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Recent Developments in SPDEs and BSDEs meet Harmonic and Functional Analysis

Organized by
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Abstract. The purpose of this workshop is the strengthening of the interaction between the fields of Stochastic Partial Differential Equations (SPDEs), Backward Stochastic Differential Equations (BSDEs), and Harmonic and Functional Analysis. We focus on the essential role of analytic techniques (including function spaces, weighted inequalities, and A_p-weights) in solving problems in critical stochastic settings. Special topics include SPDEs in critical spaces, regularization by noise, singular SPDEs, quadratic and nonlocal BSDEs.

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Introduction by the Organizers

The workshop *Recent developments in SPDEs and BSDEs meet harmonic and functional analysis*, organized by Benjamin Gess (Bielefeld/Leipzig), Stefan Geiss (Jyväskylä), Stefanie Petermichl (Würzburg), and Mark Veraar (Delft), brought together experts from the fields of stochastic analysis on the one side, and geometric and harmonic analysis on the other side.

The central theme of the meeting was the deepening interaction between the analysis of Stochastic Partial Differential Equations (SPDEs) and Backward Stochastic Differential Equations (BSDEs) on the one hand, and modern methods from Harmonic and Functional Analysis on the other. This intersection has recently yielded deep insights into phenomena that are inaccessible through deterministic methods alone. This volume collects extended abstracts from a series of talks that highlight the vitality of this field, spanning a wide spectrum of inquiry from the regularization effects of noise in fluid dynamics to the fine geometric properties of rough paths, from bounded mean oscillation and interpolation to BSDEs and SPDEs, and from the rigorous foundations of stochastic maximal regularity to sharp estimates in harmonic analysis.

SPDEs, Regularity and Approximation

A major focus of the workshop was the treatment of SPDEs with a focus on well-posedness, approximation, and regularity theory.

A crucial ingredient for local and global well-posedness theory for parabolic quasilinear SPDEs in critical spaces is the theory of *Stochastic Maximal L^p-Regularity* (SMR). Emiel Lorist explained in his lecture how SMR provides sharp regularity estimates for the linear part of an SPDE and represents a fusion of probabilistic techniques with operator theory. Participants discussed how SMR relates to functional calculus, Calderón–Zygmund theory, and Littlewood–Paley theory, and how a more probabilistic approach to harmonic analysis, such as sparse domination and A_p-weights, might resolve open problems in this area. Max Sauerbrey explained how he used the abstract local and global well-posedness theory for the thin film equations. SMR results play also a key role in the phenomenon of *regularization by noise*, as discussed in the talk of Antonio Agresti on the 3D Navier–Stokes equations and in the talk of Davide Addona regarding damped wave equations and stochastic damped Euler–Bernoulli equations.

The field of singular and conservative SPDEs, together with rough-path analysis, constituted a major focus of the workshop. Prototypical examples include the KPZ equation, models of fluctuating hydrodynamics such as the Dean–Kawasaki system, and quantized quantum field theories. These equations are generally ill-posed in the classical sense because the driving noise exhibits low regularity, necessitating renormalization and the development of nonstandard analytic frameworks. A central theme was the interaction between the theory of regularity structures, the paracontrolled distribution method, and probabilistic techniques such as martingale formulations and the analysis of Markov semigroups. Paracontrolled calculus, grounded in harmonic analysis and relying on Bony's paraproduct decomposition and Littlewood–Paley theory, yields a robust mechanism for defining nonlinear operations on distributions of negative regularity. It thereby establishes a close methodological link between harmonic-analytic tools and stochastic analysis. These topics were addressed in presentations by Sonja Cox, Máté Gerencsér, Peter Imkeller, Helena Kremp, Vitalii Konarovskyi, Adrian Martini, Jonas Sauer, and Alexander Steinicke. Moreover, Zdzisław Brzeźniak reported new results concerning stochastic heat equations under geometric constraints.

BSDEs, Quadratic Growth, and Non-Local Operators

A significant part of the workshop concerned Backward Stochastic Differential Equations (BSDEs). Originally derived for non-linear parabolic PDEs and optimal control, BSDEs have seen a surge in activity regarding their connection to harmonic analysis and structural stability. Nine talks did relate to this topic directly. A focal point was *quadratic and mean field BSDEs* (qBSDEs), where the driver has a quadratic non-linearity in the gradient. This represents a critical setting analogous to the one found in SPDEs: the second-order variation from the stochastic integration competes with the first-order driver term, leading to potential instabilities. Super quadratic BSDEs with an unbounded terminal condition and jumps were discussed by Hannah Geiss and the global stochastic maximum principle for mean-field forward-backward stochastic control systems with quadratic generators by Juan Li. The question of generalized mean-field dynamics for BSDEs was also addressed by Rainer Buckdahn who discussed viscosity solutions to the master Bellman equation in a setting where market participants have different control options. Another important class of BSDE are equations *subject to a reflection on the boundary of a domain or to a so-called switching*, where both settings may lower the regularity of a BSDE. A natural condition to study reflection is the convexity of the domain. However, in applications non-convex domains appear, which was addressed by Adrien Richou. Celine Labart presented work about numerics for optimal switching problems. Switching problems occur, for example, in models for energy markets as power plants cannot be scaled in a continuous way, but only in a discrete one. This switching behaviour generates additional issues and potential singularities in numerical algorithms as the switching occurs at random times. A further topic concerned *gradient and variational estimates* for BSDEs. Federica Masiero investigated Bismut–Elworthy type formulae for BSDEs with degenerate noise. Bismut–Elworthy type formulae provide a representation of the gradient of the generator associated to the BSDE. Closely connected, Stefan Geiss discussed gradient estimates for non-local PDEs associated to Banach space valued Lévy processes and Łukasz Leżaj considered the regularity of BSDEs driven by random measures by means of a newly introduced coupling method - which directly relates to non-local PDEs as well. Finally, Ludger Overbeck presented related stochastic representations of path-dependent non-local PDEs.

Probabilistic aspects in Harmonic Analysis

Since the late 90s, dyadic harmonic analysis has seen major changes. Radically different proofs emerged for very delicate, difficult new mathematics, that hinted at a connection to probability theory, that has recently been explored and used in various fields in analysis.

Results on functional calculus appeared in the talks by Christoph Kriegler, Floris Roodenburg, and Jan van Neerven. Recent developments in the theory of A_p-weights were presented in the lectures of Spyridon Kakaroumpas, Stefanie Petermichl, and Sandra Pott. Here, the matrix weighted questions were at a focus, with their origins in multivariate stationary stochastic processes and mixing problems. The talks featured a negative solution to a long standing conjecture in the field (the matrix A_2 conjecture) that in turn relies on a construction of a matrix valued martingale with infinitely many singularities. The aspects brought by Pott showed an operator theoretic, completely different, very meaningful approach to these questions, while the result presented by Kakaroumpas looked into the matrix A_2 conjecture with the characteristic dampened through the Poisson kernel. Discrete to continuous settings were part of many other lectures, in particular, in the presentations of Komla Domelevo on the UMD property in conjunction with the boundedness of the Hilbert transform, Johanna Fladung on the best constant in a dyadic version of Uchiyama's Lemma, and Patricia Alonso Ruiz on the algebra property of Sobolev spaces. Mateusz Kwaśnicki presented his recent results on using continuous martingale transforms to study discrete versions of well-known singular integrals arising in harmonic analysis. His important developments are a prime example of the interplay of harmonic analysis and probability, as these methods had solved a famous open question on the norm of discrete Hilbert transform, in itself a deterministic subject. In addition, Krzysztof Bogdan gave an overview about recent developments related to the Bregman variation of semimartingales and their applications to harmonic analysis, further illustrating the cross-fertilization of the fields.

In summary, the workshop served as a platform to bridge the gap between stochastic analysis and harmonic analysis. It highlighted that these fields are no longer evolving in isolation; rather, the analytical tools of weights, interpolation, and maximal regularity are now fundamental components of the stochastic analyst's toolkit, just as probabilistic challenges are driving new inquiries in pure analysis.

The organizers also made a specific effort to invite PhD students and postdocs and to give them the opportunity to speak as well. For most of them it was the first Oberwolfach workshop they attended. The organizers also identified possible female participants, thus ensuring that among the participants of the workshop there were twelve female mathematicians.

Workshop: Recent Developments in SPDEs and BSDEs meet Harmonic and Functional Analysis

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Abstracts

Pathwise uniqueness and strong existence in infinite dimension

Davide Addona
(joint work with Davide Augusto Bignamini)

1. Assumptions and definition

Let U and H be separable Hilbert spaces. We consider the Stochastic Differential Equation (SDE) evolving in H of the form

$$dX(t) = AX(t)dt + B(X(t))dt + GdW(t),\ t \in (0,T], \qquad X(0) = x \in X, \tag{1}$$

where $W = (W(t))_{t\geq 0}$ is a U-cylindrical Wiener process, i.e., for every $t \geq 0$, $W(t)$ can be written, at least formally, as $W(t) = \sum_{n\in\mathbb{N}} e_n\beta_n(t)$, $\mathbb{P}$-a.s., where $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis of U and $(\beta_n)_{n\in\mathbb{N}}$ is a family of independent standard real Brownian motions. In this talk I present some result about pathwise uniqueness for the solution to (1). I further show that, in some situation, under our assumptions, the deterministic counterpart of (1), i.e., without W, is ill-posed.

We begin by listing our hypotheses.

Hypotheses 1.1.

(i) *$A : \mathrm{Dom}(A) \subseteq H \to H$ generates the C_0 analytic semigroup $\{e^{tA}\}_{t\geq 0} \subset \mathcal{L}(H)$.*

(ii) *$G \in \mathcal{L}(U;H)$ satisfies $G = \widetilde{G}\mathcal{V}$, where $\widetilde{G} \in \mathcal{L}(U;H)$ and $\mathcal{V} \in \mathcal{L}(U;U)$. $B = \widetilde{G}\widetilde{B}$, where $\widetilde{B} \in C^\theta_b(H;U)$ for some $\theta \in (0,1)$.*

(iii) *There exists a sequence of finite-dimensional subspaces $\{H_n\}_{n\in\mathbb{N}} \subseteq H$ such that $H = \overline{\cup_{n\in\mathbb{N}} H_n}$, $H_0 := \{0\}$ and for every $n \in \mathbb{N}$ we have*

$$H_{n-1} \subseteq H_n, \qquad H_{n-1} \subseteq \mathrm{Dom}(A), \qquad A(H_n \cap H_{n-1}^\perp) \subseteq \left(H_n \cap H_{n-1}^\perp\right).$$

(iv) *For every $t > 0$ we have $e^{tA}(H) \subseteq Q_t^{\frac{1}{2}}(H)$, $\Gamma_t := Q_t^{-\frac{1}{2}} e^{tA} \in \mathcal{L}(H)$ and*

$$Q_t := \int_0^t e^{sA}GG^* e^{sA^*} ds, \qquad \int_0^t \|\Gamma_s\|^{1-\theta}_{\mathcal{L}(H)} \|\Gamma_s \widetilde{G}\|_{\mathcal{L}(U;H)} ds < \infty,$$

and $(0,t) \ni s \mapsto \|\Gamma_s\|^{1-\theta}_{\mathcal{L}(H)}$, $\|\Gamma_s\widetilde{G}\|_{\mathcal{L}(U;H)}$ are bounded from below functions.

We say that $X = (X(t))_{t\geq 0}$ is a mild solution to (1) if, for every $t \in [0,T]$,

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A}B(X(s))ds + \int_0^t e^{(t-s)A}GdW(s), \qquad \mathbb{P}\text{-a.s.} \tag{2}$$

Definition 1.2. *We say that for (1) weak mild existence holds if there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, which satisfies the usual conditions, a U-cylindrical Wiener process $W = (W(t))_{t\geq 0}$ and a process $X = (X(t))_{t\in[0,T]}$, adapted with respect to $(\mathcal{F}_t)_{t\geq 0}$, which satisfies (2).*

Definition 1.3. *Strong uniqueness holds for* (1) *if every couple of weak mild solutions* X^1, X^2*, defined on the same probability space with respect to the same cylindrical Wiener process, satisfies* $\mathbb{P}(X^1(t) = X^2(t),\ t \in [0,T]) = 1$.

2. The strategy and the main theorem

We begin by defining the finite dimensional approximations and stating their main properties. Fix a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, X, W)$ to (1). For every $n \in \mathbb{N}$, we set

$$dX_n(t) = A_n X_n(t)dt + B_n(X(t))dt + G_n dW(t), \quad t \in (0,T], \quad X_n(0) = P_n x, \tag{3}$$

where P_n is the projection on H_n, $A_n = AP_n$, $B_n = P_n B(P_n\cdot)$, $G_n = P_n G$.

Lemma 2.1. *Assume Hypotheses 1.1. Then,* $\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}[|X_n(t) - X(t)|^2] = 0$.

We prove a representation of X_n by means of a modification of the Itô–Tanaka trick. For every $n \in \mathbb{N}$, U_n is the mild solution to the backward Kolmogorov equation with $U_n(T,x) = 0$ and

$$\partial_t U_n(t,x) + \frac{1}{2}\mathrm{Tr}[G_n G_n^* D^2 U_n(t,x)] + \langle Ax, DU_n(t,x)\rangle + \langle B_n, DU_n(t,x)\rangle = -B_n(x).$$

Proposition 2.2. *Assume that Hypotheses 1.1 are satisfied. Then, for every* $n \in \mathbb{N}$ *and every* $t \in [0,T]$*, we have*

$$\begin{aligned} X_n(t) =& e^{tA_n} P_n x - U_n(t, X_n(t)) + e^{tA_n} U_n(0, P_n x) \\ &+ \int_0^t e^{(t-s)A_n}\big(B_n(X(s)) - B_n(X_n)\big)ds - A_n \int_0^t e^{(t-s)A_n} U_n(s, X_n(s))ds \\ &+ \int_0^t e^{(t-s)A_n} D_x U_n(s, X_n(s))\big(B_n(X(s)) - B_n(X_n(s))\big)ds \\ &+ \int_0^t e^{(t-s)A_n} D_x U_n(s, X(s)) G_n dW(s) + \int_0^t e^{(t-s)A_n} G_n dW(s). \end{aligned} \tag{4}$$

Theorem 2.3. *Pathwise uniqueness holds true for* (1).

Proof. We consider two weak mild solutions X^1, X^2, defined on the same probability space with respect to the same cylindrical Wiener process W, and the respective finite dimensional approximations X_n^1, X_n^2. From (4) we obtain

$$\int_0^T \mathbb{E}|X_n^1 - X_n^2|_H^2 dt \leq o(T,0) \int_0^T \mathbb{E}|X_n^1 - X_n^2|_H^2 + o(n,\infty).$$

Letting n tend to ∞ and exploiting Lemma 2.1, we get the thesis. □

3. An example

Our results apply to the following stochastic damped wave differential equation:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(t) = -\Delta y(t) - \rho\Delta^\alpha\left(\dfrac{\partial y}{\partial t}(t)\right) + C\left(y(t), \dfrac{\partial y}{\partial t}(t)\right) + \Delta^{-\gamma}\dot{W}(t),\ t \in (0,T], \\ y(0) = y_0 \in H_0^1(0,1),\ \dfrac{\partial y}{\partial t}(0) = y_1 \in L^2(0,1),\ y(t,0) = y(t,1) = 0,\ t \in (0,T], \end{cases}$$

where y is a $L^2(0,1)$-valued function, ρ, γ are positive constants, $\alpha \in (0,1)$, Δ is the realization of the Laplace operator on $L^2(0,1)$ with Dirichlet homogeneous boundary conditions and W is a $L^2(0,1)$-valued cylindrical Wiener process.

