J. Fractal Geom. 13 (2026), 113–149
DOI 10.4171/JFG/167

© 2025 European Mathematical Society
Published by EMS Press
This work is licensed under a CC BY 4.0 license

Noncommutative geometry on the Berkovich projective line

Masoud Khalkhali and Damien Tageddine

Abstract. In this paper, we construct several C^*-algebras associated to the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$. In the commutative setting, we construct a spectral triple as a direct limit over finite $\mathbb{R}$-trees. More general C^*-algebras generated by partial isometries are also presented. We use their representations to associate a Perron–Frobenius operator and a family of projection-valued measures. Finally, we show that invariant measures, such as the Patterson–Sullivan measure, can be obtained as KMS-states of the crossed product algebra with a Schottky subgroup of $\mathrm{PGL}_2(\mathbb{C}_p)$.

1. Introduction

The aim of this paper is to show that geometries over non-Archimedean fields provide natural examples of noncommutative geometries; we can refer to the work [12, 13] for similar approaches. We focus on the Berkovich projective line, $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$, as a fundamental example of Berkovich's theory. For further details on noncommutative geometry, we refer to the books [7, 8]; for an extensive exploration of the interplay between number theory and noncommutative geometry, we refer to the series of works [8–10].

The primary objective of this work is to construct and analyze several C^*-algebras and spectral triples that can be associated to the Berkovich projective line. These C^*-algebras are expected to encapsulate some essential geometric and arithmetic features of $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ in a manner analogous to how C^*-algebras encode information about classical spaces.

Let K be an algebraically closed field that is complete with respect to a nontrivial non-Archimedean absolute value $|\cdot|$. In Section 2, we begin by reviewing the definition of the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk}}(K)$ and its fundamental properties. In particular, after restriction to the case $K = \mathbb{C}_p$, we recall the classification of the points of $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ into four types according to Berkovich's classification theorem.

Mathematics Subject Classification 2020: 58B34.
Keywords: Berkovic space, non-Archimedean field, spectral triple, noncommutative geometry, p-adic, countable alphabet, C^*-algebra, Ważewski dendrite.

We focus on the $\mathbb{R}$-tree structure obtained from a projective limit of finite trees and refer to [1] for more details. We also introduce a hyperbolic metric structure and the definition of the hyperbolic space denoted by $\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)$.

In Section 3, we provide a first construction of a commutative spectral triple represented by the datum $(C_{\mathrm{Lip}}(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)), \mathcal{H}, D)$ as an inverse limit of finite spectral triples associated with finite trees. The result is summarized in Theorem 3.9. We closely follow the general construction of an inverse limit of spectral triples given in [16].

In Section 5, we propose an alternative construction of C^*-algebra associated to $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ relying on the identification of the projective line with the Ważewski universal dendrite introduced in Section 4. The C^*-algebra $\mathcal{O}_{\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)}$ is generated by partial isometries indexed by the branching points of the universal dendrite. It is in fact the full shift C^*-algebra associated to a countable alphabet, with letters given by the rational numbers in $(0, 1)$. This type of algebra bears similar properties than the Cuntz–Krieger algebras, see for instance [22, 24]. The results are summarized in Proposition 5.2 and Theorem 5.6. Using the representation space, we also define a Perron–Frobenius operator and projection-valued measures.

Finally, in Section 5.2, we use the fact that $\mathrm{PGL}_2(\mathbb{C}_p)$ is the isometry group of $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$. We exhibit a unitary representation of $\mathrm{PGL}_2(\mathbb{C}_p)$ through the left action on the projective line. This allows us to identify the boundary of $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ as the limit set of the action. The boundary coincides with the p-adic. We construct the crossed product C^*-algebras, $C_{\mathrm{Lip}}(\mathbb{P}^1(\mathbb{C}_p)) \rtimes \Gamma$ with a Schottky group. The Patterson–Sullivan measure is obtained as a KMS-state in Theorem 6.11.

The long-term goal of the project we start in this paper is to provide a unified approach to the noncommutative geometry treatment of number theory in both the Archimedean and non-Archimedean settings, as studied in the papers [9, 10, 12, 13]. This follows the general philosophy of Berkovich aiming at describing analogues of complex analytic spaces in the complex p-adic case, providing a p-adic analytic space where such a theory can be defined. Methods of noncommutative geometry, specially spectral triples and Bost–Connes systems, have proved to be powerful to capture the geometry of such *wild spaces* that often appear in number theory [11].

Apart from potential applications to number theory, the class of C^*-algebras introduced in this paper and that appears in the potential theory and dynamics of rational maps on the Berkovich projective line provide new examples of infinite graph C^*-algebras. In a different direction, we would like to mention that the present work hopefully provides a framework to test and rigorously define a version of the AdS/CFT correspondence along the lines of [19, 21].

2. The Berkovich projective line

In this section, we recall the definition of *multiplicative seminorms* and introduce the construction of Berkovich spaces.

2.1. The multiplicative seminorms and Berkovich spaces

Definition 2.1. Let K be a field with absolute value $|\cdot|$ and let A be a K-algebra. A *multiplicative seminorm* on A is a map

$$A \to \mathbb{R}_{\geq 0}, \quad a \mapsto \|a\|$$

such that

(1) $\|\cdot\|$ restricts to $|\cdot|$ on K;

(2) $\|a+b\| \leq \|a\| + \|b\|$ for all $a, b \in A$;

(3) $\|ab\| = \|a\| \cdot \|b\|$ for all $a, b \in A$.

If in addition $a = 0$ is the only element with $\|a\| = 0$, then $\|\cdot\|$ is a *multiplicative norm* (which is the same as an absolute value on A extending $|\cdot|$). We call

$$\ker\|\cdot\| = \{a \in A : \|a\| = 0\}$$

the kernel of $\|\cdot\|$; it is a prime ideal of A.

A K-algebra A with a fixed multiplicative norm such that A is complete with respect to this norm is a *Banach algebra* over K.

Definition 2.2. Let $(K, |\cdot|)$ be a field with absolute, let $a \in K$ and $r \geq 0$. Then,

$$D(a, r) := \{\xi \in K : |\xi - a| \leq r\}$$

is the closed disk of radius r around a.

Example 2.3. Let $(K, |\cdot|)$ be a complete non-Archimedean field.

(1) For any $a \in K$, the map

$$f \mapsto \|f\|_{a,0} := |f(a)|$$

is a multiplicative seminorm on $K[x]$.

(2) For any $a \in K$ and any $r > 0$, the map

$$f \mapsto \|f\|_{a,r} := |f(x+a)|_r$$

is a multiplicative (semi)norm on $K[x]$.

(3) Let (a_n) be a sequence in K and (r_n) a strictly decreasing sequence in $\mathbb{R}_{>0}$ such that $D(a_{n+1}, r_{n+1}) \subset D(a_n, r_n)$ for all n. Then

$$f \mapsto \|f\| = \lim_{n\to\infty} \|f\|_{a_n,r_n}$$

is a multiplicative seminorm on $K[x]$.

Definition 2.4. Let K be a complete and algebraically closed non-Archimedean field and let A be a finitely generated K-algebra, so that A is the coordinate ring of the affine K-variety $X = \mathrm{Spec}(A)$. Then the *Berkovich space* associated to A or X is

$$\mathrm{Berk}(A) := X_{\mathrm{Berk}} := \left\{\|\cdot\| : A \to \mathbb{R} \text{ multiplicative seminorm on } A\right\} \tag{2.1}$$

the set of multiplicative seminorms on A. The topology on X_{Berk} is the Gelfand topology, i.e., the weakest topology that makes the maps $X_{\mathrm{Berk}} \to \mathbb{R}$, $\|\cdot\| \to \|f\|$, continuous for all $f \in A$. (Concretely, this means that any open set is a union of finite intersections of sets of the form $U_{f,a,b} = \{\|\cdot\| \in X_{\mathrm{Berk}} : a < \|f\| < b\}$.)

Elements of X_{Berk} are called points. For a point $\xi \in X_{\mathrm{Berk}}$, the corresponding seminorm is denoted $\|\cdot\|_\xi$.

2.2. Berkovich's classification theorem

Theorem 2.5 (Berkovich's classification theorem). *Every point $x \in \mathbb{A}^1_{\mathrm{Berk}}(K)$ corresponds to a nested sequence $D(a_1, r_1) \supseteq D(a_2, r_2) \supseteq D(a_3, r_3) \supseteq \cdots$ of closed disks, in the sense*

$$\|f\|_x = \lim_{n\to\infty} \|f\|_{D(a_n,r_n)}.$$

Two such nested sequences define the same point of $\mathbb{A}^1_{\mathrm{Berk}}(K)$ if and only if

(a) *each has a nonempty intersection, and their intersections are the same; or*

(b) *both have empty intersection, and the sequences are cofinal.*

This brings us to Berkovich's classification of elements of $\mathbb{A}^1_{\mathrm{Berk}}(K)$ into four types of points according to the nature of $D(a, r) = \cap\, D(a_n, r_n)$:

(I) $D(a, r)$ is a point ($r = \lim r_i = 0$).

(II) $D(a, r)$ is a closed disk of radius $r = \lim r_i > 0$ belonging to the value group $|K^*|$ of K. The corresponding discs $D(a, r)$ are called *"rational"*.

(III) $D(a, r)$ is a closed disk of radius $r = \lim r_i > 0$ that does not belong to the value group $|K^*|$ of K. The corresponding discs $D(a, r)$ are called *"irrational"*.

(IV) $D(a, r)$ is empty. As noted above, necessarily $\lim r_i > 0$.

Remark 2.6. Berkovich's classification theorem is crucial to understand the action of the automorphism group of the tree. Specifically, the type of a point is invariant under the action of the group. Moreover, type II points are dense in the tree, therefore one can understand the action by restriction to the branching points.

2.3. The Berkovich affine line $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$

In this section and in the rest of the paper, we consider (unless stated otherwise) the Berkovich affine line over $\mathbb{C}_p$, $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p) = \mathrm{Berk}(\mathbb{C}_p[x])$. According to the Berkovich classification of points, the affine line $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ possesses four types of points. Type I points are the classical points $\mathbb{A}^1(\mathbb{C}_p) = \mathbb{C}_p$. Type II and III correspond to closed disks $D(a, r)$ with $r > 0$ in, respectively, not in, the value group $p^{\mathbb{Q}}$ of $\mathbb{C}_p$. We use the notation $\zeta_{a,r}$ for these points. By extension, we identify the points a of type I with $\zeta_{a,0}$.

The Berkovich affine line is endowed with a natural partial order. For any two points $\xi, \xi' \in \mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$, we say that:

$$\xi \leq \xi' \quad \text{if and only if} \quad \forall f \in \mathbb{C}_p[x] : \ \|f\|_\xi \leq \|f\|_{\xi'}.$$

In particular, for two disks D_1 and D_2, $\|\cdot\|_{D_1} \leq \|\cdot\|_{D_2}$ if and only if $D_1 \subset D_2$. The relation $\leq$ defines a partial order on the affine line. For each pair of points $\xi, \xi' \in \mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$, there is a unique least upper bound $\xi \vee \xi' \in \mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ with respect to this partial order. Moreover, if $\xi \leq \xi'$, then we write $[\xi, \xi']$ for the set of points ξ'' such that $\xi \leq \xi'' \leq \xi'$. If the points ξ and ξ' are of type I, II or III, then the set $[\xi, \xi']$ is homeomorphic to a closed interval in $\mathbb{R}$.

Proposition 2.7. *The space $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ is Hausdorff, locally compact and uniquely path-connected.*

We can now introduce different metrics on the Berkovich space $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$.

Definition 2.8. The diameter of a point $\xi \in \mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ is

$$\mathrm{diam}(\xi) := \inf\{\|x - a\|_\xi : a \in \mathbb{C}_p\}.$$

Therefore, points of types I, II and III have a diameter $\mathrm{diam}(\zeta_{a,r}) = r$, and for a type IV point ξ represented by a nested sequence of disks $D(a_n, r_n)$, we have

$$\mathrm{diam}(\xi) = \lim_{n\to\infty} r_n > 0.$$

There are two metrics that one can define using the diam map.

Definition 2.9. For any couple of points $\xi, \xi' \in \mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$, we define the *small metric* by

$$d(\xi, \xi') = 2\,\mathrm{diam}(\xi \vee \xi') - \mathrm{diam}(\xi) - \mathrm{diam}(\xi').$$

In the case that ξ, ξ' are both not of type I, the *big metric* is defined by

$$\rho(\xi, \xi') = 2\log_v \operatorname{diam}(\xi \vee \xi') - \log_v \operatorname{diam}(\xi) - \log_v \operatorname{diam}(\xi').$$

The small metric measures the total variation of diameter as we move along the path from ξ to ξ'. The big metric is the analogue of the hyperbolic metric on the upper half-plane $ds^2 = (dx^2 + dy^2)/y^2$ in $\mathbb{C}$. In fact, the *hyperbolic part* of $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ consists of points of types II, III and IV. The big metric is invariant under the action of $\mathrm{PGL}_2(\mathbb{C}_p)$, analogous to the fact that the standard hyperbolic metric is invariant under the action of $\mathrm{PSL}_2(\mathbb{R})$. We can now introduce the tree-like structure of the Berkovich line.

Definition 2.10 ($\mathbb{R}$-tree). An $\mathbb{R}$-tree is a metric space (T, d) such that for any two points $x, y \in T$, there exists a unique path $[x, y]$ in T joining x to y, which is a geodesic segment, i.e., the map $\gamma : [a, b] \to T$ giving the path can be chosen so that

$$d\big(\gamma(u), \gamma(v)\big) = |u - v| \text{ for all } u, v \in [a, b].$$

A point $x \in T$ is a *branch point* if $T \setminus \{x\}$ has at least three connected components in the metric topology. A point x is an *endpoint* if $T \setminus \{x\}$ is connected. An *ordinary* point x is a point for which $T \setminus \{x\}$ has exactly two connected components.

The metric topology on T is called the *strong topology*. The *weak topology* is defined using the *tangent direction* at $x \in T$. A *tangent direction* is an equivalence class of path $[x, y]$ with $x \neq y$, where two paths are equivalent if they share an initial segment. The connected components of $T \setminus \{x\}$ are in one-to-one correspondence with the tangent directions at x. For a tangent direction v at x, we define

$$U(x, v) = \big\{y \in T : y \neq x, [x, y] \in v\big\}.$$

The sets $U(x, v)$ are a subbasis for the *weak topology*.

