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Singularities

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Abstract. Singularity theory concerns local and global structure of singularities of algebraic varieties and maps, often focusing on the interplay between algebraic geometry, symplectic topology, algebra, combinatorics, etc.

Mathematics Subject Classification (2020): 14Bxx, 14Exx, 32Sxx, 13Dxx.

Introduction by the Organizers

This workshop fits into a long tradition of Oberwolfach *Singularities* meetings, aimed at gathering researchers working on singularities of algebraic varieties from various points of view. A recurring theme of these events is combining diverse approaches, motivations and tools. This time, it was especially visible during several talks focused around the themes of mirror symmetry, proposing deep and somewhat mysterious connections between complex and symplectic geometry, homological algebra, etc., which arise from studying singularities and their invariants.

The meeting gathered 47 participants representing a broad spectrum of interests, which was a great incentive to exchange ideas and establish new collaborations. We followed a standard schedule of five talks a day, with Wednesday afternoon free for a hike (this time, with excellent weather). Longer breaks after lunch, lively evenings and great working conditions provided a stimulating environment for mathematical discussions. It should be noted that a significant number of talks was given by young researchers.

A short summary below, and subsequent extended abstracts give an overview of diverse themes touched during the workshop, such as: birational geometry, arc spaces, non-archimedean and motivic techniques, various takes on McKay correspondence, maximal Cohen–Macaulay modules, derived categories, etc.

The meeting began with a talk of Barbara Fantechi, who introduced criteria for smoothability of projective schemes with possibly non-isolated lci singularities, thus extending Tziolas's results to non-reduced schemes by geometric methods. Smoothability of log toroidal varieties motivated the talk by Helge Ruddat, who presented an approach towards describing all smooth Fano threefolds as such smoothings, obtained in a combinatorial way from simple 0-mutable polynomials (A_n, and Reid's Tom & Jerry). In dimension 2, classification of log canonical Fanos (del Pezzo surfaces) was presented by Tomasz Pełka, following Palka's program of finding simple $\mathbb{P}^1$-fibrations of almost minimal models.

Such birational-geometric techniques are deeply related with arc spaces, studied during the talks of Tommaso de Fernex and Roi Docampo. The former explained the solution of the Nash problem for 2-dimensional rational double points in any characteristic, focusing on explicit equations rather than topological methods. The latter, motivated by deformation theory of arc spaces, outlined an ambitious program of developing derived algebraic geometry for arc spaces, using Lurie's framework of animated categories. As an application, he presented a far-fetched generalization of the Curve Selection Lemma in arc spaces over a non-perfect field.

Non-archimedean geometry was represented by a talk of Raf Cluckers, concerning new versions of Pila-Wilkie counting theorems in Hensel-minimal structures. A different, metric approach to non-archimedean problems was shown by Immanuel Halupczok, who introduced a new way of measuring non-archimedean distances to certain singular "strata" of a hypersurface over $\mathbb{C}((t))$ (taking into account the shortcomings of Whitney stratifications in this context), leading to a new "intrinsic" tropicalization of such singularities.

Metric aspects of degenerations were the main theme of the talk by Norbert A'Campo, who took this viewpoint to study monodromy groups for plane curve singularities, and suitable lengths of vanishing cycles (and showed that such lengths, despite the name, do not vanish). The talk of Baldur Sigurðsson provided explicit geometric realization of these vanishing cycles, as a spine in the Milnor fiber. This construction is rather intricate, as it involves piecewise rescaling the gradient vector field so that it has a limit at the boundary of the real oriented blowup of the resolution, with some disks collapsed to account for non-trivial monodromy.

Multi-singularities of maps were treated in the lecture of Toru Ohmoto, who presented a proof of Thom–Kazarian principle, laying the foundation for a uniform theory addressing enumerative problems for such singularities.

The classical McKay correspondence, and its modern interpretations, were a common theme of several talks. Eleonore Faber presented a version of this correspondence for order 2 reflection groups, in terms of semi-orthogonal decompositions. Arthur Forey explained how to generalize motivic McKay correspondence of Batyrev and Denef–Loeser to μ_r-actions on Artin stacks, which led to a formula for a certain zeta function in terms of a motivic integral.

Such interpretation of motivic zeta functions for hypersurface singularities, and similar interpretation of Campillo–Delgado–Gusein-Zade motivic Poincaré series for plane curves, were presented by Dimitri Wyss: combined, these results can be seen as a conjectural counterpart of the isomorphism between fixed-point and knot Floer homologies for plane curves: indeed, a bridge to symplectic geometry is provided via Arc–Floer conjecture and Gorsky–Némethi results comparing knot Floer and lattice homology.

Combinatorial framework of lattice homology and Poincaré series was used in the talk of Alex Hof, who applied them to distinguish between finite, tame and wild Cohen-Macaulay type of modules over local rings of complex-analytic curve germs. Another instance of lattice homology was presented by Gergő Schefler, who showed how to associate such an invariant to a submodule N of a finitely generated module M over any Noetherian k-algebra $\mathcal{O}$, so that it categorifies $\dim_k(M/N)$. This new general invariant allows to uniformly reinterpret Ágoston–Némethi lattice homologies of curve and surface singularities, without any assumption on the link.

Geometry of the link of quotient surface singularities, and its secondary characteristic classes, were a starting point of the talk of Agustín Romano-Velazquéz, who used them to study maximal Cohen–Macaulay modules. He showed that the latter are uniquely determined by their rank, first Chern class, and Atiyah-Patodi-Singer invariant; and explained why the analytic datum is necessary.

Contact geometry of links of more general, sandwiched surface singularities was explored in the talk of Olga Plamenevskaya. She classified their weak symplectic fillings in terms of de Jong–van Straten picture deformations, and showed that the Milnor fiber is (by far!) not the unique Stein filling. One dimension higher, symplectic geometry of Milnor fibers of cDV threefold singularities was a subject of the talk of Aline Zanardini. Explicitly computing their symplectic homology, she gave evidence towards Evans–Lekili conjecture asserting that, for a cDV singularity, admitting a small resolution is a contact invariant; and she also gave a new construction of infinitely many contact structures on the sphere S^5. It is worth noting that the computation passes through a mirror symmetry argument, and is performed on the algebraic side.

Homological mirror symmetry played an important role in the talk of Yankı Lekili, who computed deformations of Kalck–Karmazyn algebras of certain quotient surface singularities, interpreting them as endomorphisms of certain Kawamata Lagrangians in Fukaya category of a punctured torus. In higher dimension, this direction was pursued in the talk of Martin Kalck, who presented constructions of Kawamata semi-orthogonal decompositions for weighted projective spaces, and, more generally, cones over nice Fano varieties. As explained during his talk, Kawamata s.o.d.'s are suitable generalizations, to the setting of singular Gorenstein varieties, of both "tilting" and full exceptional collections. Contributing to the latter theme, Atsushi Takahashi presented formulas counting full exceptional collections in the derived category of of $\mathbb{P}^1$ with a fixed orbifold structure. He showed that these counts coincide with degrees of Lyashko–Looijenga maps — in agreement with the expectations of homological mirror symmetry.

The meeting ended with a talk of Adam Parusiński, who proposed a setting to study families of hypersurface singularities, not necessarily isolated, by means of generic projections. To do this, he introduced the notion of nested uniformly transverse Zariski equisingularity, as an appropriate version of genericity.

Furthermore, on Wednesday evening, Laurenţiu Maxim gave a memorial talk in honor of Mihai Tibăr, who passed away shortly before this meeting. This talk explained his contribution to the study of the polar degree (in particular, application of Lefschetz pencils with singularities in the axis, which led June Huh to prove Dimca–Papadima conjecture on homaloidal hypersurfaces), Euler obstruction, its applications to optimization problems, and other insights he developed.

To summarize: the meeting covered a broad range of topics, which allowed the participants to become acquainted with various active areas of research, share their independent insights, and establish new collaborations. We are very grateful to the MFO staff for their hard work maintaining the uniquely stimulating atmosphere at this Institute. We hope that it will result in new, exciting discoveries, to be shared during the next *Singularities* meeting.

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Workshop: Singularities

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Abstracts

Vanishing cycles do not vanish

Norbert A'Campo
(joint work with Pablo Portilla Cuadrado)

1. Hamiltonian vector field, vanishing cycle, geometric monodromy

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial mapping. The complex Hamiltonian vector field $\mathbb{H} = \mathbb{H}_f$ is the vector field on $\mathbb{C}^2$ defined with respect to linear coordinates (x, y) by

$$i_{\mathbb{H}} dx \wedge dy = df$$

or by

$$\mathbb{H} = (f_y, -f_x)$$

in terms of partial derivatives. The vector field $\mathbb{H}$ is tangent to the fibers of f and does not vanish at the smooth points F_t^* of a fiber $F_t := f^{-1}(t)$, $t \in \mathbb{C}$. A linear change of coordinates on $\mathbb{C}^2$ changes $dx \wedge dy$ and $\mathbb{H}_f$ by a constant non-zero factors $\kappa, 1/\kappa$ respectively. A Jonquiére change of coordinates $(x, y) \to (x, y+h(x))$ preserves the complex symplectic form $dx \wedge dy$, so also the field $\mathbb{H}_f$.

The local restriction of $\mathbb{H}_f$ near a singular point p of f is introduced as a new tool in the theory of plane curve singularities in their seminal paper [6] by Pablo Portilla Cuadrado and Nick Salter.

Let f have at $0 \in \mathbb{C}^2$ an isolated singularity with $f(0) = 0$, Lefschetz-Milnor ball B, Milnor number μ, unfolding germ $g : B \times \mathbb{U} \to \mathbb{C}$ and a discriminant $\Delta \subset \mathbb{U}$. We choose $\mathbb{U} \subset \mathbb{C}^\mu$ to be a small ball centered at the origin such that for a small δ all fibers G_λ defined by

$$G_\lambda := \{(x, y) \in B \mid g(x, y, \lambda) = t,\ |t| < \delta,\ \lambda \in \mathbb{U}\}$$

intersect transversely the boundary of B. Moreover, $G_0 = F_0$ should hold, so that the family $f_\lambda := g(?, ?, \lambda)$, $\lambda \in \mathbb{U}$ is the universal deformation of the germ at 0 of the function $f = f_0$.

Let $\mathbb{H}_\lambda$ be the vector field on G_λ defined by

$$i_{\mathbb{H}_\lambda} dx \wedge dy = df_\lambda.$$

For $\lambda \in \mathbb{U}^* := \mathbb{U} \setminus \Delta$ the fibers G_λ are smooth Riemann surfaces with boundary, so we get a locally trivial fibration of manifolds $\pi_f : G \to \mathbb{U}^*$. The fiber-wise restriction to the boundaries $\partial\pi_f : \partial G \to \mathbb{U}^*$ is a trivial fibration since the singularity of f was supposed to be isolated. An important invariant of the singularity of f is the geometric monodromy group Γ_f of π_f, i.e., the image of the geometric monodromy representation

$$T : \pi_1(\mathbb{U}^*, \lambda_0) \to MCG(G_{\lambda_0}, \partial G_{\lambda_0})$$

of the bundle π_f into the relative mapping class group of the fiber over the base point λ_0.

The description of the group Γ_f needs extra care due to the extra symmetries that occur for some singularities of type A_n, D_n or with fiber genus $\leqslant 5$. Here we simplify the presentation by stating that there exists a subgroup $\Gamma_f^+ \subset MCG(G_{\lambda_0}, \partial G_{\lambda_0})$ that contains the group Γ_f as a finite index subgroup and that this index differs from one only for some plane curve singularities as mentioned. For more details, see [6]. As examples:

A_1, A_2**:** $\Gamma_f = MCG$, $\Gamma_f = \Gamma_f^+$,
A_3**:** $\Gamma_f \neq MCG$, $\Gamma_f = \Gamma_f^+$,
A_4**:** $\Gamma_f \neq \Gamma_f^+ \neq MCG$.

An ordered system of non-separating simply closed curves $c_1, c_2, \cdots, c_\mu$ can effectively be constructed by [1, 4, 5], see also [3], such that the corresponding right Dehn twists D_{c_i} generate Γ_f and the composition in that order equals the geometric Milnor monodromy. But it remains the difficult group theoretical problem of deciding for $\phi \in MCG(G_{\lambda_0}, \partial G_{\lambda_0})$ if $\phi \in \Gamma_f$ holds.

The results of [6] give a very effective answer in terms of the winding number function $w_{\mathbb{H}} : S(G_{\lambda_0}) \to \mathbb{N}$ that assigns to a simple parametrized curve c on G_{λ_0} the winding along c of the vector field $\mathbb{H} = \mathbb{H}_{\lambda_0}$ and the speed vector $\dot{c}$.

Theorem 1.1. *(P.-S. [6]) For c a non-separating simply closed curve on G_{λ_0}:*

$$D_c \in \Gamma_f^+ \iff w_{\mathbb{H}}(c) = 0$$

Theorem 1.2. *(P.-S. [6]) For $\phi \in MCG(G_{\lambda_0}, \partial G_{\lambda_0})$:*

$$\phi \in \Gamma_f^+ \iff w_{\mathbb{H}}(c) = w_{\mathbb{H}} \circ \phi(c), \quad c \in S(G_{\lambda_0})$$

If a system as above $c_1, c_2, \cdots, c_\mu$ is at hand one has more effectively:

Theorem 1.3. *(P.-S. [6]) For $\phi \in MCG(G_{\lambda_0}, \partial G_{\lambda_0})$:*

$$\phi \in \Gamma_f^+ \iff w_{\mathbb{H}}(c) = w_{\mathbb{H}} \circ \phi(c), \quad c \in \{c_1, c_2, \cdots, c_\mu\}$$

The winding function $w_{\mathbb{H}}$ only depends upon the homotopy type $[\mathbb{H}]$ of the vector field $\mathbb{H}$. Together with the complex structure on the fibers one gets a framing $(\mathbb{H}, i\mathbb{H})$, whose homotopy type $[\mathbb{H}, i\mathbb{H}]$ depends up to homotopy of framings only on the homotopy type of the field $\mathbb{H}$. So the above theorem is formulated in [6] by

$$\phi \in \Gamma_f^+ \iff [\mathbb{H}, i\mathbb{H}] = [\phi_*\mathbb{H}, \phi_* i\mathbb{H}]$$

expressing that the group Γ_f^+ is a *framed mapping class group.*

2. The flat metric $g_t^{\mathbb{H}_f}$

The homotopy type of the above vector field $\mathbb{H}_f$ allows to grasp the monodromy group Γ_f inside the mapping class group by requiring a property. Clearly, it is also interesting to study the vector field $\mathbb{H}_f$ from a more geometric perspective. The following is such:

Theorem 2.1. *The vector field $\mathbb{H}_f$ defines on the smooth part F_t^* of the fiber $F_t := f^{-1}(t) \subset \mathbb{C}^2$ a Riemannian metric $g_t^{\mathbb{H}_f}$ by requiring that the frame $(\mathbb{H}_f, i\mathbb{H}_f)$ is $g_t^{\mathbb{H}_f}$-orthonormal. The metric $g_t^{\mathbb{H}_f}$ is flat of Gaussian curvature 0 and $\mathbb{H}_f$ trajectories are $g_t^{\mathbb{H}_f}$ geodesics. Moreover, the metric $g_t^{\mathbb{H}_f}$ is in the conformal class of the complex structure of F_t^*. The same statements hold for the unfolding of an isolated singularity of f, so for the compact smooth surfaces with non-empty boundary $G_\lambda, \lambda \in \mathbb{U}^*$, vector field $\mathbb{H}_{f_\lambda}$, and metric $g_0^{\mathbb{H}_{f_\lambda}}$.*

Proof. The metric is clearly in the conformal class of the complex structure of F_t. The non-vanishing $\mathbb{H}_{f,p} \neq 0$ at $p \in F_t^*$ holds, hence the vector field is locally at p the coordinate vector field $\frac{\partial}{\partial u}$ for a real local coordinate system (u, v) and it follows that the flow $\Phi^{\mathbb{H}_f}$ preserves locally $\mathbb{H}_f$. The flow $\Phi^{\mathbb{H}_f}$ is holomorphic since the field $\mathbb{H}_f$ is holomorphic, hence preserves also the field $i\mathbb{H}_f$. Also, the flow $\Phi^{i\mathbb{H}_f}$ preserves the field $\mathbb{H}_f$ and it follows that the Lie bracket $[\mathbb{H}_f, i\mathbb{H}_f]_{Lie}$ vanishes. The coordinates (u, v) can be chosen such that

$$\frac{\partial}{\partial u} = \mathbb{H}_f, \quad \frac{\partial}{\partial v} = i\mathbb{H}_f$$

hold and that the complex coordinate $z = u + iv$ will map isometrically a neighborhood of the point p to a neighborhood of $0 \in \mathbb{C}$ and map the framing $(\mathbb{H}_f, i\mathbb{H}_f)$ to the framing $(1, i)$ of $\mathbb{C}$. This proves that the Gaussian curvature equals 0 and that the trajectories of $\mathbb{H}_f$ are $g_t^{\mathbb{H}_f}$ geodesics. □

By a complex linear change of coordinates of determinant 1 one may suppose without any restriction that the coordinate y is a Weierstrass coordinate for the isolated singularity of f. The form $dx \wedge dy$ and consequently the vector fields $\mathbb{H}_\lambda$ do not change. By choosing the above Milnor ball B and germ $\mathbb{U}^*$ small enough one may assume with one exception that the winding angle of $\mathbb{H}_{f_\lambda}$ and tangent vectors of boundaries of the fibers G_{f_λ} vary in a strictly monotone regular way. The exception concerns the singularity type A_1, the only plane curve singularity such that the winding number $w_{\mathbb{H}}(b)$ for a link component b vanishes. We say that the field $\mathbb{H}$ is *boundary monotone.*

Theorem 2.2. *Let X be a Riemann surface, connected, compact with non-empty smooth (oriented) boundary, together with a boundary monotone, holomorphic, nowhere vanishing vector field $\mathbb{H}$ inducing as above the flat metric $g^{\mathbb{H}}$, such that the pair $(X, \mathbb{H})$ appears in the unfolding of an isolated plane curve singularity f. Then all maximal trajectories are compact, either periodic disjoint from the boundary, or one arc with distinct endpoints, or a finite chain of arcs.*

Proof. The boundary ∂X is a finite union of intervals having as endpoints the points of tangency of $\mathbb{H}$ with the boundary. Exceptionally, if X is homeomorphic to a cylinder, all trajectories are arcs with two endpoints on different boundary components or all trajectories are periodic and $w_{\mathbb{H}}(b) = 0$ for each boundary component. In all other cases, $w_{\mathbb{H}}(b) > 0$ and $\mathbb{H}$ vanishes at each boundary component b exactly $2w_{\mathbb{H}}(b) \geqslant 2$ times. Each boundary component b is the union of $2w_{\mathbb{H}}(b)$ segments where the vector field $\mathbb{H}$ is alternately inward- and outward-pointing. Let $I \in \{I_1, I_2, \cdots, I_{\chi(X)}\}$ be such a segment for which $\mathbb{H}$ is inward-pointing. The flow $\Phi^{\mathbb{H}}(t)$ will push I inwards in the beginning as a strip of constant width and length t. The $g^{\mathbb{H}}$-area grows linearly with t, so there exists $t_1 > 0$ such that the $\Phi^{\mathbb{H}}(t_1)(I)$ hits some segment J where $\mathbb{H}$ is pointing outward. Hence, we have found a flow line, in fact an arc. Let I_J be the maximal subsegment of I of initial points of trajectories that are parallel to this arc and have endpoints in an opposite subsegment. Now we remove from the boundary all the segments I_J, I running over $I_1, I_2, \cdots$, and start over with the remaining inward-pointing segments in the boundary. This gives all arcs and all chains of arcs, which will be organized in "rectangles", which have two opposite sides on the boundary ∂X and which are filled with trajectories connecting those sides. The complement in X of these rectangles is a disjoint union of cylinders filled with periodic trajectories. □

The A_1 singularity given by $f = xy$ is special, $w_{\mathbb{H}_f}(b) = 0$ for both boundary components! The vanishing cycle c_t in F_t, $t > 0$, is the closed curve

$$c_t(s) = (\sqrt{t}e^{\pi i s}, \sqrt{t}e^{-\pi i s}), \quad s \in [-1, 1]$$

One computes the $g^{\mathbb{H}_f}$-length: the lengths are independent of t and equal to π, so c_t shrinks topologically to a point but stays constant in $g^{\mathbb{H}_f}$-length!

A vanishing cycle on F_t, $t > 0$, for the A_3 singularity $f = y^2 + x^4$ is the level curve $f = t$ in the real plane. One estimates its $g^{\mathbb{H}_f}$-length: the length tends with $t \to 0$ to $+\infty$. These examples explain our title.

The singularity A_2 given by $f = y^2 - x^3$ is also special, $w_{\mathbb{H}_f}(b) = -1$, hence the smooth fibers F_λ of the unfolding

$$G(x, y, \lambda_1, \lambda_2) := y^2 - x^3 - \lambda_1 x - \lambda_2$$

can be closed by gluing a disk and extending the field $\mathbb{H}_\lambda$ to a non-vanishing holomorphic vector field. Showing *en passant* in a very elementary way that all complex elliptic curves admit a conformally equivalent flat Riemannian metric!

Especially, the curves $F_t = \{y^2 - x^3 = t\}$, $t \neq 0$, equipped with the flat metric $g^{\mathbb{H}_f}$ are isometric to the hexagonal torus from which one deletes a point. So it was interesting to find more abstractly a fibration with base space $\mathbb{C}^*$ and get an equation free approach to the Milnor fibration. The following is such a construction. Let P_t be the convex hull in $\mathbb{C}$ of the 6 solutions of $z^6 = t$, $t \neq 0$. Let $\tilde{P}_t$ the flat torus that one obtains by identifying opposite boundary edges of P_t by translations. The family $\tilde{P}_t^*$ defines such a fibration. This example has led to *tête-à-tête* monodromy, presented at a previous Oberwolfach Singularities Conference, see report [3] and [4].

The same construction with a metric correction at the center of the polygons works for all Pham-Brieskorn singularities $f = x^p + y^q$ by gluing $r = (p, q)$ flat $2pq/r$-gons with alternating angles $2\pi/p$, $2\pi/q$ and at its center a metric orbifold point. As a result, the Hamiltonian vector field and the decomposition in the above rectangles and cylinders is explicitly constructed.

The smooth part $(F_0^*, g^{\mathbb{H}_f})$ of the singular fiber in the Milnor ball of an isolated plane curve singularity $f = 0$ with r branches is a flat surface with r boundary components and r open ends. Interesting would be to understand the Busemann compactification, which is a union of cobordisms between a component of the link of the singularity and the ideal space of points at infinity that appear by compactifying the corresponding end.

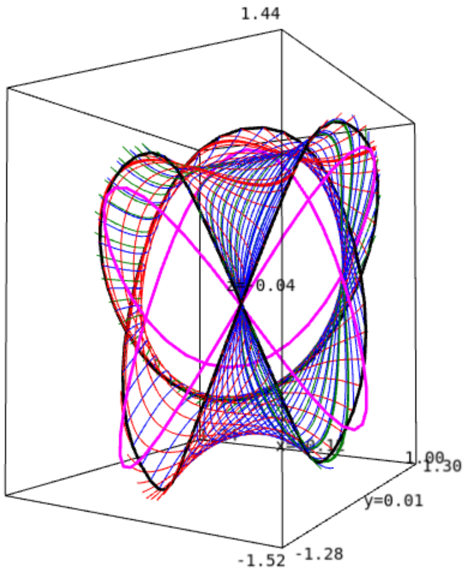

FIGURE 1. Trajectories of $\mathbb{H}$ (red, fine) all from boundary to boundary, of $i\mathbb{H}$ (blue, fine), one tangent to the boundary and many parallel closed curves (green). Boundary of singular fiber (magenta), of smooth fiber $t = 1$ (black).

3. Adjacency

A smooth injective immersed path $\gamma : [0,1] \to \mathbb{U}$ with $\gamma([0,1)) \subset \mathbb{U}^*$, $\gamma(1) \in \Delta$, $\gamma(0) = \lambda_0 \in \mathbb{U}^*$, is called with *Weierstrass ending* if there exists a holomorphically embedded disk $D \subset \mathbb{U}^*$ centered at $\gamma(1)$ and tangent to the speed vector $\dot{\gamma}(1)$, such that the disk D is the Weierstrass coordinate axis for a local equation at $\gamma(1)$ of the discriminant Δ in $\mathbb{U}$. One supposes moreover that $\gamma((1/2, 1]) \subset D$ holds and is a ray to the center of D. This path is called with *Weierstrass ending adjacency* if moreover the fiber $F_{\gamma(1)}$ has only one singularity σ.

After the choice of a monotone Milnor ball B' for the singularity σ, the local fiber $F'_{\gamma(1-\delta)} \subset B$, $0 < \delta \ll 1$, will be a subsurface in $F_{\gamma(1-\delta)}$ such that its local Hamiltonian vector field $\mathbb{H}'$ extends to the Hamiltonian vector field $\mathbb{H}$ on $F_{\gamma(1-\delta)}$. We speculate to gain obstructions to adjacency for plane curve singularities.

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Finiteness and diophantine results in Hensel minimal structures

Raf Cluckers

We present work with Halupczok, Rideau-Kikuchi, Vermeulen [8], [9], [10] which is partially still work in progress and which provides non-archimedean analogues to o-minimality and to the general Pila-Wilkie counting theorem.

