*.*

Proof. This lemma is proved by following the argument for [6, Theorem 2].

Step I. By (6.2) and Poincaré inequality,

$$\left\|\mathbf{G}^{\varepsilon,r}(x,\cdot)\right\|_{L^1(\mathcal{Q}^{,r})} < c\left\|\nabla\mathbf{G}^{\varepsilon,r}(x,\cdot)\right\|_{L^1(\mathcal{Q}^{,r})} \le c, \tag{6.7}$$

where c is a constant independent of ε, μ, ω, r, x. For each point $x \in \mathcal{Q}^{,r}$, extend $\mathbf{G}^{\varepsilon,r}(x,\cdot)$ from $\mathcal{Q}^{,r}$ to $\mathbb{R}^2$ by 0 and define

$$\psi(h) \equiv \frac{1}{2\pi h}\int_{\partial B_h(x)} |\mathbf{G}^{\varepsilon,r}(x,y)|\,dy \quad \text{for any } h > 0.$$

Suppose $t > s > 0$, by (6.2),

$$\begin{aligned}|\psi(t)-\psi(s)| &\le \int_s^t |\psi'(h)|dh \le \int_s^t \frac{1}{2\pi h}\int_{\partial B_h(x)} |\nabla\mathbf{G}^{\varepsilon,r}(x,y)|\,dydh \\ &\le \frac{1}{2\pi s}\int_{B_t(x)\setminus B_s(x)} |\nabla\mathbf{G}^{\varepsilon,r}(x,y)|dy \le c\frac{|B_t(x)\setminus B_s(x)|^{\frac{1}{2}}}{s} \le c\frac{t}{s},\end{aligned} \tag{6.8}$$

where c is independent of ε, μ, ω, r, x, s, t.

Next, we claim $\psi(h) \le c(1+|\ln h|)$, where c is independent of ε, μ, ω, r, x, h. ■

Proof of the claim. (i) If $h \in (\frac{1}{2}, 1]$, then (6.8) can be written as

$$s\psi(h) \le s\psi(s) + ch \quad \text{for } s < \frac{1}{2}. \tag{6.9}$$

Integrate (6.9) with respect to s over $(\frac{1}{4}, \frac{1}{2})$ to get, by (6.7),

$$\frac{3}{32}\psi(h) \le \int_{1/4}^{1/2} s\psi(s)ds + ch \le c + ch \le c, \tag{6.10}$$

where c is bounded independent of ε, μ, ω, r, x, h.

(ii) If $h > 1$, we find $m \in \mathbb{N}$ such that $\frac{1}{2} < \frac{h}{2^m} \le 1$. Then, m is the integer part of $\frac{\ln h}{\ln 2}$. Apply (6.8) with $t = \frac{h}{2^j}$, $s = \frac{h}{2^{j+1}}$ to see

$$\psi\left(\frac{h}{2^j}\right) - \psi\left(\frac{h}{2^{j+1}}\right) \le \left|\psi\left(\frac{h}{2^j}\right) - \psi\left(\frac{h}{2^{j+1}}\right)\right| \le c, \tag{6.11}$$

where c is independent of ε, μ, ω, r, x, h, j. Sum up (6.11) for $0 \le j \le m-1$ and apply (6.10) to get

$$\psi(h) \le \psi\left(\frac{h}{2^m}\right) + c\,m \le c(1+\ln h), \tag{6.12}$$

where c is independent of ε, μ, ω, r, x, h.

(iii) If $h \le \frac{1}{2}$, we find $m \in \mathbb{N}$ such that $\frac{1}{2} < 2^m h \le 1$. Then, m is the integer part of $-1 - \frac{\ln h}{\ln 2}$. Apply (6.8) with $t = 2^{j+1}h$, $s = 2^j h$ to see

$$\psi(2^j h) - \psi(2^{j+1}h) \le \left|\psi(2^j h) - \psi(2^{j+1}h)\right| \le c. \tag{6.13}$$

Sum up (6.13) for $0 \leq j \leq m-1$ to get

$$\psi(h) \leq c(1 + |\ln h|), \tag{6.14}$$

where c is independent of ε, μ, ω, r, x, h. Equations (6.10), (6.12), and (6.14) imply the claim.

Step II. Derive a bound on the L^∞ norm of $\mathbf{G}^{\varepsilon,r}(x,\cdot)$ in any annulus.

By the result in Step I, for any $\rho > 0$,

$$\begin{aligned}
&\|\mathbf{G}^{\varepsilon,r}(x,\cdot)\|_{L^1(B_{3\rho}(x)\setminus B_{2\rho}(x))} \\
&\quad = 2\pi \int_{2\rho}^{3\rho} h\psi(h)dh \leq c\int_{2\rho}^{3\rho} h(1+|\ln h|)dh \\
&\quad \leq \begin{cases} c\int_{2\rho}^{3\rho} h(1+\ln h)dh \leq c\rho^2(1+|\ln \rho|) & \text{if } \rho \geq \frac{1}{2}, \\ c\int_{2\rho}^{3\rho} h(1+|\ln h|)dh \leq c\rho^2(1+|\ln \rho|) & \text{if } \frac{1}{3} < \rho < \frac{1}{2}, \\ c\int_{2\rho}^{3\rho} h(1-\ln h)dh \leq c\rho^2(1+|\ln \rho|) & \text{if } \rho \leq \frac{1}{3}, \end{cases}
\end{aligned} \tag{6.15}$$

where c is independent of ε, μ, ω, r, x, ρ. Let $\widehat{\eta} \in C_0^\infty(\mathbb{R}^2)$ be a bell-shaped function satisfying $|\nabla\widehat{\eta}| \leq \frac{c}{\rho}$, $\widehat{\eta} = 0$ in $\mathbb{R}^2 \setminus B_{3\rho}(x)$, and $\widehat{\eta} = 1$ in $B_{2\rho}(x)$. By Sobolev embedding theorem [19], $\phi = \mathbf{G}^{\varepsilon,r}(x,\cdot)\,\widehat{\eta}$ satisfies

$$\begin{aligned}
&\|\mathbf{G}^{\varepsilon,r}(x,\cdot)\|_{L^2(B_{2\rho}(x)\setminus B_{\rho/2}(x))} \\
&\quad \leq c\|\phi\|_{L^2(\mathbb{R}^2)} \leq c\|\nabla\phi\|_{L^1(\mathbb{R}^2)} \\
&\quad \leq c\|\nabla\mathbf{G}^{\varepsilon,r}(x,\cdot)\|_{L^1(B_{3\rho}(x))} + \frac{c}{\rho}\|\mathbf{G}^{\varepsilon,r}(x,\cdot)\|_{L^1(B_{3\rho}(x)\setminus B_{2\rho}(x))},
\end{aligned} \tag{6.16}$$

where c is independent of ε, μ, ω, r, x, ρ. By (6.2) with $\mathbf{d} = 1$, we know

$$\|\nabla\mathbf{G}^{\varepsilon,r}(x,\cdot)\|_{L^1(B_{3\rho}(x))} \leq c\rho, \tag{6.17}$$

where c is independent of ε, μ, ω, r, x, ρ. Equation (6.15) implies

$$\rho^{-1}\|\mathbf{G}^{\varepsilon,r}(x,\cdot)\|_{L^1(B_{3\rho}(x)\setminus B_{2\rho}(x))} \leq c\rho(1+|\ln \rho|), \tag{6.18}$$

where c is independent of ε, μ, ω, r, x, ρ. Therefore, (6.16)–(6.18) imply

$$\|\mathbf{G}^{\varepsilon,r}(x,\cdot)\|_{L^2(B_{2\rho}(x)\setminus B_{\rho/2}(x))} \leq c\rho(1+|\ln \rho|), \tag{6.19}$$

where c is independent of ε, μ, ω, r, x, ρ. By Lemma 4.3, (6.19), and (6.2),

$$\begin{aligned}
\sup_{B_{3\rho/2}(x)\setminus B_\rho(x)} |\mathbf{G}^{\varepsilon,r}(x,\cdot)| &\leq \frac{c}{\rho}\|\mathbf{G}^{\varepsilon,r}(x,\cdot)\|_{L^2(B_{2\rho}(x)\setminus B_{\rho/2}(x))} \\
&\quad + c\rho^{1-\frac{2}{\mathbf{d}}}\|\nabla\mathbf{G}^{\varepsilon,r}(x,\cdot)\|_{L^{\mathbf{d}}(B_{2\rho}(x)\setminus B_{\rho/2}(x))} \leq c(1+|\ln \rho|),
\end{aligned}$$

where c is independent of ε, μ, ω, r, x, ρ. So, we prove (6.6). ∎

Remark 6.3. By Lemmas 6.1–6.2, the solution of (6.1) is the Green function of differential operator $-\nabla\cdot(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla)$ in $\mathcal{Q}^{,r}$. Also, $\mathbf{G}^{\varepsilon,r}(x,y)=\mathbf{G}^{\varepsilon,r}(y,x)$ for any $x,y\in\mathcal{Q}^{,r}$ (see [24, p. 62]).

Remark 6.4. Let us recall some results [29, pp. 1579–1580] and [12]. Let $\mathbf{D}$ be a bounded Lipschitz domain in $\mathbb{R}^2$.

(i) If $\varphi\in L^1(\mathbf{D})$, $x\in\mathbf{D}$, and $\rho>0$, define

$$\overline{\varphi}_{x,\rho}\equiv\begin{cases}(\varphi)_{B_\rho(x)} & \text{if } \operatorname{dist}(B_\rho(x),\partial\mathbf{D})>0,\\ 0 & \text{otherwise.}\end{cases}$$

(ii) BMO($\mathbf{D}$) is defined as

$$\mathrm{BMO}(\mathbf{D})\equiv\left\{\zeta\in L^1(\mathbf{D})\mid \|\zeta\|_{*,\partial\mathbf{D}}\equiv\sup_{x\in\overline{\mathbf{D}},\rho>0}\fint_{B_\rho(x)\cap\mathbf{D}}\left|\zeta-\overline{\zeta}_{x,\rho}\right|dy<\infty\right\}.$$

(iii) $\mathcal{A}\in L^\infty(\mathbf{D})$ is an atom for $\mathbf{D}$ if there is a $\rho>0$ such that

$$\begin{cases}\operatorname{supp}(\mathcal{A})\subset B_\rho(x)\cap\mathbf{D},\\ \|\mathcal{A}\|_{L^\infty(\mathbf{D})}\le\frac{1}{|B_\rho(x)\cap\mathbf{D}|},\\ \overline{\mathcal{A}}_{x,\rho}=0.\end{cases}\tag{6.20}$$

$g\in H^{1,\infty}(\mathbf{D})$ (atomic Hardy space) if $g=\sum_{i=1}^\infty\gamma_i\mathcal{A}_i$, where $\{\mathcal{A}_i\}_{i=1}^\infty$ are atoms and $\{\gamma_i\}_{i=1}^\infty\in\ell^1$ sequence space. Also, $\|g\|_{H^{1,\infty}(\mathbf{D})}\equiv\inf\sum_{i=1}^\infty|\gamma_i|$, where the infimum is taken over all representations of g.

