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A quadratic Roth theorem for sets with large Hausdorff dimensions

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Abstract. Many results in harmonic analysis and geometric measure theory ensure the existence of geometric configurations under the largeness of sets, which are sometimes specified via the ball condition and Fourier decay. Recently, Kuca, Orponen, and Sahlsten, and also Bruce and Pramanik proved Sárközy-like theorems, which remove the Fourier decay condition and show that sets with large Hausdorff dimensions contain two-point patterns. This paper explores the existence of a three-point configuration that relies solely on the Hausdorff dimension.

1. Introduction

1.1. Pattern Recognition

Finding the maximum size of a set that avoids certain patterns is an intriguing task with many applications. A well-known result by K. F. Roth [19] shows that if a set $A \subset [N] := \{1, 2, \ldots, N\}$ and A avoids any non-trivial 3-term arithmetic progressions, then $|A| = O(N/\log\log N)$. An analog question in the continuum is also studied. If a set $A \subset [0, 1]$ avoids any arithmetic progressions, it is Lebesgue null (Lemma 4.1). Many works have examined the existence and avoidance of three-term arithmetic progressions of Lebesgue null sets. A set contains a three-term arithmetic progression if it supports a measure with a ball and a Fourier decay condition.

Theorem A ([12]). *Let C_1, C_2, and B be positive real numbers and $\beta \in (\frac{2}{3}, 1]$. Then, there exists $\varepsilon_0 > 0$ such that the following statement holds. Suppose that $E \subset [0, 1]$ is a closed set and $\mu \in \mathcal{M}(E)$ has the following properties, for an $\alpha \in (1 - \varepsilon_0, 1)$:*

(1) *(ball condition) $\mu([x, x + r]) \le C_1 r^\alpha$ for all $x \in [0, 1]$ and $r \in (0, 1]$,*

(2) *(Fourier decay condition) $|\widehat{\mu}(k)| \le C_2(1 - \alpha)^{-B}|k|^{-\frac{\beta}{2}}$ for all $k \ne 0$.*

Then, E contains a non-trivial 3-term arithmetic progression.

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The number α depends on the numbers C_1, C_2, B, and β. In addition, the ball condition is stronger than $\dim_H(E) \geq \alpha$ and the Fourier decay condition is stronger than $\dim_F(E) \geq \beta$, as there are conditions on the constants in the ball and Fourier decay conditions. In contrast, some sets of full Hausdorff dimension [10] and Salem sets [21], which are traditionally considered large fractal sets, avoid any three-term arithmetic progressions.

A similar problem is finding the maximum size of the sets $A \subset [N]$ avoiding the pattern $\{x, x + z^2\}$ for some $x, z \in \mathbb{N}$ as studied by [7, 20]. One variant of this pattern is the *three-point quadratic pattern* $\{x, x + t, x + t^2\}$ in $\mathbb{R}$. Any set $E \subset \mathbb{R}$ with a positive Lebesgue measure contains $x, x + t, x + t^2$ for some $x, t \in \mathbb{R}$ with $t \neq 0$. This is a special case of a general statement in the appendix (Lemma 4.1). From [2], if $S \subset [0, N]$ with $|S| \geq \varepsilon N$, then S also contains $\{x, x + t, x + t^2\}$ for some $x, t \in \mathbb{R}$ and t greater than an absolute constant that depends on N and ε. One direction of study is to identify criteria for Lebesgue null sets to contain the same quadratic pattern. Fraser, Guo, and Pramanik [6] obtained the following result on the three-term polynomial pattern.

Theorem B ([6]). *Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial without a constant term and* $\deg P \geq 2$. *There exists $s_0 > 0$ such that the following statement holds. For C_1, C_2, B positive real numbers, and $\beta \in (1 - s_0, 1]$, there exists $\varepsilon_0 > 0$ such that if $E \subset [0, 1]$ is a closed set and $\mu \in \mathcal{M}(E)$ has the following properties:*

(1) *(ball condition) $\mu([x, x + r]) \leq C_1 r^\alpha$ for an $\alpha \in (1 - \varepsilon_0, 1)$ and all $x \in [0, 1]$, $r \in (0, 1]$,*

(2) *(Fourier decay condition) $|\widehat{\mu}(k)| \leq C_2(1 - \alpha)^{-B}|k|^{-\frac{\beta}{2}}$ for all $k \in \mathbb{Z}\backslash\{0\}$,*

then E contains three points $x, x + t, x + P(t)$ for real numbers x, t with $t \neq 0$.

Another variant is the *two-point quadratic pattern* $\{(x, y), (x + z, y + z^2)\}$ in $\mathbb{R}^2$. Kuca, Orponen, and Sahlsten [11] studied the problem of identifying two-point quadratic patterns in $\mathbb{R}^2$ for Lebesgue null sets using the Hausdorff dimension only.

Theorem C ([11]). *There is an absolute constant $\varepsilon > 0$ such that the following holds. Let $K \subset \mathbb{R}^2$ with $\dim_H(K) > 2 - \varepsilon$. Then, there exist $x, y, z \in \mathbb{R}$ with $z \neq 0$, such that*

$$\{(x, y), (x + z, y + z^2)\} \subset K.$$

The three-point quadratic pattern in $\mathbb{R}$ (Theorem B) and the two-point quadratic pattern in $\mathbb{R}^2$ (Theorem C) have different criteria to ensure their existence. We would like to understand their fundamental differences. Specifically, we investigate whether having a large Hausdorff dimension alone is sufficient to ensure the existence of the three-point quadratic pattern. Our conjecture is as follows.

Conjecture 1.1. *For all (Borel) $E \subset [0,1]$ with $\dim_H(E) > \frac{1}{2}$, there exist $x, t \in \mathbb{R}$ with $t \neq 0$ such that*

$$\{x, x+t, x+t^2\} \subset E.$$

We require $\dim_H(E) > \frac{1}{2}$ because of the following result.

Theorem D ([14]). *There exists a compact set $E \subset \mathbb{R}$ of Hausdorff dimension $\frac{1}{2}$ such that E does not contain 3 distinct points x, y, z satisfying $(z-x) = (y-x)^2$.*

In this manuscript, we present results from adapting methods in [11] and [3] that partly address Conjecture 1.1.

1.2. Main Results

The main theorem is as follows.

Theorem 1.2. *Given $q_0 \in [0,1)$, there exists $\beta = \beta(q_0) \in (0,1)$ with the following property. For every (Borel) $E \subset \mathbb{R}$ with $\dim_H(E) > \beta$, one can find $0 < a_0 < a_1$ such that for every (a, q) with $a_0 \leq |a| \leq a_1$, $|q| \leq q_0$,*

$$\left\{x, x+t, x+P_{a,q}(t)\right\} \subset E$$

for some $x, t \in \mathbb{R}$, $t \neq 0$. Here, $P_{a,q}(t) = at^2 + qt$.

Remarks. (1) As q_0 increases, β increases.

(2) The range $[a_0, a_1]$ depends on the set E, which is different from similar results [1,4,16–18] in the discrete setting, in which the coefficient of t^2 can be chosen independently from the coefficient of t or the set. The dependency of the range is further discussed in the remark at the end of Section 2.

(3) The dependence of the range $[a_0, a_1]$ is due to the configuration not being dilation invariant. Suppose $x, x+t, x+at^2+qt \in E$ for $x, t, a, q \in \mathbb{R}$ and $b > 0$. Let $y = bx, s = bt$, then $y, y+s, y+\frac{a}{b}s^2+qs \in bE$.