The original Pila-Wilkie counting theorem on rational points on definable sets in o-minimal structures states the following.

Theorem 1 ([14])**.** *Let $X \subset \mathbb{R}^n$ be definable in an o-minimal structure. Then for every $\varepsilon > 0$ there exists $c = c_\varepsilon$ such that for every $H \geqslant 1$ one has*

$$\#X^{\mathrm{trans}}(\mathbb{Q}, H) < cH^{\varepsilon}.$$

Here, $X^{\mathrm{trans}}(\mathbb{Q}, H)$ is the set of points $(x_1, \ldots, x_n)$ in $\mathbb{Q}^n$ lying on X^{trans} and with each x_i of height at most H, and X^{trans} is the set $X \setminus X^{\mathrm{alg}}$ where X^{alg} is the algebraic part of X. Recall that X^{alg} is the set of $x \in X$ for which there exists a semi-algebraic curve C lying in X which is of constant local dimension 1 and which contains x.

Pila-Wilkie's Theorem 1 plays an important role in many arithmetic applications, culminating recently in the solution of the André-Oort Conjecture [13].

A precursor to Theorem 1 is the theory of o-minimal structures, where cell decomposition plays important roles. Hensel minimality is for non-achimedean geometry what o-minimality is for the real field. In the talk I recalled the definitions of o-minimality and of Hensel minimality and I explained their similarity.

Let us right away state the non-archimedean analogue of Theorem 1 for Hensel minimal structures on p-adic fields instead of o-minimal structures, as follows. Note that 1-h-minimality is a concrete form of Hensel minimality.

Theorem 2 (Pila-Wilkie type bounds for 1-h-minimal structures, [10])**.** *Let K be a finite field extension of $\mathbb{Q}_p$ for some prime number p. Let $X \subset K^n$ be definable in a 1-h-minimal structure on K. Then for every $\varepsilon > 0$ there exists $c = c_\varepsilon$ such that for every $H \geqslant 1$ one has*

$$\#X^{\mathrm{trans}}(\mathbb{Q}, H) < cH^{\varepsilon}.$$

The notion of Hensel minimality (and its concrete instance of 1-h-minimality) is an analogue of o-minimality for the non-archimedean setting and has been recently developed in [8] and [9]. These notions of 1-h- and of o-minimality are built upon insights coming from cell decomposition results, and their definitions boil down to finiteness conditions on unary sets, implying far reaching finiteness and tameness properties for all definable sets, see [11, 8, 9].

Theorem 1 comes in many variants, like a variant for definable families, and versions with so-called blocks that allow, for example, to bound the number of points of bounded degree over $\mathbb{Q}$ (of bounded height) instead of just rational points (see [12]).

Also Theorem 2 comes in many variants, including for definable families, a block version (allowing to bound algebraic points of bounded degree), and a version which works uniformly in the p-adic field K. Such uniformity in the p-adic field implies similar counting results in large positive characteristic, counting tuples of rational functions $a(t)/b(t)$ in $\mathbb{F}_q(t)$ with a, b polynomials of bounded degree in t, lying on a definable subset of $\mathbb{F}_q(\!(t)\!)^n$ for some power q of a large prime p.

Previously, only some special variants of these results were known: for subanalytic sets [6, 7], for analytic loci [3], and for dimension one in Hensel minimal structures [9]. For curves, a generalization of Theorem 2 has recently been obtained in [15].

In order to show Theorem 2, we develop Taylor approximation results for definable functions in general dimension and up to any finite degree, allowing us to provide general parametrization results for definable sets analogous to Yomdin-Gromov parametrizations. To develop these parametrization results, we overcome three challenges (compared to the mentioned previously known cases). We use methods to pass from local to piecewise going back in the non-archimedean context to [5], and we develop new arguments to control local Taylor approximation and bounds on the derivatives without using local analyticity. (Indeed, in the Hensel minimal setting, there is no local analyticity in general.)

I also presented some finiteness results in the geometric setting from [1], namely for subanalytic sets in $\mathbb{C}(\!(t)\!)$, and a non-archimedean analogue of Wilkie's conjecture in the Pfaffian (and even Noetherian) case, recently shown in the real case [2].

Let me end this report with an open question, inspired by [15], where the one-dimensional case is treated. Suppose that X is a definable set in $\mathbb{C}_p^n$ for some Hensel minimal structure on $\mathbb{C}_p$ (the completion of an algebraic of $\mathbb{Q}_p$). Is it possible to bound the number $\#X^{\mathrm{trans}}(\mathbb{Q}, H)$ uniformly in $H \geqslant 1$, with upper bounds as in Theorem 2?

A similar talk was given at the Conference "Recent Applications of Model Theory" (June 2025) at the Institute for Mathematical Sciences of the National University of Singapore, with a similar report [4].

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The Nash Problem and Du Val singularities in positive characteristic

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(joint work with Shih-Hsin Wang)

Let X be an algebraic surface over an algebraically closed field of characteristic 0. The arc space of X, denoted by X_∞, parametrizes formal arcs $\alpha\colon \operatorname{Spec} k[\![t]\!] \to X$. It comes with a natural projection map $\pi_X\colon X_\infty \to X$ sending an arc $\alpha(t)$ to its base point $\alpha(0)$.

Let $f\colon Y \to X$ be the minimal resolution. For any exceptional prime divisor E on Y, we consider the set $N_E \subset X_\infty$ given by the closure of $f_\infty(\pi_Y^{-1}(E))$, where $f_\infty\colon Y_\infty \to X_\infty$ is the induced map. These sets N_E cover $\pi_X^{-1}(\operatorname{Sing} X)$, which consists of the arcs stemming from the singularities of X, thus every irreducible component of $\pi_X^{-1}(\operatorname{Sing} X)$ is equal to N_E for some E.

Nash asked whether, conversely, every set N_E defines an irreducible component of $\pi_X^{-1}(\operatorname{Sing} X)$ [10]. This question, which became known as the *Nash Problem*, was answered affirmatively in [5].

Theorem 1 ([5]). *The Nash Problem holds for surfaces in characteristic zero.*

We are interested in this question in positive characteristic.

The proof of Theorem 1 uses topological methods and does not generalize easily to positive characteritic. A purely algebraic proof was later given in [1]. Both proofs rely on a version of the Curve Selection Lemma holding in a non-Noetherian setting. The strategy, which goes back to [7], is that if by contradiction there are two exceptional divisors E and F on the minimal resolution such that $N_E \subset N_F$, and $\alpha_E \in N_E$ is the generic point of the smaller set, then the adjacency between these two sets can be detected by an arc the arc space

$$\Phi\colon \operatorname{Spec} K[\![s]\!] \to X_\infty$$

such that $\Phi(0) = \alpha_E$ and $\Phi(\eta) \in N_F \setminus N_E$. Here K is some field extension of the residue field at α_E. Since in general the local ring of X_∞ at α_E is not Noetherian, the existence of such an arc Φ is actually quite subtle. It was established in [14] by showing that the completion of the local ring is Noetherian. A different proof of this property can also be found in [2]. These results hold in arbitrary characteristic.

After specializing the coefficients to $k \subset K$, Φ yields a k-rational wedge on X, namely, a map (which we denote by the same symbol)

$$\Phi\colon S = \operatorname{Spec} k[\![s,t]\!] \to X$$

from a germ of a surface which does not lift to the minimal resolution. The proof of Theorem 1 given in [1] goes by considering the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\phi} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{\Phi} & X \end{array}$$

where g is the minimal sequence of point blow-ups resolving the indeterminacies of $f^{-1} \circ \Phi\colon X \dashrightarrow Y$. The contradiction is drawn by analyzing the ramification of ϕ, using the nefness of K_Y over X and the property that the special arc of the wedge, namely $\Phi(0,t)$, is the image of an arc on Y with order of contact one along the exceptional divisor of f.

Lejeune-Jalabert asked where, in fact, any such wedge should always lift to the resolution. This questions is referred to as the *Wedge Lifting Problem.* A positive answer to this question implies a positive answer to the Nash Problem. The proof of Theorem 1 given in [1] shows that this is the case in characteristic zero.

The Wedge Lifting Problem, however, is known to fail in positive characteristic. In [15], Reguera gives an example of a wedge on an E_8 singularity of characteristic 2, with the properties discussed above, which does not lift to the minimal resolution. Similar examples are given (still on an E_8 singularity) in characteristic 3 and 5, and examples can in fact be given in all positive characteristics. What breaks down the proof of [1] is the possibility of wild ramification along ϕ, which invalidates the computation of its ramification. A different approach is needed in positive characteristic.

People have looked at the Nash Problem in positive characteristics, and some cases such as minimal surface singularities and toric singularities are known [6, 15]. Adding to these cases, we settle the Nash Problem for rational double points in positive characteristics.

Theorem 2 ([4])**.** *The Nash Problem holds for two-dimensional rational double points in positive characteristic.*

It is instructive to look at the history of the Nash Problem for rational double points in characteristic zero. While the case of A_n singularities can be checked with a simple computation and was already understood by Nash, it took several decades before the remaining Du Val singularities were settled in characteristic zero [10, 12, 13, 8, 11]. Interestingly, while A_n, D_n, E_6, and E_7 singularities were understood directly from their equations, the remaining case of the E_8 singularity proved to be too hard to be dealt with the methods introduced in those works, and completely different route was taken in [11] to solve that case, along all quotient singularities, using transcendental methods (a similar route was followed in [5]).

With the approach via wedges failing in positive characteristics, we address Theorem 2 via a direct computation from the given equations. The idea is to construct the irreducible components in the space of arcs from the irreducible components at the finite jet level, by going up one level at a time. The proof is inspired by the computations of the irreducible components of the jets spaces through rational double points carried out, in characteristic zero, in [9].

In [3], we constructed a map $\Psi_{X,m}$, defined for $m \gg 1$, from the set of (non-degenerate) irreducible components of the space of arcs through the singularities of an algebraic variety X and the set of irreducible components of the space of m-jets through the singularities of X. There, we used results related to the higher dimensional Nash Problem obtained in [1] to show that this map is surjective for a certain class of singularities of characteristic zero that generalize the notion of Du Val singularities in higher dimensions.

Here, looking at two-dimensional rational double points of arbitrary characteristic, we follow an opposite approach, where we first compute the irreducible components at the jet level, and then show that map $\Psi_{X,m}$ is surjective, deducing from there the validity of the Nash Problem. A crucial part of the proof is to distinguish the components at the arc level.

In light of Reguera's example, and with Theorem 2 now settling the Nash Problem for E_8 in all characteristics, we conclude that the Nash Problem and the Wedge Lifting Problem are not equivalent in positive characteristic.

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Derived Arc Spaces

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(joint work with Lance E. Miller and C. Eric Overton-Walker)

Arc spaces and jet schemes, i.e., parameter spaces of arc and jets, form a basic tool in the study of singularities of algebraic varieties and their birational geometry. Classically, one identifies invariants of resolution of singularities, like discrepancies and log canonical thresholds, to topological invariants of jet and arc spaces, like certain asymptotic codimensions in arc spaces and dimensions of jet schemes [8]. Aiming to capture finer invariants, there has been a recent surge in the study of jet and arc spaces from the point of view of their scheme-theoretic structure [14, 15, 12, 6, 3, 4]. This leads to more natural characterizations (for example, Mather discrepancies are computed directly as embedding dimensions of certain local rings) and more effective computational tools.

An important result in this context is the main theorem of [6], which provides explicit formulas to compute and understand the structure of the sheaves differentials of jet and arc spaces. For arcs in the affine case, the formula is:

$$\Omega_{J_\infty(R)/k} \cong \Omega_{R/k} \otimes_R P_\infty(R). \tag{1}$$

Here $X = \operatorname{Spec}(R)$ is an affine k-scheme with corresponding arc space given by $J_\infty(X) = \operatorname{Spec}(J_\infty(R))$. By definition, the k-algebra $J_\infty(R)$ is characterized by the adjunction $\operatorname{Hom}_{\mathrm{Alg}_k}(J_\infty(R), S) \cong \operatorname{Hom}_{\mathrm{Alg}_k}(R, S[\![t]\!])$. In particular, it comes equipped with a morphism $R \to J_\infty(R)[\![t]\!]$, known as the universal arc. Then the module $P_\infty(R)$ appearing in (1) is given by $P_\infty(R) = \frac{J_\infty(R)(\!(t)\!)}{tJ_\infty(R)[\![t]\!]}$, considered as an R-module via the universal arc. The formulas for the case of jet schemes, and in the global case, are similar.

We are interested in exploring versions of (1) for cotangent complexes of jet and arc spaces. We arrive to this problem after encountering technical difficulties when applying (1) to the study of singularities. For example, many of the results of [6, 3, 4] assume a perfect ground field k, which seems unnecessary and unnatural. Understanding cotangent complexes should also shed light on the deformation theory of arc spaces. Naive attempts at generalizing (1) quickly encounter difficulties: the proofs in [6] rely on the universal property of $\Omega_{R/k}$, but cotangent complexes lack a universal property in the category of schemes. Instead, one should recognize that cotangent complexes are inherently the derived analogues of sheaves of differentials. We are naturally led to a more ambitious goal: laying down the foundations of the theory of jet and arc spaces in the context of derived algebraic geometry.

We adopt Lurie's foundations for derived algebraic geometry [11], and use the language of animation [2]. Given an algebraic category $\mathcal{C}$, see [10, 1], Lurie produces an associated ∞-category $\mathrm{Ani}(\mathcal{C})$, the animation of $\mathcal{C}$. And given a functor $F\colon \mathcal{C} \to \mathcal{D}$ preserving sifted colimits, we get an induced functor $\mathbf{L}F\colon \mathrm{Ani}(\mathcal{C}) \to \mathrm{Ani}(\mathcal{D})$, the left derived functor of F. When $\mathcal{C} = \mathrm{Mod}_R$ is the category of R-modules, $\mathrm{Ani}(\mathcal{C}) = \mathrm{aMod}_R = D(R)_{\geqslant 0}$ is the connective part of (the ∞-categorical incarnation of) the derived category of R. For $\mathcal{C} = \mathrm{Alg_k}$, its animation $\mathrm{Ani}(\mathcal{C}) = \mathrm{aAlg}_k$ is the ∞-category of simplicial algebras localized by inverting weak equivalences. Derived schemes are obtained by gluing affine derived schemes, and these are constructed as the opposite of aAlg_k. Similar to how classical left derived functors are computed using free resolutions, general left derived functors are computed by using what are known as cofibrant replacements. For example, consider the functor $\mathrm{Alg}_k \to \mathrm{AlgMod}_k$ giving the module of differentials, $R \mapsto (R, \Omega_{R/k})$. Its left derived functor gives the cotangent complex, $R_\bullet \mapsto (R_\bullet, \mathbf{L}_{R_\bullet/k})$. Each animated algebra $R_\bullet$ admits a weak equivalence $P_\bullet \to R_\bullet$ where each component P_n of $P_\bullet$ is a polynomial k-algebra. Such $P_\bullet$ is called a cofibrant replacement of $R_\bullet$. Then the left derived functor is computed as $\mathbf{L}_{R_\bullet/k} = \Omega_{P_\bullet/k} \otimes_{P_\bullet} R_\bullet$.

In principle, we have two routes to construct arc spaces and jet schemes in the context of derived algebraic geometry. One possibility is to study representability of the moduli problem of arcs/jets in the ∞-category of derived schemes. Alternatively, we can simply look at the left derived functors of $J_\infty(-)$ and $J_m(-)$. Our first result is that these approaches give the same answer, and that the resulting constructions share many features with the classical counterparts.

Theorem. *We denote by $\mathbf{L}J_\infty(-)$ and $\mathbf{L}J_m(-)$ the left derived functors of $J_\infty(-)$ and $J_m(-)$. They are computed as*

$$\mathbf{L}J_\infty(R_\bullet) = J_\infty(P_\bullet) \otimes_{P_\bullet} R_\bullet \qquad \text{and} \qquad \mathbf{L}J_m(R_\bullet) = J_m(P_\bullet) \otimes_{P_\bullet} R_\bullet,$$

where $P_\bullet \to R_\bullet$ is a cofibrant replacement for $R_\bullet$. We call them the derived arc/jet space functors*, and they satisfy the following properties:*

(1) $\mathrm{Maps}_{\mathrm{aAlg_k}}(\mathbf{L}J_\infty(R_\bullet), S_\bullet) \cong \mathrm{Maps}_{\mathrm{aAlg_k}}(R_\bullet, S_\bullet[\![t]\!])$ *and* $\mathrm{Maps}_{\mathrm{aAlg_k}}(\mathbf{L}J_m(R_\bullet), S_\bullet) \cong \mathrm{Maps}_{\mathrm{aAlg_k}}(R_\bullet, S_\bullet[t]/(t^{m+1}))$.
(2) $\mathbf{L}J_\infty(R_\bullet) \cong \mathrm{hocolim}_m(\mathbf{L}J_m(R_\bullet))$.
(3) $\pi_0(\mathbf{L}J_\infty(R_\bullet)) = J_\infty(\pi_0(R_\bullet))$ *and* $\pi_0(\mathbf{L}J_m(R_\bullet)) = J_m(\pi_0(R_\bullet))$.
(4) $\mathbf{L}J_1(R_\bullet) = \mathbf{L}\mathrm{Sym}_{R_\bullet}(\mathbf{L}_{R_\bullet/k})$.

Our construction is particularly effective in what is known as the quasi-smooth case. Given a polynomial algebra $k[\underline{x}] = k[x_1, \dots, x_d]$ and a sequence of elements $(\underline{f}) = (f_1, \dots, f_c)$ in $k[\underline{x}]$, we can form the quotient $k[\underline{x}]/\!\!/(\underline{f})$ in aAlg_k. This is defined using an appropriate homotopy colimit, and one immediately sees that the underlying classical algebra is the usual quotient, $\pi_0(k[\underline{x}]/\!\!/(\underline{f})) = k[\underline{x}]/(\underline{f})$. One can check that $k[\underline{x}]/\!\!/(\underline{f}) = k[\underline{x}]/(\underline{f})$ if and only if $(\underline{f})$ is a regular sequence, and more generally that the homotopy groups of the derived quotient are computed by Koszul homology: $\pi_i(k[\underline{x}]/\!\!/(\underline{f})) = H_i(K_\bullet(\underline{f}, k[\underline{x}]))$. An animated algebra $R_\bullet$ which is étale locally of the form $k[\underline{x}]/\!\!/(\underline{f})$ is called a quasi-smooth k-algebra. A classical algebra R is quasi-smooth (when considered as an animated algebra) precisely when it is a local complete intersection. Our next result asserts that quasi-smoothness is preserved by the derived jet space functors.

Theorem. *If $R_\bullet$ is a quasi-smooth k-algebra, then $\mathbf{L}J_m(R_\bullet)$ is also quasi-smooth. More precisely: $\mathbf{L}J_m(k[\underline{x}]/\!\!/(\underline{f})) = k[\underline{x}, \underline{x}', \dots, \underline{x}^{(m)}]/\!\!/(\underline{f}, \underline{f}', \dots, \underline{f}^{(m)})$, where $g^{(p)}$ denotes the p-th Hasse-Schmidt derivative of g.*

As an immediate consequence we obtain interesting results for classical local complete intersections. Combined with results of Mustaţă and collaborators [13, 9, 7, 5], we get the following.

Theorem. *Let R be a classical k-algebra, and assume that R is a reduced local complete intersection. Write $R = S/I$ where S is a smooth k-algebra, let $M =$ $\mathrm{Spec}(S)$ and $X = \mathrm{Spec}(R)$, and let c be the codimension of X in M. Then the following are equivalent:*

(1) The derived jet schemes are classical, i.e., $\mathbf{L}J_m(R) = J_m(R)$ for all m.
(2) $J_m(R)$ is a local complete intersection for all m.
(3) The pair (M, cX) is log canonical.

If X is normal, these statements are equivalent to X being log canonical itself.

From the previous two results we get many examples of varieties with non-classical derived jet schemes. We think of the homotopy groups $\pi_n(\mathbf{L}J_m(R))$ as new invariants of singularities. In the local complete intersection case they give obstructions to log canonicity, and it would be interesting to understand them in more generality.

Our next result gives the sought generalization of formula (1).

Theorem. *We have quasi-isomorphisms*

$$\mathbf{L}_{\mathbf{L}J_\infty(R_\bullet)/k} \cong \mathbf{L}_{R_\bullet/k} \otimes_{R_\bullet} P^{\mathrm{der}}_\infty(R_\bullet) \quad \textit{and} \quad \mathbf{L}_{\mathbf{L}J_m(R_\bullet)/k} \cong \mathbf{L}_{R_\bullet/k} \otimes_{R_\bullet} P^{\mathrm{der}}_m(R_\bullet),$$

where $P^{\mathrm{der}}_{\infty}(R_\bullet) = \frac{\mathbf{L}J_\infty(R_\bullet)((t))}{t\,\mathbf{L}J_\infty(R_\bullet)[\![t]\!]}$ *and* $P^{\mathrm{der}}_{m}(R_\bullet) = \frac{t^{-m}\,\mathbf{L}J_m(R_\bullet)[\![t]\!]}{t\,\mathbf{L}J_m(R_\bullet)[\![t]\!]}$. *When* $R_\bullet = R$ *is classical, the above formulas hold for the classical jet/arc spaces if and only if the derived jet/arc spaces are classical.*

The failure of the formulas when derived jet/arcs spaces are non-classical justifies the necessity of derived methods. As an application, we are able to generalize many of the results in [6, 3, 4] by removing any hypothesis on the ground field. For example, we obtain the following very general version of the curve selection lemma [15, 6, 16].

Theorem. *Let k be an arbitrary field. Let X be a reduced scheme essentially of finite type over k, and let α be a stable point in the arc space $J_\infty(X)$. Then the completed local ring $\widehat{\mathcal{O}}_{J_\infty(X),\alpha}$ is Noetherian.*

Notice that while the techniques of the proof use derived methods, the end result only concerns classical algebraic geometry. Other results in [6, 3, 4] are also generalized, like the analysis of cotangent maps at the level of arc spaces, and the finite generation of maximal ideals of local rings at stable points.

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Semiorthogonal decompositions of equivariant derived categories for some reflection groups via the McKay correspondence

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(joint work with Anirban Bhaduri, Yael Davidov, Katrina Honigs, Peter McDonald, C. Eric Overton-Walker, and Dylan Spence)

The classical McKay correspondence relates the representation theory of a finite group H in $\mathrm{SL}(2,\mathbb{C})$ and the geometry of the exceptional divisor of the minimal resolution Y of the corresponding quotient singularity $\mathbb{C}^2/H$. In particular, the irreducible representations of H are in bijection with the components of the exceptional divisor, see [Buc12] for more on history and algebraic extensions of this result. Kapranov and Vasserot [KV00] showed that the correspondence may be realized as a derived equivalence between the derived category of coherent sheaves on the minimal resolution Y, and the derived category of equivariant coherent sheaves on the two-dimensional vector space the group is acting on:

$$D^b(Y) \simeq D^H(\mathbb{C}^2)\ .$$

This result has been extended to the case of small finite subgroups G in $\mathrm{GL}(2,\mathbb{C})$, see [IU15], giving a derived version of the geometric special McKay correspondence first established by Wunram [Wun88]. Furthermore, in the seminal paper [BKR01] a derived McKay correspondence for finite subgroups of $\mathrm{SL}(3,\mathbb{C})$ was established, using equivariant Hilbert schemes.

On the other hand, for a complex reflection group $G \subseteq \mathrm{GL}(2,\mathbb{C})$ acting on $\mathbb{C}^2$ the quotient $\mathbb{C}^2/G$ is smooth by the theorem of Chevalley–Shephard–Todd. This makes the geometric picture quite different from the classical case, as there are no singularities to resolve. However, a recent algebraic version of the McKay correspondence for reflection groups [BFI20] shows a bijection of the irreducible representations of G with certain Cohen–Macaulay modules over the coordinate ring of the *discriminant* of the reflection group, a singular curve in $\mathbb{C}^2$. Furthermore, the following conjecture predicts a semiorthogonal decomposition of the equivariant category $D^G(\mathbb{C}^2)$:

Conjecture 1 (Polishchuk–Van den Bergh [PVdB19]). *Suppose that G is a finite group acting effectively on a smooth variety X and that for all $g \in G$ the geometric quotient $\bar{X}^g = X^g/C(g)$ is smooth. Then there is a semiorthogonal decomposition of $D^G(X)$ whose components $C_{[g]}$ are in bijection with conjugacy classes and $C_{[g]} \simeq D(\bar{X}^g)$.*

There is the following well-known relation between finite groups $H \subseteq SL(2,\mathbb{C})$ and finite reflection groups $G \subseteq GL(2,\mathbb{C})$ generated by order 2 reflections: $H \subseteq G$

is a subgroup of index 2. Indeed, one obtains a bijection between these groups H and G, for details see e.g. [BFI23].