(iv) BMO($\mathbf{D}$) is the dual of $H^{1,\infty}(\mathbf{D})$ [12, 29]. So, if $\zeta\in\mathrm{BMO}(\mathbf{D})$, then

$$\|\zeta\|_{*,\partial\mathbf{D}}=\sup\left\{\int_{\mathbf{D}}\zeta\,\mathcal{A}\,dy\mid\mathcal{A}\text{ is an atom for }\mathbf{D}\right\}.\tag{6.21}$$

(v) By John–Nirenberg inequality [29] and [12, Theorem 3.6],

$$\mathrm{BMO}(\mathbf{D})\subset L^q(\mathbf{D})\subset H^{1,\infty}(\mathbf{D})\quad\text{for any } q\in(1,\infty).\tag{6.22}$$

Lemma 6.5. *Assume* (A1)*,* $\omega\in(1,\infty)$*, and* $\frac{\varepsilon}{r},r,\mu\in(0,1)$*. If* $\mathcal{A}$ *is an atom for* $\mathcal{Q}^{,r}$ *and* Ψ *is a solution of*

$$\begin{cases}-\nabla\cdot\left(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla\Psi\right)=\mathcal{A} & \text{in } \mathcal{Q}^{,r},\\ \Psi=0 & \text{on } \partial\mathcal{Q}^{,r},\end{cases}\tag{6.23}$$

then $\|\Psi\|_{L^\infty(\mathcal{Q}^{,r})}\le c$*, where* c *is a constant independent of* ε*,* μ*,* ω*,* r*.*

Proof. Let p_* be same as Lemma 3.11 and c a constant independent of ε, μ, ω, r. Set $p\in(2,\infty)$, $m=\frac{2p}{2+p}$, $\frac{1}{p}+\frac{1}{p'}=1$, and $\frac{1}{m}+\frac{1}{m'}=1$. Any solution of (6.23) satisfies

$$\begin{cases}\|\nabla\Psi\|_{L^p(\mathcal{Q}^{,r})}\le c\|\mathcal{A}\|_{L^m(\mathcal{Q}^{,r})},\\ \|\Psi\|_{L^{m'}(\mathcal{Q}^{,r})}\le c\|\mathcal{A}\|_{W^{-1,p'}(\mathcal{Q}^{,r})}.\end{cases}\tag{6.24}$$

Indeed, $(6.24)_1$ is from Corollary 3.16. Equation $(6.24)_2$ is from Lemma 3.15 and Sobolev embedding theorem [19].

If $z \in \Omega^{,r}$ and $\mathcal{A}$ is supported in $B_\rho(x) \cap \Omega^{,r}$, by Hölder inequality and Sobolev embedding theorem [19] and $m = \frac{2p}{2+p}$,

$$|\Psi(z)| \le \left|\Psi(z) - (\Psi)_{B_\rho(z)\cap\Omega^{,r}}\right| + \left|(\Psi)_{B_\rho(z)\cap\Omega^{,r}}\right| \\ \le c\rho^{1-\frac{2}{p}} \|\nabla\Psi\|_{L^p(B_\rho(z)\cap\Omega^{,r})} + \rho^{\frac{2}{p}-1}\|\Psi\|_{L^{m'}(B_\rho(z)\cap\Omega^{,r})}. \tag{6.25}$$

By $(6.20)_{1,2}$ and $(6.24)_1$,

$$\|\nabla\Psi\|_{L^p(\Omega^{,r})} \le c\|\mathcal{A}\|_{L^m(B_\rho(z)\cap\Omega^{,r})} \le c\rho^{\frac{2}{p}-1}. \tag{6.26}$$

For any $\zeta \in W_0^{1,p}(\Omega^{,r})$, by $(6.20)_3$ and Sobolev embedding theorem [19],

(i) if $B_\rho(x) \Subset \Omega^{,r}$, then

$$\left|\int_{\Omega^{,r}} \mathcal{A}\,\zeta\,dz\right| = \left|\int_{B_\rho(z)} \mathcal{A}\big(\zeta - (\zeta)_{B_\rho(x)}\big)dz\right| \\ \le \|\zeta - (\zeta)_{B_\rho(x)}\|_{L^\infty(B_\rho(z))} \le \rho^{1-\frac{2}{p}}\|\nabla\zeta\|_{L^p(B_\rho(z))}; \tag{6.27}$$

(ii) if $\mathrm{dis}(B_\rho(x), \Omega^{,r}) = 0$, then

$$\left|\int_{\Omega^{,r}} \mathcal{A}\,\zeta\,dz\right| \le \|\zeta\|_{L^\infty(B_\rho(z)\cap\Omega^{,r})} \le \rho^{1-\frac{2}{p}}\|\nabla\zeta\|_{L^p(B_\rho(z)\cap\Omega^{,r})}. \tag{6.28}$$

Equations (6.27)–(6.28) and $(6.24)_2$ imply

$$\|\Psi\|_{L^{m'}(\Omega^{,r})} \le c\|\mathcal{A}\|_{W^{-1,p'}(\Omega^{,r})} \le c\rho^{1-\frac{2}{p}}. \tag{6.29}$$

The lemma follows from (6.25), (6.26), (6.29). ■

Lemma 6.6. *Assume* (A1), $\omega \in (1,\infty)$, *and* $\frac{\varepsilon}{r}, r, \mu \in (0,1)$. *The solution of* (6.1) *satisfies, for any* $x \in \Omega^{,r}$,

$$\left\|\mathbf{G}^{\varepsilon,r}(x,\cdot)\right\|_{*,\partial\Omega^{,r}} \le c, \tag{6.30}$$

where c *is a constant independent of* ε, μ, ω, r, x.

Proof. This is proved by following the argument for [29, Theorem 4.1]. Let c be a constant independent of ε, μ, ω, r, x. If $g \in L^p(\Omega^{,r})$ for $p > 1$, consider the equation

$$\begin{cases} -\nabla\cdot\left(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla\zeta\right) = g & \text{in } \Omega^{,r}, \\ \zeta = 0 & \text{on } \partial\Omega^{,r}. \end{cases} \tag{6.31}$$

For a fixed $x \in \Omega^{,r}$ and $\rho > 0$, let Ψ be the solution of (6.31) with $g = f \in L^p(\Omega^{,r})$ for $p > 1$, and let $\mathfrak{G}_\rho(x,\cdot)$ denote the solution of (6.31) with $g = \frac{\chi_{B_\rho(x)\cap\Omega^{,r}}}{|B_\rho(x)\cap\Omega^{,r}|} \in L^\infty(\Omega^{,r})$. By Green's formula, Ψ and $\mathfrak{G}_\rho(x,\cdot)$ satisfy

$$⨍_{B_\rho(x)\cap\Omega^{,r}} \Psi(y)\,dy = \int_{\Omega^{,r}} \mathfrak{G}_\rho(x,y)f(y)\,dy. \tag{6.32}$$

If f in (6.32) is an atom, then (6.21), (6.32), and Lemma 6.5 imply

$$\begin{aligned}
&\|\mathfrak{G}_\rho(x,\cdot)\|_{*,\partial\mathcal{Q}^{,r}} \\
&\leq \sup_{f\in\{\text{atom for }\mathcal{Q}^{,r}\}} \left| \int_{\mathcal{Q}^{,r}} \mathfrak{G}_\rho(x,y) f(y)dy \right| \\
&\leq \sup_{\Psi\in\{\text{solution of (6.31) with } g=f\in\{\text{atom for }\mathcal{Q}^{,r}\}\}} \left| \fint_{B_\rho(x)\cap\mathcal{Q}^{,r}} \Psi(y)dy \right| \leq c. \qquad (6.33)
\end{aligned}$$

By Alaoglu's theorem [14], there is a sequence $\{\rho_j\}$ with $\lim_{j\to\infty}\rho_j = 0$ and a function $\widetilde{\mathfrak{G}}(x,\cdot) \in \mathrm{BMO}(\mathcal{Q}^{,r})$ so that $\mathfrak{G}_{\rho_j}(x,\cdot)$ converges to $\widetilde{\mathfrak{G}}(x,\cdot)$ in the weak-star topology of $\mathrm{BMO}(\mathcal{Q}^{,r})$.

If $f \in L^p(\mathcal{Q}^{,r})$ for $p > 1$ in (6.32), the solution Ψ of (6.31) with $g = f$ is Hölder continuous by Corollary 3.16. The left-hand side of (6.32) converges to $\Psi(x)$ as $\rho \to 0$. By (6.22), equation (6.32) implies, as $\rho \to 0$,

$$\Psi(x) = \int_{\mathcal{Q}^{,r}} \widetilde{\mathfrak{G}}(x,y) f(y)\, dy \quad \text{for } x \in \mathcal{Q}^{,r}.$$

So, $\widetilde{\mathfrak{G}}(x,\cdot)$ is a Green function of (6.1) with pole at x. Unique existence of the Green function of (6.1) implies $\widetilde{\mathfrak{G}}(x,y) = \mathbf{G}^{\varepsilon,r}(x,y)$ (see [24] and Remark 6.3). So, (6.33) implies that (6.30) is true.

From the above argument, any sequence $\{\mathfrak{G}_{\rho_k}(x,\cdot)\}$ with $\lim_{k\to\infty}\rho_k = 0$ gives a subsequence converging to the Green function. Therefore, the entire sequence $\{\mathfrak{G}_\rho(x,\cdot)\}$ converges to $\mathbf{G}^{\varepsilon,r}(x,\cdot)$ in the weak-star topology of $\mathrm{BMO}(\mathcal{Q}^{,r})$ as $\rho \to 0$. ■

Lemma 6.7. *Under* (A1)–(A2) *and* $\frac{\varepsilon}{r}, r, \alpha \in (0,1)$*, the solution of* (6.1) *satisfies, for any* $x, y, z \in \mathcal{Q}^{,r}$,

$$\begin{cases} \left|\mathbf{G}^{\varepsilon,r}(x,y) - \mathbf{G}^{\varepsilon,r}(x,z)\right| \leq c\frac{|y-z|^\alpha}{|x-y|^\alpha} & \text{if } |y-z| \leq \frac{|x-y|}{4}, \\ \left|\mathbf{G}^{\varepsilon,r}(y,x)\right| = \left|\mathbf{G}^{\varepsilon,r}(x,y)\right| \leq c\frac{|\xi_r^y|^\alpha}{|x-y|^\alpha} & \text{if } \xi_r^y \leq \frac{|x-y|}{4}, \end{cases} \qquad (6.34)$$

where c is independent of $\varepsilon, \mu\ (= \omega^{-1}), r, \alpha$. Here, $\xi_r^y \equiv \mathrm{dist}(y, \partial\mathcal{Q}^{,r})$ is the distance from y to the boundary $\partial\mathcal{Q}^{,r}$.

Proof. Clearly, $(6.34)_2$ follows from $(6.34)_1$ and Remark 6.3. So, we only need to show $(6.34)_1$. Let c be a constant independent of $\varepsilon, \mu\ (= \omega^{-1}), r, \alpha$. For $x, y \in \mathcal{Q}^{,r}$, set $h \equiv |x-y|$. We consider (i) $\xi_r^y < \frac{h}{4}$ and (ii) $\xi_r^y \geq \frac{h}{4}$ separately.

(i) $\xi_r^y < \frac{h}{4}$ case. Define $\mathbb{E}(z) \equiv \mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}(y+hz)$ and $\psi(z) \equiv \mathbf{G}^{\varepsilon,r}(x, y+hz)$. Then,

$$\begin{cases} -\nabla\cdot(\mathbb{E}\nabla\psi) = 0 & \text{in } B_{3/4}(0) \cap (\mathcal{Q}^{,r} - \{y\})/h, \\ \psi = 0 & \text{on } B_{3/4}(0) \cap (\partial\mathcal{Q}^{,r} - \{y\})/h. \end{cases}$$

Since $\mathbf{G}^{\varepsilon,r}(x,\cdot) \in \mathrm{BMO}(\mathcal{Q}^{,r})$ by Lemma 6.6, $\psi \in \mathrm{BMO}((\mathcal{Q}^{,r} - \{y\})/h)$. So, (6.22) implies

$$\|\psi\|_{L^q(B_{2/3}(0)\cap(\mathcal{Q}^{,r}-\{y\})/h)} \leq c \quad \text{for any } q \in (1,\infty). \qquad (6.35)$$

If $\frac{\varepsilon}{rh} > 1$, then [19, Theorems 4.15 and 8.17], (II) of Corollary 3.4, Lemma 3.7, and (6.35) imply

$$\|\nabla\psi\|_{L^\infty(B_{1/2}(0)\cap(\mathcal{Q}^{,r}-\{y\})/h)} \le c\|\psi\|_{L^2(B_{2/3}(0)\cap(\mathcal{Q}^{,r}-\{y\})/h)} \le c. \tag{6.36}$$

If $\frac{\varepsilon}{rh} \le 1$, then Lemma 3.11 with $\tau = 0$, (A2), Lemma 4.3, (6.35), and (6.2) imply

$$\begin{aligned}
&[\psi]_{C^\alpha(B_{1/4}(0)\cap(\mathcal{Q}^{,r}-\{y\})/h)} \\
&\quad \le c\|\psi\|_{L^\infty(B_{1/2}(0)\cap(\mathcal{Q}^{,r}-\{y\})/h)} \\
&\quad \le c\|\psi\|_{L^1(B_{3/4}(0)\cap(\mathcal{Q}^{,r}-\{y\})/h)} + \|\nabla\psi\|_{L^{\mathbf{d}}(B_{3/4}(0)\cap(\mathcal{Q}^{,r}-\{y\})/h)} \le c,
\end{aligned} \tag{6.37}$$

where $\mathbf{d} \in (1,2)$. So, $(6.34)_1$ is proved for $\xi_r^y < \frac{h}{4}$ case.

