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Mini-Workshop: The Yang–Baxter Equation and Representations of Braid Groups

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Abstract. The Yang–Baxter equation is a famous equation in mathematics and mathematical physics. It plays a central role in several areas of mathematics, including algebra, topology, and quantum field theory. The aim of the workshop is to review recent developments in areas where the Yang–Baxter equation is crucial to discuss new research directions and ideas for addressing open problems.

Mathematics Subject Classification (2020): 16T25, 20F36.

Introduction by the Organizers

One of the central objects in the theory of quantum groups, braided tensor categories and low-dimensional topology is the Yang–Baxter equation (YBE). Given a vector space V, a solution to the YBE is an automorphism $R \in \mathrm{Aut}(V^{\otimes 2})$ such that

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R),$$

as endomorphisms of $V^{\otimes 3}$.

In this workshop we are interested in two broad themes:

1. Classification problems. These include, among others:

- *Set-theoretic solutions*, corresponding to permutation-type R-matrices, where R is encoded by a bijection on an underlying set.

- *Unitary, involutive or Hecke-type solutions*, where R satisfies additional constraints such as $R^\dagger R = I$, $R^2 = I$ or specific spectral properties.
- A *categorical approach*, where solutions to the YBE are understood in terms of monoidal functors from the category of braids to another monoidal category.

2. Applications. Solutions of the YBE give rise to a wide range of constructions and applications:

- *Nichols algebras*: understanding when the Nichols algebra attached to a braided vector space is finite-dimensional.
- *Representations of the braid group*: studying the images of the braid group representations arising from R-matrices and their relation to braided fusion categories and topological quantum computation.
- *Topological applications*: constructing link invariants and topological quantum field theories (TQFTs), as well as lifting braid group representations to higher-dimensional objects (e.g. loop braid groups).

Open problems and directions

We collect here a (non-exhaustive) list of open problems related to solutions of the Yang–Baxter equation, grouped by theme. The formulations are intentionally broad, aiming to indicate directions rather than precise conjectures.

Nichols algebras and braided tensor categories

(1) *Nichols algebras over charge-conserving Yang–Baxter operators.* What can be said about the structure of the Nichols algebras associated to such braidings? Are these braidings in any sense limits of rigid braidings?

(2) *Nichols algebras as objects in braided tensor categories.* What is the "right" general definition of a Nichols algebra object inside an arbitrary braided tensor category? To what extent does the usual root system and Weyl groupoid theory of Nichols algebras extend to this general categorical setting?

(3) *Yang–Baxter operators with spectral parameter.* Let V be a (possibly infinite-dimensional) vector space and

$$R : \mathbb{R} \longrightarrow \mathrm{GL}(V \otimes V)$$

be a family of operators satisfying the Yang–Baxter equation with spectral parameter

$$R(x)_1 R(x+y)_2 R(y)_1 \;=\; R(y)_2 R(x+y)_1 R(x)_2$$

for all $x, y \in \mathbb{R}$, together with suitable regularity conditions. Can one develop algebraic or categorical methods to study such spectral-parameter solutions in a unified way? How does the structure of the associated braided (or quasi-braided) categories depend on analytic properties of $R(x)$?

(4) *Sklyanin algebras and Nichols algebras.* Are Sklyanin algebras (or suitable generalizations of them) realizable as Nichols algebras with respect to Yang–Baxter operators with spectral parameter?

Geometry and classification of linear Yang–Baxter operators

(1) *Geometry of the space of Yang–Baxter operators.* For a fixed $n = \dim V$, a linear solution of the Yang–Baxter equation is an operator

$$R \in \mathrm{End}(V \otimes V) \cong \mathrm{M}_n(\mathbb{C})$$

satisfying a system of polynomial equations. Thus the set of all such R forms an algebraic variety. What can be said about the geometry of this variety (irreducible components, dimensions, singularities, etc.)?

(2) *Classification of* 3×3 *Yang–Baxter operators.* Can one classify Yang–Baxter operators up to equivalences such as Drinfeld twist, equivalence induced by isomorphic braid group representations, conditions on the spectrum of R (eigenvalues and multiplicities)? Does such classifications have an impact at a set-theoretic level?

Braid group images and quantum computational aspects

(1) *Unitary Yang–Baxter operators and virtually abelian images.* Given a unitary solution R of the Yang–Baxter equation, is the image

$$\rho_R(B_n) \subseteq \mathrm{U}(V^{\otimes n})$$

is virtually abelian, i.e. it contains an abelian normal subgroup of finite index? This would imply that unitary linear solutions are extremely rare.

(2) *Computational power of a single Yang–Baxter gate.* Let R be a fixed Yang–Baxter operator and consider the gate set consisting solely of R acting on tensor powers (i.e. the image of the associated braid group representations $\rho_R(B_n)$). What is the quantum computational power of such a gate set? For which R does one obtain universal (or "braiding universal") quantum computation? How does this relate to known models (for example, the $(2, 5)$ model appearing in the quantum Hall effect) and to braided fusion categories that might be associated with R?

Set-theoretic solutions, skew braces and associated algebras

(1) *Images of braid groups from set-theoretic solutions.* Let (X, r) be a set-theoretic solution of the Yang–Baxter equation on a set X, and consider the induced action of B_n on X^n. What can be said about the image of B_n in $\mathrm{Aut}(X^n)$?

(2) *Internal automorphism objects in the category of skew braces.* In the category of groups, split extensions

$$0 \to X \to A \to B \to 0$$

are classified by homomorphisms $B \to \mathrm{Aut}(X)$, and similarly, derivations play this role in the category of Lie algebras. Does there exist an internal Aut-object $[X]$ in the category of skew braces that classifies split extensions (or actions) via a universal property?

(3) *Indecomposable set-theoretic solutions and recomposition.* Is there a systematic way to "recompose" or stitch together indecomposable set-theoretic solutions of the YBE to obtain new ones?
(4) *Structure algebras of set-theoretic solutions.* When is the structure algebra of a finite set-theoretic solution of the YBE is left Noetherian? When it is PI? When does it contain a non-commutative free subalgebra?
(5) *Group rings of skew braces of abelian type.* Let B be a finite skew brace of abelian type. What can be said about the integral unit group of the multiplicative group of B?
(6) *Classification up to cabling.* Another natural equivalence on set-theoretic YBE solutions comes from cabling constructions. Can one classify solutions up to cabling, or at least describe invariants that distinguish cabling-equivalence classes?

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Noetherian Hopf algebras

Nicolás Andruskiewitsch

1. Introduction

There have been advances in the classification of Hopf algebras with finite Gelfand-Kirillov dimension but the analogous question for Noetherian Hopf algebras encounters at the very start difficult classical open problems, namely:

Conjecture 1. *If the enveloping algebra of a Lie algebra* $\mathfrak{g}$ *is Noetherian, then* $\mathfrak{g}$ *is finite dimensional.*

Conjecture 2. *If the group algebra of a group* G *is Noetherian, then* G *is polycyclic-by-finite.*

There has been some activity in these conjectures. Concerning Conjecture 1, a breakthrough result was obtained by Sierra and Walton in 2013.

Theorem 3. [5] *The enveloping algebras of the Witt algebra* $W(1)$ *and the positive Witt algebra* W_+ *are not Noetherian.*

We will report on the following theorem, based on [4] and Theorem 3:

Theorem 4. [2]. *The universal enveloping algebra of an infinite-dimensional simple* $\mathbb{Z}^n$*-graded Lie algebra is not Noetherian.*

As for Conjecture 2, we will discuss the basics of the following result:

Theorem 5. [6] *If* $\Bbbk G$ *is Noetherian, then the group* G *is amenable.*

The proof is strongly dependent of [3].

Recent progress on pointed Hopf algebras with finite Gelfand-Kirillov dimension is presented in [1], which also contains a list of problems including many from previous surveys by K. Brown, K. Goodearl, S. Skryabin and others.

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Recent advances on Nichols algebras over finite simple groups

Giovanna Carnovale
(joint work with N. Andruskiewitsch)

A Nichols algebra is a (bi)algebra naturally attached to a pair (V, c), called a braided vector space, where V is a (complex) vector space and $c \in \mathrm{GL}(V \otimes V)$ satisfies the braid equation (YBE):

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c) \tag{1}$$

on $V^{\otimes 3}$. Examples of such algebras are: the symmetric algebra, the exterior algebra, the positive part of a quantized enveloping algebra, some (and conjecturally all) Fomin-Kirillov algebras.

A special family of braided vector spaces arises naturally from any Yetter-Drinfeld module V over a group G. A G-graded G-module $V = \bigoplus_{g \in G} V_g$ is called a Yetter-Drinfeld module if $h \cdot V_g = V_{hgh^{-1}}$ for any $g, h \in G$. The support of the grading is thus a union of conjugacy classes. Such a module comes equipped with a linear map $c \in \mathrm{GL}(V \otimes V)$ satisfying (1), defined by setting $c(v \otimes v') = g \cdot v' \otimes v$ for any $v \in V_g$ and any $v' \in V$.

A crucial step in the classification program of finite-dimensional pointed Hopf algebras is to identify the braided vector spaces, and in particular, the Yetter-Drinfeld modules yielding a finite-dimensional Nichols algebra.

For abelian groups the problem has been solved in [14]. If V decomposes as a non-trivial direct sum of Yetter-Drinfeld modules a full answer was given in [15]. In this case there are very strong constraints on the support of V and G. A recent breakthrough was obtained in [9] where the case of solvable groups is addressed, and again the condition on V to have a finite-dimensional Nichols algebra forces very tight restrictions on G and on the support of the grading.

When G is non-abelian and simple, it is conjectured that no non-trivial Yetter–Drinfeld module yields a finite-dimensional Nichols algebra. This is confirmed for the alternating groups, [7] and for most sporadic gorups, [8, 10, 12, 13].

Simple groups of Lie type were addressed in the series of papers [1, 2, 3, 4, 5, 11, 6]. Relying on the results in [15], the conjecture was confirmed for $\mathrm{PSL}_n(q)$ for $n > 3$ or $n = 3$ and $q > 2$; for $\mathrm{PSp}_{2n}(q)$ for $n > 2$ or $q > 7$, for Suzuki and Ree groups, and for groups coming from algebraic groups in even characteristic when $w_0 = -\mathrm{id}$. In addition, it was proved that if the Nichols algebra of a Yetter-Drinfeld module over a Chevalley or a Steinberg group is finite-dimensional, then the support of V is a semisimple conjugacy class.