Using this relation and analyzing the action of the quotient G/H on the minimal resolution Y of $\mathbb{C}^2/H$ we can show the following

Theorem 2 (see [BDF+24] Theorem A and Corrollary B)**.** *Let G be a finite group contained in $\mathrm{GL}(2,\mathbb{C})$ generated by order 2 reflections and acting on $\mathbb{C}^2$. There is a semi-orthogonal decomposition of $D^G(\mathbb{C}^2)$ of the following form, where $B_1,\dots,B_r$ are the normalizations of the irreducible components of the branch divisor $\mathbb{C}^2 \to \mathbb{C}^2/G$, $E_1,\dots,E_n$ are exceptional objects and $r+n+1$ is the number of distinct irreducible representations of G:*

$$D^G(\mathbb{C}^2) \simeq \langle D^b(\mathbb{C}^2/G), D(B_1), \dots, D(B_r), E_1, \dots, E_n \rangle \ .$$

In particular, we can relate the pieces of this semiorthogonal decomposition to the derived categories of the quotients $D(\bar{X}^g)$ and thus Conjecture 1 holds for these G acting on $\mathbb{C}^2$.

The proof strategy is inspired by Potter's thesis [Pot18], who proved the analogous result for the dihedral groups $G(m,m,2)$. An essential step in our argument is to compute, for each group G appearing in Theorem 2, the action of $G/H \cong \mathbb{Z}/2\mathbb{Z}$ on the minimal resolution Y of $\mathbb{C}^2/H$. For this, we use the explicit description of Y as the H-equivariant Hilbert scheme $H-\mathrm{Hilb}(\mathbb{C}^2)$ due to Ito and Nakamura [IN99]. A crucial observation is further that the action of G/H extends to Y and the quotient $Y/(G/H)$ is smooth. The decomposition of $D^G(Y)$ is then obtained using the equivalence $D^G(Y) \simeq D^{G/H}(Y)$ and facts about root stacks, see [IU15], and Orlov's blowup formula [Orl92].

It would be interesting to further analyze the structure of $D^G(\mathbb{C}^n)$ for reflection groups of higher rank.

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Smoothing of projective schemes with lci singularities

Barbara Fantechi
(joint work with Rosa Maria Miró Roig)

Let X be a projective scheme over an algebraically closed field of characteristic 0. Assume that X has log canonical singularities, which may be non-isolated. We give sufficient conditions to ensure that X is (geometrically) smoothable, in particular providing a new and shorter proof to the geometric version by Nobile of Tziolas's formal smoothability criterion.

The main theorem is that the general fiber of a flat proper family with smooth basis and X as a special fiber is smooth under a simple assumption on $\mathcal{T}^1_X$; this is true for proper schemes and Deligne–Mumford stacks.

As a corollary, we get criteria for smoothability of singular K3 surfaces, as well as stable surfaces of general type.

An orbifold formula for Artin stacks

Arthur Forey
(joint work with François Loeser and Dimitri Wyss)

Let k be an algebraically closed field of characteristic zero. Let G be a group acting on a smooth k-variety X. The goal of this talk is to relate some invariant of singularities of the quotient $Y = X/G$ to invariants of the action of G on X.

McKay correspondence

The first result in this direction concerns finite subgroups G of $\mathrm{SL}_2(k)$, with the linear action on $\mathbb{A}^2_k$. Let $Y = \mathbb{A}^2_k/G$ be the quotient.

Theorem (McKay)**.** *The number of irreducible components of the exceptional divisor of the minimal resolution of Y is equal to the number of non-trivial irreducible representations of G.*

McKay's result is in fact more precise. It establishes a correspondence between the dual graph of the minimal resolution of Y and a graph whose vertices are the non-trivial irreducible representations of G. This correspondence has be generalized in many directions, see [5], in this talk we will pursue one in which the group G is allowed to be more and more general.

Motivic McKay correspondence

Batyrev, then Denef and Loeser have generalized this correspondence to finite subgroups G of $\mathrm{GL}_n(k)$, using the theory of motivic integration. Due to Kontsevich, then developed by Denef and Loeser, it is a form of measure theory for some subsets of arc spaces, taking values in a localization of the Grothendieck group of varieties over k.

Set $Y = \mathbb{A}^n/G$, and O the image of the origin. The quotient Y is Gorenstein with at worse canonical singularities. The stringy motive of Y at O is defined as

$$h_{st}(Y, O) = \int_{Y(k[\![t]\!])_O} |\omega_{\mathrm{orb}}|\,,$$

where $Y(k[\![t]\!])_O$ is the space of arcs centered at O. It can be explicitly computed using a resolution of singularities of Y.

Batyrev and Denef-Loeser motivic version of McKay correspondence is the following equality

Theorem (Batyrev [1], Denef-Loeser [2])**.**

$$h_{st}(Y, O) = \sum_{[\gamma]\in \mathrm{Conj}(G)} \mathbb{L}^{-w(\gamma)},$$

where $w(\gamma)$ is the weight (or age) of $\gamma \in \mathrm{GL}_n(k)$ and $\mathbb{L}$ is the class of the affine line.

General quotient

Now let X be a smooth algebraic variety and G a linear group acting on X. Consider the Artin stack $\mathcal{M} = [X/G]$ and suppose that there exists a good moduli space $p\colon \mathcal{M} \to Y$.

We further assume that p is generically an isomorphism, and that Zariski locally on X, a power of the canonical bundle admits G-invariant sections, so that it induces a motivic volume form $|\omega_{\mathrm{can}}|$ on Y.

Let μ_r be the group of r-th roots of unity and consider the cyclotomic inertia $I_{\mu_r}\mathcal{M}$, the stack representing $\mathrm{Hom}(B\mu_r, \mathcal{M})$. A point $y \in I_{\mu_r}\mathcal{M}(K)$ corresponds to a pair (x, φ) with $x \in \mathcal{M}(K)$ and $\varphi\colon \mu_r \to \mathrm{Aut}(x)$, hence endows the tangent complex $T_x\mathcal{M}$ with a μ_r-action, to which we can assign a weight $w(y) \in \mathbb{Q}$. In our context, $w\colon I_{\mu_r}\mathcal{M} \to \mathbb{Q}$ is locally constant and takes finitely many values in a finite set W, such that we can define

$$[I_{\mu_r}\mathcal{M}]^w = \sum_{v\in W} [w^{-1}(v)]\mathbb{L}^{-v}.$$

Here classes are taken in a localization of the Grothendieck group of k-varieties.

The generalization of the orbifold formula is the following equality.

Theorem (Forey-Loeser-Wyss)**.**

$$\int_{Y(k[\![t]\!])} |\omega_{\mathrm{can}}| = -\lim_{T\to+\infty} \sum_{n\geqslant 1} [I_{\mu_n}\mathcal{M}]^w\, T^n.$$

A variant in the p-adic case has been obtained in [3].

As application, we calculate the stringy motive of the moduli stack of rank two vector bundles on a smooth projective curve, initially due to Kiem and Li [4].

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Riso-trees yield an invariant of singularities

Immanuel Halupczok

(joint work with David Bradley-Williams and Pablo Cubides Kovacsics)

1. Goal

We fix a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ once and for all for the entire talk. (In the entire talk, $\mathbb{C}$ could also be replaced by any other algebraically closed field of characteristic 0). Let $V_f \subset \mathbb{A}^n_{\mathbb{C}}$ be the variety defined by f, and suppose that $0 \in V_f(\mathbb{C})$. To understand a potential singularity of V_f at 0, it is often useful to work over $\mathbb{C}[\![t]\!]$ and to describe how the valuation $v(f(\underline{x}))$ depends on $\underline{x}$, for $\underline{x} \in t\mathbb{C}[\![t]\!]^n$. The goal of this talk is to get a better understanding of this dependence. To this end, we need to work in an algebraically closed valued field, so let K be the algebraic closure of $\mathbb{C}(\!(t)\!)$, i.e., the field of Puiseux series over $\mathbb{C}$. Denote by $\mathcal{O}_K$ and $\mathcal{M}_K$ the valuation ring of K and its maximal ideal, respectively.

A note about intuition: I do *not* think of the elements of $\mathbb{C}[\![t]\!]$ (and of K) as little arcs; instead, I think of t as an infinitesimal complex number. In particular, this is how I will draw pictures: The picture of $V_f(K)$ looks exactly like the one of $V_f(\mathbb{C})$, but it is meant to contain infinitesimal elements.

In the following, $\underline{x}$ runs over $\mathcal{M}_K^n$, i.e, the infinitesimal neighbourhood of 0 (in terms of my intuition). If V_f is smooth at 0, then $v(f(\underline{x}))$ only depends on the valuative distance $\mathrm{valdist}(\underline{x}, V_f(K)) = \min\{v(\underline{x} - \underline{y}) \mid \underline{y} \in V_f(K)\}$; let us denote this valuative distance by $\lambda_{n-1}(\underline{x})$ (Figure 1, left hand side). Indeed, if for example f is reduced, one can verify that we simply have $v(f(\underline{x})) = \lambda_{n-1}(\underline{x})$.

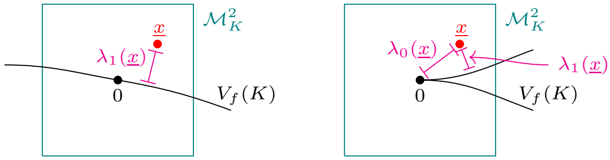

FIGURE 1. For the smooth curve, only λ_1 is relevant; for the singular curve, additionally λ_0 is relevant.

If V_f is a curve which has a singularity at 0, then $v(f(\underline{x}))$ additionally depends on the valuative distance of $\underline{x}$ to the singularity (Figure 1, right hand side); we denote that distance by $\lambda_0(\underline{x})$. The curve case of the main result of this talk says that $\lambda_0(\underline{x})$ and $\lambda_1(\underline{x})$ together "essentially" determine $v(f(\underline{x}))$, in the sense that if we fix $(\lambda_0(\underline{x}), \lambda_1(\underline{x}))$, then $v(f(\underline{x}))$ can take only finitely many different values. Moreover, this dependence is piecewise linear.

Let me state that result more precisely and in every dimension. To this end, we need to introduce maps

$$\lambda_0, \ldots, \lambda_{n-1} \colon K^n \setminus V_f(K) \to \mathbb{Q}$$

which capture the valuative distances of a point $\underline{x}$ to certain singular subsets of $V_f(K)$. Defining λ_i precisely will take most of this talk. For now, let me pretend that we have already defined them; then the main result is the following. (Recall that $f \in \mathbb{C}[x_1, \ldots, x_n]$ is fixed; the theorem is stated for the point $0 \in V_f(\mathbb{C})$).

Theorem 1 (work in progress)**.** *The set*

$$\Delta_0 := \{(\lambda_0(\underline{x}), \ldots, \lambda_{n-1}(\underline{x}), v(f(\underline{x}))) \mid \underline{x} \in \mathcal{M}_K^n \setminus V_f(K)\}$$

is the union of the graphs of finitely many linear functions $h_\ell \colon A_\ell \to \mathbb{Q}$, where each domain A_ℓ is a semi-linear subset of $\mathbb{Q}^n$, i.e., defined by linear equalities and inequalities.

One reason I find this theorem interesting is that Δ_0 (which is a combinatorial object living in $\mathbb{Q}^{n+1}$) is canonically associated to the singularity at 0. This Δ_0 has the flavour of some kind of tropicalization, but in contrast to the latter, it is independent of the choice of coordinates.

2. Riso-triviality

When $n = 3$, then according to our above definition, $\lambda_2(\underline{x})$ is the valuative distance of $\underline{x}$ to $V_f(K)$, and $\lambda_1(\underline{x})$ can tentatively be defined as the valuative distance to the singular locus $V_{f,\text{sing}}(K)$. Within $V_{f,\text{sing}}(K)$, there may be finitely many "worse" singularities. Our plan is to define $\lambda_0(\underline{x})$ to be the valuative distance to that set of worse singularities, so we have to specify what we mean by "worse" (Figure 2). Moreover, we will also need to improve the definition of $\lambda_1(\underline{x})$, as we will see in the example of Figure 4.

Whitney stratifications provide a notion of "worse singularities", but that notion is not strong enough for our purposes. (An example is given in [1, Example 5.4.6]).

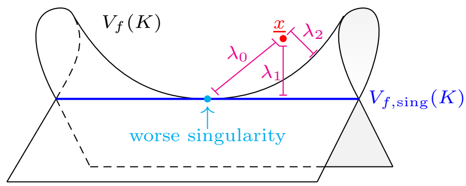

FIGURE 2. Even though the singular locus $V_{f,\mathrm{sing}}$ itself is smooth, it contains a point at which V_f is more singular than at the others.

Instead, we will use a notion we call "riso-triviality". Intuitively, "d-riso-trivial" is supposed to say something like "almost translation invariant along a d-dimensional space". Let us make this precise. For our definition to work properly, we need to work in a spherically complete valued field, so let us enlarge K accordingly: Let $\tilde{K} = \mathbb{C}((t^{\mathbb{Q}}))$ be the field of Hahn series over $\mathbb{C}$, i.e., those formal sums $\sum_{r\in\mathbb{Q}} z_r t^r$ for which the support $\{r \in \mathbb{Q} \mid z_r \neq 0\}$ is well-ordered[1].

Fix a valuative ball $B = B_\lambda(\underline{a}) = \{\underline{b} \in \tilde{K}^n \mid v(\underline{b} - \underline{a}) \geqslant \lambda\}$, for $\underline{a} \in \tilde{K}^n$ and $\lambda \in \mathbb{Q}$. (The valuation of a tuple is defined as the minimum of the valuations of its entries, so a ball is really a hypercube). We define what it means for the function $\rho\colon \underline{x} \mapsto v(f(\underline{x}))$ to be d-riso-trivial on B. This uses the notion of "risometry", which I will explain afterwards. For the moment, think of it as a small perturbation.

Definition 2 (Figure 3)**.** For $0 \leqslant d \leqslant n$, we say that ρ is ($\geqslant d$)-riso-trivial on B if there exists a risometry $\alpha\colon B \to B$ and a d-dimensional vector subspace $V \subset \tilde{K}^n$ such that the composition $\rho \circ \alpha\colon B \to \mathbb{Q} \cup \{\infty\}$ is V-translation invariant within B, i.e., such that for $\underline{a}, \underline{a}' \in B$ with $\underline{a} - \underline{a}' \in V$, we have $\rho(\alpha(\underline{a})) = \rho(\alpha(\underline{a}'))$.

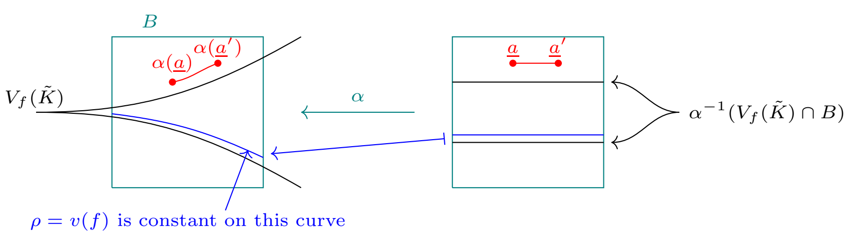

FIGURE 3. If f defines the cusp curve and B is a small valuative ball meeting both branches but not containing the origin (e.g., $B = (t + t^2\mathcal{O}_K) \times t^2\mathcal{O}_K$), then $\rho = v \circ f$ is 1-trivial on B: for a suitable risometry α, $\rho \circ \alpha$ is horizontally translation invariant.

[1]The well-orderedness assumption ensures that the product of two such series is always well-defined. One can also characterize $\mathbb{C}((t^{\mathbb{Q}}))$ as the maximal valued field extension of our field K of Puiseux series which still has residue field $\mathbb{C}$ and value group $\mathbb{Q}$.

We denote this by "($\geqslant d$)-riso-trivial" (and not just d-riso-trivial) to emphasize that ρ could be trivial in even more directions. One then defines d-riso-trivial and ($\leqslant d$)-riso-trivial as one would expect.

I now need to say what a risometry is. To this end, let me first recall what a (valuative) isometry is:

Definition 3. A bijection $\alpha\colon B \to B$ is an isometry if for all $\underline{a}, \underline{a}' \in B$, we have $v(\alpha(\underline{a}) - \alpha(\underline{a}')) = v(\underline{a} - \underline{a}')$.

Given $\underline{a} = \sum_{r\in\mathbb{Q}} \underline{z}_r t^r \in \tilde{K}^n$, we denote by $\mathrm{rv}(\underline{a})$ the leading term of $\underline{a}$, i.e., if r_0 is minimal with $\underline{z}_{r_0} \neq 0$ (meaning that $v(\underline{a}) = r_0$), then $\mathrm{rv}(\underline{a}) = \underline{z}_{r_0} t^{r_0}$. Now we simply insert "r" in various places of Definition 3:

Definition 4. A bijection $\alpha\colon B \to B$ is a risometry if for all $\underline{a}, \underline{a}' \in B$, we have $\mathrm{rv}(\alpha(\underline{a}) - \alpha(\underline{a}')) = \mathrm{rv}(\underline{a} - \underline{a}')$.

To get a geometric intuition for this notion, note that an isometry can send the difference $\underline{a} - \underline{a}'$ to any other difference of the same valuation, but maybe in a completely different direction. (Valuative isometries are much less rigid than archimedean ones.) In contrast, if α is a risometry, then $\alpha(\underline{a}) - \alpha(\underline{a}')$ and $\underline{a} - \underline{a}'$ differ only by something of smaller order of magnitude. Therefore, $\rho \circ \alpha$ being V-translation invariant implies that ρ is "V-translation invariant up to something of smaller order of magnitude".

3. The results

Before coming back to the definition of the λ_d, I want to state two key results about d-riso-triviality. The first one expresses that d-riso-triviality is, in some sense, an algebraic condition. Since valuations are involved, "algebraic" does not make sense literally; instead, one has to use the notion of definability from model theory. The second key result expresses that the locus of d-riso-triviality has dimension at most d. In the following theorem, we continue to have $f \in \mathbb{C}[x_1, \ldots, x_n]$ fixed, and we still set $\rho = v \circ f\colon \tilde{K}^n \to \mathbb{Q} \cup \{\infty\}$.

Theorem 5. *For every $0 \leqslant d \leqslant n$, the following holds:*

(1) The set

$$\{(\underline{a}, \lambda) \in \tilde{K}^n \times \mathbb{Q} \mid \rho \text{ is } d\text{-riso-trivial on } B_\lambda(\underline{a})\}$$

is definable in the language of valued fields [1, Corollary 3.1.4].

(2) There exists a d-dimensional algebraic subset $X_d(\tilde{K}) \subset \tilde{K}^n$ which has non-empty intersection with every ball $B \subset \tilde{K}^n$ on which ρ is d-riso-trivial [2, Theorem 6.6].

For readers not familiar with model theory, here is a consequence which does not need the notion of definability:

Corollary 6 ([1, Section 4.2]). *For each d, the set*

$$S_d(\mathbb{C}) := \{\underline{a} \in \mathbb{C}^n \mid \rho \text{ is } d\text{-riso-trivial on } \underline{a} + \mathcal{M}^n_{\tilde{K}}\}$$

is a constructible subset of $\mathbb{C}^n$ of dimension at most d.

A word of caution: One might hope that in Theorem 5(2), the sets X_d can be choosen canonically, in such a way that one has

$$(*) \qquad \rho \text{ is } (\leqslant d)\text{-riso-trivial on } B \textit{ if and only if } B \cap X_d(\tilde{K}) \neq \emptyset.$$

However, in general, this does not work: It can happen that ρ is (e.g.) 1-riso-trivial on some ball B, but every point in B has a (smaller) ($\geqslant 2$)-riso-trivial neighbourhood (Figure 4). In that case, there is no canonical choice of X_1.

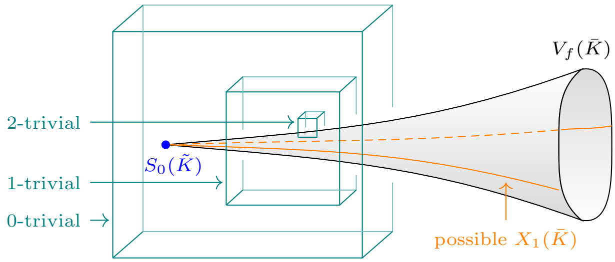

FIGURE 4. ρ is only 1-riso-trivial on the middle-sized green ball, but it is 2-riso-trivial on every smaller subball. One can choose X_1 as drawn in the picture, but this is not canonical.

We now have all the ingredients needed to define the λ_d that appear in Theorem 1. Intuitively, we would like to set $\lambda_d(\underline{x}) := \operatorname{valdist}(\underline{x}, X_d(\tilde{K}))$, but as we just saw, this makes no sense since the X_d might not be canonical. The problem can be avoided as follows: $\operatorname{valdist}(\underline{x}, X_d(\tilde{K}))$ is equal to the radius of the smallest ball B around $\underline{x}$ which has non-empty intersection with $X_d(\tilde{K})$, and if $(*)$ holds, then this intersection being non-empty is equivalent to ρ being ($\leqslant d$)-riso-trivial on B. Thus if $(*)$ holds, $\operatorname{valdist}(\underline{x}, X_d(\tilde{K}))$ is equal to the minimal λ such that ρ is ($\leqslant d$)-riso-trivial on $B_\lambda(\underline{x})$. Now we simply use this to define the λ_d even when $(*)$ does not hold:

Definition 7 (work in progress[2])**.** For $\underline{x} \in \tilde{K}^n \setminus V_f(\tilde{K})$, let $\lambda_d(\underline{x})$ be the minimal $\lambda \in \mathbb{Q}$ such that ρ is ($\leqslant d$)-riso-trivial on $B_\lambda(\underline{x})$.

In Figure 4 for example, for some $\underline{x}$ inside the smallest green ball, $\lambda_2(\underline{x})$ is the radius of the smallest green ball, $\lambda_1(\underline{x})$ is the radius of the middle one, and $\lambda_0(\underline{x})$ is the radius of the largest one.

We thus have now reached our goal of making Theorem 1 precise.

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[2]For the definition to make sense, one has to verify that the minimums exist. While this is not very difficult, we did not yet write it up; thus the "work in progress".

Cohen-Macaulay Type via Lattice Homology and the Motivic Poincaré Series

Alex Hof
(joint work with András Némethi)

The *maximal Cohen-Macaulay (MCM) modules* over a given Noetherian local ring are those with depth equal to the largest possible value, its Krull dimension d. If one wishes, these can be characterized more geometrically as the modules with local cohomology concentrated in degree d. The project of understanding such rings, and the space germs to which they correspond, through classification of their MCM modules is a longstanding one, originating in the works of Auslander, Drozd, Reiten, and Roiter [2, 3, 4, 5, 6, 7, 10] from the perspective of representation theory.

In the setting of reduced complex-analytic curve germs and the corresponding convergent power series rings, the objects can be grouped into three main classes [9]: those of *finite Cohen-Macaulay (CM) type*, which admit only finitely many indecomposable MCM modules up to isomorphism, those of *tame CM type*, which have infinitely many such modules but only finitely many 1-parameter families of them of any given rank, and those of *wild CM type*, which exhibit families of unbounded dimension in each rank. Among the main general tools for distinguishing between these types are *dominance conditions*, which characterize germs of finite and tame CM type as those which birationally dominate certain foundational plane curve germs—respectively, the *ADE* [12] and T_{pq} germs ([9])—and *overring conditions*, which give algebraic criteria (see [10, 11, 14] and [9] respectively) in terms of the interactions between various subrings of the integral closures of the local rings under consideration.

Though powerful, both types of conditions have limitations—the dominance conditions are conceptually straightforward but difficult to verify directly in practice, while the overring conditions are more computationally tractable but theoretically somewhat opaque, and both are formulated using analytic objects (subrings and overrings) which exhibit positive-dimensional moduli. Therefore, as depicted in Figure 1, we supplement them with results relating the study of MCM modules to invariants of curve germs which arise in a priori distinct contexts—the *lattice homology* [1], which categorifies the δ-invariant and has connections to low-dimensional topology, together with its *level filtration* and the corresponding spectral sequence [15], and the *motivic Poincaré series* [8], which refines the usual Poincaré series by realizing it as a specialization of a series with coefficients in a localization of the Grothendieck ring of algebraic varieties.

In particular, our lattice-homological results use the quantity $\min w_0^C$ (equal to $-\frac{1}{2}$ times the maximum grade of $\mathbb{H}_*(C,o)$) and the *modules* $\mathfrak{M}_{k,n}(C,o)$ *of minimal spectral k-cycles of weight n*, which are defined using the E^1-page of the spectral sequence arising from the level filtration. They are as follows:

Theorem 1 ([13]). *Let (C,o) be a reduced complex-analytic curve germ. Then (C,o) is of finite CM type if and only if* $\min w_0^C \geqslant -1$.

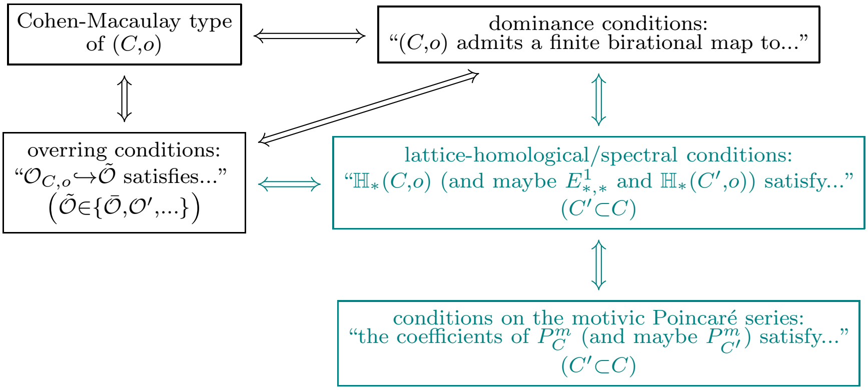

FIGURE 1. A sketch of the relationships between various criteria for determining CM type.