(ii) $\xi_r^y \ge \frac{h}{4}$ case. Define $\mathbb{E}(z) \equiv \mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}(y+hz)$ and $\psi(z) \equiv \mathbf{G}^{\varepsilon,r}(x, y+hz) - \mathbf{G}^{\varepsilon,r}(x,y)$. Then,

$$-\nabla\cdot(\mathbb{E}\nabla\psi) = 0 \quad \text{in } B_{1/4}(0)\cap(\mathcal{Q}^{,r}-\{y\})/h.$$

Repeat the arguments of (6.35)–(6.37) and apply (II) of Corollary 3.4, Lemma 3.7, Remark 5.5, Lemma 4.3, and (6.2) to get $(6.34)_1$ for $\xi_r^y \ge \frac{h}{4}$ case. ■

Lemma 6.8. *Under* (A1)–(A2) *and* $\frac{\varepsilon}{r}, r, \alpha \in (0,1)$*, the solution of* (6.1) *satisfies, for any* $x, y \in \mathcal{Q}^{,r}$,

$$\begin{cases} |\mathbf{G}^{\varepsilon,r}(x,y)| \le c\frac{|\xi_r^x|^\alpha|\xi_r^y|^\alpha}{|x-y|^{2\alpha}} & \text{if } \xi_r^x, \xi_r^y \le \frac{|x-y|}{8}, \\ |\nabla_y\mathbf{G}^{\varepsilon,r}(x,y)| \le c\frac{r}{\varepsilon}\frac{|\xi_r^x|^\alpha\max\{|\frac{\varepsilon}{r}|^\alpha, |\xi_r^y|^\alpha\}}{|x-y|^{2\alpha}} & \text{if } \xi_r^x, \xi_r^y, \frac{\varepsilon}{r} \le \frac{|x-y|}{16}, \end{cases} \tag{6.38}$$

where c *is a constant independent of* $\varepsilon, \mu\ (=\omega^{-1}), r, \alpha$*. See Lemma* 6.7 *for* ξ_r^x, ξ_r^y.

Proof. Let c be a constant independent of $\varepsilon, \mu\ (=\omega^{-1}), r, \alpha$ and define $h \equiv |x-y|$.

Step I. Proof of $(6.38)_1$. If $\tilde{x} \in B_{h/4}(x)\cap\mathcal{Q}^{,r}$, then $\xi_r^y \le \frac{h}{8}$ implies $\xi_r^y \le \frac{1}{4}|\tilde{x}-y|$. Then, $(6.34)_2$ implies

$$|\mathbf{G}^{\varepsilon,r}(\tilde{x},y)| \le c\frac{|\xi_r^y|^\alpha}{|\tilde{x}-y|^\alpha} \le c\frac{|\xi_r^y|^\alpha}{h^\alpha} \quad \text{for } \tilde{x} \in B_{h/4}(x)\cap\mathcal{Q}^{,r}. \tag{6.39}$$

Next, we trace the proof of Lemma 6.7. Let us define $\mathbb{E}(z) \equiv \mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}(x+hz)$ and $\psi(z) \equiv \mathbf{G}^{\varepsilon,r}(x+hz, y)$. Since $\xi_r^x \le \frac{h}{8}$,

$$\begin{cases} -\nabla\cdot(\mathbb{E}\nabla\psi) = 0 & \text{in } B_{1/4}(0)\cap(\mathcal{Q}^{,r}-\{x\})/h, \\ \psi = 0 & \text{on } B_{1/4}(0)\cap(\partial\mathcal{Q}^{,r}-\{x\})/h. \end{cases} \tag{6.40}$$

If $\frac{\varepsilon}{rh} > 1$, then [19, Theorem 4.15], (II) of Corollary 3.4, Lemma 3.7, and (6.39)–(6.40) imply

$$\|\nabla\psi\|_{L^\infty(B_{1/8}(0)\cap(\mathcal{Q}^{,r}-\{x\})/h)} \le c\|\psi\|_{L^\infty(B_{1/4}(0)\cap(\mathcal{Q}^{,r}-\{x\})/h)} \le c\frac{|\xi_r^y|^\alpha}{h^\alpha}. \tag{6.41}$$

If $\frac{\varepsilon}{rh} \le 1$, then Lemma 3.11 with $\tau = 0$, (A2), and (6.39)–(6.40) imply

$$[\psi]_{C^\alpha(B_{1/8}(0)\cap(\mathcal{Q}^{,r}-\{x\})/h)} \le c\|\psi\|_{L^\infty(B_{1/4}(0)\cap(\mathcal{Q}^{,r}-\{x\})/h)} \le c\frac{|\xi_r^y|^\alpha}{h^\alpha}. \tag{6.42}$$

Equation $(6.38)_1$ follows from Remark 6.3, (6.41)–(6.42), and change of variables.

Step II. Proof of $(6.38)_2$. If $\xi_r^x, \xi_r^y, \frac{\varepsilon}{r} \le \frac{h}{16}$ and $\tilde{y} \in B_{\frac{\varepsilon}{2r}}(y) \cap \mathcal{Q}^{,r}$, we see $\xi_r^x, \xi_r^{\tilde{y}} \le \frac{|x-\tilde{y}|}{8}$. For any $\alpha \in (0,1)$ and $\tilde{y} \in B_{\frac{\varepsilon}{2r}}(y) \cap \mathcal{Q}^{,r}$, by $(6.38)_1$,

$$|\mathbf{G}^{\varepsilon,r}(x,\tilde{y})| \le c\frac{|\xi_r^x|^\alpha|\xi_r^{\tilde{y}}|^\alpha}{|x-\tilde{y}|^{2\alpha}} \le \begin{cases} c\frac{|\xi_r^x|^\alpha|\xi_r^y|^\alpha}{h^{2\alpha}} & \text{if } \xi_r^y \ge \frac{\varepsilon}{r}, \\ c\left|\frac{\varepsilon}{r}\right|^\alpha\frac{|\xi_r^x|^\alpha}{h^{2\alpha}} & \text{if } \xi_r^y < \frac{\varepsilon}{r}. \end{cases}$$

(II) of Corollary 3.4 and Lemma 3.7 imply

$$\begin{aligned}\left\|\nabla_y\mathbf{G}^{\varepsilon,r}(x,\cdot)\right\|_{L^\infty(B_{\frac{\varepsilon}{4r}}(y)\cap\mathcal{Q}^{,r})} &\le c\frac{r}{\varepsilon}\|\mathbf{G}^{\varepsilon,r}(x,\cdot)\|_{L^\infty(B_{\frac{\varepsilon}{2r}}(y)\cap\mathcal{Q}^{,r})} \\ &\le c\frac{r}{\varepsilon}\frac{|\xi_r^x|^\alpha\max\{|\frac{\varepsilon}{r}|^\alpha, |\xi_r^y|^\alpha\}}{h^{2\alpha}}.\end{aligned}$$

So, $(6.38)_2$ is proved. ∎

6.2. Corrector functions

Assume (3.32)–(3.34) and $0 \in \partial\Omega$. Let $\eta \in C_0^\infty(\mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0))$ be a bell-shaped function satisfying $\eta \in [0,1]$ and $\eta = 1$ in $[-2,2]^2$. Note $\eta = 0$ on $\partial\mathcal{Q}^{,r} \cap \partial\mathcal{Q}_\dagger^{\varepsilon,r}$ (see (3.33)). If $0 < \varepsilon \le r \le 1$, find $\mathbb{W}_{\omega,\mu,2}^{\varepsilon,r} \in H^1(\mathcal{Q}^{,r})$ by solving

$$\begin{cases} -\nabla\cdot\left(\mathbf{K}_{\omega^2,\mu}^{\varepsilon,r}\left(\nabla\mathbb{W}_{\omega,\mu,2}^{\varepsilon,r} + \vec{e}_2\right)\right) = 0 & \text{in } \mathcal{Q}^{,r}, \\ \mathbb{W}_{\omega,\mu,2}^{\varepsilon,r} = \left(1-\eta\right)\mathbb{X}_{\omega,\mu,2}^{\varepsilon/r} & \text{on } \partial\mathcal{Q}^{,r}, \end{cases} \tag{6.43}$$

where $\vec{e}_2$ is the unit vector in the second coordinate direction. See (5.7) for $\mathbb{X}_{\omega,\mu,2}^{\varepsilon/r}$ and (3.33) for $\mathbf{d}_4$, $\mathbf{d}_5$.

Lemma 6.9. *Under* (A1)–(A2) *and* $\frac{\varepsilon}{r}$, $r \in (0,1)$, *a solution of* (6.43) *exists uniquely in* $H^1(\mathcal{Q}^{\varepsilon,r})$. *There is a constant c (independent of ε, μ $(=\frac{1}{\omega})$, r, $\mathbf{d}_4$, $\mathbf{d}_5$) such that*

$$\sup_{x\in\mathcal{Q}^{,r}}\left|\mathbb{W}_{\omega,\mu,2}^{\varepsilon,r}\right|(x) \le \frac{c\,\varepsilon}{r}.$$

Proof. Let c denote a constant independent of ε, μ $(=\frac{1}{\omega})$, r, $\mathbf{d}_4$, $\mathbf{d}_5$.

Step I. Unique existence of a solution of (6.43) in $H^1(\mathcal{Q}^{,r})$ is from Lax–Milgram theorem [19]. Note

$$\mathbf{K}_{\omega^2,\mu}^{\varepsilon,r}(y) = \mathbb{K}_{\omega^2,\mu}(\frac{r}{\varepsilon}y) \quad \text{in } \mathcal{Q}_\dagger^{\varepsilon,r}.$$

Define $\mathbb{Y}^{\varepsilon,r}_{\omega,\mu,2} \equiv \mathbb{W}^{\varepsilon,r}_{\omega,\mu,2} - \mathbb{X}^{\varepsilon/r}_{\omega,\mu,2}$ in $\mathcal{Q}^{\varepsilon,r}_{\dagger}$ (see (3.33)); then,

$$\begin{cases} -\nabla\cdot\left(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla\mathbb{Y}^{\varepsilon,r}_{\omega,\mu,2}\right) = 0 & \text{in } \mathcal{Q}^{\varepsilon,r}_{\dagger}, \\ \mathbb{Y}^{\varepsilon,r}_{\omega,\mu,2} \equiv \mathbb{W}^{\varepsilon,r}_{\omega,\mu,2} - \mathbb{X}^{\varepsilon/r}_{\omega,\mu,2} & \text{on } \partial\mathcal{Q}^{\varepsilon,r}_{\dagger}. \end{cases}$$

By [19, Theorem 8.1], (5.6)–(5.7), and $(3.34)_1$,

$$\sup_{\mathcal{Q}^{\varepsilon,r}_{\dagger}} |\mathbb{W}^{\varepsilon,r}_{\omega,\mu,2}| \le \frac{c\,\varepsilon}{r} + \sup_{\partial\mathcal{Q}^{\varepsilon,r}_{\dagger}\setminus\partial\mathcal{Q}^{,r}} |\mathbb{W}^{\varepsilon,r}_{\omega,\mu,2}|. \tag{6.44}$$

In (6.44), we use $(6.43)_2$ and $\eta = 0$ on $\partial\mathcal{Q}^{,r} \cap \partial\mathcal{Q}^{\varepsilon,r}_{\dagger}$. We claim

$$|\mathbb{W}^{\varepsilon,r}_{\omega,\mu,2}(x)| \le \frac{c\,\varepsilon}{r} \quad \text{for } x \in \mathcal{Q}^{,r} \setminus \mathcal{Q}^{\varepsilon,r}_{\dagger}. \tag{6.45}$$

If so, Lemma 6.9 follows from (6.44)–(6.45).