Theorem 2.11. *The following statements hold:*

(1) *The Berkovich line* $(\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p), d)$ *equipped with the small metric* d *has a* $\mathbb{R}$*-tree structure. The Berkovich topology on* $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ *is the weak topology on this* $\mathbb{R}$*-tree.*

(2) *The space* $(\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p) \setminus \mathbb{C}_p, \rho)$ *equipped with the big metric* ρ *has a* $\mathbb{R}$*-tree structure. The subspace topology on* $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p) \setminus \mathbb{C}_p$ *is the weak topology on this* $\mathbb{R}$*-tree.*

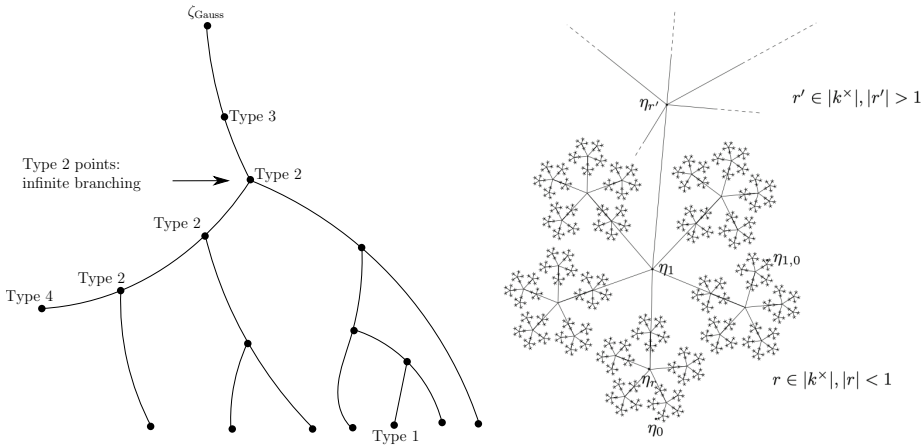

Figure 1. The Berkovich affine line. Picture on the right is taken from [26].

2.4. The Berkovich projective line $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$ and the $\mathbb{R}$-tree structure

As a set, the Berkovich projective line, denoted by $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$, is obtained by adding a type I point, denoted ∞ to the affine line $\mathbb{A}^1_{\text{Berk}}(\mathbb{C}_p)$. The set $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$ is then equipped with the one-point compactification topology.

The Berkovich hyperbolic line $\mathbb{H}_{\text{Berk}}(\mathbb{C}_p)$ is a subset of $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$ consisting of all points of type II, III, and IV. One also defines $\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p)$, the set of all type II points, and $\mathbb{H}^{\mathbb{R}}_{\text{Berk}}(\mathbb{C}_p)$ for the set of points of type II and III.

Proposition 2.12. *The subset* $\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p)$ *is dense in* $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$.

The projective line $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$ also inherits a $\mathbb{R}$-tree structure. It is the compactification of $\mathbb{A}^1_{\text{Berk}}(\mathbb{C}_p)$ seen as an $\mathbb{R}$-tree. Following the description of Baker and Rumely [1], one can *navigate* on the Berkovich projective line as follows. One starts from the so-called Gauss point $\zeta_{0,1}$, corresponding to the unit disk which is a type II point, and chooses between infinitely many (countable) branches in which to travel; there is one branch for each element of the residue field $\overline{\mathbb{F}}_p$. On the chosen direction, at each point of type II there are again infinitely many new branches in which to travel.

We can classify the points in the $\mathbb{R}$-tree structure as follows:

(1) Points of types I and IV are endpoints.

(2) Points of type II are the only branching points, with infinite (countable) tangent directions in bijection with the residue field $\mathbb{P}^1(\overline{\mathbb{F}}_p)$.

(3) Points of type III are ordinary points.

There is an important description of the Berkovich projective line as a *profinite* $\mathbb{R}$*-tree,* i.e., an inverse limit of $\mathbb{R}$-trees. Indeed, $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$ is homeomorphic to the inverse limit $\varprojlim \Gamma$ over all finite $\mathbb{R}$-trees $\Gamma \subset \mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$. More precisely, let

$$S = \{D(a_1, r_1), \dots, D(a_n, r_n)\}$$

be a finite set of (rational or irrational) discs of positive radius contained in $D(0,1)$. For simplicity, we assume that $D(0,1) \notin S$. To each disc $D(a_i, r_i)$, there is an associated point $\zeta_{a_i,r_i} \in \mathbb{A}^1_{\text{Berk}}(\mathbb{C}_p)$, which is a point of type II or III.

Define the *graph of discs* Γ_S to be the union of the associated lines of discs $[\zeta_{a_i,r_i}, \zeta_{\text{Gauss}}]$:

$$\Gamma_S = \bigcup_{i=1}^{n} [\zeta_{a_i,r_i}, \zeta_{\text{Gauss}}].$$

If S_1 and S_2 are any two finite sets of discs, then Γ_{S_1} and Γ_{S_2} are both $\mathbb{R}$-subtrees of $\Gamma_{S_1 \cup S_2}$. Moreover, the embedding of Γ_{S_i} in $\Gamma_{S_1 \cup S_2}$ is an isometry (with respect to either metric). Let $\mathcal{F}$ be the collection of all finite graphs of the form Γ_S as above. Then, $\mathcal{F}$ is a directed set under inclusion and we write $\Gamma \leq \Gamma'$ if $\Gamma \subseteq \Gamma'$ as subsets of $D(0,1)$. Thus, whenever $\Gamma \leq \Gamma'$ there is an inclusion map $i_{\Gamma,\Gamma'} : \Gamma \to \Gamma'$.

Similarly, write $\mathcal{F}^{\mathbb{Q}}$ (resp. $\mathcal{F}^{\mathbb{R}}$) for the subset of $\mathcal{F}$ consisting of graphs which are union of all arcs connecting two points in $\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}$ (resp. $\mathbb{H}^{\mathbb{R}}_{\text{Berk}}$).

There is also a retraction map $r_{\Gamma',\Gamma} : \Gamma' \to \Gamma$ defined whenever $\Gamma \leq \Gamma'$. This is a general property of $\mathbb{R}$-trees: since there is a unique path between any two points of Γ', if $x \in \Gamma'$ we can define $r_{\Gamma',\Gamma}(x)$ to be, for any $y \in \Gamma$, the first point where the unique path in Γ' from x to y intersects Γ. This definition is independent of the choice of y, and one sees from the definition that $r_{\Gamma',\Gamma}(x) = x$ if and only if $x \in \Gamma$. In particular, $r_{\Gamma',\Gamma}$ is surjective.

Theorem 2.13. *There is a canonical homeomorphism*

$$\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p) \simeq \varprojlim_{\Gamma \in \mathcal{F}} \Gamma.$$

Then $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$ *is a compact Hausdorff space identified with the inverse limit of its finite subgraphs with the inverse limit topology.*

We can also consider the direct limit with respect to the inclusion maps $i_{\Gamma,\Gamma'}$. In this case, the limit is isomorphic (as a set) to $\mathbb{H}_{\text{Berk}}(\mathbb{C}_p)$.

Theorem 2.14. *There are canonical bijections*

$$\mathbb{H}_{\text{Berk}}(\mathbb{C}_p) \simeq \varinjlim_{\Gamma\in\mathcal{F}} \Gamma$$

$$\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p) \simeq \varinjlim_{\Gamma\in\mathcal{F}^{\mathbb{Q}}} \Gamma, \quad \mathbb{H}^{\mathbb{R}}_{\text{Berk}} \simeq \varinjlim_{\Gamma\in\mathcal{F}^{\mathbb{R}}} \Gamma.$$

In other words, the branching points $\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p)$ is identified with the following set

$$\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p) = \big\{(\zeta_{a_1,r_1}, \zeta_{a_2,r_2}, \dots \zeta_{a_n,r_n}, \zeta_{a_n,r_n}, \dots) : \text{s.t. } \zeta_{a_i,r_i} \in \text{Type II for all } i\big\}.$$

We refer to [1] for more details on this construction.

3. A commutative spectral triple on $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$

3.1. Spectral triples on finite $\mathbb{R}$-trees

Consider a finite $\mathbb{R}$-tree $\Gamma \in \mathcal{F}$ such that $\Gamma = (\mathcal{V}, \mathcal{E})$. For any a vertex v, denote by n_v the number of adjacent vertices to v. We construct a spectral triple over Γ, denoted by $(A_\Gamma, \mathcal{H}_\Gamma, D_\Gamma)$. Let A_Γ be the *-algebra of (Lipschitz continuous functions) $C_{\text{Lip}}(\Gamma)$ on (Γ, ρ). The representation space is given by

$$\mathcal{H}_\Gamma = \oplus_{v\in\mathcal{V}} \mathcal{H}_v \qquad \text{such that} \qquad \mathcal{H}_v = \ell^2(\mathcal{V}) \otimes \mathbb{C}^{2n_v}. \tag{3.1}$$

We define the *-representation π_Γ of $C_{\text{Lip}}(\Gamma)$ on $\mathcal{H}_\Gamma$ by:

$$\pi(f)\psi(v) = \oplus_{v_+\sim v} \begin{pmatrix} f(v_+) & 0 \\ 0 & f(v) \end{pmatrix} \psi(v). \tag{3.2}$$

The Dirac operator D_Γ is an operator on $\mathcal{H}_\Gamma$ and is given by

$$D_\Gamma \psi(v) = \oplus_{v_+\sim v} \frac{1}{\rho(v, v_+)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi(v). \tag{3.3}$$

The grading operator γ_Γ restricted on $\mathcal{H}_v$ is given by

$$\gamma_\Gamma|_{\mathcal{H}_v} = 1_{\ell^2(\mathcal{V})} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1_{n_v}. \tag{3.4}$$

Proposition 3.1. *π is a faithful $*$-representation of $C_{\text{Lip}}(\Gamma)$.*

Proof. It follows immediately that π is a $*$-representation for $C_{\text{Lip}}(\Gamma)$. It is bounded, since f is continuous and Γ is finite. Moreover, if $\pi(f) = 0$ then f vanishes on the graph Γ. ∎

Proposition 3.2. $(C_{\mathrm{Lip}}(\Gamma), \mathcal{H}_\Gamma, D_\Gamma, \gamma_\Gamma)$ *is an even spectral triple.*

Proof. It follows by definition that D_Γ is a self-adjoint operator. We then check that $\gamma_\Gamma^* = \gamma_\Gamma$, $\gamma_\Gamma^2 = \gamma_\Gamma$, $\gamma_\Gamma D_\Gamma = -D_\Gamma \gamma_\Gamma$ and $\Gamma\pi(f) = \pi(f)\Gamma$ for all $f \in C_{\mathrm{Lip}}(\Gamma)$. ∎

3.2. Inverse limit of spectral triples

Consider two finite $\mathbb{R}$-trees Γ and Γ' such that $\Gamma \leq \Gamma'$. As mentioned, there exists a surjective map $r_{\Gamma\Gamma'} : \Gamma' \to \Gamma$. Therefore, we can define the pullback map:

$$r^*_{\Gamma\Gamma'} : C_{\mathrm{Lip}}(\Gamma) \to C_{\mathrm{Lip}}(\Gamma'), \qquad r^*_{\Gamma\Gamma'}(f)(x) = f\big(r_{\Gamma\Gamma'}(x)\big) \tag{3.5}$$

which satisfies by construction the following statement.

Proposition 3.3. *The pullback map* $r^*_{\Gamma\Gamma'}$ *is injective.*

On the other hand, the inclusion induces a pushforward map $\iota : \Gamma \to \Gamma'$ which is an inclusion of the representation spaces:

$$\iota_{\Gamma\Gamma'} : \mathcal{H}_\Gamma \to \mathcal{H}_{\Gamma'}, \qquad \iota_{\Gamma\Gamma'}(\psi)(x) = \begin{cases} \psi(x) & \text{if } x \in \iota_{\Gamma\Gamma'}(\Gamma), \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

We consider the triple $(A_\Gamma, \mathcal{H}_\Gamma, D_\Gamma)$.

Definition 3.4. A morphism between two spectral triples $(A_i, \mathcal{H}_i, D_i)$ for $i = 1, 2$ is a pair (ϕ, I) consisting of a unital $*$-homomorphism $\phi : A_1 \to A_2$ and a bounded linear operator $I : \mathcal{H}_1 \to \mathcal{H}_2$ satisfying the following conditions:

(1) $\phi(A_1^\infty) \subseteq A_2^\infty$ where A_1^∞ and A_2^∞ are defined in [16];

(2) $I\pi_1(a) = \pi_2(\phi(a))I$, for every $a \in A_1$;

(3) $I(\mathrm{Dom}(D_1)) \subseteq \mathrm{Dom}(D_2)$ and $ID_1 = D_2 I$.

A morphism (ϕ, I) is said to be isometric if ϕ is injective and I is an isometry. A morphism between two even spectral triples $(A_i, \mathcal{H}_i, D_i, \gamma_i)$ for $i = 1, 2$ is a pair (ϕ, I) satisfying the additional condition:

(4) $I\gamma_1 = \gamma_2 I$.

Proposition 3.5. *The pair* $(r^*_{\Gamma\Gamma'}, \iota_{\Gamma\Gamma'})$ *is a morphism of spectral triples between the two spectral triples* $(A_\Gamma, \mathcal{H}_\Gamma, D_\Gamma)$ *and* $(A_{\Gamma'}, \mathcal{H}_{\Gamma'}, D_{\Gamma'})$.

Proof. We check the different points given in the above definition.

(1) For any $a \in A_\Gamma$, the commutator $[D, \pi_\Gamma(a)]$ is bounded on $\mathcal{H}_\Gamma$. Similarly, the commutator $[D, \pi_{\Gamma'}(a)]$ is bounded on $\mathcal{H}_{\Gamma'}$. Therefore, $A_\Gamma^\infty = A_\Gamma$ and $A_{\Gamma'}^\infty = A_{\Gamma'}$.