In recent work we developed new tools that allowed to conclude the analysis for all finite simple Chevelley groups except from $\mathrm{PSL}_2(q)$ with q odd. The stategy involves a general method to address the modules supported on semisimple conjugacy classes in any Chevalley or Steinberg group, and a new criterion for arbitrary groups, based on [9], to give further restrictions on the support of a Yetter-Drinfeld module whose associated Nichols algebra is finite-dimensional.

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Hopf algebra perspective on set-theoretical solutions

Ilaria Colazzo

(joint work with Geoffrey Janssens)

We develop a Hopf algebraic framework for finite bijective set-theoretic solutions of the pentagon equation and show that the resulting Hopf algebras always admit a positive basis in the sense of Lu–Yan–Zhu [5]. More precisely, starting from a finite bijective set-theoretic solution of the (reversed) pentagon equation, we construct associated Hopf algebras $H_l(s)$ and $H_r(s)$ and prove that there exist bases in which all structure constants of the multiplication, comultiplication and antipode are non-negative. Our approach uses only some general results from the theory of finite set-theoretic pentagon solutions developed in [1, 2], together with the connection between pentagon solutions and Hopf algebras due to Davydov [3] and Militaru [6]. Conversely, we show that finite-dimensional Hopf algebras with the positive basis property arise from finite bijective set-theoretic solutions of the pentagon equation.

We consider the (reversed) pentagon equations

$$Z_{12}Z_{13}Z_{23} = Z_{23}Z_{12} \quad \text{(RPE)}, \qquad Z_{23}Z_{13}Z_{12} = Z_{12}Z_{23} \quad \text{(PE)},$$

for Z acting on $S \times S$ (sets), $V \otimes V$ (vector spaces), or as an element of $A \otimes A$ (Hopf algebras). A *set-theoretic* solution of PE or RPE is a pair (S, s) with a map $s\colon S \times S \to S \times S$ satisfying the corresponding identity in S^3 with the usual conventions for s_{12}, s_{13}, s_{23}.

The first ingredient is the Hopf algebra construction attached to a solution to the PE. Following Davydov [3] and Militaru [6], an algebra solution (A, R) of the RPE, with

$$R = \sum R^{(1)} \otimes R^{(2)} \in A \otimes A,$$

gives rise to two coefficient spaces

$$R_{(l)} = \Big\{ \sum a^*(R^{(2)})\, R^{(1)} \;\Big|\; a^* \in A^* \Big\}, \qquad R_{(r)} = \Big\{ \sum a^*(R^{(1)})\, R^{(2)} \;\Big|\; a^* \in A^* \Big\},$$

which can be endowed with canonical Hopf algebra structures. When (A, R) is obtained by linearising a set-theoretic solution (S, s), we denote the corresponding Hopf algebras by $H_l(s)$ and $H_r(s)$.

The second ingredient is a collection of key lemmas on finite bijective set-theoretic solutions of the pentagon equation. In [1, 2] it is shown, in particular, that if (S, s) is a finite bijective set-theoretic PE (or RPE) solution, then the underlying semigroup S is a left group, i.e. there is a decomposition

$$S \cong E \times G,$$

where G is a group and E is a set, and the maps θ_x or ψ_y when we write the solutions as

$$s(x, y) = (xy, \theta_x(y)) \quad \text{or} \quad s(x, y) = (\psi_y(x), y \circ x)$$

are either the identity or fixed-point free. These are precisely the structural results from [1, 2] that we use. This description is the starting point for controlling the combinatorics of the Hopf algebras associated to (S, s).

A Hopf algebra H over k is said to have the *positive basis property* (in the sense of Lu–Yan–Zhu [5]) if there exists a k-basis of H with respect to which all structure constants of the multiplication, comultiplication and antipode are non-negative. Lu, Yan and Zhu introduced this notion in their study of set-theoretic Yang–Baxter solutions and associated Hopf algebras, and showed that it leads to strong structural constraints. Our main result extends this perspective to the pentagon equation.

Theorem. *Let (S, s) be a finite bijective set-theoretic solution of the reversed pentagon equation. Then the associated Hopf algebras $H_l(s)$ and $H_r(s)$ admit k-bases with respect to which all structure constants of the multiplication, comultiplication and antipode are non-negative. In other words, $H_l(s)$ and $H_r(s)$ have the positive basis property.*

Finally, we relate this construction to the set-theoretical Yang–Baxter equation. The Hopf algebras $H_l(s)$ and $H_r(s)$ admit Yetter–Drinfeld modules and thus produce set-theoretical YBE solutions in the sense of Lu–Yan–Zhu. The existence of a positive basis makes these YBE solutions combinatorially accessible and suggests a close parallel between the PE and YBE cases.

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Homotopy finite spaces, once-extended TQFTs, and representations of the braid and loop braid group.

João Faria Martins

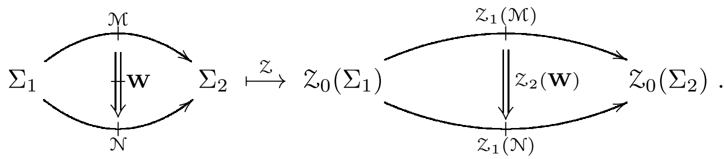
Let $n \in \mathbb{Z}_{\geq 0}$. An *$n+1$-cobordism*, $\mathcal{M}\colon \Sigma_1 \nrightarrow \Sigma_2$, in full $\mathcal{M} = (\,\Sigma_1 \xrightarrow{i_1} M \xleftarrow{i_2} \Sigma_2\,)$, is a diagram of smooth manifolds and maps, where Σ_1, Σ_2 are closed n-manifolds, M is an $n+1$-manifold, with boundary, and the resulting map $\langle i_1, i_2\rangle\colon \Sigma_1 \sqcup \Sigma_2 \to M$ factors through ∂M and gives a diffeomorphism $\Sigma_1 \sqcup \Sigma_2 \cong \partial M$.

An extended $n+2$-cobordism, $\mathbf{W}\colon \mathcal{M} \Rrightarrow \mathcal{N}$, from $n+1$-cobordism, $\mathcal{M}\colon \Sigma_1 \nrightarrow \Sigma_2$, to $\mathcal{N}\colon \Sigma_1 \nrightarrow \Sigma_2$ is a diagram, again of smooth manifolds, as in the RHS below,

$$\Sigma_1 \overset{\mathcal{M}}{\underset{\mathcal{N}}{\Downarrow \mathbf{W}}} \Sigma_2 \quad = \quad \begin{array}{ccccc} \Sigma_1 & \xrightarrow{i_1} & M & \xleftarrow{i_2} & \Sigma_2 \\ \downarrow{\scriptstyle \iota_0^{\Sigma_1}} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \iota_0^{\Sigma_2}} \\ \Sigma_1 \times I & \xrightarrow{\epsilon} & W & \xleftarrow{\omega} & \Sigma_2 \times I \\ \uparrow{\scriptstyle \iota_1^{\Sigma_1}} & & \uparrow{\scriptstyle \sigma} & & \uparrow{\scriptstyle \iota_1^{\Sigma_2}} \\ \Sigma_1 & \xrightarrow[j_1]{} & N & \xleftarrow[j_2]{} & \Sigma_2. \end{array}$$

(On the LHS we have a bicategorical notation for $\mathbf{W}$.) Here $I = [0, 1]$, $\iota_0^{\Sigma_i}(x) = (x, 0)$ and $\iota_1^{\Sigma_i}(x) = (x, 1)$, W is an $n+2$-manifold with corners, and the universally defined map, $\phi_{\mathbf{W}}\colon (\Sigma_1 \times I) \sqcup_{\Sigma_1 \sqcup \Sigma_1} (M \sqcup N) \sqcup_{\Sigma_2 \sqcup \Sigma_2} (\Sigma_2 \times I) \to W$ again gives a diffeomorphism with image ∂W.

Extended cobordisms have horizontal and vertical compositions. We have a bicategory $\mathbf{2Cob}^n$, with objects closed n-manifolds, and 1- and 2-morphisms $n+1$-cobordisms and diffeomorphism classes of $n+2$-cobordisms, which is symmetric monoidal via the disjoint union operation. A *once-extended* $n+2$*-TQFT* is then a symmetric monoidal bifunctor $\mathcal{Z}\colon \mathbf{2Cob}^n \to \mathbf{Mor}$, the bicategory of algebras, bimodules between algebras, and bimodule maps, with monoidal structure given by the appropriate tensor products. Hence $\mathcal{Z}$ is such that, on 2-morphisms,

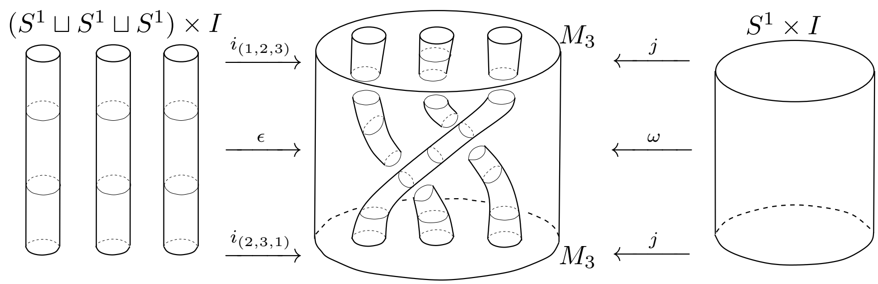


Here we have algebras, $\mathcal{Z}_0(\Sigma_1)$ and $\mathcal{Z}_0(\Sigma_2)$, as well as $\big(\mathcal{Z}_0(\Sigma_1), \mathcal{Z}_0(\Sigma_2)\big)$-bimodules, $\mathcal{Z}_1(\mathcal{M})$ and $\mathcal{Z}_1(\mathcal{N})$, and an intertwiner $\mathcal{Z}_2(\mathbf{W})\colon \mathcal{Z}_1(\mathcal{M}) \to \mathcal{Z}_1(\mathcal{N})$. This assignment preserves all compositions, and disjoint-unions / tensor products.

In [1], Tim Porter and I built a once-extended TQFT $\mathcal{Z}^{\mathbf{B}}\colon \mathbf{2Cob}^n \to \mathbf{Mor}$ given a homotopy finite space $\mathbf{B}$. This is explicitly computable in many cases.

