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Besov–Lipschitz norm and p-energy measure on scale-irregular Vicsek sets

Aobo Chen, Jin Gao, Zhenyu Yu, and Junda Zhang

Abstract. In this paper, we establish the existence of p-energy norms and the corresponding p-energy measures for scale-irregular Vicsek sets, which may lack self-similarity. We also investigate the characterization of p-energy norms in terms of Besov–Lipschitz norms, with their weak monotonicity and the corresponding Bourgain–Brezis–Mironescu convergence.

1. Introduction

The study of *nonlinear potential theory* on metric measure spaces has attracted significant attention in recent decades due to its important role in classical analysis and differential equations. Many previous studies have concentrated on the p-energy within specific classes of fractals. For example, p-energy is constructed on self-similar p.c.f. sets by Herman, Peirone, and Strichartz [22], Cao, Gu, and Qiu [8], Gao, Yu, and Zhang [17]; on the standard Sierpiński carpet by Shimizu [37], Murugan and Shimizu [36]; and on more general fractal spaces by Kigami [31]. In these previous works, the self-similarity significantly influences the construction of p-energy.

One aim of this work is trying to construct p-energy norm and p-energy measure without using the self-similar structure of the underlying fractal. The issue was first highlighted by Murugan and Shimizu in [36, Problem 12.5], posing the challenge of constructing p-energy measure on the Sierpiński carpet without using the self-similarity, and also establishing its basic properties. However, as far as the authors are concerned, many vital tools such as *Fekete's lemma* used in [8, Lemma 4.4], *combinatorial ball Loewner condition* in [36, Definition 3.1] and *Knight Move argument* in [31, 37] are no longer applicable without self-similarity. Another aim is generalizing the celebrated Bourgain–Brezis–Mironescu (BBM) convergence of p-energy semi-norms to non-self-similar fractals, where these semi-norms are supposed to have a different form from the self-similar case.

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For this reason, we concern ourselves here with the *scale-irregular Vicsek sets*, which is a class of homogeneous *Moran sets* (see, for example, [14]) and V-variable fractals (with $V = 1$, see, for example, [3]). The advantage of Vicsek sets lies in their distinctive "gradient structure", which allows us to construct p-energy norm even in the absence of self-similarity. It is worth mentioning that, when $p = 2$, the 2-energy (or Dirichlet form) may be constructed and studied by a probabilistic approach as Hambly [21], or as Barlow and Hambly [2] did on scale-irregular Sierpiński gaskets.

Our approach is mainly motivated by the works of Baudoin and Chen [5, 6]. We define p-energy norm as the limit of p-energy norms on discrete approximating graphs, since the underlying geometry structure ensures the monotonicity of discrete energy norms. To this end, for each scale-irregular Vicsek set $K^{\boldsymbol{l}}$ determined by contraction ratio sequence $\boldsymbol{l}$ (see Definition 2.1), we always equip $K^{\boldsymbol{l}}$ with the Euclidean metric d and the canonical Borel probability measure μ given by (2.5). For a Borel set B, write $⨍_B f \, \mathrm{d}\mu := \mu(B)^{-1} \int_B f \, \mathrm{d}\mu$. For a set $A \subset \mathbb{C}$ (the complex plane), denote its diameter by $\operatorname{diam}(A) := \sup_{x,y\in A} d(x, y)$.

Theorem 1.1. *Let $(K^{\boldsymbol{l}}, d, \mu)$ be a scale-irregular Vicsek set. For each $1 < p < \infty$, there exists a normed vector space $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ and a semi-norm $\mathcal{E}_p$ on $\mathcal{F}_p$ with the following properties.*

(1) *$(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ is a uniformly convex separable reflexive Banach space.*

(2) *$\mathcal{F}_p$ forms an algebra under the pointwise product, that is, $uv \in \mathcal{F}_p$ whenever $u, v \in \mathcal{F}_p$. Moreover,*

$$\mathcal{E}_p(uv) \leq 2^{p-1}\left(\|u\|^p_{C(K^{\boldsymbol{l}})}\mathcal{E}_p(v) + \|v\|^p_{C(K^{\boldsymbol{l}})}\mathcal{E}_p(u)\right) \text{ for all } u, v \in \mathcal{F}_p.$$

(3) (Regularity) *$\mathcal{F}_p \subset C(K^{\boldsymbol{l}})$ is a dense subspace of $(C(K^{\boldsymbol{l}}), \|\cdot\|_\infty)$.*

(4) (Lipschitz contractivity) *For every $u \in \mathcal{F}_p$ and every 1-Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$, we have $\varphi \circ u \in \mathcal{F}_p$ and $\mathcal{E}_p(\varphi \circ u) \leq \mathcal{E}_p(u)$.*

(5) (Spectral gap) *There is a constant $C \geq 1$ such that, for every $u \in \mathcal{F}_p$,*

$$\int_{K^{\boldsymbol{l}}} \left|u(x) - ⨍_{K^{\boldsymbol{l}}} u \, \mathrm{d}\mu\right|^p \mathrm{d}\mu(x) \leq C \operatorname{diam}(K^{\boldsymbol{l}})^{p-1}\mathcal{E}_p(u).$$

(6) (Strong locality) *If $u, v \in \mathcal{F}_p$ satisfy $\operatorname{supp}(u) \cap \operatorname{supp}(v - a\mathbb{1}_{K^{\boldsymbol{l}}}) = \emptyset$ for some $a \in \mathbb{R}$, then*

$$\mathcal{E}(u + v) = \mathcal{E}(u) + \mathcal{E}(v).$$

The explicit expressions of $\mathcal{E}_p$ and $\mathcal{F}_p$ are given in Definition 3.1. This helps to understand the dependence of the Sobolev spaces $\mathcal{F}_p$ on the exponent p. As we see in Remark 3.10 that, for $1 < p \neq q < \infty$, the intersection $\mathcal{F}_p \cap \mathcal{F}_q$ contains (many) non-constant functions.

Our next result shows the existence and some properties of the *p-energy measure* corresponding to the p-energy norm in Theorem 1.1.

Theorem 1.2. *Let $(K^{\boldsymbol{l}}, d, \mu)$ be a scale-irregular Vicsek set and $(\mathcal{E}_p, \mathcal{F}_p)$ be the p-energy in Theorem 1.1. Then there exists a family of Borel finite measures $\{\Gamma_p\langle u\rangle\}_{u\in\mathcal{F}_p}$ on $K^{\boldsymbol{l}}$ satisfying the following:*

(1) *For $u \in \mathcal{F}_p$, $\Gamma_p\langle u\rangle(K^{\boldsymbol{l}}) = \mathcal{E}_p(u)$. Moreover, $\Gamma_p\langle u\rangle = 0$ if and only if u is constant.*

(2) *For any two $u_1, u_2 \in \mathcal{F}_p$, and any non-negative Borel measurable function g on $K^{\boldsymbol{l}}$,*

$$\left(\int_{K^{\boldsymbol{l}}} g\, \mathrm{d}\Gamma_p\langle u_1 + u_2\rangle\right)^{\frac{1}{p}} \le \left(\int_{K^{\boldsymbol{l}}} g\, \mathrm{d}\Gamma_p\langle u_1\rangle\right)^{\frac{1}{p}} + \left(\int_{K^{\boldsymbol{l}}} g\, \mathrm{d}\Gamma_p\langle u_2\rangle\right)^{\frac{1}{p}}. \tag{1.1}$$

(3) *For any $f \in C^1(\mathbb{R})$ and any $u \in \mathcal{F}_p$,*

$$\mathrm{d}\Gamma_p\langle f\circ u\rangle(x) = \left|f'\big(u(x)\big)\right|^p \mathrm{d}\Gamma_p\langle u\rangle(x) \text{ for } \Gamma_p\langle u\rangle\text{-a.e. } x \in K^{\boldsymbol{l}}. \tag{1.2}$$

(4) (Energy image density property) *For any $u \in \mathcal{F}_p$, $u_*(\Gamma_p\langle u\rangle) \ll \mathscr{L}^1$. Here $u_*(\Gamma_p\langle u\rangle)$ is the push-forward measure defined by $u_*(\Gamma_p\langle u\rangle)(A) := \Gamma_p\langle u\rangle\,(u^{-1}(A))$ for all Borel subsets $A \subset \mathbb{R}$, and $\mathscr{L}^1$ is the Lebesgue measure on $\mathbb{R}$.*

Some previous works have also constructed p-energy measures and discussed their properties by utilizing the self-similarity, as Hino [23, Lemma 4.1] did for $p = 2$, or as Murugan and Shimizu [36, Section 9] did for the standard Sierpiński carpet. Motivated by [5], we use the gradient structure as an alternative approach for both construction and direct validation on the properties of the p-energy measure.

Within our framework, word spaces are applicable to construct the p-energy measure, see Proposition 3.15; however, the validation of related properties is impeded by the absence of self-similarity. This gap is bridged in Proposition 3.16 by showing the coincidence of energy measures constructed by these two different approaches, and this suggests a compatibility between the gradient structure and the fractal structure.

We define more discrete p-energy norms, denoted by $\mathcal{E}^{\beta}_{p,\infty}$ and $\mathcal{E}^{\beta}_{p,p}$ for $1 < p < \infty$ and $0 < \beta < \infty$ in Definition 4.2. Then the p-energy $\mathcal{E}_p$ constructed in Theorem 1.1 is equivalent to $\mathcal{E}^{\beta^*}_{p,\infty}$ with fixed $\beta^* > 0$. Intuitively, when $p = 2$, the norm $\mathcal{E}_2 = \mathcal{E}^{\beta^*}_{2,\infty}$ gives a strongly local Dirichlet form, while $\mathcal{E}^{\beta}_{2,2}$ gives a non-local Dirichlet form (see (4.5)). Moreover, under an additional condition that *$\boldsymbol{l}$ consists of finitely many contraction ratios*, we can describe the above norms in terms of Besov–Lipschitz norms. Let ϕ be the increasing scale function given in (4.1).

Definition 1.3. Fix $\beta^* \in (0,\infty)$. For every $1 < p < \infty$ and $0 \le \beta < \infty$, define[1]

$$\Phi_u^\beta(r) := \int_{K^{\boldsymbol{l}}} \fint_{B(x,r)} \frac{|u(x)-u(y)|^p}{\phi(r)^{\beta/\beta^*}} \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x), \tag{1.3}$$

for any $0 < r \le \mathrm{diam}(K^{\boldsymbol{l}})$, and any $u \in L^p(K^{\boldsymbol{l}},\mu)$. For $1 < q < \infty$, let

$$[u]_{B^\beta_{p,q}} := \left(\int_0^{\mathrm{diam}(K^{\boldsymbol{l}})} \left(\Phi_u^\beta(r)\right)^{q/p} \frac{\mathrm{d}r}{r}\right)^{1/q}, \; B^\beta_{p,q} := \left\{u \in L^p(K^{\boldsymbol{l}},\mu) : [u]_{B^\beta_{p,q}} < \infty\right\};$$

for $q = \infty$, let

$$[u]_{B^\beta_{p,\infty}} := \sup_{r\in(0,\mathrm{diam}(K^{\boldsymbol{l}})]} \Phi_u^\beta(r)^{1/p} \text{ and } B^\beta_{p,\infty} := \left\{u \in L^p(K^{\boldsymbol{l}},\mu) : [u]_{B^\beta_{p,\infty}} < \infty\right\}.$$

The above Besov–Lipschitz spaces were introduced by Jonsson and Wallin [27] for Euclidean spaces, by Korevaar and Schoen [32] for Riemann domains and by Jonsson [26] for the Sierpiński gasket. The main difference between Besov–Lipschitz spaces on scale-irregular fractals and those on manifolds or self-similar fractals is that ϕ is not a power function, which makes the analysis more complicated.

Our last main result is that, for β close enough to β^*, the discrete semi-norms $\mathcal{E}_p$, $\mathcal{E}^\beta_{p,\infty}$ and $\mathcal{E}^\beta_{p,p}$ in Definition 4.2 are comparable with the Besov–Lipschitz norms $[\cdot]_{B^{\beta^*}_{p,\infty}}$, $[\cdot]_{B^\beta_{p,\infty}}$ and $[\cdot]_{B^\beta_{p,p}}$, respectively. Furthermore, the weak monotonicity and BBM convergence of p-energy semi-norms are established on scale-irregular Vicsek sets.

Theorem 1.4. *If $\boldsymbol{l} = (l_n)_{n=1}^\infty$ satisfies $\sup_{n\ge1} l_n < \infty$, then for each $1 < p < \infty$*

(1) *There exists a constant $\epsilon_p \in (0,1)$ depending only on $\boldsymbol{l}$ and p, such that for every $\beta \in (\epsilon_p\beta^*,\infty)$, we have $B^\beta_{p,p} \subset B^\beta_{p,\infty} \subset C(K^{\boldsymbol{l}})$, and there exists $C \ge 1$ such that for all $u \in C(K^{\boldsymbol{l}})$,*

$$C^{-1}[u]^p_{B^\beta_{p,\infty}} \le \mathcal{E}^\beta_{p,\infty}(u) \le C[u]^p_{B^\beta_{p,\infty}},$$

$$C^{-1}[u]^p_{B^\beta_{p,p}} \le \mathcal{E}^\beta_{p,p}(u) \le C[u]^p_{B^\beta_{p,p}}.$$

In particular, when $\beta = \beta^*$, we have $\mathcal{E}_p(u) \asymp [u]_{B^{\beta^*}_{p,\infty}}$ for $u \in \mathcal{F}_p = B^{\beta^*}_{p,\infty}$ (given in Theorem 1.1).

[1] Our definition of Besov–Lipschitz spaces in Definition 1.3 may initially seem confusing, as the spaces $B^{\beta^*}_{p,q}$ do not actually depend on the choice of β^*. We use this definition to ensure the consistency with existing literature (e.g. [4]) regarding the *critical Besov exponent* in (1.4) in the self-similar case. See Remark 4.1.

(2) β^* *is the* critical Besov exponent, *that is*

$$\beta^* = \max\{\beta \in [0,\infty) : B^{\beta}_{p,\infty}(K^{\boldsymbol{l}}) \text{ contains non-constant functions}\}. \tag{1.4}$$

(3) (Weak-monotonicity property) *For all* $u \in \mathcal{F}_p = B^{\beta^*}_{p,\infty}$,

$$\sup_{r\in(0,\mathrm{diam}(K^{\boldsymbol{l}})]} \Phi^{\beta^*}_u(r) \leq C \liminf_{r\to 0} \Phi^{\beta^*}_u(r).$$

(4) (Bourgain–Brezis–Mironescu (BBM) convergence) *There exists* $C > 1$ *such that for all* $u \in \mathcal{F}_p = B^{\beta^*}_{p,\infty}$,

$$C^{-1}\mathcal{E}_p(u) \leq \liminf_{\beta\uparrow\beta^*}(\beta^*-\beta)\mathcal{E}^{\beta}_{p,p}(u) \leq \limsup_{\beta\uparrow\beta^*}(\beta^*-\beta)\mathcal{E}^{\beta}_{p,p}(u) \leq C\,\mathcal{E}_p(u) \tag{1.5}$$

and

$$C^{-1}[u]^p_{B^{\beta*}_{p,\infty}} \leq \liminf_{\beta\uparrow\beta^*}(\beta^*-\beta)[u]^p_{B^{\beta}_{p,p}} \leq \limsup_{\beta\uparrow\beta^*}(\beta^*-\beta)[u]^p_{B^{\beta}_{p,p}} \leq C[u]^p_{B^{\beta*}_{p,\infty}}. \tag{1.6}$$

The structure of the paper is as follows. In Section 2, we introduce scale-irregular Vicsek set and its measure, and discuss their properties, including the volume doubling property (Proposition 2.6), the existence of the Hausdorff measure $\mathcal{H}^{\alpha}$ (Proposition 2.7), the Ahlfors regularity (Proposition 2.9) and the non-self-similarity (Theorem 2.10). In Section 3, we construct p-energy norm and p-energy measure on scale-irregular Vicsek sets and prove Theorems 1.1 and 1.2. In Section 4, we study Besov–Lipschitz norms related to the p-energy, including the weak monotonicity and BBM convergence, and prove Theorem 1.4. Some possible extensions of our results are discussed in Section 5.

Notation 1.5. The letters C, C', C_i, C'_i, C''_i and c are universal positive constants which may vary at each occurrence. The sign $\asymp$ means that both $\leq$ and $\geq$ are true with uniform values of C depending only on $K^{\boldsymbol{l}}$. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$. We use $\#A$ for the cardinality of a set A.

2. Geometry and measure of a scale-irregular Vicsek set

The arrangement of this section is as follows. We state the definition and related notions of scale-irregular Vicsek sets in Section 2.1. The volume doubling property of the measures on scale-irregular Vicsek sets are analyzed in Section 2.2. The criterion for the existence of the Hausdorff measure $\mathcal{H}^{\alpha}$ and Ahlfors regularity, and sufficient conditions for scale-irregular Vicsek sets to be non-self-similar are given in Section 2.3.

2.1. Preliminaries

Define five points in the complex plane $\mathbb{C}$ by

$$q_0 := 0,\ q_j := \exp\big((2j-1)\pi i/4\big), \quad 1 \le j \le 4.$$

Let K_0 be the closed unit square in $\mathbb{C}$ with vertices $\{q_j\}_{j=1}^4$. Given an odd number $l \ge 3$, define

$$S_l := \left\{2nl^{-1}q_j : 0 \le n \le \frac{1}{2}(l-1),\ 1 \le j \le 4\right\}$$

so that $\#S_l = 2l-1$. We assign S_l the discrete topology for each l. For convenience, let $l_0 := 1$. For any infinite sequence $\boldsymbol{l} = (l_k)_{k=1}^\infty$, where each $l_k \ge 3$ is an odd integer for $k \ge 1$, define

$$\rho_n := 2\prod_{k=0}^{n} l_k^{-1} \text{ for } 0 \le n < \infty,$$
$$W_n^{\boldsymbol{l}} := \prod_{k=1}^{n} S_{l_k} \text{ for } 1 \le n \le \infty$$

and

$$W_*^{\boldsymbol{l}} := \bigcup_{1 \le n < \infty} W_n^{\boldsymbol{l}}.$$

We assign $W_n^{\boldsymbol{l}}$ and $W_*^{\boldsymbol{l}}$ the product topology. For each $w = w_1w_2\cdots \in W_\infty^{\boldsymbol{l}}$, we define $[w]_n := w_1w_2\cdots w_n \in W_n^{\boldsymbol{l}}$ and $[w]_n$ for $w \in W_k^{\boldsymbol{l}}$ when $k \ge n \ge 1$, similarly. For $w = w_1\cdots w_n \in W_*^{\boldsymbol{l}}$, we write

$$S(w) := \big\{v \in W_{n+1}^{\boldsymbol{l}} : [v]_n = w\big\} = \{w_1w_2\cdots w_nw_{n+1} : w_{n+1} \in S_{l_{n+1}}\}.$$

For $\alpha \in (0,1)$, we define a function δ on $W_\infty^{\boldsymbol{l}} \times W_\infty^{\boldsymbol{l}}$ by

$$\delta(w,\tau) := \begin{cases} \alpha^{\min\{n : [w]_n \ne [\tau]_n\}-1} & \text{if } w \ne \tau, \\ 0 & \text{if } w = \tau. \end{cases}$$

Then δ is a metric on $W_\infty^{\boldsymbol{l}}$ and generates the same topology on $W_\infty^{\boldsymbol{l}}$.