The proof of Theorem 1.2 follows from the proposition below, which uses the *dyadic Hausdorff content* defined as

$$\mathcal{H}^{\beta}_{\infty}(S) := \inf\left\{\sum_i l(Q_i)^{\beta} \mid S \subset \bigcup_i Q_i, \text{dyadic } Q_i \subset [0,1]\right\} \tag{1.1}$$

for a set $S \subset [0,1]$. Here, the set Q_i is dyadic if it is an interval of the form

$$\left[k_i 2^{-N_i}, (k_i+1)2^{-N_i}\right]$$

for integers $N_i \geq 0, 0 \leq k_i < 2^{N_i}$. The length of Q_i is denoted as $l(Q_i)$.

Proposition 1.3. *Let $p_0 < p_1$ be positive real numbers and $q_0 \in [0,1)$. Then, there exist $\beta \in (0,1)$ and $\delta > 0$ such that the following statement holds: for all (Borel) $E \subset [0,1]$ with $\mathcal{H}^{\beta}_{\infty}(E) \geq 1-\delta$ and for all (p,q) with $p_0 \leq |p| \leq p_1$ and $|q| \leq q_0$,*

$$\{x, x+t, x+P_{p,q}(t)\} \subset E$$

for some $x, t \in \mathbb{R}$ with $t \neq 0$. Here, $P_{p,q}(t) = pt^2 + qt$.

This is the first result confirming 3-point nonlinear patterns in sets of large Hausdorff dimensions alone. A key idea in the proof is that even though a set with a large Hausdorff dimension may not support a measure with any Fourier decay, a set with a large Hausdorff content supports a measure satisfying the spectral gap condition as stated in (3.3) below. This idea applies to many pattern recognition problems. For example, Bruce and Pramanik [3] adapted this idea to find two-point non-linear patterns for sets in $\mathbb{R}^d$ $(d \geq 2)$ with large Hausdorff dimensions.

We note that the proof strategy and many estimates are analogous to those in [11]. A key difference is that one remainder term in the proof of [11] uses the Fourier decay of measures supported on the parabola on $\mathbb{R}^2$, an application of oscillatory integrals. In contrast, in this setting, the Sobolev improving estimate (Propositions 3.1 and H) is the application of oscillatory integrals that utilize the non-vanishing Gaussian curvature of the curve $(t, P_{p,q}(t))$ in $\mathbb{R}^2$.

1.3. Application: Configuration set

An object to study in pattern recognition is the *3-point configuration set* defined as

$$\Delta_q(E) := \left\{ \frac{(z-x) - q(y-x)}{(y-x)^2} \mid x, y, z \in E, x \neq y, x \neq z \right\}.$$

The number $a \in \Delta_q(E)$ if and only if $\{x, x+t, x+at^2+qt\} \subset E$ for some $x, t \in \mathbb{R}$ with $t \neq 0$, by letting $y = x+t$ and $z = x+at^2+qt$. If E contains two points, $\Delta_q(E)$ is not empty. We remark that Conjecture 1.1 is equivalent to $1 \in \Delta_0(E)$ if $\dim_H(E) > \frac{1}{2}$. The conjecture is also equivalent to

$$(0, \infty) \subset \Delta_0(E)$$

whenever $\dim_H(E) > \frac{1}{2}$ since the Hausdorff dimension is invariant under scaling. For example, let $a > 0$ and suppose that there exist $y, s \in \mathbb{R}$ such that $\{y, y+s, y+s^2\} \subset aE$. By letting $x = a^{-1}y$ and $t = a^{-1}s$, we have $\{x, x+t, x+at^2\} \subset E$.

Greenleaf, Iosevich, and Taylor [8] studied 3-point configuration sets of general patterns for sets $E \subset \mathbb{R}^d$, where $d \geq 2$. Our Theorem 1.2 extends their result to sets in $\mathbb{R}$. One corollary of Theorem 1.2 is as follows.

Corollary 1.4. *Given* $q_0 \in [0,1)$*, there exists* $\beta = \beta(q_0) \in (0,1)$ *with the following property. If* $\dim_H(E) > \beta$*, there exist* $0 < a_0 < a_1$*, such that for all* $q \in [-q_0, q_0]$*,*

$$[-a_1, -a_0] \cup [a_0, a_1] \subset \Delta_q(E).$$

This corollary is a different generalization of [8], as it establishes the existence of the same interval in a family of configuration sets.

1.4. Notations

The length of an interval Q is denoted as $l(Q)$. The set of Schwartz functions from $\mathbb{R}$ to $\mathbb{C}$ is denoted as $\mathcal{S}(\mathbb{R})$. We denote $\mathcal{M}(S)$ as the set of Borel probability measures supported on S, and $\mathcal{L}$ as the Lebesgue measure. The support of a measure μ is

$$\operatorname{spt}\mu := \bigcap \big\{C \text{ closed} \mid \mu(C^c) = 0\big\}.$$

The s-energy integral of a measure is given by

$$I_s(\mu) = \int\int |x-y|^{-s} d\mu(x) d\mu(y) = \rho_s \int |\widehat{\mu}(\xi)|^2 |\xi|^{s-1} d\xi, \tag{1.2}$$

where

$$\rho_s = \pi^{s-1/2} \frac{\Gamma\big(\frac{1-s}{2}\big)}{\Gamma\big(\frac{s}{2}\big)},$$

and Γ is the Gamma function. We refer readers to [15] for details on the s-energy integral and its relation to the Hausdorff dimension.

The Sobolev norm is given by

$$\|\mu\|_{H^s}^2 := \int |\widehat{\mu}(\xi)|^2 \big(1 + |\xi|^2\big)^{\frac{s}{2}} d\xi \tag{1.3}$$

for a measure μ supported on $\mathbb{R}$ and $s \in \mathbb{R}$.

For two functions $f, g : D \to \mathbb{R}_{\geq 0}$ with a domain D, we write $f \lesssim g$ to denote that there exists a $c > 0$, such that for all $x \in D$, $f(x) \leq c g(x)$.

2. Proof of Theorem 1.2 given Proposition 1.3

We denote the dyadic intervals as follows. For $j \geq 0$, let $\mathcal{D}_j$ be the set of dyadic intervals $[k2^{-j}, (k+1)2^{-j}]$ where $k \in \{0, 1, \dots, 2^j - 1\}$. Let $\mathcal{D} = \cup_{j \geq 0} \mathcal{D}_j$. For an interval $Q \in \mathcal{D}_j \subset \mathcal{D}$, its diameter is $l(Q) = 2^{-j}$. The associated re-scaling map $T_Q : Q \to [0,1]$ is $T_Q(y) = 2^j(y - x_Q)$, where x_Q is the left endpoint of Q. Then $T_Q(Q) = [0,1]$.

For a measure $\mu \in \mathcal{M}([0,1])$, if $\mu(Q) > 0$, we can define $\mu_Q \in \mathcal{M}([0,1])$ to be

$$\mu_Q = \mu(Q)^{-1} T_Q\big(\mu|Q\big),$$

where $\mu|Q(S) = \mu(S \cap Q)$.