Theorem 3.1. *If $\alpha \in \left(\frac{1}{2}, 1\right)$, $\gamma \in \left(\frac{1}{4} - \frac{\alpha}{2}, \frac{1}{2} - \frac{\alpha}{2}\right) \cap [0,\infty)$, $\theta \in \left(\frac{2}{3} \cdot \frac{\gamma+1-\alpha}{1-\alpha}, 1\right)$ if $\gamma + 2\alpha < \frac{3}{2}$ and $\theta \in \left(\frac{4\gamma+2\alpha-1}{2\gamma+\alpha}, 1\right)$ if $\gamma + 2\alpha \geq \frac{3}{2}$, $C \in C_b^\theta(H;U)$, then pathwise uniqueness holds true.*

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On stochastic maximal L^p-regularity and regularization by noise of 3D Navier-Stokes equations

Antonio Agresti

Existence of global smooth solutions to the 3D Navier-Stokes equations (NSEs) is one of the biggest open problems in modern mathematics [5]. In recent years, there has been an increasing interest in the study of *physically* motivated stochastic perturbations of fluid models. Stochastic perturbations often lead to more realistic models, as the randomness can, for instance, be used to account for unmodeled small-scale behavior of fluids. Following [9], a stochastic perturbation of the NSEs can be derived by means of Newton's law, assuming that the trajectory $(\phi_t(x))_t$ of a fluid particle starting at time 0 at x follows the SDE:

$$\dot{\phi}_t(x) = u(t, \phi_t(x)) + \sigma(\phi_t(x)) \circ \dot{W}, \qquad \phi_0(x) = x, \tag{1}$$

where W is an ℓ^2-cylindrical Gaussian noise, $\circ$ denotes the Stratonovich integration, σ is a given coefficient, and u is the vector field to be determined. The above ansatz is motivated by the fact that, in certain regimes, fluids can be seen as a two-scale model: a large *regular* component $u(t, \phi_t(x))$, and a small *turbulent* component $\sigma(\phi_t(x)) \circ \dot{W}$. Under the assumption (1), one can derive the stochastic NSEs:

$$\partial_t u = -\nabla p - (u \cdot \nabla)u + \Delta u + [-\nabla\widetilde{p} + (\sigma \cdot \nabla)u] \circ \dot{W}, \qquad \nabla \cdot u = 0. \tag{2}$$

In light of the above discussion, and known regularizing effects of noise in the context of SDEs (see e.g., [6]), a natural question arises: Does there exist a noise coefficient σ for which (2) is globally well-posed for a *large* class of initial data?

Although very important steps have been made recently in understanding regularizing effects of stochastic perturbations of PDEs (see e.g., [7]), the above question remains, at the moment, unsolved. Here, we report on recent progress regarding this question obtained in [1]. In particular, we consider the following variant of the NSEs posed on the three-dimensional torus $\mathbb{T}^3$:

$$\partial_t u = -\nabla p-(u\cdot\nabla)u-(-\Delta)^\gamma u+[-\nabla\widetilde{p}+(\sigma\cdot\nabla)u]\circ\dot{W}, \quad \nabla\cdot u = 0, \quad u(0) = u_0. \tag{3}$$

where $\gamma \geq 1$, and $(-\Delta)^\gamma$ is the Fourier multiplier on $\mathbb{T}^3$ with symbol $|k|^{2\gamma}$ for $k \in \mathbb{Z}^3$. In the case $\gamma > 1$, the operator $(-\Delta)^\gamma$ is often referred to as hyperviscosity, and the system (3) is often called the hyperviscous NSEs (HNSEs). Of course, in the case $\gamma = 1$, it coincides with (2).

Theorem 1 (Informal version of Theorem 2.2 in [1])**.** *Fix $\gamma > 1$, $N \geq 1$ and $\varepsilon \in (0,1)$. Then there exists a smooth vector field $\sigma : \mathbb{T}^3 \to \ell^2$ such that, for all initial data u_0 satisfying*

$$u_0 \in H^1(\mathbb{T}^3;\mathbb{R}^3), \qquad \nabla\cdot u_0 = 0, \qquad \|u_0\|_{H^1} \leq N,$$

the stochastic 3D HNSEs (3) *has a unique global smooth (in space) solution u with lifetime τ satisfying*

$$\mathbb{P}(\tau = \infty) > 1 - \varepsilon.$$

In other words, u is smooth (in space) *and* global in time *with high probability.*

The above result holds for any $\gamma > 1$, and therefore almost reaches the Navier-Stokes case $\gamma = 1$. Interestingly, in the absence of noise, the above result is *not* known in the regime $1 < \gamma \ll \frac{5}{4}$.

The exponent $\gamma = \frac{5}{4}$ was found by J.L. Lions, and as we will discuss in a moment, it distinguishes between energy sub- and super-critical hyperviscosity. Let us emphasize that, on the one hand, the transport noise has a non-trivial effect on the dynamics of (3). On the other hand, however, it does *not* affect the energy balance. More precisely, the Itô formula ensures that any sufficiently regular solution u to (3) satisfies the energy equality: a.s. for all $t < \tau$,

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|(-\Delta)^\gamma u\|_{L^2}^2\,\mathrm{d}s = \|u_0\|_{L^2}^2. \tag{4}$$

From a scaling argument (see e.g., [1, Subsection 1.2]), the natural/critical barrier in the L^2-setting for well-posedness of the HNSEs is given by the smoothness $\frac{5}{2}-2\gamma$. In particular, the energy bound (4) is (sub-)critical if and only if $\frac{5}{2} - 2\gamma \leq 0$, i.e., $\gamma \geq \frac{5}{4}$. This is the reason why global well-posedness of the deterministic HNSEs (i.e., (3) with $\sigma = 0$) is wide open for $\gamma \ll \frac{5}{4}$ (see e.g., [4] for the case $\gamma \approx \frac{5}{4}$).

The main idea behind the result is to combine the recent results on scaling limits for SPDEs with transport noise introduced in [8] for SPDEs, and [7] for the Stokes system (as used for the HNSEs (3)), with stochastic maximal L^p-regularity techniques (see e.g., [3] and the references therein). Next, we roughly summarize the application of the above-mentioned scaling limit to the HNSEs (3) to highlight the role of the L^p-estimates with $p \gg 2$.

Interplay between scaling limit and L^p-estimates. From [7], it follows that for each $\mu > 0$, there exists a sequence of *smooth* coefficients $(\sigma^n)_n$ such that the solution u^n to (3) with σ replaced by σ^n converges to $u_{\rm det}$ that solves

(5)

$$\partial_t u_{\rm det} = -\nabla p_{\rm det} - (u_{\rm det}\cdot\nabla)u_{\rm det} - (-\Delta)^\gamma u_{\rm det} + \mu\Delta u_{\rm det}, \quad \nabla\cdot u_{\rm det} = 0, \quad u(0) = u_0.$$

Interestingly, the above limiting PDE possesses an additional (eddy) dissipation $\mu\Delta$. Although being of lower order, one can prove that if μ is large enough compared to u_0, then the solutions to (5) are globally smooth in time and space (see [1, Theorem 5.1]). Thus, if $\mu \gg 1$, then (5) is well-behaved. In particular, as $u^n \to u_{\rm det}$, one is tempted to transfer some of the regularity of $u_{\rm det}$ to u^n, provided n is large enough. In light of the criticality of the smoothness $\frac{5}{2} - 2\gamma$, this argument yields additional regularity if and only if the limit takes place in a space with higher smoothness compared to $\frac{5}{2} - 2\gamma$. Indeed, if this is not the case, then the energy equality (4), which holds for all u^n, gives already more information about u^n. Hence, one aims for a convergence of u^n towards $u_{\rm det}$ in spaces with higher regularity than $L^\infty(0, T; L^2)$. However, due to oscillatory behavior of $(\sigma^n)_n$ which allows for the extra dissipation in (5), one has

$$\text{(6)} \qquad \sup_{n\geq 1} \|\sigma^n\|_{L^\infty(\mathbb{T}^3;\ell^2)} < \infty \quad \text{and} \quad \sup_{n\geq 1} \|\sigma^n\|_{H^r(\mathbb{T}^3;\ell^2)} = \infty \ \text{ for all } r > 0.$$

In particular, no smoothness of σ^n is preserved in the limit $n \to \infty$. One of the consequences of (6) is that, as discussed in [2, Remark 3.4], u^n does *not* converge to $u_{\rm det}$ in probability in $L^2(0, T; H^\gamma)$. Hence, to improve the convergence of u^n to $u_{\rm det}$, one has to balance the lack of convergence in $L^2(0, T; H^\gamma)$ and the need for a convergence in (at least) $L^\infty(0, T; H^{\frac{5}{2}-2\gamma})$. This can be achieved using *stochastic maximal L^p-regularity* estimates. Indeed, for all $p < \infty$ (a suitable truncation of) u^n converges to $u_{\rm det}$ in probability in

$$\text{(7)} \qquad C([0, T]; H^{\gamma(1-2/p)-\varepsilon}) \cap L^p(0, T; H^{\gamma-\varepsilon}) \ \text{ for all } \varepsilon > 0,$$

where $T \gg 1$ is a finite time horizon. Once the convergence in probability in (7) is proved, then for any given $\varepsilon > 0$, one has $u^n \approx u_{\rm det}$ in $C([0, T]; H^{\gamma(1-2/p)-\varepsilon})$ with high probability. Due to the criticality of $\frac{5}{2} - 2\gamma$, the latter is useful in case

$$\text{(8)} \qquad \gamma(1 - \tfrac{2}{p}) > \tfrac{5}{2} - 2\gamma \quad \Longleftrightarrow \quad p > \tfrac{4\gamma}{6\gamma-5}.$$

On the use of $\gamma > 1$. As described in [1, Section 3], the assumption $\gamma > 1$ is crucial to prove the maximal L^p-regularity estimate leading to the convergence (7). Indeed, in the latter case, the transport noise $u \mapsto (\sigma\cdot\nabla)u \circ \dot{W}$ in (3) behaves like a lower-order term compared to the hyperviscosity $(-\Delta)^\gamma$ even for low regularity as in (6). In particular, the perturbation result in [3, Theorem 3.2] applies. In the regime $\gamma = 1$, the latter argument breaks down, and a more careful analysis of the interplay between the Laplace operator Δ and the stochastic transport $u \mapsto (\sigma\cdot\nabla)u \circ \dot{W}$ is required.

Open problems. A detailed discussion on the possibility of extending Theorem 1 by means of improving the L^p-estimates in [1, Section 3] is discussed in Section 1.2 of the aforementioned work. As a final remark, it is worth pointing out that, as in (8), one does not need the convergence in (7) for arbitrary p, but only for a single value of $p > 4$. At the moment, the latter seems a very challenging problem at the intersection between harmonic analysis, PDEs, and probability.

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Dyadic fractional Sobolev spaces and (the absence of) their algebra property

Patricia Alonso Ruiz
(joint work with Valentia Fragkiadaki)

There are a number of ways to characterize fractional Sobolev spaces depending on interest and taste: Via Fourier transform and Littlewood–Paley decomposition, by interpolation theory, through singular integrals, and also using semigroups and subordination. There is a big fractional world out there!

Motivated by the possibility to bypass the Fourier transform and to avoid abstract (yet powerful) arguments as in the case of interpolation or semigroup theory, in this talk we report a dyadic approach to these spaces and the investigation of their algebra property (or its absence) by purely dyadic methods.

To define the dyadic fractional Sobolev space, we first consider the dyadic analogue to the fractional derivative $(-\Delta)^{s/2}$, which is defined as

$$D^s_{\mathcal{D}} f := \sum_{\substack{Q\in\mathcal{D}\\ \varepsilon\neq(1,\dots,1)}} |Q|^{-\frac{s}{n}} (f, h^\varepsilon_Q) h^\varepsilon_Q(x),$$

see e.g. [1]. Here, $\mathcal{D} := \{Q \subset \mathbb{R}^n \text{ dyadic cubes of side length } 2^k,\ k \in \mathbb{Z}]\}$ and

$$\{h^\varepsilon_Q \colon Q \in \mathcal{D}, \varepsilon \in \{0,1\}^n \backslash \{(1,\dots,1)\}\}$$

build a (standard) orthonormal basis of Haar functions in $\mathbb{R}^n$, so that any function f in its span $\mathcal{S}_{\mathcal{D}}$ may be written as

$$f = \sum_{\substack{Q\in\mathcal{D}\\ \varepsilon\neq(1,\dots,1)}} (f, h^\varepsilon_Q) h^\varepsilon_Q.$$

Analogue to the classical definition of the fractional Sobolev space $H^s(\mathbb{R}^n)$ via Fourier transform, one defines the corresponding (fractional) Sobolev seminorm

$$\|f\|_{\dot H^s_{\mathcal{D}}(\mathbb{R})} := \|D^s_{\mathcal{D}} f\|_{L^2(\mathbb{R}^n, dx)} = \Bigg(\sum_{\substack{Q\in\mathcal{D}\\ \varepsilon\not\equiv 1}} |Q|^{-\frac{2s}{n}} |(f, h^\varepsilon_Q)|^2 \Bigg)^{1/2} \tag{1}$$

as well as the dyadic Sobolev space

$$H^s_{\mathcal{D}}(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n, dx) \colon D^s_{\mathcal{D}} f \in L^2(\mathbb{R}^n, dx)\} \tag{2}$$

equipped with the norm $\|f\|_{H^s_{\mathcal{D}}(\mathbb{R}^n)} := \big(\|f\|^2_{\dot H^s_{\mathcal{D}}(\mathbb{R}^n)} + \|f\|^2_{L^2(\mathbb{R}^n,dx)}\big)^{1/2}$. Note that the classical connection

$$f = C_{n,s} I_s (-\Delta)^{s/2} f,$$

where

$$I_s f(x) := \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-s}} f(y) dy, \tag{3}$$

and $C_{n,s} > 0$ is some explicit constant, also holds in the dyadic setting. In this case however, the constant is *independent* of the dimension n. The dyadic analogue to (3), given by

$$T_{\mathcal{D},s} g := \sum_{Q\in\mathcal{D}} |Q|^{\frac{s}{n}} \langle g \rangle_Q \mathbf{1}_Q,$$

has since long been object of study in the dyadic literature, see e.g. [3].

Lemma 1. *Let $0 < s < n$. For any $f \in \mathcal{S}_{\mathcal{D}}(\mathbb{R}^n)$, $f = (2^s - 1) T_{\mathcal{D},s} D^s_{\mathcal{D}} f$.*

The main novelty highlighted in this talk is the fact that, by employing *only* algebraic manipulations of the seminorm (1), the averages $\langle f \rangle_Q$ and the Haar coefficients of f^2, given by

$$(f^2, h^\eta_P) = \sum_{Q\subseteq P} \sum_{\varepsilon\not\equiv 1} (f, h^\varepsilon_Q)^2 h^\eta_P(Q) + \sum_{Q\subseteq P} \sum_{\substack{\varepsilon,\tilde\varepsilon\neq(1,\dots,1)\\ \varepsilon+\tilde\varepsilon=\eta}} (f, h^\varepsilon_Q)(f, h^{\tilde\varepsilon}_Q) + 2(f, h^\eta_P)\langle f \rangle_P$$

for any $P \in \mathcal{D}$ and $\eta \neq (1,\dots,1)$, we can fully describe (the absence of) the algebra property of the dyadic Sobolev space (2). The result recovers its smooth analogue, that is due to Strichartz [2] and reads as follows.

Theorem 2. *The space $H^s(\mathbb{R}^n)$ is an algebra if and only if $\frac{n}{2} < s < n$:*
(i) *For $0 < s < \frac{n}{2}$, the function*

$$f := \sum_{k=0}^{\infty} |Q^{(k)}|^{\alpha}\, h^{\varepsilon_0}_{Q^{(k)}}, \tag{4}$$

with $\frac{s}{n} < \alpha < \frac{s}{2n} + \frac{1}{4}$, belongs to $H^s(\mathbb{R}^n)$ while $f^2 \notin H^s(\mathbb{R}^n)$.
(ii) *For $s = \frac{n}{2}$, the function*

$$f := \sum_{k>0} \frac{1}{2^{kn/2} k^{\alpha/2}} h^{\varepsilon_0}_{Q^{(k)}} \tag{5}$$

with $1 < \alpha \leq 3/2$ belongs to $H^{\frac{n}{2}}(\mathbb{R}^n)$ while $f^2 \notin H^{\frac{n}{2}}(\mathbb{R}^n)$.
(iii) *For $\frac{n}{2} < s < n$, $f \in H^s(\mathbb{R}^n)$ implies $f^2 \in H^s(\mathbb{R}^n)$.*

In the counterexamples (4) and (5), $Q^{(k)}$ is a monotone decreasing sequence of dyadic cubes with $|Q^{(k)}| = 2^{-k}$, $k \geq 0$. The proof of the positive result in the range $\frac{n}{2} < s < n$ requires the Sobolev embedding $H^s(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n)$, which can also be proved *only* through algebraic manipulations involving Haar functions, coefficients and dyadic cubes.

