Mathematisches Forschungsinstitut Oberwolfach

Report No. 47/2025

DOI: 10.4171/OWR/2025/47

Arbeitsgemeinschaft: Combinatorial Hodge Theory

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12 October – 17 October 2025

Abstract. Combinatorial Hodge theory has undergone rapid development in recent years, revealing deep connections between matroid theory, tropical geometry, toric geometry, and convex geometry. Building on the Kähler package for matroids and the emergence of Lorentzian structures across combinatorics, the field now encompasses a broad family of Hodge-theoretic phenomena arising from purely combinatorial objects. The goal of this Arbeitsgemeinschaft was to provide participants with a structured and accessible overview of these developments, emphasizing both foundational material and current research directions. The program was organized around four major themes–matroids, Hodge theory, toric methods, and Lorentzian polynomials–with lectures highlighting topics such as Baker–Bowler framework for matroids with coefficients, Chow rings of matroids and wonderful compactifications of hyperplane arrangements, Lorentzian polynomials and volume polynomials, and matroids over triangular hyperfields. Together, these lectures aimed to articulate the unifying principles underlying the subject and to prepare participants for further research in this evolving area.

Mathematics Subject Classification (2020): 05B20, 05E14, 15A63, 14C17, 05B35, 52A39.

Introduction by the Organizers

The Arbeitsgemeinschaft *Combinatorial Hodge Theory* brought together researchers from a wide range of backgrounds–algebraic geometry, combinatorics, topology, and adjacent areas–to explore the rapid recent developments in Hodge-theoretic ideas within combinatorics. The meeting followed the classical Oberwolfach Arbeitsgemeinschaft format: the organizers prepared a detailed program structured around a sequence of expository lectures, and all talks were delivered by participants. Over the course of six days, we had seventeen one-hour lectures with ample opportunities for informal discussions, including Q&A sessions and extended conversations during meals and breaks.

The overarching theme of the workshop was how Hodge-theoretic structures — such as the hard Lefschetz theorem, Hodge–Riemann relations, and related Lorentzian properties — emerge in purely combinatorial settings. The program was organized into four broad topics: matroids, Hodge theory, toric methods, and Lorentzian polynomials. Each thematic block was introduced by talks designed to build the necessary foundations before exploring more recent developments.

Talks on matroids covered valuated matroids and the broader framework of matroids over hyperfields, and the emerging theory of foundations of matroids, which unifies realizability questions across different algebraic settings. The second thematic block introduced the Kähler package and its role in combinatorial Hodge theory. Building on the pioneering work of Adiprasito–Huh–Katz and later extensions, the talks explained how hard Lefschetz theorems and Hodge–Riemann relations arise in the Chow rings of matroids and in the intersection cohomology of arrangement Schubert varieties. Speakers highlighted shared inductive strategies underlying proofs. The third block, on toric methods, emphasized how ideas from toric geometry illuminate combinatorial Hodge theory. Lectures introduced Chow of toric varieties, Minkowski weights, and the role of positivity in understanding matroid invariants. More detailed discussions included tautological classes of matroids and the geometry of augmented wonderful compactifications. Collectively, these talks showed how toric techniques serve as a bridge between combinatorial structures and algebro-geometric tools. The final block of the workshop was devoted to the landscape of Lorentzian polynomials, a unifying framework that synthesizes log-concavity phenomena across combinatorics, convex geometry, and algebraic geometry. Talks surveyed the foundational theory of Lorentzian polynomials; their relationship to stable polynomials, Lorentzian fans, and volume polynomials; and recent advances linking Lorentzian structures to matroids over triangular hyperfields. Several expository lectures provided introductions to recent results — such as the topology of Lorentzian strata and multiplicative inequalities for Lorentzian matrices — highlighting their breadth and applications.

In addition to the formal program, the Arbeitsgemeinschaft fostered extensive informal interactions, including the excursion on Wednesday and the musical evening on Thursday. In what follows, the abstracts are arranged in the order presented during the workshop.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, “US Junior Oberwolfach Fellows”.

Arbeitsgemeinschaft: Combinatorial Hodge Theory

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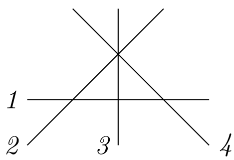
Abstracts

Wonderful varieties

Rob Silversmith

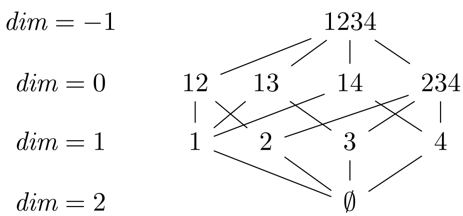
Let k be a field, and let V be a vector space of dimension r. A *hyperplane arrangement* $\mathcal{A}$ in V is a finite collection of distinct codimension-1 subspaces $L_1, \ldots, L_n \subset V$. Projectivizing gives n hyperplanes $\mathbb{P}(L_i) \subset \mathbb{P}(V)$. We work only with *essential* arrangements, i.e. those with $\bigcap_{i=1}^{n} L_i = 0$. (Otherwise we quotient V, and each L_i, by $\bigcap_{i=1}^{n} L_i = 0$ to get an essential arrangement.)

Example 1. *Here is a running example, in* $\mathbb{P}^2$*:*



Definition 2. *A* flat *of* $\mathcal{A}$ *is an intersection of subspaces* L_i*. We may speak of a flat either geometrically as a subspace* L*, or combinatorially as the set* $F = \{i \in [n] : L_i \supseteq L\}$*. Flats form a poset (actually a* lattice*) under inclusion.*

Example 3. *In the above example, the Hasse diagram of this poset is*



Given $\mathcal{A}$, the complement $\mathcal{M}_{\mathcal{A}} := \mathbb{P}(V) \setminus \mathbb{P}(\mathcal{A})$ is an affine variety over k. The geometry/topology of $\mathcal{M}_{\mathcal{A}}$ can often be expressed in terms of the lattice of flats — such as its cohomology ring when $k = \mathbb{C}$, by a theorem of Orlik-Solomon [4].

Our focus is another variety associated to $\mathcal{A}$. The (beautiful, natural) idea, due to de Concini-Procesi [1], is to blow up $\mathbb{P}(V)$ along all flats of $\mathcal{A}$, in increasing order of dimension. (This turns out to be well-defined: at each step, the centers that we wish to blow up are disjoint, because we have already blown up all flats of lower dimensions.) The result is the *wonderful variety* $Y_{\mathcal{A}}$ *of* $\mathcal{A}$. Notes:

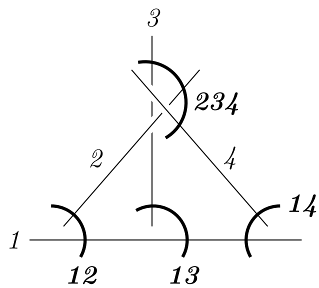
- $Y_{\mathcal{A}}$ contains a natural dense set isomorphic to $\mathcal{M}_{\mathcal{A}}$. For this reason, $Y_{\mathcal{A}}$ is often called the *wonderful compactification of* $\mathcal{M}_{\mathcal{A}}$.
- The "boundary" $Y_{\mathcal{A}} \setminus \mathcal{M}_{\mathcal{A}}$ contains an irreducible smooth divisor for each proper nonempty flat of $\mathcal{A}$. (These are the exceptional divisors of all blow-ups, plus the strict transforms of the hyperplanes.)

- For proper nonempty flats $F_1, \dots, F_k$, we have

$$\bigcap_{j=1}^{k} E_{F_j} \neq \emptyset \quad \text{if and only if, up to reordering, } F_1, \dots, F_k \text{ form a nested chain.}$$

  (Equivalently, the *boundary complex* of the compactification $\mathcal{M}_{\mathcal{A}} \subset Y_{\mathcal{A}}$ coincides with the *order complex* of the lattice of flats.)
- De Concini and Procesi proved [1] that the compactification $\mathcal{M}_{\mathcal{A}} \subset Y_{\mathcal{A}}$ is *simple normal crossings*, i.e. the boundary $Y_{\mathcal{A}} \setminus \mathcal{M}_{\mathcal{A}}$ locally resembles a union of coordinate hyperplanes.
- De Concini and Procesi discuss a more general notion of wonderful compactification, where one only blows up a *subset* of the flats. (The subset must satisfy certain properties — it must be a "building set".)

Example 4. *In our running example, the wonderful compactification is the blowup of $\mathbb{P}^2$ at 4 points, giving the following schematic picture:*



We have new exceptional divisors corresponding to the flats 12, 13, 14, *and* 234. *We can see, for example, that $E_{12} \cap E_2 \neq \emptyset$, since $2 \subseteq 12$ forms a nested chain.*

The basic structure of $Y_{\mathcal{A}}$ reflects the combinatorics of $\mathcal{A}$. A deeper geometric question: can we express the Chow ring $A_*(Y_{\mathcal{A}})$ in terms of the lattice of flats?

Some linear algebra: Consider the product of quotient maps $V \to \bigoplus_{i=1}^n V/L_i$, which is injective since $\mathcal{A}$ is essential. Projectivizing gives an embedding

$$\phi_{\mathcal{A}} : \mathbb{P}(V) \to \mathbb{P}\left(\bigoplus_{i=1}^{n} V/L_i\right) \cong \mathbb{P}^{n-1}.$$

We can alternatively define $\phi_{\mathcal{A}}$ by choosing a defining equation f_i for each hyperplane $\mathbb{P}(L_i)$, and setting $\phi_{\mathcal{A}}(P) = [f_1(P) : \cdots : f_n(P)]$ for $P \in \mathbb{P}(V)$. Note that

$$\phi_{\mathcal{A}}^{-1}(T) = \mathcal{M}_{\mathcal{A}} \qquad \text{and} \qquad \phi_{\mathcal{A}}^{-1}(H_i) = L_i,$$

where $T \subset \mathbb{P}^{n-1}$ is the torus and $H_i \subset \mathbb{P}^{n-1}$ is the ith coordinate hyperplane.

Remark 5. *The above embedding defines a point of the Grassmannian* $\mathrm{Gr}(r, n)$*. In fact, reversing the process, every point of* $\mathrm{Gr}(r, n)$ *gives rise to a hyperplane arrangement in a copy of* $\mathbb{P}^{r-1}$*. Assigning to each point of* $\mathrm{Gr}(r, n)$ *the combinatorial structure of the lattice of flats defines the "matroid stratification" of* $\mathrm{Gr}(r, n)$*, whose strata are "matroid realization spaces". These can be very nasty if $r > 2$!*

We now give an alternative characterization of $Y_{\mathcal{A}}$. Consider a special case: the wonderful variety $X_{A_{n-1}}$ associated to the "Boolean arrangement" consisting of coordinate hyperplanes in $\mathbb{P}^{n-1}$. $X_{A_{n-1}}$ is called the *nth permutohedral variety.* Note that $X_{A_{n-1}}$ is a *toric* blowup of $\mathbb{P}^{n-1}$, hence is a toric variety. The boundary divisors of $X_{A_{n-1}}$ are in natural bijection with subsets of $[n]$. The boundary strata of $X_{A_{n-1}}$, which are precisely the torus orbits, are in natural bijection with nested chains of proper subsets of $[n]$, and in particular the zero-dimensional orbits are in bijection with permutations of $[n]$.

Fact 6. *$Y_{\mathcal{A}}$ is canonically isomorphic to the strict transform of $\phi_{\mathcal{A}}(\mathbb{P}(V))$ under the blowup $X_{A_{n-1}} \to \mathbb{P}^{n-1}$.*

This can be understood as follow. $X_{A_{n-1}}$ is an iterated blow-up of $\mathbb{P}^{n-1}$, and each individual blow-up corresponds to a subset $S \subset [n]$. If S is not a flat of $\mathcal{A}$, then the blow-up center is disjoint from (the strict transform of) $\phi_{\mathcal{A}}(\mathbb{P}(V))$. On the other hand, if S is a flat of $\mathcal{A}$, then the blowup induces a blowup of $\phi_{\mathcal{A}}(\mathbb{P}(V))$ — precisely the blowup corresponding to S in the construction of $Y_{\mathcal{A}}$ as an iterated blow-up of $\mathbb{P}(V)$. In particular, $Y_{\mathcal{A}} \subset X_{A_{n-1}}$ intersects the torus orbit corresponding to a nested chain $S_1 \subset \cdots \subset S_k$ if and only if every S_i is a flat.

Feichtner and Yuzvinsky studied the open toric subvariety $X_{\mathcal{A}} \subset X_{A_{n-1}}$ obtained by deleting all torus orbits that do not intersect $Y_{\mathcal{A}}$. The above observation amounts to saying that $X_{\mathcal{A}}$ is the toric variety corresponding to the *Bergman fan* of $\mathcal{A}$, whose cones correspond to nested chains of flats[1]. In particular, they proved:

Theorem 7 ([2])**.** *The restriction map $A_*(X_{\mathcal{A}}) \to A_*(Y_{\mathcal{A}})$ is an isomorphism.*

The proof roughly goes as follows. As an iterated blowup of $\mathbb{P}(V)$, $A_*(Y_{\mathcal{A}})$ is generated by boundary divisors, which implies the restriction is surjective. On the other hand, both rings satisfy Poincaré duality. (This is not obvious.) A surjective map of Poincaré duality algebras is an isomorphism by straightforward algebra.

Chow rings of toric varieties are well-understood. For example:

Theorem 8 (Jurkiewicz, Danilov, Bifet-de Concini-Procesi, Brion)**.**

$$A_*(X_{\mathcal{A}}) \cong \frac{\mathbb{Z}[\{x_F\}_{F \text{ a proper nonempty flat}}]}{\text{(some straightforward relations)}}.$$

One can also understand $A_*(X_{\mathcal{A}})$ in terms of *Minkowski weights* [3]. The Chow ring of $Y_{\mathcal{A}}$ encodes deep geometry, and as we will see, one can therefore extract deep combinatorial properties of $\mathcal{A}$ from computations in this ring.

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The Kähler package

J. Matthew Douglass

The instructions for this talk were: Introduce the Kähler package for a graded real Artinian algebra. Define the (mixed) Kähler package, mention some of its uses in combinatorics, and provide some details about an inductive strategy to prove the Kähler package that is common to several works. The main references are the surveys [5], [4], and [6].

(Mixed) Kähler packages. Suppose $A = A^0 + A^1 + \cdots + A^r$ is a graded real vector space. For simplicity, we assume that A is finite-dimensional. Suppose also that $Q\colon A \times A \to \mathbb{R}$ is a symmetric bilinear form, and that K is an open convex cone in the vector space $\mathrm{End}_{\mathbb{R}}(A)$ consisting of commuting, graded, degree one, self-adjoint (with respect to Q) operators.

Say that the triple (A, Q, K) satisfies the *mixed Kähler package* if the following conditions hold [5]:

(PD) *Poincaré duality:* $A^0 \neq 0$, Q is non-degenerate, and the restriction of Q to $A^p \times A^q$ is equal to zero unless $p+q = r$. In particular, $Q\colon A^k \times A^{r-k} \to \mathbb{R}$ is nondegenerate for $0 \leq k \leq r$.

(MHL) *Mixed Hard Lefschetz property:* For $0 \leq k \leq r/2$ and $(L_1, \dots, L_{r-2k}) \in K^{r-2k}$, the composition $L_1 \circ \cdots \circ L_{r-2k}\colon A^k \to A^{r-k}$ is an isomorphism.

