Rev. Mat. Iberoam. (Online first)
DOI 10.4171/RMI/1618

Published by EMS Press

On the stability of free boundary minimal submanifolds in conformal domains

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Abstract. Given an n-dimensional Riemannian manifold with non-negative sectional curvature and convex boundary, that is conformal to a Euclidean convex bounded domain, we show that it does not contain any compact stable free boundary minimal submanifold of dimension $2 \leq k \leq n-2$, provided that either the boundary is strictly convex with respect to any of the two metrics or the sectional curvature is strictly positive.

Introduction

A fundamental topic in differential geometry is the study of submanifolds that are critical points of some functional within a given Riemannian manifold. The classic examples of such objects are *minimal submanifolds*, i.e., the critical points of the volume functional. Indeed, minimal submanifolds have been studied since 1760, and remain among the most active research fields in geometry, with several applications to topology and mathematical physics.

In recent years, when the compact Riemannian manifold M has non-empty boundary ∂M, critical points of the volume functional among submanifolds whose non-empty boundaries lie in ∂M attracted particular attention. Such objects are called *free boundary minimal submanifolds*, and are characterized by having vanishing mean curvature and by meeting ∂M orthogonally along their boundary. Being variational objects, it is also natural to look at the second derivative of the functional and, in particular, at the free boundary minimal submanifolds with non-negative second derivative, which are called *stable*. The free boundary minimal submanifolds that are not stable are called *unstable*.

In this context, it is well known that the positivity of the Riemann curvature tensor of M and/or some convexity of ∂M implies that free boundary minimal submanifolds must be unstable. For instance, the classical work of Simons [25] implies that in a p-convex domain of the round sphere, any k-dimensional free boundary minimal submanifold is unstable if $k \leq n - p$. More generally, one can deduce from Proposition 2 in [15] and Theorem 2 in [24] that the same holds true for Riemannian manifolds whose sectional

Mathematics Subject Classification 2020: 53A10 (primary); 58J32, 53C18 (secondary).
Keywords: minimal surfaces, free boundary problem, conformal geometry.

curvatures lie in the interval $(1/4, 1]$, called $\frac{1}{4}$-*pinched manifolds*, with p-convex boundary that are isometrically immersed as hypersurfaces of some Euclidean space (cf. [13,23] for improved results in this direction). Analogously, Fraser [7,12] proved the same conclusion for strictly p-convex Euclidean domains. The cases of geodesics, 2-dimensional disks and hypersurfaces are better understood, where Morse index estimates are known for more general ambient manifolds (cf. [1, 5, 7, 8, 12, 17–22] and references therein).

In this work, we generalise both Simons' result on domains of the round sphere and Fraser's on Euclidean domains as follows.

Theorem A. *Let $(\Omega, \tilde{g})$ be an n-dimensional Riemannian manifold with p-convex boundary and non-negative sectional curvatures that is conformal to a p-convex bounded domain (Ω, g) of $\mathbb{R}^n$. If either:*

(i) *$(\Omega, \tilde{g})$ has positive sectional curvature;*

(ii) *$\partial\Omega$ is strictly p-convex with respect to g;*

(iii) *$\partial\Omega$ is strictly p-convex with respect to $\tilde{g}$,*

then $(\Omega, \tilde{g})$ does not contain k-dimensional compact stable free boundary minimal submanifolds for all $2 \leq k \leq \min\{n-2, n-p\}$.

Remark. Note that, a priori, a compact free boundary minimal submanifold may have empty boundary, that is, it may be closed, in which case the free boundary condition is vacuous. Nevertheless, under assumption (i), namely that $(\Omega, \tilde{g})$ has positive sectional curvature, the conclusion of Theorem A remains valid. More precisely, there are no closed compact stable minimal submanifolds of dimension $2 \leq k \leq \min\{n-2, n-p\}$ in a such ambient manifold.

Remark. Under the assumptions of Theorem A, the case of geodesics (with non-empty boundary) follows from the work of Frankel (cf. paragraph (E3) Frankel in [22]). The stability inequality takes care of the case of oriented hypersurfaces (cf. Lemma 2.1 in [8]). We stated Theorem A for submanifolds of dimension between 2 and $n-2$ because our argument is only valid for these dimensions. We will emphasize in the proof of Lemma 3.4 where this dimensional condition is essential.

Similarly, we also prove:

Corollary B. *Let $(\Omega, \tilde{g})$ be an n-dimensional Riemannian manifold with non-negative sectional curvature that is conformal to a bounded domain (Ω, g) of $\mathbb{R}^n$ under the conformal transformation $\tilde{g} = e^{2u}g$. If either:*

(i) *$\partial\Omega$ is p-convex with respect to g and u is a strictly increasing function in the exterior direction of $\partial\Omega$;*

(ii) *$\partial\Omega$ is p-convex with respect to $\tilde{g}$ and u is a strictly decreasing function in the exterior direction of $\partial\Omega$,*

then $(\Omega, \tilde{g})$ does not contain k-dimensional compact stable free boundary minimal submanifolds for all $2 \leq k \leq \min\{n-2, n-p\}$.

The results of the present paper can also be interpreted in light of the (elusive) analogy between free boundary minimal submanifolds of the Euclidean ball and closed minimal submanifolds of the round sphere (cf. [9, 11, 16]). Indeed, Theorem A can be viewed as the free boundary counterpart to the result of Giada Franz and the third author [6], along with the subsequent enhancement by Hang Chen [4], which demonstrates that closed minimal submanifolds in positively curved spheres that are conformal to the round one are unstable. The idea used there is to trace the second variation over the family of conformally rescaled constant vector fields projected to the sphere. Our technique is directly inspired by this, indeed, the role of conformally rescaled constant vector fields projected to the sphere is played by conformally rescaled constant vector fields of the Euclidean space (cf. Theorem 5.1.1 in [25] and Theorem 2.2 in [22]).

It is interesting to notice that [4, 6, 13, 23, 24] fit in the context of the celebrated Lawson–Simons conjecture [15], which proposes that minimal submanifolds of closed, simply-connected, $\frac{1}{4}$-pinched Riemannian manifolds must be unstable. Observe that being $\frac{1}{4}$-pinched is motivated by the well-known fact that $\mathbb{CP}^n$ endowed with the Fubini–Study metric is a compact, simply-connected manifold with curvature between $[1/4, 1]$ admitting stable minimal submanifolds, i.e., the complex submanifolds.