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Parabolic Anderson model in bounded domains of recurrent metric measure spaces

Fabrice Baudoin
(joint work with Li Chen, Cheng Ouyang, Samy Tindel, Jing Wang)

We consider parabolic Anderson models (PAMs) on bounded domains of *recurrent metric measure spaces* equipped with a strongly local regular Dirichlet form. Recurrence is characterized by the inequality $d_h < d_w$, where d_h is the Hausdorff dimension and d_w the walk dimension. In this setting, the associated Brownian motion is strongly recurrent and admits local times, a key feature that allows for highly singular driving noises for PAMs.

The main object of study is the stochastic partial differential equation

$$\partial_t u(t,x) = \Delta u(t,x) + \beta u(t,x)\, \dot{W}_\alpha(t,x),$$

posed on a bounded domain U, with either Dirichlet or Neumann boundary conditions. The noise $\dot{W}_\alpha$ is white in time and spatially correlated, with covariance kernel given by the fractional Green function $G_{2\alpha} = (-\Delta)^{-2\alpha}$. The parameter $\alpha \geq 0$ interpolates between space–time white noise ($\alpha = 0$) and smoother spatial noises.

A first main result is the existence and uniqueness of solutions in the Itô–Skorohod sense for *all* $\alpha \geq 0$, including the space–time white noise case. This is a direct consequence of the recurrence assumption $d_h < d_w$ (equivalently, spectral dimension $d_s = 2d_h/d_w < 2$), which sharply contrasts with the Euclidean situation in dimension $d \geq 2$.

The second major contribution concerns precise moment estimates and intermittency properties. For Dirichlet boundary conditions, the p-th moments exhibit a competition between dissipation driven by the principal eigenvalue λ_1 and noise-induced growth. The long-time Lyapunov exponents are shown to depend on a geometry-dependent function Θ_α, whose asymptotic behavior reveals new intermittency exponents depending on (d_h, d_w, α). Sharp upper and lower bounds are obtained, with matching exponents in the strong disorder regime $0 < \alpha < d_s/4$.

For Neumann boundary conditions, the heat semigroup is conservative and no dissipative mechanism is present. As a consequence, the model exhibits strong disorder for any $\beta > 0$, with exponential growth of all moments. Lower bounds again reveal enhanced intermittency when the noise is sufficiently singular.

The analysis relies on a combination of chaos expansions, Feynman–Kac representations, and fine heat kernel estimates under sub-Gaussian bounds. The results apply to a broad class of spaces, including bounded intervals, metric graphs, and post-critically finite fractals such as the Sierpiński gasket, for which scaling invariance properties of the PAM are established.

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Bregman variation of semimartingales

Krzysztof Bogdan

(joint work with Dominik Kutek, Katarzyna Pietruska-Pałuba)

The talk is based on [4], where we introduce the Bregman variation of semimartingales. See also [2] and [3]. Here are basic definitions, theorems, and examples:

Young functions. A function $\phi : \mathbb{R} \to \mathbb{R}_+$ is called a *Young function* if it is even and convex and satisfies

$$\lim_{\lambda \to 0^+} \frac{\phi(\lambda)}{\lambda} = 0, \qquad \lim_{\lambda \to \infty} \frac{\phi(\lambda)}{\lambda} = \infty.$$

For example,

$$\phi(\lambda) = |\lambda|^p, \quad \phi(\lambda) = |\lambda|^p \ln^s(e+|\lambda|) \quad \text{with} \quad p>1, s\geq 0, \quad \phi(\lambda) = e^{|\lambda|}-1-|\lambda|.$$

Bregman divergence. For a convex function $\phi : \mathbb{R} \to \mathbb{R}$, the *Bregman divergence* associated with ϕ is defined by

$$F_\phi(x,y) = \phi(y) - \phi(x) - \phi'(x)(y-x), \qquad x, y \in \mathbb{R},$$

where ϕ' denotes the left derivative of ϕ. By convexity, $F_\phi(x,y) \geq 0$.

For example, $\phi(x) = |x|^p$ with $p > 1$ yields

$$F_\phi(x,y) = |y|^p - |x|^p - p|x|^{p-2}x(y-x).$$

Bregman variation of a semimartingale. Let ϕ be a C^1 Young function and let X be a semimartingale [1]. The *ϕ-variation* (Bregman variation) of X is

$$V^\phi(X)_t := \phi(X_t) - \int_0^t \phi'(X_{s-})\,\mathrm{d}X_s, \qquad t \geq 0.$$

In short, $V^\phi(X) = \phi(X) - \phi'(X_-)\cdot X$, which is again a semimartingale, with jumps

$$V^\phi(X)_t - V^\phi(X)_{t-} = F_\phi(X_{t-}, X_t), \qquad t > 0.$$

Moderate Young functions. A Young function ϕ satisfies the *doubling condition* if there exists $K < \infty$ such that

$$\phi(2\lambda) \leq K\,\phi(\lambda), \qquad \lambda > 0.$$

A Young function ϕ is called *moderate* if both ϕ and its Legendre transform ϕ^* satisfy the doubling condition.

Our first main result is the **Itô-type isometry**: If X is a càdlàg local martingale and ϕ is a C^1 moderate Young function, then the following are equivalent:

(A) X is ϕ-integrable;
(B) $\mathbb{E}\,V^\phi(X)_t < \infty$ for every $t \in \mathbb{R}_+$.

If they hold, then

$$\mathbb{E}\,\phi(X_t) = \mathbb{E}\,V^\phi(X)_t, \qquad t \in \mathbb{R}_+.$$

Furthermore, if ϕ is a C^1 moderate Young function, X is a ϕ-integrable càdlàg martingale, τ is an a.s. finite stopping time, and $\mathbb{E}V^\phi(X)_\tau < \infty$, then

$$\mathbb{E}\,\phi(X_\tau) = \mathbb{E}\,V^\phi(X)_\tau.$$

For example, $\phi(x) = x^2$ yields the usual *quadratic variation* of X,

$$[X]_t := X_t^2 - \int_0^t 2X_{s-}\,\mathrm{d}X_s, \qquad t \geq 0.$$

We denote by $[X]_t^c$ the continuous part of $[X]_t$. Then,

$$[X]_t = [X]_t^c + X_0^2 + \sum_{0<s\leq t,\, \Delta X_s \neq 0} (\Delta X_s)^2, \qquad t \geq 0.$$

Our second main result is the **representation of the Bregman variation** using the quadratic variation: Let ϕ be a moderate Young function with absolutely continuous derivative ϕ', and let $\mu_\phi = \phi''$ be the second derivative interpreted as a measure. For a semimartingale X and a stopping time $\tau < \infty$, we have

$$V^\phi(X)_\tau = \phi(X_0) + \frac{1}{2}\int_0^\tau \phi''(X_{s-})\,\mathrm{d}[X]^c_s + \sum_{0<s\le\tau,\,\Delta X_s\neq 0} F_\phi(X_{s-}, X_s).$$

The third main part of the paper are the applications to **Hardy-Stein identities**: Here is a road map. For a strong Markov process X and an open set D, the process $u(X_{t\wedge\tau_U})$ is a martingale if u is harmonic in $U \subset\subset D$. Here,

$$\tau_U := \inf\{\, t > 0 : X_t \notin U \,\}.$$

Then, to compute $\mathbb{E}^x\phi(u(X_{\tau_U}))$, we apply:

- the Itô-type isometry for the Bregman variation,
- the Itô formula for $u(X)$,
- the Lévy system for expectations of sums over the jumps of X,
- potential theory (Green function G_U) for the occupation time up to τ_U.

This yields integral formulas expressing $\mathbb{E}^x\phi(u(X_{\tau_U}))$.

For instance, here is a **semiclassical Hardy–Stein identity**, but for general moderate Young functions ϕ with absolutely continuous derivative: Consider

$$D = \{z \in \mathbb{R}^2 : |z| < 1\},$$

and integrable $g : \partial D \to \mathbb{R}$. Define the harmonic extension

$$u(z) := \int_{\partial D} g(\theta)\, P_D(z,\theta)\,\mathrm{d}\theta, \qquad P_D(z,\theta) := \frac{1}{2\pi}\,\frac{1-|z|^2}{|z-\theta|^2}.$$

Then, for a moderate C^1 Young function ϕ with absolutely continuous ϕ',

$$\frac{1}{2\pi}\int_{\partial D} \phi(g(\theta))\,\mathrm{d}\theta = \phi(u(0)) + \frac{1}{2}\int_D G_D(0,z)\,\phi''(u(z))\,|\nabla u(z)|^2\,\mathrm{d}z.$$

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Stochastic nonlinear heat equation with constraints: existence of martingale solutions and pathwise uniqueness

Zdzisław Brzeźniak
(joint work with Ashish Bawalia, Manil T. Mohan)

In my talk I reported about results from my recent two papers, almost finished, coauthored by A. Bawalia and M. Mohan. The model is a certain reaction-diffusion equation with an arbitrary (but polynomial) growth, constrained by the condition that the L^2-norm of the solution is equal to one. In [1] we consider a purely deterministic version of the problem and prove the global existence of solution and study some of the asymptotic behavior, as $t \to \infty$, of solutions. In [2] we consider a stochastic version of the problem and prove the global existence of solutions. These results generalise some papers of the speaker with J. Hussain, see e.g. [3], where similar questions were studied with an arbitrary growth in dimensions $d = 2$, but only growth of order ≤ 6 in dimension $d = 3$.

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Optimal control problems with generalized mean-field dynamics and viscosity solution to Master Bellman equation

Rainer Buckdahn
(joint work with Juan Li, Zhanxin Li)

We investigate an optimal control problem for generalized mean-field dynamics with open-loop controls, where the coefficients depend not only on the state processes and controls, but also on their joint distribution. While the value function V is defined in a conventional manner, it fails to satisfy the Dynamic Programming Principle (DPP for short). To address this issue we introduce subtly a novel value function ϑ, which is closely related to the original value function V. By characterizing ϑ as a solution to a new type of partial differential equation (PDE), we simultaneously characterize V. For this we establish the DPP for ϑ and, utilizing a tailored notion of viscosity solutions specifically constructed for our framework, demonstrate that the value function ϑ is a viscosity solution to a new type of Master Bellman equation defined on a subset of the Wasserstein space of probability measures. Furthermore, we prove the uniqueness of the viscosity solution under the assumption that the coefficients depend on time and the joint distribution of the control process and the controlled process.

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Universal approximation theorems for infinite-dimensional signatures

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(joint work with Asma Khedher, Thijs Maessen)

We show a universal approximation theorem for functions of infinite-dimensional rough paths, thereby extending recent work in the finite-dimensional setting [CGS25, CST25, CPS25]. More specifically, let f be a continuous function on the space of weakly geometric α-Hölder continuous rough paths in a Hilbert space H. Then for every every compact set K in this space and every $\epsilon > 0$ there exists an element of the (algebraic) tensor algebra $T(E)$ such that $|f(x) - \langle \ell, S(x) \rangle| < \epsilon$ for all $x \in K$. Here, $S(x)$ denotes the signature of x.

The underlying principle is of course a Stone-Weierstrass approximation argument – indeed, the fact that we are dealing with weakly geometric paths ensures that the testfunctions form an algebra.

The key question here is: what topology does "continuous" and "compact" refer to? We consider two possible choices: either we take a norm topology, which is mathematically easier to deal with but gives little control on compact sets. The other possibility we consider is a weak*-topology, so norm-bounded sets are compact.

We also address the question of what happens when H is replaced by a Banach space, in this case there does not seem to be a single, natural choice for the tensor norm defining the space of signatures, and various possibilities need to be investigated. Overal, regarding the tensor algebra structure of the signature, we rely heavily on the theory of signatures of infinite-dimensional paths as developed in [L04].

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A two sided linear estimate and a dyadic reduction of the UMD conjecture

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(joint work with Stefanie Petermichl)

UMD Banach spaces. The space X has the UMD–p property if and only if the $\{\pm 1\}$ martingale multipliers $\mathcal{T}_\alpha$'s are uniformly L^p bounded. Those are the X–valued sign toss operators acting on the Haar system via

$$\mathcal{T}_\alpha : h_I \mapsto \alpha_I h_I,$$

where $\alpha_I \in \{-1, +1\}$. The best uniform bound is the UMD–p constant of X. The precise relationship between the norm of the Hilbert transform and the UMD constant, however, remains unclear. If the L^p norm of the Hilbert transform is noted h_p and the UMD–p constant is noted m_p, then it is known that

$$m_p^{1/2} \leqslant h_p \leqslant m_p^2. \tag{1}$$

The estimate on the left hand side is due to Bourgain [1] and the estimate on the right hand side is due to Burkholder [2]. It is an open question whether these inequalities are sharp. This is the famous UMD conjecture, see for example O.6 in [4].

In the paper [3], Geiss–Montgomery-Smith–Saksman replaced the Hilbert transform by the even singular operator $\mathcal{R}_1^2 - \mathcal{R}_2^2$, the difference of squares of Riesz transforms in $\mathbb{R}^2$, and showed that the linear relation holds. If r_p denotes the L^p norm of $\mathcal{R}_1^2 - \mathcal{R}_2^2$ with values in X, they showed

$$m_p \leq r_p \leq m_p. \tag{2}$$

Their remarkable estimates rely heavily on evenness of the operator $\mathcal{R}_1^2 - \mathcal{R}_2^2$ as well as other previously observed mapping properties and relations with dyadic martingales. Other than the Hilbert transform, the difference of squares of Riesz transforms are the expectation of a martingale multiplier using a diagonal matrix, which facilitates working with the complexity 0 operators $\mathcal{T}_\alpha$.

Main result. In contrast, we work in this paper with the Hilbert transform $\mathcal{H}$ but replace $\mathcal{T}_\alpha$ by the odd dyadic shift operator of complexity 1 densely defined by

$$\mathcal{S}_0 : h_{I_\pm} \mapsto \pm h_{I_\mp}, \tag{3}$$

where $I_\pm$ denote the left and right children of I and $h_{I_\pm}$ their associated Haar functions. If $\mathcal{S}_0$ has the L^p bound s_p and $\mathcal{H}$ has the L^p bound h_p, our main results are the *two sided linear* bounds, as opposed to the quadratic bounds in (1),

Theorem. *There exists a constant* $0 < c_0 < 1$*, such that*

$$c_0 s_p \leq h_p \leq s_p. \tag{4}$$

We strongly stress the exponents 1 on s_p and remark the coefficient 1 on the right hand side. The coefficient on the left hand side is explicit as well, but smaller than 1. The estimates (4) completely reduce the UMD conjecture (its failure or its proof) to the pair of dyadic operators $\mathcal{T}_\alpha$ and $\mathcal{S}_0$.

The lower bound $c_0 s_p \leq h_p$. This estimate requires the perfect choice of dyadic model that responds to the needs of the Hilbert transform. Indeed, $\mathcal{S}_0$ mimics the Cauchy–Riemann equations 'in the probability space'. To obtain the lower bound, we add several new elements to a brilliant argument by Bourgain [1] using high frequency modulation in Fourier space. In his work, he needed to apply the Hilbert transform twice to control the martingale transforms. Our relationship of the Hilbert transform to $\mathcal{S}_0$ can be made to be much more direct and we therefore manage to only use the Hilbert transform once, yielding our linear estimate from below. Bourgain's argument uses a clever random generator to get the sign tosses of the dyadic random walks. He then increases the frequency of each increment and uses the mapping properties of the Hilbert transform to obtain the control he needs in the strong form. In our argument, we change the random generator to fit our operator, which forces it to have memory - a typical difficulty when dealing with shift operators of positive complexity. We also proceed by increasing the frequency, but use a novel striking similarity between $\mathcal{S}_0$ and the Hilbert transform in the dualized form.

