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# On some parametric elliptic systems at resonance

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**Abstract.** In this paper, we study the resonance for some parametric elliptic systems in which the nonlinearities involve convection and convolution of the solution. By applying a topological method based on fixed point theory, we establish the existence of at least one weak solution without using Landesman–Lazer-type conditions. Furthermore, we deal with a gradient-type system at resonance and prove the existence of a solution by using a variational approach.

## 1. Introduction

In recent years, the study of $(p_1, p_2)$-Laplacian systems has yielded crucial insights into nonlinear interactions between equations and the impact of coupling on system behavior. These systems offer profound mathematical implications, aiding in identifying solution existence, multiplicity, stability properties, and pattern or singularity formation. Simultaneously, their relevance extends to various physical applications. In fluid dynamics, they serve as models for intricate flows (such as multiphase flows or flows in porous media), and resonance analysis within these systems illuminates resonance-induced instabilities, vortex shedding, and wave amplification, thereby providing valuable insights into fluid flow dynamics. Similarly, in materials science and solid mechanics, these systems capture phenomena, such as crack propagation and fracture mechanics.

On the other hand, convolution is crucial for understanding how systems process input. In parametric elliptic systems with convolution and convection terms at resonance, studying convolution is vital. It reveals the intricate interplay between parameters and system dynamics, providing profound insights into the phenomena of resonance. This operation is extensively used in signal processing, particularly in convolutional neural networks for image analysis (Goodfellow et al. [11]), and in physics and engineering for solving differential equations and understanding complex systems (Strang [19] and Evans [9]).

In this paper, given a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ with Lipschitz boundary, we will study the following parametric elliptic system with convolution and convection terms

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at resonance

$$\begin{cases} -\Delta_{w_1,p_1}u - \mu_1\Delta_{w_1,q_1}u = \lambda_1 w(x)|u|^{\alpha_1-2}u|v|^{\alpha_2} \\ \qquad\qquad + f(x,\rho_1\star u,\rho_2\star v,\nabla(\rho_1\star u),\nabla(\rho_2\star v))+h_1(x) & \text{in } \Omega, \\ -\Delta_{w_2,p_2}v - \mu_2\Delta_{w_2,q_2}v = \lambda_1 w(x)|v|^{\alpha_2-2}v|u|^{\alpha_1} \\ \qquad\qquad + g(x,\rho_1\star u,\rho_2\star v,\nabla(\rho_1\star u),\nabla(\rho_2\star v))+h_2(x) & \text{in } \Omega, \\ u=v=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where, for $i\in\{1,2\}$, we have $-\Delta_{w_i,p_i}u=-\operatorname{div}(w_i(x)|\nabla u|^{p_i-2}\nabla u)$, $1<q_i<p_i<N$, $f,g:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are functions of Carathéodory satisfying some conditions that will be specified later, $w\in C(\Omega)\cap L^\infty(\Omega)$ satisfying $w^+=\max\{w(x),0\}\neq 0$, $\mu_i$ are positive constants and $h_i\in(W_0^{1,p_i}(w_i,\Omega))^\star$. Let be associated to system (1.1) the following eigenvalue problem:

$$\begin{cases} -\operatorname{div}(w_1(x)|\nabla u|^{p_1-2}\nabla u) = \lambda w(x)|u|^{\alpha_1-2}u|v|^{\alpha_2} & \text{in } \Omega, \\ -\operatorname{div}(w_2(x)|\nabla v|^{p_2-2}\nabla v) = \lambda w(x)|v|^{\alpha_2-2}v|u|^{\alpha_1} & \text{in } \Omega, \\ u=v=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where, for $i\in\{1,2\}$, $\alpha_i$ verify

$$\alpha_1,\alpha_2>0 \quad\text{and}\quad \frac{\alpha_1}{p_1}+\frac{\alpha_2}{p_2}=1 \quad \text{for all } x\in\Omega.$$

Define the functionals $\phi,\varphi: X=W_0^{1,p_1}(w_1,\Omega)\times W_0^{1,p_2}(w_2,\Omega)\to\mathbb{R}$ as

$$\begin{cases} \phi(u,v)=\dfrac{\alpha_1}{p_1}\displaystyle\int_\Omega w_1(x)|\nabla u|^{p_1}dx+\dfrac{\alpha_2}{p_2}\int_\Omega w_2(x)|\nabla v|^{p_2}dx, \\ \varphi(u,v)=\displaystyle\int_\Omega w(x)|u|^{\alpha_1-1}|v|^{\alpha_2-1}uv\,dx, \end{cases}$$

and the manifold

$$\Sigma=\{(u,v)\in X:\varphi(u,v)=1\}.$$

One can easily prove that $\phi(u,v)$ and $\varphi(u,v)$ are $(p,q)$-homogeneous, i.e.,

$$\phi(t^{1/p_1}u,t^{1/p_2}v)=t\phi(u,v)\quad\text{and}\quad\varphi(t^{1/p_1}u,t^{1/p_2}v)=t\varphi(u,v)$$

for all $t>0$ and $(u,v)\in X$. We also have $\Sigma$ as a symmetric manifold in $X$. Using the same arguments as in Zographopoulos [21], the eigenvalue problem (1.2) admits a sequence of eigenvalues where they can be variationally characterized as follows:

$$\lambda_k=\inf_{A\in\Sigma_k}\sup_{(u,v)\in A}\phi(u,v),$$

where

$$\Sigma_k=\{A\in\Sigma:\ \text{there exists a continuous, odd and surjective function } \psi:S^{k-1}\to A\},$$

with $S^{k-1}$ is the unit sphere in $\mathbb{R}^k$. We also have

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \cdots, \quad \lambda_k \to +\infty \quad \text{as } k \to +\infty.$$

Now, let us define the following:

$$\lambda_1' = \inf_{(u,v)\in\Sigma} \frac{\alpha_1}{p_1} \int_\Omega w_1(x)|\nabla u|^{p_1} dx + \frac{\alpha_2}{p_2} \int_\Omega w_2(x)|\nabla v|^{p_2} dx. \tag{1.3}$$

Hence, $\lambda_1'$ is the first eigenvalue of (1.2) as well, and taking the definition of $\lambda_1$ into account, one can easily derive that $\lambda_1' = \lambda_1$. Moreover, $\lambda_1$ is a simple, positive, and isolated principal eigenvalue of (1.2). In view of the fact that $\phi(u, v)$ and $\varphi(u, v)$ are $(p, q)$-homogeneous, the eigenfunction space corresponding to the first eigenvalue $\lambda_1$ is

$$E_{\lambda_1} = \{(u, v) \in X : \phi(u, v) = \lambda_1 \varphi(u, v)\},$$

where $(t^{1/p_1}u, t^{1/p_2}v) \in E_{\lambda_1}$ for all $t \ge 0$ and $(u, v) \in E_{\lambda_1}$. In addition, there exists $t_0 > 0$ such that

$$\|t_0^{1/p_1} u_0\|^{p_1}_{W_0^{1,p_1}(w_1,\Omega)} + \|t_0^{1/p_2} v_0\|^{p_2}_{W_0^{1,p_2}(w_2,\Omega)} = 1,$$

where $(u_0, v_0)$ is a normalized eigenvalue satisfying

$$\|u_0\|_{W_0^{1,p_1}(w_1,\Omega)} + \|v_0\|_{W_0^{1,p_2}(w_2,\Omega)} = 1.$$

It can be seen that $E_{\lambda_1}$ is not a linear subspace of $X$ of dimension one, unlike some results in the literature, see, e.g., Alves et al. [1], where this property holds true for a problem with a single equation. This distinction can be clarified by examining the structure of $E_{\lambda_1}$. Consider two elements $(t_1^{1/p_1}u, t_1^{1/p_2}v)$ and $(t_2^{1/p_1}u, t_2^{1/p_2}v)$ belonging to $E_{\lambda_1}$, where $t_1, t_2 > 0$. Their sum, given by $(t_1^{1/p_1}u + t_2^{1/p_1}u, t_1^{1/p_2}v + t_2^{1/p_2}v)$, generally fails to satisfy the defining scaling properties of $E_{\lambda_1}$ due to the lack of a single parameter $t > 0$ satisfying both components simultaneously. Similarly, applying scalar multiplication $c \in \mathbb{R}$ to $(t^{1/p_1}u, t^{1/p_2}v)$ does not yield an element of the same form unless $c = \pm 1$. These observations underscore that $E_{\lambda_1}$ does not exhibit closure under addition or scalar multiplication, as required for a linear subspace. Instead, its structure reflects a symmetric, nonlinear manifold arising from the inherent $(p, q)$-homogeneity of the functionals $\phi$ and $\varphi$.

