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Classification of decay estimate in a 4th-order quasilinear hyperbolic equation with strong damping

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Abstract. This paper deals with a 4th-order quasilinear hyperbolic equation involving strong damping and superlinear source,

$$u_{tt} - \Delta_m u + \Delta^2 u - \Delta_r u_t = |u|^{p-2}u, \quad (x,t) \in \Omega \times (0, T_{\max}),$$

subject to homogeneous Navier boundary condition, where Ω is an open bounded domain in $\mathbb{R}^n$ $(n > 2)$; $p > m \geq r \geq 2$; $\Delta_m u := \operatorname{div}(|\nabla u|^{m-2}\nabla u)$; and $\Delta_r u_t := \operatorname{div}(|\nabla u_t|^{r-2}\nabla u_t)$. For the positive initial energy case, we obtain the existence of global solutions, where the decay estimates are divided into five kinds for all the exponent regions. When the initial energy is negative, we arrive at the upper and lower bounds of blow-up time. The L^2 inner product $(u_1, u_0) > 0$ of the initial data is not a necessary condition on the existence of blow-up solutions in the region $\{p > m > 2 = r\}$.

1. Introduction

In this paper, we study the asymptotic behavior of weak solutions to an initial-boundary value problem of the 4th-order hyperbolic equation with r-Laplace damping term and superlinear source:

$$\begin{cases} u_{tt} - \Delta_m u + \Delta^2 u - \Delta_r u_t = |u|^{p-2}u, & (x,t) \in \Omega \times (0, T_{\max}), \\ u = 0, \quad \Delta u = 0, & (x,t) \in \partial\Omega \times (0, T_{\max}), \\ u(x,0) := u_0(x) \in H_0^2(\Omega) \cap W_0^{1,m}(\Omega), & x \in \Omega, \\ u_t(x,0) := u_1(x) \in L^2(\Omega), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary $\partial\Omega$; $T_{\max}(\leq +\infty)$ denotes the maximal existence time of (1.1); $p > m \geq r \geq 2$; Δ_α denotes the α-Laplace operator, that is, $\Delta_\alpha v := \operatorname{div}(|\nabla v|^{\alpha-2}\nabla v)$. High-order hyperbolic problems provide models for various phenomena in Mathematical Physics, such as the motion of elasto-plastic bars, nuclear physics, optics, geophysics, which appear naturally in inflation cosmology and super-symmetric field theory, quantum mechanics, and nuclear physics (see [1, 6]). The solutions of high-order hyperbolic partial differential equations are usually quite complex. Numerical or analytical methods are required, which vary greatly for different types

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of differential equations. Especially, the asymptotic behaviour of the solution was difficult to study, namely, the decay rate of global solutions, and the bounds of blow-up time of solutions.

Mokeddem and Mansour in [11] considered the following p-Laplacian wave equation with m-Laplacian dissipation (strong damping),

$$u_{tt} - \Delta_p u - a\Delta_m u_t = b|u|^{r-1}u, \quad (x,t) \in \Omega \times (0,+\infty),$$

subject to $u = 0$ on $\partial\Omega$, where $a, b > 0$ and $p, m, r \geq 2$. The authors extended the results by Ye (in [17, 18]) to the cases of m-Laplacian weak dissipation equations. Using the potential well theory, they obtained the existence of global solutions. By using different inequalities, they got the decay estimate of the energy.

The following nonlinear p-Laplacian wave equation with strong damping was studied in [3],

$$u_{tt} - \Delta_p u - \Delta u_t + g(x,t) = f(x), \quad (x,t) \in \Omega \times (0,T),$$

subject to $u = 0$ on $\partial\Omega$, where $2 \leq p < n$ and f, g are given functions. The authors obtained the global existence of this problem under suitable conditions on the initial data and the functions f, g. For more results, interested readers can refer to the works [9, 16, 19, 20].

In [15], Piskin studied the following quasilinear hyperbolic equation with strong damping,

$$u_{tt} - \Delta_m u - \Delta u_t + |u_t|^{q-1}u_t = |u|^{p-1}u, \quad (x,t) \in \Omega \times (0,T),$$

subject to $u = 0$ on $\partial\Omega$, where $m \geq 0$, $p, q \geq 1$. The author achieved the decay estimates of the energy function by utilizing the Nakao inequality and blow-up time estimates of solutions in different ranges.

Furthermore, the wave equations with different nonlinear damping terms and source terms,

$$u_{tt} - \Delta_\alpha u - \Delta u_t - \Delta_\beta u_t + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \quad (x,t) \in \Omega \times (0,+\infty), \quad (1.2)$$

subject to $u = 0$ on the boundary, have been discussed in [10], where $a, b, \alpha, \beta, m, p > 0$. The authors showed that if $p > \max\{\alpha, m\}$, $\alpha > \beta$, and the initial energy is negative, then any weak solution of (1.2) cannot exist for all time.

Chen and Xu in [5] considered the following 4th-order dispersive wave equation with nonlinear weak damping, linear strong damping, and logarithmic nonlinearity,

$$u_{tt} - \Delta u + \Delta^2 u - \omega(\Delta u_{tt} + \Delta u_t) + |u_t|^{r-1}u_t = u\ln|u|, \quad (x,t) \in \Omega \times (0,+\infty),$$

subject to $u = \Delta u = 0$ or $u = \frac{\partial u}{\partial v} = 0$ on the boundary $\partial\Omega$, where v is the unit outward normal on $\partial\Omega$; $\omega \in \{0, 1\}$; $r \geq 1$. On the basis of the potential well method, the authors constructed several initial data conditions that lead to the existence of global solutions or infinite time blow-up under subcritical initial energy conditions. In addition, they extended this result to the case of critical initial energy. For other relevant results, interested readers can refer to [2, 8, 14] and the references cited therein.

Motivated by the work [5, 10, 15], we consider the global existence and finite blow-up of solutions to high-order problem (1.1) involving strong damping term and nonlinear source, which has not been discussed before. We found that there was a lack of asymptotic estimates about the rates of decay solutions and blow-up time of solutions. The rest of this paper is organized as follows. In the next section, we give the definition of the weak solution and then introduce the potential well functional, the Nehari functional, and the energy functional with three preliminary lemmas. In Section 3, we prove the local existence and uniqueness of weak solutions of (1.1) by using the Banach fixed-point theorem. In Section 4, we prove that the solutions of problem (1.1) are globally bounded with positive initial energy and obtain decay estimates of weak solutions in all of the exponent regions. In Section 5, we obtain the upper and lower bounds of the blow-up time for negative initial energy.

2. Preliminaries

We denote by $\|\cdot\|_s$ the $L^s(\Omega)$ norm for $1 \leq s \leq \infty$. The norm in $H_0^2(\Omega)$ is defined by $\|u\|_{H_0^2} := \|\Delta u\|_2$, and the norm in $W_0^{1,m}(\Omega)$ is defined by $\|u\|_{W_0^{1,m}} := \|\nabla u\|_m$. The inner product in $L^2(\Omega)$ is denoted by $(f, g) := \int_\Omega f(x)g(x)\mathrm{d}x$. There are the following two different embedding inequalities, which play important roles in the proof of the main results, where m and p are the exponents in (1.1).

- If the condition
$$(\mathcal{H}_1) : p < p_1 := \begin{cases} \frac{2n}{n-4}, & n > 4, \\ +\infty, & n \leq 4, \end{cases}$$
holds, then $H_0^2(\Omega) \hookrightarrow L^p(\Omega)$, i.e., $\|u\|_p \leq S\|\Delta u\|_2$, where S is an embedding constant.
- If the condition
$$(\mathcal{H}_2) : p < p_2 := \begin{cases} \frac{mn}{n-m}, & n > m, \\ +\infty, & n \leq m, \end{cases}$$
holds, then $W_0^{1,m}(\Omega) \hookrightarrow L^p(\Omega)$, i.e., $\|u\|_p \leq \widetilde{S}\|\nabla u\|_m$, where $\widetilde{S}$ is an embedding constant.

Now, we give the definition of the weak solution of the problem (1.1).

Definition 2.1. The function $u(x,t)$ is called a weak solution of problem (1.1) on $\Omega \times [0,T](0 < T < \infty)$, if $u \in C([0,T]; H_0^2(\Omega)) \cap C([0,T]; W_0^{1,m}(\Omega))$, with

$$u_t \in L^r([0,T]; W_0^{1,r}(\Omega)) \cap L^\infty([0,T]; L^2(\Omega)), \;\; u_{tt} \in C([0,T]; (H_0^2(\Omega) \cap W_0^{1,m}(\Omega))'),$$

satisfying that $u(0) = u_0$, $u_t(0) = u_1$, and

$$\int_\Omega u_{tt}\phi \mathrm{d}x + \int_\Omega |\nabla u|^{m-2}\nabla u \cdot \nabla \phi \mathrm{d}x + \int_\Omega \Delta u \Delta \phi \mathrm{d}x + \int_\Omega |\nabla u_t|^{r-2}\nabla u_t \cdot \nabla \phi \mathrm{d}x$$
$$= \int_\Omega |u|^{p-2} u\phi \mathrm{d}x,$$

for any function $\phi \in H_0^2(\Omega) \cap W_0^{1,m}(\Omega)$ and a.e. $t \in [0, T]$.

Then, we introduce the following functionals. For $u \in H_0^2(\Omega) \cap W_0^{1,m}(\Omega)$, we define

$$\text{the potential well functional } J(u) := \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{m}\|\nabla u\|_m^m - \frac{1}{p}\|u\|_p^p, \tag{2.1}$$

$$\text{the Nehari functional } I(u) := \|\Delta u\|_2^2 + \|\nabla u\|_m^m - \|u\|_p^p. \tag{2.2}$$

Meanwhile, we introduce the energy functional

$$E(t) := \frac{1}{2}\|u_t\|_2^2 + J(u). \tag{2.3}$$

The following three lemmas will play an important role in the proof of the main results.