Given p_0, p_1, q_0 with $0 < p_0 < p_1$ and $q_0 \in [0, \infty)$, we obtain β and δ via Proposition 1.3. Now, suppose that $\dim_H(E) > \beta$. Let $H := \mathcal{H}^\beta_\infty(E) \in (0, 1]$. We claim that there exists $\mathbf{Q} \in \mathcal{D}$ depending on E, such that

$$\mathcal{H}^\beta_\infty(E \cap \mathbf{Q}) \geq (1-\delta) l(\mathbf{Q})^\beta. \tag{2.1}$$

Suppose, for a contradiction, that the claim is false. For a $\tau > 0$ sufficiently small such that $(1-\delta)(H+\tau) < H$, there exists $\{Q_j\}_{j\in\mathbb{N}} \subset \mathcal{D}$, such that $\sum_j l(Q_j)^\beta \leq H + \tau$. Since (2.1) is false for all $Q \in \mathcal{D}$,

$$H = \mathcal{H}^\beta_\infty(E) \leq \sum_j \mathcal{H}^\beta_\infty(E \cap Q_j) \leq (1-\delta) \sum_j l(Q_j)^\beta \leq (1-\delta)(H+\tau) < H.$$

Using $\mathbf{Q}$ from the claim above, we deduce that

$$\mathcal{H}^\beta_\infty\big(T_{\mathbf{Q}}(E \cap \mathbf{Q})\big) = l(\mathbf{Q})^{-\beta} \mathcal{H}^\beta_\infty(E \cap \mathbf{Q}) \geq 1 - \delta.$$

By Proposition 1.3, when $p_0 \leq |p| \leq p_1$, and $|q| \leq q_0$, there exists y, s with $s \neq 0$, such that

$$\{y, y+s, y+ps^2+qs\} \subset T_{\mathbf{Q}}(E \cap \mathbf{Q}).$$

Suppose that $l(\mathbf{Q}) = 2^{-J}$ for a $J \geq 0$. Then,

$$\big\{2^{-J}y + x_{\mathbf{Q}}, 2^{-J}(y+s) + x_{\mathbf{Q}}, 2^{-J}(y+ps^2+qs) + x_{\mathbf{Q}}\big\} \subset E \cap \mathbf{Q} \subset E.$$

Finally, we perform a substitution $x = 2^{-J}y + x_{\mathbf{Q}}, t = 2^{-J}s$ such that there exist x, t with $t \neq 0$, such that

$$\{x, x+t, x+2^J pt^2 + qt\} \subset E,$$

and the conclusion of Theorem 1.2 holds if we let $a_0 = 2^J p_0$, $a_1 = 2^J p_1$.

Remark. From the proof above, Theorem 1.2 can be stated as follows: for positive real numbers p_0, p_1 and $q_0 \in [0, 1)$, there exist $\beta \in (0, 1)$ and $\delta > 0$ such that the following holds: for $E \subset [0,1]$ and a dyadic cube $\mathbf{Q} \in \mathcal{D}$ such that (2.1) holds, let $a_0 = l(\mathbf{Q})^{-1} p_0$ and $a_1 = l(\mathbf{Q})^{-1} p_1$. Then, for every (a, q) with $a_0 \leq |a| \leq a_1$ and $|q| \leq q_0$, we have

$$\big\{x, x+t, x+P_{a,q}(t)\big\} \subset E$$

for some $x, t \in \mathbb{R}$, $t \neq 0$. Here, $P_{a,q}(t) = at^2 + qt$.

3. Proof of Proposition 1.3

The existence of a pattern is ensured by the finiteness and positivity of the configuration integral (3.2). Thus, to prove Proposition 1.3, we show that the integral in (3.2) is bounded below by a positive constant if ε is sufficiently small. To prove positivity, we decompose the configuration integral into one main term and 8 remainder terms. The main term, which consists of low-frequency parts, is bounded below as shown in Lemma 3.3. The technical bulk of the proof consists of showing that the contributions from the remainder terms, which involve medium and high-frequency parts, are small. This is achieved with the Sobolev improving estimates exploiting the non-linearity of the pattern in Proposition 3.1 and the existence of measures with a spectral gap in Lemma F.

Let $l \in \mathbb{N}$ and $\tau \in \mathcal{S}(\mathbb{R})$, where $\tau(x) = 1$ if $x \in [1, 2]$, spt $\tau \subset [2^{-1}, 2^2]$, and $\tau_l(x) = \tau(2^l x)$. We apply the following proposition, which allows us to examine pattern recognition problems with analytical tools. Let $\phi \in \mathcal{S}(\mathbb{R})$ be even, real, non-negative,

$$\widehat{\phi}(0) = \int \phi(x)dx = 1, \phi(x) \geq 2^{-1} \text{ if } |x| \leq 2^{-1}, \tag{3.1}$$

spt $\phi \subset [-1, 1]$, and $\phi_\epsilon(x) = \epsilon^{-1}\phi(\epsilon^{-1}x)$.

Proposition E ([6, Section 6]). *Suppose that for $\gamma > 0$ and $l \in \mathbb{N}$,*

$$\liminf_{\varepsilon \to 0} \int \int \mu_\varepsilon(x + t)\mu_\varepsilon\big(x + P_{p,q}(t)\big)\tau_l(t)dtd\mu(x) > 0, \tag{3.2}$$

where $\mu \in \mathcal{M}(E)$, $\mu_\epsilon = \mu * \phi_\epsilon$, and $I_{1-\gamma}(\mu)$ is finite. Then, there exist $x \in E$ and $t \in [2^{-l-1}, 2^{-l+2}]$, such that $\{x, x + t, x + P_{p,q}(t)\} \subset E$.

The integral in (3.2) is often called the configuration integral. A technical tool used to bound the configuration integral is the Sobolev improving estimate stated below. We use the Sobolev norm defined in (1.3).

Proposition 3.1 ([6]). *There exist $\gamma_0 \in (0, 1)$ and $\kappa > 0$ such that the following statement holds. Let $\gamma \in (0, \gamma_0)$, $p_0 > 0$, and $q_0 \geq 0$. Then, there exists $C_{\gamma,p_0,q_0} > 0$ such that*

$$\left|\iint f(x + t)g\big(x + P_{p,q}(t)\big)\tau_l(t)dtd\mu(x)\right| \leq C_{\gamma,p_0,q_0} 2^{\kappa l} \|f\|_{H^{-\gamma}} \|g\|_{H^{-\gamma}} \|\mu\|_{H^{-\gamma}}$$

for all $l \in \mathbb{N}$ and all $f, g \in \mathcal{S}(\mathbb{R})$ if $P_{p,q}(t) = pt^2 + qt$ for $|p| \geq p_0$ and $|q| \leq q_0$.

The constant C_{γ,p_0,q_0} increases if γ increases, p_0 decreases, or q_0 increases. We note that τ_l in the integrand of (3.2) is used to enable the application of the Sobolev improving estimate (Proposition 3.1), and the integral in (3.2) is finite for each $\varepsilon > 0$ by applying Proposition 3.1. We refer readers to Section 4.3 for more discussion.

The measure we use satisfies special properties that hold when the Hausdorff content is large. The *s-energy* I_s is provided in (1.2).

Lemma F ([11]). *Given* $0 < A < B$ *with* A *sufficiently large, there exist* $\delta = \delta(A, B)$ *and* $\beta = \beta(A, B)$*, such that the following statements hold. For* $E \subset [0, 1]$*, suppose*

$$\mathcal{H}^{\beta}_{\infty}(E) \geq 1 - \delta.$$

Then, there exists a measure $\mu \in \mathcal{M}(E)$*, such that*

(1) *If* $s < \beta$*, then*

$$I_s(\mu) \leq 1 + \frac{12Cs}{\beta - s},$$

where C *is an absolute constant from Frostman's lemma (Theorem G).*

(2) *The measure* μ *satisfies the spectral gap condition that*

$$\int_{A^{\frac{1}{5}} \leq |\xi| \leq B^2} |\widehat{\mu}(\xi)|^2 d\xi \leq \int_{A^{\frac{1}{5}} \leq |\xi| \leq B^2} |\widehat{\mu}(\xi)| d\xi \leq A^{-3}. \tag{3.3}$$

Lemma F is a one-dimensional variant of Section 3 in [11]. A proof of Lemma F is also given in Section 4.2. Hence, Proposition 1.3 follows from the lemma below.