Recently, Ouannasser and El Hachimi [18] studied a similar problem as in (1.1) for the non-resonance case, but with variable exponents. More specifically, they obtained existence and uniqueness results for the following system:

$$\begin{cases} -\Delta_{p_1(x)}u_1 - \mu_1\Delta_{q_1(x)}u_1 = f_1(x, u_1, u_2, \nabla u_1, \nabla u_2) + h_1(x) & \text{in } \Omega, \\ -\Delta_{p_2(x)}u_2 - \mu_2\Delta_{q_2(x)}u_2 = f_2(x, u_1, u_2, \nabla u_1, \nabla u_2) + h_2(x) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In addition, we also refer to another interesting paper that has dealt with non-resonance for quasilinear elliptic problems, see El Hachimi and Gossez [8]. Because of the lack of information on the first eigenvalue in the case of variable exponents when dealing with resonance, we limit ourselves to the study of problem (1.1) for constant exponents. Even when the exponents are constant, there is limited knowledge regarding resonance and non-resonance issues in the context of the Lusternik–Schnirelmann spectrum for the eigenvalue problem (1.2) of the function

$$(s,t) \mapsto \frac{f(x,s,t,\xi,\eta)s + g(x,s,t,\xi,\eta)t}{w(x)|s|^{\alpha_1}|t|^{\alpha_2}}. \tag{1.5}$$

The same observation holds true when considering gradient-type systems, and it applies to functions $f$ and $g$ that do not have a dependency on $\nabla u$ and $\nabla v$ within the function

$$(s,t) \mapsto \frac{F(x,s,t)}{w(x)|s|^{\alpha_1}|t|^{\alpha_2}},$$

where $\frac{\partial F}{\partial s}(x,s,t) = f(x,s,t)$. The study of resonance has been of a great deal of importance. In our case, it follows that the function should be, in some sense, greater than the first eigenvalue but remain below the second eigenvalue. In other words, we have

$$\begin{aligned}\lambda_1 < \limsup_{|(\rho_1\star u,\rho_2\star v)|_\infty\to\infty} & \frac{f(x,\rho_1\star u,\rho_2\star v,\nabla(\rho_1\star u),\nabla(\rho_2\star v))u}{w(x)|\rho_1\star u|^{\alpha_1}|\rho_2\star v|^{\alpha_2}} \\ &+ \frac{g(x,\rho_1\star u,\rho_2\star v,\nabla(\rho_1\star u),\nabla(\rho_2\star v))v}{w(x)|\rho_1\star u|^{\alpha_1}|\rho_2\star v|^{\alpha_2}} < \lambda_2,\end{aligned}$$

with $|(\rho_1\star u,\rho_2\star v)|_\infty = \max\{|\rho_1\star u|,|\rho_2\star v|\}$. There are many works dedicated to the study of quasilinear elliptic problems at resonance, we cite Alves et al. [1], Benouhiba and Belyacine [2], De Nápoli and Mariani [5], Haddaoui et al. [12], Ou [15, 16], Ou and Tang [17], and the references therein. Most of these studies established the existence of weak solutions under Landesman–Lazer-type conditions. For example, Ou [15] studied the following system at resonance:

$$\begin{cases} -\Delta_p u = \lambda_1 a(x)|u|^{p-2}u + \lambda_1 b(x)|u|^{\alpha}|v|^{\beta}v + g_1(x,u) - h_1(x) & \text{in } \Omega, \\ -\Delta_q v = \lambda_1 c(x)|v|^{q-2}v + \lambda_1 b(x)|u|^{\alpha}|v|^{\beta}u + g_2(x,v) - h_2(x) & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where he established the existence of solutions for the system (1.6) under Landesman–Lazer-type conditions.

In this work, we extend the aforementioned works to a parametric $(p_1,p_2)$-Laplacian system, where the nonlinearities depend on the gradient of the solution, and we obtain the existence of the solution without using Landsman–Lazer-type conditions. In addition, we adopt a distinct parameterization involving functions $\rho_1$ and $\rho_2$ in $L^1(\mathbb{R}^N)$ incorporated through convolutions $\rho_1\star u$ and $\rho_2\star v$ with the solution $(u,v)$ in $W_0^{1,p_1}(w_1,\Omega)\times W_0^{1,p_2}(w_2,\Omega)$. To facilitate our analysis, it is convenient to consider the weighted Sobolev space $W_0^{1,p_1}(w_1,\Omega)$ embedded in $W^{1,p_1}(w_1,\mathbb{R}^N)$ by associating each $u\in W_0^{1,p_1}(w_1,\Omega)$

with its extension $\tilde{u} \in W^{1,p_1}(w_1, \mathbb{R}^N)$, taking the value zero outside $\Omega$, and henceforth replicating a similar approach with $v \in W_0^{1,p_2}(w_2, \Omega)$.

The lack of variational structure in problem (1.1), caused by convection terms, prevents the use of variational methods. Instead, we employ a topological method based on the surjectivity result of pseudomonotone operators. To the best of our knowledge, this is the first paper that deals with resonance for parametric elliptic systems where the nonlinearities are dependent on the gradient and the convolution of the solution, and without using Landesman–Lazer-type conditions. Such problems bring about many issues to overcome, such as the interaction of the variational spectrum with the function given in (1.5).

Similarly, we address the scenario of a gradient-type system and establish the existence of at least one weak solution, following a variational method similar to that used by El Hachimi and de Thélin [7]. More specifically, we deal with the following system at resonance:

$$\begin{cases} -\Delta_{w_1,p_1} u - \mu_1 \Delta_{w_1,q_1} u = \lambda_1 w(x)|u|^{\alpha_1-2}u|v|^{\alpha_2} \\ \qquad\qquad + \dfrac{\partial F}{\partial u}(x, \rho_1 \star u, \rho_2 \star v) + h_1(x) & \text{in } \Omega, \\ -\Delta_{w_2,p_2} v - \mu_2 \Delta_{w_2,q_2} v = \lambda_1 w(x)|v|^{\alpha_2-2}v|u|^{\alpha_1} \\ \qquad\qquad + \dfrac{\partial F}{\partial v}(x, \rho_1 \star u, \rho_2 \star v) + h_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Many studies have dealt with similar problems to the system (1.7), studied their resonance, and obtained the existence of solutions using different methods, such as Zhao and Tang [20]. In this section, we extend the works of Lv and Ou [14], El Hachimi and de Thélin [7], Ouannasser and El Hachimi [18], Ou [16], De Nápoli, and Mariani [5], Zhao and Tang [20], and deal with a more general class of systems where we obtain the existence of a solution at resonance.

The remainder of this paper is organized as follows. In Section 2, we present the necessary elements regarding the weighted Sobolev spaces in relation to our system (1.1). In addition, we provide a few convolution properties in the case of weighted Sobolev spaces. Moreover, we establish the existence of at least one weak solution using the surjectivity result of pseudomonotone operators. Finally, in Section 3, we delve into the case of a gradient-type system at resonance and prove the existence of a solution using a variational approach.

## 2. Functional framework and main results

For $i \in \{1, 2\}$, let $w_i$ be a measurable and a.e. finite function in $\mathbb{R}^N$ such that

($a_1$) $w_i \in L^1_{loc}(\Omega)$ and $w_i^{\frac{-1}{p_i-1}} \in L^1_{loc}(\Omega)$,

($a_2$) $w_i^{-s_i} \in L^1(\Omega)$ for $s_i \in ]\frac{N}{p_i}, +\infty[ \cap [\frac{1}{p_i-1}, +\infty[$.

In what follows, let $X = W_0^{1,p_1}(w_1,\Omega) \times W_0^{1,p_2}(w_2,\Omega)$ be the product space endowed with the norm

$$\|(u,v)\|_X = \max\left\{\|u\|_{W_0^{1,p_1}(w_1,\Omega)}, \|v\|_{W_0^{1,p_2}(w_2,\Omega)}\right\},$$

where

$$\|u\|_{W_0^{1,p_1}(w_1,\Omega)} = \left(\int_\Omega w_1(x)|\nabla u|^{p_1}dx\right)^{\frac{1}{p_1}},$$
$$\|v\|_{W_0^{1,p_2}(w_2,\Omega)} = \left(\int_\Omega w_2(x)|\nabla v|^{p_2}dx\right)^{\frac{1}{p_2}}$$

for every $(u,v) \in X$. It is important to note that condition $(a_1)$ is mandatory in order for $W_0^{1,p_i}(w_i,\Omega)$ to be a Banach space, for $i \in \{1,2\}$. Furthermore, we recall a few more results related to this space, such as continuous and compact embeddings.

**Theorem 2.1** (Kufner and Opic [13]). *Assume that* $(a_1)$ *is verified. Then, for* $i \in \{1,2\}$, *the weighted Sobolev space* $W_0^{1,p_i}(w_i,\Omega)$ *is a Banach space.*

**Theorem 2.2** (Drábek et al. [6]). *Let* $\Omega \subset \mathbb{R}^N$ *be an open set and assume that* $(a_1)$ *is verified. Then, for* $i \in \{1,2\}$, *we have the following continuous and compact embeddings.*

(i) *For any* $r_i$ *verifying* $1 \le r_i \le p^*_{i,s_i} = \frac{Np_is_i}{N(s_i+1)-p_is_i}$ *for* $p_is_i < N(s_i+1)$, *and* $r_i \ge 1$ *is arbitrary for* $p_is_i \ge N(s_i+1)$, *the embedding* $W_0^{1,p_i}(w_i,\Omega) \hookrightarrow L^{r_i}(\Omega)$ *is continuous.*

(ii) *For any* $r_i$ *verifying* $1 \le r_i < p^*_{i,s_i} = \frac{Np_is_i}{N(s_i+1)-p_is_i}$, *the embedding* $W_0^{1,p_i}(w_i,\Omega) \hookrightarrow\hookrightarrow L^{r_i}(\Omega)$ *is compact.*

(iii) *In particular, if* $s_i > \frac{N}{p_i}$ *then* $p^*_{i,s_i} > p_i$. *Consequently, the embedding* $W_0^{1,p_i}(w_i,\Omega) \hookrightarrow\hookrightarrow L^{r_i}(\Omega)$ *is compact.*

**Remark 2.3.** Using Theorem 2.2, and for every $r_i \in [1, p^*_{i,s_i}]$ with $i \in \{1,2\}$, there exists a positive constant $S_{r_i}$ such that

$$\|u\|_{L^{r_i}(\Omega)} \le S_{r_i}\|u\|_{W_0^{1,p_i}(w_i,\Omega)} \tag{2.1}$$

for all $u \in W_0^{1,p_i}(w_i,\Omega)$.