(2) Using the fact that $\iota_{\Gamma\Gamma'}(\psi)(v) \neq 0$ iff $v \in \iota_{\Gamma\Gamma'}(\Gamma)$ and the definition of the representation, we check that:

$$\iota_{\Gamma\Gamma'}(\pi_1(f)\psi)(v) = \oplus_{\iota_{\Gamma\Gamma'}(v_+)\sim\iota_{\Gamma\Gamma'}(v)} \begin{pmatrix} f(\iota_{\Gamma\Gamma'}(v_+)) & 0 \\ 0 & f(\iota_{\Gamma\Gamma'}(v)) \end{pmatrix} \psi_{\iota_{\Gamma\Gamma'}}(v)$$

$$\pi_2(r^*_{\Gamma\Gamma'}(f))\iota_{\Gamma\Gamma'}(\psi)(v) = \oplus_{v_+\sim v} \begin{pmatrix} f \circ r_{\Gamma\Gamma'}(v_+) & 0 \\ 0 & f \circ r_{\Gamma\Gamma'}(v) \end{pmatrix} \psi_{\iota_{\Gamma\Gamma'}}(v)$$

and for $v \in \iota_{\Gamma\Gamma'}(\Gamma)$, we have $r_{\Gamma\Gamma'}(v) = v$.

(3) Since $\iota_{\Gamma\Gamma'}(\ell^2(\Gamma)) \subseteq \ell^2(\Gamma')$, then $\iota_{\Gamma\Gamma'}(\mathrm{Dom}(D_1)) \subseteq \mathrm{Dom}(D_2)$ and

$$\iota_{\Gamma\Gamma'}(D\psi) = D\iota_{\Gamma\Gamma'}(\psi).$$

In addition, the map $\iota_{\Gamma\Gamma'}$ is an isometry. ■

Definition 3.6. Let $(J, \leq)$ be a directed index set and $(A_j, \mathcal{H}_j, D_j)_{j\in J}$ be a family of spectral triples. Suppose that for every $j, k \in J$ and $j \leq k$ there exists an isometric morphism (ϕ_{jk}, I_{jk}) from $(A_j, \mathcal{H}_j, D_j)$ to $(A_k, \mathcal{H}_k, D_k)$ satisfying $\phi_{kl}\phi_{jk} = \phi_{jl}$ and $I_{kl}I_{jk} = I_{jl}$, for all $j, k, l \in J$ and $j \leq k \leq l$. The resulting system

$$\big\{(A_j, \mathcal{H}_j, D_j), (\phi_{jk}, I_{jk})\big\}_J$$

is called an inductive system of spectral triples.

Definition 3.7. The triple $(A, \mathcal{H}, D)$ is called the inductive realization of the inductive system and denoted by $\{(A_j, \mathcal{H}_j, D_j), (\phi_{jk}, I_{jk})\}_J$.

We consider the inverse limit system of finite trees $(\Gamma_j, r_{j,j+1})_{j\in J}$ defining the Berkovich projective space $\mathbb{P}^1_{\mathrm{Berk}}$ obtained as the inverse limit

$$\mathbb{P}^1_{\mathrm{Berk}} = \varprojlim(\Gamma_j, r_{j,j+1})_{j\in J}. \tag{3.7}$$

We define the family of spectral triples $\{(A_j, \mathcal{H}_j, D_j), (r^*_{jk}, \iota_{jk})\}_J$ with the following notation:

$$A_j := C_{\mathrm{Lip}}(\Gamma_j), \quad \mathcal{H}_j = \ell^2(\Gamma_j), \quad D_j = D_{\Gamma_j} \tag{3.8}$$

and with the isometric morphism:

$$r^*_{jk} : A_j \to A_k, \quad \iota_{jk} : \mathcal{H} \to \mathcal{H}_k. \tag{3.9}$$

Proposition 3.8. *We have the following direct limits:*

$$C_{\mathrm{Lip}}\big(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)\big) = \varinjlim C_{\mathrm{Lip}}(\Gamma), \quad \ell^2(\mathbb{P}^1_{\mathrm{Berk}}) = \varinjlim \ell^2(\Gamma), \quad \pi = \varinjlim \pi_\Gamma. \tag{3.10}$$

Therefore, π is the unique representation of the C^*-algebra A on $\mathcal{H}$ such that

$$\pi\big(r_j(a)\big)\iota_j = \iota_j \pi_j(a). \tag{3.11}$$

One can describe the inductive limit of the family of operators $\{D_j\}_{j\in J}$. Following the construction in [16] , we consider the dense domain $\mathscr{D}$ of $\mathcal{H}$,

$$\mathscr{D} = \bigcup_{j\in J} \iota_j\big(\mathrm{Dom}(D_j)\big). \tag{3.12}$$

For every vector $\psi \in \mathscr{D}$ of the form $\psi = \iota_j \psi_j$, where $\psi_j \in \mathrm{Dom}(D_j)$, define

$$D\psi = \iota_j D_j \psi_j. \tag{3.13}$$

It follows that D is a densely defined symmetric operator. Moreover, since the operators D_j are self-adjoint, we have that $\mathrm{Range}(D_j \pm i) = \mathcal{H}_j$, for every $j \in J$. Consequently, $\mathrm{Range}(D \pm i)$ is dense in $\mathcal{H}$, and thus D is essentially selfadjoint. We use the same letter D, or the symbol $D = \varinjlim D_j$, to denote the closure of this essentially selfadjoint operator, and call it the inductive limit of the family of operators $\{D_j\}_{j\in J}$. Similarly, we use the symbol γ to denote the inductive limit $\gamma = \varinjlim \gamma_j$. Putting the various strands together, we consider the following inductive system given by the data $\{(A_j, \mathcal{H}_j, D_j, \gamma_j), (\phi_{j,j+1}, I_{j,j+1})\}_{j\in J}$ and its inductive realization:

$$\big(C_{\mathrm{Lip}}(\mathbb{P}^1_{\mathrm{Berk}}), \ell^2(\mathbb{P}^1_{\mathrm{Berk}}), \pi, D, \gamma\big).$$

Theorem 3.9. *The operator D is self-adjoint with compact resolvent and $[D, \pi(f)]$ is a bounded operator for any Lipschitz continuous function $a \in C_{\mathrm{Lip}}(\mathbb{P}^1_{\mathrm{Berk}})$, i.e., the spectral triple $(C_{\mathrm{Lip}}(\mathbb{P}^1_{\mathrm{Berk}}), \ell^2(\mathbb{P}^1_{\mathrm{Berk}}), \pi, D, \gamma)$ is an even spectral triple.*

Proof. Let us recall that for an integer j, the Dirac operator D_{Γ_j} is defined by

$$D_{\Gamma_j}\psi(v) = \oplus_{v_+\sim v} \frac{1}{\rho(v, v_+)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi(v) \tag{3.14}$$

for any $v \in \Gamma_j$. Then, we can bound the operator D_j using the metric ρ on the finite graph:

$$\|D_j\| = \sup_{e\in\mathcal{E}_j} \frac{1}{\ell(e)} \tag{3.15}$$

where for an edge $e \in \mathcal{E}_j$ joining the vertices v_1 and v_2, the length $\ell(e)$ is $\ell(e) := \rho(v_1, v_2)$. We denote by $R_\lambda(D)$ the resolvent of an operator D at a point $\lambda \in \mathbb{C}\backslash\mathbb{R}$. The sequence $\{\|D_j\|\}_{j\in\mathbb{NN}}$ is unbounded, then using [16, Theorem 3.1] we have that D has compact resolvent if and only if

$$\|\iota_j \circ R_\lambda(D_j) \circ \iota_j^*\| = \sup_{k>j} \frac{1}{|\|D_k\| - \lambda|} \to 0$$

for $\lambda \in \mathbb{C}\backslash\mathbb{R}$, which is equivalent to $\|D_j\| \to 0$ as $j \to 0$. Moreover, using the Lipschitz continuity of $a \in C_{\mathrm{Lip}}(\Gamma_j)$, we have that the family of operators given by $\{[D_k, \pi_k(r_{j,k}(a))]\}_{k\geq j}$ is uniformly bounded. Then, the operator D is such that $[D, \pi(a)]$ is bounded on a dense subalgebra of $C_{\mathrm{Lip}}(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p))$. ∎

4. $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ as the universal Ważewski dendrite

We would like to focus in this work on yet another construction of $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ as an inverse limit of compact spaces, called *dendrites*. The present section is based on the work in [2] for which we refer to for proves and more details on the following statements.

Definition 4.1. A *continuum* is a compact, connected metrizable space. A *simple closed curve* in a topological space is any subspace homeomorphic to a circle. A *dendrite* is a locally connected continuum containing no simple closed curve. Dendrites may be thought of as topological generalizations of trees in which branching may occur at a dense set of points.

The definitions of branch points, ordinary points and endpoints on $\mathbb{R}$-trees, extend to dendrites. We can now introduce the so-called *Ważewski's universal dendrite*.

Theorem 4.2 (Ważewski's universal dendrite). *Up to homeomorphism, there is a unique dendrite W such that its branch points are dense in W and there are $\aleph_0$ branches at each branch point. The dendrite W is called the Ważewski's universal dendrite.*

In particular, the Ważewski's universal dendrite can be obtained from a chain of dendrites $D_1 \subseteq D_2 \subseteq D_3 \subseteq \ldots$, then defines certain bonding maps $f_n : D_{n+1} \to D_n$ and then finally obtains Ważewski's universal dendrite as

$$\varprojlim\{D_n, f_n\}_{n=1}^{\infty}.$$

Proposition 4.3. *For any dendrite D, D is homeomorphic to the Ważewski's universal dendrite if and only if its set of branching points is dense in D and each of its branching points has $\aleph_0$ branches.*

From this, one can immediately notice that $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ is a dendrite and the subset of branching points $\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)$ is dense. Therefore, the Berkovich projective line is homeomorphic to the Ważewski's universal dendrite.

If (X, d) is a compact metric space, then 2^X denotes the set of all nonempty closed subsets of X. The set 2^X can be equipped with the *Hausdorff metric, d_H* making $(2^X, d_H)$ a metric space called the *hyperspace* of the metric space (X, d).

A set valued function from X to Y is a single-valued function $f : X \to 2^Y$. The *graph* $\mathrm{Gr}(f)$ of a set-valued function $f : X \to 2^Y$ is the set of all pair of points $(x, y) \in X \times Y$ such that $y \in f(x)$. A function $f : X \to 2^Y$, where X and Y are compact metric spaces, is an *upper semi-continuous* set-valued function from X to Y (abbreviated *u.s.c.*) if for each open set $V \subseteq Y$, the set $\{x \in X \mid f(x) \subseteq V\}$ is an open set in X.

Proposition 4.4. *Let X and Y be compact metric spaces and $f : X \to 2^Y$ a set-valued function. Then f is u.s.c. if and only if its graph* $\mathrm{Gr}(f)$ *is closed in $X \times Y$.*

Definition 4.5 (Inverse sequence and limit). An *inverse sequence* of compact metric spaces X_k with u.s.c. bonding functions f_k is a sequence $\{X_k, f_k\}_{k=1}^{\infty}$, where $f_k : X_{k+1} \to 2^{X_k}$.

The *inverse limit* of an inverse sequence $\{X_k, f_k\}_{k=1}^{\infty}$ with u.s.c. bonding functions is defined as the subspace

$$\varprojlim\{X_k, f_k\}_{k=1}^{\infty} := \left\{x = (x_1, x_2, x_3, \dots) \in \prod_{k=1}^{\infty} X_k : x_k \in f(x_k)\right\}.$$

In this paper, we look at the special case of inverse limit where $X_k = [0, 1]$ and $f_k = f$ for some function $f : [0, 1] \to 2^{[0,1]}$. In such case, the inverse limit is denoted by $\varprojlim\{[0, 1], f\}_{k=1}^{\infty}$.

Let X and Y be compact sets, and a given set-valued u.s.c function $f : X \to 2^Y$ with graph $\mathrm{Gr}(f)$. Let $\pi_1 : \mathrm{Gr}(f) \to X$ and $\pi_2 : \mathrm{Gr}(f) \to Y$ be the coordinate projections. Then, we have

$$f(x) = \pi_2\big(\pi_1^{-1}(x)\big), \quad x \in X,$$

Equivalently, given a closed set $G \subseteq X \times Y$ such that $\pi_1(G) = X$, then $\pi_2 \circ \pi_1^{-1}$ defines a set-valued u.s.c. function such that $\mathrm{Gr}(\pi_2 \circ \pi_1^{-1}) = G$.

Let Δ be the diagonal subset of $[0, 1] \times [0, 1]$, i.e., the subset

$$\Delta = \big\{(t, t) \in [0, 1] \times [0, 1] : t \in [0, 1]\big\}.$$

For a fixed integer n, let $\{(a_i, b_i)\}_{i=1}^{n}$ be a finite sequence in $[0, 1] \times [0, 1]$, such that $a_i < b_i$ for each $i = 1, \dots, n$ and $a_i \neq a_j$ whenever $i \neq j$. Next, denote by $L(a_i, b_i)_{i=1}^{n}$ the union of lines

$$L(a_i, b_i)_{i=1}^{n} = \bigcup_{i=1}^{n}\big([a_i, b_i] \times \{a_i\}\big) \subseteq [0, 1] \times [0, 1].$$

Then, we define the set

$$G(a_i, b_i)_{i=1}^{n} = \Delta \cup L(a_i, b_i)_{i=1}^{n} \tag{4.1}$$

which is closed in the product $[0,1]\times[0,1]$, since it is a union of finitely many closed arcs. Furthermore, $\pi_1(G(a_i,b_i)_{i=1}^n)=\pi_2(G(a_i,b_i)_{i=1}^n)=[0,1]$. Therefore, by the closed graph theorem, there is a surjective u.s.c. function $f_{(a_i,b_i)_{i=1}^n}:[0,1]\to 2^{[0,1]}$ such that its graph $\mathrm{Gr}(f_{(a_i,b_i)_{i=1}^n})=G(a_i,b_i)_{i=1}^n$.

Definition 4.6 (Comb function). Let n be a positive integer and $\{(a_i,b_i)\}_{i=1}^n$ be a subset $[0,1]\times[0,1]$, such that $0<a_i<b_i$ for each $i=1,\dots,n$ and $a_i\neq a_j$ whenever $i\neq j$. Then, $f:[0,1]\to 2^{[0,1]}$ is called an n-comb function with respect to $\{(a_i,b_i)\}_{i=1}^n$ if $f=f_{(a_i,b_i)_{i=1}^n}$.