More specifically, we have the following equivalences:

- (C, o) *birationally dominates a simple germ of type* A_n *(for some* $n \geqslant 0$*) if and only if* $\min w_0^C = 0$. *(In fact, in this case,* (C, o) *is an* A_n *germ.)*
- (C, o) *birationally dominates a simple singularity of type* D_n *(for some* $n \geqslant 4$*), but no simple germ of type* A_n *(for any* $n \geqslant 0$*), if and only if* $\min w_0^C = -1$ *and* $\mathfrak{M}_{1,0}(C, o) \neq 0$.
- (C, o) *birationally dominates a simple singularity of type* E_6*,* E_7*, or* E_8*, but no simple germ of type* A_n *(for any* $n \geqslant 0$*) or* D_n *(for any* $n \geqslant 4$*), if and only if* $\min w_0^C = -1$ *and* $\mathfrak{M}_{1,0}(C, o) = 0$.

Theorem 2 ([13]). *Let* (C, o) *be a reduced complex-analytic curve germ, with* $(C, o) = \bigcup_{i=1}^r (C_i, o)$ *the decomposition into irreducible components. Then* (C, o) *is of tame CM type if and only if all of the following hold:*

(a) $\min w_0^C = -2$.
(b) $\mathfrak{M}_{1,-1}(C, o) \neq 0$.
(c) *For each* $1 \leqslant i \leqslant r$*,* (C_i, o) *birationally dominates a simple germ of type* A_n *(for some* $n \geqslant 0$*).*
(d) *For each* $1 \leqslant i \leqslant r$*,* $(\hat{C}_i, o) := \bigcup_{j \neq i} (C_j, o)$ *birationally dominates a simple germ of type* A_n *(for some* $n \geqslant 0$*) or* D_n *(for some* $n \geqslant 4$*).*

If these conditions hold, (C, o) *is of finite growth if and only if* $\mathfrak{M}_{1,-1}(C, o)$ *is of maximal rank.*

As mentioned, we can reformulate these in terms of the multivariable motivic Poincaré series $P^m_C(\boldsymbol{t}, q)$—in fact, we need only the data of the corresponding univariable series $P^m_C(t, q) := P^m_C(\boldsymbol{t}, q)|_{t_i \to t} = \sum_{d \in \mathbb{N}} \mathfrak{p}^m_d(q) t^d$:

Theorem 3 ([13]). *Let* $f^C(\omega) := P^m_C(\boldsymbol{t}, q)|_{t_i \to \omega^{-1},\, q \to \omega^2} = P^m_C(t, q)|_{t \to \omega^{-1},\, q \to \omega^2}$. *For convenience, we also write* $\mathfrak{p}^m_d(q) = \sum_{j \in \mathbb{N}} \pi^C_{d,j} q^j$ *and set* $\mu^C := \mathrm{ord}_t(P^m_C(t, q) - \mathfrak{p}^C_0(q)) = \min\{d \geqslant 1 \mid \mathfrak{p}^C_d(q) \neq 0\}$.

Then (C, o) *is of finite CM type if and only if* $\mathrm{ord}\, f^C \geqslant -1$. *In particular:*

- (C, o) *birationally dominates (and hence is) a simple germ of type* A_n *(for some* $n \geqslant 0$*) if and only if* $\mathrm{ord}\, f^C = 0$ *or, equivalently,* $\mu^C \leqslant 2$.
- (C, o) *birationally dominates a simple singularity of type* D_n *(for some* $n \geqslant 4$*), but no simple germ of type* A_n *(for any* $n \geqslant 0$*), if and only if* $\mathrm{ord}\, f^C = -1$ *and* $\pi^C_{3,2} \neq 0$ *(equivalently,* $\pi^C_{3,2} < 0$*).*
- (C, o) *birationally dominates a simple singularity of type* E_6, E_7, *or* E_8, *but no simple germ of type* A_n *(for any* $n \geqslant 0$*) or* D_n *(for any* $n \geqslant 4$*), if and only if* $\mathrm{ord}\, f^C = -1$ *and* $\pi^C_{3,2} = 0$.

Likewise, (C, o) *is of tame CM type if and only if the following conditions hold:*

(a) $\mathrm{ord}\, f^C = -2$,

(b) *either* $\mu^C = 3$ *and* $\pi^C_{6,3} < 0$, *or* $\mu^C = 4$ *and* $\pi^C_{4,3} \neq 0$ *(equivalently,* $\pi^C_{4,3} < 0$*),*

(c) *for each* $1 \leqslant i \leqslant r$, $\mathrm{ord}\, f^{C_i} = 0$ *(equivalently,* $\mu^{C_i} \leqslant 2$*),*

(d) *for each* $1 \leqslant i \leqslant r$, *if we let* $(\hat{C}_i, o) := \bigcup_{j \neq i} (C_j, o)$, *then either* $\mathrm{ord}\, f^{\hat{C}_i} = 0$, *or* $\mathrm{ord}\, f^{\hat{C}_i} = -1$ *and* $\pi^{\hat{C}_i}_{3,2} \neq 0$.

If conditions (a–d) hold, (C, o) *is of finite growth if and only if either* $\mu^C = 3$ *and* $\pi^C_{6,3} = -2$, *or* $\mu^C = 4$ *and* $\pi^C_{4,3} = -3$. *(Note that, in general,* $\mu^C = 3$ *implies* $\pi^C_{6,3} \geqslant -2$ *and* $\mu^C = 4$ *implies* $\pi^C_{4,3} \geqslant -3$.*)*

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Derived categories of singular varieties

Martin Kalck
(joint work with Yujiro Kawamata and Nebojsa Pavic)

1. Introduction and motivation

Throughout this text, X denotes a projective variety over $k = \mathbb{C}$.

Aim. *Describe the bounded derived category $D^b(X)$ of coherent sheaves on X using derived categories $D^b(R) := D^b(\text{mod } R)$ of finite dimensional algebras*[1] *R.*

This has been achieved for projective spaces as a first example of 'tilting theory'.

Example 1.1 (Beilinson 1978). *Let $X = \mathbb{P}^n$. There are triangle equivalences*

$$D^b(\mathbb{P}^n) \cong D^b\left(\mathrm{End}_{\mathbb{P}^n}\left(\bigoplus_{i=0}^{n}\mathcal{O}(i)\right)^{\mathrm{op}}\right) \cong \langle \mathcal{O}, \ldots, \mathcal{O}(n)\rangle \cong \left\langle D^b(\mathbb{C}), \ldots, D^b(\mathbb{C})\right\rangle,$$

describing $D^b(\mathbb{P}^n)$ as $D^b(B_n)$ ('tilting')[2] *, using a full exceptional sequence and as a semiorthogonal decomposition (S.O.D), respectively.*

Building on this, full exceptional sequences have been constructed for many smooth varieties, e.g. by Hille & Perling, Kapranov, Kawamata, Kuznetsov.

However, singular projective (Gorenstein) varieties do not admit full exceptional sequences, cf. [6] and also [11]. This motivates the following definition, which generalizes both tilting ($l = 1$) and full exceptional sequences (all $\mathcal{C}_i \cong D^b(\mathbb{C})$).

Definition 1.2 ([6]). A *Kawamata semiorthogonal decomposition (KSOD)* is an (admissible) semiorthogonal decomposition

$$\text{(KSOD)} \qquad D^b(X) = \langle \mathcal{C}_1, \ldots, \mathcal{C}_l\rangle,$$

where $\mathcal{C}_i \subseteq \mathsf{Perf}(X)$ or $\mathcal{C}_i \cong D^b(R_i)$ for *finite dimensional* algebras R_i.

[1]All these algebras arise as endomorphism algebras $\mathrm{End}_X(\mathcal{F})$ for some bounded complexes of finite rank vector bundles $\mathcal{F}$ on X. In particular, they are associative $\mathbb{C}$-algebras. However, composition of endomorphisms need not commute, so these algebras are typically non-commutative. Since X is projective, such algebras are automatically finite dimensional over $\mathbb{C}$ (Serre).

[2]Here, B_n denotes the finite dimensional algebra $\mathrm{End}_{\mathbb{P}^n}(\bigoplus_{i=0}^{n}\mathcal{O}(i))^{\mathrm{op}}$.

Remark 1.3. Kawamata initiated the study of KSODs for threefolds [8], which was the starting point for our investigations in [6]. If X is singular, at least one of the algebras R_i has infinite global dimension. The derived categories $D^b(R_i)$ of these algebras capture the singular information of X and fit into Kuznetsov & Shinder's framework of 'categorical absorption of singularities' [10, 11, 12], cf. also [5, Definition 1.6] for a refinement and for obstructions to KSODs for certain odd-dimensional isolated non-nodal hypersurface singularities. This leads to our focus on non-hypersurface singularities in this text (except for the remarks on curves).

In the sequel, we describe some known constructions of KSODs.

2. Constructions of Kawamata S.O.Ds and tilting

2.1. Curves.

Theorem 2.1 (Burban [1], cf. also [6]). *Let X be a connected nodal curve with all irreducible components isomorphic to $\mathbb{P}^1$. Then $D^b(X)$ has a tilting object if and only if the dual intersection graph of X is a tree. Moreover, this tilting object is of the form $\mathcal{O}_X \oplus \mathcal{G}$, which yields a KSOD $D^b(X) \cong \langle D^b(\mathrm{End}_X(\mathcal{G})^{\mathrm{op}}), D^b(\mathbb{C})\rangle$.*

Remark 2.2. The quiver of $R_X := \mathrm{End}_X(\mathcal{G})^{\mathrm{op}}$ is the *double quiver*[3] of the dual intersection graph of X. So the algebra corresponds nicely to the geometry.

If, moreover, the dual intersection graph is of Dynkin type A, then R_X is a so-called gentle algebra, cf. e.g. [1]. Remarkably, R_X is as unique as possible: any other $\mathbb{C}$-algebra B with equivalent derived module category $D^b(B) \cong D^b(R_X)$ (as $\mathbb{C}$-linear triangulated categories) satisfies $\mathrm{mod}\, B \cong \mathrm{mod}\, R_X$ using work of Schröer & Zimmermann combined with Avella-Alaminos & Geiß.

2.2. Higher-dimensional varieties.

KSODs for toric surfaces (not necessarily Gorenstein) have been explicitly described in [7] building on [3], cf. e.g. [2] and Lekili's talk during this workshop. Therefore, we discuss varieties X with $\dim(X) > 2$.

The following generalizations of Beilinson's Example 1.1 to weighted projective spaces have been obtained in joint work with Y. Kawamata and N. Pavic [4].

Theorem 2.3. *There is an explicit tilting object on $X = \mathbb{P}(1^d, m)$ for all $m, d \geqslant 1$.*

Theorem 2.4. *Let $X_d = \mathbb{P}(1^d, d)$. There is a Kawamata S.O.D*

$$D^b(X_d) \cong \langle D^b(R_{X_d}), D^b(\mathbb{C}), D^b(\mathbb{C})\rangle, \tag{1}$$

where the categories $D^b(\mathbb{C})$ are generated by $\mathcal{O}$ and $\mathcal{O}(H)$ (H hyperplane at infinity), respectively. Moreover, R_{X_d} fits into a split short exact sequence[4]

$$0 \to \mathrm{Hom}_{E_{d-1}}(E_{d-1}, \mathbb{S}^{-1}(E_{d-1})[d-2]) \to R_{X_d} \xrightarrow{\pi} E_{d-1} \to 0 \tag{2}$$

[3] Let G be an undirected graph. Its 'double quiver' is a directed graph obtained by replacing every edge in G by a pair of arrows (between the same vertices) pointing in opposite directions.

[4] Here, π is a split epimorphism of algebras and E_{d-1} is obtained from the algebra B_{d-1} in Example 1.1 for $Z = \mathbb{P}^{d-1}$ by removing the vertex corresponding to $\mathcal{O}_Z$ from its quiver. Moreover, the kernel of π is a square-zero ideal and $\mathbb{S}$ is the Serre functor of $D^b(E_{d-1})$.

Remark 2.5. For $d = m = p^l$ a prime power, we can show that $D^b(R_{X_d})$ cannot contain any exceptional objects. In fact, we know more: any admissible S.O.D of $D^b(R_{X_d})$ has to involve a K_0-phantom category. Also, note that $D^b(R_{X_d}) = \langle \mathcal{O}, \mathcal{O}(H)\rangle^{\perp}$ is a 'residual category' in the sense of Kuznetsov.

The singularities in Theorems 2.3 & 2.4 are examples of cone singularities over $\mathbb{P}^{d-1}$ (with respect to different embeddings). We generalize the case of anticanonical embeddings (Theorem 2.4) by replacing $\mathbb{P}^{d-1}$ with certain 'nice' Fano varieties Z.

Theorem 2.6 (K–Pavic, special case)**.** *Let X be a projective variety with* $\dim X = d$ *and* $\mathsf{Sing}(X) = \{s\}$. *Assume s is a cone singularity over a* nice[5] *smooth Fano variety Z with respect to the anticanonical embedding. Let $i\colon Z \hookrightarrow Y = \mathrm{Bl}_s(X)$ be the embedding of the exceptional divisor of the crepant resolution $Y \to X$.*

Assume the existence of a geometric exceptional sequence $\mathcal{E}_1, \ldots, \mathcal{E}_m$ of type $(d-1, m)$ (cf. Bridgeland–Stern), such that $i^!\colon \langle \mathcal{E}_1, \ldots, \mathcal{E}_m\rangle \xrightarrow{\cong} \mathcal{O}_Z^{\perp}$ as triangulated categories. Then there is a KSOD

$$D^b(X) = \langle D^b(A), \mathcal{C}\rangle$$

with $\mathcal{C} \subseteq \mathsf{Perf}(X)$ and a short exact sequence describing A as an extension

$$0 \to \mathrm{Hom}_E(E, \mathbb{S}^{-1}(E)[d-2]) \to A \xrightarrow{\pi} E \to 0. \tag{3}$$

Here, $E^{\mathrm{op}} = \mathrm{End}_Y(\bigoplus_{i=1}^m \mathcal{E}_i) \cong \mathrm{End}_Z(\bigoplus_{i=1}^m i^!(\mathcal{E}_i))$, π is a ring epimorphism (possibly non-split) and $(\ker\, \pi)^2 = 0$.

Corollary 2.7 (K–Pavic)**.** *Let X be a projective cone over a* nice *smooth Fano variety Z with respect to the anticanonical embedding. Then there is a KSOD*

$$D^b(X) = \langle D^b(A), \mathcal{O}, \mathcal{O}(H)\rangle, \tag{4}$$

where the algebra A fits into a short exact sequence (3) *such that π splits and H is the hyperplane at infinity.*

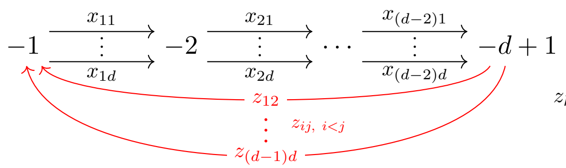
Remark 2.8. If $\dim X \leqslant 3$, the sequence (3) splits. So, in this case, the algebras A in Theorem 2.6 are (up to isomorphism) uniquely determined by the analytic type of the singularity and the choice of the exceptional sequence $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_m)$. For a fixed analytic type of singularity, different choices of $\mathcal{E}$ yield derived equivalent split extensions A.

In general, up to deformations of associative algebras, the algebras A in Theorem 2.6 are determined by the analytic type of the singularity and the choice of the exceptional sequence $\mathcal{E}$.

[5] E.g. projective spaces, quadrics, blow-ups $\mathrm{Bl}_{p_1,\ldots,p_t}(\mathbb{P}^2)$ in $t \leqslant 4$ points in general position, the threefold V_5 and finite products of these examples.

3. Explicit description of the algebras in examples

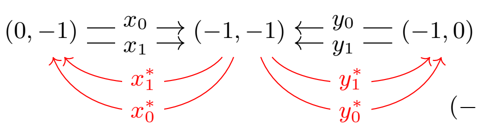
3.1. Quotient singularities $\frac{1}{d}(1^d)$.
If (3) splits, then the algebra A is given by the following quiver[6] with relations

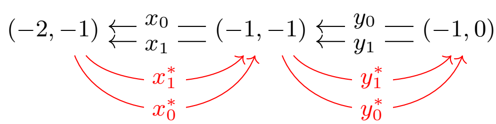


$$x_{(i+1)k}x_{ij} - x_{(i+1)j}x_{ik} = 0$$
$$x_{1j}z_{kl} + x_{1l}z_{jk} - x_{1k}z_{jl} = 0$$
$$z_{kl}x_{(d-2)j} + z_{jk}x_{(d-2)l} - z_{jl}x_{(d-2)k} = 0$$
$$(\mathbb{C}Q_d\{z_{ab} \mid 1 \leqslant a < b \leqslant d\}\mathbb{C}Q_d)^2 = 0$$
for all $1 \leqslant j < k < l \leqslant d$ and $1 \leqslant i \leqslant d-3$

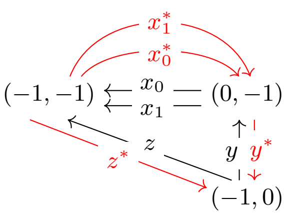
3.2. Examples with E hereditary. In the following examples, we have an inequality $\mathsf{gl.dim}\, E \leqslant 1$. This implies that the sequence (3) splits. We have $E \cong kQ$ for a finite quiver Q. Then $A \cong k\overline{Q}/I$, where $\overline{Q}$ is the double quiver[7] of Q and the two-sided ideal I is generated by all path p in $\overline{Q}$ that contain at least two of the new *-arrows, together with $\sum_{a\in Q_1}[a^*, a]$ where $[-,-]$ denotes the commutator. So, in this case, the algebras A are completely determined by Q.

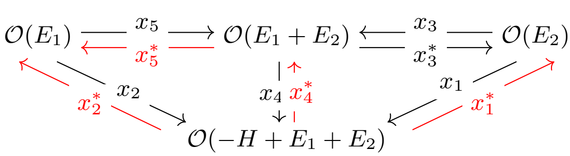
3.2.1. *Cone over $\mathbb{P}^1 \times \mathbb{P}^1$.*
We describe two non-isomorphic algebras A via their quivers $\overline{Q}$:





3.2.2. *Cones over blow-ups $\mathbb{P}^2(t) := \mathrm{Bl}_{p_1,\dots,p_t}(\mathbb{P}^2)$ in $1 \leqslant t \leqslant 2$ distinct points.*
Again, we only describe the quivers $\overline{Q}$, following [9, Propositions 6.1 & 6.2].



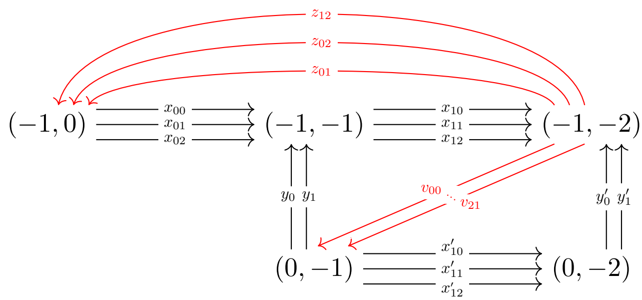


[6]Here, and in the examples below, the vertices of the quivers are labeled by certain elements in the Picard group of Z. Moreover, the black arrows generate the algebra E (we always consider opposite algebras so the direction of all arrows is opposite to the direction of morphisms between the line bundles corresponding to the vertices), whereas the red arrows generate the bimodule $\mathrm{Hom}_E(E, \mathbb{S}^{-1}(E)[d-2])$.

[7]I.e. $\overline{Q}$ is obtained from Q by adding an additional arrow $a^*\colon s(a) \leftarrow t(a) \in \overline{Q}_1$ (in the opposite direction) for every arrow $a\colon s(a) \to t(a) \in Q_1$.

3.3. **Examples with** gl.dim $E = 2$. In this case, the relations of the quiver algebra A (if (3) splits) are given by all paths containing at least two red arrows together with relations that arise as cyclic derivatives of a so-called 'superpotential'.

3.3.1. *Cone over* $\mathbb{P}^2 \times \mathbb{P}^1$.



The superpotential is given by

$$W = \sum_{i=0}^{2}\sum_{j=0}^{1} v_{ij}\rho_{v_{ij}} + \sum_{0\leqslant i<j\leqslant 2} z_{ij}\rho_{z_{ij}}, \tag{5}$$

where the $\rho_\bullet$ are the 'minimal relations' of E, i.e.

$$\rho_{v_{ij}} = y'_j x'_{1i} - x_{1i}y_j, \quad \rho_{z_{ij}} = x_{1i}x_{0j} - x_{1j}x_{0i}, \tag{6}$$

Remark 3.1. The case of threefolds $Z = S \times \mathbb{P}^1$ with S a del Pezzo surface of degree $d > 4$ and also the three dimensional quadric, and the del Pezzo threefold V_5 can be treated similarly.

More generally, if the algebras E are Koszul (which is not always the case) then it is known that the relations of the split algebras A are still given by some superpotential. However, it is more complicated to make this explicit in examples.

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Non-commutative orders from deformations of Wahl singularities

Yanki Lekili

(joint work with Jenia Tevelev)

Let $0 < a < r$ be coprime integers. Consider the singularity $\mathbb{A}^2/\mu_r$ over a field k, where the cyclic group μ_r acts on $\mathbb{A}^2$ with weights $(1, a)$. It is denoted by $\frac{1}{r}(1, a)$. In [1], Kalck–Karmazyn constructed a finite-dimensional k-algebra $R_{r,a}$ and proved a generalised Knörrer type equivalence of singularity categories:

$$D^b_{sg}(R_{r,a}) \simeq D^b_{sg}(\tfrac{1}{r}(1, a)).$$

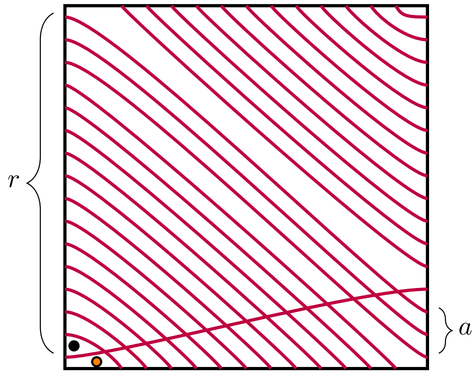
For $0 < q < n$ coprime integers, the cyclic quotient singularity $\frac{1}{n^2}(1, nq - 1)$ is called a *Wahl singularity*. It admits a 1-dimensional $\mathbb{Q}$-Gorenstein deformation:

$$\{xy = z^n + t\} \subset \tfrac{1}{n}(1, -1, q) \times \mathbb{A}^1_t,$$

where $\frac{1}{n}(1, -1, q)$ is the quotient of $\mathbb{A}^3$ by μ_n with weights $(1, -1, q)$. Recall that a matrix order $\mathcal{R}$ over $k[t]$ is a flat $k[t]$-algebra $\mathcal{R}$ such that $\mathcal{R} \otimes_{k[t]} K = \mathrm{Mat}_n(K)$, where $K = k(t)$. In [3], a matrix order $\mathcal{R}_{n,q}$ over $\mathbb{A}^1_t$ is computed such that the following equivalence holds.

Theorem. $D^b_{sg}(\mathcal{R}_{n,q}) \simeq D^b_{sg}(\frac{1}{n}(1, -1, q))$.

$\mathcal{R}_{n,q}$ is a flat deformation of the Kalck–Karmazyn algebra $(\mathcal{R}_{n,q})_{|t=0} = R_{n^2,nq-1}$. The construction of the order $\mathcal{R}_{n,q}$ comes from earlier work of Kawamata [2] and Tevelev–Urzúa [4]. However, the explicit computation was performed in [3] using mirror symmetry, where $\mathcal{R}_{n,q}$ was realized as the endomorphism algebra of a certain bulk-deformed Lagrangian in the relative Fukaya category of a punctured torus depicted in the following figure ($r = n^2, a = nq - 1$).



The construction in [4] gives a flat deformation of $R_{r,a}$ for any component of the deformation space of $\frac{1}{r}(1,a)$. Given such a component, a generators-and-relations description of the corresponding algebra deforming $R_{r,a}$ is computed in [3].

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In Memoriam: Mihai Tibăr

Laurenţiu Maxim

Our colleague and friend Mihai Tibăr passed away in September 2025, following a long struggle with a severe illness. This lecture was offered in tribute to his mathematical legacy, highlighting results and insights he developed over the last five years in relation to: (i) polar degree, and (ii) Euclidean distance degree, optimization, and linear morsification.

1. Polar degree

The *polar degree* $pol(V)$ of a projective hypersurface $V \subset \mathbb{CP}^n$ defined by a degree d homogeneous polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is the topological degree of the gradient mapping $\mathtt{grad}(f) : \mathbb{CP}^n \setminus \mathrm{Sing}(V) \to \mathbb{CP}^n$. In [2], Dolgachev conjectured that $pol(V)$ is a topological invariant, depending only on the reduced structure of V (and not on the defining polynomial f). This conjecture was proved by Dimca–Papadima in [1], who showed that

$$pol(V) = \text{rank } H_{n-1}(V \setminus H), \tag{1}$$

where $H \subset \mathbb{CP}^n$ is a general hyperplane. In fact, it is shown that $V \backslash H$ is homotopy equivalent to a bouquet of $(n-1)$-dimensional spheres, their numbers being equal to $pol(V)$. Furthermore, Dimca–Papadima proved that if the hypersurface V has at most isolated singularities, then

$$pol(V) = (d-1)^n - \sum_{p \in \mathrm{Sing}(V)} \mu_p(V), \tag{2}$$

where $\mu_p(V)$ is the Milnor number of V at a singular point $p \in \mathrm{Sing}(V)$. Therefore, the maximal value of $pol(V)$ for degree d hypersurfaces with at most isolated singularities is $(d-1)^n$, realized by the Fermat hypersurface $V_{n,d} = \{x_0^d + \cdots + x_n^d = 0\}$, where $[x_0 : \cdots : x_n]$ are homogeneous coordinates of $\mathbb{CP}^n$. In particular, the quadric $V_{n,2} \subset \mathbb{CP}^n$ is the only smooth hypersurface with polar degree equal to 1.