Applications to braid groups ensue when $n = 1$. If $p \in \mathbb{Z}_{>1}$, we have a projection $\pi\colon B_p \to S_p$, where B_p is the braid group in p-strands and S_p the symmetric group. The group B_p acts on S_p via $\sigma \triangleleft b = \sigma \cdot \pi(b)$. The action groupoid, $S_p /\!\!/ B_p$, has objects elements $\sigma \in S_p$ and morphisms have form $(b, \sigma)\colon \sigma \to \sigma \triangleleft b$, for $b \in B_p$.

Let M_p be the 2-manifold obtained from D^2 by excising p small disjoint open 2-disks, inserted along a diameter, so ∂M_p has p 'inner' S^1 components and an 'outer' component. Each $\sigma \in S_p$, gives a 2-cobordism, $\mathcal{M}_p^\sigma = (\sqcup_{i=1}^p S^1 \xrightarrow{i_\sigma} M_p \xleftarrow{j} S^1)$, where the k^{th} circle in $\sqcup_{i=1}^p S^1$ parametrises the $\sigma(k)^{th}$ inner S^1 component of ∂M_p, and $j\colon S^1 \to \partial M_p$ the outer component. A morphism, $(b, \sigma)\colon \sigma \to \sigma \triangleleft b$, in $S_p /\!\!/ B_p$ gives an extended 3-cobordism $\mathcal{F}_p(b, \sigma)\colon \mathcal{M}_p^\sigma \Rrightarrow \mathcal{M}_p^{\sigma \triangleleft b}$. This is as sketched in the figure below, in the particular case $\big(⤫, (1,2,3)\big)\colon (1,2,3) \to (2,3,1)$,



Clearly $\mathcal{F}_p$ sends the composition in $S_p /\!\!/ B_p$ to the vertical composition in $\mathbf{2Cob}^1$. Extended cobordisms, $\mathcal{F}_p(b)$ and $\mathcal{F}_{p'}(b')$, can also be composed horizontally. This is closely related with the grafting operations in the *braid operad*, as hinted in [2].

For a once-extended 3-TQFT, $\mathcal{Z}\colon \mathbf{2Cob}^1 \to \mathbf{Mor}$, let $\mathcal{A} = \mathcal{Z}_0(S^1)$, an algebra, and $\mathrm{bimod}(\mathcal{A}^{\otimes p}, \mathcal{A})$ be the category of $(\mathcal{A}^{\otimes p}, \mathcal{A})$-bimodules, and their intertwiners. Composing $\mathcal{Z}$ with $\mathcal{F}_p$ gives a functor $\mathcal{Z}'_p\colon S_p /\!\!/ B_p \to \mathrm{bimod}(\mathcal{A}^{\otimes p}, \mathcal{A})$. Considering $\mathcal{Z}^{\mathbf{B}}$, where $\mathbf{B}$ arises from a finite group, G, via classifying space, then, expectedly, $\mathcal{A}$ is the quantum double of $\mathbb{Q}G$, and we recover the quantum representations of B_p. Our extended TQFT point of view thus uncovers the compatibility of those braid group representations with the grafting operation in the braid operad.

Going up a dimension, if $n = 2$, an analogue [3] of B_p is the *loop braid group* LB_p. A once-extended 4-TQFT, similarly gives a functor $\mathcal{Z}'_p\colon S_p /\!\!/ LB_p \to \mathrm{bimod}(\mathcal{A}^{\otimes p}, \mathcal{B})$, where $\mathcal{A} = \mathcal{Z}_0(T^2)$ and $\mathcal{B} = \mathcal{Z}_0(S^2)$. Considering $\mathcal{Z}^{\mathbf{B}}$, where $\mathbf{B}$ is the homotopy finite space derived from a group G acting on an abelian group A, both finite, by automorphisms, then the associated representation of LB_p arises from the following *welded* [3] set-theoretical solution of the Yang–Baxter equation,

$$\begin{aligned}&(G^2 \times A) \times (G^2 \times A) \to (G^2 \times A) \times (G^2 \times A),\\ &\quad\big((p,g,a),(q,h,b)\big) \mapsto \Big(\big(q,h,b + a \triangleleft (p^{-1}q) - a \triangleleft (p^{-1}qh)\big), \big(qh^{-1}q^{-1}p, g, a\big)\Big).\end{aligned}$$

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Drinfeld twists for solutions to the Yang–Baxter equation

Davide Ferri

A set-theoretic solution to the Yang–Baxter equation (YBE), hereafter just a *solution*, is a set X with a map $r\colon X \times X \to X \times X$ satisfying the braid relation

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r),$$

i.e. an equality of maps $X^3 \to X^3$. A graphical interpretation is given in Figure 1 (left). The same equation can be defined in any monoidal category, up to including the associators.

These set-theoretic solutions were introduced by Drinfel'd [3], with the aim of producing linear solutions by linearization and deformation. We write

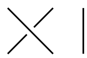
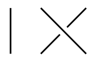
$$r(a,b) = (a \rightharpoonup b,\, a \leftharpoonup b)$$

for suitable maps $\rightharpoonup, \leftharpoonup$. We call a solution *left* (resp. *right*) non-degenerate if all maps $a \rightharpoonup \_\_$ (resp. $\_\_ \leftharpoonup a$) are bijections; it is *non-degenerate* if both conditions hold.

Solutions and representations of the braid group. A representation of a group G in a (locally small) category $\mathcal{C}$ is a homomorphism $G \to \mathrm{Aut}(X)$ for some object X in $\mathcal{C}$. Set-theoretic solutions to the YBE are connected with special representations in $\mathcal{C} = \mathrm{Set}$ of Artin's braid group $\mathbb{B}_n$ [1]. Clearly, representations in Set on the set X correspond to permutation representations in $\mathrm{Vec}_{\Bbbk}$ on the $\Bbbk$-vector space $\Bbbk X$ generated by X.

We shall use braid diagrams, read from the top down.

Remark. Let X be a set, and $r\colon X^2 \to X^2$ a map. Then r is a solution to the YBE if and only if

$\rho_r\colon \mathbb{B}_3 \to \mathrm{Aut}(X^3)$,  $\mapsto r \times \mathrm{id}_X$,  $\mapsto \mathrm{id}_X \times r$

is a representation of the braid group on three strands $\mathbb{B}_3$.

In a similar way, a representation of $\mathbb{B}_n$ for all $n \geq 3$ can be obtained.

Drinfeld twists. Drinfeld twists of solutions were introduced by Kulish and Mudrov [8]. Here we formulate the definition in Ghobadi's language [6].

Definition (Drinfeld twist). Let (X, r) be a solution to the YBE. A *Drinfeld twist* for (X, r) is the datum of bijections $F\colon X^2 \to X^2$ and $\Phi, \Psi\colon X^3 \to X^3$ satisfying

$$F_{12}\Psi = F_{23}\Phi, \qquad \Psi r_{12} = r_{12}\Psi, \qquad \Phi r_{23} = r_{23}\Phi,$$

where $F_{12} = F \times \mathrm{id}_X$, $F_{23} = \mathrm{id}_X \times F$, etc.

If r is a solution, then FrF^{-1} is again a solution [6, 8]. Classifying all set-theoretic solutions is a huge open problem, and it seems currently beyond our reach. However, we advocate a classification of solutions *up to Drinfeld twists*, which is motivated by the following observation.

Proposition. Let (X, r) be a solution, and $F\colon X^2 \to X^2$ a bijection. Suppose that FrF^{-1} is again a solution. Then the representations of the braid group on three strands ρ_r and $\rho_{FrF^{-1}}$ given above are isomorphic (as representations in Set, i.e. as permutation representations) if and only if there exist $\Phi, \Psi\colon X^3 \to X^3$ that make (F, Φ, Ψ) into a Drinfeld twist.

More precisely, isomorphisms of representations between ρ_r and $\rho_{FrF^{-1}}$ are in bijective correspondence with Drinfeld twist structures on F.

Thus, classifying solutions up to Drinfeld twists is the same as classifying them up to isomorphism class of the associated braid group representation. An analogue of this proposition holds more generally in any monoidal category.

Reflections. Given a solution (X, r), a map $k\colon X \to X$ satisfies the reflection equation (RE) with respect to r if

$$k_2 r k_2 r = r k_2 r k_2,$$

where $k_2 = \mathrm{id}_X \times k$, and in this case k is called a *reflection* for r. The set-theoretic form of the RE was defined by Caudrelier, Crampé, and Zhang [11]. A graphical interpretation is given in Figure 1 (right).

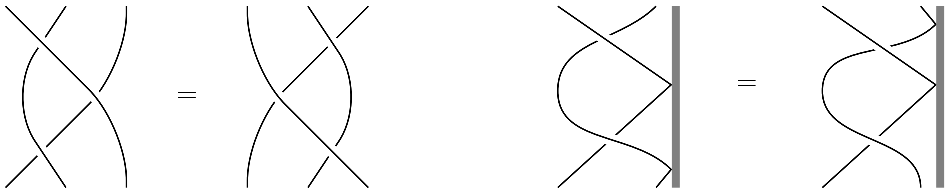


FIGURE 1. Graphical depiction of the YBE (left) and the RE (right), with each crossing representing r, and each bounce on the lateral wall representing k.

Reflections and Drinfeld twists. From now on, let the solutions be right non-degenerate. Lebed and Vendramin [10] discovered that every reflection k for (X, r) can be used to produce a new solution to the YBE, the *k-derived solution*, denoted $r^{(k)}$.

Theorem (Ferri [5]). The solution $r^{(k)}$ is obtained as a Drinfeld twist of r. In particular, it yields an isomorphic representation of the braid group.

Here the map F of the Drinfeld twist is given by the *guitar map* defined in [10].

Braided groups and structure group of solutions. A *braided group* in the sense of Lu, Yan, and Zhu [9] is a group G equipped with a left action $\rightharpoonup$ and a right action $\leftharpoonup$ of G on itself, satisfying the compatibility

$$ab = (a \rightharpoonup b)(a \leftharpoonup b)$$

for all $a, b \in G$. The map $r\colon (a, b) \mapsto (a \rightharpoonup b, a \leftharpoonup b)$ is called a *braiding* on G, and it is automatically a (non-degenerate) solution to the YBE [9].