To obtain a well-defined convolution, $\rho_1 \star u$ (resp., $\rho_2 \star v$), with $\rho_1 \in L^1(\mathbb{R}^N)$ (resp., $\rho_2 \in L^1(\mathbb{R}^N)$) and $u \in W_0^{1,p_1}(w_1,\Omega)$ (resp., $v \in W_0^{1,p_2}(w_2,\Omega)$), it should be noted that the space $W_0^{1,p_1}(w_1,\Omega)$ (resp., $W_0^{1,p_2}(w_2,\Omega)$) is embedded in $W^{1,p_1}(w_1,\mathbb{R}^N)$ (resp., $W^{1,p_2}(w_2,\mathbb{R}^N)$) by identifying every $u \in W_0^{1,p_1}(w_1,\Omega)$ (resp., $v \in W_0^{1,p_2}(w_2,\Omega)$) with its extension equal to zero outside $\Omega$. Henceforth, the convolution $\rho_1 \star u$ can be defined as follows:

$$\rho_1 \star u = \int_{\mathbb{R}^N} \rho_1(x-y)u(y)w_1(y)dy,$$

see Brezis [3, Sections 4.4 and 9.1]. Furthermore, the weak partial derivatives of $\rho_1 \star u$ are given by

$$\frac{\partial}{\partial x_j}(\rho_1 \star u) = \rho_1 \star \frac{\partial u}{\partial x_j} \in L^{p_1}(w_1, \mathbb{R}^N) \tag{2.2}$$

for all $j \in \{1, \dots, N\}$. Now, using Fubini's and Tonelli's theorems alongside Hölder's inequality, we derive

$$\|\rho_1 \star u\|_{L^{r_1}(w_1,\mathbb{R}^N)} \le \|\rho_1\|_{L^1(\mathbb{R}^N)} \|u\|_{L^{r_1}(w_1,\Omega)} \tag{2.3}$$

for $r_1 \in [1, p^*_{1,s_1}]$. In addition, we have

$$\|\rho_1 \star \frac{\partial u}{\partial x_j}\|_{L^{p_1}(w_1,\mathbb{R}^N)} \le \|\rho_1\|_{L^1(\mathbb{R}^N)} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^{p_1}(w_1,\Omega)} \tag{2.4}$$

for all $j \in \{1, \dots, N\}$. From (2.2), (2.3), and (2.4), we derive that $u \in W_0^{1,p_1}(w_1, \Omega) \mapsto \rho_1 \star u \in W^{1,p_1}(w_1, \mathbb{R}^N)$ is continuous. Now, using Minkowski's inequality, the convexity of $t \mapsto t^{p_1}$ on $\mathbb{R}^+$ alongside (2.2) and (2.4), we find

$$\begin{aligned} \|\nabla(\rho_1 \star u)\|^{p_1}_{L^{p_1}(w_1,\mathbb{R}^N)} &= \int_\Omega w_1(x)|\nabla(\rho_1 \star u)|^{p_1} dx \\ &= \int_\Omega w_1(x)\Bigg(\sum_{j=1}^N \Big(\rho_1 \star \frac{\partial u}{\partial x_j}\Big)^2\Bigg)^{\frac{p_1}{2}} dx \\ &\le \int_{\mathbb{R}^N} w_1(x)\Bigg(\sum_{j=1}^N \Big|\rho_1 \star \frac{\partial u}{\partial x_j}\Big|\Bigg)^{p_1} dx \\ &\le N^{p_1-1} \sum_{j=1}^N \left\|\rho_1 \star \frac{\partial u}{\partial x_j}\right\|^{p_1}_{L^{p_1}(w_1,\mathbb{R}^N)} \\ &\le N^{p_1-1} \|\rho_1\|^{p_1}_{L^1(\mathbb{R}^N)} \sum_{j=1}^N \left\|\frac{\partial u}{\partial x_j}\right\|^{p_1}_{L^{p_1}(w_1,\Omega)} \\ &\le N^{p_1} \|\rho_1\|^{p_1}_{L^1(\mathbb{R}^N)} \|u\|^{p_1}_{W_0^{1,p_1}(w_1,\Omega)}. \end{aligned} \tag{2.5}$$

We adopt a similar approach when $\rho_2 \in L^1(\mathbb{R}^N)$ and $v \in W_0^{1,p_2}(w_2, \Omega)$ to obtain the same estimations. Next, we define

$$\overline{N}_f : W_0^{1,p_1}(w_1, \mathbb{R}^N) \times W_0^{1,p_2}(w_2, \mathbb{R}^N) \subset L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N) \to L^{r_1'}(\mathbb{R}^N) \times L^{r_2'}(\mathbb{R}^N),$$
$$\overline{N}_g : W_0^{1,p_1}(w_1, \mathbb{R}^N) \times W_0^{1,p_2}(w_2, \mathbb{R}^N) \subset L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N) \to L^{r_1'}(\mathbb{R}^N) \times L^{r_2'}(\mathbb{R}^N),$$

the Nemytskij operators associated with $f$ and $g$. In addition, let

$$j_1^* : L^{r_1'}(\mathbb{R}^N) \times L^{r_2'}(\mathbb{R}^N) \to (W_0^{1,p_1}(w_1, \mathbb{R}^N))^* \times (W_0^{1,p_2}(w_2, \mathbb{R}^N))^*,$$
$$j_2^* : L^{r_1'}(\mathbb{R}^N) \times L^{r_2'}(\mathbb{R}^N) \to (W_0^{1,p_1}(w_1, \mathbb{R}^N))^* \times (W_0^{1,p_2}(w_2, \mathbb{R}^N))^*$$

be the adjoint operators for the embeddings

$$j_1 : W_0^{1,p_1}(w_1, \mathbb{R}^N) \times W_0^{1,p_2}(w_2, \mathbb{R}^N) \to L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N),$$
$$j_2 : W_0^{1,p_1}(w_1, \mathbb{R}^N) \times W_0^{1,p_2}(w_2, \mathbb{R}^N) \to L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N).$$

Furthermore, we define

$$\begin{aligned} N_f &:= j_1^* \circ \overline{N}_f : W_0^{1,p_1}(w_1, \mathbb{R}^N) \times W_0^{1,p_2}(w_2, \mathbb{R}^N) \\ &\quad \to (W_0^{1,p_1}(w_1, \mathbb{R}^N))^* \times (W_0^{1,p_2}(w_2, \mathbb{R}^N))^*, \\ N_g &:= j_2^* \circ \overline{N}_g : W_0^{1,p_1}(w_1, \mathbb{R}^N) \times W_0^{1,p_2}(w_2, \mathbb{R}^N) \\ &\quad \to (W_0^{1,p_1}(w_1, \mathbb{R}^N))^* \times (W_0^{1,p_2}(w_2, \mathbb{R}^N))^*. \end{aligned}$$

Let the map $\mathcal{B} : W_0^{1,p_1}(w_1, \Omega) \times W_0^{1,p_2}(w_2, \Omega) \to W^{1,p_1}(w_1, \mathbb{R}^N) \times W^{1,p_2}(w_2, \mathbb{R}^N)$, which is given by the extension outside $\Omega$ with 0, to obtain $\mathcal{B}(u, v) = (\tilde{u}, \tilde{v})$ as mentioned in Section 1. We denote by

$$\mathcal{B}^\star : (W^{1,p_1}(w_1, \mathbb{R}^N))^\star \times (W^{1,p_2}(w_2, \mathbb{R}^N))^\star \to (W_0^{1,p_1}(w_1, \Omega))^\star \times (W_0^{1,p_2}(w_2, \Omega))^\star$$

the adjoint mapping of $\mathcal{B}$, and we introduce $\mathcal{I} : X \to X^\star$ as a nonlinear operator defined by

$$\begin{aligned} \mathcal{I}(u, v) &:= \widetilde{\mathcal{I}}(u, v) - \lambda_1 A(u, v) - \mathcal{B}^\star(N_f(\rho_1 \star \tilde{u}, \rho_2 \star \tilde{v})) \\ &\quad - \mathcal{B}^\star(N_g(\rho_1 \star \tilde{u}, \rho_2 \star \tilde{v})) - \int_\Omega h_1 u dx - \int_\Omega h_2 v dx \end{aligned} \tag{2.6}$$

for all $(u, v) \in X$, where

$$\begin{aligned} \widetilde{\mathcal{I}}(u, v) &= \int_\Omega \frac{w_1(x)}{p_1} |\nabla u|^{p_1} + \frac{\mu_1 w_1(x)}{q_1} |\nabla u|^{q_1} dx \\ &\quad + \int_\Omega \frac{w_2(x)}{p_2} |\nabla v|^{p_2} + \frac{\mu_2 w_2(x)}{q_2} |\nabla v|^{q_2} dx, \end{aligned} \tag{2.7}$$

and

$$A(u, v) = \int_\Omega w(x) |u|^{\alpha_1} |v|^{\alpha_2} dx.$$

**Definition 2.4.** Let $X$ be a reflexive Banach space, $X^*$ its dual space and denote by $\langle \cdot, \cdot \rangle$ the duality pairing. Consider an application $\mathcal{I} : X \to X^*$. Then, $\mathcal{I}$ is called