Lemma 2.1. *Let u be a weak solution of problem (1.1). Then, $E(t) \leq E(0)$.*

Proof. Multiplying (1.1) by u_t and then integrating over Ω, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 - \frac{1}{m}\|\nabla u\|_m^m - \frac{1}{p}\|u\|_p^p\right) = -\|\nabla u_t\|_r^r.$$

By (2.3), we get $\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\|\nabla u_t\|_r^r \leq 0$ for $t \in [0, T]$. By integration, we have $E(t) \leq E(0)$. ∎

Lemma 2.2. *Let $(\mathcal{H}_1, \mathcal{H}_2)$ hold. Assume $u_0 \in H_0^2(\Omega) \cap W_0^{1,m}(\Omega)$, $u_1 \in L^2(\Omega)$, $E(0) > 0$, $I(u_0) \geq 0$, and $\theta_1 + \theta_2 \leq 1$, where*

$$\theta_1 := \alpha \widetilde{S}^p \left(\frac{mp}{p-m}E(0)\right)^{\frac{p-m}{m}}, \quad \theta_2 := (1-\alpha)S^p\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}$$

with $0 < \alpha < 1$. Then, $I(u(t)) \geq 0$ for any $t \in [0, T]$.

Proof. By continuity, there exists a constant t_0 such that $I(u(t)) \geq 0$ for any $t \in [0, t_0]$. Combining (2.1) with (2.2), we get, for any $t \in [0, t_0]$,

$$\begin{aligned} J(u) &= \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{m}\|\nabla u\|_m^m - \frac{1}{p}\|u\|_p^p \\ &= \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{m}\|\nabla u\|_m^m + \frac{1}{p}I(u) - \frac{1}{p}\|\Delta u\|_2^2 - \frac{1}{p}\|\nabla u\|_m^m \\ &= \frac{p-2}{2p}\|\Delta u\|_2^2 + \frac{p-m}{mp}\|\nabla u\|_m^m + \frac{1}{p}I(u). \end{aligned}$$

Hence, by (2.3), Lemma 2.1, and $I(u(t)) \geq 0$ for any $t \in [0, t_0]$, we have

$$\frac{1}{2}\|u_t\|_2^2 + \frac{p-2}{2p}\|\Delta u\|_2^2 + \frac{p-m}{mp}\|\nabla u\|_m^m \leq E(t) \leq E(0).$$

Furthermore, there are the inequalities

$$\|\Delta u\|_2^2 \leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0), \tag{2.4}$$

$$\|\nabla u\|_m^m \leq \frac{mp}{p-m}E(t) \leq \frac{mp}{p-m}E(0). \tag{2.5}$$

In addition, combining (2.4), (2.5), and $(\mathscr{H}_1)$, $(\mathscr{H}_2)$, we obtain, for $0 < \alpha < 1$,

$$\begin{aligned}\|u\|_p^p &= \alpha\|u\|_p^p + (1-\alpha)\|u\|_p^p \\ &\leq \alpha\widetilde{S}^p\|\nabla u\|_m^{p-m}\|\nabla u\|_m^m + (1-\alpha)S^p\|\Delta u\|_2^{p-2}\|\Delta u\|_2^2 \\ &\leq \alpha\widetilde{S}^p\left(\frac{mp}{p-m}E(0)\right)^{\frac{p-m}{m}}\|\nabla u\|_m^m + (1-\alpha)S^p\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}\|\Delta u\|_2^2.\end{aligned}$$

Denote $\theta_1 := \alpha\widetilde{S}^p(\frac{mp}{p-m}E(0))^{\frac{p-m}{m}}$, $\theta_2 := (1-\alpha)S^p(\frac{2p}{p-2}E(0))^{\frac{p-2}{2}}$. Then,

$$\|u\|_p^p \leq \theta_1\|\nabla u\|_m^m + \theta_2\|\Delta u\|_2^2$$

for $t \in [0, t_0]$. Since $\theta_1 + \theta_2 < 1$, we have $\|u\|_p^p \leq \|\nabla u\|_m^m + \|\Delta u\|_2^2$ for any $t \in [0, t_0]$. By using the extensibility of solutions, we obtain $I(u(t)) \geq 0$ for any $t \in [0, T]$. ■

Lemma 2.3 ([11, Lemma 2.4]). *Suppose that E is a non-increasing non-negative function on $[0, \infty)$. If there is a non-negative constant β and a positive constant A such that $\int_s^{+\infty} E^{\beta+1}(t)\mathrm{d}t \leq AE(s)$ for $0 \leq s < +\infty$, then there are the inequalities*

$$E(t) \leq \left[A\left(1+\frac{1}{\beta}\right)\right]^{\frac{1}{\beta}} t^{\frac{-1}{\beta}}, \quad \textit{for } t > 0,\ \beta > 0,$$

$$E(t) \leq E(0)\exp\left(1-\frac{t}{A}\right), \quad \textit{for } t \geq 0,\ \beta = 0.$$

3. Local existence and uniqueness of weak solution

Before giving the proof of the existence of the weak solution defined in Definition 2.1, we consider a lemma related to a linear wave equation, which can be proved by Banach's fixed-point theorem. Interested readers can refer to [4, 7, 12, 13].

Lemma 3.1. *Let $u_0 \in H_0^2(\Omega) \cap W_0^{1,m}(\Omega)$ and $u_1 \in L^2(\Omega)$. Then, for any $T > 0$, $u \in C([0,T]; H_0^2(\Omega) \cap W_0^{1,m}(\Omega))$ and $u_t \in L^r([0,T]; W_0^{1,r}(\Omega)) \cap L^\infty([0,T]; L^2(\Omega))$, there* *exists a unique function $v(x,t)$, which satisfies*

$$
\begin{aligned}
v &\in C([0,T];H_0^2(\Omega))\cap C([0,T];W_0^{1,m}(\Omega)),\\
v_t &\in L^r([0,T];W_0^{1,r}(\Omega))\cap L^\infty([0,T];L^2(\Omega)),\\
v_{tt} &\in C([0,T];(H_0^2(\Omega)\cap W_0^{1,m}(\Omega))'),
\end{aligned}
$$

and

$$
\begin{cases}
v_{tt}-\Delta_m v+\Delta^2 v-\Delta_r v_t=|u|^{p-2}u, & (x,t)\in\Omega\times(0,T),\\
v=0,\quad \Delta v=0, & (x,t)\in\partial\Omega\times(0,T),\\
v(x,0)=u_0(x),\quad v_t(x,0)=u_1(x), & x\in\Omega.
\end{cases}
\tag{3.1}
$$

Theorem 3.1. *Assume $u_0\in H_0^2(\Omega)\cap W_0^{1,m}(\Omega)$ and $u_1\in L^2(\Omega)$. If p satisfies*

$$
\begin{cases}
2<p<\frac{2n-6}{n-4}, & n>4,\\
2<p<\infty, & n\le 4,
\end{cases}
$$

then problem (1.1) admits a unique local solution u on $\Omega\times[0,T]$. If

$$T_{\max}:=\sup\{T>0:u=u(t)\text{ exists on }[0,T]\}<\infty,$$

then $\lim_{t\to T_{\max}^-}(\|u_t\|_2+\|\Delta u\|_2+\|\nabla u\|_m)=+\infty$. *In addition,* $\lim_{t\to T_{\max}^-}\|u\|_p=+\infty$.

Proof. For every $T>0$, we consider the space $\mathcal{H}:=C([0,T];H_0^2(\Omega)\cap W_0^{1,p}(\Omega))\cap C^1([0,T];L^2(\Omega))$, endowed with the norm

$$\|u\|_{\mathcal{H}}^2:=\sup\{\|u_t\|_2^2+\|\Delta u\|_2^2+\|\nabla u\|_m^2,\ t\in[0,T]\}.$$

We use Banach's fixed-point theorem on a closed $B_R(0)\subset\mathcal{H}$. By Lemma 3.1, for every u, we get that problem (3.1) admits a unique local solution $v\in\mathcal{H}$ with

$$
\begin{aligned}
v_t &\in L^r([0,T];W_0^{1,r}(\Omega))\cap L^\infty([0,T];L^2(\Omega)),\\
v_{tt} &\in C([0,T];(H_0^2(\Omega)\cap W_0^{1,m}(\Omega))').
\end{aligned}
$$

Set $S(u)=v$.

At first, we prove that S maps $\mathcal{H}$ into $\mathcal{H}$. Multiplying (3.1) by v_t and integrating over $\Omega\times[0,t]$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\|v_t\|_2^2+\frac{1}{2}\|\Delta v\|_2^2+\frac{1}{m}\|\nabla v\|_m^m+\int_0^t\int_\Omega|\nabla v_s|^r\mathrm{d}x\mathrm{d}s\\
&\quad=\frac{1}{2}\|v_1\|_2^2+\frac{1}{2}\|\Delta v_0\|_2^2+\frac{1}{m}\|\nabla v_0\|_m^m+\int_0^t\int_\Omega|u|^{p-2}uv_s\mathrm{d}x\mathrm{d}s.
\end{aligned}
\tag{3.2}
$$

Applying Young's inequality and $H_0^2(\Omega)\hookrightarrow L^{2p-2}(\Omega)$, we get

$$\left|\int_\Omega|u|^{p-2}uv_s\mathrm{d}x\right|\le\frac{\varepsilon}{4}\|v_t\|_2^2+\frac{4}{\varepsilon}\|u\|_{2p-2}^{2p-2}\le\frac{\varepsilon}{4}\|v_t\|_2^2+\frac{4S_2^{2p-2}}{\varepsilon}\|\Delta u\|_2^{2p-2},\tag{3.3}$$

where S_2 is an embedding constant and ε is a small constant. Inserting (3.3) into (3.2), we have

$$\frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\|\Delta v\|_2^2 + \frac{1}{m}\|\nabla v\|_m^m + \int_0^t \int_\Omega |\nabla v_s|^r \mathrm{d}x\mathrm{d}s \\ \leq \frac{1}{2}\|v_1\|_2^2 + \frac{1}{2}\|\Delta v_0\|_2^2 + \frac{1}{m}\|\nabla v_0\|_m^m + \frac{\varepsilon}{4}\int_0^t \|v_s\|_2^2 \mathrm{d}s + \frac{4S_2^{2p-2}}{\varepsilon}\int_0^t \|\Delta u\|_2^{2p-2}\mathrm{d}s.$$

Hence, we get

$$\frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\|\Delta v\|_2^2 + \frac{1}{2}\|\nabla u\|_m^2 \leq \frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\|\Delta v\|_2^2 + \frac{1}{m}\|\nabla v\|_m^m + \left(\frac{1}{2}m^{\frac{2}{m}}\right)^{\frac{m}{m-2}} \\ \leq \frac{1}{2}\|v_1\|_2^2 + \frac{1}{2}\|\Delta v_0\|_2^2 + \frac{1}{m}\|\nabla v_0\|_m^m + \left(\frac{1}{2}m^{\frac{2}{m}}\right)^{\frac{m}{m-2}} \\ + \frac{\varepsilon}{4}\int_0^t \|v_s\|_2^2 \mathrm{d}s + \frac{4S_2^{2p-2}}{\varepsilon}\int_0^t \|\Delta u\|_2^{2p-2}\mathrm{d}s,$$

which implies

$$\sup_{t\in[0,T]} \|v_t\|_2^2 + \sup_{t\in[0,T]} \|\Delta v\|_2^2 + \sup_{t\in[0,T]} \|\nabla u\|_m^2 \\ \leq \lambda + \frac{\varepsilon T}{2}\sup_{t\in[0,T]} \|v_t\|_2^2 + \frac{8S_2^{2p-2}T}{\varepsilon}\|u\|_{\mathcal{H}}^{2p-2},$$

where $\lambda := \frac{1}{2}\|v_1\|_2^2 + \frac{1}{2}\|\Delta v_0\|_2^2 + \frac{1}{m}\|\nabla v_0\|_m^m + (\frac{1}{2}m^{\frac{2}{m}})^{\frac{m}{m-2}}$.