(MHR) *Mixed Hodge-Riemann relations:* For $0 \leq k \leq r/2$ and a tuple $\mathcal{L} = (L_0, L_1, \dots, L_{r-2k})$ in K^{r-2k+1}, the bilinear form, $Q^k_{\mathcal{L}}\colon A^k \times A^k \to \mathbb{R}$ given by $Q^k_{\mathcal{L}}(a, b) = (-1)^k Q\big(a, L_1 \circ \cdots \circ L_{r-2k}(b)\big)$, is positive definite on the subspace $A^k \cap \ker(L_0 \circ L_1 \circ \cdots \circ L_{r-2k})$ of A^k.

Say that the triple (A, Q, K) satisfies the *unmixed Kähler package* when the linear transformations L_0, L_1, $\dots$, L_{r-2k} in (MHL) and (MHR) are all equal. Then

(HL) is the condition that if $L \in K$, then $L^{r-2k}\colon A^k \to A^{r-k}$ is an isomorphism, and

(HR) is the condition that if $L \in K$, then the bilinear form $Q^k_L\colon A^k \times A^k \to \mathbb{R}$ with $Q^k_L(a, b) = (-1)^k Q\big(a, L^{r-2k}(b)\big)$ is positive definite on the kernel $P^k_L = A^k \cap \ker L^{r-2k+1}$.

Graded algebras. Suppose A is a graded, commutative $\mathbb{R}$-algebra with identity, $\deg\colon A^r \to \mathbb{R}$ is an isomorphism, and $\mathcal{K}$ is an open convex cone in A^1. Let Q be the composition of multiplication in A, followed by projection onto A^r, followed by deg; let $L_\ell\colon A \to A$ be multiplication by ℓ for $\ell \in A^1$; and define $K = \{\, L_\ell \mid \ell \in \mathcal{K} \,\}$. Say that $(A, \deg, \mathcal{K})$ satisfies the *mixed or unmixed Kähler package* if (A, Q, K) does.

Lefschetz decomposition. Suppose (A, Q, K) satisfies the mixed Kähler package. The conditions (MHL) and (MHR) determine orthogonal decompositions of each graded piece of A. The basic result is formulated in the next lemma.

Lemma 1. *Suppose $0 \leq k \leq r/2$ and $\mathcal{L} = (L_0, L_1, \ldots, L_{r-2k})$ is a tuple of elements in K. Define $P^k_{\mathcal{L}} = A^k \cap \ker(L_0 \circ L_1 \circ \cdots \circ L_{r-2k})$ and recall the bilinear form $Q^k_{\mathcal{L}}$. Then $L_0\big(A^{k-1}\big)$ and $P^k_{\mathcal{L}}$ are orthogonal complements with respect to $Q^k_{\mathcal{L}}$. Thus, there is an orthogonal decomposition: $A^k = L_0\big(A^{k-1}\big) \perp_{Q^k_{\mathcal{L}}} P^k_{\mathcal{L}}$.*

Starting with $k = \lfloor r/2 \rfloor$, one can recursively construct orthogonal decompositions of A^k for $0 \leq k \leq r/2$ and then (MHL) and (MHR) give decompositions of A^k for $r/2 < k \leq r$. This is most easily visualized in the unmixed case. For example, for $r = 5$ and $L \in K$ the decompositions are

$$\begin{aligned}
A^5 &= L^5(P^0_L) \\
A^4 &= L^4(P^0_L) \oplus L^3(P^1_L) \\
A^3 &= L^3(P^0_L) \oplus L^2(P^1_L) \oplus L(P^2_L) \\
A^2 &= L^2(P^0_L) \oplus L(P^1_L) \oplus P^2_L \\
A^1 &= L(P^0_L) \oplus P^1_L \\
A^0 &= P^0_L,
\end{aligned}$$

where for $0 \leq k \leq r/2$, Q^k_L is positive or negative definite on each summand of A^k. For example, Q^2_L has the indicated "signature": $A^2 = \underbrace{L^2(P^0_L)}_{+\text{ def}} \perp \underbrace{L(P^1_L)}_{-\text{ def}} \perp \underbrace{P^2_L}_{+\text{ def}}$.

Application to log concavity. An important application of Kähler packages in combinatorics is as a tool to prove properties of sequences. One such application is to log concavity [5].

Theorem 2. *Suppose $(A, \deg, \mathcal{K})$ is a graded, commutative $\mathbb{R}$-algebra that satisfies the mixed Kähler package and that $a, b \in K$. Then the sequence*

$$\big(\deg(a^0 b^r), \deg(ab^{r-1}), \ldots, \deg(a^r b^0)\big)$$

is positive and log concave.

The proof requires (MHL) and (MHR), not just (HL) and (HR), but only in degrees $k = 0$ and $k = 1$. To show the sequence is positive, for $0 \leq i \leq r$, define $\mathcal{L}_i = \big(\underbrace{L_a, \ldots, L_a}_{i+1}, \underbrace{L_b, \ldots, L_b}_{r-i}\big)$ in K^{r+1}. Then by definition $Q^0_{\mathcal{L}_i}(1,1) = \deg(a^i b^{r-i})$ and (MHL) in degree 0 asserts that $Q^0_{\mathcal{L}_i}$ is positive definite on $P^0_{\mathcal{L}_i} = A^0$, so $\deg(a^i b^{r-i}) > 0$. The proof of log concavity follows from an analysis of the restriction of the form $Q^1_{\mathcal{L}'_i}$ to the subspace of A^1 spanned by $\{a, b\}$ and the Lefschetz decomposition for the tuple $\mathcal{L}'_i = \big(\underbrace{L_a, \ldots, L_a}_{i}, \underbrace{L_b, \ldots, L_b}_{r-i-1}\big)$ in K^{r-1}. The Lefschetz decomposition is $A^1 = aA^0 \perp P^1_{\mathcal{L}'_i}$. By (MHL), $Q^1_{\mathcal{L}'_i}$ is negative definite on $aA^0 = \mathbb{R} \cdot a$, and positive definite on the hyperplane $P^1_{\mathcal{L}'_i}$.

† *The rest of this abstract is the part of the talk that was not fully presented because there wasn't enough time.*

Toward proving the Kähler package. In order to be able to use results like the previous theorem, one needs to prove that triples $(A, \deg, \mathcal{K})$ satisfy the mixed Kähler package. For this, it is convenient to develop properties of Kähler packages. For example:

(1) When (PD) holds, (MHR) is equivalent to (MHL) plus a numerical condition on the signatures of the forms $Q^k_{\mathcal{L}}$ ([1, §7], [2, §5]).

A general strategy to prove that $(A, \deg, \mathcal{K})$ satisfies the mixed Kähler package is induction on the top degree, r. One possibility is that A has a presentation that is well-suited to induction: Suppose A has generators $a_1, \dots, a_n \in A^1$ such that $\mathcal{K} = \sum_i \mathbb{R}_{>0} a_i$. Let $I_i = \mathrm{ann}_A(a_i)$ denote the annihilator of a_i and consider the quotients A/I_i. Assuming $(A, \deg)$ satisfies (PD), one defines $\deg_i$ on A/I_i so that $(A/I_i, \deg_i)$ satisfies (PD) and $(A/I_i)^k = 0$ for $k > r - 1$. If one can also define a cone $\mathcal{K}_i \in (A/I_i)^1$ so that, by induction, $(A/I_i, \deg_i, \mathcal{K}_i)$ satisfies the mixed or unmixed Kähler package, then one can try to use (1) to show that $(A, \deg, \mathcal{K})$ satisfies (MHL) or (HL), respectively. With this inductive scheme, one can also try to show that (HL) implies (MHL) and (HR) implies (MHR) along the way. See [1, §7], [3, §6], and [2, §5] for examples. Thus, one is reduced to proving (HR) for $(A, \deg, \mathcal{K})$. For this, a convexity argument shows that it suffices to prove (HR) for a single element $a \in \mathcal{K}$ [1, §7]. Finally, one identifies a suitable element, or elements, and then proves that (HR) holds for such an element, or elements.

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Lorentzian polynomials

Matt Baker

The purpose of this talk is to provide a gentle and motivated introduction to the theory of Lorentzian polynomials. One goal of the theory of Lorentzian polynomials is to provide new techniques for proving that various naturally-occurring sequences $a_0, a_1, \dots, a_d$ of non-negative real numbers a_k are *log-concave*, meaning that $a_k^2 \geq a_{k-1} a_{k+1}$ for all $1 \leq k \leq d - 1$. More generally, the authors of [1, 3] consider homogeneous multivariate polynomials $p(x_1, \dots, x_n)$ of degree d with non-negative coefficients and study certain natural extensions of log-concavity to this setting. We begin by introducing two "classical" methods for proving log-concavity of the coefficients of certain polynomials.

Method #1 (Newton): Let $f(x,y) = \sum_{k=0}^{d} \frac{c_k}{k!(d-k)!} x^k y^{d-k}$, and assume that $f(x,1)$ is real-rooted. Then, by differentiating $k-1$ times with respect to x and $d-k-1$ times with respect to y, we are reduced (via Rolle's theorem) to the quadratic case $d=2$. Applying the inequality $\mathrm{Disc}(f(x,1)) \geq 0$ gives $c_k^2 \geq c_{k-1}c_{k+1}$. Log-concavity of the normalized coefficients c_k is equivalent to *ultra-log-concavity* of the non-normalized coefficients $a_k = \frac{c_k}{k!(d-k)!}$.

Method #2 (Cauchy, Sylvester): Suppose $f(x,y) = \frac{1}{2}\sum_{i=1}^{n} c_{ii}x_i^2 + \sum_{i\neq j} c_{ij}x^i y^j$ is a quadratic form with $c_{ii} > 0$ for all i and having *Lorentzian signature* $(1, n-1)$ (i.e., the symmetric matrix $C = (c_{ij})$ has one positive and $n-1$ negative eigenvalues). By Cauchy interlacing, every principal minor of C has Lorentzian signature as well. Applying this to the 2×2 principal minor C_{ij} of C, it follows that $\det(C_{ij}) < 0$, hence $c_{ij}^2 > c_{ii}c_{jj}$ for all i, j.

Note that in both the Newton and Cauchy-Sylvester methods, log-concavity is ultimately proved by reducing to the degree two case. Our next goal is to describe a general class of homogeneous polynomials for which one can reduce log-concavity considerations to the case of Lorentzian quadratic forms via partial differentiation.

Lorentian polynomials: theory

Let $\Delta_n^d := \{\alpha \in \mathbb{N}^n \mid \alpha_1 + \cdots + \alpha_n = d\}$ be the *discrete hypersimplex*, and let $\mathring{L}_n^2$ be the space of homogeneous quadratic polynomials $f \in H_n^2$ whose Hessian $\mathcal{H}_f$ has *Lorentzian signature.*

A *strictly Lorentzian polynomial* of degree d in $x_1, \dots, x_n$ is a homogeneous polynomial $f \in H_n^d$ such that $\partial^\alpha f \in \mathring{L}_n^2$ for all $\alpha \in \Delta_n^{d-2}$. A *Lorentzian polynomial* is a limit of strictly Lorentzian polynomials.

We let L_n^d denote the space of Lorentzian polynomials of degree d in $x_1, \dots, x_n$.

Example 1. *When $n=2$, a normalized homogeneous polynomial $f(x,y) = \sum_{k=0}^{d} \frac{c_k}{k!(d-k)!} x^k y^{d-k} \in H_2^d$ is Lorentzian iff its coefficient sequence $c_0, c_1, \dots, c_d$ has no internal zeroes and is log-concave.*

Example 2. *When $d=2$, a quadratic form $f(x,y) = \frac{1}{2}\sum_{i=1}^{n} c_{ii}x_i^2 + \sum_{i\neq j} c_{ij}x^i y^j \in H_n^2$ is Lorentzian iff the matrix C_{ij} has at most one positive eigenvalue.*

Example 3. *If $A_1, \dots, A_n$ are positive semi-definite matrices then*

$$f(x_1, \dots, x_n) := \det(x_1 A_1 + \cdots + x_n A_n)$$

is Lorentzian, cf. [3, p. 829].

An *M-convex set* of rank r on $[n]$ is a nonempty subset J of Δ_n^r such that for all $\alpha, \beta \in J$ and every $i \in [n]$ with $\alpha_i < \beta_i$, there exists an $j \in [n]$ with $\alpha_j > \beta_j$ such that J contains both $\alpha + \epsilon_i - \epsilon_j$ and $\beta - \epsilon_i + \epsilon_j$. An M-convex set contained in $\{0,1\}^n$ is the same thing as a *matroid.*

The first link between Lorentzian polynomials and M-convexity is given by:

Theorem 4 ([3, Theorem 2.25]). *If $f \in H_n^d$ is Lorentzian, then* $\mathrm{supp}(f) \subset \Delta_n^d$ *is an M-convex set.*

The following is one of the key results from [3]. It provides a "limit-free" characterization of Lorentzian polynomials, which in particular allows one to check the Lorentzian property in an algorithmic way.

Theorem 5 ([3, Theorem 2.25]). *Let $f \in H_n^d$. Then the following are equivalent:*

(1) *f is Lorentzian.*
(2) $\mathrm{supp}(f) \subset \Delta_n^d$ *is M-convex and the Hessian of $\partial^\alpha f$ has at most one positive eigenvalue for all $\alpha \in \Delta_n^{d-2}$.*

Using Theorem 5, Brändén and Huh show that the space L_J of Lorentzian polynomials with support J is non-empty for every M-convex set J. This is in marked contrast to what happens for stable polynomials; for example, the Fano matroid is not the support of any stable polynomial [2].

There are a number of basic operations which preserve the space of Lorentzian polynomials. For example, any nonnegative linear change of variables preserves the Lorentzian property:

Theorem 6 ([3, Theorem 2.10]). *If $f \in L_n^d$ and A is an $n \times m$ matrix with non-negative real entries, viewed as a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then $f \circ A \in L_m^d$.*

One obtains, as a non-trivial consequence, the following two results:

Corollary 7. *If $f(x_1, \ldots, x_n) \in L_n^d$ is Lorentzian, then the univariate polynomial $f(x, x, \ldots, x)$ is also Lorentzian.*

Corollary 8 ([3, Corollary 2.32]). *A product of Lorentzian polynomials is again Lorentzian.*

Lorentian polynomials: applications

Our first application is a 1972 conjecture of Mason, which established independently by Brändén-Huh [3] and Anari-Liu-Oveis Gharan-Vinzant [1].

Theorem 9. *Let M be a matroid on n elements, and let $I_k(M)$ denote the number of independent sets of size k in M. Then the sequence $I_k(M)$ is ultra-log-concave.*

A special case of this result (which seems to be no easier to prove than the general case) is the following: Let E be a finite set of vectors in some finite-dimensional vector space, and let I_k denote the number of k-element linearly independent subsets of E. Then the sequence I_k is ultra-log-concave (which is stronger than being log-concave).