(a) to verify the $(S^+)$-property if

$$u_n \rightharpoonup u \text{ in } X \text{ and } \limsup_{n\to\infty} \langle \mathcal{I} u_n, u_n - u \rangle \le 0 \text{ imply } u_n \to u \text{ in } X,$$

(b) pseudomonotone if

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } X \text{ and } \limsup_{n\to\infty} \langle \mathcal{I} u_n, u_n - u \rangle \le 0 \text{ imply } \mathcal{I} u_n \\ &\quad \rightharpoonup \mathcal{I} u \text{ and } \langle \mathcal{I} u_n, u_n \rangle \to \langle \mathcal{I} u, u \rangle, \end{aligned}$$

(c) coercive if

$$\lim_{\|u\|_X \to \infty} \frac{\langle \mathcal{I}u, u \rangle}{\|u\|_X} = \infty.$$

In order to prove the existence result, we need the following assumptions.

($H_1$) $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions such that, for $i \in \{1, 2\}$, there exist three positive continuous functions $\alpha_i^*$, $q_i^*$, and $\delta_i$, and a nonnegative function $e_i \in L^{\frac{\gamma_i}{\gamma_i - 1}}(\Omega)$ with $1 < \gamma_i < p_{i,s_i}^*$, satisfying the following conditions:

$$0 < \alpha_i^* < p_{i,s_i}^* - 1, \quad 0 < \delta_i < \frac{q_i}{(q_i^*)'}, \quad \frac{\alpha_1^*}{\gamma_1} + \frac{\alpha_2^*}{\gamma_2} = \frac{\gamma_1 - 1}{\gamma_1},$$

and nonnegative constants $a_i, b_i$, such that for a.e. $x \in \Omega$ and all $s, t \in \mathbb{R}$, we have

$$|f(x, s, t, \xi, \eta)| \le a_1 |s|^{\alpha_1^*} |t|^{\alpha_2^*} + b_1 \Big( |\xi|^{\delta_1} + |\eta|^{\frac{\delta_1 q_2}{q_1}} \Big) + e_1(x)$$

and

$$|g(x, s, t, \xi, \eta)| \le a_2 |s|^{\alpha_1^*} |t|^{\alpha_2^*} + b_2 \Big( |\xi|^{\frac{\delta_2 q_1}{q_2}} + |\eta|^{\delta_2} \Big) + e_2(x).$$

($H_2$) $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \exists C_\varepsilon \in L^1(\Omega), \forall (s, t) \in \mathbb{R}^2$ with $\max\{|s|, |t|\} > \delta_\varepsilon, \forall (\xi, \eta) \in \mathbb{R}^{2N}, \forall x \in \Omega$, we have

$$\begin{aligned} & f(x, s, t, \xi, \eta)s + g(x, s, t, \xi, \eta)t \\ & \quad \le (\lambda_2 + \varepsilon) w(x) |s|^{\alpha_1} |t|^{\alpha_2} + c(w_1(x)|\xi|^{q_1} + w_2(x)|\eta|^{q_2}) + C_\varepsilon(x), \end{aligned}$$

where the constant $c$ satisfies

$$0 < c < \frac{\min\{\mu_1, \mu_2\}}{\max\{N^{q_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{q_1}, N^{q_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{q_2}\}}.$$

The main tool in this section is based on the surjectivity result for pseudomonotone operators from Carl et al. [4, Theorem 2.99], which we now state.

**Theorem 2.5** (Carl et al. [4]). *Let $X$ be a real reflexive Banach space, and assume that $\mathcal{I} : X \to X^*$ is a bounded, pseudomonotone, and coercive operator. Then, there exists a solution to the equation $\mathcal{I}(u) = v$, where $v \in X^*$.*

**Lemma 2.6.** *The operator $\widetilde{\mathcal{I}} : X \to X^\star$ defined in (2.7) is bounded, continuous, monotone (hence maximal monotone), and of type $(S^+)$.*

*Proof.* The proof bears resemblance to the proof given by Fan and Zhang [10, Theorem 3.1] for a single equation when $\mu_i = 0$ for $i \in \{1, 2\}$ and is omitted here. ∎

**Theorem 2.7.** *Suppose that the hypotheses ($H_1$) and ($H_2$) are verified. Then, for all $(h_1, h_2) \in X^\star$, the system (1.1) admits at least one weak solution $(u, v) \in X$.*

*Proof.* First, it is worth mentioning that both the Nemytskij operators $N_f$ and $N_g$ are well-defined based on hypothesis $(H_1)$. The arguments above, along with (2.5), imply that operator $\mathcal{I}$ is bounded and continuous. Going back to (2.6), the couple $(u, v) \in X$ is said to be a weak solution for the system (1.1) if and only if

$$\langle \mathcal{I}(u, v), (\varphi, \psi) \rangle = 0 \quad \text{for all } (\varphi, \psi) \in X.$$

Hence, proving the surjectivity of the operator $\mathcal{I} : X \to X^\star$ allows us to obtain the existence of a weak solution. In this regard, in order to prove that $\mathcal{I}$ is surjective, we employ Theorem 2.5. Using hypothesis ($\mathrm{H}_1$) alongside Hölder's inequality, we find

$$\begin{aligned}
&\left| \int_\Omega f(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v)) w_1 dx \right| \\
&\leq \|e_1\|_{L^{\frac{\gamma_1}{\gamma_1 - 1}}(\Omega)} \|w_1\|_{L^{\gamma_1}(\Omega)} + a_1 \|\rho_1 \star u\|^{\alpha_1^*}_{L^{\gamma_1}(\mathbb{R}^N)} \|\rho_2 \star v\|^{\alpha_2^*}_{L^{\gamma_2}(\mathbb{R}^N)} \|w_1\|_{L^{\gamma_1}(\Omega)} \\
&\quad + b_1 \Big( \|\nabla(\rho_1 \star u)\|^{\delta_1}_{L^{q_1}(\mathbb{R}^N)} + \|\nabla(\rho_2 \star v)\|^{\delta_1 \frac{q_2}{q_1}}_{L^{q_2}(\mathbb{R}^N)} \Big) \|w_1\|_{L^{\frac{q_1}{q_1 - \delta_1}}(\Omega)}
\end{aligned}$$

for all $(u, v) \in X$, where $(u, v)$ is identified with $\mathcal{B}(u, v) = (\tilde{u}, \tilde{v})$. Moreover, using relations (2.1) and (2.3) together, we obtain

$$\begin{aligned}
&\left| \int_\Omega f(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v)) w_1 dx \right| \\
&\leq \|e_1\|_{L^{\frac{\gamma_1}{\gamma_1 - 1}}(\Omega)} \|w_1\|_{L^{\gamma_1}(\Omega)} \\
&\quad + a_1 \|\rho_1\|^{\alpha_1^\star}_{L^1(\mathbb{R}^N)} S_{\gamma_1}^{\alpha_1^\star} \|u\|^{\alpha_1^\star}_{W_0^{1,p_1}(w_1,\Omega)} \|\rho_2\|^{\alpha_2^\star}_{L^1(\mathbb{R}^N)} S_{\gamma_2}^{\alpha_2^\star} \|v\|^{\alpha_2^\star}_{W_0^{1,p_2}(w_2,\Omega)} \|w_1\|_{L^{\gamma_1}(\Omega)} \\
&\quad + b_1 \Big( \|\nabla(\rho_1 \star u)\|^{\delta_1}_{L^{q_1}(\mathbb{R}^N)} + \|\nabla(\rho_2 \star v)\|^{\delta_1 \frac{q_2}{q_1}}_{L^{q_2}(\mathbb{R}^N)} \Big) \|w_1\|_{L^{\frac{q_1}{q_1 - \delta_1}}(\Omega)}.
\end{aligned}$$

Ultimately, by relation (2.5), it follows that

$$\begin{aligned}
&\left| \int_\Omega f(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v)) w_1 dx \right| \\
&\leq \|e_1\|_{L^{\frac{\gamma_1}{\gamma_1 - 1}}(\Omega)} \|w_1\|_{L^{\gamma_1}(\Omega)} \\
&\quad + a_1 \|\rho_1\|^{\alpha_1^\star}_{L^1(\mathbb{R}^N)} S_{\gamma_1}^{\alpha_1^\star} \|u\|^{\alpha_1^\star}_{W_0^{1,p_1}(w_1,\Omega)} \|\rho_2\|^{\alpha_2^\star}_{L^1(\mathbb{R}^N)} S_{\gamma_2}^{\alpha_2^\star} \|v\|^{\alpha_2^\star}_{W_0^{1,p_2}(w_2,\Omega)} \|w_1\|_{L^{\gamma_1}(\Omega)} \\
&\quad + b_1 \Big( N^{\delta_1} \|\rho_1\|^{\delta_1}_{L^1(\mathbb{R}^N)} \|u\|^{\delta_1}_{W_0^{1,p_1}(w_1,\Omega)} \\
&\qquad + N^{\delta_1 \frac{q_2}{q_1}} \|\rho_2\|^{\delta_1 \frac{q_2}{q_1}}_{L^1(\mathbb{R}^N)} \|v\|^{\delta_1 \frac{q_2}{q_1}}_{W_0^{1,p_2}(w_2,\Omega)} \Big) \|w_1\|_{L^{\frac{q_1}{q_1 - \delta_1}}(\Omega)}.
\end{aligned} \tag{2.8}$$