By taking $\frac{\varepsilon T}{2} < 1$, we have $\|v\|_{\mathcal{H}}^2 \leq \lambda + \frac{8S_2^{2p-2}T}{\varepsilon}\|u\|_{\mathcal{H}}^{2p-2}$. We choose R large enough such that if $\lambda < R^2$ and $T < \frac{\varepsilon(R^2-\lambda)}{8S_2^{2p-2}R^{2p-2}}$, then $\|v\|_{\mathcal{H}}^2 \leq \lambda + \frac{8S_2^{2p-2}T}{\varepsilon}R^{2p-2} \leq R^2$. Hence, $S : \mathcal{H} \mapsto \mathcal{H}$.

Next, we will show that S is a contraction mapping. Let $v_1 := S(u_1)$, $v_2 := S(u_2)$ and $v := v_1 - v_2$, $u := u_1 - u_2$. Then, v is the unique solution to the equation

$$v_{tt} + \Delta^2 v - (\Delta_m v_1 - \Delta_m v_2) - (\Delta_r v_{1t} - \Delta_r v_{2t}) = |u_1|^{p-2}u_1 - |u_2|^{p-2}u_2. \quad (3.4)$$

Multiplying (3.4) by v_t and integrating on $\Omega \times (0, t)$, we get

$$\frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\|\Delta v\|_2^2 + \int_0^t \int_\Omega \left(|\nabla v_1|^{m-2}\nabla v_1 - |\nabla v_2|^{m-2}\nabla v_2\right) \cdot (\nabla v_{1s} - \nabla v_{2s})\mathrm{d}x\mathrm{d}s \\ + \int_0^t \int_\Omega \left(|\nabla v_{1s}|^{m-2}\nabla v_{1s} - |\nabla v_{2s}|^{m-2}\nabla v_{2s}\right) \cdot (\nabla v_{1s} - \nabla v_{2s})\mathrm{d}x\mathrm{d}s \\ = \int_0^t \int_\Omega \left(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\right)v_s \mathrm{d}x\mathrm{d}s.$$

By applying $(|a|^{\gamma-1}a - |b|^{\gamma-1}b)(a-b) \geq 0$ for $a, b, \in \mathbb{R}^n$, $\gamma > 0$ and

$$|\alpha - \beta|^m \leq 2^{m-2}(|\alpha|^{m-2}\alpha - |\beta|^{m-2}\beta) \cdot (\alpha - \beta)$$

for $\alpha, \beta, \in \mathbb{R}^n$, $2 \leq m < \infty$, denoting $q(u) := |u|^{p-2}u$, we get

$$\frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\|\Delta v\|_2^2 + \frac{2^{2-m}}{m}\|\nabla v\|_m^m \leq \int_0^t \int_\Omega (q(u_1) - q(u_2))v_s \mathrm{d}x\mathrm{d}s. \tag{3.5}$$

Using Lagrange's theorem, for $0 < \alpha < 1$, $\xi := \alpha u_1 + (1-\alpha)u_2$, and Young's inequality, and Hölder's inequality, we have

$$\begin{aligned}
&\int_\Omega (q(u_1) - q(u_2))v_t \mathrm{d}x \\
&= \int_\Omega q'(\xi)uv_t \mathrm{d}x \\
&\leq \frac{\delta}{2}\|v_t\|_2^2 + \frac{2}{\delta}\int_\Omega |q'(\xi)|^2|u|^2 \mathrm{d}x \\
&\leq \frac{\delta}{2}\|v_t\|_2^2 + \frac{2}{\delta}(p-1)^2\int_\Omega |\alpha u_1 + (1-\alpha)u_2|^{2(p-2)}|u|^2 \mathrm{d}x \\
&\leq \frac{\delta}{2}\|v_t\|_2^2 + \frac{2}{\delta}(p-1)^2\left(\int_\Omega |u|^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{n-2}{n}}\left(\int_\Omega |\alpha u_1 + (1-\alpha)u_2|^{n(p-2)}\mathrm{d}x\right)^{\frac{2}{n}} \\
&\leq \frac{\delta}{2}\|v_t\|_2^2 + \frac{2}{\delta}(p-1)^2\|u\|_{\frac{2n}{n-2}}^2\left(\int_\Omega |u_1 + u_2|^{n(p-2)}\mathrm{d}x\right)^{\frac{2}{n}} \\
&\leq \frac{\delta}{2}\|v_t\|_2^2 + C_\delta\|\Delta u\|_2^2(\|\Delta u_1\|_2^{2(p-2)} + \|\Delta u_2\|_2^{2(p-2)}) \\
&\leq \frac{\delta}{2}\|v_t\|_2^2 + 2C_\delta\|\Delta u\|_2^2\|u\|_{\mathcal{H}}^{2(p-2)},
\end{aligned} \tag{3.6}$$

where C_δ is a positive constant related to δ, p and the embedding constants in $H_0^2(\Omega) \hookrightarrow L^{n(p-2)}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$.

Hence, taking (3.6) into (3.5), choosing R large enough and δ small enough, we obtain

$$\frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\|\Delta v\|_2^2 + \frac{2^{2-m}}{m}\|\nabla v\|_m^m \leq 4C_\delta R^{2(p-2)}T\|u\|_{\mathcal{H}}^2.$$

Choosing T small enough, we obtain that S is a contraction mapping.

If $\|u\|_{\mathcal{H}}^2$ remains bounded, then, by the continuation argument, we get that the solution may be continued. Therefore, if $T_{\max} < \infty$, we get

$$\lim_{t \to T_{\max}^-}\|u\|_{\mathcal{H}}^2 = \lim_{t \to T_{\max}^-}(\|u_t\|_2 + \|\Delta u\|_2 + \|\nabla u\|_m) = +\infty. \tag{3.7}$$

In addition, by (2.1), (2.3), and Lemma 2.1, we have

$$\begin{aligned}
\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_m^2 &\leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{m}\|\nabla u\|_m^m + \left(\frac{1}{2}m^{\frac{2}{m}}\right)^{\frac{m}{m-2}} \\
&\leq E(0) - J(u) + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{m}\|\nabla u\|_m^m + \left(\frac{1}{2}m^{\frac{2}{m}}\right)^{\frac{m}{m-2}} \\
&= E(0) + \left(\frac{1}{2}m^{\frac{2}{m}}\right)^{\frac{m}{m-2}} + \frac{1}{p}\|u\|_p^p.
\end{aligned}$$

It can be checked from (3.7) that $\lim_{t \to T_{\max}^-}\|u\|_p = +\infty$. ∎

Remark 3.1. If $(\mathscr{H}_1)$ holds and $\lim_{t\to T_{\max}^-}\|u\|_p = +\infty$, then $\lim_{t\to T_{\max}^-}\|\Delta u\|_2 = \infty$. If $(\mathscr{H}_2)$ holds and $\lim_{t\to T_{\max}^-}\|u\|_p = +\infty$, then

$$\lim_{t\to T_{\max}^-}\|\nabla u\|_m = \infty.$$

4. Decay estimates of global solutions

Theorem 4.1. *Let $(\mathscr{H}_1, \mathscr{H}_2)$ hold. Assume $u_0 \in H_0^2(\Omega)\cap W_0^{1,m}(\Omega)$, $u_1 \in L^2(\Omega)$, $E(0) > 0$, and $I(u_0) \geq 0$. Then, the local solution of problem (1.1) is globally bounded and satisfies the following decay estimates.*

(i) *If $m = r = 2$, then there exists a constant $\omega > 0$, which is independent on $E(0)$, such that*

$$E(t) \leq E(0)\exp\left(1 - \frac{t}{\omega}\right) \quad \text{for all } t > 0.$$

(ii) *If $m > 2 = r$, then there exists a constant $\tau_1 > 0$ depending on $E(0)$ such that*

$$E(t) \leq \left(\frac{\tau_1}{t}\right)^{\frac{m}{m-2}} \quad \text{for all } t > 0.$$

(iii) *If $2 < r < 3$ and $r \leq m \leq \frac{2}{3-r}$, then there exists a constant $\tau_2 > 0$ depending on $E(0)$ such that*

$$E(t) \leq \left(\frac{\tau_2}{t}\right)^{\frac{2}{r-2}} \quad \text{for all } t > 0.$$

(iv) *If $2 < r < 3$ and $m > \frac{2}{3-r}$, then there exists a constant $\tau_3 > 0$ depending on $E(0)$ such that*

$$E(t) \leq \left(\frac{\tau_3}{t}\right)^{\frac{m(r-1)}{m-r}} \quad \text{for all } t > 0.$$

(v) *If $m \geq r \geq 3$, then there exists a constant $\tau_4 > 0$ depending on $E(0)$ such that*

$$E(t) \leq \left(\frac{\tau_4}{t}\right)^{\frac{2}{r-2}} \quad \text{for all } t > 0.$$

Remark 4.1. Theorem 4.1 provides all kinds of decay estimates for the energy function $E(t)$ of global solutions obtained in the exponent regions $m \geq r \geq 2$. Five kinds of decay estimates are shown in Figure 4.1, where the decay rates are compatible with each other on the boundaries of different exponent regions.