For each $w \in S_l$, we define a map

$$F_w^l(z) := w + l^{-1}z,\ z \in \mathbb{C},$$

and for each $w = w_1\cdots w_n \in W_*^{\boldsymbol{l}}$, define

$$F_w^{\boldsymbol{l}} := F_{w_1}^{l_1} \circ \cdots \circ F_{w_n}^{l_n}.$$

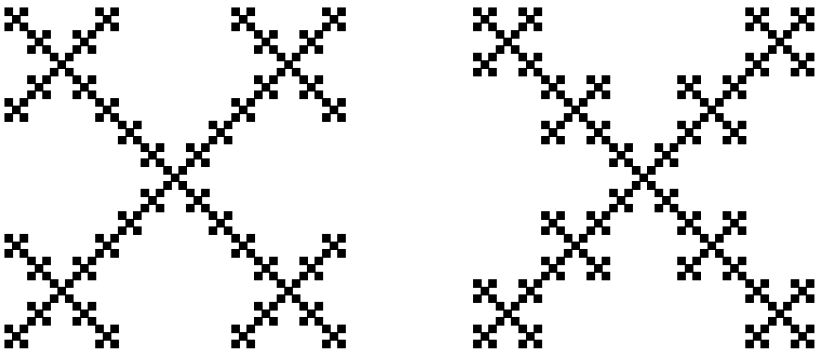
Let d be the Eucildean metric on $\mathbb{C}$. Note that for each $w = w_1 \cdots w_n \in W_*^{\boldsymbol{l}}$, the set $F_w^{\boldsymbol{l}}(K_0)$ is an isometric copy of $[0, 2^{1/2} l_1^{-1} l_2^{-1} \cdots l_n^{-1}]^2$, i.e., $F_w^{\boldsymbol{l}}(K_0)$ is a square with side length $2^{1/2} l_1^{-1} l_2^{-1} \cdots l_n^{-1}$ and

$$\operatorname{diam}\bigl(F_w^{\boldsymbol{l}}(K_0)\bigr) = 2 l_1^{-1} l_2^{-1} \cdots l_n^{-1} \le 2 \cdot 3^{-|w|}.$$

Definition 2.1 (The scale-irregular Vicsek set). For any infinite sequence $\boldsymbol{l} = (l_k)_{k=1}^{\infty}$, where each $l_k \ge 3$ is an odd integer, define $K^{\boldsymbol{l}}$ to be the non-empty compact subset of K_0 by

$$K^{\boldsymbol{l}} := \bigcap_{n=1}^{\infty} \bigcup_{w \in W_n^{\boldsymbol{l}}} F_w^{\boldsymbol{l}}(K_0).$$

The metric on $K^{\boldsymbol{l}}$ is given by the restriction of the Euclidean metric d on $\mathbb{C}$ to $K^{\boldsymbol{l}}$. We call the metric space $K^{\boldsymbol{l}}$ a *scale-irregular Vicsek set* (see Figure 1 for illustration).



(a) $K^{\boldsymbol{l}}$ at level 3, with $\boldsymbol{l} = 353\cdots$. **(b)** $K^{\boldsymbol{l}}$ at level 3, with $\boldsymbol{l} = 533\cdots$.

Figure 1. Two scale-irregular Vicsek sets $K^{\boldsymbol{l}}$.

For $w \in W_*^{\boldsymbol{l}}$, we write $K_w^{\boldsymbol{l}} := F_w^{\boldsymbol{l}}(K_0) \cap K^{\boldsymbol{l}}$. We call $K_w^{\boldsymbol{l}}$ a *level-n cell* if $w \in W_n^{\boldsymbol{l}}$.

Proposition 2.2. *For $w \in W_*^{\boldsymbol{l}}$, we have $K_w^{\boldsymbol{l}} = \bigcup_{v \in S(w)} K_v^{\boldsymbol{l}}$, namely,*

$$F_w^{\boldsymbol{l}}(K_0) \cap K^{\boldsymbol{l}} = \bigcup_{v \in S(w)} F_v^{\boldsymbol{l}}(K_0) \cap K^{\boldsymbol{l}}.$$

Proof. Since

$$F_w^{\boldsymbol{l}}(K_0) \cap \Biggl(\bigcup_{v \in W_{|w|+1}^{\boldsymbol{l}}} F_v^{\boldsymbol{l}}(K_0) \Biggr) = \bigcup_{v \in S(w)} F_v^{\boldsymbol{l}}(K_0) \text{ for any } w \in W_*^{\boldsymbol{l}},$$

the conclusion follows by taking the intersection with $K^{\boldsymbol{l}}$ on both sides. ∎

Proposition 2.3. *The map* $\chi : W_\infty^{\boldsymbol{l}} \to K^{\boldsymbol{l}}$ *defined by*

$$\{\chi(w)\} = \bigcap_{n\geq 1} F_{[w]_n}^{\boldsymbol{l}}(K_0) \text{ for all } w \in W_\infty^{\boldsymbol{l}}$$

is a continuous surjective map from $(W_\infty^{\boldsymbol{l}}, \delta)$ *to* $(K^{\boldsymbol{l}}, d)$. *Moreover,* $\#\chi^{-1}(x) \leq 2$ *for all* $x \in K^{\boldsymbol{l}}$ *and* $\#\chi^{-1}(x) = 2$ *only if* $x \in \bigcup_{n=1}^{\infty} \bigcup_{w\in W_n^{\boldsymbol{l}}} F_w^{\boldsymbol{l}}(\{q_j\}_{j=1}^4)$.

We call χ the *coding map*.

Proof. We first show that $\bigcap_{n\geq 1} F_{[w]_n}^{\boldsymbol{l}}(K_0)$ is a singleton in $K^{\boldsymbol{l}}$ so that χ is well defined. Indeed, since $F_{[w]_n}^{\boldsymbol{l}}(K_0)$ are compact and form a decreasing sequence (with respect to inclusion) as n increases, we conclude by Cantor's Intersection Theorem that $\bigcap_{n\geq 1} F_{[w]_n}^{\boldsymbol{l}}(K_0)$ is non-empty. Since $\operatorname{diam}(F_{[w]_n}^{\boldsymbol{l}}(K_0)) \to 0$ as $n \to \infty$, the set $\bigcap_{n\geq 1} F_{[w]_n}^{\boldsymbol{l}}(K_0)$ cannot contain more than two points, hence $\#\bigcap_{n\geq 1} F_{[w]_n}^{\boldsymbol{l}}(K_0) = 1$.

To see that χ is surjective, note that for any $x \in K^{\boldsymbol{l}} = \bigcap_{n=1}^{\infty} \bigcup_{w\in W_n^{\boldsymbol{l}}} F_w^{\boldsymbol{l}}(K_0)$, there exists $w_1 \in W_1^{\boldsymbol{l}}$ for each such that $x \in F_{w_1}^{\boldsymbol{l}}(K_0)$. Whenever we find w_n, we can apply Proposition 2.2 and find $w_{n+1} \in S_{l_{n+1}}$ such that $x \in F_{w_1 w_2 \cdots w_n w_{n+1}}^{\boldsymbol{l}}(K_0)$. Let $w = w_1 w_2 \cdots$, then $x \in \bigcap_{n\geq 1} F_{[w]_n}^{\boldsymbol{l}}(K_0) = \{\chi(w)\}$ by definition, showing the desired result.

To see that χ is continuous, note that when $\min\{n : [w]_n \neq [\tau]_n\} - 1 = k$, i.e., $\delta(w,\tau) = \alpha^k$, both $\chi(w)$ and $\chi(\tau)$ belong to $F_{[w]_k}^{\boldsymbol{l}}(K_0)$. Thus $d(\chi(w), \chi(\tau)) \leq \operatorname{diam} F_{[w]_k}^{\boldsymbol{l}}(K_0)$. As $\delta(w,\tau) \to 0$, we have $k \to \infty$ and thus

$$d\big(\chi(w), \chi(\tau)\big) \leq \operatorname{diam} F_{[w]_k}^{\boldsymbol{l}}(K_0) \to 0,$$

showing the desired result.

To show the final assertion, suppose that $x = \chi(w) = \chi(w')$ for two distinct infinite words $w, w' \in W_\infty^{\boldsymbol{l}}$. We can write $w = [w]_k \tau$, $w' = [w]_k \tau'$ where $[w]_k = w_1 w_2 \cdots w_k$ denotes the longest common initial word of w, w' with length k (when $k = 0$, $[w]_k$ is the empty word), so that $[\tau]_1 \neq [\tau']_1$. Clearly, we have

$$x \in F_{[w]_k[\tau]_1}^{\boldsymbol{l}}(K_0) \cap F_{[w]_k[\tau']_1}^{\boldsymbol{l}}(K_0),$$

which means that there exist two points $z_1, z_2 \in K_0$ such that

$$x = F_{w_1}^{l_1} \circ \cdots \circ F_{w_k}^{l_k} \circ F_{[\tau]_1}^{l_{k+1}}(z_1) = F_{w_1}^{l_1} \circ \cdots \circ F_{w_k}^{l_k} \circ F_{[\tau']_1}^{l_{k+1}}(z_2). \tag{2.1}$$

Since each $F_{w_j}^{l_j}$, $1 \leq j \leq k$ is invertible, we can apply $(F_{w_j}^{l_j})^{-1}$ $(1 \leq j \leq k)$ successively on both sides of (2.1) and obtain

$$F_{[\tau]_1}^{l_{k+1}}(z_1) = F_{[\tau']_1}^{l_{k+1}}(z_2). \tag{2.2}$$

Let $[\tau]_1 = 2nl_{k+1}^{-1}q$ and $[\tau']_1 = 2n'l_{k+1}^{-1}q'$ where $0 \le n \ne n' \le (l_{k+1}-1)/2$ and $q, q' \in \{q_j\}_{j=1}^4$. Then (2.2) can be interpreted as

$$2nl_{k+1}^{-1}q + l_{k+1}^{-1}z_1 = 2n'l_{k+1}^{-1}q' + l_{k+1}^{-1}z_2,$$

i.e., $z_1 - z_2 = 2(n'q' - nq)$ and therefore $d(z_1, z_2) = 2d(n'q', nq)$. We show that $d(z_1, z_2) = 2$.

(1) If $q \ne q'$, then

$$\begin{aligned} 4 \ge d(z_1, z_2)^2 &= 4d(n'q', nq)^2 \\ &= 4\big(n^2 + n'^2 - 2n'n\mathrm{Re}(q \cdot \overline{q'})\big) \ge 4\big(n^2 + n'^2\big) \ge 4 \end{aligned}$$

since $\mathrm{Re}(q \cdot \overline{q'}) \in \{0, -1\}$ and $n \ne n'$, thus $d(z_1, z_2) = 2$.

(2) If $q = q'$, then $2 \ge (z_1, z_2) = 2|n' - n| \ge 2$, so $d(z_1, z_2) = 2$.

Therefore, z_1 and z_2 must be the endpoints of the diagonal of K_0, i.e., either $\{z_1, z_2\} = \{q_1, q_3\}$ or $\{z_1, z_2\} = \{q_2, q_4\}$ and

$$\begin{aligned} \text{either } x &= F^{\boldsymbol{l}}_{[w]_k[\tau]_1}(q_1) \cap F^{\boldsymbol{l}}_{[w]_k[\tau']_1}(q_3), \\ x &= F^{\boldsymbol{l}}_{[w]_k[\tau]_1}(q_3) \cap F^{\boldsymbol{l}}_{[w]_k[\tau']_1}(q_1), \\ x &= F^{\boldsymbol{l}}_{[w]_k[\tau]_1}(q_2) \cap F^{\boldsymbol{l}}_{[w]_k[\tau']_1}(q_4), \\ \text{or } x &= F^{\boldsymbol{l}}_{[w]_k[\tau]_1}(q_4) \cap F^{\boldsymbol{l}}_{[w]_k[\tau']_1}(q_2). \end{aligned}$$

The proof is complete. ■

We state some terminologies in graph theory for scale-irregular Vicsek sets collecting from [5, 11, 35].

Definition 2.4 (Graph and Cable system). Fix $\boldsymbol{l} = (l_k)_{k=1}^{\infty}$.

(1) For each $n \ge 0$, define $V_0 := \{q_j\}_{j=0}^4$ for $n = 0$, and $V_n := \bigcup_{w \in W_n^{\boldsymbol{l}}} F_w^{\boldsymbol{l}}(V_0)$ for $n \ge 1$. Define

$$E_n := \big\{(x, y) \in V_n \times V_n : d(x, y) = l_0^{-1}l_1^{-1}l_2^{-1} \cdots l_n^{-1}\big\} \text{ for } n \ge 0,$$

so that (V_n, E_n) is a finite connected planer graph. We write $x \sim y$ if and only if $(x, y) \in E_n$ and say that x and y are *adjacent*.

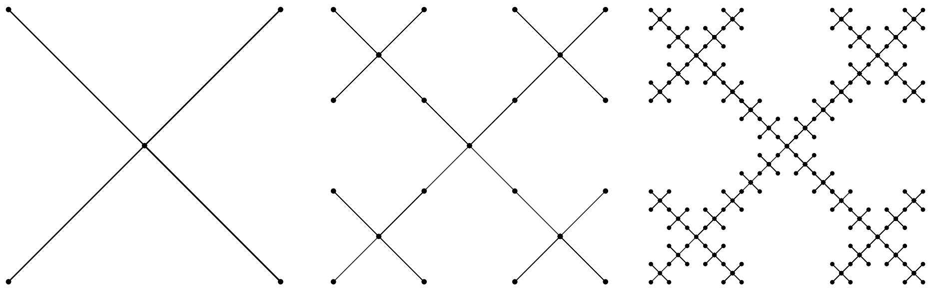
(2) For each $n \ge 0$, by replacing each edge in E_n by an isometric copy of the line segment $[0, l_0^{-1}l_1^{-1}l_2^{-1} \cdots l_n^{-1}]$ and gluing them in an obvious way at the vertices, we obtain a set $\overline{V}_n$, called the corresponding *cable system of* (V_n, E_n). With an abuse of notation, we regard $\overline{V}_n$ as a subset of $K^{\boldsymbol{l}}$ (see Figure 2).

(3) Define the *skeleton* $\mathcal{S} := \bigcup_{n=0}^{\infty} \overline{V}_n$. Let ν be the Lebesgue measure on $\mathcal{S}$, that is, ν assign $l_0^{-1}l_1^{-1}l_2^{-1}\cdots l_n^{-1}$ for each isometric copy of $[0, l_0^{-1}l_1^{-1}l_2^{-1}\cdots l_n^{-1}]$. We extend ν to $K^{\boldsymbol{l}}$ by letting $\nu(K^{\boldsymbol{l}} \setminus \mathcal{S}) = 0$.

(4) For every $n \geq 0$ and every adjacent $x, y \in V_n$, write $e(x, y)$ the geodesic in $\mathcal{S}$ connecting x and y, namely, the linear map from $[0, 1]$ to the isometric copy of $[0, l_0^{-1}l_1^{-1}l_2^{-1}\cdots l_n^{-1}]$ connecting u and v such that $e(x, y)(0) = x$ and $e(x, y)(1) = y$. Then $\bigcup_{\substack{x,y\in V_n \\ x\sim y}} e(x, y)([0, 1]) = \overline{V}_n$. With an abuse of notation, we sometimes regard $e(x, y)$ as a subset of $\mathcal{S}$.

(5) A subset $A \subset K^{\boldsymbol{l}}$ is called *convex* if for any two points $x, y \in A \cap \mathcal{S}$, the geodesic path connecting x to y is included in $A \cap \mathcal{S}$.

(6) For two adjacent x and y in V_n, we say that $x \prec y$ when the geodesic distance from 0 to x in $\overline{V}_n$ is less than the geodesic distance from 0 to y.



(a) $\overline{V_0}$ **(b)** $\overline{V_1}$ **(c)** $\overline{V_2}$

Figure 2. Cable systems $\overline{V_0}$, $\overline{V_1}$ and $\overline{V_2}$ for $K^{\boldsymbol{l}}$ with $\boldsymbol{l} = 35\cdots$.

Remark 2.5. (1) The measure ν is not a Radon measure, since the measure of any ball with positive radius is infinite.

(2) Although the image of $e(x, y)$ and $e(y, x)$ coincide for adjacent vertices x, y, we distinguish them when we take integration (along this edge) by assigning $e(x, y)$ the positive orientation when $x \prec y$. So for an integrable non-negative function g, we have $0 \leq \int_{e(x,y)} g \, \mathrm{d}\nu = -\int_{e(y,x)} g \, \mathrm{d}\nu$ when $e(x, y)$ is positively oriented.

2.2. Measure on scale-irregular Vicsek set: Doubling property

Given an odd integer $l \geq 3$, let K^l be the self-similar Vicsek set with common contraction ration l^{-1}. Then by solving Moran's equation [12, Theorem 2.7] directly, we see that K^l has Hausdorff dimension

$$\alpha_l := \dim_H(K^l) = \frac{\log(2l-1)}{\log l}. \tag{2.3}$$

Let

$$\psi(r) := \begin{cases} \prod_{k=0}^{n}(2l_k-1)^{-1} & \text{for } \rho_{n+1} < r \le \rho_n \ (n \ge 0), \\ 1 & \text{for } r \ge 2. \end{cases} \tag{2.4}$$

For each word $w \in W_m^{\boldsymbol{l}}$, we define

$$\mu(K_w^{\boldsymbol{l}}) := (\# W_m^{\boldsymbol{l}})^{-1} = \psi(\rho_m). \tag{2.5}$$

We extend μ to be a Borel measure on $K^{\boldsymbol{l}}$ by Kolmogorov's extension theorem.

If the sequence $\boldsymbol{l}$ consists of finitely many contraction ratios, i.e., $\sup_{n\ge 1} l_n < \infty$, the *volume doubling property* and *reverse volume doubling property* hold, as the following states.

Proposition 2.6. *Let $K^{\boldsymbol{l}}$ be a scale-irregular Vicsek set with the sequence $\boldsymbol{l}$ satisfying* $\sup_{n\ge1} l_n < \infty$.

(1) *There exist constants $c_1, c_2 > 0$ such that for all* $0 < r < R \le 2$*,*

$$c_1\left(\frac{R}{r}\right)^{\inf_{n\ge1}\alpha_{l_n}} \le \frac{\psi(R)}{\psi(r)} \le c_2\left(\frac{R}{r}\right)^{\sup_{n\ge1}\alpha_{l_n}}. \tag{2.6}$$

(2) *There exist constants $c_3, c_4 > 0$ such that for all $x \in K^{\boldsymbol{l}}$ and* $0 < r \le 2$*,*

$$c_3\psi(r) \le \mu\big(B(x,r)\big) \le c_4\psi(r).$$

Proof. (1) Note that $1 < \inf_{n\ge1}\alpha_{l_n} \le \sup_{n\ge1}\alpha_{l_n} \le \frac{\log 5}{\log 3}$. For $0 < r < R \le 2$, assume that $R \in (\rho_{m+1}, \rho_m]$ and $r \in (\rho_{n+1}, \rho_n]$ for some $0 \le m \le n$, then

$$\begin{aligned} \frac{\psi(R)}{\psi(r)} &= \frac{\prod_{k=0}^{m}(2l_k-1)^{-1}}{\prod_{k=0}^{n}(2l_k-1)^{-1}} = \prod_{k=m+1}^{n}(2l_k-1) = \prod_{k=m+1}^{n} l_k^{\alpha_{l_k}} \\ &\le \left(\prod_{k=m+1}^{n} l_k\right)^{\sup_{k\ge1}\alpha_{l_k}} = \left(\frac{\rho_m}{\rho_n}\right)^{\sup_{k\ge1}\alpha_{l_k}} \\ &\le (\sup_{k\ge1} l_k)^{\sup_{k\ge1}\alpha_{l_k}}\left(\frac{R}{r}\right)^{\sup_{k\ge1}\alpha_{l_k}}. \end{aligned}$$

Similarly, we also have

$$\frac{\psi(R)}{\psi(r)} \ge \left(\frac{\rho_m}{\rho_n}\right)^{\inf_{k\ge1}\alpha_{l_k}} \ge C(\sup_{k\ge1} l_k)^{-\inf_{k\ge1}\alpha_{l_k}}\left(\frac{R}{r}\right)^{\inf_{k\ge1}\alpha_{l_k}}.$$

Since $\sup_{n\geq 1} l_n < \infty$, we see that (2.6) holds.