Applying similar arguments for $g$, we derive

$$\begin{aligned}
&\left|\int_\Omega g(x,\rho_1\star u,\rho_2\star v,\nabla(\rho_1\star u),\nabla(\rho_2\star v))w_2dx\right| \\
&\leq \|e_2\|_{L^{\frac{\gamma_2}{\gamma_2-1}}(\Omega)}\|w_2\|_{L^{\gamma_2}(\Omega)} \\
&+a_2\|\rho_1\|^{\alpha_1^\star}_{L^1(\mathbb{R}^N)}S^{\alpha_1^\star}_{\gamma_1}\|u\|^{\alpha_1^\star}_{W_0^{1,p_1}(w_1,\Omega)}\|\rho_2\|^{\alpha_2^\star}_{L^1(\mathbb{R}^N)}S^{\alpha_2^\star}_{\gamma_2}\|v\|^{\alpha_2^\star}_{W_0^{1,p_2}(w_2,\Omega)}\|w_2\|_{L^{\gamma_2}(\Omega)} \qquad (2.9)\\
&+b_2\Big(N^{\delta_2\frac{q_1}{q_2}}\|\rho_1\|^{\delta_2\frac{q_1}{q_2}}_{L^1(\mathbb{R}^N)}\|u\|^{\delta_2\frac{q_1}{q_2}}_{W_0^{1,p_1}(w_1,\Omega)} \\
&\qquad+N^{\delta_2}\|\rho_2\|^{\delta_2}_{L^1(\mathbb{R}^N)}\|v\|^{\delta_2}_{W_0^{1,p_2}(w_2,\Omega)}\Big)\|w_2\|_{L^{\frac{q_2}{q_2-\delta_2}}(\Omega)}
\end{aligned}$$

for all $(u,v)\in X$.

Now, let us prove that $\mathcal{I}:X\to X^\star$ is pseudomonotone. To do this, let $(u_n,v_n)_{n\in\mathbb{N}}\subset W_0^{1,p_1}(w_1,\Omega)\times W_0^{1,p_2}(w_2,\Omega)$ be a sequence such that

$$(u_n,v_n)\rightharpoonup(u,v)\quad\text{in } X \tag{2.10}$$

and

$$\limsup_{n\to\infty}\langle\mathcal{I}(u_n,v_n),(u_n-u,v_n-v)\rangle\leq 0. \tag{2.11}$$

Because $(u_n,v_n)_{n\in\mathbb{N}}$ is bounded in $X$, then there exist two constants $C_1,C_2>0$ such that

$$\begin{cases} N\|\rho_1\|_{L^1(\mathbb{R}^N)}\|u_n\|_{W_0^{1,p_1}(w_1,\Omega)}\leq C_1,\\ N\|\rho_2\|_{L^1(\mathbb{R}^N)}\|v_n\|_{W_0^{1,p_2}(w_2,\Omega)}\leq C_2.\end{cases} \tag{2.12}$$

Using (2.8) and (2.12), we obtain

$$\begin{aligned}
&\left|\int_\Omega f(x,\rho_1\star u_n,\rho_2\star v_n,\nabla(\rho_1\star u_n),\nabla(\rho_2\star v_n))(u_n-u)dx\right| \\
&\leq\|e_1\|_{L^{\frac{\gamma_1}{\gamma_1-1}}(\Omega)}\|u_n-u\|_{L^{\gamma_1}(\Omega)}+a_1S^{\alpha_1^\star}_{\gamma_1}C_1^{\alpha_1^\star}S^{\alpha_2^\star}_{\gamma_2}C_2^{\alpha_2^\star}\|u_n-u\|_{L^{\gamma_1}(\Omega)} \\
&\quad+b_1\big(C_1^{\delta_1}+C_2^{\delta_1\frac{q_2}{q_1}}\big)\|u_n-u\|_{L^{\frac{q_1}{q_1-\delta_1}}(\Omega)}
\end{aligned}$$

for all $n\in\mathbb{N}$. Similarly for $g$, using (2.9) and (2.12), we find

$$\begin{aligned}
&\left|\int_\Omega g(x,\rho_1\star u_n,\rho_2\star v_n,\nabla(\rho_1\star u_n),\nabla(\rho_2\star v_n))(v_n-v)dx\right| \\
&\leq\|e_2\|_{L^{\frac{\gamma_2}{\gamma_2-1}}(\Omega)}\|v_n-v\|_{L^{\gamma_2}(\Omega)}+a_2S^{\alpha_1^\star}_{\gamma_1}C_1^{\alpha_1^\star}S^{\alpha_2^\star}_{\gamma_2}C_2^{\alpha_2^\star}\|v_n-v\|_{L^{\gamma_1}(\Omega)} \\
&\quad+b_1\big(C_1^{\delta_2\frac{q_1}{q_2}}+C_2^{\delta_2}\big)\|v_n-v\|_{L^{\frac{q_2}{q_2-\delta_2}}(\Omega)}
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking into account $(u_n, v_n) \rightharpoonup (u, v)$ in $X$ as well as the compact embedding of $W_0^{1,p_i}(w_i, \Omega)$ in $L^{r_i}(\Omega)$, $L^{\gamma_i}(\Omega)$ and $L^{\frac{q_i}{q_i-\delta_i}}(\Omega)$ for $i \in \{1, 2\}$, it follows that

$$\begin{cases} \lim_{n\to\infty} \int_\Omega f(x, \rho_1 \star u_n, \rho_2 \star v_n, \nabla(\rho_1 \star u_n), \nabla(\rho_2 \star v_n))(u_n - u)dx = 0, \\ \lim_{n\to\infty} \int_\Omega g(x, \rho_1 \star u_n, \rho_2 \star v_n, \nabla(\rho_1 \star u_n), \nabla(\rho_2 \star v_n))(v_n - u)dx = 0. \end{cases} \tag{2.13}$$

Using (2.13), we deduce that

$$\limsup_{n\to\infty} \langle \widetilde{\mathcal{I}}(u_n, v_n), (u_n - u, v_n - v)\rangle = \limsup_{n\to\infty} \langle \mathcal{I}(u_n, v_n), (u_n - u, v_n - v)\rangle \le 0. \tag{2.14}$$

Because $\widetilde{\mathcal{I}}$ satisfies the $(S^+)$-property as seen in Lemma 2.6, and from (2.10) and (2.14), we obtain

$$(u_n, v_n) \to (u, v) \quad \text{in } X.$$

Ultimately, as $\mathcal{I}$ is continuous, we find $\mathcal{I}(u_n, v_n) \to \mathcal{I}(u, v)$ in $X^*$. Thus, $\mathcal{I}$ is pseudomonotone.

The final step is to verify that the operator $I : X \to X^\star$ is coercive. Using hypothesis $(H_2)$ and (2.5), we find

$$\begin{aligned} &\int_\Omega f(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v))u \\ &\quad + g(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v))v dx \\ &\le \int_\Omega (\lambda_2 + \varepsilon)w(x)|\rho_1 \star u|^{\alpha_1}|(\rho_2 \star v)|^{\alpha_2} dx \\ &\quad + \int_\Omega c(w_1(x)|\nabla(\rho_1 \star u)|^{q_1} + w_2(x)|\nabla(\rho_2 \star v)|^{q_2})dx + C_\varepsilon(x) \\ &\le \int_\Omega \left(\frac{\lambda_2 + \varepsilon}{\lambda_1}\right)\left(\frac{\alpha_1}{p_1} w_1(x)|\nabla(\rho_1 \star u)|^{p_1} + \frac{\alpha_2}{p_2} w_2(x)|\nabla(\rho_2 \star v)|^{p_2}\right)dx \\ &\quad + cN^{q_1}\|\rho_1\|^{q_1}_{L^1(\mathbb{R}^N)} \int_\Omega w_1(x)|\nabla u|^{q_1} dx + cN^{q_2}\|\rho_2\|^{q_2}_{L^1(\mathbb{R}^N)} \int_\Omega w_2(x)|\nabla v|^{q_2} dx \\ &\quad + \|C_\varepsilon(x)\|_{L^1(\Omega)}. \end{aligned} \tag{2.15}$$

On the other hand, we have

$$\begin{aligned} \langle \widetilde{\mathcal{I}}(u, v), (u, v)\rangle \ge &\int_\Omega w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2} dx \\ &+ \min\{\mu_1, \mu_2\} \int_\Omega w_1(x)|\nabla u|^{q_1} + w_2(x)|\nabla v|^{q_2} dx. \end{aligned} \tag{2.16}$$