Lemma 3.2. *Given numbers* $p_0, p_1, q_0 \in \mathbb{R}_+$ *with* $q_0 < 1$*, there exist* $B > A > 0$*,* $l = \log_2 A + 3$*,* $\delta > 0$ *sufficiently small,* $\beta \in (0, 1)$*, such that when*

$$\mathcal{H}^{\beta}_{\infty}(E) \geq 1 - \delta,$$

for $E \subset [0, 1]$*, there exists a measure* $\mu \in \mathcal{M}(E)$ *from Lemma F, such that if* $\varepsilon > 0$ *is sufficiently small, then*

$$\int\int \mu_\varepsilon(x + t)\mu_\varepsilon\big(x + P_{p,q}(t)\big)\tau_l(t)\,dt\,d\mu(x) \gtrsim A^{-1} \tag{3.4}$$

for $p_0 \leq |p| \leq p_1$ *and* $|q| \leq q_0$.

3.1. Positivity of the configuration integral

The strategy to prove Lemma 3.2 is to apply

$$\mu_\epsilon = \mu_{A^{-1}} + (\mu_{B^{-1}} - \mu_{A^{-1}}) + (\mu_\epsilon - \mu_{B^{-1}}) := \mu_{A^{-1}} + \mu_{\text{mid}} + \mu_{\text{high}}$$

to rewrite the first two factors of the integrand in (3.4). Then, (3.4) is decomposed into nine terms. The main term of (3.4) is

$$\int\int \mu_{A^{-1}}(x + t)\mu_{A^{-1}}\big(x + P_{p,q}(t)\big)\tau_l(t)\,dt\,d\mu(x).$$

The rest of the terms are remainder terms. We call a remainder term Type I if the function $(\mu_\varepsilon - \mu_{B^{-1}}) = \mu_{\text{high}}$ is not part of the integrand. Otherwise, the remainder term is categorized as a Type II remainder term. There are 3 Type I remainder terms and 5 Type II remainder terms.

Table 1 records the estimates of all nine terms. Except for the main term, all bounds are upper bounds. Additionally, unless stated otherwise all remainder terms are Type II.

	$\mu_{A^{-1}}(x + P_{p,q}(t))$	$\mu_{\text{mid}}(x + P_{p,q}(t))$	$\mu_{\text{high}}(x + P_{p,q}(t))$
$\mu_{A^{-1}}(x+t)$	Main: $\gtrsim A^{-1}$ (3.5)	Type I: $2^{-l} A^{-\frac{2}{5}}$ (3.12)	$\mathbf{C}^{\frac{3}{2}} 2^{\kappa l} B^{\frac{1-s-\gamma}{10}}$ (3.13)
$\mu_{\text{mid}}(x+t)$	Type I: $2^{-l} A^{-\frac{2}{5}}$	Type I: $2^{-l} A^{-\frac{14}{5}}$	$\mathbf{C} 2^{\kappa l} A^{-\frac{3}{2}} B^{\frac{1-s-\gamma}{10}}$
$\mu_{\text{high}}(x+t)$	$\mathbf{C}^{\frac{3}{2}} 2^{\kappa l} B^{\frac{1-s-\gamma}{10}}$	$\mathbf{C} 2^{\kappa l} A^{-\frac{3}{2}} B^{\frac{1-s-\gamma}{10}}$	$\mathbf{C}^{\frac{3}{2}} 2^{\kappa l} B^{\frac{1-s-\gamma}{5}}$

Table 1. Bounds on all terms assuming $I_s(\mu), I_{1-\gamma}(\mu) \le \mathbf{C}$ for a $\mathbf{C} > 0$.

The lower bound on the main term (3.5) is shown in Lemma 3.3, and an overview of the upper bounds of all remainder terms is provided in Section 3.3, where Lemmas 3.6 and 3.7 contain detailed proofs of the bounds (3.12) and (3.13), respectively.

Proof of Lemma 3.2. We use γ and κ from Proposition 3.1. First, we choose two positive real numbers s, β' such that $0 < 1-\gamma < s < \beta' < 1$. Let $\mathbf{C} = 1 + \frac{12Cs}{\beta'-s}$.

We choose l, $A = 2^{l-3}$, such that A is sufficiently large to apply Lemma F, the lower bound of the main term (3.5) is large, and the upper bounds on the Type I remainder terms are small. Finally, we choose B so that the Type II remainder terms are small. Then, we apply Lemma F with the chosen A, B to obtain β and δ. If $E \subset [0,1]$ satisfies

$$\mathcal{H}_\infty^{\max\{\beta,\beta'\}}(E) \ge 1-\delta,$$

there is a measure $\mu \in \mathcal{M}(E)$ with $I_s(\mu), I_{1-\gamma}(\mu) \le \mathbf{C}$ that satisfies (3.3). The bound (3.3) yields Lemma 3.4 and the upper bounds of 5 error terms in Table 1 that involve μ_{mid}. Then, we use the estimates in Table 1 for a lower bound on the main term and upper bounds on the 8 remaining terms. With the appropriate choice of l, A, and B, Lemma 3.3 applies, and the norm of each remainder term is smaller than a multiple of A^{-1}. ■

Remark. For values p_0, p_1, and q_0 that bound the range of p and q, p_0 and q_0 are used in Proposition 3.1, whereas p_1 and the requirement that $q_0 < 1$ are from Lemma 3.3.

3.2. The main term

Lemma 3.3. *If* $(4A)^{-1} = 2^{1-l}$ *for an* $l \in \mathbb{N}$ *and* A *is sufficiently large such that* $|P_{p,q}(t)| \leq |t|$ *when* $|t| \leq (4A)^{-1}$*, we have*

$$\int\int \mu_{A^{-1}}(x+t)\mu_{A^{-1}}\big(x + P_{p,q}(t)\big)\tau_l(t)dtd\mu(x) \geq 2^{-10}c^2A^{-1}. \tag{3.5}$$

for a $c > 0$.

Proof. The key to obtain (3.5) is the following estimate. For $c > 0$, let

$$D_c := \big\{x \mid \exists r_x \in (0,1], \mu\big(B(x,r_x)\big) \leq cr_x\big\}. \tag{3.6}$$

We claim that there exists $c > 0$, with $\mu(D_c) \leq 2^{-1}$. Note that

$$D_c \subset \bigcup_{x \in D_c} B(x, r_x).$$

By applying the Vitali covering lemma [5, Theorem 1.24] to $\cup_{x\in D_c} B(x, r_x/5)$, there exist $\{x_j\}_{j\in\mathbb{N}} \subset D_c$ and $\{r_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_{>0}$, such that

$$D_c \subset \bigcup_{j=1}^{\infty} B(x_j, r_j),$$

and $\mu(B(x_j, r_j)) \leq cr_j$. In addition, $B(x_j, r_j/5) \cap B(x_k, r_k/5) = \emptyset$ if $j \neq k$, but

$$\bigcup_{j=1}^{\infty} B\big(x_j, r_j/5\big) \subset [-2, 2].$$

Then,

$$\begin{aligned}\mu(D_c) &\leq \sum_{j=1}^{\infty} \mu\big(B(x_j, r_j)\big) \\ &\leq \sum_{j=1}^{\infty} cr_j = \frac{5c}{2}\sum_{j=1}^{\infty} \mathcal{L}\big(B(x_j, r_j/5)\big) \\ &\leq \frac{5c}{2}\mathcal{L}\big([-2,2]\big) = 10c.\end{aligned}$$

The claim holds when $c = 20^{-1}$.