(2) For this, we only need to observe that when $r \in (\rho_{m+1}, \rho_m]$ for some m, any metric ball $B(x,r)$ in $K^{\boldsymbol{l}}$ (where $x \in K^{\boldsymbol{l}}$) contains a level-$(m+2)$ cell and can be covered by 5 level-m cells. Indeed, a level-$(m+2)$ cell has diameter smaller than ρ_{m+1}, and each level-n cell has at most 4 neighboring level-n cells, while level-n cells are separated by distance ρ_n when they are not neighbors. Thus

$$\psi(\rho_{m+2}) \leq \mu\big(B(x,r)\big) \leq 5\psi(\rho_m),$$

our assertion then follows by using (1) and the fact that $\frac{\rho_{m+2}}{\rho_m}$ is bounded (because $\sup_{n\geq 1} l_n < \infty$). ∎

2.3. Lack of Ahlfors-regularity and self-similarity

In this section, we show that scale-irregular Vicsek sets may lack Ahlfors-regularity and self-similarity. For simplicity, we make the following assumption in this section: [2]

$$\begin{aligned} &\boldsymbol{l} \text{ consists of two distinct odd numbers } a, b \geq 3, \\ &\text{and the limit } \theta := \lim_{n\to\infty} \theta_n := \lim_{n\to\infty} \frac{[n]_a}{[n]_b} \in [0,\infty) \text{ exists,} \end{aligned} \tag{2.7}$$

where $[n]_a := \sum_{j=1}^n \mathbb{1}_{\{l_j=a\}}$ and $[n]_b := \sum_{j=1}^n \mathbb{1}_{\{l_j=b\}}$ are the numbers of a and b appeared in the first n-digits of $\boldsymbol{l}$. We first analyze the behavior of the Hausdorff measure by using [25, Theorem 3.1], since scale-irregular Vicsek sets are *Moran sets* (see [25, Section 1.2] for definitions).

Proposition 2.7. *Assume* (2.7) *and define* $\eta_n := n(\theta_n - \theta)$. *Then the following holds:*

(1) *The Hausdorff dimension of* $K^{\boldsymbol{l}}$, *denoted by* α, *is given by*

$$\alpha = \frac{\theta \log(2a-1) + \log(2b-1)}{\theta \log a + \log b}. \tag{2.8}$$

(2) *When* $3 \leq a < b$, *we have the following equivalences:*

$$\begin{aligned} 0 < \mathcal{H}^\alpha(K^{\boldsymbol{l}}) < \infty \ &\text{if and only if} \ \liminf_{k\to\infty} \eta_k \in \mathbb{R}, \\ \mathcal{H}^\alpha(K^{\boldsymbol{l}}) = 0 \ &\text{if and only if} \ \liminf_{k\to\infty} \eta_k = -\infty, \\ \mathcal{H}^\alpha(K^{\boldsymbol{l}}) = \infty \ &\text{if and only if} \ \liminf_{k\to\infty} \eta_k = +\infty. \end{aligned}$$

[2] We adopt the convention $1/0 = \infty$ and the limit is considered in the extended real line $\overline{\mathbb{R}} = [-\infty, \infty]$.

(3) *When $3 \le b < a$, we have the following equivalences:*

$$0 < \mathcal{H}^\alpha(K^{\boldsymbol{l}}) < \infty \ \text{ if and only if } \limsup_{k\to\infty} \eta_k \in \mathbb{R},$$
$$\mathcal{H}^\alpha(K^{\boldsymbol{l}}) = 0 \ \text{ if and only if } \limsup_{k\to\infty} \eta_k = +\infty,$$
$$\mathcal{H}^\alpha(K^{\boldsymbol{l}}) = \infty \ \text{ if and only if } \limsup_{k\to\infty} \eta_k = -\infty.$$

Proof. (1) This is a direct application of [25, Theorem 3.1].

(2) Define

$$\begin{aligned}\xi_n :&= \sum_{w\in W_n^{\boldsymbol{l}}} \big(\operatorname{diam}(K_w^{\boldsymbol{l}})\big)^\alpha = \rho_n^\alpha\big(\psi(\rho_n)\big)^{-1} \\ &= (a^{-[n]_a} b^{-[n]_b})^\alpha (2a-1)^{[n]_a}(2b-1)^{[n]_b}.\end{aligned}$$

By applying [25, Theorem 3.1], $\mathcal{H}^\alpha(K^{\boldsymbol{l}})$ and $\liminf_{k\to\infty} \xi_k$ are simultaneously zero, finite and positive, or infinite. So by taking the logarithm, we only need to show that $\liminf_{k\to\infty} \log \xi_k$ and $\liminf_{k\to\infty} \eta_k$ are simultaneously $-\infty$, finite, or ∞, respectively. Let

$$f(x) := \frac{x\log(2a-1) + \log(2b-1)}{x\log a + \log b}, \quad x \in \mathbb{R}.$$

Then for sufficiently large n,

$$\begin{aligned}\log \xi_n &= \big([n]_a \log(2a-1) + [n]_b \log(2b-1)\big) - \alpha\big([n]_a \log a + [n]_b \log b\big) \\ &= \big([n]_a \log a + [n]_b \log b\big)\left(\frac{[n]_a \log(2a-1) + [n]_b \log(2b-1)}{[n]_a \log a + [n]_b \log b} - \alpha\right) \\ &\asymp n\big(f(\theta_n) - f(\theta)\big) \\ &\asymp f'(\theta)\eta_n,\end{aligned}$$

where we used the conclusion $\alpha = f(\theta)$ of (1) in the third line, and in the fourth line we use $3 \le a < b$ to obtain

$$\begin{aligned}f'(x) &= \frac{\log(2a-1)\log b - \log a \log(2b-1)}{(x\log a + \log b)^2} \\ &= \frac{\log a \log b}{(x\log a + \log b)^2}\left(\frac{\log(2a-1)}{\log a} - \frac{\log(2b-1)}{\log b}\right) > 0\end{aligned}$$

by noting that $\frac{\log(2l-1)}{\log l}$ strictly decreases in l. Therefore, we see that $\liminf_{k\to\infty} \log \xi_k$ and $\liminf_{k\to\infty} \eta_k$ are simultaneously $-\infty$, finite, or $+\infty$.

(3) Note that in the case of $3 \le b < a$, $\liminf_{n\to\infty} \log \xi_n \asymp f'(\theta) \limsup_{n\to\infty} \eta_n$ as $f'(\theta) < 0$. The rest of the proof is the same as (2). ∎

Remark 2.8. Assume (2.7). If the Hausdorff measure $\mathcal{H}^\alpha$ of $K^{\boldsymbol{l}}$ exists, i.e., $0 < \mathcal{H}^\alpha(K^{\boldsymbol{l}}) < \infty$, then $\mathcal{H}^\alpha$ is equivalent to μ. The reason for this equivalence is that $\mathcal{H}^\alpha(K^{\boldsymbol{l}}_w) = \mathcal{H}^\alpha(K^{\boldsymbol{l}})\mu(K^{\boldsymbol{l}}_w)$ as level-k cells are only translations of each other for all $w \in W^{\boldsymbol{l}}_k$ $(k \geq 1)$, and any measurable set of $K^{\boldsymbol{l}}$ can be approximated by some unions of cells of level k (as $k \to \infty$), which means that $\mathcal{H}^\alpha = \mathcal{H}^\alpha(K^{\boldsymbol{l}})\mu$.

In our setting, μ is not necessarily Ahlfors regular, as we see in the following. A compact set $F \subseteq \mathbb{R}^d$ is called *Ahlfors regular* if there exists a constant $c > 0$ such that

$$\frac{1}{c} r^{\dim_H F} \leq \mathcal{H}^{\dim_H F}\big(B(x,r) \cap F\big) \leq c\, r^{\dim_H F}$$

for all $x \in F$ and all $0 < r < \operatorname{diam}(F)$ (see, for example, [15, Section 6.4]).

Proposition 2.9. *Assume* (2.7)*. Let α be given by* (2.8)*. Then the following are equivalent:*

(1) *$\{\eta_n\}_n$ is bounded.*

(2) *$K^{\boldsymbol{l}}$ is Ahlfors regular.*

(3) *The measure μ on $K^{\boldsymbol{l}}$ is α-Ahlfors regular.*

Proof. (1) $\Leftrightarrow$ (3). By Proposition 2.6, μ is α-Ahlfors regular if and only if $\psi(\rho_n) \asymp \rho_n^\alpha$, i.e., the sequence $\{(\psi(\rho_n))^{-1}\rho_n^\alpha\}_n$ is bounded below from 0 and away from ∞, which, by taking the logarithm and using the definition of η_n and θ_n, is equivalent to that the sequence

$$\frac{\eta_n}{1+\theta_n}\log\Big(\frac{2a-1}{a^\alpha}\Big) + \frac{n}{1+\theta_n}\Big(\theta\log\Big(\frac{2a-1}{a^\alpha}\Big) + \log\Big(\frac{2b-1}{b^\alpha}\Big)\Big)$$

is bounded. By (2.8), we have

$$\theta\log\Big(\frac{2a-1}{a^\alpha}\Big) + \log\Big(\frac{2b-1}{b^\alpha}\Big) = 0$$

and $\log(\frac{2a-1}{a^\alpha}) \neq 0$ by $\theta \in [0,\infty)$. The conclusion immediately follows by (2.7).

(1) $\Rightarrow$ (2). The existence of the Hausdorff measure $\mathcal{H}^\alpha$ immediately follows from Proposition 2.7. Then by Remark 2.8, we just need to check the α-Ahlfors regularity of μ, which is guaranteed by (1) $\Leftrightarrow$ (3).

(2) $\Rightarrow$ (3). This immediately follows from Remark 2.8. ∎

Now we can present our sufficient conditions for $K^{\boldsymbol{l}}$ to be non-self-similar. For Moran sets, we can use a different and simpler approach from that in [13] for graph-directed attractors.

Theorem 2.10. *Assume* (2.7) *with* $a < b$. *If* $\{\eta_n\}_n$ *satisfies one of the following two conditions:*

(1) $\liminf_{n\to\infty} \eta_n = \infty$,

(2) $\liminf_{n\to\infty} \eta_n \in \mathbb{R}$ *but* $\limsup_{n\to\infty} \eta_n = \infty$,

then $K^{\boldsymbol{l}}$ *is not self-similar, i.e., it is not the attractor of any standard iterated function system.*

Proof. (1) By Proposition 2.7, $\liminf_{n\to\infty} \eta_n = \infty$ implies $\mathcal{H}^\alpha(K^{\boldsymbol{l}}) = \infty$. But it is known that $\mathcal{H}^{\dim_H(K)}(K) < \infty$ for any self-similar set K by [12, Corollary 3.3], thus $K^{\boldsymbol{l}}$ is not self-similar.

(2) By Proposition 2.7, $\liminf_{n\to\infty} \eta_n \in \mathbb{R}$ implies $0 < \mathcal{H}^\alpha(K^{\boldsymbol{l}}) < \infty$. Then by Remark 2.8, $\mathcal{H}^\alpha$ is equivalent to μ on $K^{\boldsymbol{l}}$. By Proposition 2.9,

$$\limsup_{n\to\infty} \eta_n = \infty$$

implies that μ is not Ahlfors regular, so that $\mathcal{H}^\alpha$ is not Ahlfors regular. But [1, Theorem 2.1] states that for any self-similar set K, the Hausdorff measure $\mathcal{H}^{\dim_H(K)}$ is Ahlfors regular, should it exist. This shows that $K^{\boldsymbol{l}}$ cannot be self-similar. ■

Example 2.11. We give some examples of scale-irregular Vicsek set with $a = 3$ and $b = 5$ satisfying the conditions in Theorem 2.10. The sequence

$$\boldsymbol{l} = 3353335533335553333355555\cdots$$

satisfies Theorem 2.10 (1). That is, “3” appears consecutively $k+1$ times and then “5” appears consecutively k times ($k \geq 1$), as the following shows.

$$\boldsymbol{l} = \underbrace{\overbrace{33}^{2}\ \overbrace{5}^{1}\ \overbrace{333}^{3}\ \overbrace{55}^{2}\ \overbrace{3333}^{4}\ \overbrace{555}^{3}\cdots\overbrace{3\cdots3}^{k+1}\ \overbrace{5\cdots5}^{k}}_{\substack{\text{“3” appears } (k^2+3k)/2 \text{ times} \\ \text{“5” appears } (k^2+k)/2 \text{ times}}}\overbrace{33\cdots3}^{k+2}\,\overbrace{55\cdots5}^{k+1}\cdots$$

We claim that $\{\theta_n\}_n$ has limit $\theta = 1$ and $\liminf_{n\to\infty} \eta_n = \infty$. Indeed, when $n = (k^2+2k)+t$ for some $k \geq 1$ and $1 \leq t \leq k+2$,“3” occurs $t + (k^2+3k)/2$ times and “5” occurs $(k^2+k)/2$ times in the first n-digits of $\boldsymbol{l}$, thus for $k \geq 1$ and $1 \leq t \leq k+2$,

$$\theta_{k^2+2k+t} = \frac{k^2+3k+2t}{k^2+k} \text{ and } \eta_{k^2+2k+t} = \frac{k^2+2k+t}{k(k+1)}(2k+2t).$$

When $n = (k^2+3k+2)+t$ for some $k \geq 1$ and $1 \leq t \leq k+1$, “3” occurs $(k^2+5k+4)/2$ times and “5” occurs $t + (k^2+k)/2$ times in the first n-digits of $\boldsymbol{l}$, thus for $k \geq 1$ and $1 \leq t \leq k+1$,

$$\theta_{k^2+3k+2+t} = \frac{k^2+5k+4}{k^2+k+2t} \text{ and } \eta_{k^2+3k+2+t} = \frac{k^2+3k+2+t}{k^2+k+2t}(4k+4-2t).$$

For both cases, the claim follows by noting that

$$\theta_n = 1 + O(n^{-1/2}) \text{ and } \eta_n \geq 2 \cdot 3^{-1} n^{1/2}.$$

In the same way, one can verify that the sequence $\boldsymbol{l} = 35335533355533335555\cdots$ satisfies Theorem 2.10 (2). For both sequences, their corresponding Vicsek sets cannot be self-similar.

3. p-energy and its associated energy measure

In this section, we prove Theorem 1.1 and Theorem 1.2. Throughout this section, we fix $\boldsymbol{l} = (l_k)_{k=1}^{\infty}$ where each $l_k \geq 3$ is an odd integer ($l_0 := 1$). We omit the superscript $\boldsymbol{l}$ and write $K^{\boldsymbol{l}}$ as K.

3.1. Construction of p-energy norm

We construct p-energy norms on scale-irregular Vicsek sets by extending the methods in [5].

Definition 3.1. For each convex $A \subset K$, $1 < p < \infty$, $n \geq 0$ and $u \in C(K)$, define

$$\mathcal{E}_{p,n;A}(u) := \frac{1}{2}\left(\prod_{j=0}^{n} l_j^{p-1}\right) \sum_{\substack{x,y \in A \cap V_n \\ x \sim y}} |u(x) - u(y)|^p$$

and we denote $\mathcal{E}_{p,n;K}$ as $\mathcal{E}_{p,n}$. Define the function space $\mathcal{F}_p$ by

$$\mathcal{F}_p := \left\{u \in C(K) : \sup_{n \geq 0} \mathcal{E}_{p,n}(u) < \infty\right\}; \tag{3.1}$$

and for each $u \in \mathcal{F}_p$, define

$$\begin{gathered} \mathcal{E}_{p;A}(u) := \sup_{n \geq 0} \mathcal{E}_{p,n;A}(u), \quad \mathcal{E}_p(u) := \mathcal{E}_{p;K}(u) \\ \text{and } \|u\|_{\mathcal{F}_p} := \left(\|u\|_{L^p(K,\mu)}^p + \mathcal{E}_p(u)\right)^{\frac{1}{p}}. \end{gathered} \tag{3.2}$$

It is then easy to verify that $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ is a normed real vector space. The following proposition shows that these discrete energies $\{\mathcal{E}_{p,n;A}(u)\}$ increase in n.

Proposition 3.2. *For each $0 \leq m \leq n$ and each $u \in C(K)$, we have*

$$\mathcal{E}_{p,m;A}(u) \leq \mathcal{E}_{p,n;A}(u).$$

In particular,

$$\mathcal{E}_p(u) = \sup_{n \geq 0} \mathcal{E}_{p,n}(u) = \lim_{n \to \infty} \mathcal{E}_{p,n}(u) \text{ for all } u \in C(K). \tag{3.3}$$

Proof. For two adjacent vertices x and y in V_n, by the tree structure of K, there are unique $(l_{n+1}+1)$ vertices $\{x_j\}_{j=0}^{l_{n+1}} \subset V_{n+1}$ such that $x_j \sim x_{j+1}$, $1 \le j \le l_{n+1}$ and $x_0 = x$, $x_{l_{n+1}} = y$. Thus

$$|u(x_0) - u(x_{l_{n+1}})|^p = \left| \sum_{j=1}^{l_{n+1}} \big(u(x_{j-1}) - u(x_j)\big) \right|^p \le l_{n+1}^{p-1} \sum_{j=1}^{l_{n+1}} |u(x_{j-1}) - u(x_j)|^p.$$

By adding all adjacent vertices in V_n on both sides, we conclude that

$$\mathcal{E}_{p,n;A}(u) \le \mathcal{E}_{p,n+1;A}(u)$$

for each convex $A \subset K$, which completes the proof. ■

Next, we analyze the “core” functions in the domain $\mathcal{F}_p$.

Definition 3.3 (Piecewise affine functions). A continuous function $\Psi : K \to \mathbb{R}$ is called *n-piecewise affine*, if Ψ is linear between the vertices of $\overline{V}_n$ and constant on any connected component of $\overline{V}_m \setminus \overline{V}_n$ for every $m > n$. A continuous function $\Psi : K \to \mathbb{R}$ is called *piecewise affine*, if there exists $n \ge 0$ such that Ψ is n-piecewise affine.

Proposition 3.4. *For each $u \in C(K)$ and $n \ge 0$, define $H_n u$ to be the unique n-piecewise affine function on K that coincides with u on V_n. Then $H_n u \to u$ uniformly on K as $n \to \infty$. Moreover, $H_n u \in \mathcal{F}_p$ for each $u \in C(K)$ and $n \ge 0$.*

Proof. For each word $w \in W_n$, we have

$$\min_{K_w} u \le H_n u(x) \le \max_{K_w} u, \quad \forall x \in K_w.$$

It follows that

$$|H_n u(x) - u(x)| \le \operatorname*{Osc}_{K_w} u, \quad \forall x \in K_w,$$

where $\mathrm{Osc}_{K_w} u := \max_{x \in K_w} u(x) - \min_{x \in K_w} u(x)$. The first assertion then follows from

$$\sup_{x \in K} |H_n u(x) - u(x)| \le \sup_{w \in W_n} \operatorname*{Osc}_{K_w} u \to 0 \quad (n \to \infty)$$

as u is uniformly continuous. The second assertion is obvious. ■

As in the work of Baudoin and Chen [5, Theorem 3.1], a notable characteristic of Vicsek sets lies in their distinctive geometric structure, which permits the existence of gradients. The following proposition says that [5, Theorem 3.1] also holds on scale-irregular Vicsek sets.

Proposition 3.5 ([5, Theorem 3.1]). *Let $1 < p < \infty$ and $u \in C(K)$. The following are equivalent:*

(1) $u \in \mathcal{F}_p$;

(2) *There exists* $g \in L^p(\mathcal{S}, \nu)$ *such that, for every* $n \geq 0$ *and every adjacent* $x, y \in V_n$,

$$u(y) - u(x) = \int_{e(x,y)} g \, \mathrm{d}\nu. \tag{3.4}$$

In this case, g *is unique in* $L^p(\mathcal{S}, \nu)$. *Moreover, for every* $u \in \mathcal{F}_p$ *and every convex* $A \subset K$,

$$\mathcal{E}_{p;A}(u) = \int_{A \cap \mathcal{S}} |g|^p \, \mathrm{d}\nu. \tag{3.5}$$

We denote g in this proposition by ∂u and refer it as the gradient of u.