Now, using the embedding theorem, there exist $\theta_0, \theta_1 > 0$ such that

$$\delta\Big[\int_\Omega (w_1(x)|\nabla u|^{q_1} + w_2(x)|\nabla v|^{q_2})dx\Big] \geq \theta_0[\int_\Omega (w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2})dx] - \theta_1, \tag{2.17}$$

where

$$\delta = \min\{\mu_1, \mu_2\} - c\max\Big\{N^{q_1}\|\rho_1\|^{q_1}_{L^1(\mathbb{R}^N)}, N^{q_2}\|\rho_2\|^{q_2}_{L^1(\mathbb{R}^N)}\Big\}.$$

Therefore, from (2.15) and (2.16), we deduce that

$$\begin{aligned}
&\langle \mathcal{I}(u,v),(u,v)\rangle \\
&\geq \int_\Omega w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2}dx \\
&\quad + \delta\Big(\int_\Omega w_1(x)|\nabla u|^{q_1} + w_2(x)|\nabla v|^{q_2}dx\Big) \\
&\quad - \int_\Omega \Big(\frac{\alpha_1}{p_1}w_1(x)|\nabla u|^{p_1} + \frac{\alpha_2}{p_2}w_2(x)|\nabla v|^{p_2}\Big)dx \\
&\quad - \Big(\frac{\lambda_2+\varepsilon}{\lambda_1}\Big)\int_\Omega \Big(\frac{\alpha_1}{p_1}w_1(x)|\nabla(\rho_1\star u)|^{p_1} + \frac{\alpha_2}{p_2}w_2(x)|\nabla(\rho_2\star v)|^{p_2}\Big)dx \\
&\quad - \|C_\varepsilon(x)\|_{L^1(\Omega)} - \|(u,v)\|_X\|(h_1,h_2)\|_{X^\star}.
\end{aligned}$$

Now, using (2.17), it follows that

$$\begin{aligned}
&\langle \mathcal{I}(u,v),(u,v)\rangle \\
&\geq (1+\theta_0)\Big[\int_\Omega w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2}dx\Big] - \theta_1 \\
&\quad - \frac{\min\{\alpha_1,\alpha_2\}}{\max\{p_1,p_2\}}\Big(\int_\Omega w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2}dx\Big) - \|C_\varepsilon(x)\|_{L^1(\Omega)} \\
&\quad - \Big(\frac{\lambda_2+\varepsilon}{\lambda_1}\Big)\Big[\frac{\alpha_1}{p_1}N^{p_1}\|\rho_1\|^{p_1}_{L^1(\mathbb{R}^N)}\int_\Omega w_1(x)|\nabla u|^{p_1}dx \\
&\qquad\qquad + \frac{\alpha_2}{p_2}N^{p_2}\|\rho_2\|^{p_2}_{L^1(\mathbb{R}^N)}\int_\Omega w_2(x)|\nabla v|^{p_2}dx\Big] \\
&\quad - \|(u,v)\|_X\|(h_1,h_2)\|_{X^\star} \\
&\geq \Big[(1+\theta_0) - \frac{\min\{\alpha_1,\alpha_2\}}{\max\{p_1,p_2\}} \\
&\qquad - \Big(\frac{\lambda_2+\varepsilon}{\lambda_1}\Big)\max\Big\{\frac{\alpha_1}{p_1}N^{p_1}\|\rho_1\|^{p_1}_{L^1(\mathbb{R}^N)}, \frac{\alpha_2}{p_2}N^{p_2}\|\rho_2\|^{p_2}_{L^1(\mathbb{R}^N)}\Big\}\Big] \\
&\quad \cdot\Big(\int_\Omega w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2}dx\Big) - \theta_1 - \|C_\varepsilon(x)\|_{L^1(\Omega)} \\
&\quad - \|(u,v)\|_X\|(h_1,h_2)\|_{X^\star}
\end{aligned}$$

$$
\begin{aligned}
&\geq \Bigg[(1+\theta_0) - \frac{\min\{\alpha_1, \alpha_2\}}{\max\{p_1, p_2\}} \\
&\quad - \left(\frac{\lambda_2 + \varepsilon}{\lambda_1}\right) \max\left\{\frac{\alpha_1}{p_1} N^{p_1} \|\rho_1\|^{p_1}_{L^1(\mathbb{R}^N)}, \frac{\alpha_2}{p_2} N^{p_2} \|\rho_2\|^{p_2}_{L^1(\mathbb{R}^N)}\right\}\Bigg] \\
&\cdot \left(\|u\|^{p_1}_{W_0^{1,p_1}(w_1,\Omega)} + \|v\|^{p_2}_{W_0^{1,p_2}(w_2,\Omega)}\right) - \theta_1 - \|C_\varepsilon(x)\|_{L^1(\Omega)} - \frac{\alpha_0}{\tau} \|(u,v)\|^{\tau}_X \\
&- \frac{1}{\alpha_0 \tau'} \|(h_1, h_2)\|^{\tau'}_{X^\star},
\end{aligned}
$$

where

$$
0 < \alpha_0 < \tau < \min\{p_1, p_2\} \quad \text{with } \tau' = \frac{\tau}{\tau - 1}.
$$

Thus, we obtain

$$
\begin{aligned}
&\langle \mathcal{I}(u,v), (u,v) \rangle \\
&\quad \geq \Bigg[1 + \theta_0 - \frac{\min\{\alpha_1, \alpha_2\}}{\max\{p_1, p_2\}} - \frac{\alpha_0}{\tau} \\
&\qquad - \left(\frac{\lambda_2 + \varepsilon}{\lambda_1}\right) \max\left\{\frac{\alpha_1}{p_1} N^{p_1} \|\rho_1\|^{p_1}_{L^1(\mathbb{R}^N)}, \frac{\alpha_2}{p_2} N^{p_2} \|\rho_2\|^{p_2}_{L^1(\mathbb{R}^N)}\right\}\Bigg] \|(u,v)\|^{\tau}_X \\
&\quad - \theta_1 - \|C_\varepsilon(x)\|_{L^1(\Omega)} - \frac{1}{\alpha_0 \tau'} \|(h_1, h_2)\|^{\tau'}_{X^\star}.
\end{aligned}
$$

Choosing appropriate values for $\varepsilon$ and $\alpha_0$ such that

$$
\begin{aligned}
1 + \theta_0 > &\frac{\min\{\alpha_1, \alpha_2\}}{\max\{p_1, p_2\}} \\
&+ \left(\frac{\lambda_2 + \varepsilon}{\lambda_1}\right) \max\left\{\frac{\alpha_1}{p_1} N^{p_1} \|\rho_1\|^{p_1}_{L^1(\mathbb{R}^N)}, \frac{\alpha_2}{p_2} N^{p_2} \|\rho_2\|^{p_2}_{L^1(\mathbb{R}^N)}\right\} + \frac{\alpha_0}{\tau}
\end{aligned}
$$

leads to the coerciveness of $\mathcal{I}$.

Thus, all assumptions of Theorem 2.5 are verified. Therefore, there exists $(u, v) \in X$ such that $\mathcal{I}(u, v) = 0$. This completes the proof. ■

**Example 2.8.** Let the functions $f$ and $g$ be defined by

$$
f(x,s,t,\xi,\eta) = |s|^{\alpha_1 - 2} s |t|^{\alpha_2} \left(a_1(x) + \frac{d_1(x)}{1 + |s|^{\alpha_1} |t|^{\alpha_2}} |\xi|^{q_1}\right),
$$

and

$$
g(x,s,t,\xi,\eta) = |s|^{\alpha_1} |t|^{\alpha_2 - 2} t \left(a_2(x) + \frac{d_2(x)}{1 + |s|^{\alpha_1} |t|^{\alpha_2}} |\eta|^{q_2}\right),
$$

where $\alpha_1$ and $\alpha_2$ are defined in Section 1, and $a_1, a_2, d_1, d_2 \in L^\infty(\Omega)$ with the condition $\|a_1 + a_2\|_{L^\infty(\Omega)} < \lambda_2$ are satisfied. Then, $f$ and $g$ satisfy the hypotheses in Theorem 2.7.

## 3. Case of a gradient-type system

In this section, we focus our study on a gradient-type system. In other words, we consider the following system:

$$\begin{cases} -\Delta_{w_1,p_1} u - \mu_1 \Delta_{w_1,q_1} u = \lambda_1 w(x)|u|^{\alpha_1-2}u|v|^{\alpha_2} \\ \qquad\qquad + \dfrac{\partial F}{\partial u}(x, \rho_1 \star u, \rho_2 \star v) + h_1(x) & \text{in } \Omega, \\ -\Delta_{w_2,p_2} v - \mu_2 \Delta_{w_2,q_2} v = \lambda_1 w(x)|v|^{\alpha_2-2}v|u|^{\alpha_1} \\ \qquad\qquad + \dfrac{\partial F}{\partial v}(x, \rho_1 \star u, \rho_2 \star v) + h_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Similarly, in order to prove the existence of at least one weak solution, we need the following assumptions on $F$.