Morally, r behaves like a "constraint of abelianity" for the group G: indeed, G is abelian if and only if the canonical flip is a braiding.

Braidings are set-theoretic solutions (not solutions in Gp!), but not all set-theoretic solutions defined on a group are automatically braidings. However, every solution (X, r) is functorially associated with a braided group $G(X, r)$ (see [4, 7, 9]), yielding an adjunction

$$G : \mathsf{YB}(\mathsf{Set}) \rightleftarrows \mathsf{BrGp} : \mathcal{U}$$

between the functor $G\colon (X, r) \mapsto G(X, r)$ and the forgetful functor $\mathcal{U}$; here $\mathsf{YB}(\mathsf{Set})$ is the category of set-theoretic solutions, and BrGp is the category of braided groups, both with the obvious morphisms.

Group reflections and group Drinfeld twists. There is a notion of Drinfeld twist for a braided group (G, r) which twists it into a new braided group; this was defined by Ghobadi [6].

Definition (group Drinfeld twist). Let (G, r) be a braided group. A Drinfeld twist (F, Φ, Ψ) for the associated solution r is called a *group Drinfeld twist* if moreover, for all $a, b, c \in G$,

$$\begin{aligned}
&(\text{GDT1}) \quad \Phi(1,b,c) = (1,b,c), \quad \Psi(a,b,1) = (a,b,1),\\
&(\text{GDT2}) \quad F(a,1) = (a,1), \quad F(1,b) = (1,b),\\
&(\text{GDT3}) \quad m_{23}\Phi = Fm_{23},\\
&(\text{GDT4}) \quad m_{12}\Psi = Fm_{12},
\end{aligned}$$

where $m\colon G \times G \to G$ is the multiplication.

Given a braided group $(G, m, 1, r)$ with multiplication m, neutral element 1, and braiding r, every group Drinfeld twist (F, Φ, Ψ) produces a new braided group $G^F := (G, mF^{-1}, 1, FrF^{-1})$ (note that not only the braiding, but also the group structure has changed).

De Commer [2] introduced a notion of *braided action* of a braided group (G, r) on a set X. This braided action depends on a map $k_X : G \times X \to G \times X$. When $X = \{\bullet\}$ is a singleton, the data amount to a map $k\colon G \to G$ satisfying certain conditions; this is what we call a *group reflection* [5].

Definition (group reflection). A *group reflection* for (G, r) is a braided action on the singleton; equivalently, a map $k\colon G \to G$ satisfying

$$\begin{aligned}
&(\text{BRE1}) \quad k(1) = 1,\\
&(\text{BRE2}) \quad k(ab) = (a \rightharpoonup k(b))\, k(a \leftharpoonup k(b)),\\
&(\text{BRE3}) \quad k(a) = (a \rightharpoonup b) \rightharpoonup k(a \leftharpoonup b) \quad \text{for all } a, b \in G.
\end{aligned}$$

If k is a group reflection for r, then it is also a set-theoretic reflection [2].

Theorem (Ferri [5]). Group reflections yield group Drinfeld twists. Here the map F is again the guitar map.

Addressing a problem of Ghobadi [6], we also have:

Theorem. A set-theoretic reflection k for (X, r) lifts to a group reflection for $G(X, r)$ if and only if k satisfies (BRE3).

Racks and an open question. Braided groups have been classified up to Drinfeld twists [6]. The above motivates the following problem, which is still open (and has already appeared, in equivalent forms, in multiple places in the literature).

Problem. Classify all solutions in Set up to Drinfeld twists.

As we have seen, this is equivalent to classifying solutions up to isomorphism class of the associated representation of $\mathbb{B}_3$. The following well-known class of solutions is crucial in the theory.

Definition (rack solution). A *rack solution* is a left non-degenerate solution (X, r) such that $a \leftharpoonup b = a$ for all $a, b \in X$.

A standard fact is that every right non-degenerate solution r can be Drinfeld-twisted into a rack solution. Thus classifying non-degenerate solutions up to Drinfeld twists boils down to classifying racks up to Drinfeld twists.

The first natural question is: if (X, r) and (X, s) are both rack solutions, and they are related by a Drinfeld twist, must they coincide? This is still open, and even hard to check computationally. A positive answer would imply that racks classify solutions up to Drinfeld twists.

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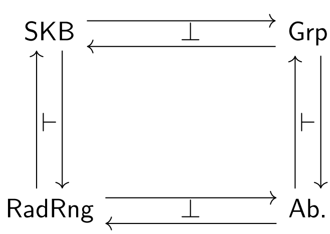
A categorical approach to skew braces and to cocommutative Hopf braces

Marino Gran

(joint work with Thomas Letourmy and Leandro Vendramin)

The variety SKB of skew (left) braces [1] contains several interesting subvarieties, such as the varieties RadRng of radical rings, Grp of groups, and Ab of abelian groups. Indeed, each of these subcategories is determined by some suitable identities that have to be added to the algebraic theory of skew braces, so that they are all subvarieties of SKB thanks to the classical Birkhoff theorem in universal algebra. These varieties are then related by the following adjunctions, where the right adjoints are all inclusion functors:



Since SKB is a variety of Ω-groups in the sense of Higgins - and therefore, in particular, a semi-abelian category - it is natural to study it from the viewpoint of categorical algebra. In this joint work with T. Letourmy and L. Vendramin [2], we have applied methods from non-abelian homological algebra to establish some new Hopf formulae for the homology of skew braces, where the coefficient functors are the left adjoint functors from the variety SKB to each of the three subvarieties RadRng, Grp and Ab, which are depicted in the diagram above. In particular these results provide a useful interpretation of the terms appearing in a Stallings–Stammbach five-term exact sequence associated with any exact sequence of skew braces, that extends the classical one in the variety of groups.

For instance, for any short exact sequence

$$0 \longrightarrow K \rightarrowtail A \xrightarrow{f} \!\!\!\!\!\rightarrow B \longrightarrow 0$$

in the variety SKB of skew braces, there is an induced exact sequence in the variety Grp of groups:

$$\mathsf{H}_2(A,I) \xrightarrow{\mathsf{H}_2(f)} \mathsf{H}_2(B,I) \longrightarrow \frac{K}{[K,A]_{\mathsf{Grp}}} \longrightarrow \mathsf{H}_1(A,I) \xrightarrow{\mathsf{H}_1(f)} \mathsf{H}_1(B,I) \longrightarrow 0.$$

Here the group $\mathsf{H}_1(A,I)$ is the quotient $I(A) = \frac{A}{[A,A]_{\mathsf{Grp}}}$ of the skew brace $(A,+,\circ)$ by the ideal $[A,A]_{\mathsf{Grp}}$ of A generated by all the elements of the form

$$a * b = -a + a \circ b - b$$

for all $a \in A$, $b \in A$ (with a similar definition for $\mathsf{H}_1(B,I)$). The ideal $[K,A]_{\mathsf{Grp}}$ is the additive subgroup of $(A,+,\circ)$ generated by the elements

$$\{k * a,\ a * k,\ c + a * k - c \mid k \in K, a \in A, c \in A\},$$

which turns out to be an ideal of A. The second homology group $\mathsf{H}_2(A,I)$ is defined by using a generalized Hopf formula for skew braces that is relative to the subvariety Grp of groups: indeed, starting from any *free presentation*

$$0 \longrightarrow K \rightarrowtail F \xrightarrow{f} \!\!\!\!\!\rightarrow A \longrightarrow 0$$

of a skew brace $(A,+,\circ)$, the expression

$$\mathsf{H}_2(A,I) \cong \frac{K \cap [F,F]_{\mathsf{Grp}}}{[K,F]_{\mathsf{Grp}}}$$

is an invariant, in the sense that it is independent of the chosen free presentation. Similarly, one defines $\mathsf{H}_2(B,I)$. Further results are obtained by choosing the left adjoint functors from the variety SKB to the subvarieties RadRng and Ab, respectively.

It would be interesting to develop a similar approach to study the (co)homological properties of cocommutative Hopf braces in the sense of [3]. On the one hand the category HopfBr of cocommutative Hopf braces is semi-abelian [4], so that the main homological lemmas hold true in it. On the other hand, the category of cocommutative Hopf braces is monadic on the category of cocommutative coalgebras, as shown in [5], yielding a canonical free presentation for any Hopf brace.

These observations should allow one to extend the results recently obtained for cocommutative Hopf algebras in [6] to the category of cocommutative Hopf braces.

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A rack perspective to conjugacy classes of $PSL(2,q)$

Istvan Heckenberger

(joint work with Fengchang Li)

In the theory of Nichols algebras, especially within the framework of the classification of finite-dimensional Nichols algebras over groups, one of the most challenging problems is to obtain significant structural information on the Nichols algebras of finite (quasi)simple groups. Nichols algebras of direct sums of Yetter-Drinfeld modules can be treated via reflection theory [1]. Simple Yetter-Drinfeld modules may have braided subspaces arising from subracks of conjugacy classes of the group, giving rise to Nichols subalgebras. However, for some conjugacy classes of some simple groups this approach provides only poor information. We therefore study the groups $PSL(2,q)$ to see more clearly the available structures and the limitations for a crucial family of examples.

Subracks of (unions of) conjugacy classes of a group arise by intersecting conjugacy classes with subgroups. We use Dickson's theorem [2, Th. 8.27] on the subgroups of $PSL(2,q)$ to identify all subracks of each conjugacy class of $PSL(2,q)$.

We call a rack (or a union of conjugacy classes of a group) *minimal non-abelian* if it is not abelian, but all proper subracks are abelian. We identify all minimal non-abelian conjugacy classes of $PSL(2,q)$. The exact list is a bit technical; it contains

- the unipotent classes of $PSL(2,q)$ for a prime q,
- the class of order 3 elements in $PSL(2,2^m)$ for an odd prime m, and
- non-split classes satisfying some technical conditions.

For the study of Nichols algebras over $PSL(2,q)$ one also needs to deal with the possible cocycles of the classes. These can be obtained from the associated group

$$As(X) = \langle g_x \mid x \in X\rangle/(g_x g_y = g_{x\triangleright y} g_x, x, y \in X).$$

We proved for the non-split semisimple classes of $PSL(2,q)$ with $q>4$ and $q\neq 9$ using Bruhat decomposition that $As(X)\cong SL(2,q)\times\mathbb{Z}$. The proof does not seem to generalize easily to other classes and other groups. We then found (a few days after the mini-workshop), that it is possible to adapt Kervaire's treatment of stem extensions of perfect groups. We proved that for generating conjugacy classes X of perfect groups G the associated group of X is a stem extension of $G\times\mathbb{Z}$ by a quotient of the Schur multiplier of G. Our results are published in arXiv [3].