The upper bound $h_p \leq s_p$. This estimate is interesting in the light that the averaging procedure used in [5] fails for $\mathcal{S}_0$ and that the coefficient in the estimate we obtain by our means is 1. However, the comparison of the two operators can be achieved through stochastic representations of the Hilbert transform using martingales driven by a two-dimensional Brownian motion. It reveals a profound connection between $\mathcal{H}$ and $\mathcal{S}_0$, which was an important motivation for us for bringing a 'dyadic Hilbert transform' into play. We show that the way $\mathcal{S}_0$ acts forces the definition of a discrete dyadic (non-Markovian) two-dimensional random walk compatible with the Cauchy-Riemann relations. Via methods in stochastic numerical analysis we show weak L^p convergence of sampled versions of our discrete martingales towards those governed by Brownian motion. Many difficulties such as the presence of non-Markovian processes, random stopping times and approximate harmonicity must be carefully handled. This approach is very different from all representation theorems of singular operators via dyadic shifts.

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The optimal bound e in a dyadic version of Uchiyama's Lemma

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(joint work with Stefanie Petermichl)

We discuss a dyadic version of Uchiyama's Lemma. In the continuous setting this lemma plays a key role in establishing the best known upper bound for the operator norm of the Carleson Embedding from the Hardy-Hilbert space H^2 into $L^2(\mu)$ with μ being a Carleson measure. It states, that for a bounded subharmonic function $\varphi \leq 0$ on the unit disc $\mathbb{D} \subseteq \mathbb{C}$ and for the measure

$$d\mu = \Delta\varphi(z) \log \frac{1}{|z|} dA(z)$$

the space $H^2(\mathbb{D})$ embeds into $L^2(\mu)$:

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq e\|\varphi\|_\infty \|f\|_{H^2}^2 \qquad \forall f \in H^2(\mathbb{D}).$$

It was conjectured by Treil that the prefactor e is sharp in the Uchiyama Lemma, see [1]. Departing from the classical case in complex analysis, we construct a model case in the dyadic filtration. Analyticity is replaced by a condition on orthonormal martingale differences of a real and an imaginary part. In this setting, a sufficiently large constant of the analogous embedding remains e. But in contrast to the continuous case, we can show that e is optimal. Moreover, we may deduce a dyadic version of the reproducing kernel thesis for the embedding theorem by utilizing the dyadic Uchiyama Lemma.

The link between dyadic filtrations and harmonic analysis has a long history. Precise models go back to at least Bourgain and Burkholder, where the so-called martingale multiplier – a sign toss on the increments – was used to model the

Hilbert transform, see [3], [5] and [4]. Since then, other models have been used or the Hilbert transform was replaced by a more appropriate operators e.g. in [6], [7], [8], [9] and [2]. These models can be more precise for the Hilbert transform, but have the drawback that they tend to come with a shift, e.g. in time, and along with it a non-Markov flavor.

We develop a model for the Hilbert transform such that the constant e is optimal for the Carleson Embedding Theorem. Both directions of the proof are facilitated by the Bellman method. For the upper bound we use a Bellman function quite similar to the one in the continuous case, which is actually a supersolution to a Bellman equation, but it nevertheless yields the optimal bound e. The lower bound is established by utilizing the so called true Bellman function, from which a suitable extremal problem arises.

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Superquadratic BSDEs with an unbounded terminal condition and jumps

Hannah Geiss

(joint work with Céline Labart, Adrien Richou, Alexander Steinicke)

Backward stochastic differential equations (BSDEs) have been studied for many years, starting with the seminal papers from Pardoux and Peng ([5] in 1990 and [4] in 1992) that consider the Brownian motion setting. Then, this class of equations was extended to the setting of random measures associated to a Lévy process by Barles, Buckdahn, and Pardoux ([2] in 1997).

Here we are interested in path-dependent forward backward stochastic differential equations driven by a Brownian motion or by a Lévy process:

$$\begin{aligned} X_t &= x + \int_0^t b(s, (X_{r\wedge s})_{r\in[0,T]})ds + \int_0^t \sigma(s)dW_s \\ &\quad + \int_{(0,t]\times\mathbb{R}_0} \rho(s,v)\tilde{\mathcal{N}}(ds,dv), \\ Y_t &= g((X_s)_{s\in[0,T]}) + \int_t^T f\left(s, (X_{r\wedge s})_{r\in[0,T]}, Y_s, Z_s, U_s\right) ds \\ &\quad - \int_t^T Z_s dW_s - \int_{(t,T]\times\mathbb{R}_0} U_s(v)\tilde{\mathcal{N}}(ds,dv), \quad 0 \le t \le T, \end{aligned}$$

where W is a Brownian motion, and $\tilde{\mathcal{N}}$ a compensated Poisson random measure, both coming from the Lévy-Itô decomposition of a one-dimensional Lévy process.

To match the settings in applications, weaker assumptions than the originally assumed Lipschitz continuity on the coefficients are often needed. For example, quadratic or super-quadratic behavior in the control variable Z are assumed in optimal stochastic control applications.

Let $D[0,T]$ be the space of càdlàg paths endowed with the Skorohod J_1-topology. We assume that $g\colon D[0,T] \to \mathbb{R}$ is locally Lipschitz as follows:

$$|g(x) - g(x')| \le c(1 + |x|_\infty^r + |x'|_\infty^r)|x - x'|_\infty, \quad x, x' \in D[0,T]$$

where $r \le \frac{1}{2}$ and $|x|_\infty := \sup_{0\le s\le T} |x(s)|$. It turns out that one needs to require additionally that g is $\mathcal{B}(D[0,T])$-measurable as this does not follow from the above Lipschitz condition.

Like in [6] we assume (super)quadratic growth in the Z process of the generator and show that this can be balanced by controlling the growth of the terminal condition, i.e. choosing r in a suitable way. A similar procedure is carried out for the growth of the U process. The proof uses Malliavin derivatives to provide a representation for the solution processes. To prove Malliavin differentiability in the path-dependent setting we exploit a recent characterisation for the Wiener Malliavin Sobolev space $\mathbb{D}_{1,2}$ which can be found in [3].

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Directional gradient estimates for functionals on Banach space valued Lévy processes

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(joint work with Nguyen Tran Thuan)

Given a separable Banach space E, an E-valued Lévy process $X = (X_t)_{t\in[0,T]}$, and a measurable function $f : E \to R$, we consider the stochastic solution $F : [0,T] \times E \to R$ to the PDIE

$$\frac{\partial F}{\partial t} + \int_E [F(t,x+z) - F(t,x)]\nu(dz) = 0,$$

where ν is the Lévy measure of X and where $F(t,x) := Ef(z+X_{T-t})$. We obtain a family of generalised gradients

$$(D_\rho^\Phi f)(t,x) := \int_{E\setminus\{0\}} \frac{F(t,x+z) - F(t,x)}{\Phi(z)}\rho(dz),$$

where the family is parametrised by a finite measure ρ on $\mathcal{B}(E\setminus\{0\})$ and a measurable function $\Phi : E \to R$ that satisfies $|\Phi(z)| = \|z\|$. Different choices of ρ and Φ allow for various applications, for example to the aforementioned PDIE, to martingale decompositions of the Lévy-Itô space, and to the study of second-order derivatives. The upper bounds for $(D_\rho^\Phi f)(t,x)$ are proven for $f \in (B_b^0(E), \mathrm{Lip}^0(E))_{\eta,q}$, where $B_b^0(E)$ is the space of bounded measurable functions $f : E \to R$ vanishing at $x = 0$, $\mathrm{Lip}^0(E)$ is the space of Lipschitz functions $f : E \to R$ vanishing at $x = 0$, and $(\cdot,\cdot)_{\eta,q}$ denotes the real interpolation method. It turns out that the interpolation parameters (η, q), the small ball behaviour of the measure ρ, and the behaviour of $\|\mathrm{law}(X_s + z) - \mathrm{law}(X_s)\|_{\mathrm{TV}}$ (here $\|\mu\|_{\mathrm{TV}}$ denotes the total variation of a signed measure on $\mathcal{B}(E)$) determine the singularity of $(D_\rho^\Phi f)(t,x)$ when $t \uparrow T$. The obtained upper bounds $(D_\rho^\Phi f)(t,x)$ are proven to be sharp with respect to the parameter η, the parameter of small ball behaviour of the measure ρ, and the behaviour of $\|\mathrm{law}(X_s + z) - \mathrm{law}(X_s)\|_{\mathrm{TV}}$.

The presented results extend work that was done in [1].

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Variance renormalisation of singular SPDEs

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(joint work with Yueh-Sheng Hsu, Fabio Toninelli)

Scaling arguments give a natural guess for the regularity condition on the noise in a stochastic PDE for a local solution theory to be possible, using the machinery of regularity structures or paracontrolled distributions. This guess of *subcriticality* is often, but not always, correct. In cases when it is not, the blowup of the variance of certain non-linear functionals of the noise necessitates a different, *multiplicative* renormalisation. This led to the first results in the case of the KPZ equation in [3], where it is shown that the KPZ equation driven by the derivative of a space-time white noise $\partial_x\zeta$ needs an additional prefactor $\varepsilon^{\frac{3}{4}}$ in front of the smoothened noise (in addition to the standard 'additive' renormalisation). That is the solutions of

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + (\partial_x h_\varepsilon)^2 + \varepsilon^{\frac{3}{4}}\partial_x\zeta_\varepsilon - C_\varepsilon\ ,$$

have a non-trivial limit in law, which turns out to be another KPZ equation. Here, the exponent $\frac{3}{4}$ is nothing but the difference between the *super-regularity* (or variance blowup) threshold ([1], [5]) and the regularity of the rough noise $\partial_x\zeta$. A similar result holds when one places the vanishing factor in front of the nonlinearity instead of the noise, with a different exponent: in [2] it is shown that the solutions of

$$\partial_t\tilde{h}_\varepsilon = \partial_x^2\tilde{h}_\varepsilon + \varepsilon^{\frac{3}{2}}(\partial_x\tilde{h}_\varepsilon)^2 + \partial_x\zeta_\varepsilon - \tilde{C}_\varepsilon\ ,$$

also converge in law, this time to a Gaussian limit, which however differs from the solution of the linear equation $\partial\tilde{h} = \partial_x^2\tilde{h} + \partial_x\zeta$. So although the nonlinearity disappears, it does have a nontrivial contribution to the limit.

A general prediction is then made in [3] that this kind of phenomenon should persist for other singular SPDEs with different non-linearities, whenever the critical exponent for variance blowup is strictly bigger than the critical exponent for scaling. Towards confirming this prediction for 'more' nonlinear equations, we consider the typical guinea pig of nonlinear singular SPDEs, the 2-dimensional generalised parabolic Anderson model (gPAM). We drive the 2-d gPAM with a derivative of the white noise, to put the noise beyond the variance blowup range, so the equation formally reads as

$$\partial_t u = \Delta u + g(u)\partial_{x_1}\xi.$$

The variance renormalisation of this equation then looks as follows.

Theorem 1 (Gerencsér, Hsu)**.** *Let ξ be a spatial white noise on $\mathbb{T}^2$ and $(\xi^\varepsilon)_{\varepsilon\in(0,1]}$ be a family of smooth approximations of it. For any $\varepsilon > 0$ consider the equation*

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + g(u_\varepsilon)\varepsilon^{\frac{1}{2}}\partial_{x_1}\xi_\varepsilon - C_\varepsilon g'g(u_\varepsilon) - \bar{C}_\varepsilon(g')^3 g(u_\varepsilon) - \hat{C}_\varepsilon g''g'g^2(u_\varepsilon) \qquad (1)$$

on $\mathbb{T}^2$ with a Hölder continuous initial condition ψ. There exists a sequence of constants $C_\varepsilon, \bar{C}_\varepsilon, \hat{C}_\varepsilon$ and a constant $c_\rho > 0$ such that the solutions u_ε to (1) *converge in law in $C([0,1]\times\mathbb{T}^2)$ to the renormalised solution u of the SPDE*

$$\partial_t u = \Delta u + c_\rho g' g(u) \eta \qquad (2)$$

on $\mathbb{T}^2$ *with initial condition* ψ*, where* η *is also a spatial white noise.*

Note that the noise $\partial_{x_1}\xi$ is too rough and the regularity structure black box does not apply to it, and indeed the renormalisation has to include the vanishing factor $\varepsilon^{\frac{1}{2}}$, where again the exponent $\frac{1}{2}$ coincides with the difference between variance blowup and the noise regularity.

To obtain the convergence to (2), we first show that the BPHZ lift $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ associated to (1) converges in law to a model $(\hat{\Pi}, \hat{\Gamma})$, which is *not* a BPHZ lift of any noise. Rather, in analogy with pure area rough paths, this limit is a nontrivial lift of the trivial 0 noise. The new white noise in (2) is obtained from the 'dumbbell' tree of the approximating equation (1) via the fourth moment theorem of [6], while to obtain limits in higher order chaoses (which are also required in the description of (2)), one needs to control a large number of Feynman graphs obtained from the covariance structure of the models $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$. The distribution of the ε factors within the Feynman diagrams and the generalised convolution bound of [4] play crucial roles in the argument.

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Geometric properties of rough curves via dynamical systems: SBR measure, local time, Rademacher chaos and number theory

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(joint work with Olivier Pamen, Frank Proske)

The research we are focused on was motivated on the one hand by the observation that our knowledge on local times of deterministic rough curves is small and has been small for a very long time, and the success of rough path analysis in recent decades on the other hand. The aim of our research is to provide criteria for the existence and regularity of local times of rough curves of the type of the well known

Takagi or Weierstrass functions. These functions are given in series decompositions of the form

$$\mathbb{T}(x) = \sum_{k=0}^{\infty} \gamma^k f(2^k x),$$

with a roughness parameter $\gamma \in]\frac{1}{2}, 1[$, and a 1-periodic function f defined on the real line. The case $f(x) = \mathrm{dist}(x, \mathbf{Z})$ corresponds to Takagi functions, while $f(x) = \cos(2\pi x)$ gives Weierstrass functions. These deterministic curves can be investigated probabilistically by embedding them into metric dynamical systems in which they emerge as pullback attractors.

To describe them, let $\Omega = \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. For $\omega \in \Omega$, write $\omega = ((\omega_{-n})_{n\geq 0}, (\omega_n)_{n\geq 1})$. Let $\mathbf{F}$ be the product σ-field on Ω, $\nu = \otimes_{n\in\mathbf{Z}}(\frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\delta_{\{1\}})$ the infinite product of Bernoulli measures. Then $\theta : \Omega \to \Omega, \omega \mapsto (\omega_{n+1})_{n\in\mathbf{Z}}$, is ν-measure preserving, and thus $(\Omega, \mathbf{F}, \nu, \theta)$ a metric dynamical system.

This dynamical system is translated onto the unit square via dyadic expansions. In fact, let

$$D : \Omega \to [0,1]^2, \quad \omega \mapsto (\sum_{n=0}^{\infty} \omega_{-n} 2^{-(n+1)}, \sum_{n=1}^{\infty} \omega_n 2^{-n}).$$

Then $\nu \circ D^{-1} = \lambda^2$ is Lebesgue measure on $[0,1]^2$, where

$$D^{-1}(\xi, x) = \big((\bar{\xi}_{-n})_{n\geq 0}, (\bar{x}_n)_{n\geq 1}\big)$$

denotes the dyadic representation of $(\xi, x) \in [0,1]^2$. We call the image of θ by D, i.e. $B = D \circ \theta \circ D^{-1}$ *baker's transformation.* The ν-invariance of θ implies B-invariance of λ^2.