Proof. We first show that (1) implies (2). If Ψ is a piecewise affine function, it is obvious that there exists a function, denoted by $\partial\Psi$, such that for every adjacent $x, y \in V_n$,

$$\Psi(y) - \Psi(x) = \int_{e(x,y)} \partial\Psi \, \mathrm{d}\nu.$$

In fact, for each adjacent $x, y \in V_n$ with $x \prec y$ (so that $e(x, y)$ is positively oriented), we can choose Ψ such that $\partial\Psi$ takes the value $(\Psi(y) - \Psi(x)) \cdot d(x, y)^{-1}$ on $e(x, y)((0, 1))$. Therefore,

$$\begin{aligned}\int_{A \cap \mathcal{S}} |\partial\Psi|^p \, \mathrm{d}\nu &= \frac{1}{2} \sum_{\substack{x,y \in A \cap V_n \\ x \sim y}} d(x, y)^{-p+1} |\Psi(x) - \Psi(y)|^p \\ &= \frac{1}{2} \left(\prod_{j=0}^{n} l_j^{p-1} \right) \sum_{\substack{x,y \in A \cap V_n \\ x \sim y}} |\Psi(x) - \Psi(y)|^p.\end{aligned}$$

For any $u \in \mathcal{F}_p$ and for each $n \geq 0$, we define $H_n u$ by Proposition 3.4. Then

$$\sup_n \int_{A \cap \mathcal{S}} |\partial(H_n u)|^p \, \mathrm{d}\nu = \sup_n \frac{1}{2} \left(\prod_{j=0}^{n} l_j^{p-1} \right) \sum_{\substack{x,y \in A \cap V_n \\ x \sim y}} |u(x) - u(y)|^p = \mathcal{E}_{p;A}(u) < \infty.$$

The reflexivity of the space $L^p(\mathcal{S}, \nu)$ and Mazur's Lemma imply that, there exists a convex combination of a subsequence of $\partial(H_n u)$ that converges in $L^p(\mathcal{S}, \nu)$ to some $g \in L^p(\mathcal{S}, \nu)$. Since $H_n u$ converges uniformly to u by Proposition 3.4, we have then for every adjacent $x, y \in V_n$,

$$u(y) - u(x) = \int_{e(x,y)} g \, \mathrm{d}\nu.$$

This proves (2). Furthermore, since a convex combination of a subsequence of $\partial(H_n u)$ converges in $L^p(\mathcal{S}, \nu)$ to some g, we have

$$\int_{A\cap\mathcal{S}} |g|^p \,\mathrm{d}\nu \le \sup_n \int_{A\cap\mathcal{S}} |\partial(H_n u)|^p \,\mathrm{d}\nu \le \mathcal{E}_{p;A}(u).$$

We then show (2) implies (1). It follows from (2) and Hölder's inequality that for $n \ge 0$,

$$\begin{aligned}\left(\prod_{j=0}^{n} l_j^{p-1}\right) \sum_{\substack{x,y\in A\cap V_n\\ x\sim y}} |u(x)-u(y)|^p &\le \left(\prod_{j=0}^{n} l_j^{p-1}\right) \sum_{\substack{x,y\in A\cap V_n\\ x\sim y}} \left(\int_{e(x,y)} |g| \,\mathrm{d}\nu\right)^p \\ &\le \sum_{\substack{x,y\in A\cap V_n\\ x\sim y}} \int_{e(x,y)} |g|^p \,\mathrm{d}\nu \le 2\int_{A\cap\mathcal{S}} |g|^p \,\mathrm{d}\nu.\end{aligned}$$

Hence

$$\mathcal{E}_{p;A}(u) = \sup_{n\ge 0} \mathcal{E}_{p,n}(u) \le \int_{A\cap\mathcal{S}} |g|^p \,\mathrm{d}\nu$$

and we deduce that $u \in \mathcal{F}_p$ with $\|u\|_{\mathcal{F}_p}^p \le \|u\|_{L^p(K,\mu)}^p + \|g\|_{L^p(\mathcal{S},\nu)}^p$.

If g_1, g_2 both satisfy (2), then $\int_{e(x,y)} (g_1 - g_2)\,\mathrm{d}\nu = 0$ for all $n \ge 0$ and every adjacent $x, y \in V_n$. Since for each $x, y \in V_n$, $e(x, y)$ is the union of l_{n+1} edges in V_{n+1}, we may apply the Lebesgue differentiation Theorem to $(e(x, y), \nu|_{e(x,y)})$ (note that $\nu|_{e(x,y)}$ is the Lebesgue measure) and conclude that $g_1 - g_2 = 0$ ν-a.e. on $e(x, y)$, thus on all $\mathcal{S}$. This proves the uniqueness. ■

Remark 3.6. The uniqueness in Proposition 3.5 tells us more: for any $u \in \mathcal{F}_p$, if a convex combination of a subsequence of $\partial(H_n u)$ converges, then it must converge to ∂u.

We recall two inequalities that are used later. These are easy extensions of [5, Theorem 3.13 and Corollary 3.14] for the scale-irregular Vicsek sets. Since the geodesic distance and the Euclidean distance are bi-Lipschitz equivalent on a scale-irregular Vicsek set, their proofs are virtually identical with that in [5] and we omit them.

Lemma 3.7 (Morrey's inequality, [5, Theorem 3.13]). *Let $1 < p < \infty$. There exists a constant $C \ge 1$ such that, for every $u \in \mathcal{F}_p$ and $x, y \in A$,*

$$|u(x) - u(y)|^p \le C d(x, y)^{p-1} \mathcal{E}_{p;A}(u).$$

Lemma 3.8 (Poincaré inequality, [5, Corollary 3.14]). *Let $1 < p < \infty$. There exists a constant $C \ge 1$ such that, for every closed convex set $A \subset K$ and every $u \in \mathcal{F}_p$,*

$$⨍_A \left| u(x) - ⨍_A u \,\mathrm{d}\mu \right|^p \mathrm{d}\mu(x) \le C \operatorname{diam}(A)^{p-1} \mathcal{E}_{p;A}(u).$$

Combining the above facts, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ be defined by (3.1) and (3.2).

(1) Let $\{u_n\}$ be a Cauchy sequence in $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$. Then $\{u_n\}$ is also a Cauchy sequence in $L^p(K,\mu)$. Assume that $u_n \to u$ in $L^p(K,\mu)$ and a subsequence $u_{n_k} \to u$ μ-a.e. Since $\mathcal{E}_p(u_n - u_m) = \|\partial u_n - \partial u_m\|_{L^p(\mathcal{S},\nu)}$, we know that $\{\partial u_n\}$ is a Cauchy sequence in $L^p(\mathcal{S},\nu)$. Assume that $\partial u_n \to g$ in $L^p(\mathcal{S},\nu)$. Fix a point $x_0 \in K$ and let $h_n = u_n - u_n(x_0)$. By Lemma 3.7, for all $m,n \geq 1$ and all $x \in K$,

$$\begin{aligned}|h_n(x) - h_m(x)|^p &\leq C d(x,x_0)^{p-1}\mathcal{E}_p(u_n - u_m)\\ &\leq C\,\mathrm{diam}(K)^{p-1}\mathcal{E}_p(u_n - u_m),\end{aligned}$$

which implies that $\{h_n\}_n$ is a Cauchy sequence in $C(K)$. Therefore, there exists $h \in C(K)$ such that $h_n \to h$ in $C(K)$, and also in $L^p(K,\mu)$ by Hölder's inequality. It is immediate that $u_n - h_n$ also converges to $u - h$ in $L^p(K,\mu)$. Thus $u_{n_k}(x_0) = u_{n_k} - h_{n_k}$ converges to $u - h$ μ-a.e. Hence $u - h$ must be a constant, say $u - h \equiv c$, and u admits a continuous μ-version on K, which are also denoted by u. Note that

$$\|u_{n_k} - u\|_{C(K)} \leq \|h_{n_k} - h\|_{C(K)} + |u_{n_k}(x_0) - c| \to 0 \quad (k \to \infty).$$

For every $n \geq 0$ and every adjacent $x, y \in V_n$, we have by Proposition 3.5 that

$$u_{n_k}(y) - u_{n_k}(x) = \int_{e(x,y)} \partial u_{n_k}\,\mathrm{d}\nu.$$

Letting $k \to \infty$, we have $u(y) - u(x) = \int_{e(x,y)} g\,\mathrm{d}\nu$ for μ-a.e. x, y. By Proposition 3.5 again, we conclude that $u \in \mathcal{F}_p$ and $\partial u = g$ ν-a.e., thus

$$\mathcal{E}_p(u_n - u) = \|\partial u_n - \partial u\|_{L^p(\mathcal{S},\nu)} \to 0,$$

and then $u_n \to u$ in $\mathcal{F}_p$.

To prove that $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ is uniformly convex, we first define the norm $\|\cdot\|_p$ on the product space $L^p(K,\mu) \times L^p(\mathcal{S},\nu)$ by

$$\|(u,v)\|_p := \left(\|u\|^p_{L^p(K,\mu)} + \|v\|^p_{L^p(\mathcal{S},\nu)}\right)^{1/p}$$

for $(u,v) \in L^p(K,\mu) \times L^p(\mathcal{S},\nu)$, and a map

$$T : (\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p}) \to \left(L^p(K,\mu) \times L^p(\mathcal{S},\nu), \|\cdot\|_p\right)$$

by $Tu := (u, \partial u)$ for all $u \in \mathcal{F}_p$. Then by Proposition 3.5 and the above paragraph, $T(\mathcal{F}_p)$ is isometric to $\mathcal{F}_p$ and is a closed subspace of $\left(L^p(K,\mu) \times L^p(\mathcal{S},\nu), \|\cdot\|_p\right)$.

Since L^p spaces are uniformly convex for $1 < p < \infty$, we conclude by [10, Theorem 1] that the product space

$$\big(L^p(K,\mu) \times L^p(\mathcal{S},\nu), \|\cdot\|_p\big)$$

is uniformly convex, and so does its closed subspace $T(\mathcal{F}_p)$. Since $\mathcal{F}_p$ and $T(\mathcal{F}_p)$ are isometric, we conclude that $\mathcal{F}_p$ is uniformly convex. The reflexivity then follows from the uniform convexity and Milman–Pettis theorem (see, for example, [7, Theorem 3.31]).

The separability is obvious since the space of all piecewise affine functions is dense in $\mathcal{F}_p$, and clearly there is a countable dense subset of all piecewise affine functions.

(2) For any $u, v \in \mathcal{F}_p$,

$$\begin{aligned}
\mathcal{E}_{p,n}(uv) &= \frac{1}{2}\Big(\prod_{j=0}^{n} l_j^{p-1}\Big) \sum_{\substack{x,y\in V_n\\ x\sim y}} |u(x)v(x) - u(y)v(y)|^p \\
&\le \frac{1}{2}\Big(\prod_{j=0}^{n} l_j^{p-1}\Big) \sum_{\substack{x,y\in V_n\\ x\sim y}} 2^{p-1}\big(\|u\|_{C(K)}^p |v(x) - v(y)|^p \\
&\qquad + \|v\|_{C(K)}^p |u(x) - u(y)|^p\big) \\
&\le 2^{p-1}\big(\|u\|_{C(K)}^p \mathcal{E}_{p,n}(v) + \|v\|_{C(K)}^p \mathcal{E}_{p,n}(u)\big).
\end{aligned}$$

Taking the supremum of n on both sides, we have

$$\mathcal{E}_p(uv) \le 2^{p-1}\big(\|u\|_{C(K)}^p \mathcal{E}_p(v) + \|v\|_{C(K)}^p \mathcal{E}_p(u)\big),$$

which means that the subset $\mathcal{F}_p \subset C(K)$ is an algebra under the product operation.

(3) The regularity follows directly from Proposition 3.4.

(4) This follows by noting that $|\varphi(u(x)) - \varphi(u(y))|^p \le |u(x) - u(y)|^p$.

(5) This follows from Lemma 3.8 with $A = K$.

(6) We may assume $a = 0$ as $\mathcal{E}(v) = \mathcal{E}(v - a\mathbb{1}_K)$ and $\mathcal{E}(u + v) = \mathcal{E}(u + v - a\mathbb{1}_K)$ by definition. Write $A = \mathrm{supp}(u)$ and $B = \mathrm{supp}(v)$. Then A and B are two disjoint compact subsets of K and thus $d(A, B) > 0$. We can find n_0 sufficiently large so that for all $n \ge n_0$, the closed subsets

$$A_n := \bigcup_{\substack{w\in W_n\\ K_w\cap A\neq\emptyset}} K_w \ \text{ and } \ B_n := \bigcup_{\substack{w\in W_n\\ K_w\cap B\neq\emptyset}} K_w$$

are also disjoint, and each $x \in A_n \cap V_n$ and $y \in B_n \cap V_n$ are not adjacent. Then

$$\begin{aligned}\mathcal{E}_{p,n}(u+v) &= \frac{1}{2}\Big(\prod_{j=0}^{n} l_j^{p-1}\Big) \sum_{\substack{x,y\in V_n\\ x\sim y}} \big|\big(u(x)-u(y)\big)+\big(v(x)-v(y)\big)\big|^p \\ &= \frac{1}{2}\Big(\prod_{j=0}^{n} l_j^{p-1}\Big)\Big(\sum_{\substack{x,y\in A_n\cap V_n\\ x\sim y}} |u(x)-u(y)|^p + \sum_{\substack{x,y\in B_n\cap V_n\\ x\sim y}} |v(x)-v(y)|^p\Big) \\ &= \mathcal{E}_{p,n}(u)+\mathcal{E}_{p,n}(v).\end{aligned}$$

Letting $n \to \infty$, we derive $\mathcal{E}_p(u+v) = \mathcal{E}_p(u) + \mathcal{E}_p(v)$. ∎

Remark 3.9. In fact, we can show the uniform convexity of the semi-normed space $(\mathcal{F}_p, \mathcal{E}_p^{1/p})$ by applying [28, Proposition 3.5] and proving the following *p-Clarkson's inequality*:

$$\begin{aligned}&\mathcal{E}_p(f+g)+\mathcal{E}_p(f-g) \geq 2\big(\mathcal{E}_p(f)^{\frac{1}{p-1}}+\mathcal{E}_p(g)^{\frac{1}{p-1}}\big)^{p-1}, \quad \text{if } p\in(1,2],\\ &\mathcal{E}_p(f+g)+\mathcal{E}_p(f-g) \leq 2\big(\mathcal{E}_p(f)^{\frac{1}{p-1}}+\mathcal{E}_p(g)^{\frac{1}{p-1}}\big)^{p-1}, \quad \text{if } p\in[2,\infty).\end{aligned} \tag{3.6}$$

Roughly speaking, since $\mathcal{E}_p(f) = \|\partial f\|^p_{L^p(\mathcal{S},\nu)}$ by Proposition 3.5, and $(\mathcal{S},\nu)$ is σ-finite, we can apply [7, equations (4) and (6) in p. 462] to the gradients of f and g to obtain inequality (3.6).

Remark 3.10. As shown in Proposition 3.4 and Theorem 1.1, for two exponents $p, q \in (1,\infty)$ and any non-constant $u \in C(K)$, $H_1 u \in \mathcal{F}_p \cap \mathcal{F}_q$. As we can easily choose u so that $H_1 u$ is non-constant, we see that $\mathcal{F}_p \cap \mathcal{F}_q$ contains non-constant functions.

We conclude this subsection by stating a consequence of Proposition 3.5 and Lemma 3.7. We use the definition of the *p-resistance* $R_p(\cdot,\cdot)$ in [41]

$$R_p(x,y) := \sup\Big\{\frac{|u(x)-u(y)|^p}{\mathcal{E}_p(u)} : u\in\mathcal{F}_p \text{ and } \mathcal{E}_p(u)>0\Big\}, \ \forall x,y\in K.$$

Proposition 3.11. *There exists $C > 1$ such that*

$$C^{-1}d(x,y)^{p-1} \leq R_p(x,y) \leq Cd(x,y)^{p-1}, \ \forall x,y\in K.$$

Proof. Lemma 3.7 immediately implies that $R_p(x,y) \leq Cd(x,y)^{p-1}$. To see the lower bound, for $n \geq 1$ and $x, y \in V_n$, we choose a continuous function $u \in \mathcal{F}_p$ satisfying that $u(x) = 1$, $u(y) = 0$, $|\partial u|$ is constant over the geodesic cables connecting x and y, and u is piecewise constant over other cables. By a direct computation, $\mathcal{E}_p(u)^{-1/(p-1)}$ equals to the geodesic distance between x and y. By the bi-Lipschitz equivalence of the Euclidean distance and the geodesic distance restricted to the skeleton, we see that

$$R_p(x,y) \geq c d(x,y)^{p-1}, \ \forall x,y \in \bigcup_n V_n. \tag{3.7}$$

The lower bound is then established by the fact that $R_p : K \times K \to [0,\infty)$ is upper semi-continuous, that is $\limsup_{n\to\infty} R_p(x_n,y_n) \leq R_p(x,y)$ if $x_n \to x$ and $y_n \to y$. Indeed, given $\epsilon > 0$, we may choose N large enough such that $d(x_n,x) \vee d(y_n,y) < \epsilon$ for all $n \geq N$. Then for any $u \in \mathcal{F}_p$ with $\mathcal{E}_p(u) > 0$, we have

$$\begin{aligned}
&\left|\frac{u(x_n)-u(y_n)}{\mathcal{E}_p(u)^{1/p}}\right| \\
&\leq \left|\frac{u(x_n)-u(x)}{\mathcal{E}_p(u)^{1/p}}\right| + \left|\frac{u(x)-u(y)}{\mathcal{E}_p(u)^{1/p}}\right| + \left|\frac{u(y)-u(y_n)}{\mathcal{E}_p(u)^{1/p}}\right| \\
&\leq C d(x_n,x)^{(p-1)/p} + C d(y_n,y)^{(p-1)/p} + \left|\frac{u(x)-u(y)}{\mathcal{E}_p(u)^{1/p}}\right| \ \text{(by Lemma 3.7)} \\
&\leq 2C\epsilon^{(p-1)/p} + \left|\frac{u(x)-u(y)}{\mathcal{E}_p(u)^{1/p}}\right|.
\end{aligned} \tag{3.8}$$

Taking the supremum over all $u \in \mathcal{F}_p$ with $\mathcal{E}_p(u) > 0$ in (3.8), we see by definition that for all $n \geq N$,

$$R_p(x_n,y_n)^{1/p} \leq 2C\epsilon^{(p-1)/p} + R_p(x,y)^{1/p},$$

which shows the upper semi-continuity. Since $\bigcup_n V_n$ is dense in K and $R_p(\cdot,\cdot)$ is upper semi-continuous, we can extend (3.7) to all $x,y \in K$, completing the proof. ∎

Some consequences of Proposition 3.11 are discussed in Section 5.

3.2. Associated p-energy measure

After constructing the p-energy norm, it is natural to consider the corresponding p-energy measure. It is shown by Murugan and Shimizu in [36, Section 9] that the p-energy measure with good properties on the standard Sierpiński carpet can be constructed, which heavily relies on self-similarity. Nevertheless, we can use the special gradient structure of scale-irregular Vicsek sets to achieve our aim. Before discussing the p-energy measure, we record some properties of the operator ∂.

Lemma 3.12. *For any $u \in \mathcal{F}_p$ and any adjacent x,y, the function $u_{e(x,y)} : (0,1) \to \mathbb{R}$ defined by $u_{e(x,y)}(t) = u(e(x,y)(t))$ belongs to $W^{1,p}((0,1))$, and its weak derivative $Du_{e(x,y)}$ can be chosen as $t \mapsto \partial u(e(x,y)(t))$.*

Proof. Fix $u \in \mathcal{F}_p$. Let $g_{e(x,y)}(t) = \partial u(e(x,y)(t))$, then $g_{e(x,y)} \in L^p((0,1))$. By the continuity of u and the density of $\bigcup_n V_n$ in K, we can extend (3.4) to

$$u_{e(x,y)}(b) - u_{e(x,y)}(a) = \int_{(a,b)} g_{e(x,y)} \, \mathrm{d}\mathscr{L}^1, \quad 0 \le a < b \le 1, \tag{3.9}$$

where $\mathscr{L}^1$ is the Lebesgue measure on $\mathbb{R}$. For any $\phi \in C_c^\infty((0,1))$, by Fubini's theorem and the fundamental theorem of calculus, we have

$$\begin{aligned}
&\int_0^1 u_{e(x,y)}(t)\phi'(t) \, \mathrm{d}\mathscr{L}^1(t) \\
&\overset{(3.9)}{=} \int_0^1 \left(\int_{(0,t)} g_{e(x,y)}(s) \, \mathrm{d}\mathscr{L}^1(s) \right) \phi'(t) \, \mathrm{d}\mathscr{L}^1(t) + u_{e(x,y)}(0) \int_0^1 \phi'(t) \, \mathrm{d}\mathscr{L}^1(t) \\
&= \int_0^1 \left(\int_{(s,1)} \phi'(t) \, \mathrm{d}\mathscr{L}^1(t) \right) g_{e(x,y)}(s) \, \mathrm{d}\mathscr{L}^1(s) \text{ (note that } \phi(1) = \phi(0) = 0 \text{)} \\
&= \int_0^1 (\phi(1) - \phi(s)) g_{e(x,y)}(s) \, \mathrm{d}\mathscr{L}^1(s) = -\int_0^1 \phi(s) g_{e(x,y)}(s) \, \mathrm{d}\mathscr{L}^1(s).
\end{aligned}$$

Thus $u_{e(x,y)} \in W^{1,p}((0,1))$ and $Du_{e(x,y)} = g_{e(x,y)}$. ∎

The following properties on ∂ are generalizations of [6, Proposition 2.2].