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Group rings and braces

Eric Jespers

Each part in this talk is well-known to the respective specialists in distinct areas. The aim is to make it accessible to all participants and to highlight the link between braces and group rings, in particular via some old open problems on the isomorphism problem of integral group rings.

Recall Problem 29 in [2] for a finite (nilpotent) group G and the group of normalised units $U_1(\mathbb{Z}[G])$ of the integral group ring $\mathbb{Z}[G]$:

(1) Does G have a normal complement in $U_1(\mathbb{Z}[G])$, i.e. does there exist a normal subgroup N of $U_1(\mathbb{Z}[G])$ such that $U_1(\mathbb{Z}[G])=N\rtimes G$?

(2) If a normal complement exists, does there exists a torsion-free normal complement?

The main reason for this question is that positive answers yield a positive solution to the isomorphism problem (ISO) for G (and this is very easy to prove). If, moreover, G is a nilpotent group then it is sufficient to prove the existence of a normal complement.

(ISO) : if H and G are finite groups such that the rings $\mathbb{Z}[G]$ and $\mathbb{Z}[H]$ are isomorphic then the groups G and H are isomorphic.

Positive answers to Problem 29 are known for several classes of groups, including finite abelian groups and finite circle groups (i.e. the multiplicative group associated to a finite radical ring). Hertweck in [1] showed that in general the isomorphism problem for integral group rings of finite groups does not hold. He constructed a counter example $G=Q\rtimes P$ of order $|G|=2^{21}97^{28}$, with Q a 97-group of nilpotency class 2 and P a 2-group which is a semidirect product of two abelian groups. So G is a solvable group of derived length 4 and G is the multiplicative group of a brace. It turns out that $\mathbb{Z}[G]\cong\mathbb{Z}[H]$ for a finite non-isomorphic group H; and H is constructed in a similar way and thus also is a brace.

Hence the isomorphism problem also does not hold if one restricts to groups that are multiplicative groups of braces. The method to construct G is based on the idea to complement the set $G-1$ in the augmentation ideal $\omega(\mathbb{Z}[G])$ of $\mathbb{Z}[G]$ by a left ideal.

The latter brings us to a link with braces via Sysak's result. Recall that if $(G,\oplus)$ is an abelian group and $(G,\cdot)$ is group so that $a(b\oplus c)=ab\ominus a\oplus ac$ for all $a,b,c\in G$, then $(G,\oplus,\cdot)$ is said to be a left brace. One calls $(G,\cdot)$ the multiplicative group of the left brace. If also $(b\oplus c)a=ba\ominus a\oplus ca$ for all $a,b,c\in G$ then $(G,\oplus,\cdot)$ is said to be a two-sided brace.

Theorem 1 (Sysak [3]). *A group G is the multiplicative group of a left brace if and only if there exists a left ideal L of $\mathbb{Z}[G]$ such that $\omega(\mathbb{Z}[G])=(G-1)+L$ and $(G-1)\cap L=\{0\}$. Moreover, $(G,\oplus,\cdot)$ is a two-sided brace if and only a two-sided ideal L exists.*

It is well-known that $(G,\oplus,\cdot)$ is a two-sided brace if and only if $(G,\oplus,*)$ is an associative ring (actually a radical ring), where $*$ is the operation defined by $a*b=ab\ominus a\ominus b$. In this case $(G,*)$ is a group, called a circle group.

Corollary 1. *Let $(G,\oplus,\cdot)$ be a finite left brace. Then*

$$U_1(\mathbb{Z}[G])=G\,((1+L)\cap U_1(\mathbb{Z}[G])\ \ \textit{and}\ G\cap(1+L)=\{1\},$$

i.e. G has a complement in $U_1(\mathbb{Z}[G])$. In particular, if L is a two-sided ideal then G has a normal complement in $U_1(\mathbb{Z}[G]$.

As an immediate consequence one recovers the result that for both a finite abelian group and a finite circle group a normal complement exists in the augmented units of its integral group ring and thus that the isomorphism problem has a positive answer for such groups.

A well-known and deep result is that the isomorphism problem has a positive answer for finite nilpotent groups. However, an answer to Problem 29 is in general unknown for finite nilpotent groups. Note that Bachiller has shown that not all finite nilpotent p-groups are braces. Sysak's result yields that one has a complement but not necessarily a normal complement. Hence one can pose the following problem.

3. Determine classes of finite groups G that are the multiplicative group of a left brace so that G has a normal complement in $U_1(\mathbb{Z}[G])$.

There has been extensive research on the algebraic structure of (finite) braces. So maybe the structural results obtained can possibly give some new insights into group rings via the Sysak result. Hence the following broadly formulated problem.

4. Do structural results of finite braces $(G,\oplus,\cdot)$ give new insights in the structure of integral group rings $\mathbb{Z}[(G,\cdot)]$, and vice-versa?

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Can we weld a YBE solution?

Victoria Lebed
(joint work with Ilaria Colazzo, João Faria Martins, Senne Trappeniers)

A set-theoretic YBE solution (X, r) gives rise to an action of every *braid group* B_n on the nth power of X, hence to a representation of B_n. It is natural to ask when such actions extend to the *welded braid groups* WB_n. The letter groups can be seen topologically as the motion groups of n unlinked unknotted oriented circles in $\mathbb{R}^3$, or algebraically as the groups of all automorphisms of the free groups on the generators $g_1, \ldots, g_n$ sending each g_i to a conjugate of some g_j. See [3] for more detail.

To get WB_n-actions out of (X, r), one needs another, involutive YBE solution s on X, compatible with r in a certain way. This idea can be traced back to [2]. In this work, we show that a WB_n representation associated to such a *welded pair* (X, r, s) is necessarily isomorphic to one coming from its *derived pair* $(X, r^{(s)}, t)$, where t is the *twist* solution $t(a, b) = (b, a)$, and $r^{(s)}$ is constructed out of r and s by an explicit formula. Thus, for representation theory purposes, only welded pairs of type (X, r, t) are relevant. This answers an open question from [1].

Question 1. Does using general second components s become relevant when one enriches the induced representations with weights computed using 2-*cocycles*, in the spirit of [4]?

The homology groups of a welded pair (X, r, s) and its derived pair $(X, r^{(s)}, t)$ are very different in general, suggesting that the corresponding invariants might indeed differ significantly.

The representation isomorphism above is established via explicit maps, which we call *violin maps*. They are the virtual versions of the *guitar maps*, used in [5] to realise an isomorphism between the B_n-representations induced by a solution (X, r), and those induced by its *derived solution* (X, r'). The latter is always of *rack type*, i.e., of the form $r'(a, b) = (b, a * b)$. Thus, for representation theory purposes, only solutions of rack type are relevant. The guitar maps are not compatible with an additional solution s on X in general, making legitimate the following question.

Question 2. Are the WB_n-representations induced by a welded pair (X, r, s) isomorphic to ones induced by a welded pair (X, r', q) for some q? Here (X, r') is the derived solution of (X, r).

Another natural question would be a characterisation of solutions (X, r) "weldable" into a welded pair (X, r, t). We show that this can be tested by checking two conditions on the *structure skew brace* $B(X, r)$. The most curious of the two conditions is that of being a *bi-skew brace*, that is, both $(B,+,\circ)$ and $(B,\circ,+)$ need to be skew braces. See [7] for a detailed study of such objects.

Question 3. What object plays for a welded pair the same role as the structure skew brace plays for a solution?

A possible candidate is a pair of skew braces acting on each other, since for a welded pair (X,r,s), at least the structure groups of (X,r) and of (X,s) act on each other in a compatible way.

In another vein, one might think of the solutions $(X,r^{(s)})$ as a family of deformations of (X,r) indexed by involutive solutions s compatible with r. This is reminiscent of the solution families $(X,r^{(k)})$ from [6] indexed by solutions k of the reflection equation associated to r.

Question 4. Is there a common framework embracing the two families of solution deformations above?

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Unitary R-matrices, mostly with two eigenvalues

Gandalf Lechner

In this talk I reviewed a classification programme for solutions to the Yang-Baxter equation. Two main ideas are that a) a classification of braided vector spaces up to isomorphism seems hopeless, but a classification up to a weaker notion of equivalence, based on braid group representations, has much better chances of success, and b) even when classifying up to such an equivalence, one wants to restrict attention to subclasses of solutions.

The class of interest discussed in this talk where the unitary R-matrices. That is, we considered finite-dimensional complex Hilbert spaces V and unitary solutions $R : V\otimes V \to V\otimes V$ to the YBE, inducing representations ρ_R of the infinite braid group B_∞ inside the infinite tensor product $\bigotimes_{n\in\mathbb{N}} \mathrm{End}(V)$ (closed in the weak operator topology given by the normalized trace τ [4]). Two solutions R,S are called equivalent, written $R\sim S$, if

$$R\sim S :\Leftrightarrow \chi_R=\chi_S,\quad \dim R=\dim S, \tag{1}$$

where $\chi_R = \tau \circ \rho_R$ is the normalized character defined by R, and $\dim R := \dim V$ the dimension of the "base space". This natural notion of equivalence has been considered by several authors [1, 5, 3].

I then reviewed what is known about the classification up to $\sim$, focussing in particular on the spectrum and the partial trace as invariants. In the subcase of unitary R-matrices with precisely two eigenvalues, one may be fixed to be -1, whereas the other eigenvalue, $q \in \mathbb{T}$, is a free parameter. Two quite different scenarios occur:

(1) ($q = +1$). This is the case of unitary involutive R-matrices. A complete classification is known [3] (many equivalence classes exist).
(2) ($q \neq +1$). This is the case of non-involutive Hecke type unitary R-matrices. It is known which values of q are possible ([Lechner, to be published], using the results of [2]). A small list of possible equivalence classes emerges, and for all but one class, representatives are known.

This setting was compared to the case of non-degenerate set-theoretic solutions and to the problem of localizing braid group representations [6]. The task of determining whether the last class exists was posed as an open problem.