Step II. Proof of (6.45). Set $\mathbb{W}^{\varepsilon,r}_{\omega,\mu,2} \equiv \mathbb{X}^{\varepsilon/r}_{\omega,\mu,2} + \mathbb{U}_1 + \mathbb{U}_2$ in $\mathcal{Q}^{,r}$, where $\mathbb{U}_1$ satisfies

$$\begin{cases} -\nabla\cdot\left(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla\mathbb{U}_1\right) = 0 & \text{in } \mathcal{Q}^{,r}, \\ \mathbb{U}_1 = -\eta\mathbb{X}^{\varepsilon/r}_{\omega,\mu,2} & \text{on } \partial\mathcal{Q}^{,r}, \end{cases}$$

and $\mathbb{U}_2$ satisfies

$$\begin{cases} -\nabla\cdot\left(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}(\nabla\mathbb{U}_2 + \nabla\mathbb{X}^{\varepsilon/r}_{\omega,\mu,2} + \vec{e}_2)\right) = 0 & \text{in } \mathcal{Q}^{,r}, \\ \mathbb{U}_2 = 0 & \text{on } \partial\mathcal{Q}^{,r}. \end{cases} \tag{6.46}$$

By (5.6) and maximal principle [19],

$$\|\mathbb{X}^{\varepsilon/r}_{\omega,\mu,2}\|_{L^\infty(\mathcal{Q}^{,r})} + \|\mathbb{U}_1\|_{L^\infty(\mathcal{Q}^{,r})} \le \frac{c\,\varepsilon}{r}. \tag{6.47}$$

Set $\mathcal{Q}^{,r}(\delta)^{\mathbf{c}} \equiv \{x \in \mathcal{Q}^{,r} \mid \beta^x_r \le \delta\}$ for $\delta > 0$, where β^x_r is the distance from x to $\partial\Omega/r$ (or see Section 2 for β^x_r). Find $\tilde{\eta} \in C^\infty(\mathcal{Q}^{,r})$ with $\tilde{\eta} \in [0,1]$, $\tilde{\eta} = 1$ in $\mathcal{Q}^{,r}(\frac{3\varepsilon}{r})^{\mathbf{c}}$, $\text{supp}(\tilde{\eta}) \subset \mathcal{Q}^{,r}(\frac{4\varepsilon}{r})^{\mathbf{c}}$, and $\|\nabla\tilde{\eta}\|_{L^\infty(\mathcal{Q}^{,r})} \le c\frac{r}{\varepsilon}$. By $(3.34)_2$,

$$\begin{cases} \text{supp}(\nabla\tilde{\eta}) \subset \mathcal{Q}^{,r}(\frac{4\varepsilon}{r})^{\mathbf{c}} \setminus \mathcal{Q}^{,r}(\frac{3\varepsilon}{r})^{\mathbf{c}} \subset \mathcal{Q}^{\varepsilon,r}_{\dagger}, \\ \mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}(y) = \mathbb{K}_{\omega^2,\mu}(\frac{r}{\varepsilon}y) \quad \text{in } \mathcal{Q}^{\varepsilon,r}_{\dagger}, \\ \mathcal{Q}^{,r} \setminus \mathcal{Q}^{\varepsilon,r}_{\dagger} \subset \mathcal{Q}^{,r}(\frac{3\varepsilon}{r})^{\mathbf{c}}. \end{cases} \tag{6.48}$$

See (3.14) for $\mathbb{K}_{\omega^2,\mu}$. $\mathcal{R}_{\mathbf{d}_8,\mathbf{d}_9}(x) \equiv [x_1 - \mathbf{d}_8, x_1 + \mathbf{d}_8] \times [x_2 - \mathbf{d}_9, x_2 + \mathbf{d}_9]$ for $x = (x_1, x_2)$ and $\mathbf{d}_8, \mathbf{d}_9 \in [\frac{64\varepsilon}{r}, \frac{68\varepsilon}{r}]$ is a rectangle such that the volume $|\mathcal{R}_{\mathbf{d}_8,\mathbf{d}_9}(x) \cap \frac{\varepsilon}{r}(\mathcal{Y} + \mathbf{j})|$ for $\mathbf{j} \in \mathcal{I}_{\varepsilon/r}$ is 0 or $|\frac{\varepsilon}{r}|^2$ for some $0 < \varepsilon \le r \le 1$. Clearly, $\partial\mathcal{R}_{\mathbf{d}_8,\mathbf{d}_9}(x) \cap \Omega^{\varepsilon}_{\mu} = \emptyset$ and $\mathbf{d}_8, \mathbf{d}_9$ depend on x, $\frac{\varepsilon}{r}$. By (6.1), Remark 6.3, $(6.48)_2$, and (5.1), solution $\mathbb{U}_2(x)$ of (6.46) for

$x \in \mathcal{Q}^{,r} \setminus \mathcal{Q}_{\dagger}^{\varepsilon,r}$ satisfies

$$\begin{aligned}|\mathbb{U}_2(x)| &= \left|\int_{\mathcal{Q}^{,r}} \nabla_y \mathbf{G}^{\varepsilon,r}(x,y)\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\left(\nabla\mathbb{X}_{\omega,\mu,2}\left(\frac{r}{\varepsilon}y\right)+\vec{e}_2\right)dy\right| \\ &\le \left|\int_{\mathcal{Q}^{,r}\cap\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)} \nabla_y \mathbf{G}^{\varepsilon,r}(x,y)\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\left(\nabla\mathbb{X}_{\omega,\mu,2}\left(\frac{r}{\varepsilon}y\right)+\vec{e}_2\right)dy\right| \\ &\quad+\left|\int_{\mathcal{Q}^{,r}(\frac{4\varepsilon}{r})^{\mathbf{c}}\setminus\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)} \nabla_y \mathbf{G}^{\varepsilon,r}(x,y)\widetilde{\eta}\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\left(\nabla\mathbb{X}_{\omega,\mu,2}\left(\frac{r}{\varepsilon}y\right)+\vec{e}_2\right)dy\right| \\ &\quad+\left|\int_{\mathrm{supp}(\nabla\widetilde{\eta})\setminus\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)} \mathbf{G}^{\varepsilon,r}(x,y)\nabla\widetilde{\eta}\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\left(\nabla\mathbb{X}_{\omega,\mu,2}\left(\frac{r}{\varepsilon}y\right)+\vec{e}_2\right)dy\right| \\ &\quad+\int_{\partial\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)\cap\mathrm{supp}(1-\widetilde{\eta})}\left|\mathbf{G}^{\varepsilon,r}(x,y)(1-\widetilde{\eta})\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\left(\nabla\mathbb{X}_{\omega,\mu,2}\left(\frac{r}{\varepsilon}y\right)+\vec{e}_2\right)\right|d\sigma_y.\end{aligned} \tag{6.49}$$

In the last term of (6.49), boundary condition $(6.1)_2$ is used. By (5.6),

$$\sup_{y\in\mathcal{Q}^{,r}}\left|\nabla\mathbb{X}_{\omega,\mu,2}\left(\frac{r}{\varepsilon}y\right)+\vec{e}_2\right|\le c. \tag{6.50}$$

By (6.50) and (6.2) with $\mathbf{d}=1$,

$$\left|\int_{\mathcal{Q}^{,r}\cap\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)} \nabla_y \mathbf{G}^{\varepsilon,r}(x,y)\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\left(\nabla\mathbb{X}_{\omega,\mu,2}\left(\frac{r}{\varepsilon}y\right)+\vec{e}_2\right)dy\right|\le c\frac{\varepsilon}{r}. \tag{6.51}$$

If $x \in \mathcal{Q}^{,r} \setminus \mathcal{Q}_{\dagger}^{\varepsilon,r}$ and $y \in \mathcal{Q}^{,r}(\frac{4\varepsilon}{r})^{\mathbf{c}} \setminus \mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)$, then $\beta_r^x, \beta_r^y, \frac{\varepsilon}{r} \le \frac{|x-y|}{16}$ by $(3.34)_2$. See Section 2 for β_r^x, β_r^y. So, $\alpha \in (\frac{1}{2}, 1)$, $(6.38)_2$, (6.50), and (A2) imply

$$\begin{aligned}&\left|\int_{\mathcal{Q}^{,r}(\frac{4\varepsilon}{r})^{\mathbf{c}}\setminus\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)} \nabla_y \mathbf{G}^{\varepsilon,r}(x,y)\widetilde{\eta}(y)\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\left(\nabla\mathbb{X}_{\omega,\mu,2}\left(\frac{r}{\varepsilon}y\right)+\vec{e}_2\right)dy\right| \\ &\quad\le c\left|\frac{\varepsilon}{r}\right|^{2\alpha-1}\int_{\mathcal{Q}^{,r}(\frac{4\varepsilon}{r})^{\mathbf{c}}\setminus\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)}\frac{\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}}{|x-y|^{2\alpha}}dy\le c\frac{\varepsilon}{r}.\end{aligned} \tag{6.52}$$

If $x \in \mathcal{Q}^{,r} \setminus \mathcal{Q}_{\dagger}^{\varepsilon,r}$ and $y \in \mathrm{supp}(\nabla\widetilde{\eta}) \setminus \mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)$, then $\beta_r^x, \beta_r^y \le \frac{|x-y|}{8}$ by $(3.34)_2$ and $(6.48)_1$. So, $\alpha \in (\frac{1}{2}, 1)$, $(6.38)_1$, (6.50), and (A2) imply

$$\begin{aligned}&\left|\int_{\mathrm{supp}(\nabla\widetilde{\eta})\setminus\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)} \mathbf{G}^{\varepsilon,r}(x,y)\nabla\widetilde{\eta}(y)\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\left(\nabla\mathbb{X}_{\omega,\mu,2}\left(\frac{r}{\varepsilon}y\right)+\vec{e}_2\right)dy\right| \\ &\quad\le c\int_{\mathrm{supp}(\nabla\widetilde{\eta})\setminus\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)}\frac{r}{\varepsilon}\frac{|\beta_r^x|^\alpha|\beta_r^y|^\alpha\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}}{|x-y|^{2\alpha}}\le c\frac{\varepsilon}{r}.\end{aligned} \tag{6.53}$$

If $x \in \mathcal{Q}^{,r} \setminus \mathcal{Q}_{\dagger}^{\varepsilon,r}$ and $y \in \partial\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)$, then $(3.34)_2$ implies $\beta_r^x \le \frac{|x-y|}{4}$. By $(6.34)_2$ and $\mathbf{G}^{\varepsilon,r}(x,y) = \mathbf{G}^{\varepsilon,r}(y,x)$,

$$\sup_{\partial\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)\cap\mathrm{supp}(1-\widetilde{\eta})}|\mathbf{G}^{\varepsilon,r}(x,\cdot)|\le c. \tag{6.54}$$

Since $\partial\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x) \cap \Omega^\varepsilon_\mu = \emptyset$, $\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}(y) = 1$ for $y \in \partial\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)$. By (6.50) and (6.54),

$$\int_{\partial\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x)\cap\mathrm{supp}(1-\widetilde{\eta})} \left|\mathbf{G}^{\varepsilon,r}(x,y)(1-\widetilde{\eta})\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\left(\nabla\mathbb{X}_{\omega,\mu,2}\left(\frac{r}{\varepsilon}y\right)+\vec{e}_2\right)\right| d\sigma_y$$
$$\leq c\left|\partial\mathcal{R}_{\mathbf{d_8},\mathbf{d_9}}(x) \cap \mathrm{supp}(1-\widetilde{\eta})\right| \leq c\frac{\varepsilon}{r}. \tag{6.55}$$

So, (6.49), (6.51)–(6.53), (6.55) imply $\|\mathbb{U}_2\|_{L^\infty(\Omega^{,r}\setminus\Omega^{\varepsilon,r}_\dagger)} \leq \frac{c\,\varepsilon}{r}$. Together with (6.47), we prove (6.45). ∎

7. Boundary Lipschitz estimate for strongly elliptic equations

Assume (3.32)–(3.34), and let $0 \in \partial\Omega$. We plan to derive the Lipschitz estimate for the solutions of strongly elliptic equations around a neighborhood of 0.