Given the aforementioned analogy between closed minimal submanifolds in sphere and free boundary minimal submanifolds in the unit ball, it would be natural to propose a free boundary version of the Lawson–Simons conjecture with some version of Theorem A as a corroborating evidence. Unfortunately, it is unclear to the authors what should be the correct curvature and boundary conditions in this context. Indeed, the standard calibration argument proving the stability of complex submanifolds in Kähler manifolds does not automatically go through when the submanifold has non-empty boundary. For instance, it is straightforward to observe from Theorem 6 in [15] that free boundary minimal submanifolds of strictly convex domains of $\mathbb{CP}^n$ are unstable. On the other hand, if we allow free boundary minimal submanifolds to have empty boundary and we allow weaker convexity assumptions on the boundary of the ambient manifold, then one can find closed complex submanifolds (hence, stable) in p-convex domains of $\mathbb{CP}^n$ with $p \geq 3$ (cf. [26] and the excellent reference [3], p. 351–352).

The paper is organized as follows. In Section 1, we fix our notations and we recall classical notions on conformal geometry and free boundary minimal submanifolds. In Section 2, we study the behavior of a certain quadratic form under conformal transformations. This quadratic form coincides with the second variation formula in the case of minimal submanifolds. In Section 3, we prove Theorem A and Corollary B.

1. Preliminaries

Let (M, g) be a fixed compact n-dimensional Riemannian manifold with Levi-Civita connection ∇.

1.1. Curvature

We define the Riemann curvature tensor R_g of (M,g) as

$$R_g(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathfrak{X}(M).$$

For any $x \in M$ and two-plane $\Pi \subset T_x M$, we denote the sectional curvature of Π, with respect to g, by $K_g(\Pi)$. If Π is generated by the vectors $X, Y \in T_x M$, then $K_g(\Pi)$ can also be indicated by $K_g(X,Y)$, and have the form

$$K_g(X,Y) = \frac{g(R_g(X,Y)X,Y)}{|X|_g^2\,|Y|_g^2 - g(X,Y)},$$

where $|\cdot|_g$ is the norm induced by g.

1.2. Submanifold theory

Let Σ be a k-dimensional submanifold of M, and let us denote by $T\Sigma$ and $N\Sigma$ the tangent bundle and the normal bundle of Σ. The tangent and normal bundles naturally admit connections compatible with the Levi-Civita connection ∇ of M, which we denote by $\nabla^\top$ and $\nabla^\perp$, respectively, and define as $\nabla_X^\top Y = (\nabla_X Y)^\top$ and $\nabla_X^\perp V = (\nabla_X V)^\perp$ for all smooth sections $X, Y \in \mathfrak{X}(\Sigma)$ and $V \in \Gamma(N\Sigma)$. Also, we consider the divergence operator $\operatorname{div}_\Sigma$, which is expressed as $\operatorname{div}_\Sigma(X) = \sum_{i=1}^k g(\nabla_{v_i} X, v_i)$ in a local orthonormal frame $\{v_1, \ldots, v_k\}$ of Σ.

Let $u\colon M \to \mathbb{R}$ be a smooth function. We denote the gradient of u with respect to g by ∇u and its Hessian by $\nabla^2 u$, which is defined, for every $X, Y \in \mathfrak{X}(M)$, as $\nabla^2 u(X,Y) = g(\nabla_X \nabla u, Y)$. We consider the tangent and normal parts of ∇u with respect to Σ respectively as $\nabla^\top u = (\nabla u)^\top$ and $\nabla^\perp u = (\nabla u)^\perp$.

Recall that the second fundamental form α of the submanifold Σ is the symmetric tensor defined by

$$\alpha(X,Y) = (\nabla_X Y)^\perp,$$

for all $X, Y \in \mathfrak{X}(\Sigma)$. The trace of α gives the mean curvature H of Σ. This means that

$$H(x) = \sum_{i=1}^k \alpha(v_i, v_i),$$

for an arbitrary orthonormal basis $\{v_1, \ldots, v_k\}$ of $T_x\Sigma$. If Σ is compact, we can compute its k-volume by

$$\operatorname{vol}_k(\Sigma) = \int_\Sigma d\mu,$$

where $d\mu$ is the volume element induced on Σ by g.

We now discuss the geometric condition we want to impose on ∂M, which we see as a hypersurface in M oriented by its inward-pointing normal η. For every integer $1 \le p \le n-1$, we say that ∂M is *p-convex* if the sum of the lowest p principal curvatures is non-negative at each point of ∂M. Equivalently, this means that at each $x \in \partial M$ for every orthonormal vectors $\{v_1, \ldots, v_p\}$ of $T_x\partial M$, we have

$$\sum_{i=1}^p g(\alpha_{\partial M}(v_i, v_i), \eta) \ge 0,$$

where $\alpha_{\partial M}$ is the second fundamental form of ∂M. If the inequality is strict, we say that ∂M is *strictly p-convex*. The special cases of $p = 1$ or $p = n-1$ correspond to *convexity* and *mean-convexity*, respectively.

1.3. Conformal geometry

We say that two Riemannian metrics $\tilde{g}$ and g on M are *conformal* if there is a smooth function $u\colon M \to \mathbb{R}$ such that $\tilde{g} = e^{2u} g$. In this case, the function e^{2u} is called the *conformal factor*. The following results collect some classical formulas relating objects of $\tilde{g}$ and g in terms of u.

Proposition 1.1 (cf. Theorem 1.159 in [2]). *Let M be an n-dimensional manifold, and let $\tilde{g}$ and g be conformal metrics on M with conformal factor e^{2u}. The following identities holds for vector fields $X, Y, Z \in \mathfrak{X}(M)$.*

(i) *The Levi-Civita connections $\widetilde{\nabla}$ and ∇ respectively of $\tilde{g}$ and g are related by*

$$\widetilde{\nabla}_X Y = \nabla_X Y + X(u)Y + Y(u)X - g(X,Y)\nabla u.$$

(ii) *The Riemann curvature tensors $R_{\tilde{g}}$ and R_g respectively of $\tilde{g}$ and g are related by*

$$\begin{aligned} R_{\tilde{g}}(X,Y)Z = R_g(X,Y)Z &+ X(u)Z(u)Y - Y(u)Z(u)X - X(u)g(Y,Z)\nabla u \\ &+ Y(u)g(X,Z)\nabla u - g(X,Z)\nabla_Y \nabla u + g(Y,Z)\nabla_X \nabla u \\ &- g(X,Z)|\nabla u|_g^2 Y + g(Y,Z)|\nabla u|_g^2 X - \nabla^2 u(X,Z)Y + \nabla^2 u(Y,Z)X. \end{aligned}$$

(iii) *The sectional curvatures $K_{\tilde{g}}$ and K_g respectively of $\tilde{g}$ and g for orthonormal X and Y with respect to g are related by*

$$e^{2u} K_{\tilde{g}}(X,Y) = K_g(X,Y) + X(u)^2 + Y(u)^2 - |\nabla u|_g^2 - \nabla^2 u(X,X) - \nabla^2 u(Y,Y).$$

(iv) *The volume elements $d\tilde{\mu}$ and $d\mu$ respectively of $\tilde{g}$ and g are related by*

$$d\tilde{\mu} = e^{nu}\, d\mu.$$

Using the formula for $\widetilde{\nabla}$ and ∇, it is direct to conclude the following.