Notation 4.7. For each positive integer j, let i_j be a nonnegative integer. We use the notation

$$(a_1^{i_1},a_2^{i_2},a_3^{i_3},\dots)$$

to represent the point $(\underbrace{a_1,a_1,\dots,a_1}_{i_1},\underbrace{a_2,a_2,\dots,a_2}_{i_2},\dots)$ and

$$(a_1^{i_1},a_2^{i_2},a_3^{i_3},\dots,a_j^{i_j},t^\infty)$$

to denote the point $(\underbrace{a_1,a_1,\dots,a_1}_{i_1},\underbrace{a_2,a_2,\dots,a_2}_{i_2},\dots,\underbrace{a_j,a_j,\dots,a_j}_{i_j},t,t,t,\dots)$.

Example 4.8. Let f be a 1-comb function with respect to the point (a,b). Then a point $x\in\varprojlim\{[0,1],f\}_{k=1}^\infty$ if and only if either $x=(t^\infty)$ for a $t\in[0,1]$, or there exists a positive integer n such that $x=(a^n,t^\infty)$ for a $t\in(a,b]$. Therefore, $\varprojlim\{[0,1],f\}_{k=1}^\infty$ is a star with center the point (a^∞) and beams

$$L_0=\{(t^\infty)\mid t\in[0,a]\},\ L_0'=\{(t^\infty)\mid t\in[a,1]\}\text{ and }L_n=\{(a^n,t^\infty)\mid t\in[a,b]\}.$$

The center (a^∞) is the only ramification point, and the maximal free arcs are the beams.

Proposition 4.9. *Let n be any positive integer and let $f:[0,1]\to 2^{[0,1]}$ be any n-comb function. Then $\varprojlim\{[0,1],f\}_{k=1}^\infty$ is a dendrite. In particular, we denote by D_n the dendrite*

$$D_n=\varprojlim\{[0,1],f_{(a_i,b_i)_{i=1}^n}\}_{k=1}^\infty.$$

Let $\{(a_n,b_n)\}_{n=1}^\infty$ be any sequence in $[0,1]\times[0,1]$ such that

(1) $a_n<b_n$ for each n;

(2) $a_i\neq a_j$ whenever $i\neq j$;

(3) $\lim_{n\to\infty}(b_n-a_n)=0$.

Then, similarly one can construct the set $G(a_n, b_n)_{n=1}^{\infty}$

$$G(a_n, b_n)_{n=1}^{\infty} = \Delta \cup L(a_n, b_n)_{n=1}^{\infty} \tag{4.2}$$

as a closed subset of $[0, 1] \times [0, 1]$. Therefore, there exists a surjective u.s.c. function $f_{(a_n,b_n)_{n=1}^{\infty}} : [0, 1] \to 2^{[0,1]}$ such that its graph is exactly $G(a_n, b_n)_{n=1}^{\infty}$. Thus, $f_{(a_n,b_n)_{n=1}^{\infty}}$ is called the comb function with respect to the set of points $\{(a_n, b_n)\}_{n=1}^{\infty}$.

Theorem 4.10. *Let $f : [0, 1] \to 2^{[0,1]}$ be any comb function with respect to an admissible sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$ such that the set $\{a_n \mid n = 1, 2, \dots\}$ is dense in $[0, 1]$. Then,*

$$\varprojlim \{[0, 1], f\}_{k=1}^{\infty} = \mathrm{Cl}\left(\bigcup_{n=1}^{\infty} D_n\right).$$

Moreover, $\varprojlim\{[0, 1], f\}_{k=1}^{\infty}$ is homeomorphic to Ważewski's universal dendrite.

Corollary 4.11. *Let $f : [0, 1] \to 2^{[0,1]}$ be the comb function with respect to a sequence $\{(a_n, b_n)_{n=1}^{\infty}\}$. Any point $x \in \varprojlim\{[0, 1], f\}_{k=1}^{\infty}$ can be classified as follows:*

(1) *If $x \in D_n$ for some positive integer n then, either $x = (t^{\infty})$, $t \in [0, 1]$, or is of the form*

$$x = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \dots, a_{i_m}^{k_m}, t^{\infty})$$

for some positive integer m and for each $\ell \le m$ it holds that $i_\ell \le n$, $k_\ell > 0$, $a_{i_\ell} < a_{i_{\ell+1}} \le b_{i_\ell}$, and $a_{i_m} \le t \le b_{i_m}$. In particular,

(a) *x is a branching point in D_n if and only if*

$$x = (x_1, x_2, x_3, \dots, x_m, a_j^{\infty}), \quad j \le n,$$

(b) *x is an endpoint in D_n for some positive n, i.e.,*

$$x = (x_1, x_2, x_3, \dots, x_m, b_i^{\infty}), \quad i \le n.$$

(2) *If $x \in \varprojlim\{[0, 1], f\}_{k=1}^{\infty} \setminus \bigcup_{n=1}^{\infty} D_n$, then it has the general form*

$$x = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \dots).$$

Notation 4.12. For any dense subset $\{a_n, n \in \mathbb{N}\}$ of the interval $(0, 1)$, there exists a comb function f such that the inverse limit $\varprojlim\{[0, 1], f\}_{k=1}^{\infty}$ is homeomorphic to the Ważewski's universal dendrite. From now on, we assume that the comb map $f_{(a_i,b_i)_{i=1}^{\infty}}$ is constructed such that the set $\{a_n : n \in \mathbb{N}\}$ is an enumeration of the rational numbers in $(0, 1)$, i.e., $\mathbb{Q} \cap (0, 1)$. In the rest of the paper, we use q_n instead of a_n to emphasize the fact that we are using rational numbers as coordinates for branch points.

Remark 4.13. In Corollary 4.11, we recognize the general structure of a *dendrite* and the classification of points of the Berkovich line, identified with the Ważewski universal dendrite. We identify the four categories of points:

(1) The branching points or Type II points:

$$\mathrm{Br}(X) = \big\{x \in X : x = (q_{i_1}^{k_1}, q_{i_2}^{k_2}, q_{i_3}^{k_3}, \dots, q_{i_{m-1}}^{k_{m-1}}, q_{i_m}^{\infty}), \\ \text{for each } \ell \le m,\ q_{i_\ell} < q_{i_{\ell+1}}\big\}.$$

(2) The regular points or Type III points:

$$\mathrm{Reg}(X) = \big\{x \in X : x = (q_{i_1}^{k_1}, q_{i_2}^{k_2}, q_{i_3}^{k_3}, \dots, q_{i_m}^{k_m}, t^{\infty}), \\ \forall \ell \le m,\ q_{i_\ell} < q_{i_{\ell+1}}, t \in (q_{i_m}, 1] \backslash (q_n)\big\}.$$

(3) The endpoints or Type IV points:

$$\mathrm{End}(X) = \big\{x \in X : x = (q_{i_1}^{k_1}, q_{i_2}^{k_2}, q_{i_3}^{k_3}, \dots, q_{i_m}^{k_m}, b_j^{\infty}), \\ \forall \ell \le m,\ q_{i_\ell} < q_{i_{\ell+1}},\ q_{i_m} < b_j\big\}.$$

(4) The limit points or Type I points:

$$\mathrm{Lim}(X) = \big\{x \in X : x = (q_{i_1}^{k_1}, q_{i_2}^{k_2}, q_{i_3}^{k_3}, \dots, q_{i_m}^{k_m}, \dots), \forall \ell,\ q_{i_\ell} < q_{i_{\ell+1}}\big\}.$$

Remark 4.14. An important property of the Ważewski dendrite $\mathcal{W}$ is the following: it is a homogeneous space in the following sense: the closure of any connected open subset of $\mathcal{W}$ is homeomorphic to $\mathcal{W}$ (see for instance [6]).

Remark 4.15. The previous results give a description of Berkovich spaces as inverse systems in the compact space $[0, 1]$. This construction allows us to introduce the *shift map* induced by the comb function f in the inverse limit. This shift map defines a dynamical system on $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$.

4.1. Two equivalent topologies on $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$

So far, we have described the topology on the Berkovich line or equivalently the Ważewski dendrite as the *weak topology*. In this short paragraph, we would like to define an equivalent topology given by the so-called cylindrical sets as a subbasis.

The weak topology on $\mathbb{R}$-trees. This topology is also known as the *Observer's topology* on a tree T. It is the topology generated (in the sense of a subbasis) by sets $U(x, v)$, for $x \in T$ and $v \in T_x$ such that $U(x, v) = \{y \in T : y \neq x, [x, y] \in v\}$. We denote this topology by τ_{O}.

Cylindrical topology. Let $q = q_{i_1}^{k_1}, q_{i_2}^{k_2}, q_{i_3}^{k_3}, \ldots, q_{i_m}^{k_m}$ be a finite word. Consider the *cylinder set* defined by

$$Z(q) := \left\{ qx \in \mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p) : x \in \mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p) \right\}. \tag{4.3}$$

Then, define the topology τ_{Cyl} generated (as a subbasis) by the cylinder sets.

Proposition 4.16. *The topologies τ_{O} and τ_{Cyl} are equivalent.*

Proof. Fix $x, y \in \mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$ and consider the direction set $U(x, [x, y])$. Let $z \in U(x, [x, y])$ such that, without loss of generality, we have $x \leq z \leq y$. By a density argument, we can find x' of Type II such that $x \leq x' \leq z \leq y$. Then, we would have $x' = q_{i_1}^{k_1}, q_{i_2}^{k_2}, q_{i_3}^{k_3}, \ldots, q_{i_n}^{\infty}$, $x = q_{i_1}^{k_1}, q_{i_2}^{k_2}, q_{i_3}^{k_3}, \ldots q_{i_m}^{k_m}, t^{\infty}$, for some $t \leq q_{i_n}$ and also $z = q_{i_1}^{k_1}, q_{i_2}^{k_2}, q_{i_3}^{k_3}, \ldots q_{i_n}^{k_n}, t^{\infty}$ for some $t \geq q_{i_n}^{k_n}$. Define the word $q = q_{i_1}^{k_1}, q_{i_2}^{k_2}, q_{i_3}^{k_3}, \ldots, q_{i_n}^{k_n}$ and the cylinder set $Z(q)$. Then, by construction $z \in Z(q)$ and any point $z' \in Z(q)$ is such that $z' \geq x'$ and thus the path $[x, z']$ is equivalent to the path $[x, y]$ as they share the common segment $[x, x']$. Hence, $z \in Z(q) \subseteq U(x, [x, y])$.

Conversely, let $q = q_{i_1}^{k_1}, q_{i_2}^{k_2}, q_{i_3}^{k_3}, \ldots, q_{i_m}^{k_m}$ and a cylinder set $Z(q)$. Fix $z \in Z(q)$. In addition, define the Type II point given by $x = q_{i_1}^{k_1}, q_{i_2}^{k_2}, q_{i_3}^{k_3}, \ldots, q_{i_m}^{\infty}$. We consider next $U(x, [x, z])$, the direction of z at x. Then, $z \in U(x, [x, z])$. Moreover, note that any $z' \in U(x, [x, z])$ is such that $z' \geq x$ (in the tree's partial order); thus z' can be written $z' = qy$ for some $y \in \mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$ and therefore, we have $z' \in Z(q)$. Hence, $z \in U(x, [x, z]) \subseteq Z(q)$. ∎

5. Noncommutative geometry on $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$

There are several ways to associate a C^*-algebra to the space $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$; we expose some of them that appear to be relevant to the construction of spectral triples. We start with C^*-algebras associated to the space of branching points $\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p)$ seen as a countable alphabet. We use the construction presented in the works [3, 4].

5.1. C^*-algebras of countable subshifts

In this paragraph, we denote by $\mathbb{Q}_1$ the set $\mathbb{Q} \cap (0, 1)$. We consider the set of Type 2 points as a countable alphabet. The *shift map* on $\mathbb{Q}_1^{\mathbb{N}}$ is the map $\sigma : \mathbb{Q}_1^{\mathbb{N}} \to \mathbb{Q}_1^{\mathbb{N}}$ given by $\sigma(x_n) = (x_{n+1})$. Now, we notice that $\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p)$ is a subset of $\mathbb{Q}_1^{\mathbb{N}}$ and is *invariant* for σ i.e. $\sigma(\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p)) \subseteq \mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p)$. For an invariant subset $\mathsf{X} \subseteq \mathcal{A}^{\mathbb{N}}$, we define $\mathcal{L}_n(\mathsf{X})$ as the set of all words of length n that appear in some sequence of X,

that is for $\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)$,

$$\mathcal{L}_n(\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)) := \big\{(a_0 \dots a_{n-1}) \in \mathbb{Q}^n : \exists\, x \in \mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p) \text{ s.t. } (x_0 \dots x_{n-1}) = (a_0 \dots a_{n-1})\big\}.$$

We set $\mathcal{L}_0(\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)) = \emptyset$. The *language* of $\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)$ is then the set

$$\mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)} := \bigcup_{n=0}^{\infty} \mathcal{L}_n\big(\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)\big)$$

consisting of all finite words that appear in some sequence in $\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)$. Given the subshift $\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)$ over the alphabet $\mathbb{Q}_1$ and $\alpha, \beta \in \mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$, we define

$$C(\alpha, \beta) := \big\{\beta x \in \mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p) : \alpha x \in \mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)\big\}.$$

In particular, $Z_\beta := C(\emptyset, \beta)$ is called the *cylinder set* of β, and $F_\alpha := C(\alpha, \emptyset)$ the *follower set* of α.

Let $\mathcal{B}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$ be the Boolean algebra of subsets of $\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)$ generated by the sets $C(\alpha, \beta)$ for any words $\alpha, \beta \in \mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$.