Hypersurfaces with polar degree 1–also known as *homaloidal hypersurfaces*–are especially interesting, since the gradient map defines in this case a polar Cremona transformation. Dolgachev [2] classified homaloidal curves, and Dimca–Papadima [1] conjectured that, although there exist many homaloidal hypersurfaces of any given degree $d \geqslant 3$ in $\mathbb{CP}^n$ for $n \geqslant 3$, none of them have only isolated singularities, except for the smooth quadric and the plane curves identified by Dolgachev. Their conjecture was confirmed by Huh [4], who relied on Tibăr's theory of *slicing by hyperplanes with singularities in the axis* [12, 13] to first establish lower bounds for the polar degree. Specifically, Huh proved that if the hypersurface $V \subset \mathbb{CP}^n$ has only isolated singularities and H_p is a general hyperplane passing *through* a singular point $p \in \mathrm{Sing}(V)$, then–assuming V is not a cone at p–one has

$$pol(V) = \mu_p^{\langle n-2 \rangle}(V) + \mathrm{rank}\ H_n(\mathbb{CP}^n \setminus V, (\mathbb{CP}^n \setminus V) \cap H_p), \tag{3}$$

where $\mu_p^{\langle n-2 \rangle}(V)$ is the Milnor number of the slice $H_p \cap V$ at p. In particular,

$$pol(V) \geqslant \mu_p^{\langle n-2 \rangle}(V), \quad \text{for all } p \in \mathrm{Sing}(V), \tag{4}$$

and this lower bound is positive.

Huh [4] also proposed a conjectural classification of projective hypersurfaces with only isolated singularities and polar degree 2. This conjecture was recently settled by Siersma–Steenbrink–Tibăr [10], using as a key tool the semicontinuity of the spectrum. They further show that no such hypersurfaces exist for $n > 3$, and confirmed Hu's finiteness conjecture for higher polar degrees.

Finally, in [11], Siersma–Tibăr generalized Huh's formula (3) to hypersurfaces with arbitrary singularities, with the goal of deriving computable lower bounds in terms of vanishing cycles, thereby generalizing inequality (4). Specifically, they expressed the polar degree of a hypersurface $V \subset \mathbb{CP}^n$ as a sum of two non-negative integers,

$$pol(V) = \alpha(V, H) + \beta(V, H), \tag{5}$$

which quantify local vanishing cycles of two different types. The numbers $\alpha(V, H)$ and $\beta(V, H)$ depend on the choice of a special "admissible" hyperplane H, with

$\alpha(V,H)$ collecting the numbers of vanishing cycles of the isolated stratified singularities of the slice $V \cap H$, while $\beta(V,H)$ counts the vanishing cycles of the singularities of the polynomial f outside V.

2. Euclidean distance degree. Optimization. Linear morsification

The *Euclidean distance degree* (ED degree) [3] of an algebraic variety $X \subset \mathbb{C}^n$ is defined as the number of critical points of the squared Euclidean distance function $d_u(x) := \sum_{i=1}^n (x_i - u_i)^2$ on the smooth locus X_{reg} of X, for a *generic* point $u = (u_1, \ldots, u_n)$. It is denoted by $\text{EDdeg}(X)$. When X is the complexification of a real algebraic model $X_{\mathbb{R}}$, the ED degree of X measures the algebraic complexity of the nearest point problem for $X_{\mathbb{R}}$. A topological interpretation of the ED degree of a complex affine variety was given in [5], expressing it in terms of the weighted Euler characteristic of MacPherson's *local Euler obstruction function*:

$$\text{EDdeg}(X) = (-1)^{\dim(X)} \cdot \chi(Eu_{X \setminus Q_c}), \tag{6}$$

where $Q_c = \{d_u(x) = c\}$ for a generic $c \in \mathbb{C}$. This formula was obtained by linearization the distance function and applying the Seade–Tibăr–Verjovsky formula [9] for linear optimization. It played a key role in proving a conjecture from [3] concerning the ED degree of the *multiview variety* arising in computer vision.

Typically, results on Euclidean distance degrees and nearest point problems assume the genericity of the data point u, or focus on the ED-discriminant locus–that is, the set of data points u for which the function d_u has a number of critical points different from $\text{EDdeg}(X)$. However, in many practical applications, the data are not generic. The results in [6] address this situation by analyzing the *limiting behavior* of critical sets obtained for generic perturbations of the data. Specifically, by adding some *noise* $\epsilon \in \mathbb{C}^n$ to an arbitrary data point u, one returns to the generic situation, and the limiting behavior of critical points of $d_{u+t\epsilon}$ on X_{reg} for $t \in \mathbb{C}^*$ (with $|t|$ very small), as $t \to 0 \in \mathbb{C}$, provides valuable information about the initial nearest point problem. Notice that one can write $d_{u+t\epsilon}(x) = d_u(x) - tl(x) + c$, with $l(x) = 2\sum_{i=1}^n \epsilon_i x_i$ and c is a constant with respect to x. Thus, the critical points of $d_{u+t\epsilon}$ coincide with those of $d_u - tl$. Since ϵ is generic, l is a generic linear function. This observation naturally leads to the *Morsification* procedure.

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function and $l\colon \mathbb{C}^n \to \mathbb{C}$ a linear one. Let $X \subset \mathbb{C}^n$ be a possibly singular closed irreducible subvariety such that f is not constant on X. Restrict f and l to X. If l is general enough, the function $f_t := f - tl$ is a holomorphic Morse function on X_{reg} (that is, it has only non-degenerate isolated critical points) for all but finitely many $t \in \mathbb{C}$. One is then interested in studying the critical set $\text{Crit}(f_t|_{X_{\text{reg}}})$ and its *limit* as $t \to 0 \in \mathbb{C}$. In more general situations, when f has a positive-dimensional stratified singular locus, the limit as $t \to 0 \in \mathbb{C}$ captures a finite set of special points in the critical locus of f (see [7] for details). When $X = \mathbb{C}^n$, the number of Morse points $\#\text{Crit}(f_t|_{X_{\text{reg}}})$ equals the total degree of the gradient of f, hence it can be viewed as an *affine polar degree* of the polynomial f.

Results from [6], based on the linear optimization formula of [9], yield the following counting formula for Morse points:

$$\#\mathrm{Crit}(f_t|_{X_{\mathrm{reg}}}) = (-1)^{\dim(X)} \cdot \chi(Eu_{X\setminus\{f_t=c\}}), \tag{7}$$

for a general choice of c. A natural question then arises: how can this number be expressed in terms of data related to f and l, as $t \to 0$? A classical result of Brieskorn states that if X is smooth and f has only isolated singularities $\{p_i\}_{i=1}^k$, then in a small neighborhood of each p_i the function f_t has μ_i Morse critical points (where μ_i is the Milnor number of f at p_i). As t approaches 0, these points coalesce at p_i. More general formulas were obtained in [6] using vanishing cycle calculations, though without accounting for the critical points of f_t which *escape at infinity* as $t \to 0$.

The subsequent projects [7, 8], developed jointly with Mihai Tibăr, further refine this picture by achieving the following goals:

- Providing a purely topological interpretation [7] of $\#\mathrm{Crit}(f_t|_{X_{\mathrm{reg}}})$ in terms of X and invariants of f and l.
- Detecting the limit points of stratified Morse trajectories and effectively computing their multiplicities [7, 8].
- Understanding the contribution of Morse points at infinity in the limit process, and identifying the corresponding attractor points at infinity [8].

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Multi-singularity Thom polynomials and algebraic cobordism

Toru Ohmoto

In my talk, I outlined a proof of the Existence Theorem of universal polynomials which express *multi-singularity loci classes* of prescribed types for any proper morphisms $f : X \to Y$ between smooth schemes over an algebraically closed field of characteristic zero – we call them *Thom polynomials for multi-singularity types of maps* [3]. It has often been referred to as the *Thom–Kazarian principle* [1], and unsolved for a long time. This result solidifies the foundation for a general enumerative theory of singularities of maps which is applicable to a broad range of problems in classical and modern algebraic geometry (a general survey on Thom polynomial theory can be found in [2, 4]). For instance,

- it extends *curvilinear multiple-point theory*, which was established in 80s by Kleiman et. al. and deals with maps $f\colon X \to Y$ with $\dim\ker df \leqslant 1$ of $\kappa = \dim Y - \dim X \geqslant 0$, to be adapted to maps with arbitrary singularities;
- it guarantees universal expressions for counting divisors with a prescribed combination of isolated singular points in a given family of divisors, e.g., *Göttsche's conjecture* counting r-nodal curves in a sub-linear system on a projective surface (known as *instanton counting*) is immediately obtained from our theorem for the simplest type A_1^r of codimension $\kappa = -1$;
- various counting problems of intersection and contact are formulated in a unified way; especially, it contributes to a satisfactory answer to an advanced form of Hilbert's 15th problem and recent questions motivated by mathematical physics.

First, we define the multi-singularity loci class for an arbitrary proper morphism by combining Intersection Theory on Hilbert schemes of points with the Thom–Mather Theory for classification of singularities of maps. Next, a main feature of our proof is a striking use of *cohomology operations* for algebraic cobordism Ω^* of Levine-Morel, recently established by A. Vishik [5] as an analogue to the well-studied notion in complex cobordism MU^* in topology. In fact, what we have proved is the existence theorem of *Thom polynomials valued in* Ω^*. In topology, René Thom simultaneously invented singularity theory of maps, characteristic classes and cobordism theory, that should have been consistent in his mind. Indeed, those together suggest the ground design of our enumerative theory, and now it is time to properly realize it in the context of *derived algebraic geometry.*

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Zariski's dimensionality type of singularities. Case of dimensionality type 2

Adam Parusiński
(joint work with Laurenţiu Păunescu)

In 1979, O. Zariski introduced a general theory of equisingularity for algebroid and algebraic hypersurfaces over an algebraically closed field k of characteristic zero. This theory is based on the notion of dimensionality type of hypersurface singularities, introduced and developed by Zariski.

Let $f \in k[\![x_1, \dots, x_{r+1}]\!]$ be reduced, let $V = f^{-1}(0) \subset \mathbb{A}_k^{r+1}$. For all $p \in V$ we define *the dimensionality type*, a lower semi-continuous invariant,

$$\mathrm{d.\,t.}(V, p) \in \mathbb{N}$$

(1) $\mathrm{d.\,t.}(V, p) = 0$ iff p is a smooth point of V.
(2) The levels of d. t. give a canonical stratification of V (independent of the system of coordinates).
(3) V is equisingular (in some sense) along each stratum of this stratification.

The dimensionality type is defined recursively by considering the discriminants loci of successive "generic" corank 1 projections.

Example (Dimensionality type 1)**.** A singularity of dimensionality type 1 is isomorphic to the total space of an equisingular family of plane curve singularities. Write such a family as

$$f_t(x, y) = y^d + \sum_{j=1}^{d} a_j(x, t) y^{d-j}.$$

Then $f_t = 0$ is equisingular (with respect to parameter $t \in k^{r-1}$) if so is the family if its discriminant i.e. $\Delta_f(x) = x^m \cdot \mathit{unit}(x, t)$. To check whether such family is equisingular it suffices to assume that the projection $(t, x, y) \to (t, x)$ is transverse to $f_t = 0$ (i.e. the kernel of projection is not tangent to $f_t = 0$).

Dimensionality type. Idea of definition. Let $\pi : (k^{r+1}, p) \to (k^r, \pi(p))$ be "generic" and let Δ be the discriminant locus of $\pi_{|V}$. If p is a singular point of V then we define

$$\mathrm{d.\,t.}(V, p) := \mathrm{d.\,t.}(\Delta, \pi(p)).$$

The notion of a generic projection plays a crucial role in Zariski's definition. For dimensionality type 1 generic means transverse. This is no longer the case for the singularities of dimensional type 2.

Example (Luengo, 1985)**.** The following family of surface singularities in $\mathbb{C}^3$ is Zariski equisingular for one transverse system of coordinates but not for a generic linear system of coordinates: $f_t(x, y, z) = x^{10} + tyz^3x^7 + y^{10} + y^6z^4 + z^{16}$.

Generic projection after Zariski.
The generic projection $\pi_u = (\pi_{u,1}(x), \dots, \pi_{u,r}(x)) : (k^*)^{r+1} \to (k)^{*r}$

$$\pi_{u,i}(x) = \sum_{d \geqslant 1} \sum_{\nu_1 + \cdots \nu_{r+1} = d} u^{(i)}_{\nu_1, \cdots, \nu_{r+1}} x^\nu,$$

where k^* is any field extension of k that contains all $u^{(i)}_{\nu_1, \cdots, \nu_{r+1}}$ as indeterminates.

Thus a generic projection of Zariski involves adding all the coefficients of a local projection as indeterminates to the ground field. As Zariski showed, a generic (in classical sense) polynomial projection is sufficient for any dimensionality type, though he gave no explicit bound on the degree of such polynomial map. This makes an algorithmic computation of Zariski's canonical stratification from the definition impossible. The algebraic case was studied by Hironaka, where the algebraic semicontinuity of such a degree is shown.

Theorem (Hironaka, 1979)**.** *For algebraic V, Zariski's canonical stratification is algebraic.*

The question whether a generic linear projection is always sufficient is still open for dimensionality type three and higher.

Open problem. Is a generic linear projection generic, i.e. can its discriminant be used to compute the dimensionality type?

The affirmative answer for the dimensionality type 1 follows from the work of Zariski. We settle the case of dimensionality type 2 by studying Zariski equisingular families of surfaces singularities, not necessarily isolated, in the 3D-space. For this we introduce the nested uniformly transverse (ν-transverse) Zariski equisingularity. Our main result is the following.

Theorem (–, L. Păunescu)**.** *Let $V \subset (k^{r+1}, 0)$ be an algebroid hypersurface and let S be nonsingular subspace of V of dimension $r-2$. The following are equivalent:*

(1) V is ν-transverse Zariski equisingular along S.
(2) For every $p \in S$ the dimensionality type of V at p is equal to 2.
(3) V is generic Zariski equisingular along S.
(4) For a local system of coordinates, V is generic linear Zariski equisingular along S.
(5) For all local systems of coordinates, V is generic linear Zariski equisingular along S.

Definition (ν-transverse system of coordinates)**.** Given $V_0 = f_0^{-1}(0) \subset k^3$, f_0 reduced. For $b \in k$ denote

$$\text{(A)} \qquad \pi_b(x, y, z) = (x, y - bz)$$

and by Δ_b the discriminant locus of $\pi_b|_{V_0}$. We say that a local system of coordinates x, y, z is *ν-transverse* for V_0 if

(1) The projection $(x,y,z) \mapsto (x,y)$ is transverse to V_0 at p;
(2) The projection $(x,y) \mapsto x$ is transverse to Δ_0 at $\pi_0(p)$;
(3) The family of plane curve singularities parameterized by Δ_b is equisingular.

Idea of proof of Theorem. We show for a family of surface singularities the implications between different notions of equisingularity

$$\text{generic linear} \quad \overset{(1)}{\Longrightarrow} \quad \nu\text{-transverse} \quad \overset{(2)}{\Longrightarrow} \quad \text{Zariski generic.}$$

The main difficulty comes from understanding how the discriminant of $\pi_{|V}$ changes where we move the projection. In particular how the Weierstrass form of a series f changes. The proof of Hironaka's theorem is based entirely on the study the Weierstrass Preparation in such context.

For the proof of $\overset{(1)}{\Longrightarrow}$ we use the Eisenstein-Rond series (ER-series). We say that $f(x,\tau) \in k[\![x,\tau]\!]$, $x = (x_1, \ldots, x_{r+1})$, is *an Eisenstein-Rond series (uniformly rational in $\tau = (\tau_1, \ldots, \tau_s)$)* if

$$f(x,\tau) = \sum_\alpha a_\alpha(\tau) x^\alpha \qquad \text{with } a_\alpha = b_\alpha / c^{|\alpha|+1}, \quad \text{where} \quad b_\alpha, c \in k[\tau].$$

Lemma. If an ER-series $f(x,\tau) \in k[\![x,\tau]\!]$ uniformly rational in τ is regular in x_{r+1} (i.e. $f(0, x_{r+1}, \tau) \not\equiv 0$), then the coefficients a_i of the Weierstrass form of f over $k(\tau)$

$$f(x,\tau) = \Big(x_{r+1}^d + \sum_i a_i(x',\tau) x_{r+1}^{d-i}\Big) \cdot \mathit{unit}(x,\tau), \quad x' = (x_1, \ldots, x_r),$$

are also uniformly rational in τ (maybe with a different denominator c).

In particular, the discriminant $\Delta_f(x',\tau)$ of f is an ER-series uniformly rational in τ.

For the proof of $\overset{(2)}{\Longrightarrow}$ we use the polar wedges of Neumann-Pichon (based on earlier work of Briançon-Henry and Teissier). For a single surface singularity $V \subset (k^3, 0)$ a polar wedge is the union over b of the polar curves given by the projections (A), with the assumption that the discriminants are equsingular. A polar wedge W_i can be parameterized by $(u,b) \mapsto (x_i(u,b), y_i(u,b), z_i(u,b))$, with x_i, y_i, z_i analytic as follows

$$\begin{aligned} x_i(u,b) &= u^n, \\ y_i(u,b) &= y_i(u,0) + b^2 u^{m_i} \cdot \mathit{unit}(u,b), \\ z_i(u,b) &= z_i(u,0) + b u^{m_i} \cdot \mathit{unit}(u,b). \end{aligned}$$

Moreover, the parameterizations of two distinct polar wedges W_i, W_j satisfy:

$$\begin{aligned} y_i(u,b) - y_j(u,b) &= u^{k_{ij}} \cdot \mathit{unit}(u,b), \\ z_i(u,b) - z_j(u,b) &= O(u^{k_{ij}}). \end{aligned}$$

We get analogous parameterizations, with a parameter t, for ν-transverse equisingular families. Finally, these parameterizations allow us to parametrize the discriminant curves for arbitrary projections close to the standard one $(x, y, z) \mapsto (x, y)$, and show that these discriminant curves form an equisingular family.

The proof of $\stackrel{(1)}{\Longrightarrow}$ works for any dimensionality type, the proof of $\stackrel{(2)}{\Longrightarrow}$ only for the dimensionality type 2 at the moment.

del Pezzo surfaces of rank one

Tomasz Pełka
(joint work with Karol Palka)

A normal surface $\bar{X}$ is *del Pezzo* if its anti-canonical divisor $-K_{\bar{X}}$ is ample. In my talk I presented the following result, classifying log canonical del Pezzo surfaces of Picard rank one, over an algebraically closed field k of arbitrary characteristic. By a *singularity type* of a normal surface we mean a weighted graph of the exceptional divisor of its minimal resolution.

Theorem 1. *Let $\mathcal{P}(\mathcal{S})$ be the set of isomorphism classes of del Pezzo surfaces of rank one and singularity type $\mathcal{S}$. If $\mathcal{P}(\mathcal{S}) \neq \emptyset$, then both the singularity type $\mathcal{S}$ and the number $\#\mathcal{P}(\mathcal{S})$ are listed in* [13, 14, 15]. *Moreover, with one exception (Example 8), there is a smooth family $f\colon (\mathcal{X}, \mathcal{D}) \to B$ such that the set of isomorphism classes of its fibers equals the set of minimal log resolutions of surfaces in $\mathcal{P}(\mathcal{S})$, and one of the following holds, where (X_b, D_b) denotes the fiber of f over $b \in B$.*

- *(1) We have $\#\mathcal{P}(\mathcal{S}) = \infty$, and B admits a finite group action such that the fibers (X_b, D_b) and $(X_{b'}, D_{b'})$ are isomorphic if and only if b and b' lie in the same orbit. The number* $\dim B$ *is listed in* [13, 14, 15].
- *(2) We have $\#\mathcal{P}(\mathcal{S}) \in \{1, 2, 3\}$, and B admits a stratification such the fibers (X_b, D_b) and $(X_{b'}, D_{b'})$ are isomorphic if and only if b and b' lie in the same stratum; and the deepest one is a point $\{\bar{b}\}$.*

Furthermore, if $\operatorname{char} k \neq 2$ *then the germ of f at any $b \in B$ in case* (1)*, or at $\bar{b}$ in case* (2)*, is a semiuniversal deformation.*

Log terminal del Pezzo surfaces of rank one are important e.g. as possible outcomes of the birational part of the minimal model program. They were investigated by many authors from various points of view. For instance, if $k = \mathbb{C}$, Gurjar–Zhang [2] proved that their smooth loci have finite fundamental groups, and Keel–McKernan [3] proved that they are covered by images of $\mathbb{A}^1$.

The techniques developed in [2] led to a classification in particular cases, see [4, 5, 1]. The approach of [3] was recently extended by Lacini [6], who arranged *all* log terminal del Pezzo surfaces of rank one in 24 (non-disjoint) series, assuming only $\operatorname{char} k \neq 2, 3$. This description is explicit enough e.g. to bound the number of singularities and discuss liftability to characteristic zero, but does not give sufficient insight towards Theorem 1. For instance the problem of uniqueness, i.e. computing $\#\mathcal{P}(\mathcal{S})$, is not addressed.

To prove Theorem 1, we took a substantially different approach, based on bounding the following new invariant.

Definition 2 (Height)**.** Let $\bar{X}$ be a normal surface, and let (X, D) be its minimal log resolution. The *height* of $\bar{X}$, denoted by $\mathrm{ht}(\bar{X})$, is the minimal number h such that there is a $\mathbb{P}^1$-fibration of X whose fiber F satisfies $F \cdot D = h$.

Theorem 3 (Palka [11], see [9])**.** *Let $\bar{X}$ be a del Pezzo surface of rank* 1*. Then* $\mathrm{ht}(\bar{X}) \leqslant 4$*, with well described exceptions in case* $\mathrm{char}\, k \in \{2, 3\}$*.*

We now outline the proof of Theorem 3. Assume that $\bar{X}$ is log terminal: this is the most important, and the most difficult case (the non–lt case can be deduced easily from [8], see [12]). We apply the logarithmic minimal model program: *not* to (X, D), as it would simply contract D back to $\bar{X}$, but rather to $(X, \frac{1}{2}D)$. This log MMP terminates at a log Mori fiber space $(X_{\min}, \frac{1}{2}D_{\min})$ whose boundary $D_{\min}$ can be nonzero, but the singularities are much simpler ($\frac{1}{2}$-dlt instead of lt, cf. [10, §6]). If the base of this Mori fibration is a curve then its fiber F satisfies

$$0 > F \cdot (K_X + \tfrac{1}{2}D) = 2p_a(F) - 2 + \tfrac{1}{2}F \cdot D, \quad \text{so} \quad F \cong \mathbb{P}^1 \text{ and } F \cdot D \leqslant 3,$$

hence $\mathrm{ht}(\bar{X}) \leqslant 3$, as needed. If the base is a point, we get a *log* del Pezzo surface of rank one, which can be further improved by playing the two-ray game as in [8]: one extracts a divisor E with smallest log discrepancy, and contracts the curve $E^{\vee}$, spanning other ray of the effective cone (see [7, p. 381] for a general principle). We illustrate it with the following example.

Example 4. Consider a $\frac{1}{2}$-dlt log del Pezzo surface $(X_{\min}, \frac{1}{2}D_{\min})$ of rank one such that $X_{\min}$ has singularity type $\mathrm{A}_1 + \mathrm{A}_4 + [3]$, and $D_{\min}$ is a rational curve such that $\mathrm{Sing}\, D_{\min} \subseteq X_{\min}^{\mathrm{reg}}$ is an ordinary cusp, and $D_{\min}$ meets $\mathrm{Sing}\, X_{\min}$ only in the A_1 point, see Figure 1 (left). It exists if $\mathrm{char}\, k = 5$, see [15].

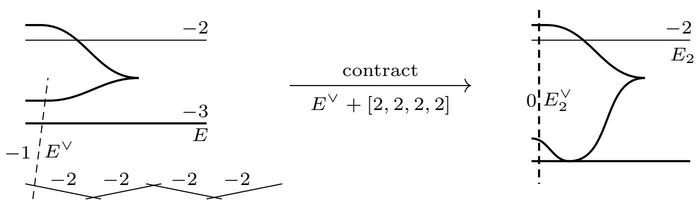


Figure 1. Minimal resolutions of surfaces in Example 4.

We extract the (-3)-curve, call it E, and compute that the proper transform of $E^{\vee}$ is a (-1)-curve as in Figure 1. Contracting it smooths the A_4 point, so our new $X_{\min}$ is of type A_1, and the new $D_{\min}$ is the cuspidal curve from before, and the image of E, which is tangent to it. Now, we extract the (-2)-curve E_2 over the A_1-point: its dual curve $E_2^{\vee}$ is a fiber of a $\mathbb{P}^1$-fibration of height 4, as needed.