Proposition 3.13 ([6, Proposition 2.2]). *Let $\partial : u \mapsto \partial u$, $(u \in \mathcal{F}_p)$ be defined as in Proposition 3.5. The following properties hold:*

(1) (Linearity) *For any two $u_1, u_2 \in \mathcal{F}_p$, $\partial(u_1 + u_2) = \partial u_1 + \partial u_2$.*

(2) (Leibniz rule) *For any two $u_1, u_2 \in \mathcal{F}_p$, $u_1 u_2 \in \mathcal{F}_p$ and*

$$\partial(u_1 u_2) = u_1 \partial u_2 + u_2 \partial u_1.$$

(3) (Chain rule) *For any $f \in C^1(\mathbb{R})$ and any $u \in \mathcal{F}_p$, $\partial(f \circ u) = f'(u)\partial u$.*

(4) (Closedness) *The operator $\partial : \mathcal{F}_p \to L^p(\mathcal{S}, \nu)$ is closed, if we view ∂ as an unbounded operator on $C(K)$.*

Proof. The assertions (1), (2) and (3) are obtained by Lemma 3.12, the linearity, Leibniz rule, and chain rule of the weak derivatives, respectively. The assertion (4) is an immediate consequence of Proposition 3.5. ∎

After these preparations, we can prove Theorem 1.2.

Proof of Theorem 1.2. Define

$$\Gamma_p\langle u\rangle(A) := \int_{A \cap \mathcal{S}} |\partial u|^p \, \mathrm{d}\nu, \text{ i.e., } \mathrm{d}\Gamma_p\langle u\rangle := |\partial u|^p \, \mathrm{d}\nu. \tag{3.10}$$

We prove that $\{\Gamma_p\langle u\rangle\}_{u\in\mathcal{F}_p}$ on K is a family of Borel finite measures having the properties stated in Theorem 1.2.

(1) $\Gamma_p\langle u\rangle(K) = \mathcal{E}_p(u)$ follows from Proposition 3.5. The “if" part in the second assertion is trivial by definition. For the converse, if $\Gamma_p\langle u\rangle = 0$, then $\mathcal{E}_p(u) = 0$. By Morrey’s inequality in Lemma 3.7, u must be constant.

(2) Note that for any non-negative Borel measurable function g on K,

$$\left(\int_K g\,\mathrm{d}\Gamma_p\langle u\rangle\right)^{1/p} = \left(\int_{\mathcal{S}} \left(|g|^{1/p}|\partial u|\right)^p \mathrm{d}\nu\right)^{1/p} = \||g|^{1/p}|\partial u|\|_{L^p(\mathcal{S},\nu)}.$$

So (1.1) holds by the Minkowski inequality of $L^p(\mathcal{S},\nu)$.

(3) The identity (1.2) holds by the definition of $\Gamma_p\langle u\rangle$ and the Leibniz rule in Proposition 3.13.

(4) The proof is essentially the same as in [9, Theorem 4.3.8] and [37, Proposition 7.6]. Since all compact subsets generate the Borel σ-algebra of $\mathbb{R}$, we only need to prove that $u_*(\Gamma_p\langle u\rangle)(F) = 0$ whenever $u \in \mathcal{F}_p$ and F is a compact subset of $\mathbb{R}$ with $\mathcal{L}^1(F) = 0$. We can choose a sequence $\{\phi_n\}_{n\geq 1} \subset C_c^\infty(\mathbb{R})$ such that $|\phi_n| \leq 1$, $\lim_{n\to\infty} \phi_n(x) = \mathbb{1}_F(x)$ for each $x \in \mathbb{R}$, and

$$\int_0^\infty \phi_n(t)\,\mathrm{d}t = \int_{-\infty}^0 \phi_n(t)\,\mathrm{d}t = 0$$

for each $n \in \mathbb{N}$. Let $\Phi_n(x) := \int_0^x \phi_n(t)\,\mathrm{d}t$ for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then we see that $\Phi_n \in C^1(\mathbb{R})$ with compact support, $\Phi_n(0) = 0$, and $|\Phi_n'(x)| \leq 1$ for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$. By the dominated convergence theorem, we know that $\lim_{n\to\infty} \Phi_n(x) = 0$ for each $x \in \mathbb{R}$ and $\Phi_n \circ u$ converges to 0 in $L^p(K,\mu)$. Since $\mathcal{E}_p(\Phi_n \circ u) \leq \mathcal{E}_p(u)$ by the Lipschitz contractivity of $\mathcal{E}_p$ in Theorem 1.1, we know that $\{\Phi_n \circ u\}_{n\geq 1}$ is bounded in $\mathcal{F}_p$. By the uniform convexity of $\mathcal{F}_p$ and [29], there exists a subsequence $\{n_k\}_{k\geq 1}$ of $\mathbb{N}$ such that the arithmetic mean $\Psi_j \circ u := \frac{1}{j}\sum_{k=1}^j \Phi_{n_k} \circ u \to 0$ in $\mathcal{F}_p$ as $j \to \infty$. For each $x \in \mathbb{R}$, since

$$\left|\mathbb{1}_F(x) - \frac{1}{j}\sum_{k=1}^j \phi_{n_k}(x)\right| = \left|\frac{1}{j}\sum_{k=1}^j \left(\mathbb{1}_F(x) - \phi_{n_k}(x)\right)\right| \to 0 \text{ as } j \to \infty,$$

we conclude by Fatou’s lemma that

$$\begin{aligned}\left(u_*\Gamma_p\langle u\rangle\right)(F) &= \int_{\mathbb{R}} \lim_{j\to\infty}\left|\frac{1}{j}\sum_{k=1}^j \Phi_{n_k}'(t)\right|^p \mathrm{d}\left(u_*\Gamma_p\langle u\rangle\right)(t) \\ &\leq \liminf_{j\to\infty} \int_K \left|\Psi_j'\left(u(x)\right)\right|^p \mathrm{d}\Gamma_p\langle u\rangle(x) \\ &= \liminf_{j\to\infty} \Gamma_p\langle \Psi_j \circ u\rangle(K) = \liminf_{l\to\infty} \mathcal{E}_p(\Psi_j \circ u) = 0. \quad \blacksquare\end{aligned}$$

Remark 3.14. (1) Theorem 1.2 shows that it is possible to define p-energy measures on some fractals without self-similarity. However, it is still an open problem to give a general procedure to define p-energy and associated energy measure on general Moran fractals, such as scale-irregular Sierpiński gaskets.

(2) By definition, $\Gamma_p\langle u\rangle \ll \nu$ for all $u \in \mathcal{F}_p$. So ν is a *minimal energy-dominant measure* in the sense of [24]. Since ν is independent of p, this gives an example of p-energy on fractals whose minimal energy-dominant measure for different exponents can be absolutely continuous (with or without self-similarity).

The approach in [36, Section 9], using the word space of a fractal, provides another way to construct energy measures, which also works in the scale-irregular Vicsek fractal setting with minor modification. Proposition 3.16 shows that the construction using the word space agree with that using the gradient structure in Theorem 1.2.

Proposition 3.15. *For any $u \in \mathcal{F}_p$ and $n \geq 1$, the measure $m_p^{(n)}\langle u\rangle$ defined by*

$$E \mapsto \sum_{w\in E} \mathcal{E}_{p;K_w}(u) =: m_p^{(n)}\langle u\rangle(E), \quad \forall E \subset W_n$$

satisfies

$$\sum_{v\in S(w)} m_p^{(n+1)}\langle u\rangle\big(\{v\}\big) = m_p^{(n)}\langle u\rangle\big(\{w\}\big).$$

Proof. This follows directly from (3.5) that

$$\begin{aligned}\sum_{v\in S(w)} \mathcal{E}_{p;K_v}(u) &= \sum_{v\in S(w)} \int_{\mathcal{S}\cap K_v} |\partial u|^p \,\mathrm{d}\nu \\ &= \int_{\mathcal{S}\cap K_w} |\partial u|^p \,\mathrm{d}\nu - \frac{1}{2}\sum_{v,v'\in S(w)} \int_{\mathcal{S}\cap K_v\cap K_{v'}} |\partial u|^p \,\mathrm{d}\nu \\ &= \int_{\mathcal{S}\cap K_w} |\partial u|^p \,\mathrm{d}\nu = \mathcal{E}_{p;K_w}(u)\end{aligned}$$

since $\bigcup_{v\in S(w)} K_v = K_w$ and ν has no atom. ∎

By the Kolmogorov's extension theorem and Proposition 3.15, we obtain a finite Borel measure $m_p\langle u\rangle$ on W_∞ such that

$$m_p\langle u\rangle\big(\{\tau \in W_\infty : [\tau]_n = w\}\big) = \mathcal{E}_{p;K_w}(u), \quad \forall n \geq 1,\ w \in W_n.$$

Clearly, $m_p\langle u\rangle(W_\infty) = \mathcal{E}_p(u)$. Moreover, $m_p\langle u\rangle$ is non-atomic because $m_p\langle u\rangle(w) \leq \mathcal{E}_{p;K_{[w]_n}}(u) \to 0$ as $n \to \infty$ for any $w \in W_\infty$. Now, we show that these two different ways give the same p-energy measure.

Recall that χ is the coding map in Proposition 2.3.

Proposition 3.16. *For any $u \in \mathcal{F}_p$, the push-forward of $m_p\langle u\rangle$ under χ coincides with $\Gamma_p\langle u\rangle$, namely, $\Gamma_p\langle u\rangle = \chi_* m_p\langle u\rangle$, where $\chi_* m_p\langle u\rangle(\cdot) := m_p\langle u\rangle(\chi^{-1}(\cdot))$.*

Proof. We first prove that for any $w \in W_*$, $\Gamma_p\langle u\rangle(K_w) = \chi_* m_p\langle u\rangle(K_w)$, namely,

$$m_p\langle u\rangle\big(\chi^{-1}(K_w)\big) = \mathcal{E}_{p;K_w}(u).$$

Assume that $w \in W_n$. Since $\{\tau \in W_\infty : [\tau]_n = w\} \subset \chi^{-1}(K_w)$, we first have

$$m_p\langle u\rangle\big(\chi^{-1}(K_w)\big) \geq \mathcal{E}_{p;K_w}(u).$$

If $\tau \in \chi^{-1}(K_w)$ but $[\tau]_n \neq w$, then by the last assertion in Proposition 2.3, $\chi(\tau)$ must belong to $K_w \cap V_n$. So the set $\chi^{-1}(K_w) \setminus \{\tau \in W_\infty : [\tau]_n = w\}$ has countably many elements. Since $m_p\langle u\rangle$ is non-atomic, we must have $m_p\langle u\rangle(\chi^{-1}(K_w)) = \mathcal{E}_{p;K_w}(u)$. We write $\mathcal{P}$ as the collection of all $K_w\,(w \in W_*)$ and all singletons in K, and $\mathcal{L}$ as the collection of all Borel subsets B of K such that $\Gamma_p\langle u\rangle(B) = \chi_* m_p\langle u\rangle(B)$. It is easy to verify that $\mathcal{P}$ forms a π-system and $\mathcal{L}$ forms a λ-system. A standard application of $\pi - \lambda$ theorem shows that $\mathcal{L}$ contains $\sigma(\mathcal{P})$, the σ-algebra generated by $\mathcal{P}$. It suffices to show that $\sigma(\mathcal{P})$ is the Borel σ-algebra of K. To show this, we only need to show that every closed subset F of K is in $\sigma(\mathcal{P})$. Let

$$F_m = \bigcup_{\substack{w \in W_m \\ K_w \cap F \neq \emptyset}} K_w$$

so that $F_m \in \sigma(\mathcal{P})$. Since $\max_{w \in W_m} \operatorname{diam}(K_w) \to 0$ as $m \to \infty$, we have $\bigcap_{m \geq 1} F_m = F$. Therefore $F \in \sigma(\mathcal{P})$ and we complete the proof. ■

We conclude this section by comparing the energy measure constructed here with that in [34]. In the case $p = 2$, Theorem 1.1 gives a regular Dirichlet form $(\mathcal{E}_2, \mathcal{F}_2)$ on a Vicsek set in the definition of [16, Chapter 1], and Theorem 1.2 gives the energy measure with respect to $(\mathcal{E}_2, \mathcal{F}_2)$ in the definition of [16, Chapter 3]. As we said in Remark 3.14, ν is a minimal energy-dominant measure for the Dirichlet form $(\mathcal{E}_2, \mathcal{F}_2)$. For every $u \in \mathcal{F}_2$, we see from (3.10) that the Radon–Nikodym derivative is

$$\frac{\mathrm{d}\Gamma_2\langle u\rangle}{\mathrm{d}\nu} = |\partial u|^2$$

and therefore, at least formally,

$$\mathcal{E}_p(u) = \int_K \left|\frac{\mathrm{d}\Gamma_2\langle u\rangle}{\mathrm{d}\nu}\right|^{p/2} \mathrm{d}\nu \text{ and } \mathrm{d}\Gamma_p\langle u\rangle = \left|\frac{\mathrm{d}\Gamma_2\langle u\rangle}{\mathrm{d}\nu}\right|^{p/2} \mathrm{d}\nu.$$

It turns out that the p-energy norm and p-energy measure in this paper are also equivalent to those in [34].

4. Besov–Lipschitz norms and their properties

Throughout this section, we fix a contraction ratio sequence $\boldsymbol{l}$ satisfying $\sup_{n\geq 1} l_n < \infty$, and fix a $\beta^* \in (0,\infty)$.

We write $K^{\boldsymbol{l}} = K$, $K^{\boldsymbol{l}}_w = K_w$ and omit the index p when no confusion is caused. We first investigate some basic properties of Besov–Lipschitz spaces in Section 4.1. The norm equivalences and the critical exponent in Theorem 1.4 is proved in Section 4.2. The weak monotonicity property and BBM convergence in Theorem 1.4 is proved in Section 4.3.

4.1. Besov–Lipschitz spaces related to the p-energy

Recall (2.4) and define a function

$$\phi(r) = \begin{cases} \rho_n^{p-1}\psi(\rho_n) & \text{for } \rho_{n+1} < r \leq \rho_n \ (n\geq 0), \\ 2^{p-1} & \text{for } r \geq 2. \end{cases} \tag{4.1}$$

Remark 4.1. If we assume that $\boldsymbol{l}$ consists of only one odd number $l \geq 3$ and choose $\beta^* := 1 + (\alpha_l - 1)/p$ in Definition 1.3, then $B^{\beta}_{p,\infty}$ (defined with respect to this particular β^*) is the same as $\mathcal{B}^{\beta,p}$ in [4].

Recall Definition 1.3. Using (2.6), the definition of ρ_n and $\sup_{n\geq 1} l_n < \infty$, there exists a constant $C > 0$ such that, for all $n \geq 0$, $\rho_{n+1} < r \leq \rho_n$ and all $u \in L^p(K,\mu)$,

$$C^{-1}\Phi^{\beta}_u(\rho_{n+1}) \leq \Phi^{\beta}_u(r) \leq C\Phi^{\beta}_u(\rho_n). \tag{4.2}$$

With these notions, we generalize the p-energy norm given in Definition 3.1.

Definition 4.2. For every $1 < p < \infty$, $0 \leq \beta < \infty$, $n \in \mathbb{N}$ and every $u \in C(K)$, define

$$\mathcal{E}^{\beta}_n(u) := \frac{1}{2}\phi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n) \sum_{\substack{x,y\in V_n \\ x\sim y}} |u(x)-u(y)|^p, \tag{4.3}$$

and

$$\mathcal{E}^{\beta}_{p,\infty}(u) := \sup_{n\geq 0}\mathcal{E}^{\beta}_n(u), \quad \mathcal{E}^{\beta}_{p,p}(u) := \sum_{n=0}^{\infty}\mathcal{E}^{\beta}_n(u).$$

Remark 4.3. (1) In view of (3.2) and (3.3), we have for each $n \in \mathbb{N}$ and all $u \in C(K)$ that $\mathcal{E}_{p,n;K}(u) = \mathcal{E}^{\beta^*}_n(u)$. Therefore,

$$\mathcal{E}_p(u) = \mathcal{E}^{\beta^*}_{p,\infty}(u) = \sup_{n\geq 0}\mathcal{E}^{\beta^*}_n(u) = \limsup_{n\to\infty}\mathcal{E}^{\beta^*}_n(u). \tag{4.4}$$

(2) We may write $\mathcal{E}^{\beta}_{p,p}(u)$ in a "non-local p-energy" manner

$$\mathcal{E}^{\beta}_{p,p}(u) = \frac{1}{2}\sum_{n=0}^{\infty} \phi(\rho_n)^{-\frac{\beta}{\beta^*}} \psi(\rho_n) \sum_{\substack{x,y\in V_n\\ x\sim y}} |u(x)-u(y)|^p$$
$$= \iint_{(K\times K)\setminus \mathrm{diag}} |u(x)-u(y)|^p \,\mathrm{d}J_\beta(x,y) \tag{4.5}$$

with the symmetric positive measure

$$\mathrm{d}J_\beta(x,y) = \frac{1}{2}\sum_{n=0}^{\infty} \sum_{\substack{v,w\in V_n\\ v\sim w}} \phi(\rho_n)^{-\frac{\beta}{\beta^*}} \psi(\rho_n)\,\mathrm{d}\delta_v(x)\,\mathrm{d}\delta_w(y)$$

defined on $(K\times K)\setminus \mathrm{diag}$, where $\mathrm{diag} := \{(x,x) : x\in K\}$ and δ_v is the Dirac measure at point v.

For any $1 < p < \infty$, we define a real number ϵ_p by

$$\epsilon_p := \left(1 + \frac{p-1}{\sup_{n\geq 1} \alpha_{l_n}}\right)^{-1} \in (0,1), \tag{4.6}$$

where α_l is given in (2.3). For each odd integer $l \geq 3$, let

$$\beta_l^{(p)} := p - 1 + \alpha_l. \tag{4.7}$$

For the non-self-similar case, we need the following estimates.

Proposition 4.4. *Let* $1 < p < \infty$.

(1) *For any* $\beta \in (\epsilon_p \beta^*, \infty)$, *we have* $\inf_{n\geq 1}(\frac{\beta}{\beta^*}\beta_{l_n}^{(p)} - \alpha_{l_n}) > 0$. *For every* δ *satisfying*

$$0 \leq \delta < \inf_{n\geq 1}\left(\frac{\beta}{\beta^*}\beta_{l_n}^{(p)} - \alpha_{l_n}\right), \tag{4.8}$$

there exists $C = C(\beta,\delta)$ *such that for any integer* $n \geq 0$,

$$\sum_{k=n}^{\infty} \phi(\rho_k)^{\frac{\beta}{\beta^*}} \psi(\rho_k)^{-1} \rho_k^{-\delta} \leq C\phi(\rho_n)^{\frac{\beta}{\beta^*}} \psi(\rho_n)^{-1} \rho_n^{-\delta}, \tag{4.9}$$

$$\sum_{k=0}^{n} \phi(\rho_k)^{-\frac{\beta}{\beta^*}} \psi(\rho_k) \rho_k^{\delta} \leq C\phi(\rho_n)^{-\frac{\beta}{\beta^*}} \psi(\rho_n) \rho_n^{\delta}. \tag{4.10}$$

Moreover, the sequence

$$\{\phi(\rho_k)^{\frac{\beta}{\beta^*}} \psi(\rho_k)^{-1} \rho_k^{-\delta}\}_{k\geq 0} \text{ decreases to } 0 \text{ as } k \to \infty. \tag{4.11}$$

(2) *For any integer* $n \geq 0$ *and any* $\delta > 0$*, we have*

$$\sum_{k=n}^{\infty} \phi(\rho_k)^{\delta} \in \left[\frac{\phi(\rho_n)^{\delta}}{1-(\sup_{n\geq 1} t_{l_n})^{-\delta}}, \frac{\phi(\rho_n)^{\delta}}{1-(\inf_{n\geq 1} t_{l_n})^{-\delta}} \right], \tag{4.12}$$

where $t_l := (2l-1)l^{p-1}$.