For $(\xi, x) \in [0,1]^2$ the explicit description of baker's transformation is given by

$$B(\xi, x) = \Big(2\xi(\mathrm{mod}1), \frac{\bar{\xi}_0 + x}{2}\Big), \quad B^{-1}(\xi, x) = \Big(\frac{\xi + \bar{x}_1}{2}, 2x(\mathrm{mod}1)\Big).$$

In terms of baker's transformation, we can interpret the Takagi curve by

$$\mathbb{T}(\xi, x) = \mathbb{T}(x) = \sum_{n=0}^{\infty} \gamma^n f(B_2^{-n}(\xi, x)).$$

The pushforward of Lebesgue measure to the stable manifolds of these random dynamical systems creates the Sinai-Bowen-Ruelle (SBR) measures of the systems. If $\kappa = \frac{1}{2\gamma}$ is the *dual roughness parameter*, the essential stable manifold function is expressed by

$$\mathbb{S}(\xi, x) = -\sum_{n=1}^{\infty} \kappa^n f'(B_2^n(\xi, x)).$$

So $\mathbb{T}$ and $\mathbb{S}$ are dual to each other, where in passing from $\mathbb{T}$ to $\mathbb{S}$ "time" in parameter space is reversed, and the generating function f gets differentiated.

This notable duality establishes a relationship between the regularity of the SBR measures on the one hand, and the smoothness of the occupation measures of the curves on the other hand, i.e. the existence and regularity of their local times. By means of various mathematical techniques we are able to investigate regularity of both SBR and occupation measures. This is a particularly intriguing and interesting subject for Takagi curves. Here it is known from old papers by Erdös that the SBR measure may be singular, if 2γ is a *Pisot number*, i.e. its minimal polynomial in $\mathbb{C}[x]$ possesses roots that, besides 2γ, are contained in the interior of the unit circle. For the study of smoothness of occupation measures we are led to mathematical tools such as Malliavin's calculus on Rademacher space, and - as shown by the link to Erdös' work - into metric number theory, around Weyl's equidistribution theorem. For Weierstrass curves regularity of both SBR and occupation measures are linked to the concept of transversality of the flow associated to the underlying random dynamical system.

Generalizations to different smooth generating functions f and to p-adic instead of dyadic curves are possible. Smoothness of occupation measures of deterministic curves allows to use them to additively regularize singular ODE. The individual trajectories of Brownian motion can be realized as randomized Takagi curves, if they are expressed along their Haar-Schauder expansion. This way we get access to the geometry of individual Brownian motion trajectories.

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Sharpness of matrix weighted L^2 estimates under the Poisson condition

Spyridon Kakaroumpas

(joint work with Komla Domelevo, Stefanie Petermichl, Sergei Treil, Alexander Volberg)

The most important operator in classical harmonic analysis is probably the Hilbert transform H, which acts on nice functions $f : \mathbb{R} \to \mathbb{C}$ by

$$Hf(x) := \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y}\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

It is a classical result that H is bounded as an operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for all $1 < p < \infty$. Questions in the theory of Hardy spaces and stationary processes lead naturally to the consideration of the boundedness of H on *weighted* Lebesgue spaces $L^p(w) := L^p(w(x)\,\mathrm{d}x)$ for $1 < p < \infty$, where the *weight* w is a locally integrable, a.e. positive function on $\mathbb{R}$. R. Hunt, B. Muckenhoupt, R. L. Wheeden [3] showed that H acts boundedly as a self-map of $L^p(w)$ if and only if the so-called *Muckenhoupt A_p characteristic*

$$[w]_{A_p} := \sup_I \left(\frac{1}{|I|}\int_I w(x)\,\mathrm{d}x\right)\left(\frac{1}{|I|}\int_I w(x)^{-1/(p-1)}\,\mathrm{d}x\right)^{p-1}$$

is finite, where the supremum is taken over all intervals $I \subseteq \mathbb{R}$. Later, S. Buckley [1] quantified this in showing that there is $\alpha(p) > 0$ such that

$$\|H\|_{L^p(w)\to L^p(w)} \leq C_p[w]_{A_p}^{\alpha(p)}, \tag{1}$$

and that any such exponent $\alpha(p)$ cannot be smaller that $\max(1, 1/(p-1))$. Finally, S. Petermichl [8] showed that (1) holds with the conjectured optimal exponent $\alpha(p) := \max(1, 1/(p-1))$ for any $1 < p < \infty$.

Questions in the theory of Toeplitz operators lead naturally to the consideration of larger, *"fattened"* A_p characteristics of the form

$$[w]_{A_p}^{\mathrm{fat}} := \sup_{\substack{z\in\mathbb{C}\\ \mathrm{Im}(z)>0}} \left(\int_{\mathbb{R}} w(x)P_z(x)\,\mathrm{d}x\right)\left(\int_{\mathbb{R}} w(x)^{-1/(p-1)}P_z(x)\,\mathrm{d}x\right)^{p-1},$$

where the probability kernel $P_z(\cdot)$ has less decay than the characteristic function of an interval. In the course of disproving a conjecture by D. Sarason, F. Nazarov [5] observed that if the weights w and $w^{-1/(p-1)}$ are sufficiently *doubling*, then the fattened characteristics are actually comparable to their usual versions. To construct counterexamples with sufficiently doubling weights, F. Nazarov [5] introduced the technique of *remodeling*. Much later, S. Kakaroumpas and S. Treil [4] introduced the method of *small step* transformations and subsequently refined Nazarov's method of remodeling to the technique of *iterated remodeling*, showing that the exponent $\alpha(p) := \max(1, /(p-1))$ in (1) remains optimal even if one replaces $[w]_{A_p}$ by its "fattened version".

Problems in the theory of multivariate stationary processes motivate the study of the boundedness of the (entrywise) extension of the Hilbert transform on *matrix weighted* Lebesgue spaces of vector valued functions. Precisely, for $1 < p < \infty$ and a vector-valued measurable function $f : \mathbb{R} \to \mathbb{C}^d$ we define

$$\|f\|_{L^p(W)} := \left(\int_{\mathbb{R}} \|W(x)^{1/p}f(x)\|^p\,\mathrm{d}x\right)^{1/p},$$

where the *$(d \times d)$ matrix weight* W is a locally integrable, a.e. positive definite $(d \times d)$ matrix valued function on $\mathbb{R}$. F. Nazarov, S. Treil and A. Volberg showed in a series of works [7, 9, 10] that for all $1 < p < \infty$, H is bounded on $L^p(W)$ if and only if the so-called *matrix Muckenhoupt A_p characteristic* $[W]_{A_p}$ is bounded, which for $p = 2$ is given by

$$[W]_{A_2} := \sup_I \left\| \left(\frac{1}{|I|}\int_I W(x)\,\mathrm{d}x\right)^{1/2} \left(\frac{1}{|I|}\int_I W(x)^{-1}\,\mathrm{d}x\right)^{1/2} \right\|^2.$$

After several works proving upper bounds for $\|H\|_{L^2(W)\to L^2(W)}$, F. Nazarov, S. Petermichl, S. Treil, A. Volberg [6] showed the up to that point best bound

$$\|H\|_{L^2(W)\to L^2(W)} \leq C[W]_{A_2}^{3/2}. \tag{2}$$

It was conjectured for some time that the exponent 3/2 in (2) could be lowered to 1, just as in the scalar case. However, very recently K. Domelevo, S. Petermichl, S. Treil, A. Volberg [2] showed that this exponent 3/2 cannot be improved. They constructed novel examples for dyadic models and achieved then the passage to the Hilbert transform through a refinement of the method of iterated remodeling of Kakaroumpas and Treil [4].

Just as in the scalar case, one can define here also larger, *"fattened"* A_2 characteristics of the form

$$[W]_{A_2}^{\mathrm{fat}} := \sup_{\substack{z \in \mathbb{C} \\ \mathrm{Im}(z)>0}} \left\| \left(\int_{\mathbb{R}} P_z(x) W(x) \, \mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}} P_z(x) W(x)^{-1} \, \mathrm{d}x \right)^{1/2} \right\|^2 .$$

It is natural to ask whether the exponent $3/2$ in (2) remains sharp if one replaces $[W]_{A_2}$ by its "fattened" version. In this work we answer this question to the positive. First, we construct examples for dyadic models with *dyadically doubling* matrix weights. Our starting point are the dyadic examples due to Domelevo–Petermichl–Treil–Volberg [2]. However, the weights there lack a sufficient dyadic doubling property. To alleviate this, we construct new examples through a generalized and refined version of the small step transform by Kakaroumpas and Treil [4]. The main challenge consists in constructing martingale random walks on barycentric subdivisions of simplices that end up with equal probability on each vertex, while preserving the estimates of our operators. Lastly, we achieve the passage from the dyadic models to the Hilbert transform in the same way as Domelevo–Petermichl–Treil–Volberg [2]. This bootstraps at the same time the dyadic doubling property of the weights to its continuous version.

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A Quantitative Central Limit Theorem for the Simple Symmetric Exclusion Process

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(joint work with Benjamin Gess)

The talk is devoted to scaling limits of interacting particle systems and to quantitative convergence results for non-equilibrium dynamics. We establish an *optimal* convergence rate in the central limit theorem (CLT) for the simple symmetric exclusion process (SSEP) on the discrete torus. The classical proof of the CLT for the SSEP, relying on compactness arguments, the martingale central limit theorem, and the Holley–Stroock theory [3], does not provide quantitative rates [5, 1]. Our results close this gap by combining

(a) a generator comparison between the fluctuation field of the SSEP and the limiting Ornstein–Uhlenbeck process,
(b) infinite-dimensional extensions of Berry–Esseen bounds for the initial fluctuations [4, 6], and
(c) precise control of approximation errors in Sobolev spaces.

More precisely, consider the SSEP on the d-dimensional discrete torus $\mathbb{T}_n^d$ with configurations $\eta \in \{0,1\}^{\mathbb{T}_n^d}$, where $\eta(x) = 1$ indicates that site x is occupied and $\eta(x) = 0$ otherwise. Its generator is given by

$$\mathcal{G}_n F(\eta) = n^2 \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} \big[F(\eta^{x \leftrightarrow x+e_j}) - F(\eta)\big],$$

for $F : \{0,1\}^{\mathbb{T}_n^d} \to \mathbb{R}$, where $\eta^{x \leftrightarrow y}$ is obtained from η by exchanging the values at x and y. Let $\eta^n = (\eta_t^n)_{t \ge 0}$ be the SSEP started from a local equilibrium profile $\rho_0|_{\mathbb{T}_n}$ for a smooth function $\rho_0 : \mathbb{T} \to [0,1]$. The density fluctuation field

$$\zeta_t^n(x) := n^{d/2}\big(\eta_t^n(x) - \mathbb{E}\eta_t^n(x)\big), \qquad x \in \mathbb{T}_n^d,$$

is known to converge to the solution ζ_t^∞ of the linear SPDE

$$d\zeta_t^\infty = \tfrac{1}{2}\Delta\zeta_t^\infty \, dt + \nabla\cdot\big(\sqrt{\rho_t^\infty(1-\rho_t^\infty)}\, dW_t\big),$$

where ρ_t^∞ solves the heat equation with initial data ρ_0.

Let H_{-I} denote the Sobolev space on $\mathbb{T}^d$. Our main quantitative CLT is as follows.

Theorem. [2] *Let I be sufficiently large and $F \in C^3(H_{-I})$. Then for every $T > 0$ there exists a constant C, independent of F and n, such that*

$$\sup_{t \in [0,T]} \big|\mathbb{E}F(\zeta_t^n) - \mathbb{E}F(\zeta_t^\infty)\big| \;\le\; \frac{C}{n^{\frac{d}{2}\wedge 1}} \|F\|_{C^3}.$$

To the best of our knowledge, this is the first quantitative CLT for a non-equilibrium interacting particle system. The results provide a quantitative foundation for effective SPDE models and open the door to refined analyses in hydrodynamic limit theory. We are further interested in extending these methods to other particle systems and SPDE limits, particularly in regimes where effective stochastic PDEs naturally arise.

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Overcoming the spatial order barrier for nonlinear SPDEs with additive space-time white noise

Helena Kremp
(joint work with Lukas Anzeletti, Máté Gerencsér)

When approximating solutions of SPDEs, a basic challenge is that the rate of convergence is often limited due to the low time and space regularity of the solution. In the paper [1], we overcome the limitation for the spatial convergence rate. We consider nonlinear SPDEs with additive space-time white noise ξ on the one-dimensional torus $\mathbb{T}$:

$$\partial_t u = \Delta u + f(u) + \xi, \qquad u(0,\cdot) = u_0. \tag{1}$$

While the temporal rate has been successfully improved in many cases ([4, 5, 3]), to our best knowledge so far no discretisation scheme of an SPDE has yet been able to achieve a spatial strong rate of convergence that is superior to the spatial regularity of the solution. Compared to the solution u, which satisfies that for any $\epsilon > 0$ almost surely it is $(\frac{1}{4} - \epsilon)$-Hölder continuous in time and $(\frac{1}{2} - \epsilon)$-Hölder in space, we note that the solution v of the remainder equation

$$\partial_t v = \Delta v + f(v + O) \qquad v(0,\cdot) = u_0 \tag{2}$$

admits much higher temporal and spatial regularities, provided that u_0 is sufficiently regular. The idea for the numerical scheme is to leverage the higher regularity of v and to simulate the stochastic convolution

$$O_t(x) = \int_0^t \int_{\mathbb{T}} p_{t-s}(x-y)\xi(ds, dy)$$

for the heat kernel p_t, explicitly on the time-space grid (t,x) using the methods from [2] (respectively, to simulate large parts of O explicitly on the time-space grid and leave out an exponentially small remainder).

In order to define the numerical scheme, we introduce the following quantities. We let Θ_N be the restriction operator to the spatial grid

$$\Pi_N = \{\frac{n}{N} \mid n = 0, \ldots, N-1\},$$

that is, for $v \in C(\mathbb{T})$, $\Theta_N v(x) := v(x), x \in \Pi_N$. For the Fourier basis (e_j), we define $e_j^N := \Theta_N e_j$ for $j \in \mathbb{Z}$ and note that $(e_j^N)_{j\in J_N}$ for $J_N := \{-\lfloor \frac{N}{2} \rfloor, \ldots, \lceil \frac{N}{2} \rceil - 1\}$ is an orthonormal basis for $L^2(\Pi_N)$ with respect to the discrete scalar product. We define the operator $\tilde{\Delta}_N$ as an approximation of the Laplacian Δ given by $\tilde{\Delta}_N e_j^N = -4\pi^2 j^2 e_j^N$ for $j \in J_N$ and let $\tilde{P}_t^N e_j^N = e^{-4\pi^2 j^2 t} e_j^N, j \in J_N$ be the semigroup of $\tilde{\Delta}_N$.

In the case of a C_b^2 Nemytskii nonlinearity f, we propose the following numerical scheme for (2), which is an exponential Euler scheme in time: Set inductively, for $k = 1, \ldots, M$ and $t_k = \frac{Tk}{M}$,

$$\begin{aligned} V_{t_k}^{M,N} &= \tilde{P}_h^N V_{t_{k-1}}^{M,N} + (\tilde{\Delta}_N)^{-1} (\tilde{P}_h^N - \mathrm{Id}) \Big(f(V_{t_{k-1}}^{M,N} + \Theta_N O_{t_{k-1}}) \Big), \\ V_0^{M,N} &= \Theta_N u_0. \end{aligned}$$

Note that $\tilde{\Delta}_N$ is invertible only on $\mathrm{span}\{e_j^N : j \in J_N \setminus \{0\}\}$, therefore we set by convention $(\tilde{\Delta}_N)^{-1}(\tilde{P}_h^N - \mathrm{Id}) e_0^N := h e_0^N$. We can rewrite $V^{M,N}$ in the mild form as

$$V_t^{M,N} = \tilde{P}_t^N \Theta_N u_0 + \int_0^t \tilde{P}_{t-s}^N \Big(f\big(V_{\kappa_M(s)}^{M,N} + \Theta_N O_{\kappa_M(s)}\big)\Big)\, ds, \tag{3}$$

where $\kappa_M(s) = \lfloor s h^{-1} \rfloor h$ and the mild form is defined for any $t \in [0,T]$. Further, we set the scheme for (1) to be

$$V_{t_k}^{M,N} + \Theta_N O_{t_k}, \qquad k = 0, \ldots, M.$$

Since for $t \in [0,T]$, $u_t = v_t + O_t$, we clearly have

$$\Theta_N u_t - (V_t^{M,N} + \Theta_N O_t) = \Theta_N v_t - V_t^{M,N},$$

and thus the order barrier for the convergence rate is overcome due to the much better regularity of v compared to u. Our main theorem can now be formulated as follows.