Proof of Theorem 4.1. By Lemma 2.2, using $E(0) > 0$ and $I(u_0) \geq 0$, we derive that $I(u(t)) \geq 0$ for any $t \in [0, T]$. By the definition of $J(u)$, $I(u)$, and $E(t)$, we have

$$E(t) \geq \frac{1}{2}\|u_t\|_2^2 + \frac{p-2}{2p}\|\Delta u\|_2^2 + \frac{p-m}{mp}\|\nabla u\|_m^m.$$

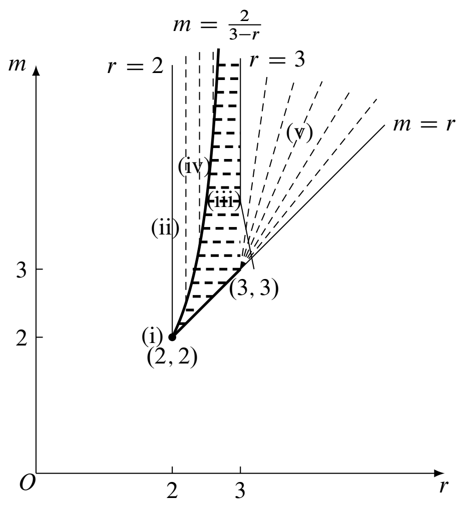

Figure 4.1. The decay estimates of energy.

Hence, by Lemma 2.1, we obtain

$$\|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_m^m \le \max\Big\{2, \frac{2p}{p-2}, \frac{mp}{p-m}\Big\} E(t) \le \bar{C} E(0) < +\infty,$$

where $\bar{C} := \max\{2, \frac{2p}{p-2}, \frac{mp}{p-m}\}$. This implies that the weak solution is global.

Next, we will give the decay estimates of the energy function of global solutions. Multiplying (1.1) by $E^q(t)u$ and integrating over $\Omega \times [S, T]$, we get

$$\int_S^T \int_\Omega E^q(t) u(u_{tt} + \Delta^2 u - \Delta_m u - \Delta_r u_t) \mathrm{d}x\mathrm{d}t = \int_S^T \int_\Omega E^q(t)|u|^p \mathrm{d}x\mathrm{d}t, \quad (4.1)$$

where $0 \le S \le T \le +\infty$ and q is a positive constant, which will be given later. Since

$$\int_S^T \int_\Omega E^q(t) u u_{tt} \mathrm{d}x\mathrm{d}t = \int_S^T E^q(t) \int_\Omega (u u_t)_t \mathrm{d}x\mathrm{d}t - \int_S^T \int_\Omega E^q(t)|u_t|^2 \mathrm{d}x\mathrm{d}t,$$

by using (4.1), we have

$$\int_S^T \int_\Omega E^q(t)(u u_t)_t \mathrm{d}x\mathrm{d}t - \int_S^T \int_\Omega E^q(t)|u_t|^2 \mathrm{d}x\mathrm{d}t + \int_S^T \int_\Omega E^q(t)|\Delta u|^2 \mathrm{d}x\mathrm{d}t$$
$$+ \int_S^T \int_\Omega E^q(t)|\nabla u|^m \mathrm{d}x\mathrm{d}t + \int_S^T \int_\Omega E^q(t)|\nabla u_t|^{r-2}\nabla u_t \cdot \nabla u \mathrm{d}x\mathrm{d}t$$
$$= \int_S^T \int_\Omega E^q(t)|u|^p \mathrm{d}x\mathrm{d}t. \quad (4.2)$$

By (4.2), we obtain

$$\begin{aligned}
&\int_S^T\int_\Omega E^q(t)(uu_t)_t\mathrm{d}x\mathrm{d}t+\int_S^T\int_\Omega E^q(t)|\nabla u_t|^{r-2}\nabla u_t\cdot\nabla u\mathrm{d}x\mathrm{d}t\\
&+\int_S^T E^q(t)[\theta_1\|\nabla u\|_m^m+\theta_2\|\Delta u\|_2^2+(2+\theta_1+\theta_2)\|u_t\|_2^2]\mathrm{d}t\\
&-(3+\theta_1+\theta_2)\int_S^T\int_\Omega E^q(t)|u_t|^2\mathrm{d}x\mathrm{d}t+(1-\theta_1)\int_S^T\int_\Omega E^q(t)|\nabla u|^m\mathrm{d}x\mathrm{d}t\\
&+(1-\theta_2)\int_S^T\int_\Omega E^q(t)|\Delta u|^2\mathrm{d}x\mathrm{d}t\\
&=\int_S^T\int_\Omega E^q(t)(uu_t)_t\mathrm{d}x\mathrm{d}t+\int_S^T\int_\Omega E^q(t)|\nabla u_t|^{r-2}\nabla u_t\cdot\nabla u\mathrm{d}x\mathrm{d}t\\
&\quad-(3-\theta_1-\theta_2)\int_S^T\int_\Omega E^q(t)|u_t|^2\mathrm{d}x\mathrm{d}t+(1-\theta_1)\int_S^T\int_\Omega E^q(t)(|\nabla u|^m+|u_t|^2)\mathrm{d}x\mathrm{d}t\\
&\quad+(1-\theta_2)\int_S^T\int_\Omega E^q(t)(|u_t|^2+|\Delta u|^2)\mathrm{d}x\mathrm{d}t\\
&=\int_S^T\big(-\theta_1\|\nabla u\|_m^m-\theta_2\|u_t\|_2^2+\|u\|_p^p\big)E^q(t)\mathrm{d}t\le 0.
\end{aligned}\tag{4.3}$$

For $k:=\min\{1-\theta_1,1-\theta_2\}$, we get

$$\begin{aligned}
k\int_S^T E^{q+1}(t)\mathrm{d}t&=k\int_S^T E^q(t)\left(\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\Delta u\|_2^2+\frac{1}{m}\|\nabla u\|_m^m-\frac{1}{p}\|u\|_p^p\right)\mathrm{d}t\\
&\le(1-\theta_1)\int_S^T E^q(t)\left(\frac{1}{2}\|u_t\|_2^2+\frac{1}{m}\|\nabla u\|_m^m\right)\mathrm{d}t\\
&\quad+(1-\theta_2)\int_S^T E^q(t)\left(\frac{1}{2}\|\Delta u\|_2^2+\frac{1}{2}\|u_t\|_2^2\right)\mathrm{d}t.
\end{aligned}$$

Using (4.3), we have

$$\begin{aligned}
k\int_S^T E^{q+1}(t)\mathrm{d}t\le&-\int_S^T\int_\Omega E^q(t)(uu_t)_t\mathrm{d}x\mathrm{d}t\\
&-\int_S^T\int_\Omega E^q(t)|\nabla u_t|^{r-2}\nabla u_t\cdot\nabla u\mathrm{d}x\mathrm{d}t\\
&+(3-\theta_1-\theta_2)\int_S^T E^q(t)\|u_t\|_2^2\mathrm{d}t.
\end{aligned}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}(E^q(t)\int_\Omega uu_t\mathrm{d}x)=E^q(t)\int_\Omega(uu_t)_t\mathrm{d}x+qE^{q-1}(t)E'(t)\int_\Omega uu_t\mathrm{d}x$, we get

$$k\int_S^T E^{q+1}(t)\mathrm{d}t\le I_1+I_2+I_3+I_4,\tag{4.4}$$

where

$$I_1 := -\int_S^T \frac{\mathrm{d}}{\mathrm{d}t}\left(E^q(t)\int_\Omega uu_t\mathrm{d}x\right)\mathrm{d}t,$$
$$I_2 := q\int_S^T E^{q-1}(t)E'(t)\int_\Omega uu_t\mathrm{d}x\mathrm{d}t,$$
$$I_3 := (3-\theta_1-\theta_2)\int_S^T E^q(t)\|u_t\|_2^2\mathrm{d}t,$$
$$I_4 := -\int_S^T\int_\Omega E^q(t)|\nabla u_t|^{r-2}\nabla u_t\cdot\nabla u\mathrm{d}x\mathrm{d}t.$$

Now, let us estimate I_1, I_2, I_3, and I_4, respectively.

$$\begin{aligned} I_1 &= -\int_S^T \frac{\mathrm{d}}{\mathrm{d}t}\left(E^q(t)\int_\Omega uu_t\mathrm{d}x\right)\mathrm{d}t \\ &\le \left|E^q(S)\int_\Omega u(S)u_t(S)\mathrm{d}x - E^q(T)\int_\Omega u(T)u_t(T)\mathrm{d}x\right| \\ &\le \left|E^q(S)\int_\Omega u(S)u_t(S)\mathrm{d}x\right| + \left|E^q(T)\int_\Omega u(T)u_t(T)\mathrm{d}x\right|. \end{aligned}$$

By Lemma 2.1, we have $\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\int_\Omega |\nabla u_t|^r\mathrm{d}x \le 0$, and hence, $E(t)$ is a non-increasing function in variable t, that is, $E(T)\le E(S)$. On the other hand, by applying Young's inequality, Lemma 2.2, and $\|u\|_2^2 \le S_3^2\|\Delta u\|_2^2$, we have

$$\begin{aligned} \left|\int_\Omega uu_t\mathrm{d}x\right| &\le \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u\|_2^2 \le \frac{1}{2}\|u_t\|_2^2 + \frac{S_3^2}{2}\|\Delta u\|_2^2 \\ &= \frac{1}{2}\|u_t\|_2^2 + \frac{S_3^2}{2}\frac{2p}{p-2}\frac{p-2}{2p}\|\Delta u\|_2^2 \le C_1E(t), \end{aligned} \tag{4.5}$$

where $C_1 := \max\left\{1, \frac{S_3^2 p}{p-2}\right\}$ is a positive constant. Hence,

$$I_1 \le C_1E^{q+1}(S) + C_1E^{q+1}(T) \le 2C_1E^{q+1}(S). \tag{4.6}$$

Also, using (4.5) and Lemma 2.1, we get

$$\begin{aligned} I_2 &\le \left|q\int_S^T E^{q-1}(t)E'(t)\int_\Omega uu_t\mathrm{d}x\mathrm{d}t\right| \le -qC_1\int_S^T E^q(t)E'(t)\mathrm{d}t \\ &= \frac{qC_1}{q+1}(E^{q+1}(S)-E^{q+1}(T)) \le 2C_2E^{q+1}(S), \end{aligned} \tag{4.7}$$

where $C_2 := \frac{qC_1}{q+1}$.