Definition 5.1. Let $\mathcal{U}$ be a Boolean algebra. The *unital subshift C^*-algebra* $\mathcal{O}_{\mathsf{X}}$ associated with X is the universal unital C^*-algebra generated by projections $\{p_A : A \in \mathcal{U}\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ subject to the relations:

(i) $p_{\mathsf{X}} = 1$, $p_{A\cap B} = p_A p_B$, $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ and $p_\emptyset = 0$, for every $A, B \in \mathcal{U}$;

(ii) $s_\beta s_\alpha^* s_\alpha s_\beta^* = p_{C(\alpha,\beta)}$ for all $\alpha, \beta \in \mathcal{L}_{\mathsf{X}}$ for all $\alpha, \beta \in \mathcal{L}_{\mathsf{X}}$, where $s_\emptyset = 1$ and, for $\alpha = \alpha_1 \dots \alpha_n \in \mathcal{L}_{\mathsf{X}}$, $s_\alpha = s_{\alpha_1} \dots s_{\alpha_n}$.

In particular, $s_\alpha^* s_\alpha = p_{C(\alpha,\emptyset)} = p_{F_\alpha}$ and $s_\beta s_\beta^* = p_{C(\emptyset,\beta)} = p_{Z_\beta}$ for all $\alpha, \beta \in \mathcal{L}_{\mathsf{X}}$.

We can now define the universal algebra $\mathcal{O}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$ associated to the subshift $\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)$, which satisfies the following properties.

Proposition 5.2. *In $\mathcal{O}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$ the following hold:*

(i) $s_q^* s_p = \delta_{q,p} p_{F_q}$ *for all* $q, p \in \mathbb{Q}_1$;

(ii) $s_\alpha^* s_\alpha$ *and* $s_\beta^* s_\beta$ *commute for all* $\alpha, \beta \in \mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$;

(iii) $s_\alpha^* s_\alpha$ *and* $s_\beta s_\beta^*$ *commute for all* $\alpha, \beta \in \mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$;

(iv) $s_\alpha s_\beta = 0$ *for all* $\alpha, \beta \in \mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$ *such that* $\alpha\beta \notin \mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$;

(v) $\mathcal{O}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$ *is generated by the set* $\{s_q, s_q^* : q \in \mathbb{Q}_1\} \cup \{1\}$.

Proof. (i) From the definition of a partial isometry, one has $s_q = p_{Z_q} s_q$ and therefore we can write the following equalities $s_q^* s_p = s_q^* p_{Z_q} p_{Z_p} s_p = \delta_{q,p} p_{F_q}$. (ii) Using the point (ii) in the definition, for $\alpha, \beta \in \mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$, we have that $s_\alpha^* s_\alpha s_\beta^* s_\beta = p_{F_\alpha} p_{F_\beta} = p_{F_\beta} p_{F_\alpha}$ from which commutativity follows. (iii) The proof of (iii) is identical. (iv) Similarly to the proof of (i), we have by definition of a partial isometry that $s_\alpha s_\beta = s_\alpha p_{F_\alpha} p_{Z_\beta} s_\beta = s_\alpha p_{F_\alpha \cap Z_\beta} s_\beta$. The last term in the previous equality vanishes whenever $\alpha\beta \notin \mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$. (v) The algebra $\mathcal{O}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$ is generated by the partial isometries s_a and the projections p_A for any $A \in \mathcal{B}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$. But the projections on the sets $C_{\alpha,\beta}$ can be written as $p_{C(\alpha,\beta)} = s_\beta s_\alpha^* s_\alpha s_\beta^*$. Moreover, the sets $C(\alpha, \beta)$ generate the Boolean algebra, from which the statement follows. ∎

Remark 5.3. The universal C^*-algebra $\mathcal{O}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$ can be equivalently constructed as an Exel-Laca graph C^*-algebra [15] as shown in [4, Proposition 4.4].

Moreover, one can associate a Hilbert space to the set $\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)$, which we denote by $\ell^2(\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p), dq)$, where the measure dq is the counting measure on $\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)$. In this section, this Hilbert space is denoted by $\mathcal{H}$. In particular, the Hilbert space $\mathcal{H}$ admits an orthonormal basis $\{e_x : x \in \mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)\}$ of Kronecker delta functions.

Proposition 5.4. *Consider the family of operators $\{S_q\}_{q\in\mathbb{Q}_1}$ and $\{P_Q\}_{Q\in\mathcal{U}}$ on the space $\ell^2(\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p))$ defined by*

$$S_q(e_x) := \begin{cases} e_{\sigma_q(x)} & \text{if } x \in F_q \\ 0 & \text{otherwise} \end{cases} \qquad P_Q(e_x) := \begin{cases} e_x & \text{if } x \in Q \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

where $\{e_x : x \in \mathsf{X}\}$ is an orthonormal basis of $\ell^2(\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p))$. Then the map $\pi : \mathcal{O}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)} \to B(\mathcal{H})$, such that $\pi(s_q) = S_q$ for all $q \in \mathbb{Q}$ and $\pi(p_Q) = P_Q$ for all $Q \in \mathcal{U}$ defines a *-representation.

Proof. We verify immediately that P_Q is a projection satisfying the following properties: $P_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)} = 1$, $P_{Q\cap R} = P_Q P_R$, $P_{Q\cup R} = P_Q + P_R - P_{Q\cap R}$ and $P_\emptyset = 0$, for every $Q, R \in \mathcal{B}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$.

Moreover, for any $q \in \mathbb{Q}_1$, S_q is a partial isometry between the following initial and final space:

$$S_q : \overline{\mathrm{span}}\{e_x : x \in F_a\} \to \overline{\mathrm{span}}\{e_x : x \in Z_a\}$$

with the adjoint given by

$$S_q^*(e_x) = \begin{cases} e_y & \text{if } x = ay \in Z_a \\ 0 & \text{otherwise.} \end{cases}$$

In addition, we can define $S_\alpha := S_{\alpha_1} \cdots S_{\alpha_n}$ for $\alpha = \alpha_1 \cdots \alpha_n \in \mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$. The partial isometries also generate the projections on the generating sets $S_\beta S_\alpha^* S_\alpha S_\beta^* = P_{C(\alpha,\beta)}$. The claim then follows from the universal property of $\mathcal{O}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$. ∎

5.2. Semibranching systems on $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$

We can extend the previous representation on the set of rational points $\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)$ to the full space $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$. We can also exploit the metric structure on the Berkovich line in order to single out invariant measures. To exhibit representation spaces of such C^*-algebras, we rely on semibranching systems; we refer to [5] for further details on such construction.

5.2.1. Invariant measure. In order to define a representation of the universal C^*-algebra $\mathcal{O}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$ using the space $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$, we need to single out a measure μ on the projective line. In order to do so, we use the isometry group $\mathrm{Iso}(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p))$.

A fact of fundamental importance is that the action of a non-constant rational map on $\mathbb{P}^1(\mathbb{C}_p)$ extends naturally to an action on $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$, and such map preserves the type of the point upon which it acts.

Let $\varphi \in \mathbb{C}_p(T)$ be a rational function of degree $d \geq 1$. The usual action on $\mathbb{P}^1(\mathbb{C}_p)$ extends to a continuous action $\varphi : \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) \to \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$.

Particularly, the maps $\gamma \in \mathrm{PGL}_2(\mathbb{C}_p)$ act transitively on type II points of $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$, and any type II point $\zeta_{a,r}$ can be written as $\zeta_{a,r} = \gamma(\zeta_{root})$, where $\gamma = \left(\begin{smallmatrix} q & a \\ 0 & 1 \end{smallmatrix}\right)$ and $|q|_v = r$.

Corollary 5.5 ([1]). $\mathrm{Aut}(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)) \simeq \mathrm{PGL}_2(\mathbb{C}_p)$, *the group of Möbius transformations (or linear fractional transformations) acting on* $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$.

The group $\mathrm{PGL}_2(\mathbb{C}_p)$ of Möbius transformations acts continuously on $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ in a natural way compatible with the usual action $\mathbb{P}^1(\mathbb{C}_p)$, and this action preserves $\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)$. Using the definition of $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ in terms of multiplicative seminorms (and extending each $[\cdot]_x$ to a seminorm on its local ring in the quotient field $\mathbb{C}_p(T)$) we have

$$[f]_{M(x)} = [f \circ M]_x \tag{5.2}$$

for each $M \in \mathrm{PGL}_2(\mathbb{C}_p)$. The action of $\mathrm{PGL}_2(\mathbb{C}_p)$ on $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ can also be described concretely in terms of Berkovich's classification theorem, using the fact that each $M \in \mathrm{PGL}_2(\mathbb{C}_p)$ takes closed discs to closed discs. An important observation is that $\mathrm{PGL}_2(\mathbb{C}_p)$ acts *isometrically* on $\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)$, i.e.,

$$\rho(Mx, My) = \rho(x, y) \tag{5.3}$$

for all $x, y \in \mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)$ and all $M \in \mathrm{PGL}_2(\mathbb{C}_p)$, where ρ is the *big metric*. This shows that the path metric ρ is "coordinate-free".

Following [17], since $(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p), \rho)$ is a compact metric space with isometry group $\text{PGL}_2(\mathbb{C}_p)$, there exists a non-zero $\text{PGL}_2(\mathbb{C}_p)$-invariant Radon measure on $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$, denoted by μ_G. Moreover, since $\text{PGL}_2(\mathbb{C}_p)$ acts transitively, then μ_G is strictly positive. We can then define the Hilbert space $L^2(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p), \mu_G)$.

5.2.2. Representation in $L^2(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p), \mu_G)$. Consider the shift map

$$\sigma : \mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p) \to \mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p).$$

It is a continuous surjection. Keeping the notations introduced in the previous section, we consider the sets

$$C(\alpha, \beta) := \left\{\beta x \in \mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p) : \alpha x \in \mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)\right\}$$

for every $\alpha, \beta \in \mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p)}$. In particular, $Z_\beta := C(\emptyset, \beta)$ is the *cylinder set* of β, and $F_\alpha := C(\alpha, \emptyset)$ the *follower set* of α. Let us define the maps σ_q for $q \in$such that

$$\sigma_q : F(q) \to Z(q), \quad \sigma_q(x) = qx. \tag{5.4}$$

Let $\mathcal{B}_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)}$ be the Boolean algebra of subsets of $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$ generated by all the sets $C(\alpha, \beta)$ for $\alpha, \beta \in \mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p)}$. similarly, one can construct the universal C^*-algebra $\mathcal{O}_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)}$.

Theorem 5.6. *The C^*-algebras $\mathcal{O}_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)}$ and $\mathcal{O}_{\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p)}$ are *-isomorphic.*

Proof. The map j defined on cylinder sets such that

$$j\left(\left\{\beta x \in \mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p) : \alpha x \in \mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p)\right\}\right) = \left\{\beta x \in \mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p) : \alpha x \in \mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)\right\}$$

extends to an isomorphism of Boolean algebras between $\mathcal{B}_{\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p)}$ and $\mathcal{B}_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)}$. The statement follows then from the universal property of $\mathcal{O}_{\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}(\mathbb{C}_p)}$. ∎

Further on, we consider $\mathcal{O}_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)}$ as the C^*-algebra associated to the Berkovich line. We now study its representation on $L^2(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p), \mu_G)$.

Definition 5.7 (Semibranch system). Consider a measure space (X, μ) and a countable family $\{\sigma_i\}_{i\in\mathbb{N}}$, of measurable maps $\sigma_i : D_i \to X$, defined on measurable subsets $D_i \subset X$. The family $\{\sigma_i\}_{i\in\mathbb{N}}$ is called *a semibranching system* if the following hold:

(1) There exists a corresponding family $\{R_i\}_{i\in\mathbb{N}}$ of measurable subsets of X with the property,

$$\mu(X \setminus \cup_i R_i) = 0, \quad \text{and} \quad \mu(R_i \cap R_j) = 0, \quad \text{for } i \neq j \tag{5.5}$$

where we denote by R_i the range $R_i = \sigma_i(D_i)$.

(2) There is a Radon–Nikodym derivative

$$\Phi_{\sigma_i} = \frac{d(\mu \circ \sigma_i)}{d\mu}$$

with $\Phi_{\sigma_i} > 0$, μ-almost everywhere on D_i.

A measurable map $\sigma : X \to X$ is called a coding map for the family $\{\sigma_i\}$ if for all $x \in D_i, \sigma \circ \sigma_i(x) = x$.

Lemma 5.8. *For every $q \in \mathbb{Q}_1$, the Radon–Nikodym derivative $\Phi_{\sigma_q} = 1$.*

Proof. The sets $F(q)$ and $D(q)$ are both non-empty compact, locally connected metric spaces with no closed curves, hence both are dendrites. Moreover, their sets of branching points are dense (by construction) with infinite branching at each point. By universality of the Ważewski dendrite $\mathcal{W}$, $F(q)$ and $D(q)$ are both homeomorphic to $\mathcal{W}$. In, addition the map $\sigma_q : F(q) \to D(q)$ is continuous, surjective and thus a homeomorphism, with inverse σ (see [6]). Thus, $\sigma_q \in \mathrm{Homeo}(\mathcal{W})$ and by identification with $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$, σ_g can be identified with an element γ_g of $\mathrm{PGL}_2(\mathbb{C}_p)$. Using the invariance of μ_G we can compute the Radon–Nikodym derivative:

$$\Phi_{\sigma_q} = \frac{d(\mu_G \circ \sigma_q)}{d\mu_G} = \frac{d(\mu_G \circ \gamma_g)}{d\mu_G} = 1. \tag{5.6}$$

■

Proposition 5.9. *The shift map $\sigma : \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) \to \mathbb{P}^1_{Berk}(\mathbb{C}_p)$ is the coding map of the semibranching function system given by the family of maps $\{\sigma_q\}_q$ such that*

$$\sigma_q : F(q) \to Z(q), \quad \sigma_q(q_1, q_2, \dots) = (q, q_1, q_2, \dots) \quad \textit{for all } q \in \mathbb{Q}_1. \tag{5.7}$$

Moreover, one can construct a family of operator $(S_q)_{q\in\mathbb{Q}}$ acting on the Hilbert space $L^2(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p), \mu_G)$:

$$(S_q\psi)(x) = \chi_{Z(q)}(x)\psi \circ \sigma(x). \tag{5.8}$$

We also denote the space $L^2(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p), \mu_G)$ by $\mathcal{H}$.