Proposition 1.2 (cf. equation (1.163) in [2]). *Let M be a manifold and let Σ be a k-dimensional submanifold of M. If $\tilde{g}$ and g are conformal metrics on M with conformal factor e^{2u}, then the second fundamental forms $\tilde{\alpha}$ and α of Σ with respect to $\tilde{g}$ and g, respectively, are related by*

$$\tilde{\alpha}(X,Y) = \alpha(X,Y) - g(X,Y)\nabla^{\perp} u$$

for all $X, Y \in \mathfrak{X}(\Sigma)$. In particular, the mean curvature vectors $\widetilde{H}$ and H of Σ with respect to $\tilde{g}$ and g, respectively, are related by

$$e^{2u}\widetilde{H} = H - k\nabla^{\perp} u.$$

1.4. Free boundary minimal submanifolds

We now assume that $\partial M \neq \emptyset$, and we consider $f\colon \Sigma \to M$ to be a compact k-dimensional submanifold of M such that $f(\Sigma) \cap \partial M = f(\partial\Sigma)$ and $\partial\Sigma \neq \emptyset$. In this context, a *variation* of Σ is a map $F\colon (-\varepsilon, \varepsilon) \times \Sigma \to M$ satisfying:

(i) F is smooth;

(ii) $F(0,x) = f(x)$ for all $x \in \Sigma$;

(iii) $F(t,y) \in \partial M$ for all $y \in \partial\Sigma$.

Every variation F induces a *variational vector field* X, which is the vector field defined, for all $x \in \Sigma$, by

$$X(x) = \frac{d}{dt} f_t(x)\Big|_{t=0},$$

where $f_t(x) = F(t,x)$. Obviously, X is tangent to ∂M along $f(\partial\Sigma)$. Conversely, from any vector field which is tangent to ∂M along $f(\partial\Sigma)$, one can construct a variation via a *twisted exponential map*, as in pp. 210–211 of [14]. This construction modifies the unit normal vector field near the boundary and extends it to a tubular neighborhood so as to preserve the tangency condition along ∂M, thereby generating admissible deformations through proper immersions.

In order to keep our notation light, we will identify Σ and $\partial\Sigma$ with their images $f(\Sigma)$ and $f(\partial\Sigma)$ from now on. In the usual way, for any variational vector field X as above, we can define the first and second variation of the volume of Σ as follows:

$$\delta\Sigma(X) = \frac{d}{dt}\mathrm{vol}_k(f_t(\Sigma))\Big|_{t=0} \quad \text{and} \quad \delta^2\Sigma(X,X) = \frac{d^2}{dt^2}\mathrm{vol}_k(f_t(\Sigma))\Big|_{t=0}.$$

A *free boundary minimal submanifold* is a critical point of the volume functional, i.e.,

$$\delta\Sigma(X) = 0$$

for every variational vector field X. In our setting, the classical first variation formula (cf. Section 2 of [22]) reads

$$\delta\Sigma(X) = -\int_\Sigma g(X,H)\,d\mu + \int_{\partial\Sigma} g(X,\nu_g)\,da,$$

where ν_g is the outward unit conormal of Σ (i.e., ν_g is the unique outer unit vector tangent to Σ and normal to $\partial\Sigma$), $d\mu$ and da are the volume elements induced by g in Σ and $\partial\Sigma$, respectively. Hence, Σ is a free boundary minimal submanifold if and only if $H \equiv 0$ and $g(X,\nu_g) \equiv 0$ for every X tangent to ∂M along $\partial\Sigma$. The latter condition is commonly referred to as the *free boundary condition*, and geometrically means that Σ meets ∂M orthogonally along $\partial\Sigma$.

A free boundary minimal submanifold Σ is *stable* if $\delta^2\Sigma(X,X) \geq 0$ for all X normal to Σ, and *unstable* if there exists a normal X such that $\delta^2\Sigma(X,X) < 0$. In this setting, the classical second variation formula proves that $\delta^2\Sigma(X,X)$ admits the following simpler form (cf. Section 2 of [22]):

$$\begin{aligned}\delta^2\Sigma(X,X) = &\int_\Sigma \Big(|\nabla^\perp X|_g^2 - \sum_{i=1}^k g(R_g(X,v_i)X,v_i) - g(\alpha,X)^2\Big)\,d\mu \\ &+ \int_{\partial\Sigma} g(\nabla_X X,\nu_g)\,da,\end{aligned}$$

where $\{v_1,\ldots,v_k\}$ is an arbitrary orthonormal frame of Σ, while

$$|\nabla^\perp X|_g^2 = \sum_{i=1}^k |\nabla^\perp_{v_i} X|_g^2 \quad \text{and} \quad g(\alpha,X)^2 = \sum_{i,j=1}^k g(\alpha(v_i,v_j),X)^2.$$

2. The second variation of the free boundary volume functional after a conformal change of metric

Let (M, g) be a compact n-dimensional Riemannian manifold with $\partial M \neq 0$, and let Σ be a k-dimensional free boundary submanifold of M such that $\Sigma \cap \partial M = \partial \Sigma$ (not necessarily minimal). We consider the quadratic form defined for every $X \in \Gamma(N\Sigma)$ by

$$Q_g(X, X) = \int_\Sigma S_g(X, X)\, d\mu + \int_{\partial\Sigma} T_g(X, X)\, da,$$

where, respectively at each point of Σ and $\partial\Sigma$, S_g and T_g are the quadratic operators given by

$$S_g(X, X) = |\nabla^\perp X|_g^2 - \sum_{i=1}^k g(R_g(X, v_i)X, v_i) - g(\alpha, X)^2, \quad T_g(X, X) = g(\nabla_X X, \nu_g).$$

As usual, $\{v_1, \dots, v_k\}$ represents a local orthonormal frame of Σ and ν_g is the outward unit conormal of Σ. Observe that since X is normal to Σ, the free boundary condition guarantees that it tangent to ∂M. Also, if Σ is a compact free boundary minimal submanifold, the operator Q_g we have just defined coincides with the stability operator $\delta^2\Sigma$.