For I_3, by the embedding $\|u\|_2^2 \le S_4^2\|\nabla u\|_r^2$, Lemma 2.1, and Young's inequality, we have

$$\begin{aligned}
I_3 &= (3-\theta_1-\theta_2)\int_S^T E^q(t)\|u_t\|_2^2 \mathrm{d}t \\
&\le (3-\theta_1-\theta_2)\int_S^T E^q(t)(S_4^r\|\nabla u_t\|_r^r)^{\frac{2}{r}}\mathrm{d}t \\
&= (3-\theta_1-\theta_2)S_4^2\int_S^T E^q(t)(-E'(t))^{\frac{2}{r}}\mathrm{d}t \\
&\le (3-\theta_1-\theta_2)S_4^2\varepsilon_1^{\frac{r}{r-2}}\frac{r-2}{r}\int_S^T E^{\frac{qr}{r-2}}(t)\mathrm{d}t + (3-\theta_1-\theta_2)S_4^2\varepsilon_1^{\frac{-r}{2}}\frac{2}{r}\int_S^T(-E'(t))\mathrm{d}t \\
&= C_3\varepsilon_1^{\frac{r}{r-2}}\int_S^T E^{\frac{qr}{r-2}}(t)\mathrm{d}t + C_4\varepsilon_1^{\frac{-r}{2}}(E(S)-E(T)) \\
&\le C_3\varepsilon_1^{\frac{r}{r-2}}\int_S^T E^{\frac{qr}{r-2}}(t)\mathrm{d}t + C_4\varepsilon_1^{\frac{-r}{2}}E(S), \qquad (4.8)
\end{aligned}$$

where $C_3 := (3-\theta_1-\theta_2)S_4^2(r-2)/r$, $C_4 := 2(3-\theta_1-\theta_2)S_4^2/r$, and $\varepsilon_1 > 0$.

For I_4, by Hölder's inequality, we have

$$\begin{aligned}
\int_\Omega |\nabla u_t|^{r-2}\nabla u_t\cdot\nabla u\mathrm{d}x &\le \left(\int_\Omega |\nabla u_t|^{\frac{m(r-1)}{m-1}}\mathrm{d}x\right)^{\frac{m-1}{m}}\left(\int_\Omega|\nabla u|^m\mathrm{d}x\right)^{\frac{1}{m}} \\
&\le \left[\left(\int_\Omega|\nabla u_t|^r\mathrm{d}x\right)^{\frac{m(r-1)}{r(m-1)}}|\Omega|^{1-\frac{m(r-1)}{r(m-1)}}\right]^{\frac{m-1}{m}}\left(\int_\Omega|\nabla u|^m\mathrm{d}x\right)^{\frac{1}{m}} \\
&= |\Omega|^{\frac{m-r}{mr}}\|\nabla u_t\|_r^{r-1}\|\nabla u\|_m. \qquad (4.9)
\end{aligned}$$

Hence, thanks to Lemma 2.1 and Lemma 2.2, using Young's inequality and (4.9), we have

$$\begin{aligned}
I_4 &\le \left|\int_S^T\int_\Omega E^q(t)|\nabla u_t|^{r-2}\nabla u_t\cdot\nabla u\mathrm{d}x\mathrm{d}t\right| \\
&\le |\Omega|^{\frac{m-r}{mr}}\int_S^T E^q(t)\|\nabla u_t\|_r^{r-1}\|\nabla u\|_m\mathrm{d}t \\
&\le |\Omega|^{\frac{m-r}{mr}}\int_S^T E^q(t)(-E'(t))^{\frac{r-1}{r}}\left(\frac{mp}{p-m}E(t)\right)^{\frac{1}{m}}\mathrm{d}t \\
&= |\Omega|^{\frac{m-r}{mr}}\left(\frac{mp}{p-m}\right)^{\frac{1}{m}}\int_S^T E^{q+\frac{1}{m}}(t)(-E'(t))^{\frac{r-1}{r}}\mathrm{d}t \\
&\le |\Omega|^{\frac{m-r}{mr}}\left(\frac{mp}{p-m}\right)^{\frac{1}{m}}\left[\frac{\varepsilon_2^r}{r}\int_S^T E^{(q+\frac{1}{m})r}(t)\mathrm{d}t + \frac{r-1}{r}\varepsilon_2^{-\frac{1}{r-1}}\int_S^T(-E'(t))\mathrm{d}t\right]
\end{aligned}$$

$$= C_5\varepsilon_2^r \int_S^T E^{(q+\frac{1}{m})r}(t)\mathrm{d}t + C_6\varepsilon_2^{-\frac{1}{r-1}} \int_S^T (-E'(t))\mathrm{d}t$$
$$\le C_5\varepsilon_2^r \int_S^T E^{(q+\frac{1}{m})r}(t)\mathrm{d}t + C_6\varepsilon_2^{-\frac{1}{r-1}} E(S), \tag{4.10}$$

where $C_5 := \frac{1}{r}|\Omega|^{\frac{m-r}{mr}}(\frac{mp}{p-m})^{\frac{1}{m}}$, $C_6 := \frac{r-1}{r}|\Omega|^{\frac{m-r}{mr}}(\frac{mp}{p-m})^{\frac{1}{m}}$, and $\varepsilon_2 > 0$.

By inserting (4.6)–(4.8), (4.10) into (4.4), we arrive at

$$k\int_S^T E^{q+1}(t)\mathrm{d}t \le (2C_1 + 2C_2)E^{q+1}(S) + C_3\varepsilon_1^{\frac{r}{r-2}} \int_S^T E^{\frac{qr}{r-2}}(t)\mathrm{d}t$$
$$+ \big(C_4\varepsilon_1^{-\frac{r}{2}} + C_6\varepsilon_2^{-\frac{r}{r-1}}\big)E(S) + C_5\varepsilon_2^r \int_S^T E^{(q+\frac{1}{m})r}(t)\mathrm{d}t. \tag{4.11}$$

Now, we give the proof of the decay estimates of global solutions. It could be checked from Figure 4.1 that the exponent regions in cases i, ii, iii, iv, and v cover the whole region $m \ge r \ge 2$. In the following, C denotes a positive constant which could be different from line to line and even in the same line.

Case i. If $m = r = 2$, we choose $q = 0$ and small ε_2. From (4.11), we get

$$\int_S^T E^{q+1}(t)\mathrm{d}t \le \omega E(S),$$

where ω is a positive constant independent of $E(0)$. Hence, by Lemma 2.3, we arrive at $E(t) \le E(0)\exp(1 - t/\omega)$ for all $t > 0$.

Case ii. If $m > 2 = r$, we choose q such that $q + 1 = 2(q + \frac{1}{m})$, that is, $q = \frac{m-2}{m}$. Choosing ε_2 small enough, from (4.11), we get

$$\int_S^T E^{q+1}(t)\mathrm{d}t \le CE^{q+1}(S) + C'E(S) \le (CE^q(0) + C')E(S),$$

where C, C' are different constants, which are independent of $E(0)$. Hence, by Lemma 2.3,

$$E(t) \le \left[(CE^q(0) + C')\left(1 + \frac{1}{q}\right)\right]^{\frac{1}{q}} t^{-\frac{1}{q}} = \left(\frac{\tau_1}{t}\right)^{\frac{m}{m-2}} \quad \text{for all } t > 0,$$

where τ_1 is a positive constant depending on $E(0)$.

Case iii. If $2 < r < 3$, $r \le m \le \frac{2}{3-r}$, we choose q such that $q + 1 = \frac{qr}{r-2}$, i.e., $q = \frac{r-2}{2}$, and choose α_1 such that $q + 1 + \alpha_1 = (q + \frac{1}{m})r$, i.e., $\alpha_1 = \frac{(r-2)(r-1)}{2} + \frac{r}{m} - 1 > 0$. Setting $\varepsilon_2 := \varepsilon E^{-\frac{\alpha_1}{r}}(0)$ and choosing ε_1, ε small enough, from (4.11), we get

$$\int_S^T E^{q+1}(t)\mathrm{d}t \le CE^{q+1}(S) + C'E(S) + C''E^{\frac{r-2}{2}+\frac{m-r}{m(r-1)}}(0)E(S)$$
$$\le (CE^q(0) + C' + C''E^{\frac{r-2}{2}+\frac{m-r}{m(r-1)}}(0))E(S),$$

where C, C', and C'' are different constants, which are independent of $E(0)$. Hence, by Lemma 2.3,

$$E(t) \le \left(CE^q(0) + C' + C''E^{\frac{r-2}{2}+\frac{m-r}{m(r-1)}}(0)\right)^{\frac{1}{q}}\left(1+\frac{1}{q}\right)^{\frac{1}{q}} t^{-\frac{1}{q}}$$
$$= \left(\frac{\tau_2}{t}\right)^{\frac{2}{r-2}} \quad \text{for all } t > 0,$$

where τ_2 is a positive constant depending on $E(0)$.

Case iv. If $2 < r < 3$ and $m > \frac{2}{3-r}$, we choose q such that $q+1 = r(q+\frac{1}{m})$, i.e., $q = \frac{m-r}{m(r-1)}$, and choose α_2 such that $q+1+\alpha_2 = \frac{qr}{r-2}$, i.e., $\alpha_2 = \frac{r(3m-mr-2)}{m(r-2)(r-1)} > 0$. Setting $\varepsilon_1 = \varepsilon E^{-\frac{r-2}{r}\alpha_2}(0)$ and choosing ε_2, ε small enough, from (4.11), we get

$$\int_S^T E^{q+1}(t)\mathrm{d}t \le CE^{q+1}(S) + C'E(S) + C''E^{\frac{r(3m-mr-2)}{2m(r-1)}}(0)E(S)$$
$$\le (CE^q(0) + C' + C''E^{\frac{r(3m-mr-2)}{2m(r-1)}}(0))E(S),$$

where C, C', and C'' are different constants independent of $E(0)$. Hence, by Lemma 2.3,

$$E(t) \le \left(CE^q(0) + C' + C''E^{\frac{r(3m-mr-2)}{2m(r-1)}}(0)\right)^{\frac{1}{q}}\left(1+\frac{1}{q}\right)^{\frac{1}{q}} t^{-\frac{1}{q}}$$
$$= \left(\frac{\tau_3}{t}\right)^{\frac{m(r-1)}{m-r}} \quad \text{for all } t > 0,$$

where τ_3 is a positive constant depending on $E(0)$.