Proof. Item (1) in the definition of a semibranching system follows from the fact that the collection $\{Z(q) : q \in \mathbb{Q}_1\}$ forms a disjoint partition of $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$. Item (2) follows from Lemma 5.8. ■

Lemma 5.10. *The adjoint of the operator S_q is given by*

$$S_q^* : \mathcal{H} \to \mathcal{H}, \quad (S_q^*\varphi)(x) = \chi_{F(q)}(x)\psi \circ \sigma_q(x). \tag{5.9}$$

Proof. For any $q \in \mathbb{Q}_1$, we can write

$$\begin{aligned}\langle S_q\psi, \varphi\rangle &= \int_{Z(q)} \psi\big(\sigma(x)\big)\varphi(x)d\mu_G(x)\\ &= \int_{F(q)} \psi(u)\varphi\big(\sigma_q(x)\big)\Phi_{\sigma_q}(u)d\mu_G(u) = \langle \psi, S_q^*\varphi\rangle\end{aligned}$$

where we used Lemma 5.8 to state that $\Phi_{\sigma_q}(u) = 1$. ∎

Consider then the C^*-algebra generated by the set of partial isometries $\{S_q, S_q^* :$ for all $\zeta \leq \zeta'\}$ and then define the concrete C^*-algebra

$$A(\mathcal{H}) := \overline{\{S_q, S_q^* : \text{for all } q \in \mathbb{Q}_1\}}^{\text{S.O.T.}} \tag{5.10}$$

Proposition 5.11. *The operators S_q and their adjoints satisfy the relation $S_qS_q^* = P_{Z(q)}$, where $P_{Z(q)}$ is the projection given by multiplication by the characteristic function $\chi_{Z(q)}$. This gives*

$$\sum_q S_qS_q^* = 1. \tag{5.11}$$

Similarly, $S_q^*S_q = P_{F(q)}$, where $P_{F(q)}$ is the projection given by multiplication by $\chi_{F(q)}$.

Proof. For any $q \in \mathbb{Q}_1$, we can write

$$S_qS_q^*\psi(x) = \chi_{Z(q)}(x)\chi_{F(q)}\big(\sigma(x)\big)\psi\big(\sigma\big(\sigma_q(x)\big)\big) = \chi_{Z(q)}(x)\chi_{F(q)}\big(\sigma(x)\big)\psi(x)$$

where we have used the fact that $\sigma(\sigma_q(x)) = x$ by definition of σ_q. Now, if $x \in Z(q)$ then $\sigma(x) \in F(q)$ and thus $\chi_{F(q)}(\sigma(x)) = 1$. Hence, we deduce that

$$S_qS_q^*\psi(x) = \chi_{Z(q)}(x)\psi(x)$$

and therefore $S_qS_q^* = P_{Z(q)}$. The proof that $S_q^*S_q = P_{F(q)}$ is identical. Finally, for any $\psi \in \mathcal{H}$ and $x \in \mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$, we have

$$\sum_{q\in\mathbb{Q}_1} S_qS_q^*\psi(x) = \sum_{q\in\mathbb{Q}_1} P_{Z(q)}\psi(x) = \psi(x)$$

since $\{Z(q) : q \in \mathbb{Q}_1\}$ forms a partition of $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$. ∎

Corollary 5.12. *The map $\pi : \mathcal{O}_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)} \to A(\mathcal{H})$ given by*

$$\pi(s_q) = S_q \tag{5.12}$$

*is a *-homomorphism and defines a representation of $\mathcal{O}_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)}$ as a concrete C^*-subalgebra of the algebra of bounded operators $B(L^2(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p), \mu_G))$.*

5.3. Projection-valued measures

In this section, we denote by $\mathcal{H}$ the Hilbert space $L^2(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p), \mu_G)$.

Definition 5.13. Let $\mathcal{H}$ be a Hilbert space and

$$\mathcal{P}_{\mathcal{H}} := \{P \in B(\mathcal{H}) : P = P^2 = P^*\}$$

be the set of all orthogonal projections on $\mathcal{H}$. Further, let $\Sigma(X)$ be the σ-algebra of a measurable space X. An operator-valued map $P : \Sigma(X) \to \mathcal{P}_{\mathcal{H}}$ defined on $\Sigma(X)$ with values in bounded linear operators on Hilbert space $\mathcal{H}$ is called a *spectral measure* or a *projection valued measure* if

(1) $P(X) = 1$ and $P(\emptyset) = 0$.

(2) If $B_1, B_2, \dots$ in $\Sigma(X)$, such that $B_i \cap B_j = \emptyset$ for $i \neq j$, one has

$$P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i) \tag{5.13}$$

in the strong topology sense.

(3) $P(E \cap F) = P(E)P(F)$ for $E, F \in \Sigma(X)$.

As an example, consider $(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p), \mu_G)$ as a σ-finite measure space and let Σ_G be the σ-algebra of μ_G-measurable subsets of $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$. For each measurable set $E \subseteq \mathbb{P}^1_{\mathrm{Berk}}$, define $P_G(E)$ the projection in $B(\mathcal{H})$ given by $P(E) = m_{\chi_E}$, that is

$$P_G(E)f = \chi_E f. \tag{5.14}$$

Then $E \to P_G(E)$ is a projection-valued measure on $\mathcal{H}$ and called the *canonical projection-valued measure* on $L^2(\mu_G)$.

Let $\Sigma(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p))$ be the *cylinder σ-algebra*, i.e., the σ-algebra generated by the cylinder sets.

Proposition 5.14. *The operator-valued map defined on the cylinders $Z(q) \mapsto S_q S_q^*$ extends to a spectral measure $P : \Sigma(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)) \to \mathcal{P}_{\mathcal{H}}$. Moreover, we have the equality*

$$P(E) = P_G(E), \qquad \forall E \in \Sigma\big(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)\big). \tag{5.15}$$

Proof. Let $f \in \mathcal{H}$, we use Kolmogorov extension theorem on the map

$$E \mapsto \langle P(E)f, f\rangle$$

defined on cylinder sets. We just have to check the consistency conditions. Consider two different words $Q = q_1 \dots q_n$ and $Q' = q'_1 \dots q'_n$ in $\mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$, then by definition

the cylinders $Z(Q)$ and $Z(Q')$ are disjoint. The projections $P_{Z(Q)} = S_Q S_Q^*$ and $P_{Z(Q')} = S_{Q'} S_{Q'}^*$ are orthogonal.

Again, by construction, for a finite word $Q = q_1 \dots q_n$, the cylinder $Z(Q)$ is the disjoint union of the cylinders $Z(q_1 \dots q_n q)$ over $q \geq q_n$. We can then write,

$$\begin{aligned} \Big\langle \sum_{q \geq q_n} P\big(Z(q_1 \dots q_n q)\big) f, f \Big\rangle &= \Big\langle \sum_{q \geq q_n} S_{Qq} S_{Qq}^* f, f \Big\rangle \\ &= \langle S_Q S_Q^* f, f \rangle = \big\langle P\big(Z(Q)\big) f, f \big\rangle. \end{aligned} \tag{5.16}$$

Therefore, the consistency relations are true for an arbitrary $f \in \mathcal{H}$ and hence P extends to a projection valued measure on $\mathcal{P}_{\mathcal{H}}$.

Finally for $f \in \mathcal{H}$ and for any $E \in \Sigma(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p))$, using the representation relations, we have $P(E) f = \chi_E f$ and thus P coincides with P_G. ∎

Lemma 5.15. *There exists a cyclic vector $f \in \mathcal{H}$ such that*

$$\mu_G(E) = P_G^f(E) := \big\langle P_G(E) f, f \big\rangle = \| P_G(E) f \|^2 \tag{5.17}$$

for every μ_G-measurable sets in $\mathbb{P}^1_{\text{Berk}}$.

Proof. Let us recall that a vector $f \in \mathcal{H}$ is called a *cyclic* vector for the operator P_G if the linear span of the vectors $P_G(E) f$, for $E \in \Sigma(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p))$, is dense in $\mathcal{H}$. Since μ_G is a finite measure and the set of simple functions is dense in $\mathcal{H}$, then the identity function 1 is a cyclic vector for P_G. ∎

Proposition 5.16. *Consider the projection-valued measure $(P, \Sigma(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)))$ with a cyclic vector $f \in \mathcal{H}$. The measure μ_G coincides withx the real valued Borel measure on the space $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$ defined by*

$$\mu_f(E) := P^f(E) := \big\langle P(E) f, f \big\rangle = \| P(E) f \|^2. \tag{5.18}$$

Moreover, the measure μ_f satisfies

$$\int_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)} \psi \, d\mu_f = \int_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)} \psi \, d\mu_G$$

and

$$\sum_{q \in \mathbb{Q}_1} \int_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)} \psi \circ \sigma_q \, d\mu_{S_q^* f} = \int_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)} \psi \, d\mu_f.$$

Proof. Let us denote by $\pi : C(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)) \to B(\mathcal{H})$ the left-multiplication representation of continuous functions on $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$. We can write

$$\begin{aligned}\sum_{q\in\mathbb{Q}_1}\int_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)} \psi\circ\sigma_q\, d\mu_{S_q^* f} &= \sum_{q\in\mathbb{Q}_1}\int_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)} \chi_{F(q)}\psi\circ\sigma_q\, d\mu_{S_q^* f}\\ &= \sum_{q\in\mathbb{Q}_1}\big\langle S_q^* f, P\big(F(q)\big)\pi(\psi\circ\sigma_q)S_q^* f\big\rangle\end{aligned}$$

Moreover, using the fact that $P(F(q)) = S_q^* S_q$, we can say that

$$P\big(F(q)\big)\pi(\psi\circ\sigma_q)S_q^* f = \chi_{F(q)}\pi(\psi\circ\sigma_q)(f\circ\sigma_q) = S_q^*\pi(\psi)f$$

and thus

$$\big\langle S_q^* f, P\big(F(q)\big)\pi(\psi\circ\sigma_q)S_q^* f\big\rangle = \big\langle f, S_q S_q^*\pi(\psi)f\big\rangle = \big\langle f, P\big(Z(q)\big)\pi(\psi)f\big\rangle.$$

Finally, we conclude that

$$\sum_{q\in\mathbb{Q}_1}\int_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)} \psi\circ\sigma_q\, d\mu_{S_q^* f} = \sum_{q\in\mathbb{Q}_1}\big\langle f, P\big(Z(q)\big)\pi(\psi)f\big\rangle = \big\langle f, \pi(\psi)f\big\rangle.$$

In other words, we can write

$$\sum_{q\in\mathbb{Q}_1}\int_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)} \psi\circ\sigma_q\, d\mu_{S_q^* f} = \int_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)} \psi\, d\mu_f. \qquad \blacksquare$$

Proposition 5.17. *There exists a unique continuous linear map*

$$\widehat{P} : L^\infty(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)) \to B(\mathcal{H})$$

with $\widehat{P}(\chi_E) = P(E)$ for $E \in \Sigma(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p))$. This map is called the spectral integral, and we also write it:

$$\widehat{P}(f) = \int_{\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)} f(\zeta)dP(\zeta). \tag{5.19}$$

This map satisfies $\widehat{P}(f)^* = \widehat{P}(\overline{f})$, $\widehat{P}(fg) = \widehat{P}(f)\widehat{P}(g)$ and $\|\widehat{P}(f)\| \le \|f\|_\infty$ for $f, g \in L^\infty(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p))$. In particular, $(\widehat{P}, \mathcal{H})$ is a representation of the C^*-algebra $L^\infty(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p))$ of all bounded measurable functions on the space $\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p)$.

Proof. The properties of the map $\widehat{P}$ are obtained from the properties of multiplication operators by functions in $L^\infty(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p))$. Uniqueness of the spectral integral follows from the density of the subspace of step functions $\{\chi_E : E \in \Sigma(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p))\}$ in $L^\infty(\mathbb{P}^1_{\text{Berk}}(\mathbb{C}_p))$. $\blacksquare$

Corollary 5.18. *Consider the projection-valued measure* $(P, \Sigma(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)))$ *with a cyclic vector* $f \in \mathcal{H}$. *Define the Hilbert space* $\mathcal{H}_f$ *such that*

$$\mathcal{H}_f := \overline{\mathrm{span}}\{S_\alpha S_\alpha^* f : \alpha \in \mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}\}. \tag{5.20}$$

Then there exists a unique isometry $W_f : L^2(\mu_f) \to \mathcal{H}_f$ *of* $L^2(\mu_f)$ *onto* $\mathcal{H}_f$ *such that*

$$W_f(1) = f, \qquad \text{and} \qquad W_f \widehat{P}(\chi_{Z(Q)}) W_f^* = S_Q S_Q^* \tag{5.21}$$

for all words $Q \in \mathcal{L}_{\mathbb{H}^{\mathbb{Q}}_{\mathrm{Berk}}(\mathbb{C}_p)}$.

Proof. To define $W_f : L^2(\mu_f) \to \mathcal{H}_f$, we set

$$W_f(1) = f \quad \text{and} \quad W_f(\chi_{Z(Q)}) = S_Q S_Q^* f$$

which then satisfies

$$\int_{\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)} |\chi_Q(x)|^2 \, d\mu_f(x) = \langle S_Q S_Q^* f, S_Q S_Q^* f \rangle = \|W_f(\chi_Q)\|^2_{\mathcal{H}_f}$$

and thus extends to an isometry $W_f : L^2(\mu_f) \to \mathcal{H}_f$. Again, uniqueness follows from density of simple functions in $L^2(\mu_f)$. ∎

5.4. Perron–Frobenius operator

Consider the *transfer operator* $T_\sigma : L^2(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p), \mu_G) \to L^2(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p), \mu_G)$ that composes with the coding map denoted by $\sigma : \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) \to \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$,

$$(T_\sigma \psi)(x) = \psi\big(\sigma(x)\big). \tag{5.22}$$

One can associate to the operator T_σ its adjoint P_σ given by

$$\int \psi P_\sigma(\xi) d\mu = \int T_\sigma(\psi) \xi \, d\mu \tag{5.23}$$

and called the *Perron–Frobenius operator*.