(F$_1$) $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t} : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions such that there exist positive continuous functions $\alpha_i^*$, nonnegative functions $e_i \in L^1(\Omega)$ verifying $1 < \alpha_i^* < p_{i,s_i}^*$, and nonnegative constants $a_i$ for $i \in \{1, 2\}$, such that for all $(s,t) \in \mathbb{R}^2$, we have

$$\left|\frac{\partial F}{\partial s}(x,s,t)\right| \leq a_1 w(x)|s|^{\alpha_1^*-1}|t|^{\alpha_2^*} + e_1(x)$$

and

$$\left|\frac{\partial F}{\partial t}(x,s,t)\right| \leq a_2 w(x)|s|^{\alpha_1^*}|t|^{\alpha_2^*-1} + e_2(x).$$

(F$_2$) There exist a nonnegative constant $C_1$ and a nonnegative function $C_2 \in L^1(\Omega)$ such that, for all $x \in \Omega$, we have

$$|F(x,s,t)| \leq C_1 w(x)|s|^{\alpha_1}|t|^{\alpha_2} + C_2(x).$$

(F$_3$) $F_\infty(x) \equiv \limsup_{|(s,t)|\to+\infty} \frac{F(x,s,t)}{w(x)|s|^{\alpha_1}|t|^{\alpha_2}} < \lambda_2$ uniformly for a.e. $x \in \Omega$.

Define

$$\mathcal{I}(u,v) = \widetilde{\mathcal{I}}(u,v) - \lambda_1 A(u,v) - \int_\Omega F(x, \rho_1 \star u, \rho_2 \star v)dx - \langle h_1, u\rangle - \langle h_2, v\rangle, \tag{3.2}$$

where

$$\begin{aligned} \widetilde{\mathcal{I}}(u,v) = &\int_\Omega \frac{w_1(x)}{p_1}|\nabla u|^{p_1} + \frac{\mu_1 w_1(x)}{q_1}|\nabla u|^{q_1}dx \\ &+ \int_\Omega \frac{w_2(x)}{p_2}|\nabla v|^{p_2} + \frac{\mu_2 w_2(x)}{q_2}|\nabla v|^{q_2}dx, \end{aligned} \tag{3.3}$$

and

$$A(u,v) = \int_\Omega w(x)|u|^{\alpha_1}|v|^{\alpha_2}dx.$$

Next, we state and prove the existence result for the system (3.1) at resonance.

**Theorem 3.1.** *Suppose that hypotheses* ($F_1$)–($F_3$) *are verified. Then, the system* (3.1) *admits at least one weak solution.*

*Proof.* In order to obtain the desired result, we need to minimize the energy functional associated with the system (3.1), that is,

$$\mathcal{I}(u,v) = \widetilde{\mathcal{I}}(u,v) - \lambda_1 A(u,v) - \int_\Omega F(x, \rho_1 \star u, \rho_2 \star v)dx - \langle h_1, u\rangle - \langle h_2, v\rangle.$$

To this end, we show that $\mathcal{I}$ is weakly lower semicontinuous and coercive. Let $(u_n, v_n)_{n\in\mathbb{N}}$ be a sequence such that

$$(u_n, v_n) \rightharpoonup (u,v) \quad \text{in } X.$$

Taking the compact embedding $W_0^{1,p_i}(w_i, \Omega) \hookrightarrow\hookrightarrow L^{r_i}(\Omega)$ into consideration, we obtain

$$(u_n, v_n) \to (u,v) \quad \text{in } L^{r_1}(\Omega) \times L^{r_2}(\Omega).$$

Since $r_i \in [1, p^*_{i,s_i}]$ for $i \in \{1,2\}$. In addition, there exists $(\theta_1, \theta_2) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ such that

$$\begin{cases} |u_n(x)| \le \theta_1(x), \\ |v_n(x)| \le \theta_2(x) \end{cases}$$

for a.e. $x \in \Omega$. Therefore, using hypothesis ($F_2$) as well as the properties of convolution and mollifiers, we obtain

$$\begin{aligned} F(x, \rho_1 \star u_n(x), \rho_2 \star v_n(x)) &\le C_1 w(x) |\rho_1 \star u_n|^{\alpha_1} |\rho_2 \star v_n|^{\alpha_2} + C_2(x) \\ &\le C_1 w(x) |\rho_1 \star \theta_1|^{\alpha_1} |\rho_2 \star \theta_2|^{\alpha_2} + C_2(x). \end{aligned}$$

Using the fact that $F(x, \rho_1 \star u_n(x), \rho_2 \star v_n(x)) \to F(x, \rho_1 \star u(x), \rho_2 \star v(x))$, for a.e. $x \in \Omega$ and applying Fatou's lemma, we derive

$$\limsup_{n\to+\infty} \int_\Omega F(x, \rho_1 \star u_n(x), \rho_2 \star v_n(x))dx \le \int_\Omega F(x, \rho_1 \star u(x), \rho_2 \star v(x))dx.$$

Thus, we find

$$\begin{aligned} \mathcal{I}(u,v) &= \widetilde{\mathcal{I}}(u,v) - \lambda_1 A(u,v) - \int_\Omega F(x, \rho_1 \star u, \rho_2 \star v)dx - \langle h_1, u\rangle - \langle h_2, v\rangle \\ &\le \liminf_{n\to+\infty} \widetilde{\mathcal{I}}(u_n, v_n) - \lambda_1 \lim_{n\to+\infty} A(u_n, v_n) \\ &\quad - \limsup_{n\to+\infty} \int_\Omega F(x, \rho_1 \star u_n, \rho_2 \star v_n)dx - \lim_{n\to+\infty} \langle h_1, u_n\rangle - \lim_{n\to+\infty} \langle h_2, v_n\rangle \\ &\le \liminf_{n\to+\infty} \Big\{ \widetilde{\mathcal{I}}(u_n, v_n) - \lambda_1 A(u_n, v_n) - \int_\Omega F(x, \rho_1 \star u_n, \rho_2 \star v_n)dx \\ &\qquad - \langle h_1, u_n\rangle - \langle h_2, v_n\rangle \Big\} \\ &\le \liminf_{n\to+\infty} \mathcal{I}(u_n, v_n). \end{aligned}$$

Thus, $\mathcal{I}$ is weakly lower semicontinuous.

The final step involves proving that $\mathcal{I}$ is coercive. Using hypotheses $(F_3)$, and for all $\varepsilon > 0$, there exist $\delta_\varepsilon > 0$ and $C_\varepsilon \in L^1(\Omega)$ such that

$$F(x, \rho_1 \star u, \rho_2 \star v) \le (F_\infty(x) + \varepsilon) w(x) |\rho_1 \star u|^{\alpha_1} |\rho_2 \star v|^{\alpha_2} + C_\varepsilon(x) \tag{3.4}$$

for every $(u, v) \in \mathbb{R}^2$ satisfying

$$|(\rho_1 \star u, \rho_2 \star v)|_\infty > \delta_\varepsilon$$

for a.e. $x \in \Omega$. On the other hand, applying the embedding theorem, there exist $\eta_0, \eta_1 > 0$ such that

$$\begin{aligned} \min\{\mu_1, \mu_2\} \int_\Omega \frac{w_1(x)}{q_1} |\nabla u|^{q_1} + \frac{w_2(x)}{q_2} |\nabla v|^{q_2} dx \\ \ge \eta_0 \int_\Omega \frac{w_1(x)}{p_1} |\nabla u|^{p_1} + \frac{w_2(x)}{p_2} |\nabla v|^{p_2} dx - \eta_1. \end{aligned} \tag{3.5}$$

Therefore, from (2.16) and (3.4), we derive

$$\begin{aligned} \mathcal{I}(u, v) \ge & \int_\Omega \frac{w_1(x)}{p_1} |\nabla u|^{p_1} + \frac{\mu_1 w_1(x)}{q_1} |\nabla u|^{q_1} dx \\ & + \int_\Omega \frac{w_2(x)}{p_2} |\nabla v|^{p_2} + \frac{\mu_2 w_2(x)}{q_2} |\nabla v|^{q_2} dx \\ & - \lambda_1 \int_\Omega w(x) |u|^{\alpha_1} |v|^{\alpha_2} dx \\ & - (\lambda_2 + \varepsilon) \int_\Omega w(x) |\rho_1 \star u|^{\alpha_1} |\rho_2 \star v|^{\alpha_2} dx \\ & - \int_\Omega C_\varepsilon(x) dx - \|(u, v)\|_X \|(h_1, h_2)\|_{X^\star}. \end{aligned}$$

It follows then

$$\begin{aligned} \mathcal{I}(u, v) \ge & \int_\Omega \left( \frac{w_1(x)}{p_1} |\nabla u|^{p_1} + \frac{w_2(x)}{p_2} |\nabla v|^{p_2} \right) dx \\ & + \min\{\mu_1, \mu_2\} \int_\Omega \left( \frac{w_1(x)}{q_1} |\nabla u|^{q_1} + \frac{w_2(x)}{q_2} |\nabla v|^{q_2} \right) dx - \|C_\varepsilon(x)\|_{L^1(\Omega)} \\ & - \int_\Omega \left( \frac{\alpha_1 w_1(x)}{p_1} |\nabla u|^{p_1} + \frac{\alpha_2 w_2(x)}{p_2} |\nabla v|^{p_2} \right) dx - \|(u, v)\|_X \|(h_1, h_2)\|_{X^\star} \\ & - \left( \frac{\lambda_2 + \varepsilon}{\lambda_1} \right) \int_\Omega \left( \frac{\alpha_1 w_1(x)}{p_1} |\nabla(\rho_1 \star u)|^{p_1} + \frac{\alpha_2 w_2(x)}{p_2} |\nabla(\rho_2 \star v)|^{p_2} \right) dx. \end{aligned}$$