Case v. If $m \ge r \ge 3$, we choose q such that $q+1 = \frac{qr}{r-2}$, i.e., $q = \frac{r-2}{2}$, and choose α_1 such that $q+1+\alpha_1 = (q+\frac{1}{m})r$, i.e., $\alpha_1 = \frac{(r-2)(r-1)}{2} + \frac{r}{m} - 1 > 0$. Setting $\varepsilon_2 = \varepsilon E^{-\frac{\alpha_1}{r}}(0)$ and choosing ε_1, ε small enough, from (4.11), we get

$$\int_S^T E^{q+1}(t)\mathrm{d}t \le CE^{q+1}(S) + C'E(S) + C''E^{\frac{r-2}{2}+\frac{m-r}{m(r-1)}}(0)E(S)$$
$$\le (CE^q(0) + C' + C''E^{\frac{r-2}{2}+\frac{m-r}{m(r-1)}}(0))E(S),$$

where C, C', and C'' are different constants, which are independent of $E(0)$. Hence, by Lemma 2.3, $E(t) \le (\frac{\tau_4}{t})^{\frac{2}{r-2}}$ for all $t > 0$, where τ_4 is a positive constant depending on $E(0)$. ∎

5. Blow-up of solutions

5.1. Negative initial energy case

In this section, we get the lower and upper bounds of the blow-up time for the negative initial energy case.

Theorem 5.1. *Let* $(\mathcal{H}_2)$ *hold. If* $u_0 \in H_0^2(\Omega) \cap W_0^{1,m}(\Omega)$, $u_1 \in L^2(\Omega)$, $E(0) < 0$, *and* $p > m > r \geq 2$, *then the weak solution* u *of* (1.1) *blows up at some finite time* $T_{\max}$ *in the sense that* $\lim_{t \to T_{\max}^-} \|u\|_p = +\infty$, *and the blow-up time* $T_{\max}$ *satisfies* $T_{\max} \leq \frac{1-\alpha}{\zeta\alpha} L^{-\frac{\alpha}{1-\alpha}}(0)$, *where* $0 < \alpha < \min\{\frac{m-r}{p(r-1)}, \frac{m-2}{2m}\}$; ζ *is a positive constant;*

$$L(0) := (-E(0))^{1-\alpha} + \varepsilon \int_\Omega u_1 u_0 \mathrm{d}x > 0,$$

where ε *is a small constant and will be given below. In addition, if* p *satisfies*

$$\begin{cases} 2 < p < \frac{2(n-2)}{n-4}, & n > 4, \\ 2 < p < \infty, & n = 3, 4, \end{cases}$$

then the blow-up time

$$T_{\max} \geq \int_{M(0)}^{+\infty} \frac{1}{s + \widetilde{C}(\frac{2}{m})^{p-2} s^{p-1}} \mathrm{d}s, \tag{5.1}$$

where $M(0) := \frac{m}{2}\|u_1\|_2^2 + \frac{m}{2}\|\Delta u_0\|_2^2 + \|\nabla u_0\|_m^m$, *and* $\widetilde{C}$ *is the embedding constant in the embedding* $H_0^2(\Omega) \hookrightarrow L^{2(p-1)}(\Omega)$.

Proof. Let $H(t) := -E(t)$. By Lemma 2.1, we get $H'(t) = -E'(t) \geq 0$. Therefore, $H(t)$ is a non-decreasing function. By the definition of $E(t)$, we obtain

$$\begin{aligned} 0 < H(0) \leq H(t) &= -\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|\Delta u\|_2^2 - \frac{1}{m}\|\nabla u\|_m^m + \frac{1}{p}\|u\|_p^p \\ &\leq \frac{1}{p}\|u\|_p^p \quad \text{for every } t \geq 0. \end{aligned} \tag{5.2}$$

We define $L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega u_t u \mathrm{d}x$, where $\varepsilon > 0$ is a small constant. Taking the derivative of $L(t)$ and using (1.1), we obtain

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega u_t^2 \mathrm{d}x + \varepsilon \int_\Omega u_{tt} u \mathrm{d}x \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega u_t^2 \mathrm{d}x - \varepsilon \int_\Omega |\Delta u|^2 \mathrm{d}x - \varepsilon \int_\Omega |\nabla u|^m \mathrm{d}x \\ &\quad + \varepsilon \int_\Omega |u|^p \mathrm{d}x - \varepsilon \int_\Omega |\nabla u_t|^{r-2} \nabla u_t \cdot \nabla u \mathrm{d}x. \end{aligned} \tag{5.3}$$

For the last term of (5.3), by applying Young's inequality,

$$\int_\Omega |\nabla u_t|^{r-2} \nabla u_t \cdot \nabla u \mathrm{d}x \leq \frac{\lambda^r}{r} \int_\Omega |\nabla u|^r \mathrm{d}x + \frac{r-1}{r} \lambda^{-\frac{r}{r-1}} \int_\Omega |\nabla u_t|^r \mathrm{d}x. \tag{5.4}$$

Inserting (5.4) into (5.3), we get

$$\begin{aligned} L'(t) &\geq (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega u_t^2 \mathrm{d}x - \varepsilon \int_\Omega |\Delta u|^2 \mathrm{d}x - \varepsilon \int_\Omega |\nabla u|^m \mathrm{d}x \\ &\quad + \varepsilon \int_\Omega |u|^p \mathrm{d}x - \varepsilon \frac{\lambda^r}{r} \int_\Omega |\nabla u|^r \mathrm{d}x - \varepsilon \frac{r-1}{r} \lambda^{-\frac{r}{r-1}} \int_\Omega |\nabla u_t|^r \mathrm{d}x. \end{aligned}$$

Taking $\lambda^{-\frac{r}{r-1}} := M_1 H^{-\alpha}(t)$, we have

$$L'(t) \geq (1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon\int_\Omega u_t^2 \mathrm{d}x-\varepsilon\int_\Omega |\Delta u|^2 \mathrm{d}x-\varepsilon\int_\Omega |\nabla u|^m \mathrm{d}x+\varepsilon\int_\Omega |u|^p \mathrm{d}x$$
$$-\varepsilon\frac{M_1^{1-r}}{r}H^{\alpha(r-1)}(t)\int_\Omega |\nabla u|^r \mathrm{d}x-\varepsilon\frac{r-1}{r}M_1 H^{-\alpha}(t)\int_\Omega |\nabla u_t|^r \mathrm{d}x.$$

Denote $M := \frac{r-1}{r}M_1$. Using $E'(t) = -\|\nabla u_t\|_r^r$, we have

$$L'(t) \geq (1-\alpha-\varepsilon M)H^{-\alpha}(t)H'(t)+\varepsilon\int_\Omega u_t^2 \mathrm{d}x-\varepsilon\int_\Omega |\Delta u|^2 \mathrm{d}x$$
$$-\varepsilon\int_\Omega |\nabla u|^m \mathrm{d}x+\varepsilon\int_\Omega |u|^p \mathrm{d}x-\varepsilon\frac{M_1^{1-r}}{r}H^{\alpha(r-1)}(t)\int_\Omega |\nabla u|^r \mathrm{d}x.$$

For some constant k, we get

$$\begin{aligned} L'(t) \geq{}& (1-\alpha-\varepsilon M)H^{-\alpha}(t)H'(t)+kH(t)+\left(\frac{k}{2}+\varepsilon\right)\int_\Omega u_t^2 \mathrm{d}x \\ &+\left(\frac{k}{2}-\varepsilon\right)\int_\Omega |\Delta u|^2 \mathrm{d}x+\left(\frac{k}{m}-\varepsilon\right)\int_\Omega |\nabla u|^m \mathrm{d}x \\ &+\left(\varepsilon-\frac{k}{p}\right)\int_\Omega |u|^p \mathrm{d}x-\varepsilon\frac{M_1^{1-r}}{r}H^{\alpha(r-1)}(t)\int_\Omega |\nabla u|^r \mathrm{d}x. \end{aligned} \tag{5.5}$$

Since $m > r$, combining (5.2) and the embedding $W_0^{1,m}(\Omega) \hookrightarrow L^p(\Omega)$, we have

$$\begin{aligned} H^{\alpha(r-1)}(t)\int_\Omega |\nabla u|^r \mathrm{d}x &\leq \left(\frac{1}{p}\right)^{\alpha(r-1)}\left(\int_\Omega |u|^p \mathrm{d}x\right)^{\alpha(r-1)}\int_\Omega |\nabla u|^r \mathrm{d}x \\ &\leq \bar{C}\left(\frac{1}{p}\right)^{\alpha(r-1)}\left(\int_\Omega |\nabla u|^m \mathrm{d}x\right)^{\frac{p\alpha(r-1)+r}{m}}, \end{aligned} \tag{5.6}$$

where $\bar{C}$ is an embedding constant. For any $z \geq 0$ and $\beta > 0$, we have

$$z^v \leq (z+1) \leq \left(1+\frac{1}{\beta}\right)(z+\beta) \quad \text{for all } z \geq 0,\ 0 < v \leq 1. \tag{5.7}$$

Since $\alpha < \frac{m-r}{p(r-1)}$, by (5.2), (5.7), we get

$$\begin{aligned} \left(\int_\Omega |\nabla u|^m \mathrm{d}x\right)^{\frac{p\alpha(r-1)+r}{m}} &\leq \left(1+\frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^m \mathrm{d}x+H(0)\right) \\ &\leq \left(1+\frac{1}{H(0)}\right)\left(\int_\Omega |\nabla u|^m \mathrm{d}x+H(t)\right) \\ &= M_2\left(\int_\Omega |\nabla u|^m \mathrm{d}x+H(t)\right), \end{aligned} \tag{5.8}$$

where $M_2 := 1 + \frac{1}{H(0)}$. Therefore, combining (5.8) and (5.6), (5.5) becomes

$$\begin{aligned}
L'(t) \geq & (1-\alpha-\varepsilon M)H^{-\alpha}(t)H'(t) + kH(t) + \left(\frac{k}{2}+\varepsilon\right)\int_\Omega u_t^2 \mathrm{d}x \\
& + \left(\frac{k}{2}-\varepsilon\right)\int_\Omega |\Delta u|^2 \mathrm{d}x + \left(\frac{k}{m}-\varepsilon\right)\int_\Omega |\nabla u|^m \mathrm{d}x + \left(\varepsilon - \frac{k}{p}\right)\int_\Omega |u|^p \mathrm{d}x \\
& - \varepsilon \frac{M_1^{1-r}\overline{C}}{r}\left(\frac{1}{p}\right)^{\alpha(r-1)} M_2\left(\int_\Omega |\nabla u|^m \mathrm{d}x + H(t)\right) \\
= & (1-\alpha-\varepsilon M)H^{-\alpha}(t)H'(t) + \left(\frac{k}{2}+\varepsilon\right)\int_\Omega u_t^2 \mathrm{d}x + \left(\frac{k}{2}-\varepsilon\right)\int_\Omega |\Delta u|^2 \mathrm{d}x \\
& + \left[\frac{k}{m} - \varepsilon - \frac{\varepsilon M_1^{1-r}\overline{C}}{r}\left(\frac{1}{p}\right)^{\alpha(r-1)} M_2\right]\int_\Omega |\nabla u|^m \mathrm{d}x + \left(\varepsilon - \frac{k}{p}\right)\|u\|_p^p \\
& + \left[k - \varepsilon \frac{M_1^{1-r}\overline{C}}{r}\left(\frac{1}{p}\right)^{\alpha(r-1)} M_2\right] H(t).
\end{aligned}$$