Suppose that x satisfies the condition that for all $r \in (0,1)$, $\mu(B(x,r)) \geq cr$. Then, for $|t| \leq (4A)^{-1}$,

$$\begin{aligned}
\mu_{A^{-1}}(x+t) &= A\int \phi\big(A(x+t-y)\big)d\mu(y) \\
&\geq \frac{A}{2}\mu\big(B(x+t,(2A)^{-1})\big), \quad \text{from (3.1)}, \phi(x) \geq 2^{-1} \text{ if } |x| \leq 2^{-1} \\
&\geq \frac{A}{2}\mu\big(B(x,(4A)^{-1})\big) \\
&\geq \frac{A}{2}c(4A)^{-1} = \frac{c}{8}
\end{aligned}$$

since $x \in (D_c)^c$.

Similarly, since $|P_{p,q}(t)| \leq |t| \leq (4A)^{-1}$, we also have $\mu_{A^{-1}}(x+P_{p,q}(t)) \geq \frac{c}{8}$. To complete the proof,

$$\begin{aligned}
&\int\int \mu_{A^{-1}}(x+t)\mu_{A^{-1}}\big(x+P_{p,q}(t)\big)\tau_l(t)dtd\mu(x) \\
&\geq \int_{(D_c)^c}\int_{t\in(0,(4A)^{-1})} \mu_{A^{-1}}(x+t)\mu_{A^{-1}}\big(x+P_{p,q}(t)\big)\tau_l(t)dtd\mu(x) \\
&\geq 2^{-1}(2^{-3}c)^2\mathcal{L}\big((0,(4A)^{-1})\cap(2^{-l},2^{-l+1})\big) \\
&= 2^{-10}c^2A.
\end{aligned}$$

■

3.3. Estimate of remainder terms

In this section, we present the estimates of the Type I and Type II remainder terms. We first record some basic estimates that will prove useful later. From our choice of ϕ in (3.1), since

$$\frac{\partial\widehat{\phi}}{\partial\xi}(0) = -\int 2\pi i x\phi(x)dx = 0$$

and $\phi \in \mathcal{S}(\mathbb{R})$, we have

$$|\widehat{\phi}(\xi) - \widehat{\phi}(0)| \lesssim |\xi|^2, \qquad \sup_{\xi\in\mathbb{R}} |\xi|^5|\widehat{\phi}(\xi)| < \infty. \tag{3.7}$$

First, since μ is a probability measure, $|\widehat{\mu}(\xi)| \leq 1$. By a change of variable, as $\widehat{\phi} \in \mathcal{S}(\mathbb{R})$,

$$\int |\widehat{\mu}(\xi)||\widehat{\phi}(A^{-1}\xi)|d\xi \lesssim A. \tag{3.8}$$

Next, from the definition of the Sobolev norm in (1.3) and the s-energy integral (1.2), we have

$$\|\mu_{A^{-1}}\|_{H^{-\gamma}}^2, \|\mu\|_{H^{-\gamma}}^2 \lesssim I_{1-\gamma}(\mu). \tag{3.9}$$

Here are estimates obtained via the spectral gap condition.

Lemma 3.4. *Assuming the spectral gap condition* (3.3)*, if A is sufficiently large, then*

$$\int |\widehat{\mu}(\xi)||\widehat{\phi}(B^{-1}\xi) - \widehat{\phi}(A^{-1}\xi)|d\xi \lesssim A^{-\frac{7}{5}}, \quad \|\mu_{B^{-1}} - \mu_{A^{-1}}\|_{H^{-\gamma}}^2 \lesssim A^{-3}. \tag{3.10}$$

Proof. To show the first equation of (3.10), we decompose the integral into three parts:

$$\int = \int_{|\xi| \le A^{\frac{1}{5}}} + \int_{A^{\frac{1}{5}} \le |\xi| \le B^2} + \int_{|\xi| \ge B^2}.$$

For the first integral, using $|\widehat{\phi}(\xi) - \widehat{\phi}(0)| \lesssim |\xi|^2$ by (3.7) and $\|\widehat{\mu}\|_{L^\infty} = 1$,

$$\begin{aligned} &\int_{|\xi| \le A^{\frac{1}{5}}} |\widehat{\mu}(\xi)||\widehat{\phi}(B^{-1}\xi) - \widehat{\phi}(A^{-1}\xi)|d\xi \\ &\quad \le \int_{|\xi| \le A^{\frac{1}{5}}} |\widehat{\phi}(B^{-1}\xi) - \widehat{\phi}(0)| + |\widehat{\phi}(0) - \widehat{\phi}(A^{-1}\xi)|d\xi \lesssim A^{-\frac{7}{5}}. \end{aligned}$$

For the second integral, by the spectral gap condition (3.3) and

$$|\widehat{\phi}(B^{-1}\xi) - \widehat{\phi}(A^{-1}\xi)| \le 2$$

by the choice of ϕ in (3.1), we have

$$\int_{A^{\frac{1}{5}} \le |\xi| \le B^2} |\widehat{\mu}(\xi)||\widehat{\phi}(B^{-1}\xi) - \widehat{\phi}(A^{-1}\xi)|d\xi \le 2\int_{A^{\frac{1}{5}} \le |\xi| \le B^2} |\widehat{\mu}(\xi)|d\xi \le 2A^{-3}.$$

For the third integral, since $|\widehat{\mu}(\xi)| \le 1$ and $|\widehat{\phi}(\xi)| \lesssim |\xi|^{-5}$ by (3.7), we have

$$\begin{aligned} &\int_{|\xi| \ge B^2} |\widehat{\mu}(\xi)||\widehat{\phi}(B^{-1}\xi) - \widehat{\phi}(A^{-1}\xi)|d\xi \\ &\quad \le \int_{|\xi| \ge B^2} |\widehat{\phi}(B^{-1}\xi)| + |\widehat{\phi}(A^{-1}\xi)|d\xi \\ &\quad \lesssim \int_{|\xi| \ge B^2} \frac{B^5 + A^5}{|\xi|^5} d\xi \lesssim B^{-3}. \end{aligned}$$

As $A < B$, we combine the above to obtain the inequality required for $A \ge 1$. The proof of the second equation of (3.10) is similar. ∎

The final estimate we need is as follows.