Proof. (1) That $\inf_{n\geq 1}(\frac{\beta}{\beta^*}\beta_{l_n}^{(p)} - \alpha_{l_n}) > 0$ follows by a simple calculation and the assumption that $\{l_n\}_{n\geq 1}$ is a finite set of odd numbers. To show the rest, let

$$a_k := \phi(\rho_k)^{\frac{\beta}{\beta^*}} \psi(\rho_k)^{-1} \rho_k^{-\delta} = \rho_k^{(p-1)\frac{\beta}{\beta^*}-\delta} \psi(\rho_k)^{\frac{\beta}{\beta^*}-1},$$

we have for each $j \geq n$,

$$\frac{a_{j+1}}{a_j} \leq \max\left\{ l^{\delta-(p-1)\frac{\beta}{\beta^*}} (2l-1)^{1-\frac{\beta}{\beta^*}} : l \in \{l_n\}_{n\geq 1} \right\} := c_0 < 1. \tag{4.13}$$

Therefore,

$$\begin{aligned} \sum_{k=n}^{\infty} \phi(\rho_k)^{\frac{\beta}{\beta^*}} \psi(\rho_k)^{-1} \rho_k^{-\delta} &= a_n + \sum_{k=n+1}^{\infty} a_n \prod_{j=n}^{k-1} \frac{a_{j+1}}{a_j} \leq a_n \sum_{k=n}^{\infty} c_0^{k-n} \\ &\leq C a_n = C \phi(\rho_n)^{\frac{\beta}{\beta^*}} \psi(\rho_n)^{-1} \rho_n^{-\delta}, \end{aligned} \tag{4.14}$$

which implies (4.9). For $0 \leq k \leq n$, let

$$b_k := a_{n-k}^{-1} = \phi(\rho_{n-k})^{-\frac{\beta}{\beta^*}} \psi(\rho_{n-k}) \rho_{n-k}^{\delta}.$$

Then

$$\frac{b_{n-i}}{b_{n-i-1}} = \frac{a_{i+1}}{a_i} \leq c_0.$$

Therefore

$$\begin{aligned} \sum_{k=0}^{n} \phi(\rho_k)^{-\frac{\beta}{\beta^*}} \psi(\rho_k) \rho_k^{\delta} &= \sum_{k=0}^{n} b_{n-k} = b_0 + \sum_{k=0}^{n-1} b_0 \prod_{j=0}^{n-k-1} \frac{b_{n-k-j}}{b_{n-k-j-1}} \\ &\leq b_0 \sum_{k=0}^{n} c_0^{n-k} \leq C b_0 = C \phi(\rho_n)^{-\frac{\beta}{\beta^*}} \psi(\rho_n) \rho_n^{\delta}, \end{aligned}$$

which implies (4.10). By (4.13) and the convergence of the series in (4.14), we see that the sequence $\{\phi(\rho_k)^{\frac{\beta}{\beta^*}} \psi(\rho_k)^{-1} \rho_k^{-\delta}\}_{k\geq 1}$ decreases to 0 as $k \to \infty$.

(2) Since $\phi(\rho_{k+1})^{\delta} \phi(\rho_k)^{-\delta}$ belong to a finite set $\{t_{l_n}^{-\delta}\}_{n\geq 1}$ for all $k \geq 0$, the proof is virtually identical with that of (1) and we omit it. ∎

In Lemma 3.7, we study the Morrey–Sobolev inequality for the space $\mathcal{F}_p$. Such inequality still holds for the Besov spaces $B^{\beta}_{p,\infty}$ and $B^{\beta}_{p,p}$ when $\beta \in (\epsilon_p \beta^*, \infty)$, where ϵ_p is given in (4.6). The proof is adapted from [18, Theorem 4.11(iii)] ($p = 2$ therein) and does not use the gradient structure as in Lemma 3.7.

Lemma 4.5 (Morrey–Sobolev inequality). *For any $\beta \in (\epsilon_p \beta^*, \infty)$ and any $u \in B^{\beta}_{p,\infty}$, there exists a continuous version $\widetilde{u} \in C(K)$ satisfying $\widetilde{u} = u$, μ-almost everywhere on K and*

$$\left|\widetilde{u}(x) - \widetilde{u}(y)\right|^p \leq C\phi\big(d(x,y)\big)^{\frac{\beta}{\beta^*}}\psi\big(d(x,y)\big)^{-1}[u]^p_{B^{\beta}_{p,\infty}}, \tag{4.15}$$

for all $x, y \in K$, where C is a positive constant.

Proof. For any $x \in K$ and $0 < r \leq 2/3$, define

$$u_r(x) := \frac{1}{\mu(B(x,r))}\int_{B(x,r)} u(\xi)\,\mathrm{d}\mu(\xi).$$

We claim that for any $u \in B^{\beta}_{p,\infty}$, and all $x, y \in K$ with $r = d(x,y) < 2/3$, the following inequality holds:

$$|u_r(x) - u_r(y)| \leq C\psi(r)^{-1/p}\phi(r)^{\frac{\beta}{p\beta^*}} \sup_{r\in(0,3d(x,y)]}\left(\Phi^{\beta}_u(r)\right)^{1/p}. \tag{4.16}$$

Indeed, letting $B_1 = B(x,r)$, $B_2 = B(y,r)$, we have

$$u_r(x) = \frac{1}{\mu(B_1)}\int_{B_1} u(\xi)\,\mathrm{d}\mu(\xi) = \frac{1}{\mu(B_1)\mu(B_2)}\int_{B_1}\int_{B_2} u(\xi)\,\mathrm{d}\mu(\eta)\,\mathrm{d}\mu(\xi),$$

and

$$u_r(y) = \frac{1}{\mu(B_1)\mu(B_2)}\int_{B_1}\int_{B_2} u(\eta)\,\mathrm{d}\mu(\eta)\,\mathrm{d}\mu(\xi).$$

Assume that $\rho_{k+1} \leq 3r < \rho_k$, by the Hölder's inequality,

$$\begin{aligned}
|u_r(x) - u_r(y)|^p &= \left(\frac{1}{\mu(B_1)\mu(B_2)}\int_{B_1}\int_{B_2}\big(u(\xi) - u(\eta)\big)\,\mathrm{d}\mu(\eta)\,\mathrm{d}\mu(\xi)\right)^p \\
&\leq \frac{1}{\mu(B_1)\mu(B_2)}\int_{B_1}\int_{B_2}|u(\xi) - u(\eta)|^p\,\mathrm{d}\mu(\eta)\,\mathrm{d}\mu(\xi) \\
&\leq C_1\psi(r)^{-2}\int_K\left[\int_{B(\xi,3r)}|u(\xi) - u(\eta)|^p\,\mathrm{d}\mu(\eta)\right]\mathrm{d}\mu(\xi) \\
&\leq C_2\psi(r)^{-1}\phi(r)^{\frac{\beta}{\beta^*}} \sup_{r\in(0,3d(x,y)]}\Phi^{\beta}_u(r),
\end{aligned}$$

thus showing (4.16).

Next, let L be the set of μ-Lebesgue points of u, and fix $x \in L$. Define $r_0 = r$, $r_k = \rho_k r$, we have $r_k + r_{k+1} = r_k + l_{k+1}^{-1} r_k < 2r_k$ for $k \geq 0$. Similar to the above arguments for (4.16), we have for any $u \in B^{\beta}_{p,\infty}$,

$$|u_{r_k}(x) - u_{r_{k+1}}(x)|^p \leq C_3 \psi(r_k)^{-1} \phi(r_k)^{\frac{\beta}{\beta^*}} \sup_{r \in (0, 3d(x,y)]} \Phi^{\beta}_u(r).$$

Therefore,

$$\begin{aligned} |u(x) - u_r(x)| &\leq \sum_{k=0}^{\infty} |u_{r_k}(x) - u_{r_{k+1}}(x)| \\ &\leq C_3 \sum_{k=0}^{\infty} \phi(r_k)^{\frac{\beta}{p\beta^*_p}} \psi(r_k)^{-\frac{1}{p}} \sup_{r \in (0,3d(x,y)]} \left(\Phi^{\beta}_u(r)\right)^{\frac{1}{p}} \\ &\leq C_4 \phi(r)^{\frac{\beta}{p\beta^*}} \psi(r)^{-\frac{1}{p}} \sup_{r \in (0,3d(x,y)]} \left(\Phi^{\beta}_u(r)\right)^{\frac{1}{p}} \text{ (similar to (4.9)).} \end{aligned} \tag{4.17}$$

Similar inequality holds for $|u(y) - u_r(y)|$. Combining (4.16) and (4.17), we have

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u_r(x)| + |u_r(x) - u_r(y)| + |u_r(y) - u(y)| \\ &\leq C_5 \phi(r)^{\frac{\beta}{p\beta^*}} \psi(r)^{-\frac{1}{p}} \sup_{r \in (0,3d(x,y)]} \left(\Phi^{\beta}_u(r)\right)^{\frac{1}{p}} \\ &\leq C_5 \phi(r)^{\frac{\beta}{p\beta^*}} \psi(r)^{-\frac{1}{p}} [u]_{B^{\beta}_{p,\infty}} \end{aligned}$$

for all $x, y \in L$.

Finally, as $\phi(r)^{\frac{\beta}{\beta^*}} \psi(r)^{-1} \to 0$ as $r \to 0$ by (4.11) with $\delta = 0$, we can use the standard procedure as in [17, Lemma 2.1] to obtain a continuous version $\widetilde{u} \in C(K)$ for any $u \in B^{\beta}_{p,\infty}$ and the desired inequality (4.15). ∎

Proposition 4.6. *There exists $C > 1$ such that for any $\beta \in [0, \infty)$ and any $u \in L^p(K, \mu)$,*

$$C^{-1} \sum_{n=0}^{\infty} \Phi^{\beta}_u(\rho_n) \leq [u]^p_{B^{\beta}_{p,p}} \leq C \sum_{n=0}^{\infty} \Phi^{\beta}_u(\rho_n). \tag{4.18}$$

In particular, $B^{\beta}_{p,p}$ is continuously embedded into $B^{\beta}_{p,\infty}$, namely, there exists $C > 0$ such that $[u]^p_{B^{\beta}_{p,\infty}} \leq C[u]^p_{B^{\beta}_{p,p}}$ for all $u \in L^p(K, \mu)$.

Proof. Splitting the integral domain $(0, 2]$ into $(\rho_{n+1}, \rho_n]$ $(n \geq 0)$, by (4.2),

$$\begin{aligned} [u]^p_{B^{\beta}_{p,p}} &= \int_0^2 \Phi^{\beta}_u(r) \frac{\mathrm{d}r}{r} = \sum_{n=0}^{\infty} \int_{\rho_{n+1}}^{\rho_n} \Phi^{\beta}_u(r) \frac{\mathrm{d}r}{r} \\ &\leq C \sum_{n=0}^{\infty} \int_{\rho_{n+1}}^{\rho_n} \Phi^{\beta}_u(\rho_n) \frac{\mathrm{d}r}{r} \leq C \log\big(\sup_{n \geq 1} l_n\big) \sum_{n=0}^{\infty} \Phi^{\beta}_u(\rho_n) \end{aligned} \tag{4.19}$$

and

$$[u]^p_{B^\beta_{p,p}} = \sum_{n=0}^{\infty} \int_{\rho_{n+1}}^{\rho_n} \Phi^\beta_u(r) \frac{\mathrm{d}r}{r}$$
$$\geq C^{-1} \sum_{n=0}^{\infty} \int_{\rho_{n+1}}^{\rho_n} \Phi^\beta_u(\rho_{n+1}) \frac{\mathrm{d}r}{r} \geq C^{-1} \log\big(\inf_{n\geq 1} l_n\big) \sum_{n=1}^{\infty} \Phi^\beta_u(\rho_n). \tag{4.20}$$

Since K satisfies the chain condition (see [18, Definition 3.4]), we can use the argument in [39, Corollary 2.2] to obtain that there exists some constant $C(n) > 0$ depending on $n \geq 1$ such that

$$\Phi^\beta_u(\rho_n) \geq C(n)^{-1} \int_K \int_K |u(x) - u(y)|^p \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) = C(n)^{-1} \Phi^\beta_u(\rho_0).$$

It follows by (4.20) that

$$[u]^p_{B^\beta_{p,p}} \geq c \sum_{n=0}^{\infty} \Phi^\beta_u(\rho_n) \tag{4.21}$$

for some constant $c > 0$. We obtain (4.18) by (4.19) and (4.21). By (4.18) and (4.2), we see that there exists $C > 0$ such that $[u]^p_{B^\beta_{p,\infty}} \leq C[u]^p_{B^\beta_{p,p}}$ and consequently $B^\beta_{p,p} \hookrightarrow B^\beta_{p,\infty}$. ∎

Combining Lemma 4.5 and the above continuous embedding $B^\beta_{p,p} \hookrightarrow B^\beta_{p,\infty}$, we immediately derive the following Morrey–Sobolev inequality for $B^\beta_{p,p}$.

Corollary 4.7. *For $\beta \in (\epsilon_p \beta^*, \infty)$ and $u \in B^\beta_{p,p}$, there exists a continuous version $\widetilde{u} \in C(K)$ such that*

$$|\widetilde{u}(x) - \widetilde{u}(y)|^p \leq C\phi\big(d(x,y)\big)^{\frac{\beta}{\beta^*}} \psi\big(d(x,y)\big)^{-1} [u]^p_{B^\beta_{p,p}}$$

for all $x, y \in K$, where C is a positive constant.

Remark 4.8. In view of Lemma 4.5 and Corollary 4.7, we always regard $B^\beta_{p,\infty}$ and $B^\beta_{p,p}$ as subsets of $C(K)$ whenever $\beta \in (\epsilon_p \beta^*, \infty)$. That is, we represent every function u in $B^\beta_{p,\infty}$, $B^\beta_{p,p}$ by its continuous version. In particular, for such u, the energies in Definitions 3.1 and 4.2 are well defined.

4.2. Norm equivalences and critical Besov exponent

For any positive integer m, let μ_m be the Borel measure on V_m given by

$$\mu_m := \frac{1}{\# V_m} \sum_{a \in V_m} \delta_a.$$

Technically, we use the discrete measures μ_m to approximate μ and *convert the estimates of the integrations in* (1.3) *to the estimates of discrete sums*. To be precise, denote the *ball-energy*

$$I_{m,n}(u) := \int_K \int_{B(x,\rho_n)} |u(x)-u(y)|^p \,\mathrm{d}\mu_m(y)\,\mathrm{d}\mu_m(x).$$

Since μ_m weak $*$-converges to μ when $m \to \infty$, we have the weak $*$-convergence of $\mu_m \times \mu_m$ to $\mu \times \mu$. For any $u \in C(K)$, the set of discontinuity points of $(x,y) \mapsto \mathbb{1}_{B(x,\rho_n)}|u(x)-u(y)|^p$ is $\mu \times \mu$-null. By an argument similar to [20, Remark 1], we have that for any $\beta \in (\epsilon_p \beta^*, \infty)$ and any $u \in C(K)$, the following limit exists

$$I_{\infty,n}(u) := \lim_{m\to\infty} I_{m,n}(u) = \int_K \int_{B(x,\rho_n)} |u(x)-u(y)|^p \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x). \tag{4.22}$$

Indeed, the weak $*$-convergence of $\mu_m \times \mu_m$ to $\mu \times \mu$ implies that the sequence $\int_{K\times K} f \,\mathrm{d}(\mu_m \times \mu_m)$ converges to $\int_{K\times K} f \,\mathrm{d}(\mu\times\mu)$ for any measurable function f as long as the set of discontinuous points of f have μ-measure 0. For each $n \in \mathbb{N}$, we define

$$f_n(x,y) := \mathbb{1}_{B(x,\rho_n)}|u(x)-u(y)|^p, \ \forall x,y \in K.$$

By Proposition 2.6 we know that $\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} =: \alpha' > 1$ and therefore, the Hausdorff dimension of any measurable set with positive μ-measure is at least $\alpha' > 1$ (see [12, Proposition 2.3]). Since every $\partial B(x,r) \subset \mathbb{R}^2$ has Hausdorff dimension no greater than 1, we know that $\mu(\partial B(x,r)) = 0$ for all $x \in K$ and all $r > 0$. Since u is continuous, we know that the set of discontinuous points of f_n has $\mu\times\mu$-measure 0, showing (4.22).

With this notion, we see that

$$\Phi_u^\beta(r) \asymp \phi(\rho_n)^{-\frac{\beta}{\beta_p^*}} \psi(\rho_n)^{-1} I_{\infty,n}(u) \ \text{ for } \rho_{n+1} < r \le \rho_n. \tag{4.23}$$

The following two lemmas compare the energies in Definitions 1.3 and 4.2.

Lemma 4.9. *For any $\beta \in (\epsilon_p \beta^*, \infty)$, there exists $C > 0$ such that*

$$I_{\infty,n}(u) \le C\phi(\rho_n)^{\frac{\beta}{\beta_p^*}} \psi(\rho_n) \sup_{k\ge n} \mathcal{E}_k^\beta(u) \ \textit{ for all } u \in C(K). \tag{4.24}$$

In other words,

$$\Phi_u^\beta(\rho_n) \le C \sup_{k\ge n} \mathcal{E}_k^\beta(u) \ \textit{ for all } u \in C(K). \tag{4.25}$$

Proof. We first estimate $I_{m,n}(u)$ for all integers $m > n \ge 0$. For any $x, y \in K$ with $d(x,y) \le \rho_n$ and $x \in K_w$ for some $w \in W_n$, due to the geometry of K, there exists a word $\widetilde{w} \in W_n$ (not necessarily distinct from w) such that $y \in K_{\widetilde{w}}$ and $K_{\widetilde{w}} \cap K_w \neq \emptyset$. Therefore,

$$\begin{aligned} I_{m,n}(u) &\leq \sum_{\substack{w,\widetilde{w}\in W_n \\ K_{\widetilde{w}}\cap K_w \neq \emptyset}} \int_{K_w} \int_{K_{\widetilde{w}}} |u(x) - u(y)|^p \, \mathrm{d}\mu_m(y) \, \mathrm{d}\mu_m(x) \\ &= \sum_{\substack{w,\widetilde{w}\in W_n \\ K_{\widetilde{w}}\cap K_w \neq \emptyset}} \sum_{x\in K_w\cap V_m} \sum_{y\in K_{\widetilde{w}}\cap V_m} \frac{1}{\# V_m^2} |u(x) - u(y)|^p. \end{aligned}$$

For every pair $(w, \widetilde{w}) \in W_n \times W_n$ with $K_{\widetilde{w}} \cap K_w \neq \emptyset$, there exists a common vertex $z \in K_w \cap K_{\widetilde{w}} \cap V_n$ (as in the proof of Proposition 2.3). Moreover, since

$$|u(x) - u(y)|^p \leq 2^{p-1}\big(|u(x) - u(z)|^p + |u(z) - u(y)|^p\big),$$

we have

$$\begin{aligned} I_{m,n}(u) &\leq 2^{p-1} \sum_{w,\widetilde{w}\in W_n} \sum_{x\in K_w\cap V_m} \sum_{y\in K_{\widetilde{w}}\cap V_m} \frac{1}{\# V_m^2} \big(|u(x) - u(z)|^p + |u(z) - u(y)|^p\big) \\ &\leq C_1 \sum_{w\in W_n} \sum_{x\in K_w\cap V_m} \sum_{z\in K_w\cap V_n} \frac{\# K_{\widetilde{w}} \cap V_m}{\# V_m^2} |u(x) - u(z)|^p \\ &\leq C_1 \psi(\rho_n)\psi(\rho_m) \sum_{w\in W_n} \sum_{x\in K_w\cap V_m} \sum_{z\in K_w\cap V_n} |u(x) - u(z)|^p, \end{aligned}$$

where we have used $\# V_m \asymp \psi(\rho_m)^{-1}$ and $\# K_{\widetilde{w}} \cap V_m \asymp \psi(\rho_m)^{-1}\psi(\rho_n)$ in the third line.