Let $\tilde{g}$ be a metric conformal to g with conformal factor e^{2u}. Similar to [6], we now explain how $Q_{\tilde{g}}$ (in particular, $S_{\tilde{g}}$ and $T_{\tilde{g}}$) can be expressed in terms of Q_g (in particular, S_g and T_g). Note that $\Gamma(N\Sigma)$ and the free boundary condition are preserved under conformal change of metrics, hence, it makes sense to consider $Q_{\tilde{g}}$ and Q_g for the given submanifold Σ. The formulas for the interior terms S_g and $S_{\tilde{g}}$ can be recovered from Proposition 4.4 and Corollary 4.5 in [6]. For the sake of completeness, we include the computations here as well.

Lemma 2.1. *Let (M, g) be a compact n-dimensional Riemannian manifold, and let $\tilde{g}$ be a Riemannian metric conformal to g with conformal factor e^{2u}. If Σ is a k-dimensional free boundary minimal submanifold of $(M, \tilde{g})$, then*

$$S_{\tilde{g}}(X, X) = S_g(X, X) + (\nabla^\top u)(|X|_g^2) + |X|_g^2 \operatorname{div}_\Sigma(\nabla u) + k|X|_g^2 |\nabla u|_g^2 + k\nabla^2 u(X, X),$$
$$T_{\tilde{g}}(X, X) = e^u(T_g(X, X) - |X|_g^2 \nu_g(u)),$$

for each vector field $X \in \Gamma(N\Sigma)$. In particular, if $\widetilde{X} = e^{-u}X$, then

$$\begin{aligned} e^{2u} S_{\tilde{g}}(\widetilde{X}, \widetilde{X}) &= S_g(X, X) - |X|_g^2 |\nabla^\top u|_g^2 + |X|_g^2 \operatorname{div}_\Sigma(\nabla u) + k|X|_g^2 |\nabla u|_g^2 \\ &\quad + k\nabla^2 u(X, X), \\ e^u T_{\tilde{g}}(\widetilde{X}, \widetilde{X}) &= T_g(X, X) - |X|_g^2 \nu_g(u). \end{aligned}$$

Proof. Fix $\{\tilde{v}_1, \dots, \tilde{v}_k\}$ a local orthonormal frame of Σ with respect to $\tilde{g}$, which induces the local orthonormal frame of Σ with respect to g: $\{v_1, \dots, v_k\}$, where $v_i = e^u \tilde{v}_i$ for all $i = 1, \dots, k$.

We first deal with $S_{\tilde{g}}(X, X)$, which we recall to be

$$S_{\tilde{g}}(X, X) = |\widetilde{\nabla}^\perp X|_{\tilde{g}}^2 - \sum_{i=1}^k \tilde{g}(R_{\tilde{g}}(X, \tilde{v}_i)X, \tilde{v}_i) - \tilde{g}(\tilde{\alpha}, X)^2.$$

Here, $\tilde{\alpha}$ denotes the second fundamental form of Σ with respect to $\tilde{g}$. Since (cf. item (i) in Proposition 1.1)

$$\tilde{\nabla}^{\perp}_{\tilde{v}_i} X = e^{-u}\big(\nabla_{v_i} X + v_i(u)X\big)^{\perp} = e^{-u}\big(\nabla^{\perp}_{v_i} X + v_i(u)X\big),$$

we obtain the following formula for the first term of $S_{\tilde{g}}(X,X)$:

$$\begin{aligned}|\tilde{\nabla}^{\perp} X|^2_{\tilde{g}} &= \sum_{i=1}^{k} g(\nabla^{\perp}_{v_i} X, \nabla^{\perp}_{v_i} X) + 2\sum_{i=1}^{k} v_i(u) g(\nabla^{\perp}_{v_i} X, X) + \sum_{i=1}^{k} v_i(u)^2 g(X,X)\\ &= |\nabla^{\perp} X|^2_g + (\nabla^{\top} u)(|X|^2_g) + |X|^2_g |\nabla^{\top} u|^2_g.\end{aligned}$$

The second term can be rewritten in the following form using item (ii) in Proposition 1.1:

$$\begin{aligned}\sum_{i=1}^{k} \tilde{g}(R_{\tilde{g}}(X,\tilde{v}_i)X,\tilde{v}_i) &= \sum_{i=1}^{k} \big(\tilde{g}(R_g(X,\tilde{v}_i)X,\tilde{v}_i) + X(u)^2 + \tilde{v}_i(u)|X|^2_g\, \tilde{g}(\nabla u, \tilde{v}_i)\big)\\ &\quad - \sum_{i=1}^{k} \big(|X|^2_g\, \tilde{g}(\nabla_{\tilde{v}_i} \nabla u, \tilde{v}_i) + |X|^2_g |\nabla u|^2_g + \nabla^2 u(X,X)\big)\\ &= \sum_{i=1}^{k} g(R_g(X,v_i)X,v_i) + kX(u)^2 + |X|^2_g |\nabla^{\top} u|^2_g\\ &\quad - |X|^2_g \operatorname{div}_{\Sigma}(\nabla u) - k|X|^2_g |\nabla u|^2_g - k\nabla^2 u(X,X).\end{aligned}$$

For the last term, we first use Proposition 1.2 to compute

$$\tilde{g}(\tilde{\alpha}(\tilde{v}_i,\tilde{v}_j),X) = g(\alpha(v_i,v_j) - g(v_i,v_j)\nabla^{\perp} u, X) = g(\alpha(v_i,v_j),X) - g(v_i,v_j)X(u),$$

which also implies

$$\begin{aligned}\tilde{g}(\tilde{\alpha},X)^2 &= \sum_{i,j=1}^{k} (g(\alpha(v_i,v_j),X) - g(v_i,v_j)X(u))^2\\ &= \sum_{i,j=1}^{k} g(\alpha(v_i,v_j),X)^2 - 2X(u)\sum_{i=1}^{k} g(\alpha(v_i,v_i),X) + X(u)^2 \sum_{i=1}^{k} g(v_i,v_i)^2\\ &= g(\alpha,X)^2 - 2X(u)g(X,H) + kX(u)^2,\end{aligned}$$

where H is the mean curvature of Σ with respect to g. The minimality of Σ with respect to $\tilde{g}$ implies (cf. Proposition 1.2) $g(X,H) = kX(u)$, from which we conclude that

$$\tilde{g}(\tilde{\alpha},X)^2 = g(\alpha,X)^2 - kX(u)^2.$$

Replacing such identities in $S_{\tilde{g}}$ we obtain the desired formula:

$$S_{\tilde{g}}(X,X) = S_g(X,X) + (\nabla^{\top} u)(|X|^2_g) + |X|^2_g \operatorname{div}_{\Sigma}(\nabla u) + k|X|^2_g |\nabla u|^2_g + k\nabla^2 u(X,X).$$