Lemma 7.1. *Under* (A1)–(A2) *and* $\alpha, \frac{\varepsilon}{r}, r \in (0,1)$*, there exist constants* $\widetilde{\theta}(<\theta)$ *and* $\widetilde{\varepsilon}_0(<\varepsilon_0)$ *such that if* $\mu, \frac{\varepsilon}{r} < \widetilde{\varepsilon}_0$,

$$\begin{cases} -\nabla\cdot(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla\mathbb{V}) = 0 & \text{in } B_1(0)\cap\Omega/r, \\ \mathbb{V} = \mathbb{V}_b & \text{on } B_1(0)\cap\partial\Omega/r, \end{cases} \tag{7.1}$$

and

$$\begin{cases} \mathbb{V}_b(0) = \partial_T\mathbb{V}_b(0) = 0, \\ \|\mathbb{V}\|_{L^\infty(B_1(0)\cap\Omega/r)},\ [\nabla\mathbb{V}_b]_{C^\alpha(B_1(0)\cap\Omega/r)} \leq 1, \end{cases} \tag{7.2}$$

then

$$\sup_{z\in B_{\widetilde{\theta}}(0)\cap\Omega/r} \left|\mathbb{V}(z) - \left(z_2 + \mathbb{W}^{\varepsilon,r}_{\omega,\mu,2}(z)\right)\mathbf{d}_{\omega,\varepsilon,r}\right| \leq \widetilde{\theta}^{1+\frac{\alpha}{2}}. \tag{7.3}$$

Here, θ, ε_0 *are from Lemma 5.2;* $z = (z_1, z_2)$*;* $\partial_T\mathbb{V}_b$ *(or* $\partial_1\mathbb{V}_b$*) is the tangential derivative of* $\mathbb{V}_b$*;* $\mathbf{d}_{\omega,\varepsilon,r}$ *is the second component of* $\mathcal{K}^{-1}_{\omega,\mu}(\mathbf{K}^{\varepsilon,r}_{\omega^2,\mu}\nabla\mathbb{V})_{B_{\widetilde{\theta}}(0)\cap\Omega/r}$*; and* $\mathcal{K}^{-1}_{\omega,\mu}$ *is the inverse matrix of* $\mathcal{K}_{\omega,\mu}$.

Proof. Assume $r \in [0,1]$ and $\mathbb{V}$, $\mathbb{V}_b$ satisfy

$$\begin{cases} -\Delta\mathbb{V} = 0 & \text{in } B_{4/5}(0)\cap\Omega/r, \\ \mathbb{V} = \mathbb{V}_b & \text{on } B_{4/5}(0)\cap\partial\Omega/r, \\ \mathbb{V}_b \in C^{1,\alpha}(B_{4/5}(0)\cap\Omega/r) & \text{with } \mathbb{V}_b(0) = \partial_T\mathbb{V}_b(0) = 0. \end{cases} \tag{7.4}$$

See Section 2 and (3.33)–(3.34) for Ω/r. By [19, Theorem 4.16], there exist $\widetilde{\theta} \in (0, \frac{4}{5})$ and $\alpha' \in (\frac{\alpha}{2}, \alpha)$ such that

$$\sup_{z\in B_{\widetilde{\theta}}(0)\cap\Omega/r} \left|\mathbb{V}(z) - z_2\,(\partial_2\mathbb{V})_{B_{\widetilde{\theta}}(0)\cap\Omega/r}\right|$$
$$\leq \widetilde{\theta}^{1+\alpha'}\left(\|\mathbb{V}\|_{L^\infty(B_{4/5}(0)\cap\Omega/r)} + [\nabla\mathbb{V}_b]_{C^\alpha(B_{4/5}(0)\cap\Omega/r)}\right), \tag{7.5}$$

where $\partial_2\mathbb{V}$ is the partial derivative with respect to z_2 variable.

We claim (7.3). If not, there is a sequence $\{\varepsilon, \mu_\varepsilon, \omega_\varepsilon, r_\varepsilon, \mathbb{V}_\varepsilon, \mathbb{V}_{b,\varepsilon}, \mathcal{K}_{\omega_\varepsilon,\mu_\varepsilon}\}$ satisfying (7.1)–(7.2) and

$$\begin{cases} \mu_\varepsilon, \frac{\varepsilon}{r_\varepsilon} \to 0, \quad r_\varepsilon \to r \in [0,1], \\ \widetilde{\mathcal{K}} = \lim_{\varepsilon\to 0} \mathcal{K}_{\omega_\varepsilon,\mu_\varepsilon}, \\ \sup_{z\in B_{\tilde\theta}(0)\cap\Omega/r_\varepsilon} \left|\mathbb{V}_\varepsilon(z) - \left(z_2 + \mathbb{W}^{\varepsilon,r_\varepsilon}_{\omega_\varepsilon,\mu_\varepsilon,2}(z)\right)\mathbf{d}_{\omega_\varepsilon,\varepsilon,r_\varepsilon}\right| > \tilde\theta^{1+\frac{\alpha}{2}}. \end{cases} \tag{7.6}$$

See (5.4) for $\mathcal{K}_{\omega_\varepsilon,\mu_\varepsilon}$. Equation $(7.6)_2$ is due to (5.5). By Lemma 3.6, (A2), and (7.2),

$$\left\|\mathbf{K}^{\varepsilon,r_\varepsilon}_{\omega_\varepsilon^2,\mu_\varepsilon}\nabla\mathbb{V}_\varepsilon\right\|_{L^2(B_{4/5}(0)\cap\Omega/r_\varepsilon)} \le c\|\nabla\mathbb{V}_\varepsilon\|_{L^2(B_{5/6}(0)\cap\Omega/r_\varepsilon)} \le c, \tag{7.7}$$

where c is independent of ε, μ_ε, ω_ε, r_ε. By (7.7), Lemma 3.11, Sobolev embedding theorem [19], (7.2), [7, Theorem 1], and remark in [21, pp. 43–44], there is a subsequence (same notation for subsequence) of $\{\mathbb{V}_\varepsilon, \mathbb{V}_{b,\varepsilon}\}$ such that

$$\begin{cases} \|\mathbb{V}_\varepsilon - \mathbb{V}\|_{L^\infty(B_{4/5}(0)\cap\Omega/r_\varepsilon\cap\Omega/r)} \to 0 \\ \|\mathbb{V}_{b,\varepsilon} - \mathbb{V}_b\|_{C^1(B_{4/5}(0)\cap\Omega/r_\varepsilon\cap\Omega/r)} \to 0 \\ \|\mathbb{V}\|_{L^\infty(B_1(0)\cap\Omega/r)},\ [\nabla\mathbb{V}_b]_{C^\alpha(B_1(0)\cap\Omega/r)} \le 1 \\ \mathbf{K}^{\varepsilon,r_\varepsilon}_{\omega_\varepsilon^2,\mu_\varepsilon}\nabla\mathbb{V}_\varepsilon \to \widetilde{\mathcal{K}}\nabla\mathbb{V} \quad \text{in } L^2(B_{4/5}(0)\cap\Omega/r) \text{ weakly} \end{cases} \quad \text{as } \frac{\varepsilon}{r_\varepsilon} \to 0, \tag{7.8}$$

where $\widetilde{\mathcal{K}}$ is a positive constant satisfying $(5.5)_1$. Also, the limit functions $\mathbb{V}$, $\mathbb{V}_b$ satisfy (7.4). By $(7.6)_2$ and $(7.8)_4$,

$$\lim_{\frac{\varepsilon}{r_\varepsilon}\to 0} \mathcal{K}^{-1}_{\omega_\varepsilon,\mu_\varepsilon} \fint_{\mathbf{S}^{r_\varepsilon}_{\tilde\theta}(0)} \mathbf{K}^{\varepsilon,r_\varepsilon}_{\omega_\varepsilon^2,\mu_\varepsilon}\nabla\mathbb{V}_\varepsilon\,dx = (\nabla\mathbb{V})_{\mathbf{S}^r_{\tilde\theta}(0)}. \tag{7.9}$$

By $(7.6)_3$, $(7.8)_1$, Lemma 6.9, (7.9), and (7.5),

$$\begin{aligned} \tilde\theta^{1+\alpha/2} &\le \lim_{\frac{\varepsilon}{r_\varepsilon}\to 0} \sup_{z\in\mathbf{S}^{r_\varepsilon}_{\tilde\theta}(0)} \left|\mathbb{V}_\varepsilon(z) - \left(z_2 + \mathbb{W}^{\varepsilon,r_\varepsilon}_{\omega_\varepsilon,\mu_\varepsilon,2}(z)\right)\mathbf{d}_{\omega_\varepsilon,\varepsilon,r_\varepsilon}\right| \\ &= \sup_{z\in\mathbf{S}^r_{\tilde\theta}(0)} \left|\mathbb{V}(z) - z_2(\partial_2\mathbb{V})_{\mathbf{S}^r_{\tilde\theta}(0)}\right| \le \tilde\theta^{1+\alpha'/2}\left(\|\mathbb{V}\|_{L^\infty(\mathbf{S}^r_{4/5}(0))} + [\nabla\mathbb{V}_b]_{C^\alpha(\mathbf{S}^r_{4/5}(0))}\right). \end{aligned}$$

We get contradiction if $\tilde\theta$ is small enough. So, (7.3) holds. ■

Lemma 7.2. *Let $\tilde\theta$, $\tilde\varepsilon_0$ be the same as Lemma 7.1. If* (A1)–(A2), *$\alpha \in (0,1)$, $\varepsilon, \mu \in (0, \tilde\varepsilon_0)$, and*

$$\begin{cases} -\nabla\cdot(\mathbf{K}^\varepsilon_{\omega^2,\mu}\nabla V) = 0 & \text{in } B_1(0)\cap\Omega, \\ V = 0 & \text{on } B_1(0)\cap\partial\Omega, \end{cases} \tag{7.10}$$

then, for any $k \in \mathbb{N}$ with $\frac{\varepsilon}{\tilde\varepsilon_0} < \tilde\theta^k$, there are constants $\mathbf{d}^{\varepsilon,k-1}_{\omega,\mu}$ satisfying

$$\begin{cases} |\mathbf{d}^{\varepsilon,k-1}_{\omega,\mu}| \le c\tilde J, \\ \sup_{z\in B_{\tilde\theta^k}(0)\cap\Omega}\left|V(z) - \sum_{j=0}^{k-1}\tilde\theta^{\frac{j\alpha}{2}}\left(z_2 + \tilde\theta^j\,\mathbb{W}^{\varepsilon,\tilde\theta^j}_{\omega,\mu,2}\left(\frac{z}{\tilde\theta^j}\right)\right)\mathbf{d}^{\varepsilon,j}_{\omega,\mu}\right| \le \tilde\theta^{k(1+\frac{\alpha}{2})}\tilde J, \end{cases} \tag{7.11}$$

where $z = (z_1, z_2)$, $\tilde J \equiv \|V\|_{L^\infty(B_1(0)\cap\Omega)}$, and c is independent of ε, μ $(= \omega^{-1})$.