Then we estimate $|u(x) - u(z)|^p$. For every $w \in W_n$, every $x \in K_w \cap V_m$ and every $z \in K_w \cap V_n$, we pick (and fix) a decreasing sequence of cells $\{K_{w_k}\}_{k=n}^m$ such that $w_k \in W_k$ with $z \in K_{w_n} \cap V_n$, $x \in K_{w_m} \cap V_m$. Then we obtain a sequence of vertices $\{z = x_n, x_{n+1}, \cdots, x_m = x\}$ such that

$$x_k \in K_{w_k} \cap V_k \text{ for } k = n, \dots, m. \tag{4.26}$$

By Hölder's inequality,

$$\begin{aligned} &|u(z) - u(x)|^p \\ &\quad \leq \Big(\sum_{k=n}^{m-1} \big(\psi(\rho_n)^{-1}\psi(\rho_k)\big)^{q/p}\Big)^{p/q} \Big(\sum_{k=n}^{m-1} \psi(\rho_n)\psi(\rho_k)^{-1} |u(x_k) - u(x_{k+1})|^p\Big) \\ &\quad \leq C_2 \sum_{k=n}^{m-1} \psi(\rho_n)\psi(\rho_k)^{-1} |u(x_k) - u(x_{k+1})|^p. \end{aligned}$$

Note that the cardinality of $(x,z) \in (K_w \cap V_m) \times (K_w \cap V_n)$ with $(s,t) = (x_k, x_{k+1})$ is no greater than $C'\psi(\rho_k)\psi(\rho_m)^{-1}$ for some $C' > 0$, since the cardinality of $K_w \cap V_n$ is uniformly bounded and the cardinality of $K_w \cap V_m$ is equivalent to the total number of level-m cells located in the level-k cells containing x_k. Hence,

$$
\begin{aligned}
&I_{m,n}(u) \\
&\le C_3\psi(\rho_n)\psi(\rho_m) \sum_{w\in W_n} \sum_{\substack{x\in K_w\cap V_m \\ z\in K_w\cap V_n}} \sum_{k=n}^{m-1} \psi(\rho_n)\psi(\rho_k)^{-1}|u(x_k)-u(x_{k+1})|^p \\
&\le C_3\psi(\rho_n)^2\psi(\rho_m) \sum_{w\in W_n} \sum_{k=n}^{m-1} \sum_{\substack{w'\in W_k \\ K_{w'}\subset K_w}} \sum_{\substack{(x,z)\in(K_w\cap V_m)\times(K_w\cap V_n) \\ (s,t)\in(K_{w'}\cap V_k)\times(K_{w'}\cap V_{k+1})}} \frac{1}{\psi(\rho_k)}|u(s)-u(t)|^p \\
&\le C_4\psi(\rho_n)^2\psi(\rho_m) \sum_{w\in W_n} \sum_{k=n}^{m-1} \sum_{\substack{w'\in W_k \\ K_{w'}\subset K_w}} \sum_{s,t\in K_{w'}\cap V_{k+1}} \psi(\rho_m)^{-1}|u(s)-u(t)|^p \\
&= C_4\psi(\rho_n)^2 \sum_{k=n}^{m-1} \sum_{w\in W_n} \sum_{\substack{w'\in W_k \\ K_{w'}\subset K_w}} \sum_{s,t\in K_{w'}\cap V_{k+1}} |u(s)-u(t)|^p \\
&\le C_5\psi(\rho_n)^2 \sum_{k=n}^{m-1} \sum_{\substack{s,t\in V_{k+1} \\ s\sim t}} |u(s)-u(t)|^p \\
&\le 2C_5\psi(\rho_n)^2 \sum_{k=n}^{m} \phi(\rho_k)^{\frac{\beta}{\beta^*}}\psi(\rho_k)^{-1} \sup_{k\ge n} \mathcal{E}_k^{\beta}(u) \ \text{ (by (4.3))},
\end{aligned} \tag{4.27}
$$

where in the second inequality we have used the fact that $(s,t) = (x_k, x_{k+1}) \in (K_{w'} \cap V_k) \times (K_{w'} \cap V_{k+1})$ with $w' = w_k$ by (4.26), and in the third inequality we use $(K_{w'} \cap V_k) \subset (K_{w'} \cap V_{k+1})$. Therefore, we obtain by (4.27) and (4.9) with $\delta = 0$ that

$$
I_{m,n}(u) \le C_6\psi(\rho_n)^2\phi(\rho_n)^{\frac{\beta}{\beta^*}}\psi^{-1}(\rho_n) \sup_{k\ge n} \mathcal{E}_k^{\beta}(u) = C_6\phi(\rho_n)^{\frac{\beta}{\beta^*}}\psi(\rho_n) \sup_{k\ge n} \mathcal{E}_k^{\beta}(u).
$$

Letting $m \to \infty$, we complete the proof by (4.22). ■

Denote the *ring-energy* by

$$
I_n(u) := \int_K \int_{\{\rho_{n+1}\le d(x,y)<\rho_n\}} |u(x)-u(y)|^p \, \mathrm{d}\mu(y)\, \mathrm{d}\mu(x).
$$

Lemma 4.10. *For $\beta \in (\epsilon_p\beta^*, \infty)$ and $u \in C(K)$, we have*

$$
\mathcal{E}_n^{\beta}(u) \le C \sup_{k\ge n} \Phi_u^{\beta}(\rho_k). \tag{4.28}
$$

Proof. For $s,t \in V_n$ with $s \sim t$, there exists some $w \in W_n$ such that $s,t \in K_w \cap V_n$. Note that for any $x \in K_w$,

$$|u(s)-u(t)|^p \le 2^{p-1}\big(|u(s)-u(x)|^p + |u(x)-u(t)|^p\big).$$

Integrating with respect to x and dividing by $\mu(K_w)$, we have

$$\begin{aligned}\sum_{\substack{s,t\in V_n\\ s\sim t}} |u(s)-u(t)|^p &= \sum_{w\in W_n}\sum_{s,t\in K_w\cap V_n} |u(s)-u(t)|^p \\ &\le 2^{p-1}\sum_{w\in W_n}\sum_{s,t\in K_w\cap V_n} ⨍_{K_w}\big(|u(s)-u(x)|^p + |u(x)-u(t)|^p\big)\,\mathrm{d}\mu(x) \\ &\le C_1 \sum_{w\in W_n}\sum_{s\in K_w\cap V_n} ⨍_{K_w} |u(s)-u(x)|^p\,\mathrm{d}\mu(x). \end{aligned} \tag{4.29}$$

For every $m \ge n+1$ and $s \in K_w \cap V_n$ with $w \in W_n$, we can find (and fix) a decreasing sequence of cells $\{K_{w_k}\}_{k=n}^m$ such that $s \in \bigcap_{k=n}^m K_{w_k}$ with $w_n = w$, $w_k \in W_k$. Let

$$\delta \in \Big(0, \inf_{n\ge 1}\Big(\frac{\beta}{\beta^*}\beta_{l_n}^{(p)} - \alpha_{l_n}\Big)\Big), \tag{4.30}$$

where $\beta_{l_n}^{(p)}$ is given in (4.7). By Hölder's inequality, we have for all $x_k \in K_{w_k}$ that

$$\begin{aligned} &|u(s)-u(x_n)|^p \\ &\quad\le 2^{p-1}|u(s)-u(x_m)|^p \\ &\qquad + 2^{p-1}\Big(\sum_{k=n}^{m-1}\big(\rho_n^{-\delta}\rho_k^{\delta}\big)^{q/p}\Big)^{p/q}\sum_{k=n}^{m-1}\rho_n^{\delta}\rho_k^{-\delta}|u(x_k)-u(x_{k+1})|^p \\ &\quad\le 2^{p-1}|u(s)-u(x_m)|^p + C_2\Big(\sum_{k=n}^{m-1}\rho_n^{\delta}\rho_k^{-\delta}|u(x_k)-u(x_{k+1})|^p\Big). \end{aligned} \tag{4.31}$$

Integrating (4.31) with respect to $x_k \in K_{w_k}$ and dividing by $\mu(K_{w_k})$ for all $n \le k \le m$ successively, then combining with (4.29), we have for $m \ge n+1$ that

$$\begin{aligned} &\sum_{\substack{s,t\in V_n\\ s\sim t}} |u(s)-u(t)|^p \\ &\le C_3\sum_{w\in W_n}\sum_{s\in K_w\cap V_n} ⨍_{K_{w_m}} |u(s)-u(x_m)|^p\,\mathrm{d}\mu(x_m) \\ &+ C_3\sum_{w\in W_n}\sum_{s\in K_w\cap V_n}\sum_{k=n}^{m-1}\rho_n^{\delta}\rho_k^{-\delta} ⨍_{K_{w_k}} ⨍_{K_{w_{k+1}}} |u(x_k)-u(x_{k+1})|^p\,\mathrm{d}\mu(x_{k+1})\,\mathrm{d}\mu(x_k) \\ &:= C_3\big(J_1(n,m) + J_2(n,m)\big). \end{aligned} \tag{4.32}$$

For $s, x_m \in K_{w_m}$,

$$|u(s) - u(x_m)| \le \operatorname*{Osc}_{K_{w_m}} u,$$

which implies

$$\begin{aligned} J_1(n,m) &= \sum_{w \in W_n} \sum_{s \in K_w \cap V_n} \frac{1}{\mu(K_{w_m})} \int_{K_{w_m}} |u(s) - u(x_m)|^p \, \mathrm{d}\mu(x_m) \\ &\le |V_n| \big(\operatorname*{Osc}_{K_{w_m}} u \big)^p \to 0 \text{ as } m \to \infty, \end{aligned} \tag{4.33}$$

since u is uniformly continuous on the compact set K.

On the other hand, for all $x_k \in K_{w_k}, x_{k+1} \in K_{w_{k+1}}$, we have $|x_k - x_{k+1}| \le \rho_k$, thus

$$\begin{aligned} J_2(n,m) &= \sum_{w \in W_n} \sum_{s \in K_w \cap V_n} \sum_{k=n}^{m-1} \frac{\rho_n^{\delta} \rho_k^{-\delta}}{\mu(K_{w_k}) \mu(K_{w_{k+1}})} \\ &\quad \times \int_{K_{w_k}} \int_{K_{w_{k+1}}} |u(x_k) - u(x_{k+1})|^p \, \mathrm{d}\mu(x_{k+1}) \, \mathrm{d}\mu(x_k) \\ &\le C_4 \sum_{k=n}^{\infty} \rho_n^{\delta} \rho_k^{-\delta} \psi(\rho_k)^{-2} \int_K \int_{B(x,\rho_k)} |u(x) - u(y)|^p \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x) \end{aligned} \tag{4.34}$$

$$\begin{aligned} &\le C_4 \sum_{k=0}^{\infty} \rho_n^{\delta} \rho_{n+k}^{-\delta} \psi(\rho_{n+k})^{-2} \cdot \sum_{l=k}^{\infty} I_{n+l}(u) \\ &= C_4 \sum_{l=0}^{\infty} \Big(\sum_{k=0}^{l} \rho_n^{\delta} \rho_{n+k}^{-\delta} \psi(\rho_{n+k})^{-2} \Big) I_{n+l}(u) \\ &\le C_5 \sum_{k=0}^{\infty} \rho_n^{\delta} \rho_{n+k}^{-\delta} \psi(\rho_{n+k})^{-2} I_{n+k}(u), \end{aligned} \tag{4.35}$$

where in the last inequality we have used the fact that

$$\begin{aligned} \sum_{k=0}^{l} \frac{\rho_{n+k}^{-\delta} \psi(\rho_{n+k})^{-2}}{\rho_{n+l}^{-\delta} \psi(\rho_{n+l})^{-2}} &= \sum_{k=0}^{l} \left(\frac{\rho_{n+l}}{\rho_{n+k}} \right)^{\delta} \left(\frac{\psi(\rho_{n+l})}{\psi(\rho_{n+k})} \right)^2 \\ &\le \sum_{k=0}^{l} \big(\inf_{n \ge 1} l_n \big)^{-\delta(l-k)} \big(2 \inf_{n \ge 1} l_n - 1 \big)^{-2(l-k)} \\ &< \big(1 - \big(\inf_{n \ge 1} l_n \big)^{-(\delta+2)} \big)^{-1}. \end{aligned}$$

By the definition of $\mathcal{E}_n^{\beta}(u)$ and (4.32), we have for $m \geq n+1$ that

$$\mathcal{E}_n^{\beta}(u) = \frac{1}{2}\phi(\rho_n)^{-\frac{\beta}{\beta_p^*}}\psi(\rho_n) \sum_{x,y\in V_n, x\sim y} |u(x)-u(y)|^p$$
$$\leq C_3\phi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\big(J_1(n,m)+J_2(n,m)\big).$$

Letting $m\to\infty$, we see by (4.33) that

$$\begin{aligned}\mathcal{E}_n^{\beta}(u) &\leq C_3\phi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\sup_{m\geq n+1} J_2(n,m) \qquad (4.36)\\ &\leq C_6\phi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\sum_{k=0}^{\infty}\rho_n^{\delta}\rho_{n+k}^{-\delta}\psi(\rho_{n+k})^{-2}I_{n+k}(u) \text{ (by (4.35))}\\ &\leq C_7\sum_{k=0}^{\infty}\left(\frac{\phi(\rho_n)}{\phi(\rho_{n+k})}\right)^{-\frac{\beta}{\beta^*}}\left(\frac{\psi(\rho_n)}{\psi(\rho_{n+k})}\right)\left(\frac{\rho_n}{\rho_{n+k}}\right)^{\delta}\sup_{k\geq n}\Phi_u^{\beta}(\rho_k)\\ &\leq C_8\sup_{k\geq n}\Phi_u^{\beta}(\rho_k) \text{ (by (4.9))},\end{aligned}$$

where the third inequality used the fact that $I_{k+n}(u) \leq I_{\infty,k+n}(u)$ and (4.23). The proof is complete. ■

The following lemma shows the equivalence between $[\cdot]^p_{B^{\beta}_{p,p}}$ and $\mathcal{E}^{\beta}_{p,p}$.

Lemma 4.11. *If $\beta \in (\epsilon_p\beta^*, \infty)$, then there exists $C>0$ such that for all $u \in C(K)$,*

$$C^{-1}[u]^p_{B^{\beta}_{p,p}} \leq \mathcal{E}^{\beta}_{p,p}(u) \leq C[u]^p_{B^{\beta}_{p,p}}. \qquad (4.37)$$

Proof. Using the sixth line in (4.27), for $l \in \mathbb{Z}^+$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\phi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}I_{n+l,n}(u)\\ &\quad\leq C_1\sum_{n=0}^{\infty}\phi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\sum_{k=n}^{n+l}\sum_{\substack{x,y\in V_k\\ x\sim y}}|u(x)-u(y)|^p\\ &\quad= C_1\sum_{k=0}^{\infty}\sum_{n=0}^{k}\phi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\sum_{\substack{x,y\in V_k\\ x\sim y}}|u(x)-u(y)|^p\\ &\quad\leq C_2\sum_{k=0}^{\infty}\phi(\rho_k)^{-\frac{\beta}{\beta^*}}\psi(\rho_k)\sum_{\substack{x,y\in V_k\\ x\sim y}}|u(x)-u(y)|^p \text{ (by (4.10) with } \delta=0)\\ &\quad= 2C_2\mathcal{E}^{\beta}_{p,p}(u). \qquad (4.38)\end{aligned}$$

Letting $l \to \infty$ and applying Fatou's lemma in (4.38), the left-hand inequality of (4.37) then follows using (4.21).

On the other hand, fix δ as required in (4.30), then

$$\begin{aligned}
&\mathcal{E}^{\beta}_{p,p}(u)\\
&\quad= \sum_{n=0}^{\infty} \mathcal{E}^{\beta}_{n}(u) \le C_3 \sum_{n=0}^{\infty} \phi(\rho_n)^{-\frac{\beta}{\beta^*}} \psi(\rho_n) \sup_{m\ge n+1} J_2(n,m) \ \ \text{(by (4.36))}\\
&\quad\le C_3 \sum_{n=0}^{\infty} \phi(\rho_n)^{-\frac{\beta}{\beta^*_p}} \psi(\rho_n) \sum_{k=n}^{\infty} \rho_n^{\delta} \rho_k^{-\delta} \psi(\rho_k)^{-2}\\
&\qquad \times \int_K \int_{B(x,\rho_k)} |u(x)-u(y)|^p \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) \ \ \text{(by (4.34))}\\
&\quad\le C_4 \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} \frac{\phi(\rho_n)^{-\beta/\beta^*} \psi(\rho_n) \rho_n^{\delta}}{\phi(\rho_k)^{-\beta/\beta^*} \psi(\rho_k) \rho_k^{\delta}} \right) \Phi^{\beta}_{u}(\rho_k) \ \text{(by (4.23))}\\
&\quad\le C_5 \sum_{k=0}^{\infty} \Phi^{\beta}_{u}(\rho_k) \le C_5 [u]^p_{B^{\beta}_{p,p}} \ \ \text{(by (4.10) and (4.19))},
\end{aligned}$$

showing the right-hand inequality of (4.37). ∎

Proof of Theorem 1.4 (1) *and* (2).

(1) Taking limsup and sup (of n) on both sides of (4.25) and (4.28), respectively, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \mathcal{E}^{\beta}_{n}(u) \asymp \limsup_{n\to\infty} \Phi^{\beta}_{u}(\rho_n) \text{ and}\\
&\mathcal{E}^{\beta}_{p,\infty}(u) = \sup_{n\to\infty} \mathcal{E}^{\beta}_{n}(u) \asymp \sup_{n\to\infty} \Phi^{\beta}_{u}(\rho_n) = [u]^p_{B^{\beta}_{p,\infty}}.
\end{aligned} \tag{4.39}$$

The assertion then follows by noting Lemma 4.11 and (4.4).

(2) Given any $\beta > \beta^*$ and any $u \in B^{\beta}_{p,\infty}$, we first have

$$\sup_{r\in(0,2]} \Phi^{\beta^*}_{u}(r) = \sup_{r\in(0,2]} \phi(r)^{\frac{\beta}{\beta^*}-1} \Phi^{\beta}_{u}(r) \le C \sup_{r\in(0,2]} \Phi^{\beta}_{u}(r) = C[u]^p_{B^{\beta}_{p,\infty}},$$

which implies $u \in B^{\beta^*}_{p,\infty}$. Since $\beta^* \in (\epsilon_p \beta^*, \infty)$, it follows from (4.4), (4.39) and (4.2) that

$$\begin{aligned}
\sup_{r\in(0,2]} \Phi^{\beta^*}_{u}(r) &\asymp \sup_{n\ge 0} \Phi^{\beta^*}_{u}(\rho_n) \asymp \sup_{n\to\infty} \mathcal{E}^{\beta^*}_{n}(u) = \limsup_{n\to\infty} \mathcal{E}^{\beta^*}_{n}(u)\\
&\asymp \limsup_{n\to\infty} \Phi^{\beta^*}_{u}(\rho_n) \asymp \limsup_{r\to 0} \Phi^{\beta^*}_{u}(r).
\end{aligned} \tag{4.40}$$

Consequently,

$$\begin{aligned}\sup_{r\in(0,2]} \Phi_u^{\beta^*}(r) &\le C \limsup_{r\to 0} \Phi_u^{\beta^*}(r)\\ &= C \limsup_{r\to 0} \phi(r)^{\frac{\beta}{\beta^*}-1} \Phi_u^{\beta}(r)\\ &\le C \sup_{r\in(0,2]} \Phi_u^{\beta}(r) \cdot \limsup_{r\to 0} \phi(r)^{\frac{\beta}{\beta^*}-1}\\ &= C [u]^p_{B^{\beta}_{p,\infty}} \cdot \limsup_{r\to 0} \phi(r)^{\frac{\beta}{\beta^*}-1} = 0.\end{aligned}$$

Therefore, $\Phi_u^{\beta^*}(r) \equiv 0$ for all $r > 0$ which implies that u is constant by its continuity. On the other hand, Lemma 3.4 shows that $\mathcal{F}_p = B^{\beta^*}_{p,\infty}$ contains all piecewise affine functions, which can be chosen to be non-constant, proving (1.4). ■

4.3. Weak monotonicity property and BBM convergence

We need the following key lemma that gives the upper bound of semi-norms by the lower limit of $\Phi_u^{\beta^*}(r)$.