Lemma 3.5. *For B sufficiently large, given $1 > s > 1 - \gamma > 0$ from the proof of Lemma* 3.2*,*

$$\|\mu_\varepsilon - \mu_{B^{-1}}\|_{H^{-\gamma}}^2 \lesssim B^{\frac{1-s-\gamma}{5}} I_s(\mu). \tag{3.11}$$

Proof. To show (3.11), we decompose

$$\|\mu_\varepsilon - \mu_{B^{-1}}\|_{H^{-\gamma}}^2 = \int |\widehat{\mu_\varepsilon}(\xi) - \widehat{\mu_{B^{-1}}}(\xi)|^2(1+|\xi|^2)^{-\frac{\gamma}{2}}d\xi$$
$$= \int |\widehat{\mu}(\xi)|^2|\widehat{\phi}(\varepsilon\xi) - \widehat{\phi}(B^{-1}\xi)|^2\big(1+|\xi|^2\big)^{-\frac{\gamma}{2}}d\xi$$

into two parts

$$\int = \int_{|\xi|\le B^{\frac{1}{5}}} + \int_{|\xi|\ge B^{\frac{1}{5}}}.$$

For the first integral, using $|\widehat{\phi}(\xi) - \widehat{\phi}(0)| \lesssim |\xi|^2$ by (3.7) and $\|\widehat{\mu}\|_{L^\infty} = 1$,

$$\int_{|\xi|\le B^{\frac{1}{5}}} |\widehat{\mu}(\xi)|^2|\widehat{\phi}(\varepsilon\xi) - \widehat{\phi}(B^{-1}\xi)|^2\big(1+|\xi|^2\big)^{-\frac{\gamma}{2}}d\xi$$
$$\le \int_{|\xi|\le B^{\frac{1}{5}}} |\widehat{\phi}(\varepsilon\xi) - \widehat{\phi}(B^{-1}\xi)|^2 d\xi \lesssim B^{-3}.$$

For the second integral, since $1-s-\gamma<0$,

$$\int_{|\xi|\ge B^{\frac{1}{5}}} |\widehat{\mu}(\xi)|^2|\widehat{\phi}(\varepsilon\xi) - \widehat{\phi}(B^{-1}\xi)|^2\big(1+|\xi|^2\big)^{-\frac{\gamma}{2}}d\xi$$
$$\le 4\int_{|\xi|\ge B^{\frac{1}{5}}} |\widehat{\mu}(\xi)|^2\big(1+|\xi|^2\big)^{-\frac{\gamma}{2}}d\xi$$
$$\le 4B^{\frac{1-s-\gamma}{5}}\int_{|\xi|\ge B^{\frac{1}{5}}} |\widehat{\mu}(\xi)|^2|\xi|^{s-1}d\xi = 4B^{\frac{1-s-\gamma}{5}}I_s(\mu).$$

We note that $1-s-\gamma > -\gamma > -1$ from Proposition 3.1. Thus, for B sufficiently large, $\|\mu_\varepsilon - \mu_{B^{-1}}\|_{H^{-\gamma}}^2 \lesssim B^{\frac{1-s-\gamma}{5}}I_s(\mu)$. ∎

Therefore, we can deduce the following Type I remainder term estimate.

Lemma 3.6. *Under the spectral gap condition* (3.3),

$$\left|\int\int \mu_{A^{-1}}(x+t)\mu_{\mathrm{mid}}\big(x+P_{p,q}(t)\big)\tau_l(t)dt\,d\mu(x)\right| \lesssim 2^{-l}A^{-\frac{2}{5}}. \tag{3.12}$$

Proof of Lemma 3.6. By applying the Fourier transform (or (4.1)), we have

$$\left|\int \mu_{A^{-1}}(x+t)\mu_{\mathrm{mid}}\big(x+P_{p,q}(t)\big)\tau_l(t)dt\,d\mu(x)\right|$$
$$= 2^{-l}\left|\int\int \widehat{\mu}(\xi+\eta)\overline{\widehat{\mu}(\xi)\widehat{\phi}(A^{-1}\xi)\widehat{\mu}(\eta)(\widehat{\phi}(B^{-1}\eta) - \widehat{\phi}(A^{-1}\eta))}\right.$$
$$\left.\int e^{-2\pi i(2^{-l}t\xi + P_{p,q}(2^{-l}t)\eta)}\tau(t)dt\,d\xi\,d\eta\right|$$
$$\le 2^{-l}\|\tau\|_{L^1}\int |\widehat{\mu}(\xi)||\widehat{\phi}(A^{-1}\xi)|d\xi\int |\widehat{\mu}(\eta)||\widehat{\phi}(B^{-1}\eta) - \widehat{\phi}(A^{-1}\eta)|d\eta$$
$$\lesssim 2^{-l}A^{-\frac{2}{5}}$$

by (3.8) and Lemma 3.4. ■

The estimates of the other error Type I remainder terms are similar. All Type II remainder terms' estimates are obtained by applying Proposition 3.1. Here is one example.

Lemma 3.7. *Assuming the spectral gap condition (3.3) and $I_s(\mu), I_{1-\gamma}(\mu) \le C$,*

$$\left|\int\int \mu_{A^{-1}}(x+t)(\mu_\varepsilon - \mu_{B^{-1}})(x+t^2)\tau_l(t)dt\,d\mu(x)\right| \lesssim \mathbf{C}^{\frac{3}{2}} 2^{\kappa l} B^{\frac{1-s-\gamma}{10}}. \quad (3.13)$$

Proof. Using Proposition 3.1,

$$\begin{aligned}
&\left|\int\int \mu_{A^{-1}}(x+t)(\mu_\varepsilon - \mu_{B^{-1}})(x+t^2)\tau_l(t)dt\,d\mu(x)\right| \\
&\qquad \le C_{\gamma,p_0,q_0} 2^{\kappa l}\|\mu_{A^{-1}}\|_{H^{-\gamma}}\|\mu_\varepsilon - \mu_{B^{-1}}\|_{H^{-\gamma}}\|\mu\|_{H^{-\gamma}} \\
&\qquad \lesssim 2^{\kappa l}\big[I_{1-\gamma}(\mu)\big]^{\frac{1}{2}}\big[B^{\frac{1-s-\gamma}{5}} I_s(\mu)\big]^{\frac{1}{2}}\big[I_{1-\gamma}(\mu)\big]^{\frac{1}{2}}
\end{aligned}$$

by (3.9) and Lemma 3.5. ■

4. Appendix

The following identity is used to study the configuration integral in the frequency space.

$$\begin{aligned}
&\int\int f(x+t)g\big(x+P_{p,q}(t)\big)(x)\tau_l(t)dt\,d\mu(x) \\
&\quad = \int\int \widehat{\mu}(\xi+\eta)\overline{\widehat{f}(\xi)\widehat{g}(\eta)}\int e^{-2\pi i(t\xi + P_{p,q}(t)\eta)}\tau_l(t)dt\,d\xi\,d\eta \qquad (4.1) \\
&\quad = 2^{-l}\int\int \widehat{\mu}(\xi+\eta)\overline{\widehat{f}(\xi)\widehat{g}(\eta)}\int e^{-2\pi i(2^{-l}t\xi + P_{p,q}(2^{-l}t)\eta)}\tau(t)dt\,d\xi\,d\eta.
\end{aligned}$$

4.1. Positive Lebesgue measure sets

In the introduction, we state that every set with positive Lebesgue measure contains the pattern $\{x, x+t, x+t^2\}$. We prove a more general statement below.