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Proving the logarithmic Kazhdan Lusztig conjecture

Simon Lentner

A conformal quantum field theory assigns to to a Riemann surface Σ a space of states $\mathcal{Z}(\Sigma)$, typically the space of solutions of some partial differential equation.

Example (Knizhnik–Zamolodchikov equation). *Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra. To any surface with punctures $z_1, \ldots, z_n$ and $\mathfrak{g}$-representations $V_1, \ldots, V_n$ we can consider the partial differential equation*

$$\kappa \frac{\partial}{\partial z_i} f(z_1, \ldots, z_n) = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} f(z_1, \ldots, z_n)$$

where f takes values in $V_1 \otimes \cdots \otimes V_n$, the scalar parameter κ is called shifted level and Ω_{ij} is the Casimir of $\mathfrak{g}$ acting on the factors $V_i \otimes V_j$. It is multivalued around singularities at $z_i = z_j$, hence the pure braid group acts on the space of solutions.

As a toy example take $\mathfrak{g}$ abelian and $V_i = V_{\lambda_i}$, then the solutions are

$$f(z_1, \ldots, z_n) = \prod_{i<j} (z_i - z_j)^{(\lambda_i, \lambda_j)/\kappa}$$

For general $\mathfrak{g}$, the celebrated Drinfeld-Kohno Theorem *relates this to the braid group representation for the quantum group $U_q(\mathfrak{g})$. On highest weight vectors, this is seen by explicit integrals called* screening operators *related to E, F, see eg.* [1].

In my current work I study similar connections between conformal field theory, more concretely vertex algebras, and quantum groups at root of unity, or more general Nichols algebras. The picture that conjecturally emerges is roughly [2]:

- On an arbitrary vertex algebra $\mathcal{V}$, with braided tensor category of representations $\mathcal{C}$, any set of screening operators fulfills the relations of a corresponding Nichols algebra in $\mathcal{C}$. For free field theories I proved this.
- The kernel of screening operators $\mathcal{W} \subset \mathcal{V}$ has a category of representations related to the relative Drinfeld center of this Nichols algebra. The main examples are small quantum group, where this was the longstanding *logarithmic Kazhdan-Lusztig conjecture*. In my talk I present a proof. A main step is essentially proving that braided tensor categories with large commutative algebras often look like Drinfeld centers of Nichols algebras.

This general picture suggests many other interesting cases, for example for $\mathcal{C}$ the category related to an affine Lie algebra at integer level or to another W-algebra, for example in the context of inverse Hamiltonial reduction.

In the theory of Nichols algebras, this naturally poses the problem of establishing the root system theory and give a classification in such more general settings.

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Skew braces, indecomposability and Dehornoy's representations for solutions to the Yang-Baxter equation

Silvia Properzi

(joint work with C. Dietzel, E. Feingesicht and C. Dietzel, S. Trappeniers)

In recent years skew braces have become a central algebraic framework for studying set-theoretical solutions to the Yang-Baxter equation. Properties of a solution are reflected in its permutation skew brace, its structure brace and the associated retraction tower. Furthermore, the structure group of a solution admits monomial representations introduced by Dehornoy, which reveal deep links between algebraic structures and combinatorics (e.g. indecomposability).

A *set-theoretical solution to the Yang-Baxter equation* consists of a non-empty set X and a bijection

$$r : X^2 \to X^2, \qquad r(x,y) = (\lambda_x(y), \rho_y(x)),$$

satisfying the braid relation

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r).$$

We work with finite, involutive and non-degenerate solutions, i.e. X is a finite set, $r^2 = \mathrm{id}$ and all λ_x, ρ_x are bijections. In this case $\rho_y = \lambda^{-1}_{\lambda_x(y)}(x)$, thus the solution is entirely determined by the maps λ_x, for $x \in X$.

The permutation group of a solution (X, r) is the finite group

$$\mathcal{G}(X) = \langle \lambda_x \mid x \in X \rangle \leq \mathrm{Sym}_X$$

that encodes the internal symmetries of X. The image of the natural map

$$\mathrm{Ret} : X \to \mathcal{G}(X), \qquad x \mapsto \lambda_x$$

defines a solution $\mathrm{Ret}(X)$, called *retraction.* A solution (X, r) is *retractable* if $|\mathrm{Ret}(X)| < |X|$, otherwise it is called *irretractable.*

A solution (X, r) is *indecomposable* if no non-trivial partition $X = A \sqcup B$ gives $r(A^2) = A^2$ and $r(B^2) = B^2$. Moreover, indecomposability is equivalent to transitivity of the $\mathcal{G}(X)$-action on X. Indecomposable solutions are building blocks of general solutions, but turn out to be significantly more well-behaved, as illustrated by the following data from [1, 10].

$\lvert X\rvert$	2	3	4	5	6	7	8	9	10	11
# Sols	2	5	23	88	595	3456	34530	321931	4895272	77182093
# Ind Sols	1	1	5	1	10	1	100	16	36	1

Table 1. Number of solutions of size ≤ 11 up to isomorphism.

Indeed, up to isomorphism, there is a unique indecomposable solution of cardinality p for p prime [8], and retraction preserves indecomposability [3].

A skew brace is a triple $(B, +, \circ)$ where $(B, +)$ and $(B, \circ)$ are groups satisfying

$$a \circ (b + c) = a \circ b - a + a \circ c,$$

equivalently $(B, +)$ and $(B, \circ)$ are groups with a homomorphism

$$\boldsymbol{\lambda} : (B, \circ) \to \mathrm{Aut}(B, +), \qquad \boldsymbol{\lambda}_a(b) = -a + a \circ b.$$

For a solution (X, r) the group $(\mathcal{G}(X), \circ)$ inherits a second operation

$$\lambda_x + \lambda_y = \lambda_x \circ \lambda_{\lambda_x^{-1}(y)},$$

giving the *permutation skew brace* $(\mathcal{G}(X), +, \circ)$. The action $\boldsymbol{\lambda}$ extends the natural action of $\mathcal{G}(X)$ on the retraction given by the application of the λ_x of the solution.

Classification for size p^2. In joint work with Dietzel and Trappeniers [7] we obtain a complete classification of indecomposable solutions of size p^2, p prime. Two main cases occur: retractable and irretractable solutions.

In the retractable case, such solutions have multipermutation level at most 2, i.e. $|\mathrm{Ret}(\mathrm{Ret}(X))| = 1$. Their structure and enumeration follow from results of Jedlička and Pilitowska in [9].

Theorem 1. *If X is a retractable indecomposable solution of size p^2, it is isomorphic to one of the following:*

(1) $X = \mathbb{Z}_{p^2}$*, with* $r(x, y) = (y + 1, x - 1)$.

(2) $X = \mathbb{Z}_p \times \mathbb{Z}_p$*, with*

$$\lambda_{(a,x)}(b, y) = (b - 1, y - SI_0(b - 1) - \Phi(b - 1 - a)).$$

where I_0 is the indicator function of 0, $S \in \mathbb{Z}_p$ and $\Phi : \mathbb{Z}_p \to \mathbb{Z}_p$ non-constant and $\Phi(0) = 0$.

In the irretractable case, we first classify solutions with a p-group as permutation group. Finally we obtain all irretractable solutions as a deformation of an irretractable solution with a p-group as permutation group.

Theorem 2. *Let X be an irretractable solution of size p^2. Then it is isomorphic to a solution on $\mathbb{Z}_p \times \mathbb{Z}_p$, with*

$$\lambda_{(a,x)}(b, y) = (\alpha^{-1}b - x, \alpha^{-1}y - \Phi(\alpha^{-1}b - x - a)).$$

where $\Phi : \mathbb{Z}_p \to \mathbb{Z}_p$ is a non-constant map with $\Phi(x) = \Phi(-x)$ and $\alpha \in \mathbb{Z}_p^*$ is such that $\Phi(\alpha x) = \alpha\Phi(x)$ for all $x \in \mathbb{Z}_p$.

This classification allows comparison with the simple solutions introduced by Cedó and Okniński, and shows that all irretractable indecomposable solutions of size p^2 are simple, settling [2, Question 7.4].

Dehornoy's representations. The *structure group* of a solution (X, r) is

$$G(X, r) = \langle X \mid xy = \lambda_x(y)\rho_y(x)\rangle,$$

and it carries a canonical skew brace structure with addition extending

$$x + y = x\,\lambda_x^{-1}(y), \qquad \text{for } x, y \in X.$$

The permutation skew brace $(\mathcal{G}(X, r), +, \circ)$ then appears as a natural finite quotient. Structure groups behave like Artin–Tits. They are Garside groups [4], Dehornoy [5] constructed Coxeter-like finite quotients $\overline{G}_\ell(X, r) = G(X, r)/(\ell dG)$, where d is the Dehornoy class, which is the exponent of $(\mathcal{G}(X, r), +)$, together with a faithful monomial representation

$$\Theta : G(X, r) \to M_X(\mathbb{C}(q)), \qquad g = \sum_{x\in X} g_x x \mapsto \left(\prod_{x\in X} D_x^{g_x}\right) P_{\lambda_g},$$

where D_x is the diagonal matrix $\mathrm{diag}(1, ..., 1, q, 1, ..., 1)$ with a q on the x-coordinate and P_{λ_g} is the permutation matrix associated to the action of λ_g on X.

Moreover for every positive integer ℓ, the evaluation q at an ℓd-th primitive root of unity yields a faithful representation

$$\overline{\Theta}_\ell : \overline{G}_\ell(X,r) \to M_X(\mathbb{C}).$$

In joint work with Dietzel and Feingesicht [6], using brace theoretic tools, we obtain a precise characterisation of irreducibility.

Theorem 3. *Let (X,r) be a solution of Dehornoy class d. Then*

(1) *(X,r) is indecomposable if and only if Θ is irreducible.*
(2) *If $\ell > 1$ or $d > 2$, then (X,r) is indecomposable if and only if $\overline{\Theta}_\ell$ is irreducible.*
(3) *If (X,r) is indecomposable, $\overline{\Theta}_1$ is irreducible if and only if $d > 2$ or $d = 2$ and $|\mathcal{G}(X,r)| < 2^{\frac{n}{2}}$.*

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Categorical braid group actions and higher idempotent completion

Isabela Recio

The categories of Soergel bimodules and singular Soergel bimodules are rich mathematical objects, with connections to algebraic combinatorics, geometric representation theory and link homology theories, among others. Soergel bimodules assemble into a monoidal category – a 2-category with a single object – that categorifies the Hecke algebra and can be used to construct homological link invariants. They admit a 2-categorical generalization which captures richer structure related to parabolic induction and restriction in the form of the 2-category of singular Soergel bimodules. We establish a relation between the two 2-categories in terms of higher idempotent completion, which allows us to talk about categorified idempotents and categorified idempotent splittings.