Theorem 1 ([1])**.** *Let $p \geq 1$ and $\epsilon > 0$ small. Let the initial condition u_0 be sufficiently regular and $f \in C_b^2(\mathbb{R})$. Then there exists a constant $C = C(p, \epsilon, T, ||f||_{C_b^2}, u_0)$, such that*

$$\big(\mathbb{E} \sup_{t\in[0,T]} ||\Theta_N u_t - (V_t^{M,N} + \Theta_N O_t)||_{L^2(\Pi_N)}^p\big)^{1/p} \leq C(M^{-\frac{3}{4}+\epsilon} + N^{-\frac{3}{2}+\epsilon}). \tag{4}$$

Remark 2. *In* [1]*, we further consider a one-sided Lipschitz nonlinearity f, which grows at most polynomial (Example: $f(x) = x - x^3$ Allen-Cahn). In this case the scheme has to be modified to an appropriate splitting scheme in order to ensure $L^p(\mathbb{P})$-convergence. Then the same rates as in the theorem above hold true in that more general setting. Furthermore one can weaken the strong regularity assumptions on u_0.*

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The harmonic oscillator on the Moyal-Groenewold plane

Christoph Kriegler

(joint work with Cédric Arhancet, Lukas Hagedorn, Pierre Portal)

We study the abstract harmonic oscillator $L = A_1^2 + \cdots + A_n^2$ associated with a Θ-Weyl tuple $(A_1, \ldots, A_n)$, which means that the operators $\mathrm{i}A_1, \ldots, \mathrm{i}A_n$ generate strongly continuous groups on a UMD Banach space X and satisfy the Θ-commutation relations encoded by a real skew-symmetric matrix Θ. For each such matrix Θ, we construct a universal Θ-Weyl tuple on the Bochner space $\mathrm{L}^p(\mathbb{R}^n, X)$ and show that the functional calculus of this universal model transfers to any Θ-Weyl tuple on X. Our method relies on twisted convolution operators and a transference principle that links harmonic analysis on Lie groups to our operator-theoretic framework. As a consequence, we prove that, after shifting away the spectral gap, L admits a bounded $\mathrm{H}^\infty(\Sigma_\omega)$ functional calculus for every $\omega \in (\frac{\pi}{2}, \pi)$. These results provide a broad extension of the Hieber–Prüss theorem [2] and reveal a unified framework encompassing both the classical Hermite operator and genuinely noncommutative models.

More precisely, let $L = -\Delta + |x|^2$ be the harmonic oscillator which generates a submarkovian semigroup on $L^p(\mathbb{R}^d)$. It is well-known that its spectral multipliers $m(L)$ are bounded on L^p, $1 < p < \infty$, provided that $m : (0, \infty) \to \mathbb{C}$ is $\lfloor \frac{d}{2} \rfloor + 1$ times differentiable with a certain L^2 type control on its derivatives (Hörmander functional calculus).

On the other hand, L can be written as a square sum

$$L = \sum_{k=1}^{d} (i\partial_k)^2 + \sum_{l=1}^{d} x_l^2$$

where the operators $iA_k = -\partial_k$ and $iB_l = ix_l$ generate bounded (translation and modulation) c_0-groups on L^p, that obey simple, so-called canonical, commutation relations (CCR).

Recently, van Neerven, Portal and Sharma showed in a series of papers [3, 4, 5] via a transference technique and square function estimates, that square sum operators (as L above) for quite general bounded c_0-groups with CCR inherit the Hörmander functional calculus from L.

We show that this also holds for c_0-groups acting on noncommutative L^p-spaces. Along the way of proof, an important step is a new method to generate square function estimates on noncommutative L^p spaces.

An application are L^p bounded Hörmander spectral multipliers of the harmonic oscillator on the Moyal plane (also called quantum euclidean space), which are particular noncommutative pseudo-differential operators studied by González-Pérez, Junge and Parcet. The corresponding article is the preprint [1] on H^∞ calculus, which will soon be followed by a second part on Hörmander functional calculus by the same authors.

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Continuous martingale transforms and discrete singular integrals

Mateusz Kwaśnicki

(joint work with Rodrigo Bañuelos, Daesung Kim)

The use of martingale methods in harmonic analysis is now classical, and the link between singular integrals and martingale transforms is well-understood. First order Riesz transforms provide a good illustration of the method: given a sufficiently regular function $f : \mathbb{R}^d \to \mathbb{R}$, the Riesz transform $R_k f$ can be constructed in the following, probabilistic way.

Let u be a bounded harmonic function in the half-space $\mathbb{R}^d \times [0, \infty)$, with boundary values f. Thus, u is the Poisson integral of f. Let $Z_t = (X_t, Y_t)$ be the

$(d+1)$-dimensional Brownian motion started at a distant point $(0, y_0)$, and let $T = \inf\{t \geq 0 : Y_t = 0\}$ be the hitting time of the boundary of the half-space. By Itô's lemma,

$$M_t = u(X_{t\wedge T}, Y_{t\wedge T}) = u(0, y_0) + \int_0^{t\wedge T} \nabla u(Z_s) \cdot dZ_s$$

defines a (stopped) martingale with terminal value $M_T = f(X_T)$. If A is an $(d+1) \times (d+1)$ matrix, then

$$\tilde{M}_t = \int_0^{t\wedge T} A\nabla u(Z_s) \cdot dZ_s$$

is said to be the *martingale transform* of M. Gundy and Varopoulous proved that if we choose $A_{k,d+1} = -A_{d+1,k} = 1$ and $A_{i,j} = 0$ otherwise, then

$$R_k f(x) = \lim_{y_0\to\infty} \mathbb{E}\big[\tilde{M}_T \,\big|\, X_T = x\big].$$

Furthermore, $\tilde{M}$ is *differentially subordinate* to M:

$$d\langle\tilde{M}\rangle_t \leq d\langle M\rangle_t,$$

and additionally M and $\tilde{M}$ are *orthogonal martingales:*

$$d\langle M, \tilde{M}\rangle_t = 0dt.$$

By Burkholder's inequality for differentiable subordinate martingales, for $p \in (1, \infty)$ the inequality

$$\mathbb{E}\big[|\tilde{M}_T|^p\big] \leq C_p^p \, \mathbb{E}\big[|M_T|^p\big]$$

holds with constant C_p equal to $\max\{p-1, \frac{p}{p-1}\}$. Bañuelos and Wang improved the constant when M and $\tilde{M}$ are orthogonal to

$$C_p = \max\{\tan \tfrac{\pi}{2p}, \cot \tfrac{\pi}{2p}\}.$$

By Jensen's inequality,

$$\mathbb{E}\Big[\big|\mathbb{E}\big[\tilde{M}_T \,\big|\, X_T\big]\big|^p\Big] \leq C_p^p \, \mathbb{E}\big[|M_T|^p\big].$$

If we multiply both sides by y_0 and pass to the limit as $y_0 \to \infty$, we obtain the classical L^p bound for R_k due to Iwaniec and Martin:

$$\|R_k f\|_p \leq C_p \, \|f\|_p.$$

Furthermore, the constant C_p is best possible; that is, C_p is the operator norm of R_k on $L^p(\mathbb{R}^d)$.

The question whether one can discretise a singular integral in a way that preserves its norm on L^p goes back to M. Riesz and E.C. Titchmarsh, who considered the one-dimensional case: continuous and discrete Hilbert transforms. Even in this simplest setting the problem remained open until ten years ago. The affirmative answer, given in [1], involved a stochastic representation of the discrete transform in terms of continuous martingales, very similar to the one described above. The main difference lies in the underlying process: in [1], Z is the Brownian motion conditioned to exit the half-space through a lattice point. This change leads to various technical difficulties, but eventually one gets an optimal ℓ^p estimate for a certain discrete operator. With further effort, one obtains a similar estimate for the standard discrete Hilbert transform.

The optimal constant C_p is specific to Burkholder's inequality for orthogonal martingales, which only applies to continuous martingales. This is why the somewhat artificial construction described above seems necessary. There are various ways to represent the discrete Hilbert (or Riesz) transform in terms of discontinuous martingales, in either discrete or continuous time, but none of them leads to the ℓ^p bound with the correct constant C_p.

In my talk I will describe our more recent work [2], where we apply the method of [1] in higher dimensions. We find a variant of the discrete Riesz transform which does not increase the L^p norm. More precisely, we construct a *probabilistic discrete Riesz transform* $\mathcal{R}_k$, acting on lattice functions $a : \mathbb{Z}^d \to \mathbb{R}$, with the following two properties:

- the operator norm of $\mathcal{R}_k$ on $\ell^p(\mathbb{Z}^d)$ is equal to the operator norm of R_k on $L^p(\mathbb{R}^d)$ (both are equal to C_p);
- if f is a Schwartz function and $a(n) = f(\delta n)$ is its discretisation, then $\mathcal{R}_k a(n)$ is asymptotically equal to $R_k f(\delta n)$.

In fact, the kernel of $\mathcal{R}_k$ is asymptotically equal (but not exactly equal) to the kernel of R_k evaluated at lattice points. The question whether the *natural discretisation* of the Riesz transform R_k preserves its norm on L^p remains open.

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A numerical scheme for optimal switching problems based on a randomization method

Céline Labart

(joint work with Marie-Amélie Morlais)

Optimal switching is a specific management problem which consists in looking for optimal management strategies. Those strategies are pairs of random dates and regimes (adapted to some underlying filtration) standing for the best dates when the controller should switch (i.e the dates he/she should change the regime of the system) and the best regime he should select. We assume that some switching costs have to be paid when changing from one regime to another. A common field of application is the energy field: for instance, the management of a nuclear power plant with two regimes (on and off). The manager has to decide when and how to generate electricity according to market demand. For example, operate at full capacity for high demand and shut down the generators for low demand. We consider fixed positive costs attached with these switchings.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and W is a one dimensional Brownian motion $\mathcal{F}$-adapted. In finite horizon setting, the controller tries to maximize the following reward function

$$J_{t,x}(\alpha) = g(X_T^{t,x,\alpha}, \alpha_T) + \int_t^T f(\alpha_s, X_s^{t,x,\alpha})ds - \sum_{n\geq 1} c(\eta_n, X_{\eta_n}^{t,x,\alpha}, \beta_{n-1}, \beta_n)\mathbf{1}_{\eta_n<T}$$

where

- $\alpha = (\eta_n, \beta_n)_n$ is the strategy of the controller : (η_n) is a nondecreasing sequence of random dates and $(\beta_n)_n$ is the regime chosen at time η_n taking values in the finite set A, β_j is $\mathcal{F}_{\eta_j}$-adapted,
- $X = (X^{t,x,\alpha})$ is the forward process (in practice X can modelize the dynamics of gas/electricity which influences the profit functional J)
$$dX_s^{t,x,\alpha} = b(s, X_s^{t,x,\alpha}, \alpha_s)ds + \sigma(s, X_s^{t,x,\alpha}, \alpha_s)dW_s,\ t \leq s \leq T,$$
with a given initial condition $X_t^{t,x,\alpha} = x \in \mathbb{R}$,
- (f, g, c) stands for the triple of data associated with OSP: g is the terminal reward, f the instantaneous reward and c the switching cost functional.

The value of the optimal switching problem (OSP) is defined as follows

$$v_i(t,x) = \sup_{\alpha\in\mathcal{A}_{t,i}} J_{t,x}(\alpha) \tag{1}$$

where the set $\mathcal{A}_{t,i}$ of admissible strategies starting from state i at time t is defined by

$$\mathcal{A}_{t,i} = \{\alpha = (\eta_j, \beta_j)_j \mid \eta_0 = t, \beta_0 = i, \mathbb{E}(|C_{t,T}^{\alpha}|^2) < \infty\},$$

and $C_{t,T}^{\alpha} := \sum_{n\geq 1} c(\eta_n, X_{\eta_n}^{t,x,\alpha}, \beta_{n-1}, \beta_n)\mathbf{1}_{t\leq\eta_n\leq T}$ represents the sum of the switching costs up to time T.
Using the randomization method (see [3]), we link the value function (1) to a non standard BSDE.

Randomization method. Let us introduce a Poisson random measure $\mu(dt, di)$ on $\mathbb{R}_+ \times A$ with finite intensity measure λ on A ($\lambda(i) > 0$ for all $i \in A$) associated with the marked point process $(\tau_i, \xi_i)_i$, independent of W, and consider the pure jump process $(I_t)_t$ valued in A defined by

$$I_t = I_0\mathbf{1}_{[0,\tau_1)}(t) + \sum_{i\geq 1} \xi_i \mathbf{1}_{[\tau_i,\tau_{i+1})}(t)\mathbf{1}_{\tau_{i+1}\leq T},\quad 0 \leq t \leq T.$$

I is interpreted as a randomization of the control process α. We also consider the uncontrolled forward regime switching diffusion process

$$dX_t^{I_t} = b(X_t^{I_t}, I_t)dt + \sigma(X_t^{I_t}, I_t)dW_t,\ X_0 = x_0$$

and the following non linear jump constrained BSDE

$$\begin{cases} Y_t = g(X_T^{I_T}, I_T) + \int_t^T f(s, X_s^{I_s}, I_s)ds + K_T - K_t \\ \qquad - \int_t^T Z_s dW_s - \int_t^T \int_A U_s(i)\tilde{\mu}(ds, di), \\ U_t(i) \leq c(t, X_t, I_{t^-}, i), \forall t \in [0,T], \forall i \in A \end{cases}$$

with the forward Markov regime-switching diffusion process (I, X^I) and $\tilde{\mu}$ the compensated Poisson measure. The following Proposition provides a link between the value function given by (1) and the solution Y of the previous BSDE.

Proposition 1. *Combining* [1] *and* [2] *we have that* $Y_t = v_{I_t}(t, X_t)$ *for all* $t \in [0, T]$.

We aim to propose a numerical scheme combining a backward discretization and a least square regression method to approximate the non standard BSDE.

Discretization of the BSDE. Let $\overline{X}$ denote the standard Euler scheme associated with X. We consider the following discretization of the BSDE

$$(\mathcal{E}_1)\begin{cases} \hat{Y}_{t_N} = \mathcal{Y}_{t_N} = g(\overline{X}_{t_N}^{\theta_{t_{N-1}}}, I_{t_{N-1}}), \\ \text{ and for all } k \in \{N-1, \cdots, 0\} \\ \mathcal{Y}_{t_k}(j) = \mathcal{P}_{k,j}(\hat{Y}_{t_{k+1}}) + hf(t_k, \overline{X}_{t_k}^{\theta_{t_{k-1}}}, j) \\ \hat{Y}_{t_k} = \max_{j\in A}\{\mathcal{Y}_{t_k}(j) - c(t_k, \overline{X}_{t_k}^{\theta_{t_{k-1}}}, I_{t_{k-1}}, j)\} \end{cases}$$

Let U denote a $\mathcal{F}_{t_{k+1}}^{W,\mu}$-measurable r.v.. $\mathcal{P}_{k,j}(U)$ represents the approximation of the conditional expectation $\mathbb{E}(U|\overline{X}_{t_k}^{\theta_{t_{k-1}}}, I_{t_k} = j)$ by using a least square regression method based on indicator functions on $[-R + x_0, x_0 + R]$ of edge δ :

$$\mathcal{P}_{k,j}(U) := \sum_{\ell=1}^{K} \hat{\lambda}_{k,j}^{\ell}(U) \cdot \mathbf{1}_{\overline{X}_{t_k}^{\theta_{t_{k-1}}} \in D_\ell}, \ \hat{\lambda}_{k,j}^{\ell}(U) = \mathbb{E}(U|\overline{X}_{t_k}^{\theta_{t_{k-1}}} \in D_\ell, I_{t_k} = j)$$

where K is such that $2R = K\delta$ and $D_\ell = [-R + x_0 + \ell\delta, -R + x_0 + (\ell+1)\delta]$.