Proof. When $k=1$, (7.11) holds by Lemma 7.1 with $r=1$ and $\mathbb{V}=\frac{V}{\widetilde{J}}$. $\mathbf{d}^{\varepsilon,0}_{\omega,\mu}$ is the second component of $\mathcal{K}^{-1}_{\omega,\mu}(\mathbf{K}^{\varepsilon}_{\omega^2,\mu}\nabla V)_{B_{\widetilde{\theta}}(0)\cap\Omega}$. By Lemma 3.6 and (A2), we see $|\mathbf{d}^{\varepsilon,0}_{\omega,\mu}|\le c\widetilde{J}$, where c is independent of $\varepsilon,\mu(=\frac{1}{\omega})$. Suppose that (7.11) holds for some $k\in\mathbb{N}$ with $\frac{\varepsilon}{\widetilde{\varepsilon}_0}<\widetilde{\theta}^k$; define, in $B_1(0)\cap\Omega/\widetilde{\theta}^k$,

$$\begin{cases}\mathbb{V}(z)\equiv\frac{1}{\widetilde{J}\,\widetilde{\theta}^{k(1+\frac{\alpha}{2})}}\Big(V\big(\widetilde{\theta}^k z\big)-\sum_{j=0}^{k-1}\widetilde{\theta}^{\frac{j\alpha}{2}}\big(\widetilde{\theta}^k z_2+\widetilde{\theta}^j\,\mathbb{W}^{\varepsilon,\widetilde{\theta}^j}_{\omega,\mu,2}\big(\frac{\widetilde{\theta}^k z}{\widetilde{\theta}^j}\big)\big)\mathbf{d}^{\varepsilon,j}_{\omega,\mu}\Big),\\ \mathbb{V}_b(z)\equiv\frac{-1}{\widetilde{J}\,\widetilde{\theta}^{k(1+\frac{\alpha}{2})}}\sum_{j=0}^{k-1}\widetilde{\theta}^{\frac{j\alpha}{2}}\,\widetilde{\theta}^k\,z_2\mathbf{d}^{\varepsilon,j}_{\omega,\mu}.\end{cases}$$

Note $[\nabla\mathbb{V}_b]_{C^\alpha(B_1(0)\cap\Omega/\widetilde{\theta}^k)}=0$. Functions $\mathbb{V}$, $\mathbb{V}_b$ satisfy (7.1)–(7.2) with $r=\widetilde{\theta}^k$. Apply Lemma 7.1 to get

$$\sup_{B_{\widetilde{\theta}}(0)\cap\Omega/\widetilde{\theta}^k}\Big|\mathbb{V}(z)-\big(z_2+\mathbb{W}^{\varepsilon,\widetilde{\theta}^k}_{\omega,\mu,2}(z)\big)\mathbf{d}_{\omega,\varepsilon,\widetilde{\theta}^k}\Big|\le\widetilde{\theta}^{1+\frac{\alpha}{2}},\tag{7.12}$$

where $\mathbf{d}_{\omega,\varepsilon,\widetilde{\theta}^k}$ is the second component of $\mathcal{K}^{-1}_{\omega,\mu}(\mathbf{K}^{\varepsilon,\widetilde{\theta}^k}_{\omega^2,\mu}\nabla\mathbb{V})_{B_{\widetilde{\theta}}(0)\cap\Omega/\widetilde{\theta}^k}$. By Lemma 3.6 and (A2), $|\mathbf{d}_{\omega,\varepsilon,\widetilde{\theta}^k}|$ are bounded uniformly in $\varepsilon,\mu(=\frac{1}{\omega}),\widetilde{\theta}^k$. Rewrite (7.12) in terms of V in $B_{\widetilde{\theta}^{k+1}}(0)$ to obtain

$$\sup_{B_{\widetilde{\theta}^{k+1}}(0)\cap\Omega}\Bigg|V(z)-\sum_{j=0}^{k-1}\widetilde{\theta}^{\frac{j\alpha}{2}}\big(z_2+\widetilde{\theta}^j\,\mathbb{W}^{\varepsilon,\widetilde{\theta}^j}_{\omega,\mu,2}(z/\widetilde{\theta}^j)\big)\mathbf{d}^{\varepsilon,j}_{\omega,\mu}\\ -\widetilde{\theta}^{\frac{k\alpha}{2}}\,\widetilde{J}\big(z_2+\widetilde{\theta}^k\,\mathbb{W}^{\varepsilon,\widetilde{\theta}^k}_{\omega,\mu,2}(z/\widetilde{\theta}^k)\big)\mathbf{d}_{\omega,\varepsilon,\widetilde{\theta}^k}\Bigg|\le\widetilde{\theta}^{(k+1)(1+\frac{\alpha}{2})}\,\widetilde{J}.$$

If $\mathbf{d}^{\varepsilon,k}_{\omega,\mu}\equiv\widetilde{J}\,\mathbf{d}_{\omega,\varepsilon,\widetilde{\theta}^k}$, then (7.11) holds for $k+1$. ■

Lemma 7.3. *Let $\widetilde{\varepsilon}_0$ be the same as Lemma 7.2. If (A1)–(A2) and $\varepsilon,\mu\in(0,\widetilde{\varepsilon}_0)$, there is a constant c independent of $\varepsilon,\mu(=\frac{1}{\omega})$ such that any solution of (7.10) satisfies*

$$\big\|\mathbf{K}^{\varepsilon}_{\omega^2,\mu}\nabla V\big\|_{L^\infty(B_{1/2}(0)\cap\Omega)}\le c\|V\|_{L^\infty(B_1(0)\cap\Omega)}.\tag{7.13}$$

Proof. By (3.32), $0\in\partial\Omega$ and there is a local coordinate $z=(z_1,z_2)$ so that

$$B_1(0)\cap\Omega=B_1(0)\cap\big\{(z_1,z_2)\in\mathbb{R}^2\mid z_1\in\mathbb{R},z_2>\Upsilon(z_1)\big\}.$$

We claim

$$\sup_{(0,z_2)\in B_{1/2}(0)\cap\Omega}\big|\mathbf{K}^{\varepsilon}_{\omega^2,\mu}\nabla V(0,z_2)\big|\le c\|V\|_{L^\infty(B_1(0)\cap\Omega)}.\tag{7.14}$$

■

Proof of claim. Let $\widetilde{\theta}$, $\widetilde{J}$, α be the same as Lemma 7.2; c is a constant independent of $\varepsilon,\mu\,(=\frac{1}{\omega})$; $k\in\mathbb{N}$ satisfies $\widetilde{\theta}^{k+1}\le\frac{\varepsilon}{\widetilde{\varepsilon}_0}<\widetilde{\theta}^k$. For any $z\equiv(0,z_2)\in B_{1/2}(0)\cap\Omega$, we have either (i) $\frac{1}{2}\widetilde{\theta}^{\ell+1}\le z_2<\frac{1}{2}\widetilde{\theta}^{\ell}$ for $0\le\ell\le k$ or (ii) $0\le z_2<\frac{1}{2}\widetilde{\theta}^{k+1}$.

For case (i). (i.e., $z \equiv (0, z_2)$, $\frac{1}{2}\widetilde{\theta}^{\ell+1} \le z_2 < \frac{1}{2}\widetilde{\theta}^{\ell}$ for $0 \le \ell \le k$). By Lemma 7.2,

$$\sup_{x=(x_1,x_2)\in B_{\widetilde{\theta}^\ell}(0)\cap\Omega} \left| V(x) - \sum_{j=0}^{\ell-1} \widetilde{\theta}^{\frac{j\alpha}{2}} \left(x_2 + \widetilde{\theta}^j\, \mathbb{W}^{\varepsilon,\widetilde{\theta}^j}_{\omega,\mu,2}\left(\frac{x}{\widetilde{\theta}^j}\right)\right) \mathbf{d}^{\varepsilon,j}_{\omega,\mu} \right| \le c\, \widetilde{\theta}^{\ell(1+\frac{\alpha}{2})}\, \widetilde{J}. \tag{7.15}$$

By Lemma 6.9, $(3.32)_4$, $(7.11)_1$, and (7.15),

$$\sup_{B_{\widetilde{\theta}^\ell}(0)\cap\Omega} |V| \le c\,\widetilde{J}\big(\widetilde{\theta}^{\ell(1+\frac{\alpha}{2})} + (\beta^z + \varepsilon)\big) \le c\,\beta^z\,\widetilde{J}. \tag{7.16}$$

See Section 2 for β^z. Here, $x_2 \le \widetilde{\theta}^\ell \le c z_2 \le c\,\beta^z$ and $\varepsilon \le \widetilde{\varepsilon}_0\,\widetilde{\theta}^k \le 2\widetilde{\varepsilon}_0 \frac{1}{2}\widetilde{\theta}^\ell \le 2\widetilde{\varepsilon}_0\, z_2 \le c\widetilde{\varepsilon}_0\,\beta^z$ in (7.16). We see that if $z \equiv (0, z_2) \in \Omega$ satisfies $\frac{1}{2}\widetilde{\theta}^{\ell+1} \le z_2 < \frac{1}{2}\widetilde{\theta}^\ell$, then

$$\sup_{B_{\beta^z/2}(z)} |V| \le c\,\beta^z\,\widetilde{J}. \tag{7.17}$$

Define

$$\begin{cases} \mathbb{E}^{\varepsilon,\beta^z}_{\omega^2,\mu}(y) \equiv \mathbf{K}^{\varepsilon}_{\omega^2,\mu}(z+\beta^z\, y) \\ \mathbb{V}(y) \equiv \frac{V(z+\beta^z\, y)}{\beta^z\,\widetilde{J}} \end{cases} \quad \text{in } B_{1/2}(0).$$

Then, $\mathbb{V}$ satisfies, by (7.17),

$$\begin{cases} -\nabla\cdot\big(\mathbb{E}^{\varepsilon,\beta^z}_{\omega^2,\mu}\nabla\mathbb{V}\big) = 0 \quad \text{in } B_{1/2}(0), \\ \|\mathbb{V}\|_{L^\infty(B_{1/2}(0))} \le c. \end{cases}$$

Remark 5.5 implies

$$\big\|\mathbf{K}^{\varepsilon}_{\omega^2,\mu}\nabla V\big\|_{L^\infty(B_{\beta^z/4}(z))} \le c\,\widetilde{J}.$$

This proves (7.14) for case (i).

For case (ii). (i.e., $0 \le z_2 < \frac{1}{2}\widetilde{\theta}^{k+1}$). By Lemma 7.2,

$$\sup_{x=(x_1,x_2)\in B_{\widetilde{\theta}^k}(0)\cap\Omega} \left| V(x) - \sum_{j=0}^{k-1} \widetilde{\theta}^{\frac{j\alpha}{2}} \left(x_2 + \widetilde{\theta}^j\, \mathbb{W}^{\varepsilon,\widetilde{\theta}^j}_{\omega,\mu}\left(\frac{x}{\widetilde{\theta}^j}\right)\right) \mathbf{d}^{\varepsilon,j}_{\omega,\mu} \right| \le c\, \widetilde{\theta}^{k(1+\frac{\alpha}{2})}\, \widetilde{J}.$$

Lemma 6.9 and $(7.11)_1$ imply

$$\sup_{B_{\widetilde{\theta}^k}(0)\cap\Omega} |V| \le c\varepsilon\widetilde{J}. \tag{7.18}$$

Define

$$\mathbb{V}(y) \equiv V(\varepsilon\, y)/\varepsilon\,\widetilde{J} \quad \text{in } B_1(0)\cap\Omega/\varepsilon.$$

By (7.18),

$$\|\mathbb{V}\|_{L^\infty(B_1(0)\cap\Omega/\varepsilon)} \le c.$$

Then, $\mathbb{V}$ satisfies

$$\begin{cases} -\nabla \cdot \big(\mathbf{K}^{\varepsilon,\varepsilon}_{\omega^2,\mu} \nabla \mathbb{V}\big) = 0 & \text{in } B_1(0) \cap \Omega/\varepsilon, \\ \mathbb{V} = 0 & \text{on } B_1(0) \cap \partial\Omega/\varepsilon. \end{cases} \tag{7.19}$$

Lemma 3.7 and classical regularity result [19] imply $\|\mathbf{K}^{\varepsilon,\varepsilon}_{\omega^2,\mu} \nabla \mathbb{V}\|_{L^\infty(B_{1/2}(0)\cap\Omega/\varepsilon)} \le c$. So, we prove (7.14) for case (ii).