Theorem 5.19. *Let* $\{\sigma_q\}$ *be the semibranching system defined associated to the coding map given by* $\sigma : \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) \to \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$. *Then, the Perron–Frobenius operator* P_σ *is of the form*

$$(P_\sigma \xi)(x) = \sum_q \chi_{Z(q)} \xi\big(\sigma_q(x)\big). \tag{5.24}$$

Proof. In the Hilbert space $L^2(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p), \mu_G)$, we can write

$$\begin{aligned}\langle T_\sigma(\psi), \xi\rangle &= \int_{\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)} \overline{\psi\big(\sigma(x)\big)}\xi(x)\, d\mu(x)\\ &= \sum_{q\in\mathbb{Q}_1} \int_{F(q)} \overline{\psi(u)}\xi\big(\sigma_q(u)\big)\, d\mu(u)\\ &= \Big\langle \psi, \sum_{q\in\mathbb{Q}_1} \chi_{F(q)}\xi\circ\sigma_q\Big\rangle.\end{aligned}$$

The left-hand side is recognized as the Perron–Frobenius operator. ∎

Since S_q^* is defined as

$$S_q^*\xi(x) = \chi_{F(q)}\xi\circ\sigma_q(x)$$

we immediately deduce the following corollary.

Corollary 5.20. *Let $\{\sigma_q\}$ be the semibranching system defined associated to the coding map given by $\sigma : \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) \to \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$. Then, the Perron–Frobenius operator P_σ is of the form*

$$P_\sigma = \sum_q S_q^* \tag{5.25}$$

and P_σ belongs to $A(\mathcal{H})$.

6. KMS states and invariant measures

6.1. Unitary representations of $\mathrm{PGL}_2(\mathbb{C}_p)$

We would like to start this section with some remarks on representations of $\mathrm{PGL}_2(\mathbb{C}_p)$ as a discrete group in the Hilbert space $L^2(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p), \mu_G)$.

Gromov hyperbolic space. Let x, y, z be points of $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$, not all equal. We define the *Gromov product* denoted $(x|y)_z$ by

$$(x|y)_z = \rho(w, z), \tag{6.1}$$

where w is the first point where the unique paths from x to z and y to z intersect. By convention, we set $(x|y)_z = +\infty$ if $x = y$ and x is a point of type I, and we set $(x|y)_z = 0$ if $x = z$ or $y = z$.

If $x, y, z \in \mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)$, then one checks easily that

$$(x|y)_z = \frac{1}{2}\big(\rho(x, z) + \rho(y, z) - \rho(x, y)\big). \tag{6.2}$$

This is the usual definition of the Gromov product in Gromov's theory of δ-hyperbolic spaces, with $\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)$ being an example of a 0-hyperbolic space.

Proposition 6.1 ([14]). $\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)$ *is Gromov* 0*-hyperbolic.*

Let ζ_G be the Gauss point of $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$. Define the *fundamental potential kernel relative*, written $\kappa_z(x, y)$, and the *canonical distance relative to* z, written $[x, y]_z$, by setting

$$\kappa_z(x, y) = -\log_v[x, y]_z = (x|y)_\zeta - (x|z)_\zeta - (y|z)_\zeta. \tag{6.3}$$

In particular, if one takes $z = \zeta_G$, then the fundamental kernel simplifies:

$$\kappa_G(x, x) = -\log_v[x, x]_G = -\log_v\big(\mathrm{diam}(x)\big). \tag{6.4}$$

Then, we define the following measure on $\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)$

$$d\mu(x) = p^{-\kappa_G(x,x)} d\mu_G(x) = \mathrm{diam}(x) d\mu_G(x). \tag{6.5}$$

Since the measure μ_G is $\mathrm{PGL}_2(\mathbb{C}_p)$-invariant, the measure μ is quasi-invariant under the action

$$\sigma_g : \mathrm{PGL}_2(\mathbb{C}_p) \times \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) \to \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p), \qquad \sigma_g(x) = \begin{pmatrix} z & a \\ 0 & 1 \end{pmatrix} \cdot x. \tag{6.6}$$

The corresponding Radon–Nikodym derivatives are given by

$$\delta(g)(x) = \frac{d(\sigma_g)_*\mu}{d\mu}(x) = \frac{\mathrm{diam}(g \cdot x)}{\mathrm{diam}(x)} = |z|. \tag{6.7}$$

Proposition 6.2. *The following maps*

$$(U_s(g)f) := e^{is}\sqrt{|z|}\, f(g \cdot x) \tag{6.8}$$

define a family of unitary representation of the group $\mathrm{PGL}_2(\mathbb{C}_p)$ *on* $L^2(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p), \mu)$*, parametrized by* $s \in \mathbb{R}$.

Proof. Clearly, $U_s(g)f$ is measurable, and we also find

$$\begin{aligned}
\|U_s(g)f\|_2^2 &= \int_{\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)} |z||f(g \cdot x)|^2 \, d\mu(x) \\
&= \int_{\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)} |f(g \cdot x)|^2 \, \frac{d(\sigma_g)_*\mu}{d\mu} d\mu(x) \\
&= \int_{\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)} |f(g \cdot x)|^2 \, d(\sigma_g)_*\mu(x) \\
&= \int_{\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)} |f(x)|^2 \, d\mu(x) = \|f\|^2.
\end{aligned}$$

Thus, $U_s(g)f$ defines an isometry of $L^2(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p), \mu)$. We also observe that, for $g, h \in \mathrm{PGL}_2(\mathbb{C}_p)$, we can write

$$U_s(gh)f = e^{2is}\sqrt{|z_g|}\sqrt{|z_h|}f(gh \cdot x) = U_s(g)\big(U_s(h)f\big). \tag{6.9}$$

In particular, we see that each isometry $U_s(g)$ is surjective with

$$U_s(g^{-1}) = U_s(g)^{-1}. \qquad \blacksquare$$

Let us recall the definition of a covariant representation of a triple C^*-dynamical system.

Definition 6.3. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C^*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is a pair (v, π) consisting of a unitary representation $v : G \to U(H)$ (the unitary group of H) and a representation $\pi : A \to B(H)$, satisfying the *covariant condition*

$$v(g)\pi(a)v(g)^* = \pi\big(\alpha_g(a)\big)$$

for all $g \in G$ and $a \in A$. It is called *nondegenerate* if π is nondegenerate.

In the present context, we consider the C^*-algebra $C_0(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p))$ of continuous functions on the hyperbolic Berkovich line. We consider the representation by left-multiplication:

$$\pi : C_0\big(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)\big) \to B\big(L^2\big(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p), \mu\big)\big), \quad \pi(f)\psi(x) = f(x)\psi(x) \tag{6.10}$$

and we denote the Hilbert space $L^2(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)$ by $\mathcal{H}$.

Theorem 6.4. *For $s \in \mathbb{R}$, the pair (U_s, π) is a covariant representation of the following triple* $(\mathrm{PGL}_2(\mathbb{C}_p), C_0(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)), \sigma)$.

Proof. We verify that the pair (U_s, π) satisfies the covariant condition:

$$\begin{aligned}\big(U_s(g)\pi(a)U_s(g)^*\big)f(x) &= \big(U_s(g)\pi(a)\big)\big(e^{is}|z|^{-\frac{1}{2}}f(g^{-1} \cdot x)\big)\\ &= U_s(g)\big(e^{is}|z|^{-\frac{1}{2}}a(x)f(g^{-1} \cdot x)\big) = a(g \cdot x)f(x)\end{aligned}$$

for $a \in C_0(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p))$ and $g \in \mathrm{PGL}_2(\mathbb{C}_p)$. $\blacksquare$

Let us denote by ν the Haar measure on $\mathrm{PGL}_2(\mathbb{C}_p)$. We let

$$C_c\big(\mathrm{PGL}_2(\mathbb{C}_p), C_0\big(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)\big), \sigma\big)$$

be the $*$-algebra of compactly supported continuous functions

$$f : \mathrm{PGL}_2(\mathbb{C}_p) \to C_0\big(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)\big),$$

with pointwise addition and scalar multiplication. We define a norm $\|\cdot\|_1$ on the algebra $C_c(\mathrm{PGL}_2(\mathbb{C}_p), C_0(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)), \sigma)$ by

$$\|f\|_1 = \int_{\mathrm{PGL}_2(\mathbb{C}_p)} \|f(g)\| d\nu(g).$$

Then, the space $L^1(\mathrm{PGL}_2(\mathbb{C}_p), C_0(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)), \sigma)$ is the Banach $*$-algebra obtained by completion of the algebra $C_c(\mathrm{PGL}_2(\mathbb{C}_p), C_0(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)), \sigma)$ with respect to $\|\cdot\|_1$.

We can define a representation $\alpha : C_c(\mathrm{PGL}_2(\mathbb{C}_p), C_0(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)), \sigma) \to B(H)$ given by

$$\begin{aligned}\alpha_s(f)\psi(x) &= \int_{\mathrm{PGL}_2(\mathbb{C}_p)} \pi(f_g) U_s(g)\psi d\nu(g) \\ &= \int_{\mathrm{PGL}_2(\mathbb{C}_p)} f_g(x)\psi(g\cdot x)e^{is}\sqrt{|z|}d\nu(g) \qquad (6.11)\end{aligned}$$

and define the *crossed product* C^*-algebra $C_0(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)) \rtimes_\sigma \mathrm{PGL}_2(\mathbb{C}_p)$ to be the norm closure of the image $\alpha_s(L^1(\mathrm{PGL}_2(\mathbb{C}_p), C_0(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)), \sigma))$.

Proposition 6.5. *The pair (α_s, H) defines a representation of the crossed-product C^*-algebra $C_0(\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)) \rtimes_\sigma \mathrm{PGL}_2(\mathbb{C}_p)$.*

6.2. *K*-cycle and dynamical system

In the previous section, we have constructed a C^*-algebra on the hyperbolic projective line $\mathbb{H}_{\mathrm{Berk}}(\mathbb{C}_p)$. If we see $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ as a tree, then we studied crossed product C^*-algebra on the interior of the tree. In this last section, we show that crossed product C^*-algebra can also be used to study the boundary $\partial\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ which is identified with $\mathbb{P}^1(\mathbb{C}_p)$.

Consider Γ be a finitely generated subgroup of $\mathrm{PGL}_2(\mathbb{C}_p)$. Moreover, let Γ be torsion-free, in which case it is called a (p-adic) *Schottky group*. In fact, a Schottky group is a free group [25].

Definition 6.6 (Limit set, [20]). Let (X, d) be a metric space and $\Gamma \subset \mathrm{Iso}(X)$ be a discrete subgroup. The *limit set* of Γ is the subset $\Lambda = \Lambda(\Gamma)$ of ∂X of points which are accumulation points of orbits in X. That is

$$\Lambda := \{y \in \partial X \mid y = \lim\{\gamma_m(x)\} \text{ for some } x \in X \text{ and } \{\gamma_m\} \text{ a sequence in } \mathrm{Iso}(X)\}.$$

Since $\mathrm{PGL}_2(\mathbb{C}_p)$ acts transitively on type II points in $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$, then the limit set of Γ is $\mathbb{P}^1(\mathbb{C}_p)$. Starting from the commutative algebra of Lipschitz-continuous functions $C(\mathbb{P}^1(\mathbb{C}_p))$ over $\mathbb{P}^1(\mathbb{C}_p)$, we consider the *reduced cross-product* for which

we briefly recall the construction. Consider the integrated representation α defined in the previous section.

$$\alpha : L^1\big(\mathrm{PGL}_2(\mathbb{C}_p), C(\mathbb{P}^1), \sigma\big) \to B(\mathcal{H}), \qquad \alpha(f)\psi(\xi) = \sum_{\gamma\in\Gamma} f_\gamma(\xi)(U_\gamma\psi)(\xi).$$

The *reduced cross-product C^*-algebra* $C_r^*(\Gamma, \mathbb{P}^1(\mathbb{C}_p))$ is the completion of the Banach algebra $L^1(\mathrm{PGL}_2(\mathbb{C}_p), C(\mathbb{P}^1(\mathbb{C}_p)))$ with respect to the operator norm

$$f \mapsto \|\alpha(f)\|_{B(\mathcal{H})}. \tag{6.12}$$

Implicitly, the definition of $C_r^*(\Gamma, \mathbb{P}^1(\mathbb{C}_p))$ is a representation of the Banach algebra $L^1(\mathrm{PGL}_2(\mathbb{C}_p), C(\mathbb{P}^1(\mathbb{C}_p)))$ and therefore of the full cross-product $C(\mathbb{P}^1(\mathbb{C}_p)) \rtimes_\sigma \mathrm{PGL}_2(\mathbb{C}_p)$. Thus, there is a surjective *-homomorphism between the crossed product algebras $C(\mathbb{P}^1(\mathbb{C}_p)) \rtimes_\sigma \mathrm{PGL}_2(\mathbb{C}_p) \to C_r^*(\Gamma, \mathbb{P}^1(\mathbb{C}_p))$.

Time evolution. One can define the time evolution as the one-parameter family acting on the reduced C^*-algebra $C_r^*(\Gamma, \mathbb{P}^1(\mathbb{C}_p))$ given by

$$\alpha_t\left(\sum_\gamma f_\gamma(\xi)U_\gamma\right) = \sum_\gamma p^{itB_\xi(\zeta,\gamma\cdot\zeta)} f_\gamma(\xi)U_\gamma \tag{6.13}$$

where $B(\zeta, \gamma\zeta, \xi)$ is the *Busemann function* based at the Gauss point ζ and, defined by:

$$B(\zeta, \gamma\zeta, \xi) = \lim_{\substack{x\to\xi \\ x\in\mathbb{H}^1_{\mathrm{Berk}}}} (\zeta, x)_{\gamma\zeta} - (\gamma\zeta, x)_\gamma = \lim_{\substack{x\to\xi \\ x\in\mathbb{H}^1_{\mathrm{Berk}}}} \log_v\left(\frac{\mathrm{diam}(\gamma^{-1}x)}{\mathrm{diam}(x)}\right) \tag{6.14}$$

for all $\xi \in \mathbb{P}^1(\mathbb{C}_p)$, using the Gromov product.