In the first part of the talk, we discuss Rouquier's categorified braid group action [5] and its corresponding generalization to colored braid groups as in [3]. These constructions assign to braid group generators, chain complexes of Soergel bimodules and singular Soergel bimodules of type A. For example, to the single crossing on two strands one associates the following chain complex of Soergel bimodules:

$$:= 0 \to \underline{B_s} \to R \to 0$$

The bimodule B_s denotes the indecomposable Soergel bimodule associated to a simple transposition – an object in SBim – while the analogue chain complex for colored braid groups is valued in Hom-categories of singular Soergel bimodules. We introduce the notion of idempotent completion for n-categories, following [2] [1], which was developed in the context of condensed matter physics and topological quantum field theory. The $n = 2$ case was explored in depth in [6]. The following is a 2-categorical idempotent.

Definition 1. *A* 2-categorical idempotent *in* $\mathcal{C}$ *consists of*

- *an object* $c \in \mathcal{C}$*;*
- *a 1-morphism* $e\colon c \to c$ *and*
- *two 2-morphisms* $\mu\colon e \circ e \Rightarrow e$, $\delta\colon e \Rightarrow e \circ e$

such that the pair (e, μ) *is a (non-unital) associative algebra, the pair* (e, δ) *is a (non-counital) coassociative coalgebra,* δ *is an e-bimodule map and* $\mu \cdot \delta = id_e$*.*

To a 2-category $\mathcal{C}$, the construction associates a 2-category denoted by $\mathrm{Kar}^2\mathcal{C}$ where all 2-categorical idempotents *split* in a suitable way. We discuss the problem of describing explicitly the 2-categories Kar^2SBim and Kar^2sSBim, where the challenge is to find and classify objects and data as in Definition 1. We state the following theorem from [4], whereby we relate the two completions.

Theorem 1. *The inclusion of Soergel bimodules into singular Soergel bimodules yields a commutative diagram*

$$\begin{array}{ccc} \mathrm{SBim} & \xrightarrow{\iota} & \mathrm{sSBim} \\ {\scriptstyle \iota_{\mathrm{SBim}}}\downarrow & & \downarrow{\scriptstyle \iota_{\mathrm{sSBim}}} \\ \mathrm{Kar}^2\mathrm{SBim} & \xrightarrow[\simeq]{} & \mathrm{Kar}^2\mathrm{sSBim} \end{array}$$

such that the bottom horizontal map is an equivalence of 2-categories.

To prove the theorem, one first identifies 2-categorical idempotents coming from the longest elements of parabolic subgroups, both for Soergel bimodules and their singular counterparts. The result then follows from [6, Lemma A.2.4.], which characterizes 2-functors that become 2-equivalences upon 2-idempotent completion.

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Braiding Circuits, Localisation, and Property F

Eric C. Rowell

In topological quantum computation (TQC) the circuits are obtained by braiding anyons (Fig. 1), see [1].

Mathematically, anyons are modeled by simple objects in a braided fusion category $\mathcal{C}$. A simple object $X \in \mathcal{C}$ produce a tower of unitary braid group representations

$$\rho_{X,n} \colon \mathbb{C}B_n \longrightarrow \mathrm{End}\big(\mathcal{H}_n(X)\big), \qquad \mathcal{H}_n(X) := \bigoplus_Y \mathrm{Hom}_{\mathcal{C}}\big(Y, X^{\otimes n}\big)\,,$$

with σ_i acting on $\mathcal{H}_n(X)$ by composition with $Id_X^{\otimes i-1} \otimes c_{X,X} \otimes Id_X^{\otimes n-i-1}$. The images of these representations yield the braiding circuits. We define $A_{X,n} = \rho_{X,n}(\mathbb{C}B_n)$ and note that we have the obvious injective algebra maps $\Phi_n : A_n \to A_{n+1}$.

A key difference between the topological set-up and the more standard quantum circuit model is that the underlying Hilbert spaces $\mathcal{H}_n(X)$ is not of the form $V^{\otimes n}$ for some fixed vector space V, as in the quantum circuit model. This non-locality of TQC is the key to its potential for fault-tolerance, as quantum errors are typically local. However, when comparing the models, we can ask:

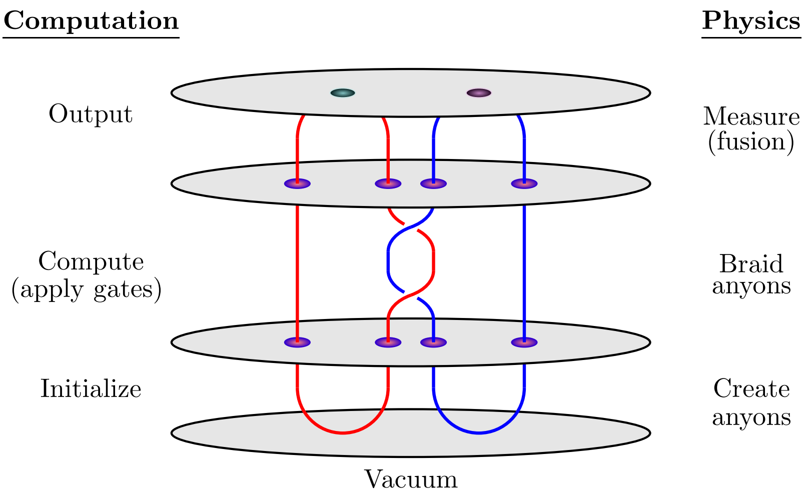

FIGURE 1. One iteration of a TQC protocol: initialise pairs, braid to compute, then fuse/measure.

When can braiding circuits be precisely simulated on a quantum (qudit) circuit model, where $d = \dim(V)$*?*

An obvious source of 'local' representations of B_n are from solutions to the Yang-Baxter equation. Given such a solution (R, V) we have $\rho_R : \mathbb{C}B_n \to End(V^{\otimes n})$ given by

$$\rho_R(\sigma_i) = Id_V^{\otimes i} \otimes R \otimes Id_V^{\otimes n-i-1}.$$

While these two sources of representations look similar, the key difference is that here the σ_i act directly on $V^{\otimes n}$, whereas $\rho_{X,n}(\sigma_i)$ is obtained by composition.

They fit into the general categorical set-up as follows:

Definition 1 (Category of braided representation systems)**.** *Let* B_n *be the braid group on* n *strands and* $\mathbb{C}B_n$ *its group algebra. Recall the* shifted embedding

$$\iota_n : \mathbb{C}B_n \to \mathbb{C}B_{n+1}, \qquad \iota_n(\sigma_i) = \sigma_{i+1} \quad (1 \le i \le n-1).$$

Objects. *An object of the category* BrSys *is a triple*

$$\mathbb{V} = \big(\{\rho_n\}_{n\ge2}, \{V_n\}_{n\ge2}, \{\phi_n\}_{n\ge2}\big),$$

where for each $n \ge 2$*:*

- $\rho_n : \mathbb{C}B_n \to \mathrm{End}(V_n)$ *is the linearisation of a* B_n *representation;*
- $A_n := \rho_n(\mathbb{C}B_n) \subseteq \mathrm{End}(V_n)$ *is the image algebra;*
- $\phi_n : A_n \hookrightarrow A_{n+1}$ *is an injective algebra homomorphism*

such that the following square commutes*:*

$$\begin{array}{ccc} \mathbb{C}B_n & \xrightarrow{\rho_n} & A_n \\ {\scriptstyle \iota_n}\big\downarrow & & \big\downarrow{\scriptstyle \phi_n} \\ \mathbb{C}B_{n+1} & \xrightarrow[\rho_{n+1}]{} & A_{n+1} \end{array} \qquad \textit{i.e.} \quad \phi_n\big(\rho_n(x)\big) = \rho_{n+1}\big(\iota_n(x)\big) \quad \forall\, x \in \mathbb{C}B_n.$$

Morphisms. *Given objects* $\mathbb{V} = (\rho_n, V_n, \phi_n)$ *and* $\mathbb{V}' = (\rho'_n, V'_n, \phi'_n)$*, a morphism* $\Psi : \mathbb{V} \to \mathbb{V}'$ *is a family of algebra maps*

$$\Psi = \{\psi_n : A_n \to A'_n\}_{n\geq 2}$$

such that for every $n \geq 2$*:*

(1) Intertwining on images: $\psi_n \circ \rho_n = \rho'_n$ *as maps* $\mathbb{C}B_n \to A'_n$.

(2) Compatibility with inclusions:

$$\begin{array}{ccc} A_n & \xrightarrow{\psi_n} & A'_n \\ {\scriptstyle \phi_n}\big\downarrow & & \big\downarrow{\scriptstyle \phi'_n} \\ A_{n+1} & \xrightarrow[\psi_{n+1}]{} & A'_{n+1} \end{array} \qquad \textit{commutes, i.e.} \quad \phi'_n \circ \psi_n = \psi_{n+1} \circ \phi_n.$$

(Thus morphisms act only on the image algebras A_n*, not on the spaces* V_n*.)*

Composition and identities. *Composition is levelwise:* $(\psi'_n)\circ(\psi_n) := (\psi'_n \circ \psi_n)$*. The identity on* $\mathbb{V}$ *is* $(\mathrm{id}_{A_n})_{n\geq 2}$*. With these,* BrSys *is a well-defined category.*

Note that $(\rho_{n,X}, \mathcal{H}_n(X), \Phi_n)$ as well as $(\rho_R, V^{\otimes n}, f_n)$ where $f_n : g \mapsto g \otimes Id_V$ are in BrSys.

Localisation. Following joint work with Wang, a tower $\{\rho_{X,n}\}_{n\geq 2}$ is *localisable* if it can be uniformly realised inside a local qudit circuit model generated by a fixed unitary solution to the Yang–Baxter equation (a unitary R-matrix) [3]. In the categorical language, this is to require an injective morphism from (ρ_n, V_n, ϕ_n) to one of the form $(\rho_R, V^{\otimes n}, f_n)$.