When $\tilde{X} = e^{-u}X$, the only non-tensorial terms in the previous formula to adjust are $(\nabla^{\top} u)(|\tilde{X}|^2_g)$ and $|\nabla^{\perp} \tilde{X}|^2_g$.

The former transforms as

$$(\nabla^\top u)(|\widetilde{X}|_g^2) = (\nabla^\top u)(e^{-2u}|X|_g^2) = e^{-2u}\left((\nabla^\top u)(|X|_g^2) - 2|X|_g^2\,|\nabla^\top u|_g^2\right),$$

while the latter as

$$\begin{aligned}|\nabla^\perp \widetilde{X}|_g^2 &= \sum_{i=1}^k |\nabla^\perp_{v_i}(e^{-u}X)|_g^2 = \sum_{i=1}^k |e^{-u}\nabla^\perp_{v_i}X - e^{-u}v_i(u)X|_g^2 \\ &= e^{-2u}\sum_{i=1}^k \left(|\nabla^\perp_{v_i}X|_g^2 - 2v_i(u)\,g(\nabla^\perp_{v_i}X, X) + |X|_g^2\,v_i(u)^2\right) \\ &= e^{-2u}\left(|\nabla^\perp X|_g^2 - (\nabla^\top u)(|X|_g^2) + |X|_g^2\,|\nabla^\top u|_g^2\right).\end{aligned}$$

We now turn our attention to the boundary term $T_{\tilde g}(X,X) = \tilde g(\widetilde{\nabla}_X X, \nu_{\tilde g})$. Since $\nu_{\tilde g} = e^{-u}\nu_g$, we can use item (i) of Proposition 1.1 to deduce

$$\begin{aligned}T_{\tilde g}(X,X) = \tilde g(\widetilde{\nabla}_X X, \nu_{\tilde g}) &= e^u g(\nabla_X X + 2X(u)X - |X|_g^2\nabla u, \nu_g) \\ &= e^u(T_g(X,X) - |X|_g^2\,\nu_g(u)).\end{aligned}$$

Since $T_{\tilde g}$ is tensorial, it is straightforward to compute $T_{\tilde g}(\widetilde{X}, \widetilde{X})$. ∎

3. Proof of the main results

In this section, we prove Theorem A and Corollary B. Hence, from now on, (Ω, g) will denote a bounded domain of $\mathbb{R}^n$ endowed with the Euclidean metric and $\tilde g$ will be a metric conformal to g with conformal factor e^{2u}.

As a first step, for *any* k-dimensional free boundary submanifold Σ of Ω (not necessarily minimal), we compute the trace of the quadratic form Q_g, restricted to constant vector fields of $\mathbb{R}^n$ projected to the normal bundle of Σ. The free boundary condition guarantees that such vector fields are in $\Gamma(N\Sigma)$, i.e., are tangential to $\partial\Omega$ along $\partial\Sigma$ and so admissible for our problem.

Lemma 3.1. *Let Σ be a k-dimensional free boundary submanifold of (Ω, g). Then at each point of Σ and $\partial\Sigma$, respectively, we have*

$$\sum_{\ell=1}^n S_g(E_\ell^\perp, E_\ell^\perp) = 0 \quad \text{and} \quad \sum_{\ell=1}^n T_g(E_\ell^\perp, E_\ell^\perp) = \sum_{\ell=1}^n g(\alpha_{\partial\Omega}(E_\ell^\perp, E_\ell^\perp), \nu_g),$$

where $\{E_1, \dots, E_n\}$ is an arbitrary orthonormal basis of $\mathbb{R}^n$ and $\alpha_{\partial\Omega}$ is the second fundamental form of $\partial\Omega$ in $\mathbb{R}^n$.

Proof. Given $x \in \Sigma$, let $\{v_1, \dots, v_n\}$ be a local orthonormal frame such that $v_1, \dots, v_k$ are tangent to Σ and $v_{k+1}, \dots, v_n$ are normal to Σ. Since

$$\sum_{\ell=1}^n S_g(E_\ell^\perp, E_\ell^\perp) \quad \text{and} \quad \sum_{\ell=1}^n T_g(E_\ell^\perp, E_\ell^\perp)$$

are independent of the orthonormal basis $\{E_1, \dots, E_n\}$, we can assume that $E_\ell(x) = v_\ell(x)$ for all $\ell = 1, \dots, n$.

For any constant vector field $E \in \mathbb{R}^n$ and v_i as above with $i = 1, \ldots, k$, there holds

$$\nabla^{\perp}_{v_i} E^{\perp} = (\nabla_{v_i} E - \nabla_{v_i} E^{\top})^{\perp} = -\alpha(v_i, E^{\top}).$$

From this, since the Riemann curvature tensor of $\mathbb{R}^n$ is zero, we obtain that

$$S_g(E^{\perp}, E^{\perp}) = |\nabla^{\perp} E^{\perp}|_g^2 - g(\alpha, E^{\perp})^2 = \sum_{i=1}^{k} |\alpha(v_i, E^{\top})|_g^2 - \sum_{i,j=1}^{k} g(\alpha(v_i, v_j), E^{\perp})^2.$$

Therefore, at x, the trace for S_g becomes

$$\begin{aligned}\sum_{\ell=1}^{n} S_g(E_\ell^{\perp}, E_\ell^{\perp}) &= \sum_{\ell=1}^{n}\sum_{i=1}^{k} |\alpha(v_i, v_\ell^{\top})|_g^2 - \sum_{\ell=1}^{n}\sum_{i,j=1}^{k} g(\alpha(v_i, v_j), v_\ell^{\perp})^2 \\ &= \sum_{i,j=1}^{k} |\alpha(v_i, v_j)|_g^2 - \sum_{i,j=1}^{k} |\alpha(v_i, v_j)|_g^2 = 0.\end{aligned}$$