Now, using (3.5), we find

$$
\begin{aligned}
\mathcal{I}(u,v) \geq & \frac{(1+\eta_0)}{\max\{p_1,p_2\}} \int_\Omega (w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2})dx - \eta_1 - \|C_\varepsilon(x)\|_{L^1(\Omega)} \\
& - \frac{\min\{\alpha_1,\alpha_2\}}{\max\{p_1,p_2\}} \int_\Omega (w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2})dx - \|(u,v)\|_X \|(h_1,h_2)\|_{X^\star} \\
& - \left(\frac{\lambda_2+\varepsilon}{\lambda_1}\right)\Bigg[\frac{\alpha_1}{p_1} N^{p_1} \|\rho_1\|^{p_1}_{L^1(\mathbb{R}^N)} \int_\Omega w_1(x)|\nabla u|^{p_1} dx \\
& \qquad + \frac{\alpha_2}{p_2} N^{p_2} \|\rho_2\|^{p_2}_{L^1(\mathbb{R}^N)} \int_\Omega w_2(x)|\nabla v|^{p_2} dx\Bigg] \\
\geq & \Bigg[\frac{(1+\eta_0)}{\max\{p_1,p_2\}} - \frac{\min\{\alpha_1,\alpha_2\}}{\max\{p_1,p_2\}} \\
& - \left(\frac{\lambda_2+\varepsilon}{\lambda_1}\right) \max\left\{\frac{\alpha_1}{p_1} N^{p_1}\|\rho_1\|^{p_1}_{L^1(\mathbb{R}^N)}, \frac{\alpha_2}{p_2} N^{p_2}\|\rho_2\|^{p_2}_{L^1(\mathbb{R}^N)}\right\}\Bigg] \\
& \cdot \left(\|u\|^{p_1}_{W_0^{1,p_1}(w_1,\Omega)} + \|v\|^{p_2}_{W_0^{1,p_2}(w_2,\Omega)}\right) - \eta_1 - \|C_\varepsilon(x)\|_{L^1(\Omega)} - \frac{\beta_0}{\rho}\|(u,v)\|_X^\rho \\
& - \frac{1}{\beta_0 \rho'} \|(h_1,h_2)\|^{\rho'}_{X^\star},
\end{aligned}
$$

where $0 < \beta_0 < \rho < \min\{p_1, p_2\}$, with $\rho' = \frac{\rho}{\rho-1}$. Thus, we obtain

$$
\begin{aligned}
\mathcal{I}(u,v) \geq & \Bigg[\frac{(1+\eta_0) - \min\{\alpha_1,\alpha_2\}}{\max\{p_1,p_2\}} \\
& - \left(\frac{\lambda_2+\varepsilon}{\lambda_1}\right) \max\left\{\frac{\alpha_1}{p_1} N^{p_1}\|\rho_1\|^{p_1}_{L^1(\mathbb{R}^N)}, \frac{\alpha_2}{p_2} N^{p_2}\|\rho_2\|^{p_2}_{L^1(\mathbb{R}^N)}\right\} - \frac{\beta_0}{\rho}\Bigg]\|(u,v)\|_X^\rho \\
& - \|C_\varepsilon(x)\|_{L^1(\Omega)} - \frac{1}{\beta_0\rho'}\|(h_1,h_2)\|^{\rho'}_{X^\star} - \eta_1.
\end{aligned}
$$

Choosing appropriate values for $\varepsilon$ and $\beta_0$ such that

$$
\frac{(1+\eta_0) - \min\{\alpha_1,\alpha_2\}}{\max\{p_1,p_2\}} > \left(\frac{\lambda_2+\varepsilon}{\lambda_1}\right) \max\left\{\frac{\alpha_1}{p_1} N^{p_1}\|\rho_1\|^{p_1}_{L^1(\mathbb{R}^N)}, \frac{\alpha_2}{p_2} N^{p_2}\|\rho_2\|^{p_2}_{L^1(\mathbb{R}^N)}\right\} + \frac{\beta_0}{\rho},
$$

leads to the coerciveness of $\mathcal{I}$.

Hence, $\mathcal{I}$ is weakly lower semicontinuous and coercive, then $\mathcal{I}$ admits a minimum point $(u, v)$ in $X$. Owing to hypothesis $(F_1)$, $\mathcal{I}$ is a $C^1$-functional. Furthermore, $(u, v) \in X$ is a weak solution for the system (3.1) if and only if $(u, v)$ is a critical point. This completes the proof. ■

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## References


[1] C. O. Alves, P. C. Carrião, and O. H. Miyagaki, Multiple solutions for a problem with resonance involving the $p$-Laplacian. *Abstr. Appl. Anal.* **3** (1998), no. 1-2, 191–201 Zbl 0968.35047 MR 1700284

[2] N. Benouhiba and Z. Belyacine, On the solutions of the $(p, q)$-Laplacian problem at resonance. *Nonlinear Anal.* **77** (2013), 74–81 Zbl 1255.35036 MR 2988761

[3] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*. Universitext, Springer, New York, 2011 Zbl 1220.46002 MR 2759829

[4] S. Carl, V. K. Le, and D. Motreanu, *Nonsmooth variational problems and their inequalities: Comparison principles and applications*. Springer Monogr. Math., Springer, New York, 2007 Zbl 1109.35004 MR 2267795

[5] P. L. de Nápoli and M. C. Mariani, Quasilinear elliptic systems of resonant type and nonlinear eigenvalue problems. *Abstr. Appl. Anal.* **7** (2002), no. 3, 155–167 Zbl 1005.35036 MR 1896978

[6] P. Drábek, A. Kufner, and F. Nicolosi, *Quasilinear elliptic equations with degenerations and singularities*. De Gruyter Ser. Nonlinear Anal. Appl. 5, Walter de Gruyter, Berlin, 1997 Zbl 0894.35002 MR 1460729

[7] A. El Hachimi and F. de Thélin, Non résonance près de la première valeur propre d'un système elliptique quasilinéaire de type potentiel. *Publ. Mat.* **39** (1995), no. 2, 393–404 Zbl 0857.35100 MR 1370895

[8] A. El Hachimi and J.-P. Gossez, A note on a nonresonance condition for a quasilinear elliptic problem. *Nonlinear Anal.* **22** (1994), no. 2, 229–236 Zbl 0816.35031 MR 1258959

[9] L. C. Evans, *Partial differential equations*. 2nd edn., Grad. Stud. Math. 19, American Mathematical Society, Providence, RI, 2010 Zbl 1194.35001 MR 2597943

[10] X.-L. Fan and Q.-H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem. *Nonlinear Anal.* **52** (2003), no. 8, 1843–1852 Zbl 1146.35353 MR 1954585

[11] I. Goodfellow, Y. Bengio, and A. Courville, *Deep learning*. Adapt. Comput. Mach. Learn., MIT Press, Cambridge, MA, 2016 Zbl 1373.68009 MR 3617773

[12] M. Haddaoui, H. Lebrimchi, B. Ouhamou, and N. Tsouli, Existence of solutions for a nonlinear problem at resonance. *Demonstr. Math.* **55** (2022), no. 1, 482–489 Zbl 1498.35313 MR 4485974

[13] A. Kufner and B. Opic, How to define reasonably weighted Sobolev spaces. *Comment. Math. Univ. Carolin.* **25** (1984), no. 3, 537–554 Zbl 0557.46025 MR 0775568

[14] Y. Lv and Z.-Q. Ou, Existence of weak solutions for a class of $(p, q)$-Laplacian systems. *Bound. Value Probl.* **2017** (2017), article no. 168 Zbl 1379.35150 MR 3722282

[15] Z.-Q. Ou, Existence of weak solutions for a class of $(p, q)$-Laplacian systems on resonance. *Appl. Math. Lett.* **50** (2015), 29–36 Zbl 1329.35142 MR 3378945

[16] Z.-Q. Ou, $(p, q)$-Laplacian elliptic systems at resonance. *Electron. J. Differential Equations* **2016** (2016), article no. 163 Zbl 1383.35083 MR 3522218

[17] Z.-Q. Ou and C.-L. Tang, Resonance problems for the $p$-Laplacian systems. *J. Math. Anal. Appl.* **345** (2008), no. 1, 511–521 Zbl 1147.35307 MR 2422668

[18] A. Ouannasser and A. E. Hachimi, Solvability of parametric elliptic systems with variable exponents. *Moroccan J. Pure Appl. Anal.* **9** (2023), no. 3, 311–330 Zbl 7836911

[19] G. Strang, *Introduction to applied mathematics*. Wellesley-Cambridge Press, Wellesley, MA, 1986 Zbl 0618.00015 MR 0870634

[20] X.-X. Zhao and C.-L. Tang, Resonance problems for $(p, q)$-Laplacian systems. *Nonlinear Anal.* **72** (2010), no. 2, 1019–1030 Zbl 1180.35223 MR 2579366

[21] N. B. Zographopoulos, $p$-Laplacian systems on resonance. *Appl. Anal.* **83** (2004), no. 5, 509–519 Zbl 1096.35052 MR 2054643

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