Here is a brief outline of the proof, following a hybrid of [3] and [1]. Let $\mathcal{I}$ be the collection of independent sets of M, and let $g_M(y, z_1, \ldots, z_n)$ be the homogenous generating function of $\mathcal{I}$, i.e., $g_M(y, z_1, \ldots, z_n) = \sum_{I \in \mathcal{I}} y^{n-|I|} \prod_{i \in I} z_i$. One shows that g_M is Lorentzian by first using a deletion-contraction argument to reduce to the rank 2 case and then applying Cauchy-Schwartz to verify the Lorentzian property of the resulting quadratic form. Setting all z_i equal to z, we see from Corollary 7 that the bivariate polynomial $g_M(y,z) = \sum_{I\in\mathcal{I}} y^{n-|I|}\prod_{i\in I} z^{|I|} = \sum_{k=0}^{n} I_k(M)y^{n-k}z^k$ is Lorentzian, and in particular its coefficients $I_k(M)$ form an ultra-log-concave sequence.

Here is another application of Lorentzian polynomials. Let λ be a partition of n, identified with its corresponding Young diagram, and let $\mathcal{T} = \mathcal{T}_\lambda$ be the collection of all semistandard Young tableaux with shape λ and entries in $[n]$. For $T \in \mathcal{T}$, let $\mu_i(T)$ be the number of appearances of i among the entries of T. The classical Schur polynomial corresponding to λ is generating function $s_\lambda(x_1,\ldots,x_n) = \sum_{T\in\mathcal{T}} x^{\mu(T)} = \sum_\mu K_{\lambda\mu}x^\mu$. The coefficients $K_{\lambda\mu}$ are called *Kostka numbers*; they are special cases of Littlewood-Richardson coefficients. The normalized Schur polynomial is the corresponding exponential generating function $S_\lambda(x_1,\ldots,x_n) = \sum_\mu K_{\lambda\mu}\frac{x^\mu}{\mu!}$. The following result is proved in [4]:

Theorem 10. *For every partition λ, the normalized Schur polynomial $S_\lambda(x_1,\ldots,x_n)$ is Lorentzian. In particular, for any weight μ and any $i,j\in[n]$ we have*

$$K_{\lambda\mu}^2 \geq K_{\lambda(\mu+e_i-e_j)}K_{\lambda(\mu-e_i+e_j)}.$$

This result verifies a special case of a conjecture of a conjecture of Okounkov about log-concavity of Littlewood-Richardson coefficients.

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Stable polynomials

Jonathan Leake

1. Basic Properties and Examples

A polynomial $p \in \mathbb{R}[x_1,\ldots,x_n]$ is called **real stable** if $p(z_1,\ldots,z_n) \neq 0$ whenever $\Im(z_i) > 0$ for all i. That is, p never vanishes when all inputs are in the complex upper half-plane. Note that when n (the number of variables) is 1, then a real stability is equivalent to real-rootedness. The famous Newton's inequalities give the most basic connection between real stability and log-concavity.

Theorem 1 (Newton's inequalities)**.** *If $f(t) = \sum_{k=0}^{d}\binom{d}{k}c_k t^k$ is real-rooted, then $c_k^2 \geq c_{k-1}c_{k+1}$ for all k.*

1.1. **Preservers.** A **real stability preserver** is a linear map T such that $T[p]$ is real stable whenever p is real stable. Given real stable polynomials $p, q \in \mathbb{R}[x_1, \dots, x_n]$, we have that

(1) $\partial_{x_i} p = \frac{\partial p}{\partial x_i}$ is real stable,
(2) $p(Ax)$ is real stable for any $A \in \mathbb{R}^{n \times m}_{\geq 0}$ and $x = (x_1, \dots, x_m)$,
(3) $p \cdot q$ is real stable,
(4) $p(x_1, \dots, x_{i-1}, r, x_{i+1}, \dots, x_n)$ is real stable for $r \in \mathbb{R}$, and
(5) $\mathrm{MAP}(p)$ is real stable, where MAP ("multiaffine part") is unique such that it preserves square-free monomials and zeros out the others.

Combining (2) and (4), the univariate polynomial $f(t) = p(u \cdot t + v)$ is real-rooted for all real stable $p \in \mathbb{R}[x_1, \dots, x_n]$, $u \in \mathbb{R}^n_{>0}$, and $v \in \mathbb{R}^n$.

1.2. **Examples.**

1.2.1. *Products of linear forms.* Given a matrix $M \in \mathbb{R}^{m \times n}_{\geq 0}$, the polynomial

$$p(x_1, \dots, x_n) = \prod_{i=1}^{m} \sum_{j=1}^{n} m_{ij} x_j$$

is real stable. It is interesting to note that when $m = n$, the polynomial p has the **permanent** of M as one of its coefficients.

1.2.2. *Determinantal polynomials.* Given positive semi-definite (possibly Hermitian) matrices $P_1, \dots, P_n$, the polynomial

$$p(x_1, \dots, x_n) = \det(x_1 P_1 + \cdots + x_n P_n)$$

is real stable.

1.2.3. *Elementary symmetric polynomials.* Given any n and $k \leq n$, the elementary symmetric polynomial

$$e_k(x_1, \dots, x_n) = \sum_{\substack{S \subseteq [n] \\ |S| = k}} x^S$$

is real stable, where $x^S := \prod_{i \in S} x_i$.

1.2.4. *Matching polynomials.* Given any graph $G = (V, E)$, a **matching** of G is a collection of edges $M \subseteq E$, no two of which are incident to the same vertex. For example if G is a 4-cycle so that $V = \{1, 2, 3, 4\}$ and $E = \{12, 23, 34, 14\}$, then G has 7 distinct matchings: $\emptyset$, $\{12\}$, $\{23\}$, $\{34\}$, $\{14\}$, $\{12, 34\}$, $\{23, 14\}$. With this, the polynomial

$$p_G\left((x_v)_{v \in V}\right) = \sum_{M} \prod_{uv \in M} -x_u x_v$$

is real stable, where the sum is over all matchings M of G. To see this, one can use the MAP operator on the product of $(1 - x_u x_v)$ over every edge $uv \in E$.

1.3. **Real Stable Polynomials are Lorentzian.** Here we sketch a proof of the following important fact.

Theorem 2. *Every d-homogeneous real stable polynomial $p \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ is Lorentzian.*

To prove Theorem 2, we use the following characterization of Lorentzian polynomials: Given a d-homogeneous polynomial $p \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$, p is Lorentzian if and only if for all $D_1, \dots, D_{d-2} \in \{\partial_{x_1}, \dots, \partial_{x_n}, \nabla_1$ the Hessian of $D_1 \cdots D_{d-2}p$ has at most one positive eigenvalue. Here $\nabla_u = \sum_i u_i \partial_{x_i}$ is the directional derivative with respect to u.

The idea is first to reduce to the $d = 2$ case by showing that ∇_1 is a real stability preserver. From there, we use real-rootedness of all linear restrictions of p to show that H (the Hessian of p) must be negative semidefinite on the hyperplane $(H1)^\perp$.

2. Rayleigh Inequalities

In this section we will present two conjectures on **multiaffine** polynomials $p \in \mathbb{R}^{\mathrm{MA}}[x_1, \dots, x_n]$, which are polynomials with only square-free terms. We will consider certain collections of inequalities called **(strongly) Rayleigh inequalities**. Given $p \in \mathbb{R}[x_1, \dots, x_n]$, $c > 0$, and $S \subseteq \mathbb{R}^n$, we will denote by $\mathcal{R}(c, S)$ the statement

$$c \cdot \partial_{x_i} p(x) \cdot \partial_{x_j} p(x) - p(x) \cdot \partial_{x_i} \partial_{x_j} p(x) \geq 0 \qquad \forall i, j \in [n], \ \forall x \in S.$$

Multiaffine real stable polynomials have a characterization in terms of Rayleigh inequalities, which we give now.

Theorem 3 (Brändén [1])**.** *Given a polynomial $p \in \mathbb{R}^{\mathrm{MA}}[x_1, \dots, x_n]$, p is real stable if and only if $\mathcal{R}(1, \mathbb{R}^n)$ holds for p.*

While Theorem 2 implies the support of a homogeneous multiaffine real stable polynomial is the set of bases of a matroid, Brändén used Theorem 3 to show that the bases of the Fano matroid is not the support of any real stable polynomial.

We now recall that Lorentzian polynomials satisfy Rayleigh inequalities as well, just of a weaker form.

Theorem 4 (Brändén–Huh [2])**.** *If $p \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ is Lorentzian, then $\mathcal{R}(2, \mathbb{R}^n_{\geq 0})$ holds for p. If p is of degree d, then this can be strengthened to $\mathcal{R}\big(2(1 - \frac{1}{d}), \mathbb{R}_{\geq 0}\big)$.*

2.1. **Two open problems.** The remainder of this section will then be devoted to describing two open problems which ask whether or not certain classes of polynomials "in between" Lorentzian and real stable polynomials satisfy some Rayleigh inequalities "in between" $\mathcal{R}(2, \mathbb{R}^n_{\geq 0})$ and $\mathcal{R}(1, \mathbb{R}^n)$.

Before stating the problems, let's look at a specific inequality one can obtain from the Rayleigh inequalities. Let $\mathcal{S} \subseteq 2^{[n]}$ be a collection of subsets, and define

$$p_{\mathcal{S}}(x) = \sum_{S \in \mathcal{S}} x^S \in \mathbb{R}^{\mathrm{MA}}[x_1, \dots, x_n].$$

The condition that $\mathcal{R}(c,\{1\})$ holds for $p_{\mathcal{S}}$ is then equivalent to

$$c \cdot |\{S \in \mathcal{S} : i \in S\}| \cdot |\{S \in \mathcal{S} : j \in S\}| \geq |\mathcal{S}| \cdot |\{S \in \mathcal{S} : i,j \in S\}|$$

for all $i,j \in [n]$ such that $i \neq j$. With this, we can now state the open problems.

Conjecture 5 (Huh–Schröter–Wang [3]). *If $\mathcal{M}$ is the set of bases of a matroid on ground set E, then $\mathcal{R}(\frac{8}{7},\{1\})$ holds for $p_{\mathcal{M}}$ where*

$$p_{\mathcal{M}}(x) = \sum_{B \in \mathcal{M}} x^B \in \mathbb{R}^{\mathrm{MA}}[(x_e)_{e\in E}].$$

The same statement is open if only considering matroids representable over $\mathbb{F}_2$.

Conjecture 6 (Kahn, Grimmett–Winkler; see [4]). *Given a graph $G = (V,E)$, let $\mathcal{F}$ be the set of forests of G, considered as subsets of E. Then $\mathcal{R}(1,\{1\})$ holds for $p_{\mathcal{F}}$ where*

$$p_{\mathcal{F}}(x) = \sum_{F \in \mathcal{F}} x^F \in \mathbb{R}^{\mathrm{MA}}[(x_e)_{e\in E}].$$

I am unaware of any counterexample to the analogous statements with $\mathcal{R}(c,\{1\})$ replaced by $\mathcal{R}(c,\mathbb{R}^n_{\geq 0})$. One general approach for both of these open problems is to find some new class of polynomials, possibly containing real stable polynomials and being contained in Lorentzian polynomials (or a closely related class), for which $\mathcal{R}(c,\mathbb{R}^n_{\geq 0})$ holds for some desired constant $c \in [1,2]$.

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Baker–Bowler theory

Jacob Matherne

We present a brief overview of a portion of the Baker–Bowler theory as developed in [2]. The idea is to develop the theory of matroids over hyperfields (more generally over tracts), which will include as special cases subspaces, matroids, oriented matroids in the sense of [4], and valuated matroids in the sense of [5]. This report (and the associated talk) follows [1].

Grassmannians and Plücker functions. We now review some aspects of Grassmannians and Plücker functions with an eye toward deducing Observation 2 below. See, for example, [6, Chapter 9] for a detailed treatment of these notions.

Let K be a field, and consider the Grassmannian of d-dimensional subspaces of K^n. Any such d-dimensional subspace can be gotten by taking the row span of a $d \times n$ rank d matrix, and two such matrices have the same row span if and only if they differ by left multiplication by an invertible $d \times d$ matrix.

We set $E = [n]$, and view it as an indexing set for the columns of $d \times n$ matrices. For any $d \times n$ rank d matrix A, we may consider the Plücker coordinate $\Delta_A \colon E^d \to K$ sending a d-tuple to the $d \times d$ minor of A on the columns indexed by the d-tuple. For any invertible $d \times d$ matrix g, by the multiplicativity of the determinant it follows that $\Delta_{g \cdot A}(i_1, \ldots, i_d) = \det(g)\Delta_A(i_1, \ldots, i_d)$. Hence, if two matrices A and A' have the same row span, then there is some nonzero $\alpha \in K$ such that $\Delta_A(i_1, \ldots, i_d) = \alpha\Delta_{A'}(i_1, \ldots, i_d)$.

It is well known that the function Δ_A satisfies the following properties:

- (NZ) $\Delta_A \neq 0$ (since A has rank d).
- (Alt) Δ_A is alternating. That is,
  - $\Delta_A(i_1, \ldots, i_k, \ldots, i_\ell, \ldots, i_d) = -\Delta_A(i_1, \ldots, i_\ell, \ldots, i_k, \ldots, i_d)$, and
  - $\Delta_A(i_1, \ldots, i_d) = 0$ if $i_k = i_\ell$ for some $k \neq \ell$.
- (SGP) For any two subsets $\{i_1, \ldots, i_{d+1}\}$ and $\{j_1, \ldots, j_{d-1}\}$ of E,

$$\sum_{k=1}^{d+1} (-1)^k \Delta_A(i_1, i_2, \ldots, \widehat{i_k}, \ldots, i_{d+1})\Delta_A(i_k, j_1, \ldots, j_{d-1}) = 0.$$

The following definition will be the key to the "basis definition" of a (strong) matroid over a tract.

Definition 1. *A* Grassmann–Plücker function *is any function* $\varphi \colon E^d \to K$ *that satisfies (NZ), (Alt), and (SGP). Two Grassmann–Plücker functions* φ *and* φ' *are called* projectively equivalent *if there exists some nonzero* $\alpha \in K$ *such that* $\varphi' = \alpha\varphi$.

It is also well known that any Grassmann–Plücker function is of the form Δ_A for some $d \times n$ rank d matrix A and Δ_A is projectively equivalent to $\Delta_{A'}$ if and only if A and A' have the same row span. Our discussion so far yields the following.

Observation 2. *A projective equivalence class of Grassmann–Plücker functions* $\varphi \colon E^d \to K$ *is the same as a* d*-dimensional subspace of* K^n.

We seek to make a version of Observation 2 when the field K is replaced by a more general object called a hyperfield (even more generally a tract). This will lead to the notion of a (strong) matroid over a hyperfield (see Definition 5 below). We now introduce hyperfields, in the sense of Krasner, along with some examples.

Matroids over hyperfields. A *hyperoperation* on a set S is a map $\boxplus \colon S \times S \to \mathcal{P}(S) \setminus \{\varnothing\}$, where $\mathcal{P}(S)$ denotes the power set of S. For $A, B \in \mathcal{P}(S) \setminus \{\varnothing\}$, define $A \boxplus B := \bigcup_{a \in A, b \in B} a \boxplus b$.