We now consider $x \in \partial\Omega$. By the free boundary condition, we must have that

$$T_g(E^{\perp}, E^{\perp}) = g(\nabla_{E^{\perp}} E^{\perp}, \nu_g) = g(\alpha_{\partial\Omega}(E^{\perp}, E^{\perp}), \nu_g),$$

for any contant vector field $E \in \mathbb{R}^n$. In particular,

$$\sum_{\ell=1}^{n} T_g(E_\ell^{\perp}, E_\ell^{\perp}) = \sum_{\ell=1}^{n} g(\alpha_{\partial\Omega}(E_\ell^{\perp}, E_\ell^{\perp}), \nu_g),$$

and the result follows. ■

Remark 3.2. By [10, 11], for any constant vector field $E \in \mathbb{R}^n$, the second variation of a compact k-dimensional free boundary minimal submanifold Σ of the Euclidean unit ball B^n in the direction of $E^{\perp}$ is

$$Q_g(E^{\perp}, E^{\perp}) = \delta^2\Sigma(E^{\perp}, E^{\perp}) = -k \int_{\Sigma} |E^{\perp}|_g^2 \, d\mu. \tag{3.1}$$

For a compact free boundary submanifold, which is not necessarily minimal, this formula may not always be valid. However, Lemma 3.1 with $\Omega = B^n$ recovers (3.1) traced over constant vector fields of $\mathbb{R}^n$. Indeed, for Σ' generic, Lemma 3.1 with $\Omega = B^n$ implies

$$\sum_{i=1}^{n} Q_g(E_i^{\perp}, E_i^{\perp}) = -(n-k) \cdot \mathrm{vol}_{k-1}(\partial\Sigma'),$$

while, for Σ minimal, (3.1) gives

$$\sum_{i=1}^{n} Q_g(E_i^{\perp}, E_i^{\perp}) = -k(n-k) \cdot \mathrm{vol}_k(\Sigma) = -(n-k) \cdot \mathrm{vol}_{k-1}(\partial\Sigma),$$

where the last equality follows from Proposition 2.4 in [16]. This observation is analogous to the one described in Theorem 5.1 of [6] for the round sphere.

Given Lemmas 2.1 and 3.1, it is natural to consider rescaled constant vector fields of $\mathbb{R}^n$ as competitors for the stability of minimal submanifolds in conformal domains.

Lemma 3.3. *Let (Ω, g) be a bounded domain of $\mathbb{R}^n$, and let $\tilde{g}$ be a Riemannian metric conformal to g with conformal factor e^{2u}. Let Σ be a k-dimensional free boundary minimal submanifold of $(\Omega, \tilde{g})$. Then, respectively at each point of Σ and $\partial\Sigma$, we have*

$$e^{2u}\sum_{\ell=1}^{n} S_{\tilde{g}}(\tilde{E}_\ell^\perp, \tilde{E}_\ell^\perp) = k|\nabla^\perp u|_g^2 - e^{2u}K_{\tilde{g}}(T\Sigma, N\Sigma),$$

$$e^{u}\sum_{\ell=1}^{n} T_{\tilde{g}}(\tilde{E}_\ell^\perp, \tilde{E}_\ell^\perp) = -(n-k)\nu_g(u) + \sum_{\ell=1}^{n} g(\alpha_{\partial\Omega}(E_\ell^\perp, E_\ell^\perp), \nu_g),$$

where $\{E_1, \dots, E_n\}$ is an arbitrary orthonormal basis of $\mathbb{R}^n$, $\tilde{E}_\ell = e^{-u}E_\ell$ for $\ell = 1, \dots, n$, and $\alpha_{\partial\Omega}$ is the second fundamental form of $\partial\Omega$ in $\mathbb{R}^n$. Here, $K_{\tilde{g}}(T\Sigma, N\Sigma)$ denotes

$$K_{\tilde{g}}(T\Sigma, N\Sigma) = \sum_{i=1}^{k}\sum_{r=k+1}^{n} K_{\tilde{g}}(\tilde{v}_i, \tilde{v}_r),$$

where $\{\tilde{v}_1, \dots, \tilde{v}_k, \tilde{v}_{k+1}, \dots, \tilde{v}_n\}$ is any orthonormal frame of $(\Omega, \tilde{g})$ such that $\tilde{v}_1, \dots, \tilde{v}_k$ are tangent to Σ and $\tilde{v}_{k+1}, \dots, \tilde{v}_n$ are normal to Σ.

Proof. Given $x \in \Sigma$, let $\{v_1, \dots, v_n\}$ be a local orthonormal frame with respect to g such that $v_1, \dots, v_k$ are tangent to Σ and $v_{k+1}, \dots, v_n$ are normal to Σ. This local frame induces an orthonormal frame with respect to $\tilde{g}$ with the same properties via $\tilde{v}_\ell = e^{-u}v_\ell$. Since $\sum_{\ell=1}^n S_{\tilde{g}}(\tilde{E}_\ell^\perp, \tilde{E}_\ell^\perp)$ and $\sum_{\ell=1}^n T_{\tilde{g}}(\tilde{E}_\ell^\perp, \tilde{E}_\ell^\perp)$ are independent of the orthonormal basis $\{E_1, \dots, E_n\}$, we can assume that $\tilde{E}_\ell(x) = \tilde{v}_\ell(x) = e^{-u}v_\ell(x)$ for all $\ell = 1, \dots, n$.

Combining Lemmas 2.1 and 3.1, it is straightforward to verify that

$$e^{2u}\sum_{\ell=1}^{n} S_{\tilde{g}}(\tilde{E}_\ell^\perp, \tilde{E}_\ell^\perp) = -(n-k)|\nabla^\top u|_g^2 + (n-k)\operatorname{div}_\Sigma(\nabla u) + k(n-k)|\nabla u|_g^2 + k\sum_{r=k+1}^{n} \nabla^2 u(v_r, v_r).$$