Theorem 1. *Let us assume that* $b, \sigma, f, g,$ *and* c *are Lipschitz functions in* x *and* g *and* c *are Lipschitz in* i. *For all* $p > 2$, *we have*

$$\mathbb{E}|Y_{t_k} - \hat{Y}_{t_k}|^2 \leq K(1 + |x_0|^5)h\left(\log\left(\frac{2T}{h}\right) + 1\right) + C(R^2\frac{\delta^2}{h^2} + \frac{C_p}{R^{p-2}}).$$

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Coupling of BSDEs driven by random measures

Łukasz Leżaj

(joint work with Stefan Geiss)

In the recent monograph [2], the authors developed the so-called *coupling* technique for the Wiener space, which can be roughly described as follows. Given two d-dimensional Brownian motions W, W' defined on two complete probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$, we consider the canonical product space $\overline{\Omega} := \Omega \times \Omega'$, $\overline{\mathbb{P}} := \mathbb{P} \otimes \mathbb{P}'$ and $\overline{\mathcal{F}} := \overline{\mathcal{F} \times \mathcal{F}'}^{\overline{\mathbb{P}}}$, extend canonically W and W' to $\overline{\Omega}$ and denote by $\overline{W} := (W, W')$ the $2d$-dimensional Brownian motion. As a toy example, one may take an interval $(a,b] \subseteq (0,T]$ and consider the Brownian motion $W^{(a,b]} = (W_t^{(a,b]})_{t\in[0,T]}$, where the increments on the interval $(a,b]$ are replaced by those from the independent copy, i.e.

$$\begin{cases} W_t, & 0 \le t \le a, \\ W_a + (W'_t - W'_a), & a < t \le b, \\ W_a + (W'_b - W'_a) + (W_t - W_b), & b < t \le T. \end{cases}$$

In other words, the Gaussian structure on the interval $(a,b]$ is replaced by an independent copy. The case described above is the special type of coupling called the *on-off* coupling, but the developed theory allows for a general case given by a measurable map $\varphi\colon [0,T] \mapsto [0,1]$ and a Brownian motion $W^\varphi = (W_t^\varphi)_{t\in[0,T]}$ defined by

$$W_t^\varphi = \int_0^t \sqrt{1-\varphi^2(s)}\, \mathrm{d}W_s + \int_0^t \varphi(s)\, \mathrm{d}W'_s,$$

which gives rise to the $\overline{\mathbb{P}}$-augmented filtration $\mathcal{F}^\varphi$ of W^φ. Considering the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, \mathrm{d}s - \int_t^T Z_s\, \mathrm{d}W_s,$$

where ξ is the terminal condition and f is a predictable random generator, if one provides the construction which allows to extend the mapping $W \mapsto W^\varphi$ and transform all objects in the equation above *along the same map*, we end up with the transformed BSDE

$$Y_t^\varphi = \xi^\varphi + \int_t^T f^\varphi(s, Y_s^\varphi, Z_s^\varphi)\, \mathrm{d}s - \int_t^T Z_s^\varphi\, \mathrm{d}W_s^\varphi, \quad t \in [0,T].$$

It turns out that this approach leads to many interesting results, such as more general description of the fractional smoothness in terms of generalised (anisotropic) Besov spaces, regularity of solutions of BSDEs including its L_p variation, and many more, see e.g. [1, 2, 3, 4].

In this talk, we extend the coupling method developed on the Wiener space in [2] to BSDEs driven by Poisson random measures associated with Lévy processes. The cornerstone is to we develop a way to transform random processes with càdlàg paths and random variables taking values in Banach spaces from one stochastic basis to another so that the corresponding classes of processes are mapped onto each other. In this way we may treat fully path-dependent terminal conditions and general random generators of the BSDE of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s - \int_{(t,T]\times(\mathbf{R}^d\setminus\{0\})} U_s(x)\,\widetilde{N}(\mathrm{d}s, \mathrm{d}x),$$

where $\widetilde{N}$ is the compensated Poisson random measure associated with Lévy process $X = (X_t)_{t\in[0,T]}$. Let us note that the above transference operation from one stochastic basis to another does not require any uniqueness of the equation under consideration and it applies to any type of coupling.

In applications, we consider various types of couplings. First, we obtain a general comparison result which measures in quantitative way the impact of coupling on the solution of the BSDE in terms of its influence on the initial data. In particular, in the *on-off* coupling case, this procedure works for all Lévy processes and in that situation we provide the L_2 regularity estimate of the solution of the above BSDE. We also give the description of the *on-off* and, in the α-stable case, the so-called *isonormal* coupling in terms of the chaos decomposition.

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A Global Stochastic Maximum Principle for Mean-Field Forward-Backward Stochastic Control Systems with Quadratic Generators

Juan Li

(joint work with Rainer Buckdahn, Yanwei Li, Yi Wang)

We study a recursive mean-field stochastic control problem motivated by Jiang, Li, and Wei [2]. The system is governed by

$$\begin{cases} dX_t^u = b(t, X_t^u, P_{X_t^u}, u_t)dt + \sigma(t, X_t^u, P_{X_t^u}, u_t)dB_t, \\ dY_t^u = -f(t, X_t^u, Y_t^u, Z_t^u, P_{(X_t^u, Y_t^u)}, u_t)dt + Z_t^u dB_t, \\ X_0^u = x_0, \quad Y_T^u = \Phi(X_T^u, P_{X_T^u}), \end{cases}$$

with cost functional $J(u) = Y_0^u$. The generator f exhibits quadratic growth in Z_t^u, and all coefficients depend on the joint law $P_{(X_t^u, Y_t^u)}$, allowing fully nonlinear dependence on (X, Y, Z). The existence and uniqueness of the solution of this system can be found in [2].

The primary analytical difficulty stems from the quadratic growth in Z, which lies beyond the classical Lipschitz BSDE framework and necessitates BMO martingale techniques. A key distinction from the classical case is that the adjoint equations arising from the Taylor expansion of the cost functional are linear mean-field BSDEs whose drivers satisfy a stochastic Lipschitz condition with BMO-type coefficients in Z. This structure is intrinsic to the quadratic setting and does not appear when f is Lipschitz in Z. The BMO property of the adjoint processes is essential to control the growth of the solutions and to handle the unbounded terms arising from the spike variation method.

To establish the stochastic maximum principle (SMP), we first develop a priori estimates and well-posedness for such linear mean-field BSDEs (Section 3), which serve as the foundation for the adjoint analysis. Our results extend beyond the classical setting where coefficients are bounded or of deterministic Lipschitz type; here, the stochastic Lipschitz condition introduces significant technical challenges that require new martingale inequalities and careful handling of the mean-field interactions.

Our main result is a global SMP for controls taking values in an arbitrary (possibly non-convex) subset $U \subset \mathbb{R}^n$. The proof is mainly based on the spike variation method. To control the critical term $\mathbb{E}[(\int_0^T |Z_t^*||\bar{Z}_t|\mathbf{1}_{E_\varepsilon}(t)dt)^p]$, we introduce the deterministic set $\Gamma_M := \{t \in [0,T] : \mathbb{E}[|Z_t^*|^2] \leq M\}$ and redefine the variation on $E_\varepsilon \cap \Gamma_M$. This avoids pointwise a.e. restrictions and yields an SMP that holds for all $t \in \Gamma_M$; letting $M \to \infty$ gives the SMP for almost every $t \in [0,T]$.

In addition to the necessary condition (SMP), we also establish a sufficient condition for optimality under convexity assumptions on U and the Hamiltonian (Section 7). This result provides a complete characterization of optimal controls in suitable cases. The sufficient condition relies on the convexity of the Hamiltonian in (X, Y, Z, u) and the terminal cost, together with the SMP. The mean-field dependence introduces additional coupling between the state and its law, which leads to a richer structure of the optimal control compared to the classical non-mean-field case.

Our framework applies to mean-field games with recursive utilities, such as portfolio optimization and asset pricing models with large interacting populations. The results generalize those of Hu, Ji, and Xu [1] to the mean-field setting with joint law dependence, and provide a foundation for future studies on convergence of N-player games to mean-field equilibria in the presence of quadratic costs.

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Extrapolation of stochastic maximal regularity

Emiel Lorist
(joint work with Mark Veraar)

Parabolic SPDEs can often be expressed as stochastic evolution equations in the form

$$\begin{cases} \frac{\partial u}{\partial t} + Au + F(u) &= \big(Bu + G(u)\big)\frac{\partial W}{\partial t}, \\ u(0) &= u_0, \end{cases} \tag{1}$$

where (A, B) are differential operators, F and G are suitable nonlinearities and u_0 is the initial condition. Furthermore, W denotes a (cylindrical) Brownian motion. Properties of parabolic SPDEs of the form (1) can be obtained from sharp estimates for the linearized equation, see e.g. the recent survey [1].

Sharp estimates for solutions to *linear* SPDEs	$\Longrightarrow$	Existence, uniqueness, and smoothness for solutions to *nonlinear* SPDEs

More mathematically precise, consider the linear stochastic Cauchy problem

$$\begin{cases} \mathrm{d}u + Au\,\mathrm{d}t + f = (Bu + g)\,\mathrm{d}W(t), \\ u(0) = 0, \end{cases} \tag{2}$$

where A is a sectorial operator on a Banach space X_0 with domain $D(A) =: X_1$, W is a (cylindrical) Brownian motion and f, g are adapted forcing terms. The simplest example to keep in mind is $A = -\Delta$ on $X = L^q(\mathbb{R}^d)$, in which case (2) is the stochastic heat equation.

For $f \in L^p((0,T);X)$ and $g \in L^p((0,T);X_{\frac{1}{2}})$ with $X_{\frac{1}{2}} := [X_0, X_1]_{\frac{1}{2}}$ the optimal regularity of the solution u to (2) is $u \in L^p((0,T);X_1)$, which is called *stochastic maximal L^p-regularity*. In case $B = 0$, the mild solution u to (2) is given by

$$u(t) = -\int_0^t \mathrm{e}^{-(t-s)A} f(s)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A} g(s)\,\mathrm{d}W(s), \qquad t \in [0,T].$$

Hence, stochastic maximal L^p-regularity is in this case equivalent to L^p-boundedness of

$$T_K f(t) := \int_0^T K(t,s) f(s)\,\mathrm{d}s, \qquad t \in [0,T],$$

$$S_K g(t) := \int_0^T K(t,s) g(s)\,\mathrm{d}W(s), \qquad t \in [0,T],$$

where $K(t,s) := \mathrm{e}^{-(t-s)A}\chi_{s<t}$. The kernel K has a singularity in $t = s$:

$$\|K(t,s)\|_{X_0 \to X_1} \lesssim \frac{1}{|t-s|},$$

$$\|K(t,s)\|_{X_{\frac{1}{2}} \to X_1} \lesssim \frac{1}{|t-s|^{\frac{1}{2}}},$$

which is why T_K and S_K are called a singular (stochastic) integral operators.

Singular integral operators T_K have been systematically studied in harmonic analysis since the seminal work of Calderón and Zygmund [2]. In stark contrast, singular *stochastic* integral operators S_K are yet to be thoroughly understood from a harmonic analytic viewpoint. Although singular stochastic integral operators resemble singular integral operators, the stochastic integral introduces fundamentally different cancellative behavior compared to the deterministic integral.

In [3, 4], we established extrapolation of L^p-boundedness of singular stochastic integral operators with Muckenhoupt weights in time, using a modern technique in harmonic analysis called *sparse domination*. These works can be seen as initial steps towards a comprehensive harmonic analytic toolbox to study parabolic SPDEs. Important current open problems in this direction include:

- The extrapolation of L^p-boundedness of singular stochastic integral operators without a kernel K, as the solution operator to (2) with $B \neq 0$ (relevant for SPDEs with transport noise) does not have a kernel representation.
- Temporal Muckenhoupt weights are fully understood in the context of stochastic maximal regularity, but Muckenhoupt weights in the probability space have so far not yet been incorporated or exploited.

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An Additive-Noise Approximation to Keller–Segel–Dean–Kawasaki Dynamics

Adrian Martini

(joint work with Avi Mayorcas)

The goal of this work is to understand the behaviour of the so-called *Keller–Segel–Dean–Kawasaki* equation,

$$(1) \quad (\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho\nabla\Phi_\rho) - \varepsilon^{1/2}\nabla\cdot(\sqrt{\rho}\boldsymbol{\xi}), \qquad -\Delta\Phi_\rho = \rho - \langle\rho, 1\rangle_{L^2(\mathbb{T}^d)},$$

defined on the $d \in \mathbb{N}$-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, where $\chi \in \mathbb{R}$ is the *chemotactic sensitivity*, $\boldsymbol{\xi}$ denotes a vector-valued space-time white noise and $\varepsilon > 0$ the *noise intensity*. Assume that the initial data ρ_0 is non-negative and of unit mass.

The interest in (1) stems from its connection to a system of overdamped Langevin diffusions that are weakly interacting through the gradient $\nabla\mathscr{G}$ of the Green function of the mean-free Laplacian on $\mathbb{T}^d$ (so that $\Phi_\rho = \mathscr{G} * \rho$.) Assume there are $N \in \mathbb{N}$ particles of mass $1/N$. Passing to the limit $N \to \infty$, the system is a stochastic particle approximation to the solution ρ_{det} of the deterministic (parabolic-elliptic) *Keller–Segel* PDE of chemotaxis [14] which is obtained by setting $\varepsilon = 0$ in (1), see [9, 18, 1]. On the other hand, for $\varepsilon = 2/N$, equation (1) describes the continuum fluctuations of the particles through the theory of fluctuating hydrodynamics (FHD) [3, 12].

The problem with stochastic PDEs driven by *Dean–Kawasaki noise* $\nabla \cdot (\sqrt{\rho}\boldsymbol{\xi})$ is that they are unstable. Indeed upon replacing $\mathscr{G}$ by a smooth function, it has been shown in [13], that a solution (in the sense of a martingale problem) exists if and only if the initial data is an empirical measure and $\varepsilon = 2/N$. Even though there are singular interactions that recover well-posedness (see e.g. the references in [13]), we do not expect the situation to be improved in the case of (1).

In recent years there has been great progress in finding good approximations to equations driven by Dean–Kawasaki noise [8, 4, 19, 2, 7, 6]. However, the case of interactions that are both non-incompressible and singular beyond the Ladyzhenskaya–Prodi–Serrin condition, such as $\mathscr{G}$, is still open.

Inspired by the physics literature [15], we take a first step in this direction by considering an *additive-noise approximation* $\rho_\delta^{(\varepsilon)}$ given by

$$(2)\quad (\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho\nabla\Phi_\rho) - \varepsilon^{1/2}\nabla\cdot(\sqrt{\rho_{\mathrm{det}}}\boldsymbol{\xi}^\delta), \qquad \rho|_{t=0} = \rho_{\mathrm{det}}|_{t=0} = \rho_0,$$

where $\delta > 0$ denotes the *correlation length*, ψ_δ a smooth spatial mollifier of scale δ, $\boldsymbol{\xi}^\delta := \psi_\delta * \boldsymbol{\xi}$ and ρ_{det} the solution to the deterministic Keller–Segel equation, i.e. (1) with $\varepsilon = 0$.

The strength of (2) lies in its non-linear dynamics in the presence of noise. Neither the zeroth-order approximation ρ_{det} nor the first-order approximation provided by *linear fluctuating hydrodynamics* can model noise-induced aggregation when started from uniform initial data. On the other hand, it is clear that (2) can blow up to a Dirac mass due to the fluctuations interacting with the non-linearity, even when started from uniformity.

Our main result is that the additive-noise approximation (2) is precise up to first order and that it is possible to quantify the approximation error through a large deviation principle (cf. Theorem 1 below.)

In the following, due to reasons of scaling sub-criticality, we restrict our attention to two dimensions. As state space for our solution we shall work in a space of functions continuous in time and taking values in a Hölder–Besov space of regularity less than -1 that are allowed to blow up in finite time but once blown up are not allowed to return. For the precise details, see [17, 16].