Definition 3. *A* commutative hyperring *is a set R with*

- *a commutative associative binary operation $\odot$,*
- *a commutative associate binary hyperoperation $\boxplus$, and*
- *distinct elements $0, 1 \in R$,*

such that

- *$(R, \odot, 1)$ is a commutative monoid,*
- *$0 \odot x = 0$ and $0 \boxplus x = \{x\}$ for all $x \in R$,*
- *for all $x \in R$, there is a unique element $-x \in R$ such that $0 \in x \boxplus -x$, and*
- *$a \odot (x \boxplus y) = (a \odot x) \boxplus (a \odot y)$ for all $a, x, y \in R$.*

A hyperfield *F is a hyperring such that every nonzero element of F has a multiplicative inverse.*

Example 4. *In the examples of hyperfields below, we only specify the operation $\boxplus$ for values that are not determined by the axioms of a hyperfield.*

(i) *fields K with $a \odot b = ab$ and $a \boxplus b = \{a + b\}$ for all $a, b \in K$.*
(ii) *the Krasner hyperfield $\mathbb{K} = \{0, 1\}$ with the usual multiplication and with $1 \boxplus 1 = \{0, 1\}$. The hyperfield $\mathbb{K}$ encodes the arithmetic of zeroness and nonzeroness.*
(iii) *the hyperfield of signs $\mathbb{S} = \{1, -1, 0\}$ with the usual multiplication and with $-1 \boxplus -1 = \{-1\}$, $1 \boxplus 1 = \{1\}$, and $1 \boxplus -1 = \{1, -1, 0\}$. The hyperfield $\mathbb{S}$ encodes the arithmetic of positive, negative, and zero.*
(iv) *the tropical hyperfield $\mathbb{T} = \mathbb{R}_{\geq 0}$ with the usual multiplication and with*

$$a \boxplus b = \begin{cases} \{\max(a, b)\} & \text{if } a \neq b \\ [0, a] & \text{if } a = b. \end{cases}$$

(v) *the triangular hyperfield $\mathbb{V} = \mathbb{R}_{\geq 0}$ with the usual multiplication and with*

$$a \boxplus b = \left\{ c \in \mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{l} \text{there is a (possibly degenerate)} \\ \text{Euclidean triangle with side lengths } a, b, c \end{array} \right\}.$$

We now note that (NZ), (Alt), (SGP), and the notion of projective equivalence all make sense suitably interpreted over an arbitrary hyperfield F. For example, the identity in (SGP) has the following analogue over a hyperfield:

(SGP') $$0 \in \boxplus_{k=1}^{d+1} (-1)^k \Delta_A(i_1, i_2, \ldots, \widehat{i_k}, \ldots, i_{d+1}) \odot \Delta_A(i_k, j_1, \ldots, j_{d-1}).$$

Thus, we make the following definition.

Definition 5 (Baker–Bowler [2])**.** *Let E be a finite set and F be a hyperfield. A* strong *F*-matroid *of rank d on E is a projective equivalence class of Grassmann–Plücker functions $\varphi\colon E^d \to F$.*

The following result is what we have been working towards.

Theorem 6 (Baker–Bowler [2])**.** *A strong F-matroid is the same as a*

(a) *d-dimensional subspace of K^n when $F = K$.*
(b) *rank d matroid on E when $F = \mathbb{K}$.*
(c) *rank d oriented matroid on E when $F = \mathbb{S}$.*
(d) *rank d valuated matroid on E when $F = \mathbb{T}$.*

Closing thoughts. A few comments are in order. As we have alluded before, a more general version of strong F-matroid can be given: indeed, Baker and Bowler define a strong matroid over a tract F as in Definition 5, but where the notion of "projective equivalence class of Grassmann–Plücker function" is interpreted appropriately over F. Tracts include as special cases both hyperfields in the sense of Krasner and partial fields in the sense of Semple and Whittle [7], both of which generalize fields. We point to [3, Figure 1] for a figure depicting the relationships among various hyperstructures.

The reader may have noticed the use of the adjective "strong"; indeed, Baker and Bowler also define the notion of weak matroid over a tract F where, essentially, one replaces (SGP') with the 3-term Grassmann–Plücker relations. The notions of weak F-matroid and strong F-matroid coincide when F is a so-called doubly distributive partial hyperfield. Indeed, the hyperfields in Example 4 (i)–(iv) are doubly distributive, and hence Theorem 6 can equivalently be stated for weak F-matroids. But, the notions of strong and weak F-matroid differ already when $F = \mathbb{V}$, the triangular hyperfield from Example 4 (v).

Definition 5 and its corresponding weak analogue can both be viewed as "basis definitions" of matroids over tracts. We encourage the reader to peruse [2] for more of the foundational theory of matroids over tracts, including other cryptomorphic definitions as well as operations on them.

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Foundations of matroids

Lukas Kühne

The foundation of a matroid is a novel algebraic object recently introduced by Baker and Lorscheid [1]. It not only governs the representations of a matroid over fields but for instance also its orientations and valuations. The goal of this talk is to give a gentle introduction to the foundation of a matroid following [1] and [2].

As foundations are pastures we start by defining these algebraic structures:

Definition 1. *A* pasture P *is a multiplicative monoid with zero such that* $P^\times = P \setminus \{0\}$ *is an abelian group with an involution* $x \mapsto -x$ *on* P *fixing* 0 *and a subset* $N_P \subset P^3$ *such that*

(1) N_P *is invariant under the action of the symmetric group* S_3*,*
(2) N_P *is invariant under the diagonal action of* $P^\times$ *and*
(3) $(0, x, y) \in N_P$ *if and only if* $y = -x$*.*

We think of the set N_P as the triples in P that sum to 0. So a field F is a pasture by collecting all triples in F that sum to 0. It is however more general as it also encompasses partial field and hyperfields (in the former case, not all elements can be added and in the latter case the addition might be multivalued).

An important example is the *Krasner hyperfield* $\mathbb{K}$: Here $\mathbb{K} = \{0, 1\}$ with the usual multiplicative relation but with the addition $0 + x = x$, $1 + 1 = 1$ and $1 + 1 = 0$, i.e., both $(1, 1, 1)$ and $(1, 1, 0)$ and their permutations are in $N_{\mathbb{K}}$. This is a hyperfield as the addition of $1 + 1$ is thus $\mathbb{K}$.

Other prominent examples beyond fields are the regular partial field, the sign hyperfield and the tropical hyperfield.

A representation of a matroid over a pasture is defined via Plücker functions:

Definition 2. *Let* M *be a matroid of rank* r *on a finite set* E *and let* P *be a pasture. A* P*-representation of* M *is a function* $\Delta : E^r \to P$ *such that*

(1) $\Delta(e_1, \ldots, e_r) \neq 0$ *if and only if* $\{e_1, \ldots, e_r\}$ *is a basis of* M*,*
(2) $\Delta(\sigma(e_1), \ldots, \sigma(e_r)) = \mathrm{sign}(\sigma)\Delta(e_1, \ldots, e_r)$ *for all permutations* $\sigma \in S_r$*,*
(3) Δ *satisfies the* 3*-term Plücker relations, which means that certain triples are in* N_P*.*

Note that we do not demand that the Plücker relations on more terms vanish.

Definition 3.

(1) M *is* representable *over* P *if there is at least one* P*-representation of* M*.*
(2) Δ *and* Δ' *are* rescaling equivalent *if if there exists* $c \in P^\times$ *and a map* $d : E \to P^\times$ *such that* $\Delta'(e_1, \ldots, e_r) = c \cdot d(e_1) \cdots d(e_r) \cdot \Delta(e_1, \ldots, e_r)$ *for all* $(e_1, \ldots, e_r) \in E^r$*.*
(3) *We denote by* $X_M^R(P)$ *the set of rescaling classes of* P*-representations of* M*.*

Baker and Lorscheid connect their theory of matroid representations over pastures to the classical theory of matroids by proving the following facts [1]:

(1) In the case of a field F, F-representations of a field are exactly matroid representations over F in the usual sense.
(2) Every matroid has a unique representation over the Krasner hyperfield.
(3) A matroid is orientable if and only if it representable over the sign hyperfield.
(4) A matroid is regular if and only if it representable over the regular partial field.

For fixed M, the map taking a pasture P to the set $X_M^R(P)$ is a functor. One of the main results in [1] is that this functor is representable in the sense of algebraic geometry:

Theorem 4. *Given a matroid M, the functor taking a pasture P to the set $X_M^R(P)$ is representable by a pasture F_M, called the* foundation *of M, i.e., there is an isomorphism*

$$\mathrm{Hom}(F_M, -) \simeq X_M^R(P).$$

Thus a matroid M is representable over a pasture P if and only if there is a pasture homomorphism from F_M to P. The realization space of a matroid over $\mathbb{C}$, i.e., the matroid stratum in the Grassmannian modulo the torus action, is then $\mathrm{Hom}(F_M, \mathbb{C})$ as a set.

In [2], the authors study generators and relations of a foundation. They prove that the foundation is generated by the so-called universal cross-ratios stemming from $U_{2,4}$ minors. The relations amongst these cross-ratios stem from matroid minors on at most seven elements.

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Chow ring of matroids

Raluca Vlad

The purpose of this talk is to introduce the Chow ring associated to a matroid and prove that it satisfies the Kähler package, following Adiprasito, Huh and Katz's paper [1].

Let N be a finitely generated abelian group. Let $\Sigma \subseteq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ be a unimodular fan and let $V_\Sigma \subset N$ be the set of primitive ray generators of Σ. The *Chow ring of* Σ is the graded $\mathbb{R}$-algebra

$$A^\bullet(\Sigma) = \mathbb{R}[x_e \colon e \in V_\Sigma] \big/ (I_\Sigma + J_\Sigma),$$

where the ideals I_Σ and J_Σ are given by

$$I_\Sigma = \left\langle \prod_{e \in S} x_e \ : \ S \subseteq V_\Sigma \text{ do not form a cone in } \Sigma \right\rangle$$

and

$$J_\Sigma = \left\langle \sum_{e \in V_\Sigma} m(e) \cdot x_e \ : \ m \in N^\vee \right\rangle.$$

By work of Billera [2], the Chow ring $A^\bullet(\Sigma)$ is isomorphic to the ring of piecewise polynomial functions on Σ, modulo linear functions on the underlying real vector space $N_{\mathbb{R}}$. In particular, the degree 1 piece $A^1(\Sigma)$ is equal to the real vector space of piecewise linear functions on Σ, modulo linear functions on $N_{\mathbb{R}}$. We define the *ample cone* $\mathcal{K}_\Sigma \subseteq A^1(\Sigma)$ to be the open convex cone consisting of classes of *strictly convex* piecewise linear functions, where a function l is strictly convex if, for every cone σ in Σ, the function l is equivalent, up to the addition of a linear function on $N_{\mathbb{R}}$, to a function on Σ that is zero on σ and strictly positive on all the rays of the fan that are not in σ but form a cone with σ.

The Chow ring of a matroid is the Chow ring of an appropriately chosen fan called the Bergman fan, which we now define. Let M be a loopless matroid of rank $r+1$ on the ground set $E = \{0, 1, \ldots, n\}$. Let $\mathcal{P}(M)$ be the poset of non-empty proper flats of M, ordered by inclusion, and let $\mathcal{P} \subseteq \mathcal{P}(M)$ be an upwards-closed order filter. Given a flag of flats

$$\mathcal{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\},$$

we define its minimum $\min \mathcal{F}$ to be either F_1 if the flag is non-empty, or E otherwise. Given such a flag $\mathcal{F}$ and a subset $I \subseteq E$, we say that I is compatible with $\mathcal{F}$, and write $I < \mathcal{F}$, if $I \subsetneq \min \mathcal{F}$. Furthermore, given a subset $I \subseteq E$, we write

$$e_I := \sum_{i \in I} e_i \ \in \ \mathbb{Z}^E.$$

Finally, let us fix the lattice $N = \mathbb{Z}^E / \langle e_E \rangle$.

Definition 1. *The* Bergman fan *associated to* $(M, \mathcal{P})$ *is the unimodular fan* $\Sigma_{M,\mathcal{P}} \subseteq N_{\mathbb{R}}$ *consisting of the polyhedral cones*

$$\sigma_{I<\mathcal{F}} := \mathbb{R}_{\geq 0} \langle e_i, e_F \ : \ i \in I, F \in \mathcal{F} \rangle$$

for all compatible pairs of subsets $I \subseteq E$ *with* $\mathrm{cl}(I) \notin \mathcal{P} \cup \{E\}$ *and flags* $\mathcal{F} \subseteq \mathcal{P}$*.*

The Chow ring $A^\bullet(M, \mathcal{P})$ associated to the pair $(M, \mathcal{P})$ is then defined to be the Chow ring of the Bergman fan $\Sigma_{M,\mathcal{P}}$. When $\mathcal{P} = \mathcal{P}(M)$ is the entire poset of flats, we drop the filter from the notation and call $A^\bullet(M) = A^\bullet(M, \mathcal{P}(M))$ the *Chow ring of the matroid* M. The following is the main result of [1].

Theorem 2. *For any loopless matroid* M *and order filter* $\mathcal{P} \subseteq \mathcal{P}(M)$*, the Chow ring* $A^\bullet(M, \mathcal{P})$ *satisfies the Kähler package with respect to the ample cone* $\mathcal{K}_{\Sigma_{M,\mathcal{P}}}$*.*

The geometric motivation behind Theorem 2 comes from the realizable case. More precisely, if M is realizable over a field k, then Feichtner and Yuzvinsky [4] prove that the Chow ring $A^\bullet(M)$ is isomorphic to the Chow ring of the wonderful compactification associated to a realization space of M. Since this wonderful compactification is a smooth projective k-variety, the Kähler package follows from classical algebraic geometry when $k = \mathbb{C}$. In fact, in [1], the authors prove the converse statement – namely, that $A^\bullet(M)$ is isomorphic to the Chow ring of a smooth projective k-variety only when M is realizable over k. The fact that Theorem 2 holds for all matroids, even in the non-realizable case, constitutes the ingenious progress made by [1].

Remark 3. *In* [1]*, the authors prove Theorem 2 using simultaneous induction on the rank of the matroid M and the size of the order filter $\mathcal{P}$, with $\mathcal{P} = \varnothing$ being the base case. Braden, Huh, Matherne, Proudfoot and Wang* [3] *give a simplified proof of the fact that the Chow ring $A^\bullet(M)$ satisfies the Kähler package using a direct sum decomposition of $A^\bullet(M)$ in terms of the Chow ring of the deletion matroid $M \setminus i$ for an element $i \in E$.*

Adiprasito, Huh and Katz [1] use Theorem 2 to prove several log-concavity conjectures. Their method relies on the following general result. Consider a graded $\mathbb{R}$-algebra $A^\bullet$ with non-zero graded pieces in degrees $0, 1, \ldots, r$. If $A^\bullet$ satisfies the Kähler package with respect to an ample cone $\mathcal{K} \subseteq A^1$, and α, β are two elements in the Euclidean closure $\overline{\mathcal{K}} \subseteq A^1$, then the degrees $\deg(\alpha^k \beta^{r-k})$ form a log-concave sequence as k ranges over $0, 1, \ldots, r$.

For example, consider the classes $\alpha, \beta \in \overline{\mathcal{K}}_{\Sigma_{M,\mathcal{P}(M)}} \subseteq A^1(M)$ determined by the linear expressions

$$\sum_{F \ni i} x_F \qquad \text{and} \qquad \sum_{F \not\ni i} x_F,$$

respectively, where the sums are taken over all the flats of the matroid M containing or not containing a fixed element $i \in E$. In this case, the intersection degrees $\deg(\alpha^k \beta^{r-k})$ form a convolution of the sequence of signless coefficients of the characteristic polynomial of M, thus implying the following corollary.