Finally, we repeat the argument for (7.14) (i.e., (7.15)–(7.19)) by varying the origin along the boundary $B_1(0) \cap \partial\Omega$ and by adjusting the constant c. We conclude that (7.13) is true. ∎

Remark 7.4. Let $\widetilde{\varepsilon}_0$ be the same as in Lemma 7.3. If $\mu \in [\widetilde{\varepsilon}_0, 1]$, (A2) implies that equation (7.10) is a uniform elliptic equation. By [32], we know that the following.

Under (A1)–(A2) *and* $\mu \in [\widetilde{\varepsilon}_0, 1]$, *any solution of* (7.10) *satisfies* (7.13).

By Lemma 3.7 and classical regularity results [19], we have the following.

Under (A1)–(A2) *and* $\varepsilon \in [\varepsilon_0, 1]$, *any solution of* (7.10) *satisfies* (7.13).

Combining with Lemma 7.3, we conclude the following.

Under (A1)–(A2), *any solution of* (7.10) *satisfies* (7.13).

8. Proof of Theorem 2.2

For any $x \in \Omega$, consider the following equation:

$$\begin{cases} -\nabla_y \cdot \big(\mathbf{K}^{\varepsilon}_{\omega^2,\mu} \nabla_y \mathscr{G}^\varepsilon(x,y) + \frac{x-y}{2\pi|x-y|^2}\big) = 0 & \text{in } \Omega, \\ \mathscr{G}^\varepsilon(x,\cdot) = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.1}$$

Tracing the proofs of Lemmas 6.1, 6.2, 6.5, and 6.6, we see the following.

Lemma 8.1. *Assume* (A1), $\omega \in (1,\infty)$, $\varepsilon, \mu \in (0,1)$, *and* $x \in \Omega$. *A unique* $L^{2,\infty}(\Omega)$ *solution of* (8.1) *exists; it is the Green's function of* $\nabla_y \cdot (\mathbf{K}^{\varepsilon}_{\omega^2,\mu} \nabla_y)$ *in* Ω, *and*

$$\begin{cases} \big\|\mathscr{G}^\varepsilon(x,\cdot)\big\|_{*,\partial\Omega} \le c, \\ \big\|\mathbf{K}^{\varepsilon}_{\omega^2,\mu} \nabla \mathscr{G}^\varepsilon(x,\cdot)\big\|_{L^{\mathrm{d}}(\mathcal{O}\cap\Omega)} < c|\mathcal{O}\cap\Omega|^{\frac{1}{\mathrm{d}}-\frac{1}{2}}, \\ \mathscr{G}^\varepsilon(x,y) = \mathscr{G}^\varepsilon(y,x) \quad \textit{for any } x, y \in \Omega, \end{cases} \tag{8.2}$$

where $\mathbf{d} \in [1,2)$, $\mathcal{O}$ *is any open set and* c *is independent of* ε, ω, μ, x, $\mathcal{O}$.

Following the proof of Lemma 6.7 and applying Lemma 8.1, we see the following.

Lemma 8.2. *Suppose* (A1)–(A2)*; the solution of* (8.1) *satisfies*

$$\begin{cases} \left|\mathbf{K}^{\varepsilon}_{\omega^2,\mu}(y)\nabla_y \mathcal{G}^{\varepsilon}(x,y)\right| \le \frac{c}{|x-y|} \\ \left|\mathbf{K}^{\varepsilon}_{\omega^2,\mu}(x)\nabla_x \mathcal{G}^{\varepsilon}(x,y)\right| \le \frac{c}{|x-y|} \end{cases} \quad \textit{for any } x, y \in \Omega, \tag{8.3}$$

where c is a constant independent of ε, μ $(=\omega^{-1})$.

Proof. Let c denote a constant independent of ε, μ $(=\omega^{-1})$. For any $x, y \in \Omega$, we set $h \equiv \frac{1}{2}|x-y|$, $\mathbb{E}^{\varepsilon,h}_{\omega^2,\mu}(z) = \mathbf{K}^{\varepsilon}_{\omega^2,\mu}(y+hz)$, and $\psi(z) \equiv \mathcal{G}^{\varepsilon}(x, y+hz)$. Then,

$$\begin{cases} -\nabla\cdot\left(\mathbb{E}^{\varepsilon,h}_{\omega^2,\mu}\nabla\psi\right) = 0 & \text{in } B_1(0)\cap(\Omega-\{y\})/h, \\ \psi = 0 & \text{on } B_1(0)\cap(\partial\Omega-\{y\})/h. \end{cases}$$

Since $\mathcal{G}^{\varepsilon}(x,\cdot) \in \mathrm{BMO}(\Omega)$ by $(8.2)_1$, $\psi \in \mathrm{BMO}(B_1(0)\cap(\Omega-\{y\})/h)$. So, $(8.2)_1$ and (6.22) imply

$$\|\psi\|_{L^q(B_1(0)\cap(\Omega-\{y\})/h)} \le c \quad \text{for any } q \in (1,\infty). \tag{8.4}$$

If $h < \varepsilon\mu$ and $\mathbf{d} \in [1,2)$, [19, Theorems 4.15 and 8.17], (II) of Corollary 3.4, and $(8.2)_2$ imply

$$\left\|\mathbb{E}^{\varepsilon,h}_{\omega^2,\mu}\nabla\psi\right\|_{L^\infty(B_{1/2}(0)\cap(\Omega-\{y\})/h)} \le c\left\|\mathbb{E}^{\varepsilon,h}_{\omega^2,\mu}\nabla\psi\right\|_{L^{\mathbf{d}}(B_1(0)\cap(\Omega-\{y\})/h)} \le c.$$

If $h \in [\varepsilon\mu, \varepsilon)$, [19, Theorems 4.15 and 8.17], Lemma 3.7, and (8.4) imply

$$\left\|\mathbb{E}^{\varepsilon,h}_{\omega^2,\mu}\nabla\psi\right\|_{L^\infty(B_{1/2}(0)\cap(\Omega-\{y\})/h)} \le c\|\psi\|_{L^2(B_1(0)\cap(\Omega-\{y\})/h)} \le c.$$

If $\frac{\varepsilon}{h} \le 1$, Remarks 5.5 and 7.4, (A2), Lemma 4.3, (8.4), and $(8.2)_2$ imply

$$\begin{aligned} &\left\|\mathbb{E}^{\varepsilon,h}_{\omega^2,\mu}\nabla\psi\right\|_{L^\infty(B_{1/4}(0)\cap(\Omega-\{y\})/h)} \\ &\le c\|\psi\|_{L^\infty(B_{1/2}(0)\cap(\Omega-\{y\})/h)} \\ &\le c\|\psi\|_{L^1(B_1(0)\cap(\Omega-\{y\})/h)} + \|\nabla\psi\|_{L^{\mathbf{d}}(B_1(0)\cap(\Omega-\{y\})/h)} \le c, \end{aligned}$$

where $\mathbf{d} \in (1,2)$. So, we prove $(8.3)_1$. Equation $(8.3)_2$ follows from $(8.3)_1$ and $(8.2)_3$. ∎

Now, we are ready to prove Theorem 2.2. Suppose that Φ is a solution of (2.1); then,

$$\left|\mathbf{K}^{\varepsilon}_{\omega^2,\mu}\nabla\Phi(x)\right| = \left|\int_\Omega \mathbf{K}^{\varepsilon}_{\omega^2,\mu}(x)\nabla_x\mathcal{G}^{\varepsilon}(x,y)G(y)dy\right|.$$

If $p > 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then $p' \in (1,2)$. Lemma 8.2 implies

$$\begin{aligned} \left|\mathbf{K}^{\varepsilon}_{\omega^2,\mu}\nabla\Phi(x)\right| &\le \int_\Omega \left|\mathbf{K}^{\varepsilon}_{\omega^2,\mu}(x)\nabla_x\mathcal{G}^{\varepsilon}(x,y)G(y)\right|dy \\ &\le \left(\int_\Omega \frac{c}{|x-y|^{p'}}dy\right)^{1/p'} \|G\|_{L^p(\Omega)} \le c\|G\|_{L^p(\Omega)}. \end{aligned}$$

So, we obtain $\|\mathbf{K}^{\varepsilon}_{\omega^2,\mu}\nabla\Phi\|_{L^\infty(\Omega)} \le c\|G\|_{L^p(\Omega)}$, and the proof of Theorem 2.2 is complete.

9. Proof of (3.18)$_1$ in Lemma 3.7

Let $\Gamma(z-y)$ denote the fundamental solution of the Laplace equation in $\mathbb{R}^2$ [19] and $B_r \subset \mathbb{R}^2$ denote a disc centered at 0 with radius r. Define single-layer and double-layer potentials as, for any smooth function ζ on the boundary ∂B_r,

$$\begin{cases} \mathcal{S}_{\partial B_r}(\zeta)(z) \equiv \int_{\partial B_r} \Gamma(z-y)\,\zeta(y)\,d\sigma_y \\ \mathcal{D}_{\partial B_r}(\zeta)(z) \equiv \int_{\partial B_r} \nabla_y \Gamma(z-y)\vec{\mathbf{n}}_y\,\zeta(y)\,d\sigma_y \end{cases} \quad \text{for } z \in \partial B_r,$$

where $\vec{\mathbf{n}}_y$ is the unit vector outward normal to ∂B_r. By [16, pp. 148–151], [13, pp. 226–227], and a similar proof as [33, Lemma 3.2], we see the following.

Lemma 9.1. *For any $\alpha \in (0,1)$, the linear operators*

$$\begin{cases} \mathcal{S}_{\partial B_1} : C^{\alpha}(\partial B_1) \to C^{1,\alpha}(\partial B_1) \\ \mathcal{D}_{\partial B_1} : C^{\alpha}(\partial B_1) \to C^{1,\alpha}(\partial B_1) \end{cases}$$

are bounded; $I - \mathbf{d}_{10}\mathcal{D}_{\partial B_1}$ for $\mathbf{d}_{10} \in [-2,2]$ are invertible in $C^{1,\alpha}(\partial B_1)$; and

$$\|\zeta\|_{C^{1,\alpha}(\partial B_1)} \le c\|(I - \mathbf{d}_{10}\mathcal{D}_{\partial B_1})(\zeta)\|_{C^{1,\alpha}(\partial B_1)},$$

where I is the identity operator and c is a constant independent of $\mathbf{d}_{10}$.

If $\vec{\mathbf{n}}$ is the unit vector outward normal to $\partial\mathcal{Y}_\mu$, we define, for any $y \in \partial\mathcal{Y}_\mu$ and any function ζ on $\mathcal{Y}$,

$$\begin{cases} \zeta_\pm(y) \equiv \lim_{t\to 0^+} \zeta(y \pm t\vec{\mathbf{n}}), & \lfloor \zeta \rfloor(y) \equiv \zeta_+(y) - \zeta_-(y), \\ \partial^\pm_{\mathbf{n}}\zeta \equiv \nabla\zeta_\pm \cdot \vec{\mathbf{n}}, & \lfloor \partial_{\mathbf{n}}\zeta \rfloor(y) \equiv \partial^+_{\mathbf{n}}\zeta(y) - \partial^-_{\mathbf{n}}\zeta(y). \end{cases} \tag{9.1}$$

Proof of (3.18)$_1$ *in Lemma* 3.7. Define $\alpha \equiv \frac{q-2}{q}$ for $q \ge 2$,

$$\check{J} \equiv \|\Psi\|_{L^2(\mathcal{Y}\setminus B_{1/4})},$$

and let c be a constant independent of μ, ω. By [19, Theorem 4.15], any solution Ψ of (3.17) satisfies

$$\|\Psi\|_{C^{1,\alpha}(B_{9/20}\setminus B_{7/20})} \le c\check{J}. \tag{9.2}$$

Next, we find $\zeta \in C^{1,\alpha}(B_{2/5})$ by solving

$$\begin{cases} -\Delta\zeta = 0 & \text{in } B_{2/5}, \\ \zeta = \Psi & \text{on } \partial B_{2/5}. \end{cases}$$