Lemma 4.1. *Let $f_j : \mathbb{R}^d \to \mathbb{R}^d$ be continuous functions with $f_j(0) = 0$ for $j \in \{1,\dots,n\}$. If $E \subset \mathbb{R}^d$ and the Lebesgue measure $\mathcal{L}(E) > 0$, then there exist $x, t \in \mathbb{R}^d$, $t \ne 0$, such that $\{x + f_j(t)\}_{j=1}^n \subset E$.*

Proof. Let

$$I(t) = \mathcal{L}\Big(\bigcap_{j=1}^n (E - f_j(t))\Big).$$

Then, for $t \in \mathbb{R}^d$, $I(t) \le I(0) = \mathcal{L}(E) > 0$. In addition,

$$I(0) - I(t) \le \mathcal{L}\Big(E \backslash \bigcap_{j=1}^{n} (E - f_j(t))\Big) \le \sum_{j=1}^{n} \mathcal{L}\big[E \backslash (E - f_j(t))\big].$$

We note that by the L^1-modulus of continuity,

$$\int_{\mathbb{R}^d} |f(x+h) - f(x)| dx \to 0 \text{ as } h \to 0$$

for $f \in L^1(\mathbb{R}^d)$. Therefore, by applying the above with $f = \mathbf{1}_E$, we have

$$\mathcal{L}\big[E \backslash (E - f_j(t))\big] \to 0$$

as $t \to 0$. ∎

4.2. Existence of a measure with a spectral gap

We present the one-dimensional version of Section 3 in [11], where dyadic rectangles of size $2^{-T} \times 4^{-T}$ are examined for a $T \ge 1$. For $\mathbb{R}$, we use dyadic intervals of length 2^{-T}.

Proof of Lemma F. Here, we use the dyadic intervals as stated in the proof of Theorem 1.2 in Section 2. Let

$$\varphi \in C^\infty(\mathbb{R}), \text{spt}\,\varphi \subset (0,1), \int \varphi(x)dx = 1, \text{ and } \|\varphi\|_{L^\infty} \le 2. \tag{4.2}$$

For $N \in \mathbb{N}$ large, there exists C_N, such that $|\xi|^N |\widehat{\varphi}(\xi)| \le C_N$. Hence,

$$\int_{|\xi| \ge A^{\frac{1}{5}}} |\widehat{\varphi}(\xi)| d\xi \le C_N \int_{|\xi| \ge A^{\frac{1}{5}}} |\xi|^{-N} d\xi = \frac{2C_N}{N-1} A^{\frac{1-N}{5}}.$$

If $N = 21$, $\frac{C_{21}}{5} \le A$, then $\int_{|\xi| \ge A^{\frac{1}{5}}} |\widehat{\varphi}(\xi)| d\xi \le 2^{-1} A^{-3}$.

Let $T \in \mathbb{N}$ such that

$$2^{-T+2} \pi B^4 \le 2^{-1} A^{-3}.$$

Suppose that $\mathcal{H}^\beta_\infty(E) \ge 1 - \delta$, where $\delta = 2^{-3T-3}$. We denote $\text{ch}(\mathbf{Q}) \subset \mathcal{D}$ to be the generation-T children of $\mathbf{Q} = [0,1]$. Equivalently, $\text{ch}(\mathbf{Q}) = \mathcal{D}_T$. We claim that if $1 - \beta$ is small, $\mathcal{H}^\beta_\infty(E \cap Q) \ge 2^{-1} l(Q)^\beta$ for all $Q \in \text{ch}(\mathbf{Q})$.

To see that the claim holds, let

$$\mathcal{G} = \big\{Q \in \text{ch}(\mathbf{Q}) \mid \mathcal{H}^\beta_\infty(E \cap Q) \ge 2^{-1} l(Q)^\beta\big\}.$$

If $\mathcal{G} \subsetneq \mathrm{ch}(\mathbf{Q})$,

$$\begin{aligned} 1 - 2^{-3T-3} =& 1 - \delta \le \mathcal{H}^{\beta}_{\infty}(E \cap \mathbf{Q}) \\ &\le \sum_{Q \in \mathcal{G}} \mathcal{H}^{\beta}_{\infty}(E \cap Q) + \sum_{Q \in \mathrm{ch}(\mathbf{Q}) \setminus \mathcal{G}} \mathcal{H}^{\beta}_{\infty}(E \cap Q) \\ &\le \sum_{Q \in \mathcal{G}} l(Q)^{\beta} + (1 - 2^{-1}) \sum_{Q \in \mathrm{ch}(\mathbf{Q}) \setminus \mathcal{G}} l(Q)^{\beta} \\ &\le \sum_{Q \in \mathrm{ch}(\mathbf{Q})} l(Q)^{\beta} - 2^{-1} 2^{-T\beta}, \qquad \text{since } |\mathrm{ch}(\mathbf{Q}) \setminus \mathcal{G}| \ge 1 \\ =& 2^{-T\beta}(2^T - 2^{-1}). \end{aligned}$$

However, for β sufficiently close to 1, $1 - 2^{-3T-3} > 2^{-T\beta}(2^T - 2^{-1})$, which contradicts the inequality above.

We apply Frostman's lemma (Theorem G) to $E \cap Q$ for each $Q \in \mathrm{ch}(\mathbf{Q})$ to obtain a Borel measure μ^0_Q, where $\mathrm{spt}\, \mu^0_Q \subset E \cap Q$. The measure μ^0_Q satisfies the following:

(1) The ball condition holds, where $\mu^0_Q(B(x,r)) \le C r^{\beta}$ for all $x \in \mathbb{R}$, $r > 0$, and C does not depend on Q.

(2) The measure μ^0_Q is not the zero measure, where

$$\mu^0_Q(Q) \ge \mathcal{H}^{\beta}_{\infty}(E \cap Q) \ge 2^{-1} l(Q)^{\beta}. \tag{4.3}$$

To construct the measure μ supported on E, we first normalize each μ^0_Q according to φ. Let

$$\mu_Q = \frac{w(Q)}{\mu^0_Q(Q)} \mu^0_Q, \text{ where } w(Q) = \int_Q \varphi(x) dx.$$

Then by (4.2),

$$\mu_Q(Q) = w(Q) \le \|\varphi\|_{L^\infty} l(Q) \le 2 l(Q) = 2^{1-T}. \tag{4.4}$$

In addition, by (4.3),

$$\mu_Q\big(B(x,r)\big) \le \frac{2 l(Q)}{2^{-1} l(Q)^{\beta}} C r^{\beta} \le 2^{2+T(\beta-1)} C r^{\beta} \le 4 C r^{\beta}. \tag{4.5}$$

Let $\mu = \sum_{Q \in \mathrm{ch}(\mathbf{Q})} \mu_Q$. Then, $\mathrm{spt}\, \mu \subset E$. We claim that μ is the desired measure. First, the total mass is

$$\mu(\mathbf{Q}) = \sum_{Q \in \mathrm{ch}(\mathbf{Q})} w(Q) = \sum_{Q \in \mathrm{ch}(\mathbf{Q})} \int_Q \varphi(x) dx = 1. \tag{4.6}$$

Second, μ is a β-Frostman's measure.