Corollary 4 ([1, Theorem 9.9])**.** *For M a loopless matroid, the signless coefficients of its characteristic polynomial form a log-concave sequence.*

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Brunn–Minkowski theory

Léo Mathis

Brunn Minkowski Theory can be defined as the study of interactions between volume and Minkowski sum. The purpose of this talk is to familiarize with the objects involved and to show the connections with Lorentzian polynomials and the Kähler package. For more details and proofs of the statements in convex geometry see [3].

Given two convex bodies (i.e. non empty convex compact subsets) K and L one can define their *Minkowski sum* to be the convex body

$$K + L := \{x + y \,|\, x \in K, y \in L\}\,.$$

See Figure 1. Minkowski then proved that, given a tuple of convex bodies

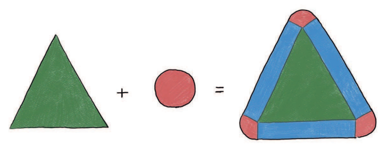

FIGURE 1. The Minkowski sum of a triangle and a disc

$K_1, \ldots, K_k$, and for any $t_1, \ldots, t_k \geq 0$, the function $(t_1, \ldots, t_k) \mapsto \mathrm{vol}(t_1K_1 + \cdots + t_kK_k)$ is a *polynomial.* This gives rise to one of the most important notion in Brunn Minkowski Theory, namely the *mixed volume* which is the unique continuous, multilinear, totally symmetric map V from n-tuples of convex bodies in $\mathbb{R}^n$ to $\mathbb{R}$ such that:

$$\mathrm{vol}(t_1K_1 + \cdots + t_kK_k) = \sum_{i_1,\ldots,i_n=1}^{k} t_{i_1} \cdots t_{i_n} \mathrm{V}(K_{i_1}, \ldots, K_{i_n}). \tag{1}$$

The mixed volume can also be seen as a polarization of the "n-form" volume. For example in dimension $n = 2$ we have $\mathrm{V}(K_1, K_2) = (\mathrm{vol}(K_1 + K_2) - \mathrm{vol}(K_1) - \mathrm{vol}(K_2))/2$. See again Figure 1 where the mixed volume of the triangle and the disc is the part of the Minkowski sum colored in blue (divided by 2).

The polynomials obtained in (1) are called *volume polynomials* or *convex volume polynomials.* Branden and Huh proved in [2] that these are Lorentzian.

An important particular case is the one of the Minkowski sum of a convex body K and B_n the unit ball of $\mathbb{R}^n$. In that case we obtain *Steiner's polynomial*:

$$\mathrm{vol}(K + tB_n) = \sum_{i=1}^{n} t^{n-i} \kappa_{n-i}\, \mathrm{V}_i(K) \tag{2}$$

where κ_d denotes the (d-dimensional) volume of the unit ball of $\mathbb{R}^d$ and V_i is the *i-th intrinsic volume.* Once again, the reader is invited to have a look at Figure 1 and identify the different monomials of the Steiner polynomial.

The i-th intrinsic volume is an $i-$homogeneous geometric quantity associated to the convex body K. In some cases it has a simple geometric interpretation. For example one can easily see taking limits in $t \to 0$ and $+\infty$ that we have $\mathrm{V}_n(K) = \mathrm{vol}(K)$ and $\mathrm{V}_0(K) = 1$ for all K. Moreover we also have $\mathrm{V}_{n-1}(K) = \mathrm{vol}_{n-1}(\partial K)$ the $(n-1)$-Hausdorff measure of the boundary ∂K of K.

These intrinsic volumes satisfy the *isoperimetric inequality* which says that with a fixed surface area, the (here convex) shape with the biggest volume is the

Euclidean ball:

$$\frac{\mathrm{V}_{n-1}(K)^n}{\mathrm{V}_n(K)^{n-1}} \geq \frac{\mathrm{V}_{n-1}(B_n)^n}{\mathrm{V}_n(B_n)^{n-1}} \tag{3}$$

for every convex body K with non empty interior.

This geometric inequality and many more can be obtained from the *Brunn-Minkowski* inequality which states that the function $\mathrm{vol}(\cdot)^{\frac{1}{n}}$ is concave. In other words for all convex bodies K_1, K_2 and $t \in [0,1]$, we have:

$$\mathrm{vol}\left((1-t)K_1 + tK_2\right)^{\frac{1}{n}} \geq (1-t)\,\mathrm{vol}\left(K_1\right)^{\frac{1}{n}} + t\,\mathrm{vol}\left(K_2\right)^{\frac{1}{n}}. \tag{4}$$

Finally, maybe the most important inequality in Brunn Minkowski Theory is the *Alexandrov–Fenchel inequality*. In this context it states that for every convex bodies $K_1, \ldots, K_n$ we have

$$\mathrm{V}\left(K_1, K_2, K_3, \ldots, K_n\right)^2 \geq \mathrm{V}\left(K_1, K_1, K_3, \ldots, K_n\right) \mathrm{V}\left(K_2, K_2, K_3, \ldots, K_n\right). \tag{5}$$

It can be proved that (5)⇒(4)⇒(3) and Alexandrov–Fenchel is of course a conscequence of the fact that volume polynomials are Lorentzian. However Branden and Huh use the Brunn-Minkowski inequality in their proof in [2]. But of course one can prove Alexandrov–Fenchel independently, once again see [3].

A recent result of Bernig, Kotrbatý and Wannerer establish a Kähler package in the context of *smooth valuations* [1]. A smooth valuation of degree k is a function ϕ on convex bodies with values in $\mathbb{R}$ such that there exists convex bodies $C_1^{(i)}, \ldots, C_{n-k}^{(i)}$ with smooth, positively curved boundary, and real numbers $\alpha^{(i)}$ for $i = 1, \ldots, N$ such that

$$\phi(K) = \sum_{i=1}^{N} \alpha^{(i)} \mathrm{V}(K[k], C_1^{(i)}, \ldots, C_{n-k}^{(i)})$$

where $K[k]$ indicates that it is repeated k times in the argument. We denote by A^k the space of smooth valuations of degree $n-k$. This clearly form a vector space. We now define a convolution product $* : \mathrm{A}^k \times \mathrm{A}^{k+l} \to \mathrm{A}^{k+l}$ characterized by:

$$\mathrm{V}(\cdot[n-k], C_1, \ldots, C_k) * \mathrm{V}(\cdot[n-l], C'_1, \ldots, C'_l) = \mathrm{V}(\cdot[n-k-l], C_1, \ldots, C_k, C'_1, \ldots, C'_l)$$

for any tuple of smooth positively curved convex bodies $C_1, \ldots, C_k, C'_1, \ldots, C'_l$.

Theorem 1 ([1])**.** *The graded algebra $\left(\bigoplus_{k=0}^n \mathrm{A}^k, +, *\right)$ satisfies a Kähler package for the cone*

$$\mathcal{K} := \{V(\cdot[n-1], C) \,|\, C \textit{ smooth pos. curved}\} \subset \mathrm{A}^1.$$

This is, to our knowledge, the only established case of a Kähler package in an infinite dimensional setting.

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Matroid Schubert varieties

Yairon Cid-Ruiz

The purpose of this talk is to present a proof (as self-contained as possible) of the result of Huh and Wang [5] establishing that the cohomology ring of a matroid Schubert variety coincides with the graded Möbius algebra.

Let M be a simple matroid on the set $[n] = \{1, \ldots, n\}$ which is *realizable* over $\mathbb{C}$. This means that there exists a linear subspace $L \subset \mathbb{C}^n$ such that the rank function of M is given by

$$\operatorname{rank}_M(S) \;=\; \dim\left(\Pi_S(L)\right) \qquad \text{for all} \qquad S \subseteq [n],$$

where $\Pi_S : \mathbb{C}^n = \bigoplus_{j\in[n]} \mathbb{C}\cdot \mathbf{e}_j \to \mathbb{C}^S = \bigoplus_{j\in S} \mathbb{C}\cdot\mathbf{e}_j$ is the natural projection and $\mathbf{e}_j = (0, \ldots, 1, \ldots, 0)$ is the j-th elementary basis vector. As introduced by Ardila and Boocher [1], the *matroid Schubert variety* $Y_M = Y_{M,L}$ of M is the closure of L under the natural inclusions

$$L \;\hookrightarrow\; \mathbb{C}^n = \mathbb{C}^1 \times \cdots \times \mathbb{C}^1 \;\hookrightarrow\; \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 = \left(\mathbb{P}^1\right)^n.$$

Let $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}[\mathbf{x}, \mathbf{z}] = \mathbb{C}[x_1, \ldots, x_n, z_1, \ldots, z_n]$ be the coordinate rings $\mathbb{C}^n$ and $\left(\mathbb{P}^1\right)^n$. Let $I(L) \subset \mathbb{C}[\mathbf{x}]$ be the vanishing ideal of the linear subspace $L \subset \mathbb{C}^n$. The vanishing ideal of Y_M can be computed by the multi-homogenization

$$I(Y_M) \;=\; \Big(f^h \;\mid\; f \in I(L)\Big) \;\subset\; \mathbb{C}[\mathbf{x}, \mathbf{z}].$$

For any $f \in \mathbb{C}[\mathbf{x}]$, the multi-homogenization f^h is obtained by substituting $x_i \mapsto \frac{x_i}{z_i}$ and then clearing out denominators.

Remark 1. *For any circuit C of the matroid M, there is a linear form $\sum_{c\in C} a_c x_c$ in $I(L)$, which is unique up to multiplication by a nonzero scalar.*

The following important result of Ardila and Boocher shows that the equations of Y_M are completely determined by the circuits of the matroid M.

Theorem 2 ([1, Theorem 1.3(a)])**.** *$Y_M \subset \left(\mathbb{P}^1\right)^n$ is defined by the multi-homogenization of the circuits of M. More precisely, we have*

$$Y_M \;=\; V\Big(\; \sum_{c\in C} a_c x_c \prod_{d \in C\setminus\{c\}} z_d \;\Big|\; C \text{ is a circuit of } M \;\Big).$$

Let $\mathcal{L}^\bullet_M$ be the lattice of flats of M. For each flat $F \in \mathcal{L}^\bullet_M$, we introduce the symbol y_F. Consider the graded free $\mathbb{Z}$-module

$$B^\bullet(M) := \bigoplus_{i\geq 0} B^i(M) \quad \text{where} \quad B^i(M) := \bigoplus_{F\in\mathcal{L}^i_M} \mathbb{Z}\cdot y_F.$$

We endow $B^\bullet(M)$ with the structure of a commutative graded algebra over $\mathbb{Z}$ by setting

$$y_{F_1}y_{F_2} = \begin{cases} y_{F_1\vee F_2} & \text{if } \operatorname{rank}_M(F_1) + \operatorname{rank}_M(F_2) = \operatorname{rank}_M(F_1\vee F_2) \\ 0 & \text{otherwise,} \end{cases}$$

and extending this by linearity. To simplify notation, we write $y_1, \dots, y_n$ instead of $y_{\{1\}}, \dots, y_{\{n\}}$. Under the above product operation, $y_\emptyset = 1$ is the identity element and the equality $y_F = \prod_{i\in I_F} y_i$ holds for any basis I_F of the flat F. Therefore, we can see $B^\bullet(M)$ as a quotient of the polynomial ring $\mathbb{Z}[y_1, \dots, y_n]$.

We are interested on the following remarkable result.

Theorem 3 (Huh-Wang [5, Theorem 14])**.** *For a realizable matroid simple M, we have the isomorphism of graded $\mathbb{Z}$-algebras*

$$B^\bullet(M) \xrightarrow{\cong} \mathrm{H}^{2\cdot\bullet}(Y_M, \mathbb{Z}), \quad y_i \mapsto h_i,$$

where h_i denotes the first Chern class of the line bundle $\mathcal{O}_{Y_M}(\mathbf{e}_i)$.

One nontrivial aspect of the above theorem is that matroid Schubert varieties typically have singularities, as the example below shows.

Example 4. *Consider the uniform matroid $U_{2,3}$ of rank two on three elements. This matroid is realized by the linear space $L = V(x_1 + x_2 + x_3) \subset \mathbb{C}^3$. From Theorem 2, we obtain*

$$Y_{U_{2,3}} = V\left(x_1z_2z_3 + x_2z_1z_3 + x_3z_1z_2\right) \subset \left(\mathbb{P}^1\right)^3.$$

Then we can check that $Y_{U_{2,3}}$ is singular at the (all-infinity) point $\underline{\infty} = \infty\times\infty\times\infty = V(z_1, z_2, z_3) \subset \left(\mathbb{P}^1\right)^3$.

Remark 5. *The graded Möbius algebra played a pivotal role in the proof of the Dowling–Wilson Top-Heavy Conjecture. The realizable case was established by Huh and Wang* [5]*, while the general case was proved by Braden, Huh, Matherne, Proudfoot, and Wang* [2]*.*

Remark 6. *For the case of polymatroids, Crowley, Simpson and Wang* [4] *gave a suitable generalization of the theorem above by utilizing the notions of* polymatroid Schubert varieties *and* combinatorial flats *(which they introduced).*

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Augmented wonderful varieties

Teddy Gonzales

We review Huh and Wang's proof of the top heavy conjecture for realizable matroids in [2] via augmented wonderful varieties. Let M be a simple matroid on $E = \{1, \dots, n\}$ of rank r, and let $\mathcal{L}$ be its lattice of flats.

A version of the top heavy conjecture states that for $p < r/2$, we have $|\mathcal{L}^p| \leq |\mathcal{L}^{r-p}|$ witnessed by an inclusion $\iota : \mathcal{L}^p \to \mathcal{L}^{r-p}$ satisfying $x \leq \iota(x)$. The main theorem of [2] is that there is a Lefschetz-type element of the graded Möbius algebra of M implying the existence of an ι for each such p. The graded Möbius algebra of M is

$$B^\bullet(M) = \bigoplus_{i=0}^{r} \bigoplus_{F \in \mathcal{L}^i} \mathbb{Q} y_F,$$

with multiplication defined as

$$y_{F_1} y_{F_2} = \begin{cases} y_{F_1 \vee F_2} & \text{if } \mathrm{rank}_M(F_1) + \mathrm{rank}_M(F_2) = \mathrm{rank}_M(F_1 \vee F_2) \\ 0 & \text{else.} \end{cases}$$

Theorem 1 (Huh-Wang [2, Theorem 6])**.** *Let $L = \sum_{i \in E} y_i$. Then for $p < r/2$,*

$$L^{r-2p} : B^p(M) \to B^{r-p}(M)$$

is injective when M is realizable over some field.

This implies the cardinality part of the top heavy conjecture since $\dim(B^p(M)) = |\mathcal{L}^p|$. The existence of ι comes from the matrix of L^{r-2p} in the standard bases of $B^p(M)$ and $B^{r-p}(M)$ by finding a maximal non-vanishing minor and noting that an entry is zero whenever the pair of indexing flats are incomparable.