By [19, Theorems 4.15 and 4.16],

$$\|\zeta\|_{C^{1,\alpha}(B_{2/5})} \le c\check{J}. \tag{9.3}$$

Recall $\mathcal{Y}_\mu = B_{\mu/4}(0)$. Define $\phi \equiv \Psi - \zeta$ in $B_{2/5}$ and $\widehat{\phi}(y) \equiv \phi(\frac{\mu}{4}y)$, $\widehat{\zeta}(y) \equiv \zeta(\frac{\mu}{4}y)$ in $B_{8/5\mu}$. Then,

$$\begin{cases} -\Delta\widehat{\phi} = 0 & \text{in } B_{\frac{8}{5\mu}} \setminus \partial B_1, \\ \lfloor \widehat{\phi} \rfloor = 0 & \text{on } \partial B_1, \\ \lfloor \mathbb{K}^{4/\mu}_{\omega^2,\mu} \nabla\widehat{\phi} \rfloor \cdot \vec{\mathbf{n}}_y = \mathbb{G} & \text{on } \partial B_1, \\ \widehat{\phi} = 0 & \text{on } \partial B_{\frac{8}{5\mu}}, \end{cases} \tag{9.4}$$

where $\vec{\mathbf{n}}_y$ is the unit vector normal to ∂B_1 and $\mathbb{G} = -\lfloor \mathbb{K}^{4/\mu}_{\omega^2,\mu} \nabla\widehat{\zeta} \rfloor \cdot \vec{\mathbf{n}}_y$. See (3.14) for $\mathbb{K}^{4/\mu}_{\omega^2,\mu}$ and (9.1) for $\lfloor \widehat{\phi} \rfloor$, $\lfloor \mathbb{K}^{4/\mu}_{\omega^2,\mu} \nabla\widehat{\phi} \rfloor$. Note, by (9.2)–(9.3) and definition of single-layer potential,

$$\begin{cases} \|\mathbb{G}\|_{C^\alpha(\partial B_1)} \le c\omega^2\mu \check{J}, \\ \big\|\mathcal{S}_{\partial B_{8/5\mu}}\big(\partial_{\mathbf{n}}\widehat{\phi}|_{\partial B_{8/5\mu}}\big)\big\|_{C^{1,\alpha}(\partial B_1)} \le c\check{J}|\ln\mu|, \end{cases} \tag{9.5}$$

where $\partial_{\mathbf{n}}\widehat{\phi}|_{\partial B_{8/5\mu}}$ is the normal derivative of $\widehat{\phi}$ on $\partial B_{8/5\mu}$. By Green's formula, (9.4), and [16, pp. 148–151],

$$\begin{cases} \frac{\widehat{\phi}}{2} + \mathcal{D}_{\partial B_1}(\widehat{\phi}) = \mathcal{S}_{\partial B_1}\big(\nabla\widehat{\phi}_- \cdot \vec{\mathbf{n}}_y|_{\partial B_1}\big) \\ \frac{\widehat{\phi}}{2} - \mathcal{D}_{\partial B_1}(\widehat{\phi}) = -\mathcal{S}_{\partial B_1}\big(\nabla\widehat{\phi}_+ \cdot \vec{\mathbf{n}}_y|_{\partial B_1}\big) + \mathcal{S}_{\partial B_{8/5\mu}}\big(\partial^-_{\mathbf{n}}\widehat{\phi}|_{\partial B_{8/5\mu}}\big) \end{cases} \quad \text{on } \partial B_1.$$

Therefore,

$$\Big(I - \frac{2(1-\omega^2)}{1+\omega^2}\mathcal{D}_{\partial B_1}\Big)\widehat{\phi} = \frac{2}{1+\omega^2}\big(\mathcal{S}_{\partial B_1}(-\mathbb{G}) + \mathcal{S}_{\partial B_{8/5\mu}}(\partial^-_{\mathbf{n}}\widehat{\phi}|_{\partial B_{8/5\mu}})\big),$$

where I is the identity matrix. Apply (9.5) and Lemma 9.1 to see

$$\begin{aligned} \|\widehat{\phi}\|_{C^{1,\alpha}(\partial B_1)} &\le \frac{c}{\omega^2}\big(\|\mathbb{G}\|_{C^\alpha(\partial B_1)} + \big\|\mathcal{S}_{\partial B_{8/5\mu}}\big(\partial_{\mathbf{n}}\widehat{\phi}|_{\partial B_{8/5\mu}}\big)\big\|_{C^{1,\alpha}(\partial B_1)}\big) \\ &\le c\check{J}\big(\mu + \omega^{-2}|\ln\mu|\big). \end{aligned} \tag{9.6}$$

By maximal principle, (9.4), and (9.6),

$$\|\widehat{\phi}\|_{W^{1,\infty}(B_1)} + \|\widehat{\phi}\|_{W^{1,\infty}(B_{8/5\mu}\setminus B_1)} \le c\check{J}\big(\mu + \omega^{-2}|\ln\mu|\big). \tag{9.7}$$

By assumptions, (9.7), and the definition of $\widehat{\phi}$, we see $\|\nabla\phi\|_{L^\infty(B_{2/5})} \le c\check{J}$, which implies $(3.18)_1$ in Lemma 3.7. ■

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References

[1] A. A. Arkhipova and O. Erlhamahmy, Regularity of solutions to a diffraction-type problem for nondiagonal linear elliptic systems in the Campanato space. *J. Math. Sci. (New York)* **112** (2002), no. 1, 3944–3966 Zbl 1113.35321 MR 1946069

[2] M. Avellaneda and F.-H. Lin, Compactness methods in the theory of homogenization. *Comm. Pure Appl. Math.* **40** (1987), no. 6, 803–847 Zbl 0632.35018 MR 0910954

[3] M. Avellaneda and F.-H. Lin, Homogenization of elliptic problems with L^p boundary data. *Appl. Math. Optim.* **15** (1987), no. 2, 93–107 Zbl 0644.35034 MR 0868901

[4] E. S. Bao, Y. Y. Li, and B. Yin, Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions. *Comm. Partial Differential Equations* **35** (2010), no. 11, 1982–2006 Zbl 1218.35230 MR 2754076

[5] J. Bao, H. Li, and Y. Li, Gradient estimates for solutions of the Lamé system with partially infinite coefficients. *Arch. Ration. Mech. Anal.* **215** (2015), no. 1, 307–351 Zbl 1309.35160 MR 3296149

[6] X. Blanc, F. Legoll, and A. Anantharaman, Asymptotic behavior of Green functions of divergence form operators with periodic coefficients. *Appl. Math. Res. Express. AMRX* (2013), no. 1, 79–101 Zbl 1308.35069 MR 3040889

[7] M. Briane, Nonlocal effects in two-dimensional conductivity. *Arch. Ration. Mech. Anal.* **182** (2006), no. 2, 255–267 Zbl 1142.35381 MR 2259333

[8] M. Briane, Y. Capdeboscq, and L. Nguyen, Interior regularity estimates in high conductivity homogenization and application. *Arch. Ration. Mech. Anal.* **207** (2013), no. 1, 75–137 Zbl 1269.78021 MR 3004768

[9] M. Briane and J. Casado-Díaz, Asymptotic behaviour of equicoercive diffusion energies in dimension two. *Calc. Var. Partial Differential Equations* **29** (2007), no. 4, 455–479 Zbl 1186.35012 MR 2329805

[10] M. Briane and J. Casado-Díaz, Uniform convergence of sequences of solutions of two-dimensional linear elliptic equations with unbounded coefficients. *J. Differential Equations* **245** (2008), no. 8, 2038–2054 Zbl 1155.35047 MR 2446184

[11] L. A. Caffarelli and I. Peral, On $W^{1,p}$ estimates for elliptic equations in divergence form. *Comm. Pure Appl. Math.* **51** (1998), no. 1, 1–21 Zbl 0906.35030 MR 1486629

[12] D.-C. Chang, The dual of Hardy spaces on a bounded domain in $\mathbf{R}^n$. *Forum Math.* **6** (1994), no. 1, 65–81 Zbl 0803.42014 MR 1253178

[13] G. Chen and J. Zhou, *Boundary element methods with applications to nonlinear problems*. 2nd edn., Atlantis Stud. Math. Eng. Sci. 7, Atlantis Press, Paris; World Scientific, Hackensack, NJ, 2010 Zbl 1207.65001 MR 2683625

[14] J. B. Conway, *A course in functional analysis*. Grad. Texts in Math. 96, Springer, New York, 1985 Zbl 0558.46001 MR 0768926

[15] G. Dolzmann and S. Müller, Estimates for Green's matrices of elliptic systems by L^p theory. *Manuscripta Math.* **88** (1995), no. 2, 261–273 Zbl 0846.35040 MR 1354111

[16] L. Escauriaza and M. Mitrea, Transmission problems and spectral theory for singular integral operators on Lipschitz domains. *J. Funct. Anal.* **216** (2004), no. 1, 141–171 Zbl 1081.35020 MR 2091359

[17] L. C. Evans, *Weak convergence methods for nonlinear partial differential equations*. CBMS Reg. Conf. Ser. Math. 74, American Mathematical Society, Providence, RI, 1990 Zbl 0698.35004 MR 1034481

[18] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*. Ann. of Math. Stud. 105, Princeton University Press, Princeton, NJ, 1983 Zbl 0516.49003 MR 0717034

[19] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*. 2nd edn., Grundlehren Math. Wiss. 224, Springer, Berlin, 1983 Zbl 0361.35003 MR 0737190

[20] J. Douglas Jr., T. Arbogast, P. J. Paes-Leme, J. L. Hensley, and N. P. Nunes, Immiscible displacement in vertically fractured reservoirs. *Transport in Porous Media* **12** (1993), 73–106

[21] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of differential operators and integral functionals*. Springer, Berlin, 1994 Zbl 0838.35001 MR 1329546

[22] C. E. Kenig, F. Lin, and Z. Shen, Homogenization of elliptic systems with Neumann boundary conditions. *J. Amer. Math. Soc.* **26** (2013), no. 4, 901–937 Zbl 1277.35166 MR 3073881

[23] C. E. Kenig and Z. Shen, Homogenization of elliptic boundary value problems in Lipschitz domains. *Math. Ann.* **350** (2011), no. 4, 867–917 Zbl 1223.35139 MR 2818717

[24] W. Littman, G. Stampacchia, and H. F. Weinberger, Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **17** (1963), 43–77 Zbl 0116.30302 MR 0161019

[25] W. Niu and J. Zhuge, Compactness and stable regularity in multiscale homogenization. *Math. Ann.* **385** (2023), no. 3-4, 1431–1473 Zbl 1516.35046 MR 4566686

[26] W. Niu and J. Zhuge, Uniform Calderón–Zygmund estimate in multiscale elliptic homogenization. 2024, arXiv:2405.15149v1

[27] H. Salimi and H. Bruining, Upscaling in vertically fractured oil reservoirs using homogenization. *Transp. Porous Media* **84** (2010), no. 1, 21–53 MR 2719443

[28] Z. Shen, Large-scale Lipschitz estimates for elliptic systems with periodic high-contrast coefficients. *Comm. Partial Differential Equations* **46** (2021), no. 6, 1027–1057 Zbl 1470.35040 MR 4267502

[29] J. L. Taylor, S. Kim, and R. M. Brown, The Green function for elliptic systems in two dimensions. *Comm. Partial Differential Equations* **38** (2013), no. 9, 1574–1600 Zbl 1279.35021 MR 3169756

[30] Z. Wu, J. Yin, and C. Wang, *Elliptic & parabolic equations*. World Scientific, Hackensack, NJ, 2006 Zbl 1108.35001 MR 2309679

[31] Q. Xu, Uniform regularity estimates in homogenization theory of elliptic systems with lower order terms on the Neumann boundary problem. *J. Differential Equations* **261** (2016), no. 8, 4368–4423 Zbl 1353.35047 MR 3537832

[32] L.-M. Yeh, Elliptic equations in highly heterogeneous porous media. *Math. Methods Appl. Sci.* **33** (2010), no. 2, 198–223 Zbl 1180.35221 MR 2597205

[33] L.-M. Yeh, Non-uniform elliptic equations in convex Lipschitz domains. *Nonlinear Anal.* **118** (2015), 63–81 Zbl 1317.35022 MR 3325606

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