We need two more definitions:

Property F. A simple object X has *Property F* if for every n the image $\rho_{X,n}(B_n)$ is finite [4].

Weakly Integral. A simple object X is *weakly integral* if $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$ [2].

The circle of conjectures. Guided by examples and partial results, we propose the equivalences

$$\textbf{(A)} \quad X \text{ is weakly integral: } \mathrm{FPdim}(X)^2 \in \mathbb{N}$$

$$\Updownarrow$$

$$\textbf{(B)} \quad \text{the tower } \{\rho_{X,n}\} \text{ is localisable by a unitary YBE solution } R \text{ [3]}$$

$$\Updownarrow$$

$$\textbf{(C)} \quad X \text{ has Property F [4].}$$

Evidence and examples.

- In many weakly integral settings, explicit unitary R-matrix realisations (Gaussian/Clifford-like) yield localisation and finite braid images; see, e.g., doubles of finite groups [6], extraspecial 2-group images [7], and property F for $SO(N)_2$ [11].
- For $\mathrm{FPdim}(X)^2 \notin \mathbb{N}$ it has been verified for many cases that the B_n images are infinite, and even dense in $SU(N)$, see [5, 9, 10].

Decategorified Conjecture. Combining the above conjectures we obtain a category-free statement [3]: Let (R, V) be a *unitary* solution to the YBE. Then $\rho_R(B_n)$ is a virtually abelian group for all n, i.e., $\rho_R(B_n)$ has a normal abelian subgroup of finite index.

As a classification of (unitary) solutions to the Yang-Baxter equation only exists for $\dim(V) = 2$, this may seem like a reckless conjecture. However, there is ample evidence. For example, it has been proved for group-type solutions and Gaussian solutions [12].

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A categorical perspective on braid representations

Fiona Torzewska
(joint work with Paul P. Martin, Eric Rowell)

There has been interest in braids and their relevance to the physical universe for millennia (see e.g. [1, 2] for examples from the 7th century B.C.E. philosopher Gārgī Vāchaknavī and the 19th century mathematician Carl Friedrich Gauss). Mathematically, braids on n strands form a group B_n. It is known that this group is isomorphic to a group defined abstractly via generators $\sigma_1, \ldots, \sigma_{n-1}$ satisfying relations

B1 $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq n-2$
B2 $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \neq 1$.

(the isomorphism takes one of the elementary exchanges of the first two strands to σ_1, and so on). An important manifestation of braids in computational physics is through the constant Yang-Baxter equation:

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$$

where R is an invertible operator on $V^{\otimes 2}$ for some finite dimensional complex vector space V, and $I = \mathrm{Id}_V$ (and we have passed to the usual strictification $V^{\otimes 3}$). In practice $V = \mathbb{C}^N$ for some N, we can fix the standard ordered basis, and then R is a matrix. Originally, such Yang-Baxter operators R were used in statistical mechanics to describe the interaction of particles in physical systems with computationally favourable scattering properties. Moreover, given such an R one obtains, for each $n \in \mathbb{N}$, a representation ρ_n^R of the braid group B_n on n strands, on $V^{\otimes n}$, by defining

$$\rho_n^R(\sigma_i) = I_V^{\otimes i-1} \otimes R \otimes I_V^{\otimes n-i-1}.$$

More recently, aspects of category theory have been embraced by physicists [3].[1] One key reason that category theory is an appropriate framework for physics was observed by Kapranov and Voevodsky [5]: *In any category, it is unnatural and undesirable to speak about equality of two objects*–just as different particles are never equal, but can usefully be regarded as equivalent if having equal responses to certain measurements. Computations in quantum physics are typically linear algebraic, so that the category Mat of matrices is of particular utility. This is a category whose objects are natural numbers $n \geq 1$ and the morphisms from m to n can be taken to be $n \times m$ matrices. The category Mat has a natural strict monoidal structure: on objects this is multiplication and on morphisms it is the Kronecker product. This corresponds to independent event probabilities being composed multiplicatively. Thus categories provide a structure in which to 'do' physics.

Incorporating braids into this framework is facilitated by the braid category B [6]. This is the category whose objects are natural numbers $k \geq 0$ with morphisms

[1] Although somewhat unenthusiastically at first, as Moore and Seiberg [4] describe category theory as "an esoteric subject noted for its difficulty and irrelevance."

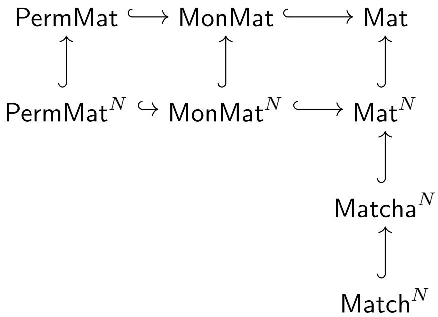

FIGURE 1. A lattice of subcategories of Mat.

from k to k consisting of the braid group B_k on k strands, and no morphisms between distinct $k_1 \neq k_2$. This category is strictly monoidal, with tensor product on objects given by $+$ and on morphisms by a chosen juxtaposition of braids.

For any $N^2 \times N^2$ matrix solution R to the Yang-Baxter equation we obtain a functor $F_R : \mathsf{B} \to \mathsf{Mat}$ by setting $F_R(1) = N$ and setting $F_R(\sigma) = R$. Of course it follows that $F_R(k) = N^k$ and $F_R(\beta) = \rho_n^R(\beta)$ for $\beta \in B_n$. The functor F_R has an additional feature: it is a (strict) monoidal functor –*i.e.* we have $F_R(j) \cdot F_R(k) = F_R(j+k)$. Such a functor $F : \mathsf{B} \to \mathsf{Mat}$ is called a *braid representation.* So classifying matrix solutions to the Yang–Baxter equation is equivalent to classifying braid representations.

An immediate simplification is to only consider the subcategory of *strict* monoidal functors, and monoidal natural transformations which is denoted $\mathsf{MonFun}(\mathsf{B}, \mathsf{Mat})$. This is justified by the fact that every monoidal functor from B to Mat is naturally isomorphic to a strict one (although the isomorphism is not itself monoidal in general).

Having cast the problem in a categorical framework leads naturally to restricting the targets of such functors to families of matrices. For example, we may consider monoidal subcategories of Mat, as in Figure 1. Indeed, the main inspiration for this article is the recent work [7] classifying $N^2 \times N^2$ charge conserving solutions to the Yang-Baxter for all N using this categorical perspective. This motivates the organisational problem:

Problem 1. *Fix a subcategory* Y *of* Mat*. Classify functors in* $\mathsf{MonFun}(\mathsf{B}, \mathsf{Y})$*, up to appropriate equivalences.*

The category $\mathsf{MonFun}(\mathsf{B}, \mathsf{Mat})$ has additional categorical structure that facilitates a more general perspective of the problem above.

Theorem 2. *The category* $\mathsf{MonFun}(\mathsf{B}, \mathsf{Mat})$ *is a strict monoidal category under the lashing product.*

There is also a natural notion of subobjects and quotient objects in the category $\mathsf{MonFun}(\mathsf{B}, \mathsf{Mat})$, which satisfy the following.

Theorem 3. *Objects* $\mathsf{MonFun}(\mathsf{B},\mathsf{Mat})$ *that are both subobjects and quotient objects of* $(N,R) \in \mathsf{MonFun}(\mathsf{B},\mathsf{Mat})$ *correspond to rank* M *endomorphisms* $A \in \mathrm{End}(N,R)$.

Thus, one could start with any object $F \in \mathsf{MonFun}(\mathsf{B},\mathsf{Mat})$ a monoidal functor and consider the monoidal subcategory generated by F and its subobjects. This example, and other categories of functors with non-categorical targets motivates the broader:

Problem 4. *Find subcategories of* $\mathsf{MonFun}(\mathsf{B},\mathsf{Mat})$ *amenable to classification.*

For example, functors associated with *group-type* solutions [8] to the Yang-Baxter equation provide such a subcategory, although they are not closed under composition. Similarly involutive (i.e., $R^2 = \mathrm{Id}$) solutions do not form a subcategory of Mat, but the associated monoidal functors form a tensor subcategory of $\mathsf{MonFun}(\mathsf{B},\mathsf{Mat})$ and they have been classified [9].

The main goal of this work is to lay the groundwork for applying categorical techniques to the classification problem. We also have the following results.

Theorem 5. *The inner autoequivalences of* $\mathsf{MonFun}(\mathsf{B},\mathsf{Mat}^N)$ *of the form* $Q \otimes Q$ *that restrict to an autoequivalence of natural elements of* $\mathsf{MonFun}(\mathsf{B},\mathsf{Match}^N)$ *are those with* Q *an invertible* $N \times N$ *monomial matrix.*

Theorem 6. *Let* $(N,R) \in \mathsf{MonFun}(\mathsf{B},\mathsf{Mat}^N)$. *Suppose that there is some matrix* Q *satisfying* $(Q \otimes Q)R(Q^{-1} \otimes Q^{-1}) = R$, *and define* $S = (\mathrm{Id}_N \otimes Q)R(\mathrm{Id}_N \otimes Q^{-1})$. *Then* R *and* S *are connected by a natural transformation of functors.*

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One is never too old to learn about Nichols algebras

Leandro Vendramin
(joint work with A. Andruskiewitsch, I. Heckenberger, E. Meir)

In this talk, I will review basic aspects of the theory of Nichols algebras over groups. I will present one of the definitions, give examples, and mention several classification results, including four different joint works written with Heckenberger and Meir [3] and with Heckenberger [4, 5]. I will also briefly mention a conjecture of [1] concerning Nichols algebras over simple groups:

Conjecture 1. *Let G be a finite simple group and V a non-zero Yetter-Drinfeld module over $\mathbb{C}G$. Then* $\dim \mathcal{B}(V) = \infty$.

The conjecture remains open. In her talk, Giovanna Carnovale will explain possible approaches to this conjecture and describe the current state of the art.

On the other hand, in a recent joint work with Andruskiewitsch and Heckenberger [2] we provide a complete classification of complex finite-dimensional Nichols algebras over finite solvable groups. In fact, no new examples arise: all the Nichols algebras appearing in this context are somewhat known.

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