The main idea of the proof is to realize L as the action of an ample line bundle on intersection cohomology, which has hard Lefschetz, and deduce injectivity by showing $B^\bullet(M)$ is contained in the intersection cohomology. To parallel the proof in the non-realizable case in [4], we summarize the argument in [2] using the stellahedral variety, which was introduced in [3].

The stellahedral variety X_E is a toric variety equipped with canonical morphisms $\pi_\Delta : X_E \to \mathbb{P}^n$ and $\pi_\square : X_E \to (\mathbb{P}^1)^n$. The map π_Δ is an iterated blowup of $\mathbb{P}^n$ of toric strata in the hyperplane at infinity, and the map $\pi_\square$ is an iterated blowup of $\mathbb{P}^1$ of toric strata containing the point ∞^n. A realization of M in the form of a surjection $\mathbb{C}^E \to V^\vee$ onto an r dimensional vector space V gives a closed embedding $\mathbb{P}(V \oplus \mathbb{C}) \to \mathbb{P}(\mathbb{C}^E \oplus \mathbb{C}) = \mathbb{P}^n$ whose strict transform under π_Δ is the augmented wonderful variety X_V, which, via the map $\pi_\square$, is a resolution of singularities of the matroid Schubert variety of the realization V.

As in the case of the wonderful variety of a realization of M, the chow ring $A^\bullet(X_V)$ only depends on the matroid M and was introduced in [3] as the augmented chow ring of M. It is shown in [3] that the subring of $A^\bullet(X_V)$ generated by the hyperplane pullback classes coming from $(\mathbb{P}^1)^n$ is isomorphic to $B^\bullet(M)$, where the y_i are identified with these hyperplane pullbacks. One checks that the cycle class map to singular cohomology is an isomorphism for the relevant varieties above. The main theorem is obtained from the following.

Let $f : X_1 \to X_2$ be a morphism of projective varieties with X_1 being r dimensional and smooth and f birational onto its image. Let $B_f^\bullet$ be the image of $H^\bullet(X_2)$ in $H^\bullet(X_1)$ under the pullback of f in singular cohomology.

Proposition 2 (Huh-Wang [2, Proposition 12])**.** *For any ample L on X_2, the multiplication map*

$$L^{r-2p} : B_f^p \to B_f^{r-p}$$

is injective for $p < r/2$.

This is shown using the decomposition theorem in [1]. One concludes by applying this proposition to the map $f : X_V \to (\mathbb{P}^1)^n$ using that $B_f^\bullet$ is identified with $B^\bullet(M)$ and the action of L, the sum of the hyperplane classes on $(\mathbb{P}^1)^n$, is identified with multiplication by $\sum_{i\in E} y_i \in B^\bullet(M)$.

Remark 3. *While we state the result using singular cohomology, one replaces singular cohomology with étale cohomology to prove the result for matroids realized over an arbitrary field.*

Remark 4. *In the non-realizable case, the injectivity statement for L on $B^\bullet(M)$ is proved in* [4] *using a combinatorial version of intersection cohomology that one can define for any matroid via the augmented chow ring of M.*

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Lorentzian fans

Tracy Chin

In this talk, we survey the theory of Lorentzian fans established by Ross in [2] and the connection to the theory of Lorentzian polynomials on cones established by Brändén and Leake in [1].

First, we describe the set up for Lorentzian fans outlined in [2]. Let $\Sigma \subseteq V$ be a d-dimensional simplicial fan with a marking $u = (u_\rho : \rho \in \Sigma(1))$, where each u_ρ is a point in relint(ρ). Then the *Chow ring* is given by

$$A^\bullet(\Sigma, u) = \frac{\mathbb{R}[x_\rho : \rho \in \Sigma(1)]}{I_\Sigma + J_{\Sigma,u}},$$

where

$$I_\Sigma = \langle x_{\rho_1} \dots x_{\rho_k} | \{\rho_1, \dots, \rho_k\} \not\subseteq \sigma \text{ for any } \sigma \in \Sigma \rangle,$$

$$J_{\Sigma,u} = \left\langle \sum_{\rho \in \Sigma(1)} \phi(u_\rho) x_\rho \middle| \phi \in V^\vee \right\rangle.$$

We note that this is a graded ring, since both of these ideals are homogeneous. Moreover, $A^1(\Sigma, u)$ is isomorphic to the space of piecewise linear maps on Σ modulo the space of linear functions, and we refer to this as the space of *tropical divisors* on Σ, denoted $D(\Sigma, u)$. We say a divisor D is (strictly) convex if for every cone $\tau \in \Sigma$, there exists a piecewise linear representative of D that vanishes on τ and is nonnegative (resp. positive) on $\sigma \setminus \tau$ whenever $\sigma \succ \tau$. The set of strictly convex divisors on Σ form a convex cone which we denote $K(\Sigma)$.

For each graded piece, the dual vector space $A^k(\Sigma, u)^\vee$ is isomorphic to the space of *Minkowski k-weights.* These can also be described as maps $\omega : \Sigma(k) \to \mathbb{R}$ satisfying a balancing condition: for all $\tau \in \Sigma(k-1)$,

$$\sum_{\substack{\sigma \in \Sigma(k) \\ \sigma \succ \tau}} \omega(\sigma) u_{\sigma \setminus \tau} \in \text{span}(\tau).$$

We call a Minkowski weight positive (resp. nonnegative) if it takes positive (resp. nonnegative) values. A *tropical d-fan* is a marked simplicial d-fan together with a positive Minkowski d-weight.

There is a natural map $A^1(\Sigma, u) \times A^k(\Sigma, u)^\vee \to A^{k-1}(\Sigma, u)^\vee$ defined by $(x, f) \mapsto f(x \cdot -)$. Passing through the isomorphisms above gives a map $D(\Sigma, u) \times MW_k(\Sigma, u) \to MW_{k-1}(\Sigma, u)$. If we have d divisors and a Minkowski d-weight, this gives us a Minkowski 0-weight, which is just a map $\{0\} \to \mathbb{R}$. Hence, a tropical d-fan (Σ, u, ω) comes with a *mixed degree map*

$$\deg_{\Sigma,u,\omega}(D_1, \dots, D_d) = (D_1 \dots D_d \cdot \omega)(0) \in \mathbb{R}.$$

Given a tropical d-fan (Σ, u, ω) and a cone $\tau \in \Sigma$, we define the *star of Σ with respect to τ*, which is a tropical fan of dimension $d^\tau := d - \dim(\tau)$ with fan structure given by the quotient

$$\Sigma^\tau = \frac{\text{subfan of } \Sigma \text{ generated by } \{\sigma \in \Sigma(d) : \sigma \succ \tau\}}{\text{span}(\tau)}$$

and with markings u^τ and a positive Minkowski d^τ-weight ω^τ inherited from Σ.

Equipped with these definitions, we are finally ready to define Lorentzian fans. We say that a tropical d-fan is *Lorentzian* if $K(\Sigma) \neq \emptyset$ and for all $\tau \in \Sigma$ and all $D_3, \dots, D_{d^\tau} \in K(\Sigma^\tau)$, the quadratic form $D(\Sigma) \times D(\Sigma) \to \mathbb{R}$ defined by $(D_1, D_2) \mapsto \deg_{\Sigma,u,\omega}(D_1 D_2 D_3 \dots D_{d^\tau})$ has exactly one positive eigenvalue.

Next, we describe the theory of Lorentzian polynomials on cones, as established in [1]. Recall that Lorentzianity can also be characterized in terms of directional derivatives: a d-homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ is Lorentzian if and only if for all $v_1, \dots, v_d \in \mathbb{R}^n_+$, we have that $\partial_{v_1} \dots \partial_{v_d} f > 0$ and the quadratic form $(x, y) \mapsto \partial_x \partial_y \partial_{v_3} \dots \partial_{v_d} f$ has one positive eigenvalue.

This version of the definition allows us to generalize Lorentzianity to other cones. Let $\mathcal{K} \subseteq \mathbb{R}^n$ be an open convex cone. We say a d-homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ is $\mathcal{K}$-*Lorentzian* if for all $v_1, \dots, v_d \in \mathcal{K}$, we have $\partial_{v_1} \dots \partial_{v_d} f > 0$ and the quadratic form $(x, y) \mapsto \partial_x \partial_y \partial_{v_3} \dots \partial_{v_d} f$ has one positive eigenvalue.

Astonishingly, Lorentzian fans and $\mathcal{K}$-Lorentzian polynomials, despite having seemingly disparate definitions, are actually very closely related. To describe the connection explicitly, we need to define the volume polynomial for tropical fans.

For $\rho \in \Sigma(1)$, we define $D_\rho \in D(\Sigma, u)$ to be the divisor class of the piecewise linear function ϕ taking values $\phi(u_\nu) = 1$ if $\nu = \rho$ and 0 otherwise. Then we can define the *volume polynomial* $\mathrm{Vol}_{\Sigma,u,\omega} : \mathbb{R}^{\Sigma(1)} \to \mathbb{R}$, where

$$\mathrm{Vol}_{\Sigma(u)}(z_\rho) = \deg_{\Sigma,u,\omega}\left(\left(\sum_{\rho \in \Sigma(1)} z_\rho D_\rho\right)^d\right).$$

This polynomial is the key to the bridge between Lorentzian fans and $\mathcal{K}$-Lorentzian polynomials.

Theorem 1 ([2, Proposition 3.3])**.** *If $\Sigma(u) = (\Sigma, u, \omega)$ is a tropical d-fan, then $\Sigma(u)$ is Lorentzian if and only if for all $\tau \in \Sigma$, the volume polynomial* $\mathrm{Vol}_{\Sigma(u)^\tau}$ *is $K(\Sigma^\tau)$-Lorentzian.*

Lorentzian fans also echo some other nice properties of Lorentzian polynomials. For example, we can characterize Lorentzian polynomials by checking a connectedness condition on their support and the Lorentzian signature condition on their quadratic partial derivatives.

Similarly, Lorentzian fans can be characterized in terms of a connectedness assumption and a quadratic condition.

Theorem 2 ([2, Theorem 4.1])**.** *Let $\Sigma(u)$ be a tropical d-fan such that $K(\Sigma) \neq \emptyset$. Then $\Sigma(u)$ is Lorentzian if and only if*

(i) *for all $\tau \in \Sigma$ such that* $\dim(\tau) \leq d - 2$, $|\Sigma^\tau| \setminus \{0\}$ *is connected, and*
(ii) *$\Sigma(u)^\tau$ is Lorentzian for all $\tau \in \Sigma(d-2)$.*

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Dually Lorentzian polynomials

Lukas Grund

There is a standard notion of duality for discrete polymatroids. The goal of this talk is to generalize this notion to Lorentzian polynomials. This is the work of Süß-Ross-Wannerer [1]. We consider the dual pair of polynomial rings $\mathbb{R}[x_1, \ldots, x_n]$ and $\mathbb{R}[\partial_1, \ldots, \partial_n]$ with the following action for $\alpha, \beta \in \mathbb{N}^n$

$$\partial^\alpha \circ x^\beta = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} x^{\beta-\alpha} & \text{if } \alpha \leq \beta, \\ 0 & \text{if otherwise,} \end{cases}$$

where $\alpha \leq \beta$ means that their components satisfy $\alpha_i \leq \beta_i$. We want to characterize polynomials $\partial_s \in \mathbb{R}[\partial_1, \ldots, \partial_n] \setminus \{0\}$ s.t. the action

$$\partial_s = s(\partial_{x_1}, \ldots, \partial_{x_n}) : \mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[x_1, \ldots, x_n]$$

preserves Lorentzian polynomials. For $\kappa \in \mathbb{N}^n$ we set

$$\mathbb{R}_\kappa[x_1, \ldots, x_n] = \{\text{polynomials of multidegree at most } \kappa\}.$$

Let $x = (x_1, x_2, \ldots x_n)$ and $y = (y_1, y_2, \ldots, y_m)$ be two sets of variables, and let T be a linear operator

$$T : \mathbb{R}_\kappa[x_1, \ldots, x_n] \longrightarrow \mathbb{R}_\gamma[y_1, \ldots, y_m].$$

We suppose that T is homogeneous, that is, $\deg T(x^\alpha) - \deg x^\alpha \in \mathbb{Z}$ does not depend on $\alpha \leq \kappa$. We will now assign a polynomial to every such operator T that provides us with a sufficiency criterion for T preserving Lorentzian polynomials.

Definition 1. *The* symbol *of T is the homogeneous polynomial*

$$sym_T(x, y) = \sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} T(x^\alpha) x^{\kappa - \alpha}.$$

The symbol theorem for Lorentzian polynomials [2, Theorem 3.2] states that, if the symbol of T is a Lorentzian polynomial, then T sends Lorentzian polynomials to Lorentzian polynomials. Denote with $N(x^\alpha) = \frac{1}{\alpha!} x^\alpha$ the normalization operator. The following definition can be seen as a generalization of polymatroid duality.

Definition 2. *For an element $s \in \mathbb{R}_\kappa[x_1, \ldots, x_n]$ we set*

$$s^\vee = N(x^\kappa s(x_1^{-1}, \ldots, x_n^{-1})).$$

If $s^\vee$ is Lorentzian, then we call s dually Lorentzian.

Remark 3. *This property is independent of the choice of κ, as can be seen with* [4, Lemma 7].

This operation can be seen as a generalization of polymatroid duality. The following theorem characterizes dually Lorentzian polynomials as differential operators acting on Lorentzian polynomials. It can be shown using the symbol theorem.

Theorem 4 (Ross-Süß-Wannerer [1, Theorem 4.4])**.** *Let $s \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial. Then ∂_s preserves Lorentzian polynomials $f \in \mathbb{R}[x_1, \dots, x_n]$ if and only if s is dually Lorentzian.*

The set of dually Lorentzian polynomials is closed under multiplication, which follows from the equality $\partial_{fg} = \partial_f \cdot \partial_g$. Several other operations preserve dually Lorentzian polynomials, such as truncations, partial derivatives, anti-derivatives, and non-negative linear change of variables. A natural class of dually Lorentzian polynomials coming from combinatorics are Schubert polynomials. This can be shown by constructing them as covolume polynomials [4, Theorem 6].

The following theorem shows that Lorentzian polynomials also act on dually Lorentzian polynomials.

Theorem 5 (Ross-Süß-Wannerer [1, Proposition 5.4] and [3, Remark 3.5])**.** *Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial. Then ∂_f preserves dually Lorentzian polynomials $s \in \mathbb{R}[x_1, \dots, x_{2n}]$ if and only if f is Lorentzian.*

The $2n$ variables in this theorem are needed, as can be seen by doing a computation in the bivariate case $n = 2$ [3, Remark 3.4]. Lastly we want to mention a possible application of dually Lorentzian polynomials.

Remark 6. *Given convex bodies $X, Y, C_1, \dots, C_{d-2}$ in $\mathbb{R}^d$ recall the Alexandrov-Fenchel inequality from convex geometry*

$$V_d(X, Y, C_1, \dots, C_{d-2})^2 \geq V_d(X, X, C_1, \dots, C_{d-2}) \cdot V_d(Y, Y, C_1, \dots, C_{d-2}).$$

For $\alpha \in \mathbb{N}^n$ with $\sum_i \alpha_i = d - 2$ and convex bodies $C_1, \dots, C_n \subset \mathbb{R}^d$ we set $V_d(X, Y, C^\alpha) = V_d(X, Y, C_1, \dots, C_1, \dots, C_n, \dots, C_n)$ and extend this linearly to homogeneous polynomials. Given a dually Lorentzian polynomial $s \in \mathbb{R}[x_1, \dots, x_n]$ of degree $d - 2$ we get the following generalized Alexandrov-Fenchel inequality [1, Theorem 8.8]

$$V_d(X, Y, s(C_1, \dots, C_n))^2 \geq V_d(X, X, s(C_1, \dots, C_n)) \cdot V_d(Y, Y, s(C_1, \dots, C_n)).$$

This generalized Alexandrov-Fenchel inequality holds similarly for nef divisors on a projective variety or for mixed discriminants.