Setting $k = \varepsilon p$, we have

$$\begin{aligned}
L'(t) \geq & (1-\alpha-\varepsilon M)H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2 \mathrm{d}x + \varepsilon\left(\frac{p}{2}-1\right)\int_\Omega |\Delta u|^2 \mathrm{d}x \\
& + \varepsilon\left[\frac{p}{m} - 1 - \frac{M_1^{1-r}\overline{C}}{r}\left(\frac{1}{p}\right)^{\alpha(r-1)} M_2\right]\int_\Omega |\nabla u|^m \mathrm{d}x \\
& + \varepsilon\left[p - \frac{M_1^{1-r}\overline{C}}{r}\left(\frac{1}{p}\right)^{\alpha(r-1)} M_2\right] H(t).
\end{aligned}$$

We choose M_1 large enough so that

$$L'(t) \geq (1-\alpha-\varepsilon M)H^{-\alpha}(t)H'(t) + \varepsilon\gamma(H(t) + \|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_m^m),$$

where γ is a positive constant.

By choosing $\varepsilon \leq \frac{1-\alpha}{M}$, we get $L(0) = H^{1-\alpha}(0) + \int_\Omega u_0 u_1 \mathrm{d}x > 0$, and

$$L'(t) \geq \varepsilon\gamma(H(t) + \|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_m^m), \tag{5.9}$$

Hence, $L(t) \geq L(0) > 0$ for $t \geq 0$.

Now, we estimate $L^{\frac{1}{1-\alpha}}(t)$. It is clear that

$$L^{\frac{1}{1-\alpha}}(t) \leq 2^{\frac{1}{1-\alpha}}\left\{H(t) + \varepsilon^{\frac{1}{1-\alpha}}\left(\int_\Omega u_t u \mathrm{d}x\right)^{\frac{1}{1-\alpha}}\right\}. \tag{5.10}$$

Since $m > 2$, using Hölder's inequality, we get

$$\begin{aligned}
\left|\int_\Omega u_t u \mathrm{d}x\right|^{\frac{1}{1-\alpha}} & \leq \left(\int_\Omega u^2 \mathrm{d}x\right)^{\frac{1}{2(1-\alpha)}}\left(\int_\Omega u_t^2 \mathrm{d}x\right)^{\frac{1}{2(1-\alpha)}} \\
& \leq \left(\int_\Omega |u|^m \mathrm{d}x\right)^{\frac{1}{m(1-\alpha)}}\left(\int_\Omega u_t^2 \mathrm{d}x\right)^{\frac{1}{2(1-\alpha)}}.
\end{aligned}$$

By applying Young's inequality, we obtain

$$\begin{aligned}\left|\int_\Omega u_t u \mathrm{d}x\right|^{\frac{1}{1-\alpha}} &\leq \frac{1}{\mu}\Big(\int_\Omega |u|^m \mathrm{d}x\Big)^{\frac{\mu}{m(1-\alpha)}} + \frac{1}{\theta}\Big(\int_\Omega u_t^2 \mathrm{d}x\Big)^{\frac{\theta}{2(1-\alpha)}} \\ &\leq C_*\Big\{\Big(\int_\Omega |u|^m \mathrm{d}x\Big)^{\frac{\mu}{m(1-\alpha)}} + \Big(\int_\Omega u_t^2 \mathrm{d}x\Big)^{\frac{\theta}{2(1-\alpha)}}\Big\}, \qquad (5.11)\end{aligned}$$

where $C_* := \max\{\frac{1}{\mu}, \frac{1}{\theta}\}$ and $\frac{1}{\mu} + \frac{1}{\theta} = 1$. Choosing $\theta := 2(1-\alpha)$, we have $\mu = \frac{2(1-\alpha)}{1-2\alpha}$. Applying Poincaré's inequality, (5.11) becomes

$$\left|\int_\Omega u_t u \mathrm{d}x\right|^{\frac{1}{1-\alpha}} \leq C\Big\{\Big(\int_\Omega |\nabla u|^m \mathrm{d}x\Big)^{\frac{2}{m(1-2\alpha)}} + \int_\Omega u_t^2 \mathrm{d}x\Big\}.$$

Since $\alpha \leq \frac{m-2}{2m}$, i.e., $\frac{2}{(1-2\alpha)m} \leq 1$, using (5.7) again, we have

$$\Big(\int_\Omega |\nabla u|^m \mathrm{d}x\Big)^{\frac{2}{m(1-2\alpha)}} \leq M_2\Big(\int_\Omega |\nabla u|^m \mathrm{d}x + H(t)\Big).$$

Therefore,

$$\left|\int_\Omega u_t u \mathrm{d}x\right|^{\frac{1}{1-\alpha}} \leq C\big(H(t) + \|\Delta u\|_2^2 + \|u_t\|_2^2 + \|\nabla u\|_m^m\big). \qquad (5.12)$$

Hence, combining (5.12) and (5.10), we have

$$L^{\frac{1}{1-\alpha}}(t) \leq C\big(H(t) + \|\Delta u\|_2^2 + \|u_t\|_2^2 + \|\nabla u\|_m^m\big). \qquad (5.13)$$

Combining (5.9) and (5.13), we get $L'(t) \geq \zeta L^{\frac{1}{1-\alpha}}(t)$ for all $t \geq 0$. Integrating this inequality with respect to t from 0 to t, we get

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\zeta\alpha}{1-\alpha}t}.$$

Hence, $L(t)$ blows up at finite time $T_{\max}$ and $T_{\max} \leq \frac{1-\alpha}{\zeta\alpha L^{\frac{\alpha}{1-\alpha}}(0)}$.

Next, it remains to prove that $\lim_{t\to T_{\max}} L(t) = +\infty$ implies $\lim_{t\to T_{\max}} \|u\|_p = +\infty$.

Case i: $\int_\Omega u u_t \mathrm{d}x \to +\infty$. Since $\int_\Omega u u_t \mathrm{d}x \leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u\|_2^2 \leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2$, recalling Lemma 2.1, we have

$$\begin{aligned}\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 &\leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{m}\|\nabla u\|_m^m \\ &= E(t) + \frac{1}{p}\|u\|_p^p \leq E(0) + \frac{1}{p}\|u\|_p^p.\end{aligned}$$

Hence, if $\int_\Omega u u_t \mathrm{d}x \to +\infty$, then $\lim_{t\to T_{\max}} \|u\|_p = +\infty$.

Case ii: $H(t) \to +\infty$. By (5.2), we get $H(t) \le \frac{1}{p}\|u\|_p^p$. Hence, if $H(t) \to +\infty$, then $\lim_{t\to T_{\max}} \|u\|_p = +\infty$.

Now, we will give the lower bound of $T_{\max}$. Define $M(t) := \frac{m}{2}\|u_t\|_2^2 + \frac{m}{2}\|\Delta u\|_2^2 + \|\nabla u\|_m^m$. Taking derivative, we have

$$\begin{aligned} M'(t) &= m\int_\Omega u_t u_{tt}\mathrm{d}x + m\int_\Omega \Delta u \Delta u_t \mathrm{d}x + m\int_\Omega |\nabla u|^{m-2}\nabla u\cdot\nabla u_t\mathrm{d}x \\ &= m\int_\Omega u_t(-\Delta^2 u + \Delta_m u + \Delta_r u_t + |u|^{p-2}u)\mathrm{d}x \\ &\quad + m\int_\Omega \Delta u\Delta u_t\mathrm{d}x + m\int_\Omega |\nabla u|^{m-2}\nabla u\cdot\nabla u_t\mathrm{d}x \\ &= -m\|\nabla u_t\|_r^r + m\int_\Omega |u|^{p-1}u_t\mathrm{d}x \le m\int_\Omega |u|^{p-1}u_t\mathrm{d}x. \end{aligned}$$

By applying Young's inequality and $H_0^2(\Omega) \hookrightarrow L^{2(p-1)}(\Omega)$, we have

$$\int_\Omega |u|^{p-1}u_t\mathrm{d}x \le \frac{1}{2}\|u_t\|_2^2 + \frac{\widetilde{C}}{2}\|\Delta u\|_2^{2(p-1)} \le \frac{1}{m}M(t) + \frac{\widetilde{C}}{2}\left(\frac{2}{m}M(t)\right)^{p-1}.$$

Hence, we get $M'(t) \le M(t) + \widetilde{C}(\frac{2}{m})^{p-2}M^{p-1}(t)$, where $\widetilde{C}$ is the best embedding constant in the embedding $H_0^2(\Omega) \hookrightarrow L^{2(p-1)}(\Omega)$. Integrating from 0 to t, we have

$$\int_0^t \frac{1}{M(s) + \widetilde{C}\left(\frac{2}{m}\right)^{p-2}M^{p-1}(s)}\mathrm{d}M(s) \le t.$$

Since u will blow up in finite time, $\lim_{t\to T_{\max}^-} M(t) = +\infty$. We get the lower bound (5.1) of $T_{\max}$. ■

5.2. Positive initial energy case

Now, we consider the existence of blow-up of solutions with positive initial energy.

Theorem 5.2. *Let $p > m = r \ge 2$ or $p > m > r = 2$. If $u_0 \in H_0^2(\Omega) \cap W_0^{1,m}(\Omega)$, $u_1 \in L^2(\Omega)$, and*

$$0 < E(0) < \frac{m\varepsilon}{m-1}\int_\Omega u_1 u_0\mathrm{d}x,$$

where ε is a sufficiently small constant, then the weak solution u of (1.1) blows up in finite time.