The situation as in Example 4, when the 2-ray game ends with a $\mathbb{P}^1$-fibration of height $\leqslant 4$, is typical. A notable special case occurs when $X_{\min}$ is canonical, and $D_{\min}$ is a singular member of $|-K_{X_{\min}}|$ contained in $X_{\min}^{\mathrm{reg}}$. Such pairs $(X_{\min}, D_{\min})$ are well described, cf. [12, Proposition 1.5], and give rise to series of del Pezzo surface classified in [13, Theorem E]. Del Pezzo surfaces in such series are of height at most $\mathrm{ht}(X_{\min})+2$: indeed, a fiber F as in Definition 2 satisfies $F\cdot D \leqslant \mathrm{ht}(X_{\min}) + F\cdot \tilde{D}_{\min}$, where $\tilde{D}_{\min}\in|-K_X|$ is the proper transform of $D_{\min}$, so $F\cdot\tilde{D}_{\min}=2$ by adjunction. It is not hard to prove that this inequality is in fact an equality, with few exceptions. Here is an example of such a series.

Example 5. Let $\ell_1+\ell_2+\ell_3\subseteq\mathbb{P}^2$ be a triangle. Blow up at each $p_i := \ell_i\cap\ell_{i+1}$ and its infinitely near point on the proper transform of ℓ_i, where $\ell_4=\ell_1$. We get a minimal resolution of $X_{\min}\in\mathcal{P}(3\mathrm{A}_2)$, $\mathrm{ht}(X_{\min})=2$.

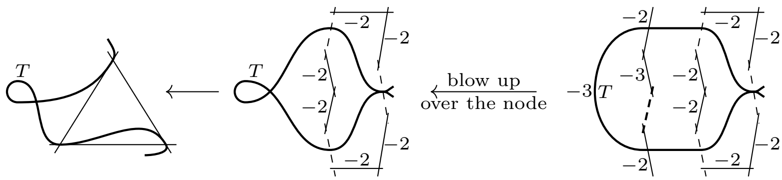


Figure 2. Del Pezzo surface of type $3\mathrm{A}_2$ and a nodal member of $|-K|$ yield a series of del Pezzo surfaces of rank 1.

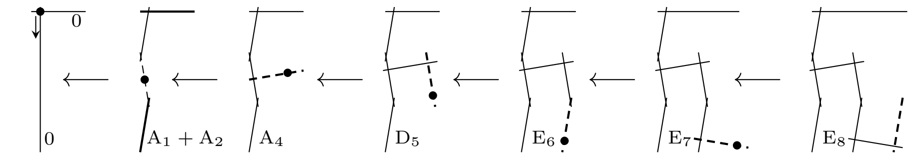
Assume $\operatorname{char} k\neq 3$. Then there is a cubic $T\subseteq\mathbb{P}^2$ with a node $q\neq p_i$ such that $(T\cdot\ell_i)_{p_i}=2$. Its proper transform $D_{\min}$ on $X_{\min}$ is a singular member of $|-K_{X_{\min}}|$. Blowing up further over the node $q\in T$, as long as the total transform of C is circular, produces a series of (minimal log resolutions of) del Pezzo surfaces of rank one and height $\mathrm{ht}(X_{\min})+2=4$, see Figure 2.

Assume $\operatorname{char} k = 3$. We can perform the above construction, but $q\in C$ is necessarily a cusp, so blowing up over q we get different singularity types. Still, blowing up twice over q we get $X'_{\min}\in\mathcal{P}(4\mathrm{A}_2)$, $\mathrm{ht}(X'_{\min})=4$. Since $\operatorname{char} k=3$, we can find another cubic T' with a cusp at $q'\neq p_i, q$ such that $(T'\cdot\ell_i)_{p_i}=2$, $(T'\cdot T)_q=3$. Its proper transform on $X'_{\min}$ is a cuspidal member of $|-K_{X'_{\min}}|$ contained in $(X'_{\min})^{\mathrm{reg}}$, so we can blow up over the cusp and produce a series of del Pezzo surfaces of rank 1 and height $\mathrm{ht}(X'_{\min})+2=6$.

The above outline of the proof of Theorem 3 shows that all del Pezzo surfaces of rank one and height $\geqslant 4$ can be reconstructed from a simple list of minimal models $(X_{\min},\frac{1}{2}D_{\min})$. In case height $\leqslant 3$, we restrict the list of such basic surfaces by a careful choice of $\mathbb{P}^1$-fibration realizing the minimimum in Definition 2 of the height. This way, we arrange all del Pezzo surfaces of rank 1 and height $\leqslant 3$ in finitely many series, obtained by inductively blowing up within the fibers. This process is easy to describe combinatorially, but not unique in general. In fact, the family f in Theorem 1 parametrizes the possible choices. Let us see some examples.

Example 6. Blow up a point of $\mathbb{P}^1\times\mathbb{P}^1$, and its infinitely near point on the proper transform of a fiber. The resulting surface is a minimal resolution of a surface in $\mathcal{P}(\mathrm{A}_1+\mathrm{A}_2)$. This construction involves no choices, so $\#\mathcal{P}(\mathrm{A}_1+\mathrm{A}_2)=1$.

We now blow up inductively on the last exceptional (-1)-curve minus the previous ones, call it A°. This way, we get minimal log resolutions (X,D) of surfaces



$\mathcal{S}$	A_1+A_2	A_4	D_5		E_6		E_7			E_8		
char			$\neq 2$	$= 2$	$\neq 2,3$	$= 2,3$	$\neq 2,3$	$= 3$	$= 2$	$\neq 2,3$	$= 3$	$= 2$
$\#\mathcal{P}$	1	1	1	2	1	2	1	2	3	2	3	3
B				$\mathbb{A}^1$		$\mathbb{A}^1$		$\mathbb{A}^1$	$\mathbb{A}^2$	$\mathbb{A}^1$	$\mathbb{A}^2$	$\mathbb{A}^2$
h^1	0	0	0	1	0	1	0	1	2	1	2	3

FIGURE 3. A series of del Pezzo surfaces of rank 1 and height 1.

in $\mathcal{P}(\mathcal{S})$ for $\mathcal{S} = A_4, D_5, E_6, E_7, E_8$, see Figure 3. Each $\mathcal{P}(\mathcal{S})$ is parametrized by the previous one times and by A°/G, where $G = \mathrm{Aut}(X, D)$.

For the first blowup, we check that G acts transitively on $A^\circ \cong \mathbb{A}^1_*$, so $\#\mathcal{P}(A_4) = \#\mathcal{P}(A_1 + A_2) = 1$. For the next one, the same holds if $\operatorname{char} k \neq 2$, so $\#\mathcal{P}(D_5) = 1$, but if $\operatorname{char} k = 2$ then G has two orbits on A°, so $\#\mathcal{P}(D_5) = 2$, and $A^\circ \cong \mathbb{A}^1$ is the basis B of a stratified family as in Theorem 1(2). Similarly, we get:

The number $h^1 := h^1(\mathcal{T}_X(-\log D))$ is the dimension of the base of the semi-universal deformation of the central fiber. This base has a natural map to B, which stops being an isomorphism only if G acts transitively on A°, but not on its tangent space. This can happen only if $\operatorname{char} k > 0$, and does happen for the last blowup if $\operatorname{char} k = 2$, see the last part of Theorem 1.

Example 7. Let $\ell \subseteq \mathbb{P}^2$ be a line, and let $\ell_1, \ell_2, \ell_3, \ell_4$ be lines passing through a point $p \notin \ell$. Write $\{q_i\} = \ell_i \cap \ell$. Blowing up at q_1, q_2, q_3, p and their infinitely near points on the proper transforms of ℓ_i yields the minimal log resolution of a surface in $\mathcal{P}(2D_4)$, of height 2, see Figure 4.

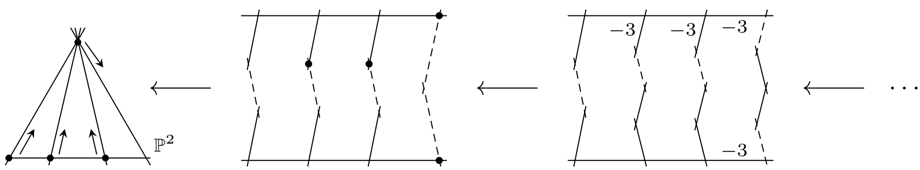


FIGURE 4. A series of del Pezzo surfaces of rank 1 and height 2.

Blowing up further within the fibers, we get surfaces of height 2 in $\mathcal{P}(\mathcal{S})$, where $\mathcal{S}$ consists of two forks and possibly one chain. Note that the shape of $\mathcal{S}$ forces all blowups to be centered at the *singular* points of the total transform of $\ell + \sum_i \ell_i$. Such a blowup is unique – there is no continuous choice as in Example 6. We conclude that $\mathcal{P}(\mathcal{S})$ is parametrized by the cross-ratio of (q_1, q_2, q_3, q_4), so the base B in Theorem 1(1) is $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Example 8. Let $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be the sum of 3 horizontal and 3 vertical lines. Blowing up within the fibers, we get plenty of log terminal del Pezzo surfaces of rank one. As in Example 7, we see that log terminality forces all blowups to be

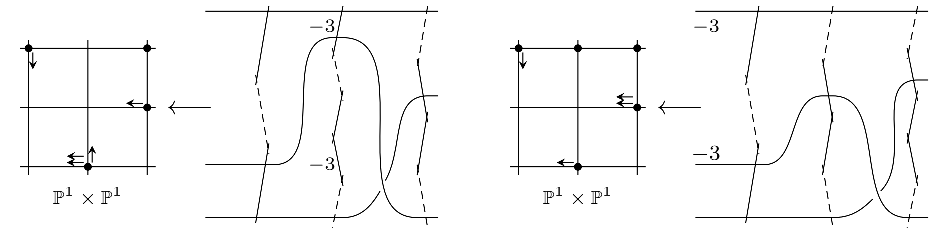


FIGURE 5. The exception in Theorem 1.

centered at the common points of two components of the total transform of D, so within one sequence of blowups, we do not have choices like in Example 6.

Nonetheless, it can happen that two different sequences of blowups lead to the same singularity type $\mathcal{S}$. We find it amusing that, among a vast number of singularity types, this phenomenon happens *exactly once*: the set $\mathcal{P}([2,2,3,(2)_5]+[3,2])$ consists of two surfaces whose minimal log resolutions are shown in Figure 5. They are rigid, so they cannot fit in a family as in Theorem 1. In fact, if $k=\mathbb{C}$ then the smooth loci of these surfaces are not even homeomorphic, as $H_1(\,\cdot\,,\mathbb{Z})$ is trivial for one, and $\mathbb{Z}/3\mathbb{Z}$ for the other.

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Sandwiched singularities and symplectic fillings

Olga Plamenevskaya
(joint work with Márton Beke and Laura Starkston)

We discuss the interplay of deformation theory of sandwiched complex surface singularities and symplectic topology of fillings of 3-dimensional contact manifolds that arise as links of these singularities. We report on the results of [6, 7] (joint with Starkston) and [1] (with Beke–Starkston).

Given a normal complex surface singularity $(X, 0) \subset (\mathbb{C}^N, 0)$, consider the link $Y = X \cap S^{2n-1}_\epsilon$ with its canonical contact structure given by the complex tangencies, $\xi = TY \cap JTY$. For small $\epsilon > 0$, the contact 3-manifold (Y, ξ) is independent of choices, and moreover, depends only on the topological rather than the analytic type of $(X, 0)$. (The topological property fails in higher dimensions.) One can also construct (Y, ξ) as the boundary of the symplectic plumbing arising from the minimal resolution of $(X, 0)$.

Milnor fibers of smoothings of singularities in the given topological type provide a collection of Stein fillings of (Y, ξ). A deformation of the minimal resolution yields another Stein filling (in the rational case, this filling also arises as the Milnor fiber of the Artin smoothing). From the perspective of low-dimensional and symplectic topology, one can ask about classification of general Stein fillings and wonder if the "expected" fillings coming from smoothings and the resolution generate all possible Stein fillings of (Y, ξ). This question falls into the general framework of comparing and contrasting algebro-geometric and symplectic properties. Indeed, sometimes familiar algebro-geometric facts extend to the symplectic setting, often requiring new tools and leading to deep results. When symplectic topology and algebraic geometry diverge, it is also interesting to understand why this happens.

It is known that all Stein (in fact, minimal symplectic) fillings arise from the Milnor fibers and the resolution for certain classes of "uncomplicated" singularities. This is true for cyclic quotient singularities (the links are lens spaces), [5], as well as for some other classes such as simple and simple elliptic singularities. When one poses no restrictions on the singularity whatsoever, Stein fillings generally become much more abundant than Milnor fibers (for example, one can construct Stein fillings violating Steenbrink's property $b_1 = 0$ that holds for Milnor fibers). Thus, the point of the question is to find out how far the correspondence between Milnor fibers and Stein fillings can be stretched, and what properties of the singularity make it work. The challenge comes from both sides, deformation theory and symplectic topology: understanding smoothings and fillings are highly non-trivial tasks, with no hope for direct classification outside of the simplest cases.

For sandwiched singularities, deformation theory is well understood due to de Jong–van Straten work [2]. A sandwiched singularity $(X, 0)$ can be described via an associated (typically reducible) singular plane curve germ C with a decoration w. Smoothings of $(X, 0)$ are then in 1-to-1 correspondence with *picture deformations* of (C, w), namely δ-constant deformations C^t of C with only transverse multipoints as allowed singularities, with multipoints and additional marked points on each component, subject to weight constraints given by w. Using picture deformations, one can immediately compute the dimensions of smoothing components for $(X, 0)$, as well as reconstruct Milnor fibers. Sandwiched singularities are a subclass of rational singularities, and are in a sense "the next simplest case" where one could still expect the correspondence Stein fillings = Milnor fibers. We were able to extend de Jong–van Straten's construction to the symplectic setting, and yet to show that there are additional Stein fillings:

Theorem 1 ([6, 7]).

(1) Let (Y, ξ) be the contact link of a sandwiched singularity $(X, 0)$. Then all minimal weak symplectic fillings of (Y, ξ) arise from certain immersed disk arrangements with marked points in the 4-ball, in close analogy with the de Jong–van Straten algebraic deformation theory.

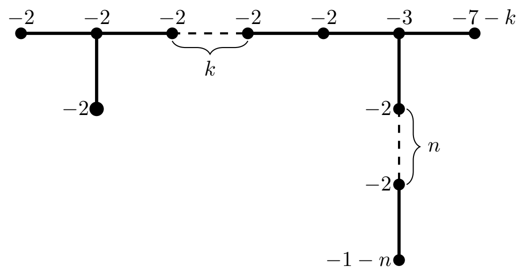
(2) However, there exists a plethora of unexpected *fillings not homeomorphic to any Milnor fibers, even for contact 3-manifolds as simple as Seifert fibered spaces over S^2.*

The class of immersed disk arrangements that we consider is motivated by picture deformations: under a generic projection, normalizations of the components of C^t are branched coverings, with transverse intersections between components. Our immersed disk arrangements are defined to have similar properties. In particular, after blowing up, the strict transforms of the components are branched coverings of the base of the standard Lefschetz fibration $D_x \times D_y \# \overline{\mathbb{CP}}^2 \to D_x$. In de Jong–van Straten setting, Milnor fibers are given by the complement of the strict transforms in this blow-up; in our generalized setting, the same construction yields Stein fillings, and morever, all minimal symplectic fillings can be generated this way. The key symplectic input for our theorem comes from spinal open books and nearly Lefschetz fibrations; we use an important recent result of Min–Roy–Wang [4].

A priori, immersed disk arrangements should be less rigid than picture deformations, although it is difficult to see whether the isotopy type of a given arrangement can be realized by a deformation. Using certain combinatorial properties, we constructed "unexpected" arrangements producing unexpected fillings not realizable by Milnor fibers. We also developed certain moves that allow to modify arrangements (*rel* boundary) to yield new fillings of the given contact fillings. One can use this method to generate fillings with topological invariants different from those of all Milnor fibers.

For an interesting illustration, we consider the question of existence of rational homology disk smoothings. Surface singularity that admits a $\mathbb{Q}$HD smoothing must be rational. Stipsicz–Szabo–Wahl [8] used Donaldson's theorem to obtain strong topological necessary conditions: if there is a $\mathbb{Q}$HD smoothing, the dual resolution graph must belong to one of several specific families. Bhupal–Stipsicz refined this result using symplectic methods. For many of the graphs in these families, it is not known where $\mathbb{Q}$HD smoothing exists (only partial results have been established); in particular, it has been conjectured that no $\mathbb{Q}$HD smoothings are possible if the minimal dual resolution graph has two or more nodes. Further, since Stipsicz–Szabo–Wahl and Bhupal–Stipsicz results use topological and symplectic methods, one could conjecture that $\mathbb{Q}$HD minimal symplectic fillings can exist only when smoothings do. However, we proved

Theorem 2 ([1])**.** *There is an infinite family of singularities whose minimal resolution graphs (shown below) have two nodes, and the contact link has a Stein filling which is a rational homology disk.*



Singularities with the resolution graphs as above are known to have no $\mathbb{Q}$HD smoothings, [3, 9].

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Reflexive modules on quotient surface singularities

Agustín Romano-Velázquez
(joint work with José Antonio Arciniega Nevárez and José Luis Cisneros-Molina)

1. Motivation

The McKay correspondence [6] establishes a bijection between the nontrivial irreducible representations of a finite subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ and the irreducible components of the exceptional divisor in the minimal resolution of the Kleinian singularity $X = \mathbb{C}^2/\Gamma$. A geometric understanding was later provided by Gonzalez–Sprinberg and Verdier [5]: to each nontrivial irreducible representation ρ they associate an indecomposable reflexive $\mathcal{O}_X$-module M. Its full sheaf $\mathcal{M} := \pi^*M/\mathrm{tor}$ in the sense of Esnault [4], on the minimal resolution $\pi : \widetilde{X} \to X$, is locally free, and $c_1(\mathcal{M})$ is the Poincaré dual of a curvette meeting exactly one irreducible component of the exceptional divisor. In particular, $c_1(\mathcal{M})$ identifies the component and hence determines M and ρ. Artin and Verdier [1] later gave a conceptual proof that c_1 determines M.

In the case of normal rational surface singularities, Esnault [4] constructed a quotient singularity admitting both an invertible full sheaf and an indecomposable rank two full sheaf with the same first Chern class, and asked whether the Chern polynomial (rank and first Chern class) distinguishes isomorphism classes of indecomposable reflexive modules. Wunram [7] answered negatively by producing two distinct full sheaves on a quotient singularity with the same rank and the same first Chern class, leaving the classification problem open.

Question. *Which invariant do we need to add to the Chern polynomial to get a complete invariant for reflexive modules (full sheaves)?*

The main difficulty is that the classification problem of full sheaves requires an invariant that determines the holomorphic structure. It is important to remark that even though there is no moduli space for line bundles in quotient singularities, we have found moduli for full sheaves of rank higher than one. This is the main difficult part of this problem. This motivates seeking an analytic invariant extracted from the link of the singularity that controls the holomorphic structure of full sheaves.

2. Results

In this section, we give a general idea of our main results. These can be found in [3]. Let (X, x) be a quotient surface singularity. Recall that (X, x) is isomorphic to $(\mathbb{C}^2/\Gamma, 0)$, where Γ is a small finite subgroup of $\mathrm{GL}(2,\mathbb{C})$. Since every finite subgroup of $\mathrm{GL}(2,\mathbb{C})$ is conjugate to a finite subgroup of $\mathrm{U}(2)$, these subgroups fix the 3-dimensional spheres in $\mathbb{C}^2$ centered at the origin. This implies that the link L of the quotient singularity $\mathbb{C}^2/\Gamma$ is diffeomorphic to the spherical 3-manifold $\mathbb{S}^3/\Gamma$. Since the links of quotient singularities are spherical 3-manifolds, it is a natural question to verify if the converse is also true, i.e., if every spherical 3-manifold is the link of a quotient surface singularity. Our first result is a positive answer to this question.

Theorem 1. *Every spherical 3-manifold with non-trivial fundamental group appears as the link of a quotient surface singularity.*

Our next goal is to classify flat vector bundles over spherical 3-manifolds using Cheeger-Chern-Simons classes, which are secondary characteristic classes. Let us recall the basics on secondary characteristic classes: Let L be a smooth manifold. Let (E, L, ∇) be a vector bundle of rank n over L with a connection ∇. Denote by $c_k^{dR}(E) = [c_k(E, \nabla)] \in H^{2k}(L; \mathbb{C})$ its de Rham cohomology class. We have the following exact sequence

$$\cdots \to H^{2k-1}(L; \mathbb{C}/\mathbb{Z}) \xrightarrow{q} H^{2k}(L; \mathbb{Z}) \xrightarrow{r} H^{2k}(L; \mathbb{C}) \xrightarrow{p_{\mathbb{Z}}} \cdots$$
$$c_k(E) \mapsto c_k^{dR}(E)$$

induced by the short exact sequence of coefficients

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}/\mathbb{Z} \to 0.$$

Now, suppose that (E, L, ∇) is a flat vector bundle. Under this hypothesis $c_k^{dR}(E) = 0$. The k-th secondary characteristic class $\widehat{c}_k(E)$ is the canonical preimage of the Chern class $c_k(E)$. Our second main theorem is the following:

Theorem 2. *Let M be a spherical 3-manifold. Indecomposable flat vector bundles over M are classified by their rank, first and second Cheeger-Chern-Simons classes.*

Now, denote by $\pi\colon \tilde{X} \to X$ its minimal resolution of singularities. The exceptional set is denoted by E, and its decomposition into irreducible components is $E = \cup_i E_i$. Let M be a reflexive $\mathcal{O}_X$-module (equivalently a maximal Cohen-Macaulay module). Its associated *full* $\mathcal{O}_{\tilde{X}}$-module is $\mathcal{M} := \pi^* M/\mathrm{tor}$.

Remark. Let us emphasize the main difficulty for classifying full sheaves: In [7, pp. 597] Wunram noticed that on the quotient singularity $(X, x) = (\mathbb{C}^2/\mathbb{I}_7, 0)$ there are two different full sheaves $\mathcal{M}_1$ and $\mathcal{M}_2$ with the same rank and the same first Chern class:

$$r = \operatorname{rank} \mathcal{M}_1 = \operatorname{rank} \mathcal{M}_2 \quad \text{and} \quad c_1(\mathcal{M}_1) = c_1(\mathcal{M}_2). \tag{1}$$

Denote by $\mathcal{L}_1 = \det \mathcal{M}_1$ and $\mathcal{L}_2 = \det \mathcal{M}_2$ the determinant bundles of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Since $c_1(\mathcal{M}_1) = c_1(\mathcal{L}_1)$ and $c_1(\mathcal{M}_2) = c_1(\mathcal{L}_2)$. Then, by (1) we get $c_1(\mathcal{L}_1) = c_1(\mathcal{L}_2)$. Since (X, x) is a rational singularity, $\mathcal{L}_1 \cong \mathcal{L}_2$. It is not hard to prove that the full sheaves $\mathcal{M}_1$ and $\mathcal{M}_2$ appear as an extension of its determinant bundle and a trivial bundle, i.e.,

$$0 \to \mathcal{O}_{\tilde{X}}^{r-1} \to \mathcal{M}_i \to \mathcal{L}_i \to 0, \quad i = 1, 2. \tag{2}$$

Note that (2) it is an exact sequence of vector bundles. In the differential category, every short exact sequence of differential vector bundles split. Therefore, as C^∞-vector bundles, we get

$$\mathcal{M}_1 \cong \mathcal{O}_{\tilde{X}}^{r-1} \oplus \mathcal{L}_1 \quad \text{and} \quad \mathcal{M}_2 \cong \mathcal{O}_{\tilde{X}}^{r-1} \oplus \mathcal{L}_2.$$

Since $\mathcal{L}_1 \cong \mathcal{L}_2$, we get that as C^∞-vector bundles $\mathcal{M}_1$ is diffeomorphic to $\mathcal{M}_2$. Nevertheless, by Wunram [7, p. 597] we know that as holomorphic vector bundles $\mathcal{M}_1 \ncong \mathcal{M}_2$. Thus, the classification problem of full sheaves requires an invariant that determines the holomorphic structure.

In our work, we prove that for quotient surface singularities, the Atiyah-Patodi-Singer analytic invariant $\tilde{\xi}$ of the Dirac operator D on the link of (X, x) is the invariant that we required to complete the classification. It is important to recall that there is a natural relation between secondary characteristic classes and the $\tilde{\xi}$-invariant; in the case of topologically trivial bundles, the $\tilde{\xi}$ -invariant coincides with the second Cheeger–Chern–Simons class (see [2] for more details). Our last and main theorem is

Theorem 3. *Let (X, x) be the germ of a quotient surface singularity. Isomorphism classes of indecomposable reflexive $\mathcal{O}_X$-modules are determined by their rank, first Chern class and $\tilde{\xi}$-invariant.*

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Smoothing Fano Varieties

Helge Ruddat

A Fano manifold is a compact complex manifold with ample anticanonical bundle.

dim	description
1	$\mathbb{P}^1$
2	either $\mathbb{P}^1 \times \mathbb{P}^1$, or a blowup of $\mathbb{P}^2$ in $k \leqslant 8$ points in general position
3	105 deformation types: *unwieldy to communicate what they are precisely*
4	finitely many deformation types, unknown how many

1. Arriving at a single construction that makes them all

Working hypothesis, Corti's proposal. *All Fano 3-folds are smoothings of singular toric Fano 3-folds*

Note that this gives structure and organization to the zoo in dimension 3 and it might work similarly in dimension 4. I want to use log geometry to prove it.