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Volume and covolume polynomials

Matěj Doležálek

Definition 1. *Let us fix an algebraically closed field k. Given a collection $D = (D_1, \dots, D_n)$ of Cartier divisors on a d-dimensional irreducible projective variety Y over k, we may define its volume polynomial as*

$$f_D(x) := \frac{1}{d!} \deg_Y \left((x_1 D_1 + \cdots + x_n D_n)^d \right),$$

where the d-th power is to be understood as an intersection product.

A homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ of degree d is said to be a realizable volume polynomial*, if $f = \lambda f_D$ for some $\lambda \in \mathbb{Q}_{>0}$ and a collection of* semiample *divisors D, i.e. those where there exist positive integers m_i such that each $m_i D_i$ is basepoint-free. Further, f is said to be a* volume polynomial*, if it is a limit of realizable volume polynomials.*

By reducing to the Hodge index theorem [3, Theorem V.1.9], one may prove:

Theorem 2 ([1, Theorem 4.6])**.** *Any volume polynomial is Lorentzian.*

Volume polynomials coincide with Lorentzian polynomials in $n \leq 2$ variables [6, Theorem 21] or in degree $d \leq 2$ [4, Theorem 1.8], but in general, they are a strictly smaller class. An example of an inequality satisfied by volume polynomials but not by all Lorentzian polynomials is the *reverse Khovanskii-Teissier* inequality (see e.g. [5, Example 14]).

To define covolume polynomials, we consider two polynomial rings in countably many variables $\mathbb{R}[x] := \mathbb{R}[x_1, x_2, \dots]$ and $\mathbb{R}[\partial] := \mathbb{R}[\partial_1, \partial_2, \dots]$. We denote monomials in these rings using multiindices, e.g. $x^\alpha = \prod_i x_i^{\alpha_i}$, where α is an infinite sequence of nonnegative integers with finitely many positive entries, and denote the *normalized monomials* $x^{[\alpha]} := \frac{x^\alpha}{\alpha!}$, where $\alpha! := \prod_i \alpha_i!$. With these notations, we let $\mathbb{R}[x]$ and $\mathbb{R}[\partial]$ act on one another via

$$\partial^\alpha \circ x^{[\beta]} := \begin{cases} x^{[\beta-\alpha]}, & \text{if } \alpha \leq \beta, \\ 0, & \text{else,} \end{cases} \qquad x^\alpha \cdot \partial^{[\beta]} := \begin{cases} \partial^{[\beta-\alpha]}, & \text{if } \alpha \leq \beta, \\ 0, & \text{else} \end{cases}$$

and define:

Definition 3. *A polynomial $g \in \mathbb{R}[\partial]_{\leq \mu} = \operatorname{span}\{\partial^\alpha \mid \alpha \leq \mu\}$ is a* realizable covolume polynomial *if $g(\partial) \circ x^{[\mu]}$ is a realizable volume polynomial. Further, g is a* covolume polynomial*, if it is a limit of realizable covolume polynomials.*

A later characterization of (realizable) covolume polynomials justifies that the above definition does not depend on the choice of μ.

Construction 4. *A collection of semiample divisors $D = (D_1, \dots, D_n)$ on Y may be used to construct a map $\phi : Y \to \mathbb{P}^\mu := \mathbb{P}^{\mu_1} \times \cdots \times \mathbb{P}^{\mu_n}$, denoting X the image of Y under ϕ. We view the Chow ring of $\mathbb{P}^\mu$ using the distinguished basis consisting of classes of smaller products of projective spaces $[\mathbb{P}^\alpha]$, or equivalently, intersections of pullbacks h_i of hyperplane classes from the i-th factor, denoting the collection $h = (h_1, \dots, h_n)$:*

$$A^\bullet(\mathbb{P}^\mu) = \bigoplus_{\alpha \le \mu} \mathbb{Z}[\mathbb{P}^\alpha] = \bigoplus_{\alpha \le \mu} \mathbb{Z} h^{\mu - \alpha}.$$

Then, reading the class $[X]$ with respect to the basis $[\mathbb{P}^\alpha]$ and identifying $x^{[\alpha]} \approx [\mathbb{P}^\alpha]$ yields a polynomial f which is a positive-rational multiple of f_D, hence a realizable volume polynomial. Similarly, reading $[X]$ with the convention $h^\alpha \approx \partial^\alpha$ yields a polynomial $g(\partial)$ which satisfies $g(\partial) \circ x^{[\mu]} = f(x)$, making it a realizable covolume polynomial.

Thus, one may consider realizable volume and covolume polynomials simply as those which appear – under an appropriate identification – as classes of irreducible *subvarieties in the Chow ring of some $\mathbb{P}^\mu$.*

Theorem 5 ([2, Theorems 1.3, 1.5, 1.9, Corollaries 1.4 1.6, 1.10, 2.9 and Proposition 2.8])**.** *(Realizable) volume and covolume polynomials satisfy the following:*

- *$g(\partial)$ is a (realizable) covolume polynomial if and only if $g(\partial) \circ -$ preserves (realizable) volume polynomials.*
- *$f(x)$ is a (realizable) volume polynomial if and only if $f(x) \cdot -$ preserve (realizable) covolume polynomials.*
- *(Realizable) covolume polynomials are closed under multiplication.*
- *(Realizable) volume polynomials are closed under multiplication.*
- *If $g(\partial)$ is a (realizable) covolume polynomial and A is a matrix with non-negative (rational) entries, then $g(A\partial)$ is again a (realizable) covolume polynomial.*
- *If $f(x)$ is a (realizable) volume polynomial and A is a matrix with non-negative (rational) entries, then $f(Ax)$ is again a (realizable) volume polynomial.*

The main idea of proofs of these statements is to mimic an arithmetic operation with a calculation in the Chow ring, the hard part then being to show the result may be represented, up to a positive rational scalar, by an irreducible variety. The following Lemma is crucial in facilitating this:

Lemma 6 ([2, Lemma 2.1], cf. [4, Lemma 2.6])**.** *Let X be an irreducible projective variety with a transitive action of a connected algebraic group G, let Y, Z be irreducible varieties and $\phi : Y \to X$, $\psi : Z \to X$ proper morphisms. Then for a general point $g \in G$, the irreducible components of the fiber product $gY \times_X Z$*

defined via

$$\begin{array}{ccc} gY \times_X Z & \longrightarrow & Z \\ \downarrow & & \downarrow \psi \\ gY & \xrightarrow{g\phi} & X \end{array}$$

are algebraically equivalent to each other in $Y \times Z$*. Here, we use* $g\phi : gY \to X$ *to denote the composition* $Y \xrightarrow{\phi} X \xrightarrow{g} X$*.*

As an example of applying this Lemma, when ϕ, ψ are closed immersions, the fiber product $gY \times_X Z$ becomes just $gY \cap Z$, hence the Lemma implies that the product $[Y][Z]$ is represented by any irreducible component of $gY \cap Z$ (for a general g) up to a positive rational scalar. Using this on a $Y \subset X := \mathbb{P}^\mu$ witnessing a realizable covolume polynomial and a $Z \subset X$ witnessing a realizable volume polynomial proves the first item of Theorem 5, since after identifying $x^{[\alpha]} \approx [\mathbb{P}^\alpha]$ and $\partial^\beta \approx h^\beta$, the action $\circ$ turns into the usual product in $A^\bullet(\mathbb{P}^\mu)$.

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Tautological classes

Sharon Robins

Although matroids are inherently combinatorial in nature, they admit useful interpretations using algebraic geometry, and many combinatorial properties can be studied through these geometric models. In this talk, we discuss the recent geometric model introduced by Berget, Eur, Spink, and Tseng [2], called *tautological classes of matroids*, which uses vector bundles on the permutohedral variety. The central idea is to construct natural classes in the equivariant K-ring and the Chow ring that depend only on the matroid. This new approach extends and unifies several existing geometric methods used to study matroids.

When a matroid M of rank r on the ground set $E = \{0, 1, \dots, n\}$ is realizable by a linear subspace $L \subset \mathbb{C}^E$, there is a natural morphism

$$\varphi_L : X_E \longrightarrow \mathrm{Gr}(r, E),$$

where X_E is the n-dimensional permutohedral variety and $\mathrm{Gr}(r, E)$ is the Grassmannian of r-planes in $\mathbb{C}^E$. Pulling back the universal subbundle and quotient bundle on the Grassmannian gives vector bundles S_L and Q_L on X_E, whose classes in the equivariant K-ring are called the *tautological classes*. A key point is that these classes depend only on the underlying matroid M, not on the specific realization L, and thus extend to a definition for non-realizable matroids by defining corresponding classes $[S_M]$ and $[Q_M]$ combinatorially.

One can also study the Chern classes of these vector bundles in the Chow ring of X_E and compare them with previously known geometric models. In the realizable case, the relevant geometric object is the *wonderful variety* W_L [3] of the associated hyperplane arrangement, while in the general case, one has the *Bergman fan* Σ_M [1]. Both of these define Chow classes in $A^{n-r}(X_E)$. A central theorem of [2] identifies these Chow classes with the top Chern class of the tautological quotient bundle:

$$c_{n-r}([Q_M]) = [\Sigma_M],$$

and in particular, when M is realizable, this equality becomes

$$c_{n-r}(Q_L) = [W_L].$$

Using the Chern classes of the tautological bundles and intersection formulas, we obtain the following log-concavity result.

Theorem 1 ([2]). *Let $T_M(x, y)$ be the Tutte polynomial of a matroid of rank r with ground set E. Then the coefficients of the transformation*

$$t_M(x, y, z, w) = (x+y)^{-1}(y+z)^r(x+w)^{|E|-r} T_M\left(\frac{x+y}{y+z}, \frac{x+y}{x+w}\right)$$

form a log-concave unbroken array.

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Bounded ratios for Lorentzian matrices

Chayim Lowen

The goal of this talk was to give an exposition of a very recent result of Huang, Huh, Soskin and Wang on *bounded ratios* for Lorentzian matrices (see [1]). Their proof of this result was inspired by the developments in [3] and [4] linking Lorentzian polynomials with Grassmanians over triangular hyperfields.

In this abstract, the term a_{ij} will always refer to the ij-th entry of an $n \times n$ symmetric matrix (with n possibly unspecified). We write $\mathbb{L}_n^+$ for the set of positive $n \times n$ Lorentzian matrices. The object of study is the set of *bounded ratios* for $\mathbb{L}_n^+$. These are by definition the Laurent monomials in the entries of an $n \times n$ symmetric matrix which are bounded above on the set of all $n \times n$ Lorentzian matrices. The simplest example of a bounded ratio is $a_{11}a_{22}a_{12}^{-2}$. Its boundedness follows from the *Alexandrov-Fenchel inequality* for Lorentzian matrices

$$a_{11}a_{22} \leq a_{12}^2$$

Remark 1. *Any inequality for $n \times n$ Lorentzian matrices automatically gives inequalities for all larger matrices by considering $n \times n$ principal submatrices. Thus it is not necessary to specify n when giving describing a (potential) bounded ratio.*

The natural row-column action of the symmetric group S_n on symmetric matrices induces an action on bounded ratios. E.g. $a_{33}a_{77}a_{37}^{-2}$ is also a bounded ratio.

To study bounded ratios, it is convenient to allow all real exponents.

Definition 2. *For any set X of symmetric $n \times n$ matrices, we define the* cone of bounded ratios $\mathrm{BR}(X)$ *of X to be the set of symmetric $n \times n$ real "exponent" matrices $(\epsilon_{ij})_{1\leq i,j\leq n}$ for which the monomial*

$$\prod_{1\leq i,j\leq n} a_{ij}^{\epsilon_{ij}}$$

is bounded above on X.

It is not hard to see that $\mathrm{BR}(X)$ is a convex cone in $\mathbb{R}^{n\times n}$. Bounded ratios for Lorentzian matrices correspond precisely to integral points of $\mathrm{BR}(\mathbb{L}_n^+)$. The authors of [1] give a characterization of this cone which we will now describe. A remarkable consequence of this characterization is that $\mathrm{BR}(\mathbb{L}_n^+)$ is a *rational polyhedral cone*. This means that it is the conical hull of the exponent vectors of finitely many bounded ratios.

By considering the natural action of $(\mathbb{R}_{>0})^n$ on (Lorentzian) matrices, one shows that any exponent matrix $(\epsilon_{ij})_{1\leq i,j\leq n}$ must satisfy the homogeneity condition

$$(*) \qquad \sum_{j=1}^{n} \epsilon_{ij} = 0 \qquad \text{for all } 1 \leq i \leq n$$

Observe that every exponent matrix can be *uniquely* made to satisfy $(*)$ by a modification of its diagonal entries.

Definition 3. *A* cut $S \mid T$ *of* n *is a partition*

$$\{1, \ldots, n\} = S \sqcup T$$

For a cut $S \mid T$ *we define the symmetric* $n \times n$ cut *matrix* $\delta_{S|T}$ *by*

$$(\delta_{S|T})_{ij} = \begin{cases} 0 & \text{if } \{i,j\} \subseteq S \text{ or } \{i,j\} \subseteq T \\ 1 & \text{otherwise} \end{cases}$$

The cut cone Cut_n *is the conical hull of all matrices* $\delta_{S|T}$ *over all cuts* $S \mid T$ *of* n.

The main theorem of Huang-Huh-Soskin-Wang is then

Theorem 4 ([1, Theorem B])**.** *An exponent matrix* $(\epsilon_{ij})_{1 \le i,j \le n}$ *satisfying* $(*)$ *lies in* $\mathrm{BR}(\mathbb{L}_n^+)$ *if and only if it lies in the dual cone of* Cut_n. *This means*

$$\sum_{1 \le i,j \le n} \epsilon_{ij} x_{ij} \le 0 \quad \text{for all } (x_{ij})_{1 \le i,j \le n} \in \mathrm{Cut}_n$$

A number of ingredients go into the proof. One is the result of Brändén-Huh that the set $\mathrm{PolyGr}(2,n)(\mathbb{T}_0)$ of polymatroids over the tropical hyperfield $\mathbb{T}_0$ naturally sits inside the space $\mathbb{L}_n$ of $n \times n$ Lorentzian matrices ([2, Corollary 3.16]). (This is a very special case of their result.) This formulation (in the style of [5]) is not the original one, but it allows allows one to state a kind of converse. Namely, the set $\mathbb{L}_n$ of Lorentzian matrices naturally sits inside the space $\mathrm{PolyGr}(2,n)(\mathbb{T}_2)$ of polymatroids over the triangular hyperfield $\mathbb{T}_2$.

$$\mathrm{PolyGr}(2,n)(\mathbb{T}_0) \subseteq \mathbb{L}_n \subseteq \mathrm{PolyGr}(2,n)(\mathbb{T}_2)$$

The notation suggests that $\mathrm{PolyGr}(2,n)(\mathbb{T}_0)$ and $\mathrm{PolyGr}(2,n)(\mathbb{T}_2)$ are very close. A concrete form of this claim can be deduced from a theorem of Gromov. To state it, we need the following definition

Definition 5. *Let* E *be a finite set and* $\delta \ge 0$ *a real number. A function* $f : E \times E \to \mathbb{R}$ *is said to be a* δ*-approximate tree metric if it satisfies:*

(nonnegativity)	$f(x,y) + \delta \ge 0$ *for all* $x, y \in E$
(positivity)	$f(x,y) + \delta > 0$ *for all* distinct $x, y \in E$.
(zero)	$f(x,x) \le \delta$ *for all* $x \in E$.
(symmetry)	$f(x,y) - f(y,x) \le \delta$ *for all* $x, y \in E$.
(triangle inequality)	$f(x,z) \le f(x,y) + f(y,z) + \delta$ *for all* $x, y, z \in E$.
(four-point condition)	$f(x,y) + f(z,w) \le$ $\le \max(f(x,z) + f(y,w), f(x,w) + f(y,z)) + \delta$ *for all* $x, y, z, w \in E$.