Lemma 4.12. *There exists $C > 0$ such that for all $u \in \mathcal{F}_p = B^{\beta^*}_{p,\infty}$,*

$$\mathcal{E}_p(u) = \mathcal{E}^{\beta^*}_{p,\infty}(u) = \lim_{n\to\infty} \mathcal{E}_n^{\beta^*}(u) \le C \liminf_{n\to\infty} \Phi_u^{\beta^*}(\rho_n). \tag{4.41}$$

Proof. The proof is based on the monotonicity of discrete p-energy norms. Firstly,

$$\begin{aligned} I_n(u) &\le I_{\infty,n}(u)\\ &\le C_1\phi(\rho_n)\psi(\rho_n) \sup_{k\ge n} \mathcal{E}_k^{\beta^*}(u) \text{ (by (4.24))}\\ &= C_1\phi(\rho_n)\psi(\rho_n) \lim_{n\to\infty} \mathcal{E}_n^{\beta^*}(u) \text{ (by (3.3))}. \end{aligned} \tag{4.42}$$

Since $\beta^* \in (\epsilon_p\beta^*, \infty)$ and $\beta_l^{(p)} - \alpha_l = p-1$, we fix $\delta \in (0, p-1)$ as in (4.8) with $\beta = \beta^*$. Using the fact that $\phi(\rho_n)\psi(\rho_n)^{-1} = \rho_n^{p-1}$ and (4.42), we see that for any $L \ge 1$,

$$\begin{aligned}\sum_{k=L+1}^{\infty} \psi(\rho_{n+k})^{-2}\rho_{n+k}^{-\delta} I_{n+k}(u) &\le C_1 \sum_{k=L+1}^{\infty} \phi(\rho_{n+k})\psi(\rho_{n+k})^{-1}\rho_{n+k}^{-\delta} \lim_{n\to\infty} \mathcal{E}_n^{\beta^*}(u)\\ &= C_1 \sum_{k=L+1}^{\infty} \rho_{n+k}^{p-1-\delta} \lim_{n\to\infty} \mathcal{E}_n^{\beta^*}(u)\\ &\le C_2 \rho_{n+L+1}^{p-1-\delta} \lim_{n\to\infty} \mathcal{E}_n^{\beta^*}(u),\end{aligned}$$

where in the last inequality we have used (4.9) with $\beta = \beta^*$. Therefore,

$$\begin{aligned} \rho_n^{1-p+\delta} \sum_{k=L+1}^{\infty} \psi(\rho_{n+k})^{-2} \rho_{n+k}^{-\delta} I_{n+k}(u) &\le C_2 \left(\frac{\rho_{n+L+1}}{\rho_n} \right)^{p-1-\delta} \liminf_{n\to\infty} \mathcal{E}_n^{\beta^*}(u) \\ &\le C_2 \big(\inf_{n\ge 1} l_n \big)^{(-p+1+\delta)(L+1)} \liminf_{n\to\infty} \mathcal{E}_n^{\beta^*}(u). \end{aligned} \tag{4.43}$$

We know by (4.36) that,

$$\begin{aligned} \mathcal{E}_n^{\beta^*}(u) &\le C_3 \phi(\rho_n)^{-1} \psi(\rho_n) \rho_n^{\delta} \sum_{k=0}^{\infty} \psi(\rho_{n+k})^{-2} \rho_{n+k}^{-\delta} I_{k+n}(u) \\ &= C_3 \rho_n^{1-p+\delta} \left(\sum_{k=0}^{L} + \sum_{k=L+1}^{\infty} \right) \psi(\rho_{n+k})^{-2} \rho_{n+k}^{-\delta} I_{k+n}(u) \\ &\le C_3 \rho_n^{1-p+\delta} \sum_{k=0}^{L} \psi(\rho_{n+k})^{-2} \rho_{n+k}^{-\delta} I_{n+k}(u) \\ &\quad + C_3 C_2 \big(\inf_{n\ge 1} l_n \big)^{(-p+1+\delta)(L+1)} \liminf_{n\to\infty} \mathcal{E}_n^{\beta^*}(u), \end{aligned}$$

where (4.43) is used in the last inequality. Taking $\liminf_{n\to\infty}$ on the right-hand side above, we have

$$C_4 \liminf_{n\to\infty} \mathcal{E}_n^{\beta^*}(u) \le \liminf_{n\to\infty} \rho_n^{1-p+\delta} \sum_{k=0}^{L} \psi(\rho_{n+k})^{-2} \rho_{n+k}^{-\delta} I_{n+k}(u),$$

where $C_4 = \frac{1}{C_3} - C_2(\inf_{n\ge 1} l_n)^{-(p-1-\delta)(L+1)}$. Fix L large enough such that $C_4 > 0$, then for every $0 \le k \le L$,

$$\left(\frac{\psi(\rho_n)}{\psi(\rho_{n+k})} \right)^2 \left(\frac{\rho_n}{\rho_{n+k}} \right)^{\delta} \le \left(\frac{\psi(\rho_n)}{\psi(\rho_{n+L})} \right)^2 \left(\frac{\rho_n}{\rho_{n+L}} \right)^{\delta} \le 4^L \big(\sup_{n\ge 1} l_n \big)^{(2+\delta)L}$$

and therefore

$$\begin{aligned} C_4 \liminf_{n\to\infty} \mathcal{E}_n^{\beta^*}(u) &\le \liminf_{n\to\infty} \left(\frac{\rho_n^{1-p}}{\psi(\rho_n)^2} \sum_{k=0}^{L} \left(\frac{\psi(\rho_n)}{\psi(\rho_{n+k})} \right)^2 \left(\frac{\rho_n}{\rho_{n+k}} \right)^{\delta} I_{n+k}(u) \right) \\ &\le C_5 \liminf_{n\to\infty} \phi(\rho_n)^{-1} \psi(\rho_n)^{-1} \sum_{k=0}^{L} I_{n+k}(u) \\ &\le C_5 \liminf_{n\to\infty} \phi(\rho_n)^{-1} \psi(\rho_n)^{-1} I_{\infty,n}(u) \\ &\le C_6 \liminf_{n\to\infty} \Phi_u^{\beta^*}(\rho_n) \ \ \text{(by (4.23))}. \end{aligned}$$

Thus (4.41) holds with $C = C_6/C_4$. ∎

We are now ready to prove the weak monotonicity property and BBM convergence.

Proof of Theorem 1.4 (3) *and* (4).

(3) For any $u \in B^{\beta^*}_{p,\infty} = \mathcal{F}_p$, we have

$$\begin{aligned}\sup_{r\in(0,2]} \Phi_u^{\beta^*}(r) &\le C_1 \sup_{n\ge 0} \mathcal{E}_n^{\beta^*}(u) \text{ (by (4.40))}\\ &= C_1 \liminf_{n\to\infty} \mathcal{E}_n^{\beta^*}(u) \text{ (by (3.3))}\\ &\le C_2 \liminf_{r\to 0} \Phi_u^{\beta^*}(r) \text{ (by (4.41))}.\end{aligned}$$

(4) We first show (1.5). Fix a function $u \in B^{\beta^*}_{p,\infty} = \mathcal{F}_p$. By (4.12), for any $\beta < \beta^*$,

$$\begin{aligned}(\beta^*-\beta)\mathcal{E}^{\beta}_{p,p}(u) &= (\beta^*-\beta)\sum_{n=0}^{\infty}\phi(\rho_n)^{1-\frac{\beta}{\beta^*}}\mathcal{E}_n^{\beta^*}(u)\\ &\le (\beta^*-\beta)\sum_{n=0}^{\infty}\phi(\rho_n)^{1-\frac{\beta}{\beta^*}}\mathcal{E}^{\beta^*}_{p,\infty}(u)\\ &\le \beta^*\left(1-\frac{\beta}{\beta^*}\right)\left(1-\left(\inf_{n\ge 1} t_{l_n}\right)^{-1+\frac{\beta}{\beta^*}}\right)^{-1}\mathcal{E}^{\beta^*}_{p,\infty}(u),\end{aligned}$$

where $t_l = l^{p-1}(2l-1)$. Therefore,

$$\limsup_{\beta\uparrow\beta^*}(\beta^*-\beta)\mathcal{E}^{\beta}_{p,p}(u) \le \frac{\beta^*}{\log(\inf_{n\ge 1} t_{l_n})}\mathcal{E}^{\beta^*}_{p,\infty}(u). \tag{4.44}$$

On the other hand, for any $A < \mathcal{E}^{\beta^*}_{p,\infty}(u)$, there exists an integer $N \ge 1$ such that $\mathcal{E}_n^{\beta^*}(u) > A$ for all $n > N$. Similarly, by (4.12) again,

$$\begin{aligned}(\beta^*-\beta)\sum_{n=0}^{\infty}\phi(\rho_n)^{1-\frac{\beta}{\beta^*}}\mathcal{E}_n^{\beta^*}(u) &> (\beta^*-\beta)\sum_{n=N+1}^{\infty}\phi(\rho_n)^{1-\frac{\beta}{\beta^*}}A\\ &\ge (\beta^*-\beta)\left(1-\left(\sup_{n\ge 1} t_{l_n}\right)^{\frac{\beta}{\beta^*}-1}\right)^{-1}\phi(\rho_{N+1})^{1-\frac{\beta}{\beta^*}}A\\ &\ge \beta^*\left(1-\frac{\beta}{\beta^*}\right)\frac{(\sup_{n\ge 1} t_{l_n})^{(N+1)(\frac{\beta}{\beta^*}-1)}}{1-(\sup_{n\ge 1} t_{l_n})^{\frac{\beta}{\beta^*}-1}}A.\end{aligned}$$

Therefore,

$$\liminf_{\beta\uparrow\beta^*}(\beta^*-\beta)\mathcal{E}^{\beta}_{p,p}(u) \ge \frac{\beta^*}{\log(\sup_{n\ge 1} t_{l_n})}A,$$

for any $A < \mathcal{E}^{\beta^*}_{p,\infty}(u)$. Letting $A \to \mathcal{E}^{\beta^*}_{p,\infty}(u)$, we obtain

$$\liminf_{\beta \uparrow \beta^*}(\beta^* - \beta)\mathcal{E}^{\beta}_{p,p}(u) \geq \frac{\beta^*}{\log(\sup_{n\geq 1} t_{l_n})}\mathcal{E}^{\beta^*}_{p,\infty}(u). \tag{4.45}$$

Then (1.5) follows by combining (4.45) with (4.44). The convergence in (1.6) is a consequence of (1.5) and Theorem 1.4 (1). ■

Remark 4.13. If we define the *Korevaar–Schoen* norms by

$$\|u\|_{\mathrm{KS}^{\beta}_{p,\infty}} := \limsup_{r\to 0} \Phi^{\beta}_{u}(r),$$

then its corresponding BBM convergence, that is, (1.6) with $\|\cdot\|_{\mathrm{KS}^{\beta^*}_{p,\infty}}$ in place of $\|\cdot\|_{B^{\beta^*}_{p,\infty}}$, also holds by (4.39) and (4.4).

5. Further discussions

We discuss some possible extensions of our results and assume that $\boldsymbol{l} = (l_n)_{n\geq 1}$ satisfies $\sup_{n\geq 1} l_n < \infty$.

In a very recent paper, Yang [41] shows that the *p-resistance estimate* is equivalent to the conjunction of *p-Poincaré inequality* and *p-cutoff Sobolev inequality* under the so-called *slow volume regular condition*. Yang also verifies the 2-cutoff Sobolev inequality in [40] for standard self-similar Vicsek sets. These conditions are also satisfied for scale-irregular Vicsek sets. To see this, we first construct a strictly increasing version of ψ, termed $\widetilde{\psi}$, as follows:

$$\widetilde{\psi}(r) := \begin{cases} \frac{\psi(\rho_n)-\psi(\rho_{n+1})}{\rho_n-\rho_{n+1}} \cdot (r-\rho_{n+1}) + \psi(\rho_{n+1}), & \text{for } \rho_{n+1} < r \leq \rho_n \ (n \geq 0), \\ \frac{1}{2}r, & \text{for } r \geq 2; \end{cases}$$

and we define

$$\widetilde{\phi}(r) = r^{p-1}\widetilde{\psi}(r).$$

It is easy to check that $\widetilde{\psi}$ and $\widetilde{\phi}$ are strictly increasing, thus their inverses $\widetilde{\psi}^{-1}$ and $\widetilde{\phi}^{-1}$ exist. By a direct computation, we see that for all $0 < r \leq 2$,

$$(2\sup_n l_n - 1)^{-1}\psi(r) \leq \widetilde{\psi}(r) \leq \psi(r),$$
$$(\sup_n l_n)^{-p+1}(2\sup_n l_n - 1)^{-1}\phi(r) \leq \widetilde{\phi}(r) \leq \phi(r).$$

By Proposition 2.6-(2) and the fact that

$$\frac{\widetilde{\psi}(R)}{\widetilde{\psi}(r)} = \left(\frac{r}{R}\right)^{p-1}\frac{\widetilde{\phi}(R)}{\widetilde{\phi}(r)} \text{ for any } 0 < r \leq R < 2,$$

we know that the slow volume regular condition $\mathrm{SVR}(\widetilde{\psi}, \widetilde{\phi})$ in [41] holds. By Proposition 3.11, the resistance estimate $\mathrm{R}(\widetilde{\psi}, \widetilde{\phi})$ in [41] holds. An application of [41, Theorem 2.3] immediately gives the next proposition.

Proposition 5.1. *Assume that $\boldsymbol{l} = (l_n)_{n\geq 1}$ satisfies $\sup_{n\geq 1} l_n < \infty$. Then for any $p \in (1, \infty)$, the p-Poincaré inequality $\mathrm{PI}(\widetilde{\phi})$ and the p-cutoff Sobolev inequality $\mathrm{CS}(\widetilde{\phi})$ hold for $(\mathcal{E}_p, \mathcal{F}_p)$ on the scale-irregular Vicsek set $K^{\boldsymbol{l}}$.*

In the special case $p = 2$, by combining Proposition 5.1 with [19, Theorem 1.2], or by combining Proposition 3.11 with [33, Theorem 3.1] (see also [30, Theorem 15.10]), we have the sub-Gaussian type heat kernel estimate for the strongly local, regular Dirichlet form $(\mathcal{E}_2, \mathcal{F}_2)$ on $L^2(K^{\boldsymbol{l}}, \mu)$, or equivalently, for the associated *Hunt process*, which is a diffusion on $K^{\boldsymbol{l}}$. More precisely, there exist constants $C, c, c', \delta > 0$ and a jointly continuous heat kernel $p_t(x, y)$ satisfying the two-sided estimate

$$\begin{aligned} &\frac{c'}{\mu(B(x, \widetilde{\phi}^{-1}(t)))} \mathbb{1}_{\{d(x,y)\leq \delta\widetilde{\phi}^{-1}(t)\}} \leq p_t(x, y) \\ &\quad \leq \frac{C}{\mu(B(x, \widetilde{\phi}^{-1}(t)))} \exp\Big(-\frac{1}{2} t\, \Phi\Big(c \frac{d(x, y)}{t}\Big)\Big), \end{aligned} \tag{5.1}$$

for any $x, y \in K^{\boldsymbol{l}}$ and any $t > 0$, where

$$\Phi(s) := \sup_{r>0} \Big\{ \frac{s}{r} - \frac{1}{\widetilde{\phi}(r)} \Big\}.$$

Let us sketch some other possible extensions of some recent related works.

(1) (*Heat-kernel based p-energy norms*) By [16, Lemma 1.3.4], it is known that

$$\mathcal{E}_2(u) = \lim_{t\to 0+} \frac{1}{2t} \int_{K^{\boldsymbol{l}}} \int_{K^{\boldsymbol{l}}} |u(x) - u(y)|^2 p_t(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}\mu(x).$$

One can define an alternative p-energy functional $\widetilde{\mathcal{E}}_p^{\beta}$ based on the heat kernel associated with $(\mathcal{E}_2, \mathcal{F}_2)$, given formally by

$$\widetilde{\mathcal{E}}_p^{\beta}(u) := \sup_{t\in(0,2^{p-1})} \frac{1}{t^{\beta/\beta^*}} \int_{K^{\boldsymbol{l}}} \int_{K^{\boldsymbol{l}}} |u(x) - u(y)|^p p_t(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}\mu(x)$$

with domain consisting of all continuous functions u for which $\widetilde{\mathcal{E}}_p^{\beta}(u) < \infty$. Using the cake-layer decomposition

$$K^{\boldsymbol{l}} = \bigcup_{n=1}^{\infty} \big(B\big(x, 2^n\widetilde{\phi}^{-1}(t)\big) \setminus B\big(x, 2^{n-1}\widetilde{\phi}^{-1}(t)\big)\big) \cup B\big(x, \widetilde{\phi}^{-1}(t)\big)$$

and the two-sided heat kernel bounds (5.1), one can show that

$$\widetilde{\mathcal{E}}_p^{\beta}(u) \asymp [u]^p_{B^{\beta}_{p,\infty}}.$$

Thus $\widetilde{\mathcal{E}}_p^{\beta^*}$ is equivalent to the energy $\mathcal{E}_p$ introduced in Theorem 1.1 by Theorem 1.4 (1), in the sense that the two functionals have the same domain and comparable energy norms.

(2) (*Gradient estimates and Hodge structure*) Recall the operator ∂ in Proposition 3.5. Based on the two-sided bounds of heat kernel (5.1), we expect that, on scale-irregular Vicsek sets, gradient estimates for the associated heat kernel and the corresponding Hodge structure can be developed by extending the approach in [6].

(3) (*p-Laplacian and PDEs*) Moreover, as in [6, Lemma 3.9], the (2-)Laplacian (or generator) Δ_2 for $(\mathcal{E}_2, \mathcal{F}_2)$ can be expressed by ∂ and its formal adjoint ∂^*. For the nonlinear case $p \neq 2$, the Fréchet differentiability of $\mathcal{E}_p$ (see [28, Theorem 3.7] and Remark 3.9) ensures that

$$\mathcal{E}_p(f; g) := \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(f + tg)\Big|_{t=0} \in \mathbb{R} \text{ exists for all } f, g \in \mathcal{F}_p.$$

This naturally suggests a definition of the *p-Laplacian operator* Δ_p on scale-irregular Vicsek sets, following the framework of Strichartz and Wong [38]. It is an intriguing question whether Δ_p can also be expressed in terms of by ∂ and ∂^*. Furthermore, a corresponding *variational principle* for p-energies could be established in analogy with [38, Theorem 3.1]. In particular, the presence of a gradient operator ∂ is expected to give finer analysis of p-harmonic functions and regularity properties of solutions to nonlinear PDEs. These investigations could be pursued within the function spaces established in Theorem 1.1, Definition 1.3, and Definition 4.2, using methods adapted from the theory of nonlinear elliptic PDEs under the fractal setting.

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Aobo Chen
Department of Mathematical Sciences, Tsinghua University, 100084 Beijing, P. R. China; cab21@mails.tsinghua.edu.cn

Jin Gao
Department of Mathematics, Hangzhou Normal University, 310036 Hangzhou, P. R. China; gaojin@hznu.edu.cn

Zhenyu Yu
Department of Mathematics, College of Science, National University of Defense Technology, 410073 Changsha, P. R. China; yuzy23@nudt.edu.cn

Junda Zhang
School of Mathematics, South China University of Technology, 510641 Guangzhou, P. R. China; summerfish@scut.edu.cn