Proof. We only prove the case $p > m = r \ge 2$. The other case $p > m > r = 2$ can be proved similarly.

Step 1. We first prove

$$\int_\Omega uu_t\mathrm{d}x - \frac{r-1}{r\varepsilon}E(t) \ge \left(\int_\Omega u_1u_0\mathrm{d}x - \frac{r-1}{r\varepsilon}E(0)\right)\mathrm{e}^{At} > 0,$$

for any $t \in [0, T]$, where A is shown in (5.14). For $0 < \eta < 1$,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega uu_t\mathrm{d}x \\
&= \|u_t\|_2^2 + \int_\Omega uu_{tt}\mathrm{d}x \\
&= \|u_t\|_2^2 - \|\Delta u\|_2^2 - \|\nabla u\|_m^m - \int_\Omega |\nabla u_t|^{r-2}\nabla u_t \cdot \nabla u\mathrm{d}x + \|u\|_p^p \\
&= -p(1-\eta)E(t) + p(1-\eta)\left(\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{m}\|\nabla u\|_m^m - \frac{1}{p}\|u\|_p^p\right) \\
&\quad - \|\Delta u\|_2^2 - \|\nabla u\|_m^m - \int_\Omega |\nabla u_t|^{r-2}\nabla u_t \cdot \nabla u\mathrm{d}x + \|u\|_p^p \\
&= -p(1-\eta)E(t) + \left[\frac{p(1-\eta)}{2} + 1\right]\|u_t\|_2^2 + \left[\frac{p(1-\eta)}{2} - 1\right]\|\Delta u\|_2^2 \\
&\quad + \left[\frac{p(1-\eta)}{m} - 1\right]\|\nabla u\|_m^m + \eta\|u\|_p^p - \int_\Omega |\nabla u_t|^{r-2}\nabla u_t \cdot \nabla u\mathrm{d}x.
\end{aligned}
$$

For the last term, by using Young's inequality with small ε, we get

$$
\left|\int_\Omega |\nabla u_t|^{r-2}\nabla u_t \cdot \nabla u\mathrm{d}x\right| \le \frac{r-1}{r\varepsilon}\|\nabla u_t\|_r^r + \frac{\varepsilon^{r-1}}{r}\|\nabla u\|_r^r.
$$

Hence,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega uu_t\mathrm{d}x - \frac{r-1}{r\varepsilon}E(t)\right) \\
&\quad \ge -p(1-\eta)E(t) + \left[\frac{p(1-\eta)}{2} + 1\right]\|u_t\|_2^2 + \left[\frac{p(1-\eta)}{2} - 1\right]\|\Delta u\|_2^2 \\
&\qquad + \left[\frac{p(1-\eta)}{m} - 1\right]\|\nabla u\|_m^m + \eta\|u\|_p^p - \frac{\varepsilon^{r-1}}{r}\|\nabla u\|_r^r.
\end{aligned}
$$

For $m = r$, using $\|\Delta u\|_2^2 \ge \lambda_1\|u\|_2^2$, where λ_1 is a constant, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega uu_t\mathrm{d}x - \frac{m-1}{m\varepsilon}E(t)\right) \\
&\quad \ge -p(1-\eta)E(t) + \left[\frac{p(1-\eta)}{2} + 1\right]\|u_t\|_2^2 + \lambda_1\left[\frac{p(1-\eta)}{2} - 1\right]\|u\|_2^2 \\
&\qquad + \left[\frac{p(1-\eta)}{m} - 1 - \frac{\varepsilon^{m-1}}{m}\right]\|\nabla u\|_m^m.
\end{aligned}
$$

Choosing η, ε small enough such that $\frac{p(1-\eta)}{m} - 1 - \frac{\varepsilon^{m-1}}{m} > 0$, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega uu_t\mathrm{d}x - \frac{m-1}{m\varepsilon}E(t)\right) \\
&\quad \ge -p(1-\eta)E(t) + \left[\frac{p(1-\eta)}{2} + 1\right]\|u_t\|_2^2 + \lambda_1\left[\frac{p(1-\eta)}{2} - 1\right]\|u\|_2^2.
\end{aligned}
$$

It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_\Omega uu_t\mathrm{d}x-\frac{m-1}{m\varepsilon}E(t)\bigg)\\
&\geq -p(1-\eta)E(t)+\sqrt{4\lambda_1\bigg[\frac{p(1-\eta)}{2}+1\bigg]\bigg[\frac{p(1-\eta)}{2}-1\bigg]}\int_\Omega uu_t\mathrm{d}x\\
&=A\bigg(\int_\Omega uu_t\mathrm{d}x-BE(t)\bigg), \qquad (5.14)
\end{aligned}$$

where $A:=\sqrt{4\lambda_1[\frac{p(1-\eta)}{2}+1][\frac{p(1-\eta)}{2}-1]}$, $B:=\frac{p(1-\eta)}{A}$. Note that, for small ε, $B\leq\frac{m-1}{m\varepsilon}$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_\Omega uu_t\mathrm{d}x-\frac{m-1}{m\varepsilon}E(t)\bigg)\geq A\bigg(\int_\Omega uu_t\mathrm{d}x-\frac{m-1}{m\varepsilon}E(t)\bigg),$$

for sufficiently small ε. Hence, by applying Grönwall's inequality, we have

$$\int_\Omega uu_t\mathrm{d}x-\frac{r-1}{r\varepsilon}E(t)\geq\bigg(\int_\Omega u_1u_0\mathrm{d}x-\frac{r-1}{r\varepsilon}E(0)\bigg)\mathrm{e}^{At}>0.$$

Step 2. Now, let us prove this theorem by contradiction. Assume that u is a global solution of problem (1.1). Since $m=r\geq 2$, combining Lemma 2.1, Hölder's inequality, and Poincaré's inequality, we get

$$\begin{aligned}
\|u\|_2^2&=\bigg\|u_0+\int_0^t u_s\mathrm{d}s\bigg\|_2\leq\|u_0\|_2+\int_0^t\|u_s\|_2\mathrm{d}s\\
&\leq\|u_0\|_2+\int_0^t\|u_s\|_m\mathrm{d}s\leq\|u_0\|_2+(1+|\Omega|)\int_0^t\|\nabla u_s\|_m\mathrm{d}s\\
&=\|u_0\|_2+(1+|\Omega|)\int_0^t(-E'(s))^{\frac{1}{m}}\mathrm{d}s\\
&\leq\|u_0\|_2+(1+|\Omega|)t^{\frac{m-1}{m}}\bigg(\int_0^t(-E'(s))\mathrm{d}s\bigg)^{\frac{1}{m}}\\
&=\|u_0\|_2+(1+|\Omega|)t^{\frac{m-1}{m}}\Big(E(0)-E(t)\Big)^{\frac{1}{m}}. \qquad (5.15)
\end{aligned}$$

Next, we prove that since $u(x,t)$ was a global solution of problem (1.1), then $E(t)\geq 0$ for any $t\geq 0$. In fact, by contradiction, there exists $t_0\in[0,+\infty]$ such that $E(t_0)\leq 0$. Choosing $u(x,t_0)$ as the new initial datum, Theorem 5.1 indicates that $u(x,t)$ blows up in finite time, which is a contradiction. Thus, we get $0\leq E(t)\leq E(0)$. Hence, (5.15) becomes

$$\|u\|_2\leq\|u_0\|_2+(1+|\Omega|)t^{\frac{m-1}{m}}E^{\frac{1}{m}}(0). \qquad (5.16)$$

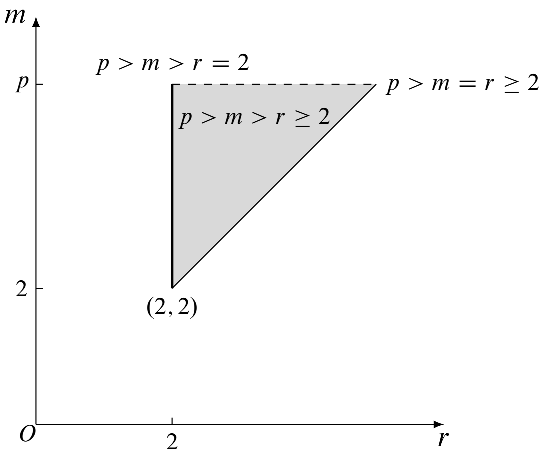

Figure 5.1. Blow-up results.

On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 = 2\int_\Omega u u_t \mathrm{d}x \geq 2\Big[\frac{m-1}{m\varepsilon}E(t) + \Big(\int_\Omega u_0 u_1 \mathrm{d}x - \frac{m-1}{m\varepsilon}E(0)\Big)e^{At}\Big].$$

Integrating the above inequality from 0 to t, we obtain

$$\begin{aligned}\|u\|_2^2 &\geq \|u_0\|_2^2 + 2\int_0^t \frac{m-1}{m\varepsilon}E(s)\mathrm{d}s + 2\int_0^t \Big(\int_\Omega u_0u_1\mathrm{d}x - \frac{m-1}{m\varepsilon}E(0)\Big)e^{As}\mathrm{d}s \\ &\geq \|u_0\|_2^2 + \frac{2}{A}(\mathrm{e}^{At}-1)\Big(\int_\Omega u_0u_1\mathrm{d}x - \frac{m-1}{m\varepsilon}E(0)\Big),\end{aligned}$$

which contradicts (5.16) for sufficiently large t. ∎

Remark 5.1. Theorem 5.1 shows that if the initial energy $E(0) < 0$ and $(-E(0))^{1-\alpha} + \varepsilon\int_\Omega u_1u_0\mathrm{d}x > 0$, where ε is a small constant, then there are blow-up solutions of (1.1) in the exponent region $\{p > m > r \geq 2\}$. In the exponent region $\{p > m = r \geq 2\}$ or $\{p > m > r = 2\}$, if $0 < E(0) < \frac{m\varepsilon}{m-1}\int_\Omega u_1u_0\mathrm{d}x$, then there are blow-up solutions of (1.1), too. It can be checked from the following Figure 5.1 that $\int_\Omega u_1u_0\mathrm{d}x > 0$ is not a necessary condition on the existence of blow-up solutions in the region $\{p > m > 2 = r\}$.

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