When $\delta = 0$, *this recovers the definition of a* tree metric.

The following theorem is due to Gromov. See [7, Section 6.1]. See also [6, Chapter 2] for an exposition.

Theorem 6. *A* δ*-approximate tree metric on a set of size* n *is* $C(n)\delta$*-close (e.g. in the* L^∞*-norm) to some tree metric, for some constant* $C(n)$ *depending only on* n.

Proof. We apply Gromov's theorem to the function $\tilde{f}$ defined by

$$\tilde{f}(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{2}(f(x,y) + f(y,x)) + \delta & \text{otherwise} \end{cases}$$

which is a $3\delta/2$-hyperbolic metric in the sense of Gromov. □

The elements of $\mathrm{PolyGr}(2,n)(\mathbb{T}_0)$ give rise to tree metrics on $E = \{1,\dots,n\}$, as explained in [8, Section 4.3]. In a similar sense, the elements of $\mathrm{PolyGr}(2,n)(\mathbb{T}_2)$ give rise to δ-approximate tree-metrics on E for $\delta = 2\log 2$ (see [1, Corollary 3.4]). Applying Gromov's theorem, one sees that the sets $\mathrm{PolyGr}(2,n)(\mathbb{T}_0)$ and $\mathrm{PolyGr}(2,n)(\mathbb{T}_2)$ have the same bounded ratios. Since $\mathbb{L}_n$ lies between the two, it also has the same bounded ratios. This chain of reasoning concludes with the observation that the cut cone is precisely the convex hull of the set of all tree metrics on $\{1,\dots,n\}$. This leads to a proof of 4.

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Triangular hyperfields I

Sergio Cristancho

The goal of this talk is to understand the topology of Lorentzian polynomials and their relation to Grassmannians through the lens of triangular hyperfields. The content of this talk is primarily based on the recent preprint of Baker, Huh, Kummer and Lorscheid [1].

Throughout, the space of square-free homogeneous polynomials of degree d in n variables and the space of square-free Lorentzian polynomials within will be denoted by $\mathrm{H}(d,n)_{\boxtimes}$ and $\mathrm{L}(d,n)_{\boxtimes}$ respectively. Brändén [2, Thm. 3.2] proved that the *projectivization* $\mathbb{P}\mathrm{L}(d,n)_{\boxtimes}$ is homeomorphic to a closed Euclidean ball of dimension $\dim \mathbb{P}\mathrm{H}(d,n)$. Furthermore, the space $\mathrm{L}(d,n)_{\boxtimes}$ has a decomposition

$\mathrm{L}(d,n)_{\boxtimes} = \sqcup_M \mathrm{L}_M$ into *matroid strata*, where M iterates over all matroids of rank d on n elements. For M is a matroid of rank d on n elements with set of bases $\mathcal{B}(M)$ is defined as

$$\mathrm{L}_M = \left\{ f = \sum_{|S|=d} a_S x^S \in \mathrm{L}(d,n)_{\boxtimes} \colon a_S \neq 0 \iff S \in \mathcal{B}(M) \right\},$$

where $x^S = \prod_{i \in S} x_i$. One of the main objectives of [1] is to understand the topology of the projectivization $\mathbb{P}\mathrm{L}_M$ of the stratum L_M.

It is convenient to study $\mathbb{P}\mathrm{L}_M$ in log-coordinates, i.e. the image $\log \mathbb{P}\mathrm{L}_M$ under the coordinate-wise logarithm map $\log|\cdot| : \mathbb{P}\mathrm{H}_M \to \mathbb{R}^{\mathcal{B}(M)}/\mathbb{R}\mathbf{1}$. By a theorem of Brändén [4], the space $\log \mathbb{P}\mathrm{L}_M$ is contained in a linear subspace $V_M/\mathbb{R}\mathbf{1}$ which depends only on M. The stratum $\log \mathbb{P}\mathrm{L}_M$ contains a distinguished piecewise linear subset: the *Dressian* $\mathrm{Dr}_M \subset \mathbb{R}^{\mathcal{B}(M)}/\mathbb{R}\mathbf{1}$ of classes of *valuated matroids* supported on M up to addition of a constant. Concretely, by Brändén and Huh [5, Thm. 3.14], if $\nu : \mathcal{B}(M) \to \mathbb{R}$ is a valuated matroid, the polynomial $f_\nu(x) = \sum_S e^{\nu(S)} x^S$ is a Lorentzian polynomial, and the projective class of f_ν depends only on ν up to addition of a constant.

Theorem 1 (Thm. 1.7, [1])**.** *For any matroid M, let B is the closed Euclidean ball in $V_M/\mathbb{R}\mathbf{1}$. Then $\mathbb{P}\mathrm{L}_M$ is homeomorphic to $B \backslash (\partial B \cap \mathrm{Dr}_M)$. In particular, $\mathbb{P}\mathrm{L}_M$ is a manifold with boundary of dimension* $\dim V_M - 1$.

Now, the decomposition into matroid strata of $\mathbb{P}\mathrm{L}(d,n)_{\boxtimes}$ is analogous to the decomposition of the Grassmannian $\mathrm{Gr}(d,n)(k)$ over a field k into *thin Schubert cells* Gr_M. In contrast to $\mathbb{P}\mathrm{L}_M$, the thin Schubert cell Gr_M is non-empty if and only if M is realizable over k, and can exhibit arbitrary singularities (see [3]). Nevertheless, one can concretely relate $\mathbb{P}\mathrm{L}_M$ to a thin Schubert cell by allowing the coefficients to take values on *hyperfields*.

For a parameter $0 < q < \infty$, the *q-triangular hyperfield* $\mathbb{T}_q = (\mathbb{R}_{\geq 0}, \cdot, \boxplus)$ is the set $\mathbb{R}_{\geq 0}$ endowed with the usual multiplication, making $\mathbb{R}_{>0}$ its group of units, and the hyperadditive structure:

$$a_1 \boxplus_q \cdots \boxplus_q a_n = 0 \iff a_1^{1/q}, \ldots, a_n^{1/q} \text{ form a (possibly degenerate) } n\text{-agon.}$$

The family of triangular hyperfields $(\mathbb{T}_q)_{0<q<\infty}$ was introduced by Viro [6] with a view towards tropicalization. When $p < q$, we have that $\mathbb{T}_p \hookrightarrow \mathbb{T}_q$ in the category of hyperfields, making it into a directed system with two limiting objects: the familiar *tropical hyperfield* $\mathbb{T}_0$, and the *degenerate triangular hyperfield* $\mathbb{T}_\infty$. Also, for any $0 < q < \infty$, $\mathbb{T}_1$ is isomorphic to $\mathbb{T}_q$ by means of the map $a \mapsto a^q$.

In this context, we can define the Grassmannian over $\mathbb{T}_q$ for $0 \leq q \leq \infty$ as:

$$\mathrm{Gr}(d,n)(\mathbb{T}_q) = \left\{ (p_S)_S \in \mathbb{R}_{\geq 0}^{\binom{n}{d}} \colon (p_S)_S \text{ satisfies the } \textit{Plücker relations} \text{ over } \mathbb{T}_q \right\} / \mathbb{R}_{>0},$$

and the thin Schubert cell $\mathrm{Gr}_M(\mathbb{T}_q)$ supported on a matroid M as

$$\mathrm{Gr}_M(\mathbb{T}_q) = \{ (p_S)_S \in \mathrm{Gr}(d,n)(\mathbb{T}_q) \colon p_S \neq 0 \iff S \in \mathcal{B}(M) \}.$$

Baker and Bowler [7] proved that the Dressian supported on M is precisely $\mathrm{Gr}_M(\mathbb{T}_0)$ in log-coordinates, in symbols $\log \mathrm{Gr}_M(\mathbb{T}_0) = \mathrm{Dr}_M$. One can also see that the vector space $V_M/\mathbb{R}\mathbf{1}$ happens to be equal to $\log \mathrm{Gr}_M(\mathbb{T}_\infty)$, hence:

Theorem 2 (Thm 1.8, [1]). *For every matroid M, we have natural inclusions*

$$\mathrm{Gr}_M(\mathbb{T}_0) \hookrightarrow \mathbb{P}\mathrm{L}_M \hookrightarrow \mathrm{Gr}_M(\mathbb{T}_\infty).$$

Since $\mathbb{T}_0 \hookrightarrow \mathbb{T}_q \hookrightarrow \mathbb{T}_\infty$ for any $0 < q < \infty$, we always have that

$$\mathrm{Gr}_M(\mathbb{T}_0) \hookrightarrow \mathrm{Gr}_M(\mathbb{T}_q) \hookrightarrow \mathrm{Gr}_M(\mathbb{T}_\infty).$$

From this perspective, $\log \mathrm{Gr}_M(\mathbb{T}_q)$ is an *amoeba* over Dr_M, the tropicalization of the thin Schubert cell supported on M. As it turns out the matroid stratum is homeomorphic to any one of these amoebas.

Theorem 3 (Thm. 1.9, [1]). *For every $0 < q < \infty$ and for any matroid M, the spaces $\mathbb{P}\mathrm{L}_M$ and $\mathrm{Gr}_M(\mathbb{T}_q)$ are homeomorphic.*

Remark 4. *All the results here also hold for general Lorentzian polynomials, not only square-free, in which case matroids are replaced with* polymatroids*, also known as M-convex sets, and the Grassmannian is replaced with a representation space for polymatroids.*

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Triangular hyperfields II

Daoji Huang

This is the second talk of the two-part presentations on Triangular Hyperfields and Lorentzian Polynomials, based on the preprint by Matthew Baker, June Huh, Mario Kummer, and Oliver Lorscheid [1]. In Triangular Hyperfields I, we have seen that the projective space $\mathbb{P}\mathrm{L}_M$ of Lorentzian polynomials supported on a matroid M is homeomorphic to $B \setminus (\partial B \cap \mathrm{Dr}_M)$, where B is a closed ball in the vector space $V_M/(1, \ldots, 1)$ where $V_M = \log \mathrm{Gr}_M(\mathbb{T}_\infty)$, and the $\mathrm{Dr}_M = -\log \mathrm{Gr}_M(\mathbb{T}_0)$ is the Dressian of the matroid M. In other words, there is a compatification of $\mathbb{P}\mathrm{L}_M$

such that the result is a closed Euclidean ball. In contrast to this compatification, one may take the closure $\overline{\mathbb{PL}}_M$ of $\mathbb{PL}_M$ inside the projective space $\mathbb{PH}_{d,n}$ of homogeneous polynomials of degree $d = \mathrm{rk}M$ and n variables, where n is the size of the ground set of M. Petter Brändén proved that when M is a uniform matroid, the space $\overline{\mathbb{PL}}_M$ is homeomorphic to a closed Euclidean ball, and the same is true when restricted to stable polynomials [2]. He then asked the question whether this is true for arbitrary matroids. The authors answered this question to the negative.

Theorem 1. *The following statements hold.*

1. *When* $\underline{\mathrm{Gr}}_M(\mathbb{T}_\infty)$ *is a singleton,* $\overline{\mathbb{PL}}_M$ *is homeomorphic to a ball.*
2. *When M is a* rigid *matroid, i.e.,* $\underline{\mathrm{Gr}}_M(\mathbb{T}_0)$ *is a singleton, the topological Euler characteristic* $\chi(\overline{\mathbb{PL}}_M)$ *is 1.*
3. *There exists a matroid M with* $\dim \underline{\mathrm{Gr}}_M(\mathbb{T}_0) = 1$ *and* $\chi(\overline{\mathbb{PL}}_M) \neq 1$.
4. *There is a rigid matroid M such that the closure of the space of stable polynomials* $\overline{\mathbb{PS}}_M$ *is nonempty and not homeomorphic to a ball.*

One example of a matroid for part (3) of the theorem is the elliptic matroid $\mathcal{T}_{11}$, the rank 3 matroid on $\{1, 2, \ldots, 11\}$ with $\{i, j, k\}$ form a basis whenever $i + j + k \not\equiv 0 \pmod{11}$. One example of a matroid for part 4 is the Betsy Ross matroid, a matroid of rank 3 on 11 elements.

Part(2) and (3) of Theorem 1 were derived from an explicit formula computing the topological Euler characteristic of the space $\overline{\mathbb{PL}}_M$ for certain matroids M.

Recall that the matroid base polytope BP_M for a matroid M is the convex hull of the vectors $\{\sum_{i\in B} e_i : B \text{ is a basis of } M\}$. A matroid M' is an initial matroid of M if it corresponds to a cell of a regular matroid polytope subdivision of M.

Theorem 2. *When* $\log \underline{\mathrm{Gr}}_M(\mathbb{T}_0)$ *consists of m rays, namely* $\dim \log \underline{\mathrm{Gr}}_M(\mathbb{T}_0) = 1$, *we have*

$$\chi(\overline{\mathbb{PL}}_M) = \sum_{i=0}^{n} (-1)^i \left(g_i + \sum_{j=0}^{m} (1-j) f_{ij} \right),$$

where g_i is the number of initial matroids M' of M such that $\dim \mathrm{BP}_{M'} = i$ *and* $\mathrm{BP}_{M'}$ *is not a face of* BP_M, *and f_{ij} is the number of initial matroids M' such that* $\dim \mathrm{BP}_{M'} = i$ *and the image of the restriction map* $\mathrm{res} : \log \underline{\mathrm{Gr}}_M(\mathbb{T}_0) \to \log \underline{\mathrm{Gr}}_{M'}(\mathbb{T}_0)$ *that maps ρ to $\rho|_{M'}$ by forgetting the terms of ρ that are not supported on the bases of M' contains j rays.*

To apply the formula in Theorem 2, we compute the set of all possible matroid polytope subdivisions of M. Every matroid polytope subdivision is induced by a point in the reduced Dressian. We therefore first compute the reduced Dressian of M, and then for each ray the corresponding matroid polytope subdivision. Finally, for every initial matroid M', we record the dimension of $\mathrm{BP}_{M'}$ and if $\mathrm{BP}_{M'}$ is a face of BP_M, the number of rays in the image of the restriction map.

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