Furthermore, Proposition 1.1 (iii) implies that for all $\ell, m = 1, \dots, n$ with $\ell \neq m$, we have

$$e^{2u}K_{\tilde{g}}(E_\ell, E_m) = E_\ell(u)^2 + E_m(u)^2 - |\nabla u|_g^2 - \nabla^2 u(E_\ell, E_\ell) - \nabla^2 u(E_m, E_m),$$

from which we deduce

$$\begin{aligned} e^{2u}\sum_{i=1}^{k}\sum_{r=k+1}^{n} K_{\tilde{g}}(\tilde{E}_i, \tilde{E}_r) &= e^{2u}\sum_{i=1}^{k}\sum_{r=k+1}^{n} K_{\tilde{g}}(E_i, E_r) \\ &= (n-k)|\nabla^\top u|_g^2 + k|\nabla^\perp u|_g^2 - k(n-k)|\nabla u|_g^2 \\ &\quad - (n-k)\operatorname{div}_\Sigma(\nabla u) - k\sum_{r=k+1}^{n} \nabla^2 u(E_r, E_r). \end{aligned} \tag{3.2}$$

In particular, at x, we obtain the desired formula

$$\sum_{\ell=1}^{n} S_{\tilde{g}}(\tilde{E}_\ell^\perp, \tilde{E}_\ell^\perp) = -\sum_{i=1}^{k}\sum_{r=k+1}^{n} K_{\tilde{g}}(\tilde{v}_i, \tilde{v}_r) + ke^{-2u}|\nabla^\perp u|_g^2.$$

Using again Lemmas 2.1 and 3.1, but now at $x \in \partial\Sigma$, we see that

$$e^u \sum_{\ell=1}^n T_{\tilde g}(\tilde E_\ell^\perp, \tilde E_\ell^\perp) = \sum_{\ell=1}^n T_g(E_\ell^\perp, E_\ell^\perp) - (n-k)\,\nu_g(u)$$
$$= \sum_{\ell=1}^n g(\alpha_{\partial\Omega}(E_\ell^\perp, E_\ell^\perp), \nu_g) - (n-k)\,\nu_g(u),$$

from which we conclude. ∎

We now show how to control $\sum_{\ell=1}^n \int_\Sigma S_{\tilde g}(\tilde E_\ell^\perp, \tilde E_\ell^\perp)\, d\tilde\mu$ in terms of boundary elements.

Lemma 3.4. *Let (Ω, g) be a bounded domain of $\mathbb{R}^n$, and let $\tilde g$ be a Riemannian metric conformal to g with conformal factor e^{2u} with non-negative sectional curvatures. Let Σ be a compact k-dimensional free boundary minimal submanifold of $(\Omega, \tilde g)$ with $2 \le k \le n-2$. Then*

$$\sum_{\ell=1}^n \int_\Sigma S_{\tilde g}(\tilde E_\ell^\perp, \tilde E_\ell^\perp)\, d\tilde\mu \le 2 \int_{\partial\Sigma} \nu_{\tilde g}(u)\, d\tilde a,$$

where $\{E_1, \dots, E_n\}$ is an arbitrary orthonormal basis of $\mathbb{R}^n$, and $\tilde E_\ell = e^{-u} E_\ell$ for every $\ell = 1, \dots, n$. Moreover, if the conformal metric g has positive sectional curvature, then the inequality is strict.

Proof. Given $x \in \Sigma$, let $\{v_1, \dots, v_n\}$ be a local orthonormal frame with respect to g such that $v_1, \dots, v_k$ are tangent to Σ and such that $v_{k+1}, \dots, v_n$ are normal to Σ. Since $\sum_{\ell=1}^n S_{\tilde g}(\tilde E_\ell^\perp, \tilde E_\ell^\perp)$ is independent of the orthonormal basis $\{E_1, \dots, E_n\}$, we can assume that $E_\ell(x) = v_\ell(x)$ for all $\ell = 1, \dots, n$.

As a first step, we observe that (3.2) can be rewritten in the following form:

$$e^{2u} \sum_{i=1}^k \sum_{r=k+1}^n K_{\tilde g}(E_i, E_r) = -(k-1)(n-k)\,|\nabla^\top u|_g^2 - k(n-k-1)\,|\nabla^\perp u|_g^2$$
$$- (n-k)\,\mathrm{div}_\Sigma(\nabla u) - k \sum_{r=k+1}^n \nabla^2 u(E_r, E_r).$$

Moreover, by minimality of Σ with respect to $\tilde g$, one can easily verify that

$$\mathrm{div}_\Sigma(\nabla u) = -g(\nabla u, H) + \mathrm{div}_\Sigma(\nabla^\top u) = -k|\nabla^\perp u|_g^2 + \mathrm{div}_\Sigma(\nabla^\top u),$$

and thus

$$e^{2u} \sum_{i=1}^k \sum_{r=k+1}^n K_{\tilde g}(E_i, E_r) = -(k-1)(n-k)\,|\nabla^\top u|_g^2 + k|\nabla^\perp u|_g^2 - (n-k)\,\mathrm{div}_\Sigma(\nabla^\top u) - k \sum_{r=k+1}^n \nabla^2 u(E_r, E_r). \tag{3.3}$$

Similarly, as $k \le n-2$, we can sum over $r, s = k+1, \dots, n$,, and obtain

$$e^{2u} \sum_{r \neq s} K_{\tilde{g}}(E_r, E_s)$$
$$= -(n-k-1)\Big((n-k)|\nabla^\top u|_g^2 + (n-k-2)|\nabla^\perp u|_g^2 + 2 \sum_{r=k+1}^{n} \nabla^2 u(E_r, E_r)\Big),$$

which, in particular, implies

$$\sum_{r=k+1}^{n} \nabla^2 u(E_r, E_r) = -\frac{1}{2}\Big((n-k)\,|\nabla^\top u|_g^2 + (n-k-2)\,|\nabla^\perp u|_g^2 + \frac{e^{2u}}{n-k-1} \sum_{r \neq s} K_{\tilde{g}}(E_r, E_s)\Big). \tag{3.4}$$

Inserting (3.4) into (3.3), we can directly compute

$$e^{2u} \sum_{i=1}^{k} \sum_{r=k+1}^{n} K_{\tilde{g}}(E_i, E_r) = \frac{1}{2}\big(-(k-2)(n-k)\,|\nabla^\top u|_g^2 + k(n-k)\,|\nabla^\perp u|_g^2\big)$$
$$- (n-k)\operatorname{div}_\Sigma(\nabla^\top u) + \frac{k e^{2u}}{2(n-k-1)} \sum_{r \neq s} K_{\tilde{g}}(E_r, E_s).$$