Basic idea.

(1) Use that a toric Fano variety degenerates canonically to a toroidal crossing variety Y,
(2) then use an adaptation of

Theorem (Felten–Filip–R. [1]). *Assume Y is a proper normal crossing variety, Y_{sing} is projective, $\mathcal{T}^1_Y = \mathcal{E}xt(\Omega_Y, \mathcal{O}_Y)$ is generated by global sections, and ω_Y^{-1} is effective. Then Y has a smoothing.*

... this was about 6 years ago ...

2. Lattice polygons and mutations

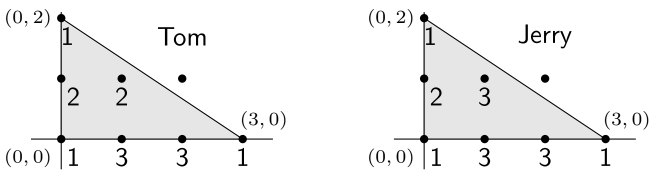
Convention. We write a Laurent polynomial $f = x + y + \frac{1}{xy}$ as $z^{(1,0)} + z^{(0,1)} + z^{(-1,-1)}$, so the monomials are z^n for $n \in N = \mathbb{Z}^2$.

The general form is then $f = \sum_{n\in N}^{\text{finite}} a_n z^n$, $a_n \in \mathbb{C}$. We define its Newton polygon Newton(f) as the convex hull of $\{n : a_n \neq 0\}$.

If we decorate the lattice points in the Newton polygon with the Laurent polynomial coefficients, we have a convenient visual representation of the polynomial.

Examples.

(1) $f_{\text{Tom}} = (1+x)^3 + 2y(1+x) + y^2$
(2) $f_{\text{Jerry}} = (1+y)^2 + 3x(1+y) + 3x^2 + x^3$
(3) $f_{A_k} = (1+x)^{k+1} + y$



Multiplying f by a monomial z^n affects Newton(f) by the translation by n. We consider polynomials equivalent is they relate by this operation.

Let $M = \text{Hom}(N, \mathbb{Z})$ be the dual, with natural pairing $\langle\cdot,\cdot\rangle \colon N \otimes M \to \mathbb{Z}$.

Definition. Given an affine integral function, that is, a map of the form

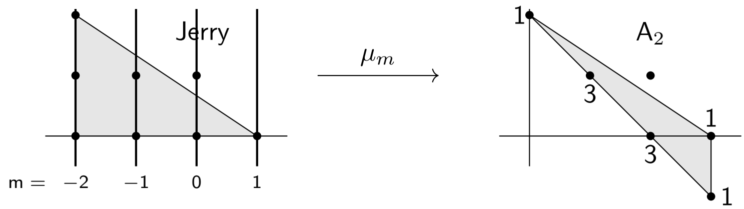
$$m \colon N \longrightarrow \mathbb{Z}, \quad m(n) = \langle n, m_0\rangle + a \quad \text{for some } m_0 \in M,\ a \in \mathbb{Z},$$

the *mutation* of $f = \sum_n a_n z^n$ with respect to m is defined as

$$\mu_m(f) = \sum_n a_n (1 + z^{n_0})^{m(n)} \cdot z^n$$

where $n_0 \in N$ is primitive with $\langle n_0, m_0\rangle = 0$, and thus can be fixed by choosing an orientation of M. We say f is *m-mutable* if $\mu_m(f)$ is a Laurent polynomial.

Example. We can mutate f_{Jerry} with $m(n) = \langle n, (1,0)\rangle - 2$ to arrive at the A_2 Laurent polynomial. After applying a coordinate transformation, this new triangle is Newton$((1+x)^3 + y)$. That, in turn, can be mutated by the integral affine function $m = \langle n, (0,1)\rangle - 1$ to a binomial, which can be mutated to a monomial.



Definition. A Laurent polynomial that permits a finite sequence of mutations into a monomial is called *zero-mutable.* Examples include f_{Tom}, f_{Jerry} and f_{A_n}.

Conjecture (Gräfnitz-R., maybe soon a theorem)**.** *Every zero-mutable Laurent polynomial in 2-variables is a cluster variable in for a suitable quiver.*

3. Singular Log Structures and Smoothings of Toroidal Crossing Fano Varieties

Exercise in toric geometry. For an affine toric variety $U_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$:

$$\begin{aligned} U_\sigma \text{ is Gorenstein} \Leftrightarrow \quad & \exists m \in \sigma^\vee \cap M : V(z^m) = \text{toric boundary} \\ & \Leftrightarrow \sigma \text{ is a cone over a polytope, that is } N = N \oplus \mathbb{Z}, \exists \Delta \subset N_\mathbb{R}, \\ & \qquad \sigma = \operatorname{Cone}(\Delta) = \{(n,r) \mid n \in r\Delta, r \geqslant 0\}. \end{aligned}$$

The moment polytope P of a Gorenstein Fano variety X_P reflexive, so P has a unique interior lattice point. We let that point be the center of a star subdivision as shown below on the right. The subdivision defines a graded Stanley Reisner ring whose Proj is a degeneration Y of X_P and is what we call a Gorenstein toroidal crossing scheme, i.e., a reducible scheme, locally isomorphic to the boundary divisor in a Gorenstein toric variety.

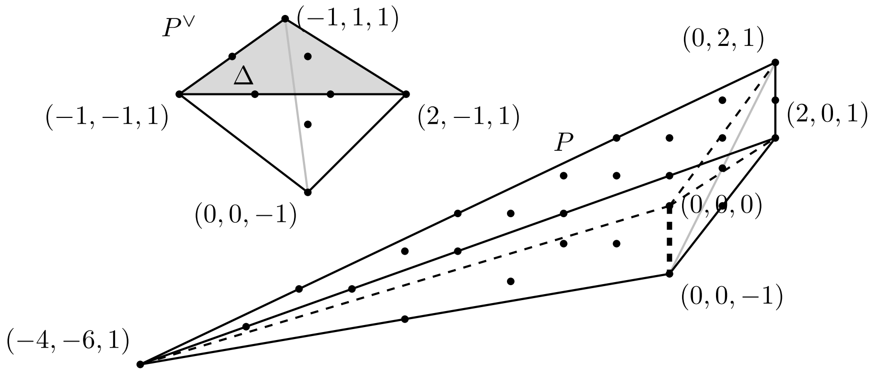


FIGURE 1. Reflexive polytope $P^\vee$ with a facet Δ and its dual P, centrally subdivided, with toroidal crossing space stratum dual to Δ marked bold dashed.

Proposition.

(a) A choice of zero-mutable Laurent polynomial on a facet Δ of the dual $P^\vee$ of P gives a singular generically log smooth structure at the one-dimensional stratum of Y corresponding to Δ.
(b) A compatible collection of zero-mutable Laurent polynomials for all facets of $P^\vee$ gives a global generically log smooth structure on Y.

Conjecture (Corti-R.)**.** *Log schemes obtained as in (b) of the Proposition permit a log crepant log resolution and also a smoothing.*

Theorem (solving Section 1 above)**.** *All 105 smooth Fano threefolds can be obtained by smoothing a toroidal crossing Fano Y obtained from a central subdivision of a reflexive polytope P by using for its log structure only products of the polynomials f_{Tom}, f_{Jerry} and f_{A_n} on the facets of $P^\vee$.*

Proof. By database, yet to be improved. □

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Categorification with Lattice Homology

Gergő Schefler
(joint work with András Némethi)

1. Introduction to lattice homology

The theory of *lattice homology* is a connecting bridge between complex analytic singularity theory and several other topics. It encompasses a family of invariants, making connections to low-dimensional topology and commutative algebra.

The *topological* version is defined combinatorially from the resolution graph of a normal surface singularity [4, 5]. It is, in fact, identical with the Heegaard Floer homology of its link [8]. On the other hand, the *analytic* version helps to distinguish the possible different analytic structures supported on a fixed topological type and categorifies the geometric genus [1, 5]. For reduced curve singularities the *analytic lattice homology* is a categorification of the delta invariant [2]. It is conjectured to be functorial with respect to flat deformations.

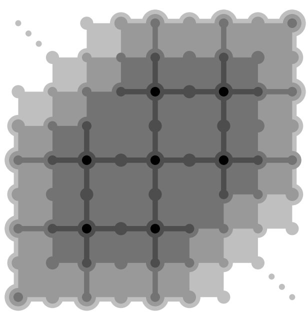
The general construction can be broadly generalized and used in different algebraic or geometric situations, providing computable and interesting invariants. In this talk I presented such a construction for integrally closed submodules and ideals, which generalizes all analytic lattice homology theories.

Lattice homology associated with a weight function [4]**.** The general construction assigns to a lattice $\mathbb{Z}^r$ with fixed basis $\{e_v\}_{v=1}^r$ (which induces a natural cubical decomposition of $\mathbb{R}^r = \mathbb{Z}^r \otimes \mathbb{R}$) and a weight function $w : \mathbb{Z}^r \to \mathbb{Z}$ a bigraded $\mathbb{Z}[U]$-module

$$\mathbb{H}_*(\mathbb{R}^r, w) = \bigoplus_{q\geqslant 0} \bigoplus_{n\in\mathbb{N}} \mathbb{H}_{q,2n}(\mathbb{R}^r, w), \quad \text{with} \quad \mathbb{H}_{q,-2n} = H_q(S_n, \mathbb{Z}),$$

where S_n is the union of those cubes which have all vertices with w-weight at most $n \in \mathbb{Z}$. The U-action is given by the inclusion maps $H_q(S_n, \mathbb{Z}) \to H_q(S_{n+1}, \mathbb{Z})$. An example with a rank 2 lattice and weight function is presented below.

4	3	2	1	0	1	0	1	0
3	2	1	0	−1	0	−1	0	1
2	1	0	−1	−2	−1	−2	−1	0
1	0	−1	0	−1	0	−1	0	1
0	−1	−2	−1	−2	−1	−2	−1	0
1	0	−1	0	−1	0	−1	0	1
0	−1	−2	−1	−2	−1	0	1	2
1	0	−1	0	−1	0	1	2	3
0	1	0	1	0	1	2	3	4



$S_{\geqslant 2}$
$\mathbb{H}_{0,\leqslant -4} = \mathbb{Z},\ \mathbb{H}_{\geqslant 1,\leqslant -4} = 0$
$S_{\geqslant 1}$
$\mathbb{H}_{0,-2} = \mathbb{Z},\ \mathbb{H}_{\geqslant 1,-2} = 0$
S_0
$\mathbb{H}_{0,0} = \mathbb{Z}^3,\ \mathbb{H}_{\geqslant 1,0} = 0$
S_{-1}
$\mathbb{H}_{0,2} = \mathbb{Z},\ \mathbb{H}_{1,2} = \mathbb{Z}^2$
S_{-2}
$\mathbb{H}_{0,4} = \mathbb{Z}^7,\ \mathbb{H}_{\geqslant 1,4} = 0$

The different versions of lattice homology differ in the way one associates the lattice and the weight function to the given geometric or algebraic situation.

2. The new construction [6]

Lattice homology of realizable submodules. Consider the following setup: let k be any field, $\mathcal{O}$ a Noetherian k-algebra and M a finitely generated $\mathcal{O}$-module.

Definition 1. Given a nontrivial discrete valuation $\mathfrak{v} : \mathcal{O} \to \mathbb{N} \cup \{\infty\}$, with the prime $\{f \in \mathcal{O} \,:\, \mathfrak{v}(f) = \infty\}$ not necessarily minimal, we call a nontrivial map $\mathfrak{v}^M : M \to \overline{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$ its *'extension'*, if it satisfies the following:

(a) $\mathfrak{v}^M(fm) = \mathfrak{v}(f) + \mathfrak{v}^M(m)$ for all $f \in \mathcal{O},\ m \in M$, and
(b) $\mathfrak{v}^M(m + m') \geqslant \min\{\mathfrak{v}^M(m), \mathfrak{v}^M(m')\}$ for all $m, m' \in M$.

If M is not torsion, such extensions exist, even more, there are infinitely many.

To a finite collection $\mathcal{D} = \{(\mathfrak{v}_v, \mathfrak{v}_v^M)\}_{v=1}^r$ of such extended discrete valuations we associate the multifiltrations:

$$\mathbb{Z}^r \ni \ell = \sum_v \ell_v e_v \mapsto \begin{cases} \mathcal{F}_{\mathcal{D}}(\ell) := \{f \in \mathcal{O} \,:\, 2 \cdot \mathfrak{v}_v(f) \geqslant \ell_v \ \forall v\} \triangleleft \mathcal{O}; \\ \mathcal{F}^M_{\mathcal{D}}(\ell) := \{m \in M \,:\, 2 \cdot \mathfrak{v}^M_v(m) \geqslant \ell_v \ \forall v\} \leqslant M. \end{cases}$$

The multiplication by 2 in these definitions serves technical purposes. In the sequel we only consider extended valuations satisfying the assumption that both $\mathcal{F}_{\mathcal{D}}(\ell)$ in $\mathcal{O}$, and $\mathcal{F}^M_{\mathcal{D}}(\ell)$ in M, are of finite k-codimension.

Definition 2. We call a finite k-codimensional submodule $N \leqslant M$ *'realizable'* if some finite collection of extended valuations $\mathcal{D}$ satisfies $N = \mathcal{F}^M_{\mathcal{D}}(0)$. We call such a $\mathcal{D}$ a *'realization of N'*.

Proposition 3. If $N \leqslant M$ is finite codimensional and *integrally closed* (in the sense of [7]), then it is realizable.

Given a realization $\mathcal{D}$ of N, on the lattice $\mathbb{Z}^{|\mathcal{D}|} = \mathbb{Z}^r$ we define the functions

$$\mathfrak{h}_{\mathcal{D}} : \ell \mapsto \dim_k \mathcal{O}/\mathcal{F}_{\mathcal{D}}(\ell) \text{ and } \mathfrak{h}^{\circ}_{\mathcal{D}} : \ell \mapsto \dim_k M/\mathcal{F}^M_{\mathcal{D}}(-\ell)$$

and the weight function $w_{\mathcal{D}}(\ell) = \mathfrak{h}_{\mathcal{D}}(\ell) + \mathfrak{h}^{\circ}_{\mathcal{D}}(\ell) - \mathfrak{h}^{\circ}_{\mathcal{D}}(0)$.

Theorem 4 (Independence Theorem)**.** *The lattice homology module $\mathbb{H}_*(\mathbb{R}^r, w_{\mathcal{D}})$ associated with this setup is independent of the realization $\mathcal{D}$ chosen, it only depends on the isomorphism type of the finite module M/N above the k-algebra $\mathcal{O}/\mathrm{Ann}_{\mathcal{O}}(M/N)$. We will denote it by $\mathbb{H}_*(N \hookrightarrow_{\mathcal{O}} M)$.*

Proposition 5. $\mathbb{H}_*(N \hookrightarrow_{\mathcal{O}} M)$ can be calculated on a finite subcomplex and has normalized Euler characteristic $\dim_k(M/N)$, i.e., it is its categorification.

Theorem 6 (Nonpositivity Theorem [3])**.** *If $k = \overline{k}$ and $\mathcal{O}$ is local, then for all $n > 0$: $\mathbb{H}_{*,-2n,red}(N \hookrightarrow_{\mathcal{O}} M) = 0$, i.e., the spaces $\{S_n\}_{n>0}$ are contractible.*

Symmetric lattice homology of integrally closed ideals. If $M = \mathcal{O}$, then the realizability of a finite k-codimensional ideal $\mathcal{I} \triangleleft \mathcal{O}$ (i.e., $\mathcal{I} = \mathcal{F}_{\mathcal{D}}(\mathbf{d})$ for some collection $\mathcal{D}$ of discrete valuations and lattice point $\mathbf{d} \in \mathbb{Z}^{|\mathcal{D}|}$) is equivalent to it being integrally closed.

Proposition 7. If $M = \mathcal{O}$ and $\mathcal{I} = \mathcal{F}_{\mathcal{D}}(\mathbf{d})$ for some $\mathcal{D} = \{\mathfrak{v}_v\}_{v=1}^r$, then the weight function is symmetric with respect to $\mathbf{d} \in \mathbb{Z}^r$. Hence, we denote the *lattice homology of* $\mathcal{I}$ by $\mathbb{SH}_*(\mathcal{I} \lhd \mathcal{O})$, which is a categorification of its codimension. In fact, it only depends on the Artin k-algebra $\mathcal{O}/\mathcal{I}$.

Remark 8. We call a k-algebra A *'integrally reduced'* if it can be presented as $\mathcal{O}/\mathcal{I}$ with $\mathcal{I}$ integrally closed. Then for any presentation $A \cong k[x_1, \ldots, x_m]/\mathcal{I}_m$ we have $\overline{\mathcal{I}_m} = \mathcal{I}_m$. Moreover, if A is Artin, then $\mathbb{SH}_*(\mathcal{I}_m \lhd k[x_1, \ldots, x_m]) \cong \mathbb{SH}_*(\mathcal{I} \lhd \mathcal{O})$.

Corollary 9. *By virtue of the construction (a compact subspace of* $\mathbb{R}^r$ *cannot have nontrivial* $H_{\geqslant r}$*), the cardinality of any realization of* $\mathcal{I}$ *is bounded from below by the 'homological dimension'* $\max\{q : \mathbb{SH}^{q-1}(\mathcal{I} \lhd \mathcal{O}) \neq 0\}$.
We conjecture *that this bound is sharp for realizable monomial ideals of* $k[x_1, x_2]$.

3. Applications

The Independence Theorem presents the possibility of categorifying numerical invariants defined as codimensions of realizable submodules.

Normal surface singularities. Originally defined by Ágoston and Némethi in [1], we can reinterpret the *analytic lattice homology* $\mathbb{H}_{an,*}(X, o)$ of a normal surface singularity in our setting. Fix a good resolution $\phi : \widetilde{X} \to X$ with reduced exceptional divisor $E = \phi^{-1}(o) = \cup_{v \in \mathcal{V}} E_v$.

Theorem 10. $\mathbb{H}_{an,*}(X, o) \cong \mathbb{H}_*\Big(H^0\big(\widetilde{X}, \Omega^2_{\widetilde{X}}\big) \hookrightarrow_{H^0\left(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\right)} H^0\big(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}\setminus E}\big)\Big)$, *where the quotient module* $H^0\big(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}\setminus E}\big)/H^0\big(\widetilde{X}, \Omega^2_{\widetilde{X}}\big)$ *is independent of the resolution. The Euler characteristic is the geometric genus* $p_g(X, o)$.

Our definition has the advantage, that it is can be applied to any normal surface singularity (without any restriction on the link), and that it can easily be generalized to give categorifications of the irregularity $q(X, o)$ and the various plurigenera and even for higher dimensional singularities as well.

Reduced curve singularities. Originally defined by Ágoston and Némethi in [2], we can reinterpret the *analytic lattice homology* $\mathbb{H}_{an,*}(C, o)$ of a reduced curve singularity in our setting. Indeed, if we denote by $\overline{C}$ the normalization, by ω_C^R the module of Rosenlicht's regular differential forms and by $\Omega^1_{\overline{C}}$ the submodule of germs of holomorphic differential forms on $\overline{C}$, then

Theorem 11. $\mathbb{H}_{an,*}(C, o) \cong \mathbb{H}_*(\Omega^1_{\overline{C}} \hookrightarrow_{\mathcal{O}_{C,o}} \omega_C^R)$ *with Euler characteristic* $\delta(C, o)$. *In the Gorenstein case* $\mathbb{H}_{an,*}(C, o) \cong \mathbb{SH}_*(\mathfrak{c} \lhd \mathcal{O}_{C,o})$*, where* $\mathfrak{c}$ *is the conductor ideal.*

Corollary 12. *'Ideal constant deformation': if along a deformation of Gorenstein curve singularities* $\mathcal{O}_{C_{t\neq 0},o}/\mathfrak{c}_{t\neq 0} \cong \mathcal{O}_{C_0,o}/\mathfrak{c}_0$*, then* $\mathbb{H}_{an,*}(C_{t\neq 0}, o) \cong \mathbb{H}_{an,*}(C_0, o)$. *A similar statement holds for Gorenstein normal surface singularities.*

Proposition 13. For a Newton nondegenerate singularity $(\{f = 0\}, 0) \subset (\mathbb{C}^n, 0)$ $\mathbb{H}_{an,*}(\{f = 0\}, 0)$ is combinatorial, in fact, it agrees with $\mathbb{SH}_*(\mathrm{adj}(\mathcal{M}) \lhd \mathcal{O}_{\mathbb{C}^n,0})$, where $\mathcal{M}$ is the monomial ideal generated by the monomials in the support of f.

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The total spine of the Milnor fibration of a plane curve singularity

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(joint work with Pablo Portilla Cuadrado)

We consider the germ of a holomorphic function $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ in two variables, defining a plane curve singularity $(C, 0) \subset (\mathbb{C}^2, 0)$, and a Milnor fibration over a punctured disk $D^*_\eta \subset \mathbb{C}$ of radius η, with Milnor fibers F_t. We assume only that $f(0, 0) = 0$, and that $f \not\equiv 0$. In particular, $(C, 0)$ does not have to be be irreducible or reduced. By [Łoj84], trajectories of the vector field

$$\xi^{\text{Ł}} = -\nabla |f|^2 \tag{1}$$

do not escape a small neighborhood of the origin, and have a limit there, which necessarily lies on C. As a result, we have a well defined map, *the collapsing map*

$$\rho : (\mathbb{C}^2, 0) \to (C, 0), \qquad \rho(x, y) = \lim_{s\to+\infty} \gamma_{(x,y)}(s)$$

where $\gamma_{(x,y)}$ is a trajectory of $\xi^{\text{Ł}}$ passing through (x, y). We also set $\rho_t = \rho|_{F_t}$ for $|t| < \eta$. The *total spine* S and the *spine* $S_t \subset F_t$, for $|t| < \eta$, are obtained by looking at which trajectories converge to the origin

$$S = \rho^{-1}(0), \qquad S_t = \rho_t^{-1}(0).$$

Observe that we are only interested in the trajectories of the vector field (1). We get the same spine if we use its positive real multiple of it, such as the vector field

$$\xi = -\nabla \log |f| = \left(\frac{\overline{-f_x}}{\overline{f}}, \frac{\overline{-f_y}}{\overline{f}} \right),$$

or the lifting of the inwards pointing radial vector field on D^*_η via the *symplectic connection*, i.e. the isomorphism between $T_{f(x,y)}\mathbb{C}$ and the symplectic orthogonal to $T_{(x,y)}F_{f(x,y)}$ in $T_{(x,y)}\mathbb{C}^2$, using the usual symplectic structure.

- ❀ If $C = \cup_{i=1}^{r} C_i$ are the branches of C, corresponding to a factorization $f = f_1^{a_1} \cdots f_r^{a_r}$ of f, then the restriction $F_t \setminus S_t \to C \setminus \{(0,0)\}$ is a disjoint union of coverings of degree a_i over C_i. Since C is the cone over the link, we find that $F_t \setminus S_t \cong \partial F \times [0,1)$ is a collar neighborhood of the boundary. This is our definition of a *spine*.
- ❀ The spine $S_t \subset F_t$ does not contain any nonempty open subset of F_t. This follows from work of A'Campo [A'C18]. Since S_t is closed in F_t, it is in fact nowhere dense in F_t.

Theorem A. *Assuming that the metric on $\mathbb{C}^2$ is chosen generically, i.e. that it is the standard metric, in generically chosen linear coordinates, the total spine S is a disjoint union of locally closed $\mathcal{C}^\infty$ submanifolds of dimension 2 (punctured disks) and 3 (solid tori and Klein bottles). As a result, the spine $S_t \subset F_t$ is the disjoint union of a finite number of points and locally closed segments.*

Remark. We would like to see this union of points and segments as an embedded graph in the Riemann surface F_t. For this to be true, the segments need to be embedded in a *nice* way, i.e. to have well defined limits at their end points, and this limit should be one of the finite number of points. We can prove that this is indeed the case, under certain genericity assumptions on the analytic type of f, but we expect this property to hold for all spines.

In a forthcoming manuscript, we describe the action of the monodromy, as well as the variation map, at the level of the chain complex associated with such a CW complex.

In order to describe the differentiable pieces in Theorem A, consider first a blow-up $(Y_0, D_0) \to (\mathbb{C}^2, 0)$ at the origin, and the strict transform $\tilde{C}$ of C in Y_0. In particular, we have a diffeomorphism $Y_0 \setminus (D_0 \cup \tilde{C}) \to \mathbb{C}^2 \setminus C$, by which we pull back the vector field ξ. It has poles along D_0, but we show that we can extend its scaling $\|(x,y)\|^2 \xi$ over $D_0 \setminus \tilde{C}$. In particular, this extension has the same trajectories as ξ outside D_0, and new ones on D_0. Under certain genericity conditions on the metric in $\mathbb{C}^2$, we show that this extension has a finite number of singularities on $Y_0 \setminus \tilde{C}$, and that these singularities are *elementary*, i.e. their Hessian is diagonalizable with nonzero real eigenvalues. Furthermore, the index of these singularities (number of negative eigenvalues) is either 2 or 3. By the Grobman-Hartman theorem, they have stable and unstable manifolds. In the case of index 2, the unstable manifold is D_0, and the stable manifold is a transversal real surface, consisting of trajectories converging to this point, hence providing a punctured disk to the total spine. In the case of index 3, the unstable manifold consists of two trajectories contained in D_0, and the stable manifold is a three dimensional real submanifold intersecting D_0 transversally in two trajectories, contributing a solid torus (of dimension 3) in the total spine.