- If $r \le 2^{-T}$, then $B(x,r)$ intersects at most three $Q \in \mathrm{ch}(\mathbf{Q})$, so

$$\mu\big(B(x,r)\big) \le 12Cr^{\beta}$$

from (4.5).
- If $r \ge 1$, $\mu(B(x,r)) \le 1 \le r^{\beta}$ from (4.6).
- If $2^{-T} \le r \le 1$, then $B(x,r)$ intersects at most $(\frac{2r}{2^{-T}}+1)$ $Q \in \mathrm{ch}(\mathbf{Q})$. From (4.4),

$$\begin{aligned}\mu\big(B(x,r)\big) \le & \sum_{Q\in \mathrm{ch}(\mathbf{Q}),\, Q \bigcap B(x,r)\neq\emptyset} \mu_Q(Q)\\ \le & \Big(\frac{2r}{2^{-T}}+1\Big)2^{1-T} \le 6r \le 6r^{\beta}.\end{aligned}$$

Third, μ satisfies the spectral gap condition. For each $Q \in \mathrm{ch}(\mathbf{Q})$,

$$\mu(Q) = w(Q) = \int_Q \varphi(x)dx.$$

Therefore, for c_Q, the center of the cube Q,

$$\int_Q e^{-2\pi i c_Q \xi} d\mu(x) = \int_Q e^{-2\pi i c_Q \xi}\varphi(x)dx.$$

For a $\xi \in \mathbb{R}$, the map $x \to e^{-2\pi i x\xi}$ is $2\pi|\xi|$-Lipschitz. Since $|x-c_Q| \le 2^{-T}$ if $x \in Q$,

$$\begin{aligned}|\widehat{\mu}(\xi)-\widehat{\varphi}(\xi)| = & \left|\int e^{-2\pi i x\xi}d\mu(x) - \int e^{-2\pi i x\xi}\varphi(x)dx\right|\\ \le & \sum_{Q\in \mathrm{ch}(\mathbf{Q})} \int_Q |e^{-2\pi i x\xi} - e^{-2\pi i c_Q\xi}|d\mu(x)\\ & + \int_Q |e^{-2\pi i x\xi} - e^{-2\pi i c_Q\xi}|\varphi(x)dx\\ \le & 2^{-T+2}\pi|\xi|.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{A^{\frac15}\le|\xi|\le B^2} |\widehat{\mu}(\xi)|d\xi \le & \int_{A^{\frac15}\le|\xi|\le B^2} |\widehat{\mu}(\xi)-\widehat{\varphi}(\xi)|d\xi + \int_{A^{\frac15}\le|\xi|\le B^2} |\widehat{\varphi}(\xi)|d\xi\\ \le & 2^{-T+2}\pi \int_{|\xi|\le B^2} |\xi|d\xi + \int_{|\xi|\ge A^{\frac15}} |\widehat{\varphi}(\xi)|d\xi\\ \le & 2^{-T+2}\pi B^4 + 2^{-1}A^{-3} \le A^{-3}\end{aligned}$$

by our choice of T.

Finally, we apply Lemma 4.2 to obtain a bound on the s-energy integral of μ. ■

Lemma 4.2 ([15, Section 2.5]). *Suppose that $\nu \in \mathcal{M}([0,1])$, and there exists $C > 0$ such that $\nu(B(x,r)) \le Cr^{\beta}$ for any $x \in [0,1], r > 0$. Then, for $s < \beta$, $I_s(\nu) \le 1 + \frac{Cs}{\beta - s}$.*

The version of Frostman's Lemma we use in the proof of Lemma F is as follows.

Theorem G ([15, Section 2.5]). *Let $0 \le s \le 1$. For a compact Borel set $A \subset [0,1]$, if $\mathcal{H}^s_\infty(A) > 0$, there exist an absolute constant $C > 0$ and a measure μ supported on A, such that*

(1) $\mu(B(x,r)) \le Cr^s$ *for all $x \in \mathbb{R}$ and $r > 0$,*

(2) $\mu(\mathbb{R}) \ge \mathcal{H}^s_\infty(A)$.

Proof. The proof strategy is the same as that in [15], and we highlight how the constant C and (2) arise, which follow from the original proof but are not stated explicitly in [15].

The measure μ is a weak limit of measures μ_k defined as follows. We use dyadic intervals of side length 2^{-k} in a standard partitioning of $\mathbb{R}$, which are of the form $[2^{-k}n, 2^{-k}(n+1)]$ for $n \in \mathbb{Z}$. We let $\mu_{k,1}$ be a constant multiple of the Lebesgue measure on each such interval Q. For an interval Q, if $A \cap Q \neq \emptyset$, we normalize the Lebesgue measure on Q so that $\mu_{k,1}(Q) = l(Q)^s$. If $A \cap Q = \emptyset$, we let $\mu_{k,1}$ be the zero measure on Q. The measure $\mu_{k,1}$ satisfies $\mu_{k,1}(Q') \le l(Q')^s$ if Q' is a dyadic interval with $l(Q') < 2^{-k}$.

Next, to construct $\mu_{k,2}$, we examine dyadic intervals of side length 2^{1-k}. For each such cube Q, we let $\mu_{k,2}$ be $\mu_{k,1}$ if $\mu_{k,1}(Q) \le l(Q)^s$. Otherwise, we normalize $\mu_{k,1}$ on Q so that $\mu_{k,2}(Q) = l(Q)^s$. We continue this process after k steps to obtain $\mu_{k,k}$, which we denote as μ_k. The measure μ_k satisfies the following two properties.

(1) $\mu_k(Q) \le l(Q)^s$ for all dyadic intervals. Therefore,

$$\mu_k\big(B(x,r)\big) \le Cr^s,$$

for a constant $C > 0$.

(2) For all $x \in A$, there exists a dyadic interval Q, such that $x \in Q$, $l(Q) \ge 2^{-k}$, and

$$\mu_k(Q) = l(Q)^s.$$

There exist finitely many maximal and disjoint such cubes Q_j that cover A. Therefore,

$$\mu_k(\mathbb{R}) = \sum_j \mu_k(Q_j) = \sum_j l(Q_j)^s \ge \mathcal{H}^s_\infty(A).$$

Let μ be a weak limit of a converging subsequence of $\{\mu_k\}$. Then, spt $\mu \subset A$ as A is compact, and both (1) and (2) are satisfied. ■

4.3. Sobolev improving estimate

Proposition 3.1 is deduced from the following proposition.

Proposition H ([6]). *There exist $\gamma_0 \in (0,1)$ and $\kappa > 0$ such that the following statement holds. Let $\gamma \in (0,\gamma_0)$, $p_0 > 0$, and $q_0 \geq 0$. Then, there exists $C_{\gamma,p_0,q_0} > 0$ such that*

$$\left|\iint f(x+t)g\big(x+P_{p,q}(t)\big)h(x)\tau_l(t)\,dt\,dx\right| \leq C_{\gamma,p_0,q_0}2^{\kappa l}\|f\|_{H^{-\gamma}}\|g\|_{H^{-\gamma}}\|h\|_{H^{-\gamma}}$$

for all $l \in \mathbb{N}$ and all $f, g, h \in \mathcal{S}(\mathbb{R})$ if

$$P_{p,q}(t) = pt^2 + qt$$

for $|p| \geq p_0$ and $|q| \leq q_0$.

We refer readers to the proof in Section 2 of [6] that yields Proposition H, which is stronger than the version in that paper for quadratics of the form $P_{p,q}(t) = pt^2 + qt$. The proof starts by applying (4.1) and Littlewood–Paley decomposition. Then, the methods of stationary phase and estimates in [13] and [9] are used.

Proof of Proposition 3.1 given Proposition H. The function $T : \mathbb{R} \to \mathbb{C}$ given by

$$T(x) = \int f(x+t)g\big(x+P_{p,q}(t)\big)\tau_l(t)\,dt$$

is continuous. Since $\mu_\varepsilon \to \mu$ weakly as $\varepsilon \to 0$,

$$\begin{aligned}&\left|\int\int f(x+t)g\big(x+P_{p,q}(t)\big)\tau_l(t)\,dt\,d\mu(x)\right|\\ &\quad= \lim_{\varepsilon\to 0}\left|\int T(x)\mu_\varepsilon(x)\,dx\right|\\ &\quad\leq \lim_{\varepsilon\to 0} C_{\gamma,p_0,q_0}2^{\kappa l}\|f\|_{H^{-\gamma}}\|g\|_{H^{-\gamma}}\|\mu_\varepsilon\|_{H^{-\gamma}}.\end{aligned}$$

■

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