Isolating $k|\nabla^\perp u|^2$ in the above equality, we can use its expression in Lemma 3.3 to obtain

$$\sum_{\ell=1}^{n} S_{\tilde{g}}(\tilde{E}_\ell^\perp, \tilde{E}_\ell^\perp) = e^{-2u}\left((k-2)\,|\nabla^\top u|_g^2 + 2\operatorname{div}_\Sigma(\nabla^\top u)\right)$$
$$- \frac{1}{n-k}\Big((n-k-2) \sum_{i=1}^{k} \sum_{r=k+1}^{n} K_{\tilde{g}}(v_i, v_r) + \frac{k}{n-k-1} \sum_{r \neq s} K_{\tilde{g}}(v_r, v_s)\Big).$$

Observe that the second term does not depend on the choice of the orthonormal frame and that the first term is in divergence form; indeed,

$$\operatorname{div}_\Sigma(e^{(k-2)u} \nabla^\top u) = (k-2)\, e^{(k-2)u}\, |\nabla^\top u|_g^2 + e^{(k-2)u} \operatorname{div}_\Sigma(\nabla^\top u).$$

Finally, since the sectional curvatures of $\tilde{g}$ are non-negative and $d\tilde{\mu} = e^{ku} d\mu$, the divergence theorem implies

$$\sum_{i=1}^{n} \int_\Sigma S_{\tilde{g}}(\tilde{E}_i^\perp, \tilde{E}_i^\perp)\, d\tilde{\mu} \le 2 \int_\Sigma \operatorname{div}_\Sigma(e^{(k-2)u} \nabla^\top u)\, d\mu = 2 \int_{\partial\Sigma} e^{(k-2)u}\, \nu_g(u)\, da$$
$$= 2 \int_{\partial\Sigma} \nu_{\tilde{g}}(u)\, d\tilde{a},$$

where, in the first inequality, we use that $2 \le k \le n-2$, that $K_{\tilde{g}} \ge 0$ and that

$$(k-2)\,|\nabla^\top u|_g^2 \le 2(k-2)\,|\nabla^\top u|_g^2.$$

The last equality follows from $\nu_{\tilde{g}}(u) = e^{-u}\, \nu_g(u)$ and $d\tilde{a} = e^{(k-1)u}\, da$. ∎

We can finally use the computations that we have carried out to prove our main results: Theorem A and Corollary B.

Proof of Theorem A *and Corollary* B. Let Σ be a free boundary minimal submanifold of $(\Omega, \tilde{g})$ with conformal factor e^{2u}. All we need to show is that $\tilde{\delta}^2\Sigma(X, X) = Q_{\tilde{g}}(X, X) < 0$ for some vector field $X \in \Gamma(N\Sigma)$, where $\tilde{\delta}^2\Sigma$ is the second variation of Σ as a submanifold of $(\Omega, \tilde{g})$. Obviously, it is enough to show that

$$\sum_{\ell=1}^{n} \tilde{\delta}^2\Sigma(\tilde{E}_\ell^\perp, \tilde{E}_\ell^\perp) < 0,$$

where $\{E_1, \ldots, E_n\}$ is an arbitrary orthonormal basis of $\mathbb{R}^n$ and $\tilde{E}_\ell = e^{-u}E_\ell$.

We first prove Theorem A. Combining Lemmas 3.3 and 3.4, we obtain

$$\begin{aligned} \sum_{\ell=1}^{n} \tilde{\delta}^2\Sigma(E_\ell^\perp, E_\ell^\perp) \leq -(n-k-2) \int_{\partial\Sigma} \nu_{\tilde{g}}(u)\, d\tilde{a} \\ + \sum_{\ell=1}^{n} \int_{\partial\Sigma} e^{-u} g(\alpha_{\partial\Omega}(E_\ell^\perp, E_\ell^\perp), \nu_g)\, d\tilde{a}. \end{aligned} \tag{3.5}$$

Now, we can estimate the first term using Lemma 3.3 and the $(n-k)$-convexity of $\partial\Omega$ with respect to $\tilde{g}$ (note that $p \leq n-k$, hence p-convexity implies $(n-k)$-convexity). Indeed, these give

$$\begin{aligned} 0 &\geq \sum_{\ell=1}^{n} \tilde{g}(\tilde{\nabla}_{\tilde{E}_\ell^\perp} \tilde{E}_\ell^\perp, \nu_{\tilde{g}}) \\ &= \sum_{\ell=1}^{n} T_{\tilde{g}}(\tilde{E}_\ell^\perp, \tilde{E}_\ell^\perp) = -(n-k)\,\nu_{\tilde{g}}(u) + \sum_{\ell=1}^{n} e^{-u} g(\alpha_{\partial\Omega}(E_\ell^\perp, E_\ell^\perp), \nu_g), \end{aligned} \tag{3.6}$$

which implies

$$\sum_{\ell=1}^{n} \tilde{\delta}^2\Sigma(E_\ell^\perp, E_\ell^\perp) \leq \frac{2}{n-k} \sum_{\ell=1}^{n} \int_{\partial\Sigma} e^{-u} g(\alpha_{\partial\Omega}(E_\ell^\perp, E_\ell^\perp), \nu_g)\, d\tilde{a} \leq 0.$$

If $\tilde{g}$ has positive sectional curvature or $\partial\Omega$ is strictly $(n-k)$-convex with respect to g or $\tilde{g}$, then the previous inequality is strict. This concludes the proof of Theorem A.

We now prove Corollary B. If $\partial\Omega$ is $(n-k)$-convex with respect to g and $\nu_{\tilde{g}}(u) = e^{-u}\nu_g(u) > 0$, it is straightforward to see from (3.6) that $\partial\Omega$ is strictly $(n-k)$-convex with respect to $\tilde{g}$. In the same way, if $\partial\Omega$ is $(n-k)$-convex with respect to $\tilde{g}$ and $\nu_{\tilde{g}}(u) = e^{-u}\nu_g(u) < 0$, then $\partial\Omega$ is strictly $(n-k)$-convex with respect to g. ■

Acknowledgements. The authors would like to thank Giada Franz and Celso Viana for many helpful comments and discussions. Federico Trinca would also like to express his gratitude to Alessandro Carlotto for introducing him to the work of Lawson–Simons and the study of free boundary minimal submanifolds, and to Jesse Madnick for pointing out the reference [3].

Funding. Alcides de Carvalho was partially supported by CNPq grant 150285/2023-0 as well as FAPEAL under the process E:60030.0000000513/2023. Roney Santos was partially supported by grant 2023/14796-3, São Paulo Research Foundation (FAPESP). Federico Trinca was partially supported by the Royal Society University Research Fellowship Renewal 2022 URF\R\221030.

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Received January 13, 2025; revised January 15, 2026.

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