Unfortunately, this process cannot be continued inductively. Pulling back the vector field ξ to an embedded resolution $(Y, D) \to (\mathbb{C}^2, 0)$, we obtain a vector field which cannot, in general, be extended over the whole exceptional divisor. It will have a *radial* weight τ_i and a *polar* weight ϖ_i along each component D_i of the exceptional divisor D, and we can extend a positive multiple of the pullback to a vector field which is not everywhere zero on D_i if and only if $\varpi_i = 0$. We call such components D_i *invariant*, otherwise, *noninvariant*. We can, however, extend a scaled pull-back of ξ over the boundary of the *real oriented blow-up* of Y along D, and we can describe this extension locally at each point. We find that trajectories converge to singularities of this vector field precisely at intersection points with the strict transform of certain polar curves. Such trajectories constitute a *center-stable manifold*, whose existence is guaranteed by results of Kelley [Kel67], providing three dimensional pieces in the total spine. Such a singularity lies on the preimage by the real oriented blow up of some exceptional component D_i, and the resulting piece is a solid torus if ϖ_i is even, and a solid Klein bottle if ϖ_i is odd. Finally, a global topological argument shows that the spine contains precisely the pieces described so far, and no other trajectories.

One might hope that the spine $S_t \subset F_t$ is the union of stable manifolds of the gradient of some Morse function on the Milnor fiber, e.g. the restriction of the squared distance function to the origin. This is *not* the case, as we have found examples where edges in this embedded graph intersect tangentially at a vertex. As we have extended the vector field ξ over the boundary of the real oriented blow-up of a resolution (up to a scalar factor), however, we have a well defined vector field on the Milnor fiber *at radius zero*. This Milnor fiber is a union of pieces, the interior of each of which covers the smooth part of an exceptional component. We show that the union of pieces corresponding to noninvariant exceptional divisors is a union of disks. After contracting these disks, and defining a suitable differentiable structure on the quotient of the Milnor fiber at radius zero, we show that the resulting vector field ξ^{inv} on this *invariant Milnor fiber* F_θ^{inv} at radius zero and angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, has a potential $\phi : F_\theta^{\mathrm{inv}} \to \mathbb{R}$ which locally looks like the absolute value squared of a holomorphic function.

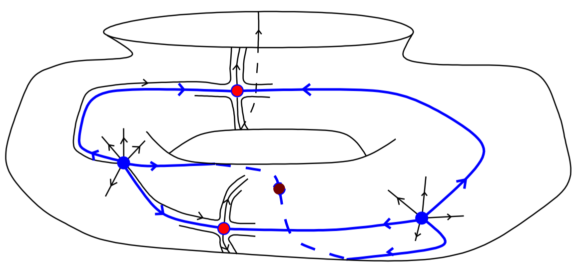

Figure 1. Dynamics of the vector field ξ^{inv} in the case of a cusp $f(x, y) = y^2 + x^3 + x^2y$. The invariant spine in blue, other partial trajectories in black. We have 2 repellers and 3 saddle points.

Theorem B. *The strict transform of the total spine intersects the invariant Milnor fiber at radius zero in the union of the stable sets of the singularities of* ξ^{inv}. *These singularities are of the following type:*

- *Repeller, with stable set a single point,*
- *Saddle point, with stable set two trajectories forming a smooth submanifold,*
- *Multipronged (or monkey-saddle) points, with stable set a union of trajectories with well defined and distinct oriented tangents.*

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Set of full exceptional collections and mirror symmetry

Atsushi Takahashi

For ADE singularities, Deligne gave a characterization of sets of distinguished bases and a recursion relation for their cardinalities in terms of the corresponding root systems to these singularities, and proved Looijenga's conjecture on their coincidence with the degrees of Lyashko-Looijenga maps associated to their universal unfolding. Briefly, these can be stated as follows:

Proposition 1 ([1])**.** Let $\overline{Q}$ be a Coxeter-Dynkin diagram for a finite simply-laced root system $\mathcal{R} = (\mathcal{N}, I, \Delta_{re})$ of rank μ with respect to a simple root basis and set

$$e(\overline{Q}) := |B/(\mathbb{Z}/2\mathbb{Z})^\mu|,$$

where

$$B = \{(\alpha_1, \ldots, \alpha_\mu) \in \Delta^\mu \mid r_{\alpha_1} \cdots r_{\alpha_{\mu_f}} = c\}.$$

We have

$$e(\overline{Q}) = \frac{h}{2} \sum_{v \in \overline{Q}_0} e(\overline{Q}^{(v)}), \quad e(A_1) = e(\circ) = 1,$$

where $\overline{Q}^{(\nu)}$ is the sub-diagram of $\overline{Q}$ given by removing the vertex v and edges connecting with v, and h is the order of the Coxeter element.

Corollary 2 ([1])**.** *The integer $e(\overline{Q})$ coincides with the degree of the Lyashko-Looijenga map. Namely,*

$$e(\overline{Q}) = \frac{\mu!}{d_1 \cdots d_\mu} h^\mu = \deg \mathrm{LL}.$$

Here, $2 \leqslant d_1 \leqslant \cdots \leqslant d_\mu = h$ are degrees of algebraically independent invariants of the Weyl group of $\mathcal{R}$.

Categorifying distinguished bases into full exceptional collections in derived directed Fukaya categories motivated by the idea of homological mirror symmetry, we naturally generalize Deligne's recursion and comparison of degrees of Lyashko-Looijenga map with numbers of full exceptional collections.

Let $\mathbb{P}^1_{A,\Lambda}$ be an orbifold projective line with $A = (a_1, ..., a_r)$ and $\Lambda = (\lambda_1, ..., \lambda_r)$ where a_i denotes the order of the isotropy group at the orbifold point λ_i on $\mathbb{P}^1$. Set

$$\mu_A := 2 + \sum_{i=1}^{r} (a_i - 1), \quad \chi_A := 2 + \sum_{i=1}^{r} \left(\frac{1}{a_i} - 1\right).$$

Denote by $\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})$ be the bounded derived category of coherent sheaves on $\mathbb{P}^1_{A,\Lambda}$ and by $\mathrm{ST}(\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda}))$ the subgroup of the group of auto-equivalences of $\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})$ generated by the 1-spherical twist functors.

Consider a set $\mathrm{FEC}(\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda}))$ of full exceptional collections in $\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})$ as a categorification of the set of distinguished basis. Based on the idea of homological mirror symmetry, the cardinality

$$e(\mathbb{P}^1_{A,\Lambda}) := |\mathrm{ST}(\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda}))\backslash \mathrm{FEC}(\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda}))/\mathbb{Z}^{\mu}|$$

is a natural extension of the $e(\overline{Q})$ above.

The main results are the following two theorems.

Theorem 3 (Otani–Shiraishi–Takahashi [5]). *Suppose $\chi_A > 0$. Then we have*

$$e(\mathbb{P}^1_{A,\Lambda}) = \frac{1}{\chi_A} \sum_{v \in (Q_A)_0} e(Q_A^{(v)}) + \sum_{i=1}^{3} a_i \sum_{j=1}^{a_i - 1} \binom{\mu_A - 1}{j-1} \cdot e(\mathbb{P}^1_{A_{(i,j)},\Lambda}) \cdot e(\vec{A}_{j-1}),$$

where Q_A denotes an extended Dynkin quiver such that $\mathcal{D}(Q_A) \cong \mathcal{D}^b(\mathbb{P}^1_{A,\Lambda})$, $Q_A^{(v)}$ is the subquiver of Q_A given by removing the vertex v and arrows connecting with v, $A_{(i,j)} := (a_1, \ldots, a_{i-1}, a_i - j, a_{i+1}, \ldots, a_r)$; $\vec{A}_{j-1}$ denotes the Dynkin quiver of type A_{j-1} and $e(\vec{A}_0) := 1$.

In particular, we have

$$e(\mathbb{P}^1_{A,\Lambda}) = \frac{\mu_A!}{a_1! a_2! a_3! \chi_A} a_1^{a_1} a_2^{a_2} a_3^{a_3} = \deg \mathrm{LL}_A,$$

where L_A is the Lyashko-Looijenga map associated to the universal unfolding of the affine cusp polynomial

$$F_A := x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - s_{\mu_A}^{-1} x_1 x_2 x_3 + s_1 + \sum_{i=1}^{3} \sum_{j=1}^{a_i - 1} s_{i,j} x_i^j,$$

over the base space $\mathbb{C}^{\mu_A - 1} \times (\mathbb{C} \setminus \{0\})$, whose degree $\deg \mathrm{LL}_A$ is calculated by Dubrovin–Zhang [2] and Ishibashi–Shiraishi-Takahashi [4].

Theorem 4 (Takahashi–Zhang [6]). *Suppose that $\chi_A = 0$. Then we have*

$$\begin{aligned} e(\mathbb{P}^1_{A,\Lambda}) &= \sum_{i=1}^{r} a_i \sum_{j=1}^{a_i - 1} \binom{\mu_A - 1}{j-1} \cdot e(\mathbb{P}^1_{A_{(i,j)},\Lambda}) \cdot e(\vec{A}_{j-1}) \cdot \frac{[\overline{\Gamma} : \overline{\Gamma(\ell_A)}]}{\ell_A} \\ &= \frac{\mu_A!}{a_1! \cdots a_r!} a_1^{a_1} \cdots a_r^{a_r} \cdot \frac{\sum_{i=1}^{r} a_i^2 (a_i - 1)}{2\mu_A} \cdot \frac{[\overline{\Gamma} : \overline{\Gamma(\ell_A)}]}{\ell_A}, \end{aligned}$$

where $\ell_A = 2, 3, 4, 6$ *for* $A = (2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ *and* $(2,3,6)$, *and* $\overline{\Gamma(\ell_A)}$ *is the principal congruence subgroup of* $\overline{\Gamma} := \mathrm{SL}(2;\mathbb{Z})/\{\pm 1\}$.

In particular, for $A = (3,3,3), (2,4,4)$ *and* $(2,3,6)$, *we have*

$$|\overline{\Gamma(2)}\backslash \mathrm{FEC}(\mathcal{D}^b(\mathbb{P}^1_{A,\Lambda}))/\mathbb{Z}^{\mu_A}| = \frac{e(\mathbb{P}^1_{A,\Lambda})}{[\overline{\Gamma} : \overline{\Gamma(\ell_A)}]} \cdot [\overline{\Gamma} : \overline{\Gamma(2)}] = \mathrm{degLL}_{E_k^{(1,1)}},$$

where $\mathrm{LL}_{E_k^{(1,1)}}$ *is the Lyashko-Looijenga map associated to the universal unfolding of the simple elliptic singularity in Legendre normal form:*

$$F_{E_6^{(1,1)}} := x_2(x_2 - x_1)(x_2 - \lambda x_1) + x_3^3 + s_1 + \sum_{i=2}^{7} s_i \phi_i(x_1, x_2, x_3), \;\; A = (3,3,3),$$

$$F_{E_7^{(1,1)}} := x_1 x_2(x_2 - x_1)(x_2 - \lambda x_1) + x_3^2 + s_1 + \sum_{i=2}^{8} s_i \phi_i(x_1, x_2, x_3), \;\; A = (2,4,4),$$

$$F_{E_8^{(1,1)}} := x_2(x_2 - x_1^2)(x_2 - \lambda x_1^2) + x_3^2 + s_1 + \sum_{i=2}^{9} s_i \phi_i(x_1, x_2, x_3) \;\; A = (2,3,6),$$

whose degree $\mathrm{degLL}_{E_k^{(1,1)}}$ *is calculated by Hertling–Roucairol* [3].

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Discriminants and motivic integration

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(joint work with Oscar Kivinen and Alexei Oblokmov)

This note is based on our recent preprint [1], motivated by several intriguing connections between algebro-geometric invariants of singularities and knot invariants [4], fixed point Floer homology [5] and the monodromy of Milnor fibers [2].

We start by recalling the last one on this list. Let $f \in \mathbb{C}[x_1, \ldots, x_m]$ be a non-constant polynomial. For any $n \geqslant 1$ consider the n-th restricted contact locus $X^0_{f,n}$ defined as

$$X^0_{f,n} = \{x \in \mathbb{A}^m(\mathbb{C}[\![t]\!]/t^{n+1})_0 \mid f(x) = t^n \in \mathbb{C}[\![t]\!]/t^{n+1}\},$$

where $\mathbb{A}^m(\mathbb{C}[\![t]\!]/t^{n+1})_0$ denotes the locus of n-jets reducing to 0 modulo t. Then the compactly supported Euler characteristics of $X^0_{f,n}$ satisfy

$$\zeta_{f,0}(T) = \exp\left(\sum_{n\geqslant 1} \frac{\chi(X^0_{f,n})}{n} T^n\right), \tag{1}$$

where $\zeta_{f,0}(T)$ denotes the zeta function of the monodromy acting on the Milnor fiber at 0.

Our results in [1] can be seen as an attempt to refine both sides of (1) by replacing integers with classes in the localized Grothendieck ring of varieties $\mathcal{C}_{\mathbb{C}} = K_0(Var_{\mathbb{C}})[\mathbb{L}^{-1}, (1-\mathbb{L}^i)^{-1} : i \geqslant 1]$. We obtain these classes from considering motivic integrals in the sense of Cluckers-Loeser [3] on relative symmetric powers of the $\mathbb{C}[\![t]\!]$-scheme

$$\mathcal{X}_f = \mathrm{Spec}(\mathbb{C}[\![t]\!][x_1, \dots, x_m]/(f-t)).$$

Concretely, the n-fold product of the $m-1$-form $\omega_f = dx_1 \wedge \cdots \wedge dx_m/df$ on $\mathcal{X}_f$ descends up to a sign to a so-called orbifold form ω_{orb} on $\mathrm{Sym}^n \mathcal{X}_f$. Its motivic integral over the set of arcs

$$\mathrm{Sym}^n \mathcal{X}^0_f = \{x \in \mathrm{Sym}^n \mathcal{X}_f(\mathbb{C}[\![t]\!]) \mid x_{|t=0} = n[0]\}$$

gives a refinement of the right hand side of (1) as follows.

Theorem 1. *For any non-constant $f \in \mathbb{C}[x_1, \dots, x_m]$ we have*

$$\sum_{n\geqslant 0} \int_{\mathrm{Sym}^n \mathcal{X}^0_f} |\omega_{orb}|^{1/2} T^n = \mathrm{Exp}\left(\mathbb{L}^{-\frac{m-1}{2}} Z_f(\mathbb{L}^{-\frac{m-3}{2}} T)\right),$$

where $\mathrm{Exp} : T\mathcal{C}_{\mathbb{C}}[\mathbb{L}^{1/2}][\![T]\!] \to 1 + T\mathcal{C}_{\mathbb{C}}[\mathbb{L}^{1/2}][\![T]\!]$ *denotes the plethystic exponential and* $Z_f(T) = \sum_{n\geqslant 1}[X^0_{f,n}]\mathbb{L}^{-mn}T^n$ *the motivic Igusa zeta function.*

In the curve case, i.e. $m = 2$, we can also refine the left hand side of (1). Let $\Delta_f \subset \mathrm{Sym}^n \mathcal{X}_f$ be the discriminant divisor and define the Gelfand form on $\mathrm{Sym}^n \mathcal{X}_f$ as $|\omega_{Gel}| = \mathbb{L}^{-\mathrm{ord}_{\Delta_f}/2}|\omega_{orb}|^{1/2}$.

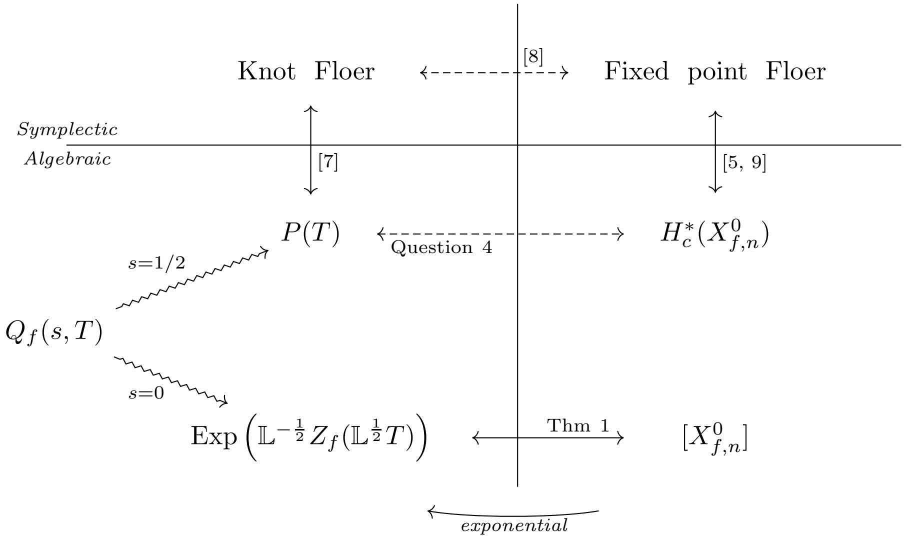
Theorem 2. *For $f \in \mathbb{C}[x, y]$ a reduced curve we have*

$$\sum_{n\geqslant 0} \int_{\mathrm{Sym}^n \mathcal{X}^0_f} |\omega_{Gel}| T^n = \sum_{n\geqslant 0} \mathbb{L}^{-n}[\mathrm{Hilb}^1_n(f)_0]T^n \in \mathcal{C}_{\mathbb{C}},$$

where $\mathrm{Hilb}^1_n(f)_0$ *parametrizes principal ideals of colength n in $\mathbb{C}[\![x, y]\!]/f(x, y)$.*

The series $P(T) = \sum_{n\geqslant 0} \mathbb{L}^{-n}[\mathrm{Hilb}^1_n(f)_0]T^n$ agrees with the motivic Poincaré series of Campillo–Delgado–Gusein-Zade [6] and in particular recovers the monodromy zeta function $\zeta_{f,0}(T)$ by passing to Euler characteristics. Furthermore, by a result of Gorsky-Némethi [7] the series $P(T)$ also agrees with the Poincaré series of the knot Floer homology of the link defined by $(f, 0)$.

One can arrange all of this in the following somewhat speculative diagram:



In order to relate the series $P(T)$ and $\mathrm{Exp}\left(\mathbb{L}^{-\frac{1}{2}}Z_f(\mathbb{L}^{\frac{1}{2}}T)\right)$ it seems natural to consider

$$Q_f(s,T) = \sum_{n\geqslant 0}\int_{\mathrm{Sym}^n\,\mathcal{X}_f^0} \mathbb{L}^{-s\cdot \mathrm{ord}_{\Delta_f}}|\omega_{orb}|^{1/2}T^n \in \mathcal{C}_{\mathbb{C}}[\mathbb{L}^{1/2}][\![T,\mathbb{L}^{-s}]\!],$$

for some formal parameter s. Unfortunately, already for a smooth curve f, computing $Q_f(s,T)$ amounts to compute the naive Igusa zeta function for the discriminant of monic degree n polynomials for all n. To the best of our knowledge, such a computation is not in the literature, although some related work in the p-adic setting by G-Wei-Yin [10] was recently used to settle a conjecture by Bhargava-Cremona-Fisher-Gajović on factorization densities of p-adic polynomials.

We finish by formulating two questions that we believe are worth investigating.

Question 3. *Compute $Q_f(s,T)$ for a smooth curve f and eventually for any reduced plane curve.*

Question 4. *For a reduced plane curve f, find an 'exponential' relation between the (cohomology of) restricted contact loci $X^0_{f,n}$ and principal Hilbert schemes* $\mathrm{Hilb}^1_n(f)_0$.

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A glimpse into the birational geometry of quasi-homogeneous cA_n singularities

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(joint work with Nikolas Adaloglou and Federica Pasquotto)

For decades, the interplay between the algebraic geometry of a complex isolated hypersurface singularity $(X, \underline{0})$ and the smooth topology of its link L has been extensively studied from different perspectives, leading to remarkable results. In recent years, contributions by various authors have further revealed a beautiful, albeit intricate, interaction between the birational geometry of $(X, \underline{0})$ and the contact topology of L. For instance, the realisation of the minimal discrepancy and the log canonical threshold as contact invariants in the works of McLean [6, 7] and a recent conjecture by Evans and Lekili [2, Conjecture 1.4], which predicts that, for an isolated compound Du Val (cDV) singularity, the existence of a small resolution can be detected by the symplectic cohomology of its Milnor fibre F.

Isolated cDV singularities are terminal threefold hypersurface singularities and form the only class of Gorenstein singularities in dimension three that can, in fact, admit a small resolution. Their links are index positive by [6] and thus the symplectic cohomology of their Milnor fibres in negative degrees is a contact invariant [2]. The precise statement of the conjecture by Evans and Lekili mentioned above is as follows.

Conjecture 1. *Let F be the Milnor fibre of an isolated cDV singularity. Then, the singularity admits a small resolution such that the exceptional set has ℓ irreducible components if and only if the symplectic cohomology of the Milnor fibre $SH^*(F)$ has rank ℓ in every negative degree.*

In [1], we give new evidence to support Conjecture 1. Using the effective algorithm outlined in [2], we provide explicit formulas for the ranks of the symplectic cohomology of the Milnor fibre of any quasihomogeneous cA_n singularity, including those which do not admit a small resolution. These singularities arise as the double suspension of curve singularities as they can be described by a polynomial of the form $p(x, y, z, w) = x^2 + y^2 + g(z, w)$, where g is a so-called invertible polynomial. We call these **suspended polynomials**.

During my talk, I illustrated our results by presenting the following theorem.

Theorem 2. *Let $g(z, w) = z(z^{a-1} + w^b)$ and let F denote the Milnor fibre of the singularity defined by the suspended polynomial $x^2 + y^2 + g(z, w)$. Then, for any integer $k \leqslant 0$, we have that*

$$\dim SH^{2k}(F) = \dim SH^{2k+1}(F) = $$
$$= \begin{cases} \gcd(a-1, b) & \text{if } q = 0 \\ q & \text{if } 1 \leqslant q \leqslant \min\{a-1, b\} \\ \min\{a-1, b\} & \text{if } \min\{a-1, b\} < q \leqslant \max\{a-1, b\} \\ (a-1+b) - q & \text{if } \max\{a-1, b\} < q \leqslant (a-1+b) - 1 \end{cases},$$

where $0 \leqslant q < a - 1 + b$ is such that $(a-1)(1-k) \equiv q \mod (a-1+b)$.

As a direct consequence, we immediately deduce that for all integers $k \leqslant 0$

$$\dim SH^{2k}(F) = \dim SH^{2k+1}(F) = \gcd(a-1, b)$$

precisely when $\min\{a-1, b\} = \gcd(a-1, b)$. It is well known [3, 4] that this is equivalent to the existence of a small resolution with $\gcd(a-1, b)$ exceptional curves. In [1], similar formulas are also obtained when $g(z, w) = zw(z^{c-1} + w^{d-1})$ and $g(z, w) = zw(z^e + w^f)$, leading us to verify that Conjecture 1 holds for all quasihomogeneous cA_n singularities.

Using these formulas and our computations, we further succeed in addressing a stronger version of Conjecture 1, namely [5, Conjecture 5.3], and show that this refined version also holds for all the singularities we consider in [1]. Our findings can be summarised as follows.

Theorem 3. *Let $p(x, y, z, w) = x^2 + y^2 + g(z, w)$ be a suspended polynomial and let F (resp. L) denote the corresponding Milnor fibre (resp. link). Consider a small $\mathbb{Q}$-factorialization of the threefold singularity $(p = 0, \underline{0})$, and let $F_1, \dots, F_m$ be the Milnor fibers of the resulting $\mathbb{Q}$-factorial singularities. Then, for any integer $k \leqslant 0$, we have that*

$$\dim SH^{2k}(F) = \sum_{i=1}^{m} \dim SH^{2k}(F_i) + b_2(L).$$

In a different direction, we further use our results to prove that the links of any two quasihomogeneous cA_n singularities are contactomorphic if and only if the two singularities are deformation equivalent, as in [1, Definition 5.1]. In particular, we prove the following.

Theorem 4. *The links associated with two Fermat-type polynomials*

$$p_i = x^2 + y^2 + z^{e_i} + w^{f_i} = 0,$$

with $i = 0, 1$ *and* $\gcd(e_i, f_i) = 1$*, are contactomorphic if and only if the corresponding singularities have the same Milnor number. That is, if and only if up to a change of coordinates* $p_0 = p_1$*.*

Note that this allows us to provide a new proof of the existence of infinitely many different contact structures on the sphere S^5 – a result first proved by Ustilovsky in [8]. We prove Theorem 4 by systematically keeping track of a bigrading on $SH^{\leqslant 1}(F)$ that comes from the Gerstenhaber bracket and yields some useful contact invariants of the link. We show that there always exist a degree κ and a bigrading $(\kappa - \sigma, \sigma)$ that detect whether the links of any two quasihomogeneous cA_n singularities are contactomorphic or not. It is intriguing to observe that, for all the suspended polynomials we consider, the rational number $\frac{1-\kappa}{\sigma+1}$ is precisely the log canonical threshold of the plane curve singularity defined by $g(z, w) = 0$. A question that remains to be answered in a future project is: *Why?*

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