Lemma 6.7. *α is a strongly-continuous one parameter group of $*$-automorphisms of the cross-product algebra $C(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)) \rtimes \Gamma$.*

Proof. Let $\gamma' \mapsto (\xi \mapsto h(\xi)U_{\gamma'}) \in \mathcal{H}$, then define the unitary operator $v(t, \xi)$ by

$$v(t, \xi)h(\xi)U_{\gamma'} = p^{itB_\xi(\zeta,\gamma'\cdot\zeta)}h(\xi)U_{\gamma'}. \tag{6.15}$$

Then, consider the operators $v(t, \xi)f(\xi)U_\gamma v(t, \xi)^*$ which acts on the Hilbert space $\mathcal{H}$ as follows

$$\begin{aligned}\big[v(t, \xi)f(\xi)U_\gamma v(t, \xi)^*\big]h(\xi)U_{\gamma'} &= \big[v(t, \xi)f(\xi)\big]p^{-itB_{\gamma^{-1}\xi}(\zeta,\gamma'\cdot\zeta)}h(\gamma^{-1}\cdot\xi)U_{\gamma\gamma'} \\ &= p^{it(B_\xi(\zeta,\gamma\gamma'\cdot\zeta)-B_{\gamma^{-1}\xi}(\zeta,\gamma'\cdot\zeta))}f(\xi)h(\gamma^{-1}\cdot\xi)U_{\gamma\gamma'}.\end{aligned}$$

From the definition of the Busemann function, we can show that

$$-B_{\gamma^{-1}\xi}(\zeta,\gamma'\cdot\zeta)=B_{\gamma^{-1}\xi}(\gamma'\cdot\zeta,\zeta),\quad B_{\gamma^{-1}\xi}(\zeta,\gamma'\cdot\zeta)=B_{\xi}(\gamma\zeta,\gamma\gamma'\cdot\zeta)$$

and $B_\xi(\zeta,\gamma\gamma'\cdot\zeta)+B_\xi(\gamma\gamma'\cdot\zeta,\gamma\zeta)=B_\xi(\zeta,\gamma\zeta)$, from which we deduce that

$$\begin{aligned}\big[v(t,\xi)f(\xi)U_\gamma v(t,\xi)^*\big]h(\xi)U_{\gamma'}&=p^{itB_\xi(\zeta,\gamma\cdot\zeta)}f(\xi)h(\gamma^{-1}\cdot\xi)U_{\gamma\gamma'}\\&=\alpha_t\big(f(\xi)U_\gamma\big)h(\xi)U_{\gamma'}.\end{aligned}$$

Thus, the map α_t coincides on the reduced C^*-algebra $C_r^*(\Gamma,\mathbb{P}^1(\mathbb{C}_p))$ with the conjugate by the unitary operator $v(t,\xi)$. Hence, $t\mapsto\alpha_t$ is a strongly continuous $*$-automorphism which clearly satisfies for any time $t,t'\in\mathbb{R}$, $\alpha_t\circ\alpha_{t'}=\alpha_{t+t'}$. ■

The Patterson–Sullivan measure on $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$. Let us recall the definition of the Patterson–Sullivan measure. In addition to the Busemann function $B(\zeta,\gamma\zeta,\xi)$, consider the function

$$j_\gamma(\xi)=p^{B(\zeta,\gamma\zeta,\xi)}=\lim_{\substack{x\to\xi\\ x\in\mathbb{H}^1_{\mathrm{Berk}}}}\frac{\mathrm{diam}(\gamma^{-1}x)}{\mathrm{diam}(x)}. \tag{6.16}$$

Definition 6.8. Let $D\geq 0$ and μ a finite (and non-trivial) measure on $\mathbb{P}^1(\mathbb{C}_p)$. The measure μ is called Γ-quasiconformal of dimension D if the pullback measures $\gamma^*\mu$ ($\gamma\in\Gamma$) are pairwise absolutely continuous and there exists $C\geq 1$ such that

$$C^{-1}j_\gamma(\xi)^D\leq\frac{d(\gamma^*\mu)}{d\mu}\leq Cj_\gamma^D,\qquad \mu-a.e. \tag{6.17}$$

Let us now recall the definition of the *critical exponent* of Γ as the number

$$\delta(\Gamma)=\limsup_{R\to\infty}\frac{1}{R}\log\big\{\gamma\in\Gamma:\rho(\zeta,\gamma\zeta)\leq R\big\}. \tag{6.18}$$

It is equivalently defined as the abscissa of convergence of the Poincaré series

$$\wp_\Gamma(s)=\sum_{\gamma\in\Gamma}e^{-s\mathrm{diam}(\gamma\zeta)}. \tag{6.19}$$

Proposition 6.9 ([14, Theorem 5.4]). *If $\delta(\Gamma)<\infty$, then there exists a (unique) measure Γ-quasiconformal of dimension $\delta(\Gamma)$ with support coinciding with $\mathbb{P}^1(\mathbb{C}_p)$. This measure is called the Patterson–Sullivan measure and denoted by $\mu_{PS,\zeta}$.*

We can then define the positive functional $\tau:C_r^*(\Gamma,\mathbb{P}^1(\mathbb{C}_p))\to\mathbb{C}$ by

$$\tau(f)=\int_{\mathbb{P}^1(\mathbb{C}_p)}f_e(\xi)d\mu_{PS,\zeta}(\xi). \tag{6.20}$$

KMS$_\beta$*-states.* Consider a C^*-dynamical, i.e., a pair (A, α) with A a C^*-algebra and $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ a time evolution.

Definition 6.10 (KMS state). A state φ on A satisfies the Kubo–Martin–Schwinger (KMS) condition with respect to α at inverse temperature $\beta \in [0, \infty)$ (φ is a α-KMS$_\beta$ state), if

$$\varphi(ab) = \varphi\big(b\alpha_{i\beta}(a)\big) \tag{6.21}$$

for all element a, b in a dense subalgebra $A^{an} \subset A$ of *analytic elements*.

Then, we can show that the C^*-dynamical $(C_r^*(\Gamma, \mathbb{P}^1(\mathbb{C}_p)), \alpha_t)$ admits a unique KMS-state that is exactly given by the Patterson–Sullivan measure on the boundary of the Berkovich projective line.

Proposition 6.11 ([23, Proposition 5.11]). *The reduced cross-product C^*-algebra*

$$C_r^*\big(\Gamma, \mathbb{P}^1(\mathbb{C}_p)\big)$$

has a unique KMS$_\beta$ *state with inverse temperature $\beta = \delta(\Gamma)$ obtained by integration with the Patterson–Sullivan measure:*

$$\varphi_{\beta,\zeta}\left(\sum_\gamma f_\gamma(\xi)U_\gamma\right) = \int_{\mathbb{P}^1(\mathbb{C}_p)} f_e(\xi)d\mu_{PS,\zeta}(\xi). \tag{6.22}$$

Hamiltonian operator. If we denote the family Radon–Nikodym derivatives as,

$$\delta(\gamma) = \frac{d\gamma^*\mu_{PS}}{d\mu_{PS}} \tag{6.23}$$

then. as previously. the family of operators acting on $L^2(\mathbb{P}^1, \mu_{PS})$:

$$U(\gamma)f(\xi) = \sqrt{\delta(\gamma)}f(\gamma \cdot \xi) \tag{6.24}$$

defines a unitary representation of the Schottky group Γ. Again, we denote by π the representation by left-multiplication of $C(\mathbb{P}^1(\mathbb{C}_p))$ on $L^2(\mathbb{P}^1(\mathbb{C}_p), \mu_{PS})$.

Theorem 6.12. *The pair (π, U) of maps defines a regular covariant representation of the crossed product C^*-algebra $C(\mathbb{P}^1(\mathbb{C}_p)) \rtimes \Gamma$. In this representation, the time evolution is implemented by the Hamiltonian*

$$Hf(\xi)U(\gamma) = B(\zeta, \gamma\zeta, \xi)f(\xi)U(\gamma) \quad \textit{such that} \quad \alpha_t(a) = e^{itH}ae^{-itH}. \tag{6.25}$$

Proof. From Lemma 6.7, we have shown the time evolution α_t is given as the conjugate of a unitary operator $v(t, \xi)$ such that

$$\alpha_t(a)(\xi) = v(t, \xi)a(\xi)v(t, \xi)^*.$$

Then, the statement follows from the fact that there exists a selfadjoint operator H acting on the Hilbert space $\mathcal{H}$ such that $v(t, \xi) = e^{itH}$. ∎

Following [18], we can then introduce on the boundary $\partial\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_{\mathbb{p}})$ a (twisted) spectral triple given by $(C_r^*(\Gamma, \mathbb{P}^1(\mathbb{C}_p)), \mathcal{H}, D)$ where the Dirac operator D is such that $|D| = H$.

References

[1] M. Baker and R. S. Rumely, *Potential theory and dynamics on the Berkovich projective line*. Math. Surveys Monogr. 159, American Mathematical Society, Providence, RI, 2010 Zbl 1196.14002 MR 2599526

[2] I. Banič, M. Črepnjak, M. Merhar, U. Milutinović, and T. Sovič, Ważewski's universal dendrite as an inverse limit with one set-valued bonding function. *Glas. Mat. Ser. III* **48(68)** (2013), no. 1, 137–165 Zbl 1276.54025 MR 3064249

[3] G. Boava, G. G. De Castro, D. Gonçalves, and D. W. Van Wyk, Algebras of one-sided subshifts over arbitrary alphabets. *Rev. Mat. Iberoam.* **40** (2024), no. 3, 1045–1088 Zbl 07858193 MR 4739831

[4] G. Boava, G. G. de Castro, D. Gonçalves, and D. W. van Wyk, C^*-algebras of one-sided subshifts over arbitrary alphabets. *Discrete Contin. Dyn. Syst.* **45** (2025), no. 9, 3275–3295 Zbl 08037072 MR 4893141

[5] O. Bratteli and P. E. T. Jorgensen, Iterated function systems and permutation representations of the Cuntz algebra. *Mem. Amer. Math. Soc.* **139** (1999), no. 663, 1–89 Zbl 0935.46057 MR 1469149

[6] W. J. Charatonik and A. Dilks, On self-homeomorphic spaces. *Topology Appl.* **55** (1994), no. 3, 215–238 Zbl 0788.54040 MR 1259506

[7] A. Connes, *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994 Zbl 0818.46076 MR 1303779

[8] A. Connes and C. Consani, The scaling site. *C. R. Math. Acad. Sci. Paris* **354** (2016), no. 1, 1–6 Zbl 1348.18004 MR 3439716

[9] A. Connes and C. Consani, Geometry of the arithmetic site. *Adv. Math.* **291** (2016), 274–329 Zbl 1368.14038 MR 3459019

[10] A. Connes and M. Marcolli, From physics to number theory via noncommutative geometry. In: *Frontiers in number theory, physics, and geometry. I*, pp. 269–347. Springer, Berlin, 2006 Zbl 1126.58006 MR 2261099

[11] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives*. Amer. Math. Soc. Colloq. Publ., American Mathematical Society, Providence, RI, 2008 Zbl 1209.58007 MR 2371808

[12] C. Consani and M. Marcolli, Spectral triples from Mumford curves. *Int. Math. Res. Not.* **2003** (2003), no. 36, 1945–1972 Zbl 1072.11045 MR 1991179

[13] C. Consani and M. Marcolli, Noncommutative geometry, dynamics, and ∞-adic Arakelov geometry. *Selecta Math. (N.S.)* **10** (2004), no. 2, 167–251 Zbl 1112.14023 MR 2080121

[14] M. Coornaert, Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov. *Pacific J. Math.* **159** (1993), no. 2, 241–270 Zbl 0797.20029 MR 1214072

[15] R. Exel and M. Laca, Cuntz-Krieger algebras for infinite matrices. *J. Reine Angew. Math.* **512** (1999), 119–172 Zbl 0932.47053

[16] R. Floricel and A. Ghorbanpour, On inductive limit spectral triples. *Proc. Amer. Math. Soc.* **147** (2019), no. 8, 3611–3619 Zbl 1429.58015 MR 3981138

[17] D. H. Fremlin, *Measure theory. Vol. 4. Topological measure spaces. Part II*. Torres Fremlin, Colchester, 2006 Zbl 1166.28001 MR 2462372

[18] M. Greenfield, M. Marcolli, and K. Teh, Twisted spectral triples and quantum statistical mechanical systems. *p-Adic Numbers Ultrametric Anal. Appl.* **6** (2014), no. 2, 81–104 Zbl 1315.58006 MR 3200942

[19] S. S. Gubser, M. Heydeman, C. Jepsen, M. Marcolli, S. Parikh, I. Saberi, B. Stoica, and B. Trundy, Edge length dynamics on graphs with applications to p-adic AdS/CFT. *J. High Energy Phys.* **2017** (2017), no. 6 Zbl 1380.83066 MR 3681882

[20] S. Hersonsky and J. Hubbard, Groups of automorphisms of trees and their limit sets. *Ergodic Theory Dynam. Systems* **17** (1997), no. 4, 869–884 Zbl 0996.37059 MR 1468105

[21] M. Heydeman, M. Marcolli, I. Saberi, and B. Stoica, Tensor networks, p-adic fields, and algebraic curves: arithmetic and the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. *Adv. Theor. Math. Phys.* **22** (2018), no. 1, 93–176 Zbl 1414.81207 MR 3858021

[22] P. E. T. Jorgensen and A. M. Paolucci, States on the Cuntz algebras and p-adic random walks. *J. Aust. Math. Soc.* **90** (2011), no. 2, 197–211 Zbl 1250.47039 MR 2821778

[23] J. Lott, Limit sets as examples in noncommutative geometry. *K-Theory* **34** (2005), no. 4, 283–326 Zbl 1106.46055 MR 2242562

[24] M. Marcolli and A. M. Paolucci, Cuntz-Krieger algebras and wavelets on fractals. *Complex Anal. Oper. Theory* **5** (2011), no. 1, 41–81 Zbl 1225.46042 MR 2773056

[25] B. Maskit, A characterization of Schottky groups. *J. Analyse Math.* **19** (1967), 227–230 Zbl 0168.06201 MR 0220929

[26] J. Poineau and D. Turchetti, Berkovich curves and Schottky uniformization I: The Berkovich affine line. In: *Arithmetic and geometry over local fields—VIASM 2018*, pp. 179–223. Lecture Notes in Math. 2275, Springer, Cham, 2021 Zbl 1458.14002 MR 4238489

Received 18 November 2024.

Masoud Khalkhali
Department of Mathematics, The University of Western Ontario, 1151 Richmond Street, London, N6A 5B7, Canada; masoud@uwo.ca

Damien Tageddine
Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street, Montreal, H3A 2K6, Canada; damien.tageddine@mail.mcgill.ca