Theorem 1 (LLN, CLT & LDP, [16, Thms. 1.3, 1.4 & 1.5])**.** *Let* $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ *be such that* $\rho_0 > 0$ *and* $\langle \rho_0, 1\rangle_{L^2(\mathbb{T}^2)} = 1$. *As* $\varepsilon \to 0$, *assume that* $\delta(\varepsilon) \to 0$ *and* $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$. *Then it follows that the sequence* $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ *converges to* ρ_{det} *in probability as* $\varepsilon \to 0$ *and satisfies an LDP with speed* ε *and good rate function*

$$(3)\quad \mathscr{I}(\rho) := \inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : (\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho\nabla\Phi_\rho) - \nabla\cdot(\sqrt{\rho_{\mathrm{det}}}h)\Big\}.$$

If, furthermore, $\varepsilon^{1/2}\log(\delta(\varepsilon)^{-1}) \to 0$ *as* $\varepsilon \to 0$*, then* $\varepsilon^{-1/2}(\rho^{(\varepsilon)}_{\delta(\varepsilon)} - \rho_{\mathrm{det}}) \to v$ *as* $\varepsilon \to 0$*, where* v *is a generalized Ornstein–Uhlenbeck (OU) process.*

Sketch of Proof. The proof of the LLN proceeds in three steps. First, we pass to paracontrolled decomposition [10], which allows us to consider the solution as a continuous map of a stochastic process known as the *noise enhancement*. Second, we use the relative scaling on $(\varepsilon, \delta(\varepsilon))$ to show that the noise enhancement vanishes in probability as $\varepsilon \to 0$. Third, we apply the continuous mapping theorem to pass to the limit. The proof of the CLT follows by the same continuity. The proof of the LDP follows by combining the paracontrolled decomposition with a generalization of Freidlin and Wentzell's theory due to Hairer and Weber [11]. □

In two dimensions and for $\chi < 8\pi$, it is known [9, 18, 1] that the underlying particle system exhibits mean-field convergence towards ρ_{det}. While the CLT for the particle system is to our knowledge still open, there are related systems for which a CLT is known with limiting fluctuations given by a generalized OU process [20]. Consequently, Theorem 1 leads us to expect that our additive-noise approximation is precise up to first order.

Further, we find that the approximation error lies in the so-called skeleton equation appearing in the rate function (3). The skeleton equation for our additive-noise approximation is additive in the Cameron–Martin element h, whereas the skeleton for the underlying particle system is expected to be non-linear in h, see [5].

One of the major modelling errors of the additive-noise approximation (2) is the possibility of observing negative values. However, one can control the probability of this event and show that it is exponentially small in $\varepsilon^{-1}(1+\delta(\varepsilon)^{-2})^{-2}$, provided one imposes the more restrictive relative scaling that $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\delta(\varepsilon)^{-2} \to 0$ as $\varepsilon \to 0$, see [16, Thm. 1.6]. Under the same restrictive relative scaling, one can also show an LLN & LDP in spaces of higher regularity, see [16, Thms. 1.1 & 1.2].

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Bismut-Elworthy type formulae for BSDEs with degenerate noise

Federica Masiero
(joint work with Davide Addona, Enrico Priola)

We consider at first a stochastic differential equation (SDE) in a real and separable Hilbert space H:

$$\begin{cases} dX_t = AX_t dt + B(t,X_t)dt + G(t,X_t)dW_t, & t\in[s,T]\subset[0,T], \\ X_\tau = x, & \tau\in[0,s], \end{cases} \tag{1}$$

where

- A is the generator of a C_0 semigroup in H,
- W_t is a cylindrical Wiener process in H,
- B, G are Lipschitz continuous and differentiable w.r. to x,
- G is invertible and $|G^{-1}(t,x)|_{L(H,H)} \le C$.

The transition semigroup $P_{s,t} : B_b(H) \to B_b(H)$ associated to X is given by

$$P_{s,t}f(x) = \mathbb{E}[f(X_t^{s,x})],\ \forall f\in B_b(H),\ x\in H,$$

We recall that under these assumptions a Bismut-Elworthy formula holds true: $\forall\, f \in B_b(H)$

$$\begin{aligned}&\langle (\nabla_x P_{s,t} f)(x), h\rangle_H\\ =&\mathbb{E}[f(X_t^x)\frac{1}{t-s}\int_s^t \left\langle G(r,X_r^{s,x})^{-1}\nabla_x X_r^{s,x}\, h,\, dW_s\right\rangle], \qquad t\in(s,T],\end{aligned}$$

leading to

$$\begin{aligned}&|\langle (\nabla_x P_{s,t} f)(x), h\rangle_H|\\ \le&\|f\|_\infty \mathbb{E}[|\frac{1}{t-s}\int_s^t \left\langle G(r,X_r^{s,x})^{-1}\nabla_x X_r^{s,x}\, h,\, dW_s\right\rangle|] \le \frac{C\|f\|_\infty |h|_H}{(t-s)^{1/2}}.\end{aligned}$$

We refer to this formula as linear Bismut formula since $u(s,x) = \mathbb{E}[\phi(X_T^{s,x})] = P_{s,T}[\phi](x)$ is the solution of the linear Kolmogorov equation

$$\begin{cases} -\frac{\partial v}{\partial s}(s,x) = \mathcal{L}_s v\,(s,x)\,, & s\in[0,T]\,,\ x\in H,\\ v(T,x) = \phi\,(x)\,, \end{cases}$$

where $\mathcal{L}_s$ is at least formally the generator of $P_{s,\cdot}$.

If we turn to semilinear Kolmogorov equations in H of the form

$$\begin{cases} -\frac{\partial v}{\partial s}(s,x) = \mathcal{L}_s v\,(s,x) + \psi\,(s,x,v(s,x),\nabla v(s,x)G(s,x))\,,\ s\in[0,T]\,,\\ v(T,x) = \phi\,(x)\,, \end{cases} \tag{2}$$

it is well known that with ψ Lipschitz continuous in v and in $\nabla v G$, there exists a unique mild solution

$$u(s,x) = \mathbb{E}[\phi(X_T^{s,x}) + \int_s^T \psi(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x})\,dr].$$

Such solution u can be defined by means of backward stochastic differential equations (BSDEs). Namely we consider a BSDE coupled with an SDE

$$\begin{cases} dX_t = AX_t dt + B(t,X_t)dt + G(t,X_t)dW_t, & t\in[s,T]\subset[0,T],\\ X_s = x, & \tau\in[0,s],\\ -dY_t = \psi(t,X_t,Y_t,Z_t)\;dt - Z_t\;dW_t, & t\in[0,T],\\ Y_T = \phi(X_T), \end{cases}$$

and it turns out the solution of equation (2) can be defined as $u(s,x) = Y_s^{s,x}$. Moreover in [3] the authors proved a Bismut formula

$$\mathbb{E}\nabla_x Y_t^{s,x} h = \mathbb{E}\left[\int_t^T \psi(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x})U_r^{h,s,x}dr + \phi(X_T)U_T^{h,s,x}\right],$$

where

$$U_t^{h,s,x} = \frac{1}{t-s}\int_s^t \left\langle G(r,X_r^{s,x})^{-1}\nabla_x X_r^{s,x}\, h,\, dW_r\right\rangle.$$

Motivated by the regularizing properities of the transition semigroup of the stochastic wave equations, studied in [4], and of the stochastic damped wave equation, first studied in [1] and next also in [2], we consider an analogous of the aforementioned linear and semilinear Bismut formulae for SDE like (1) with non invertible

diffusion G. Indeed we notice that the stochastic (damped) wave equation can be reformulated as a stochastic evolution equation with degenerate noise, see [4] and [2].
We underline the fact that in [2] we formulate the results in an abstract way and then we apply to the motivating models.

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On the H^∞-calculus of vector-valued Ornstein-Uhlenbeck operators

Jan van Neerven

In this talk, we survey some known results and open problems concerning the optimal angles of holomorphy and boundedness of the H^∞-calculus of the Ornstein–Uhlenbeck semigroup and its generator.

Let γ_d denote the standard Gaussian measure on $\mathbb{R}^d$, given by

$$\gamma_d(B) = \frac{1}{(2\pi)^{d/2}} \int_B \exp(-\tfrac{1}{2}|x|^2)\,\mathrm{d}x$$

for Borel sets $B \in \mathscr{B}(\mathbb{R}^d)$. For $t \geq 0$ and $f \in L^p(\mathbb{R}^d, \gamma_d)$ with $1 \leq p < \infty$, define

$$(P_t f)(x) := \int_{\mathbb{R}^d} f\big(e^{-t}x + \sqrt{1-e^{-2t}}y\big)\,\mathrm{d}\gamma_d(y), \qquad x \in \mathbb{R}^d.$$

The following basic properties are easy consequences of the definition:

- (Positivity) $f \geq 0$ implies $P_t f \geq 0$;
- (Markovianity) $P_t\mathbf{1} = \mathbf{1}$;
- (Contractivity) $\|P_t f\|_{L^p(\mathbb{R}^d,\gamma_d)} \leq \|f\|_{L^p(\mathbb{R}^d,\gamma_d)}$;
- (Semigroup property) $P_{s+t}f = P_s P_t f$ for $s, t \geq 0$;
- (Strong continuity) $\lim_{t\downarrow 0} P_t f = f$ with convergence in $L^p(\mathbb{R}^d, \gamma_d)$

Positivity and Markovianity are clear. Contractivity follows from Jensen's inequality. To prove the semigroup property, write

$$P_s(P_t f)(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(e^{-t}(e^{-s}x + \sqrt{1-e^{-2s}}z) + \sqrt{1-e^{-2t}}y)\,\mathrm{d}\gamma_d(y)\,\mathrm{d}\gamma_d(z),$$

and use that if Y, Z are independent standard Gaussian random variables, and W is another standard Gaussian random variable, then $e^{-t}(e^{-s}x + \sqrt{1-e^{-2s}}Z) + \sqrt{1-e^{-2t}}Y$ and $e^{-(s+t)}x + \sqrt{1-e^{-2(s+t)}}W$ are identically distributed. Finally, strong continuity is evident for functions $f \in C_c(\mathbb{R}^d)$ by dominated convergence, and the general result follows from this by approximation. The infinitesimal generator is the operator L obtained as the closure of

$$\Delta - x \cdot \nabla,$$

initially defined on the domain $C_c^2(\mathbb{R}^d)$.

It is a classical observation that if T is a bounded operator on a space $L^p(S,\mu)$ and X is a Banach space, then $T \otimes I_X$, intitially defined on the algebraic tensor product $L^p(S,\mu) \otimes X$, extends to a bounded operator, also denoted by $T \otimes I_X$, on the Bochner space $L^p(S,\mu;X)$, with norm $\|T \otimes I_X\| = \|T\|$. Applying this to the Ornstein–Uhlenbeck operators P_t, we obtain a C_0-contraction semigroup $(P_t \otimes I_X)_{t\geq 0}$ on $L^p(\mathbb{R}^d, \gamma_d; X)$ for every $1 \leq p < \infty$. To simplify notation we write $P_t^X := P_t \otimes I_X$ for this semigroup, and denote its generator by L^X.

Building on earlier results of Weissler [11], Epperson [2] proved that for all $1 < p < \infty$ the semigroup P extends to an analytic C_0 semigroup of contractions on $L^p(\mathbb{R}^d, \gamma_d)$ of angle θ_p, where

$$\cos \theta_p = \Big|\frac{2}{p} - 1\Big|,$$

and that the angle θ_p is the optimal angle for analyticity. A probabilistic proof of this result was given subsequently by Janson [7].

Another proof of analyticity of P with optimal angle was given by [9]. This proof was simplified by Harris [4], and his argument extends to the vector-valued setting. It implies that for all $1 < p < \infty$ the semigroup P^X extends to a bounded analytic C_0 semigroup on $L^p(\mathbb{R}^d, \gamma_d; X)$ of angle θ_p. Let us introduce the boundedness constants

$$M_{p,d,\theta}^X := \sup_{z\in\Sigma_\theta} \|P_z^X\|_{\mathscr{L}(L^p(\mathbb{R}^d,\gamma_d;X))}.$$

Here,

$$\Sigma_\theta = \{z \in \mathcal{C} \setminus \{0\} : |\arg(z)| < \theta\}$$

is the open sector of angle θ about the positive real axis. Adapting an argument of [4], one obtains constants M_{p,d,θ_p}^X that grow exponentially in the dimension d. This prompts the following question for the vector-valued case:

Problem 1. *Is it true that for all $1 < p < \infty$ one has dimension-independent constants, that is,*

$$\sup_{d\geq 1} M_{p,d,\theta_p}^X < \infty\,?$$

Pisier [10] proved that the Banach space X is K-convex if and only if there exists an angle $0 < \theta \leq \theta_p$ such that for some all (equivalently: for some) $1 < p < \infty$ one has

$$\sup_{d\geq 1} M_{p,d,\theta}^X < \infty.$$

Problem 2. *If X is K-convex, is it true that for all $1 < p < \infty$ one has*

$$\sup_{d\geq 1} M^X_{p,d,\theta} < \infty$$

for all $0 < \theta < \theta_p$, *or even*

$$\sup_{d\geq 1} M^X_{p,d,\theta_p} < \infty\,?$$

Similar questions may be asked with regard to the optimal angle for boundedness of the H^∞-calculus of $-L^X$. It was shown in [3] (see also [1, 4, 5]) that $-L$ has a bounded $H^\infty(\Sigma_{\theta_p})$-calculus on $L^p(\mathbb{R}^d, \gamma_d)$ for all $d \geq 1$. Since the boundedness of the H^∞-calculus is closely related to the existence of a certain singular integral, and since X-valued integrals tend to be bounded on $L^p(\mathbb{R}^d, \gamma_d; X)$ for UMD spaces X, one expects a bounded H^∞-calculus for $-L^X$ if X is UMD. By a result of Kalton and Weis [8] (see also [6, Theorem 10.7.13]), this is indeed true, but with sub-optimal angle: if X is a UMD space, then $-L^X$ has a bounded $H^\infty(\Sigma_\theta)$-calculus for *some* angle $0 < \theta < \theta_p$.

Problem 3. *If X is UMD, Is it true that for all $1 < p < \infty$ that $-L^X$ has a bounded $H^\infty(\Sigma_\theta)$-calculus for all $0 < \theta < \theta_p$? Or even a bounded $H^\infty(\Sigma_{\theta_p})$-calculus?*

Preliminary investigations supported by GPT5.1 suggest some possible lines of attack to this problem that are currently being explored.

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Path-dependent BSDEs with jumps and their connections to PPIDEs

Ludger Overbeck
(joint work with Eduard Kromer, Jasmin Röder)

We study path-dependent backward stochastic differential equations (BSDEs) with jumps. In this context path-dependence of a BSDE is the dependence of the BSDE-terminal condition and the BSDE-generator of a path of a cadlag process. We study the path-differentiability of BSDEs of this type and establish a connection to path-dependent partial-integro-differential equations in terms of the existence of a viscosity solution and the respective Feynman-Kac theorem. In more detail we first look at a path-dependent BSDE of the following type:

For any $\gamma \in D([0,T],\mathbb{R}^d)$ and a fixed $t \in [0,T]$ we are looking for a unique solution to the following path-dependent BSDE

$$
\begin{aligned}
Y_{\gamma_t}(s) &= \Phi(X_{\gamma_t,T}) + \int_s^T f(r, X_{\gamma_t,r-}, Y_{\gamma_t}(r-), Z_{\gamma_t}(r), U_{\gamma_t}(r))dr \\
&\quad - \int_s^T Z_{\gamma_t}(r)dW(r) - \int_s^T \int_{\mathbb{R}^l} U_{\gamma_t}(r,x)\tilde{N}(dr,dx), \qquad (1)
\end{aligned}
$$

with

$$
\begin{aligned}
X_{\gamma_t,s-}(u) :=&\gamma(u)\mathbf{1}_{[0,t)}(u) + (\gamma(t) + X(u) - X(t))\mathbf{1}_{[t,s)}(u) \\
&+ (\gamma(t) + X(s) - X(t))\mathbf{1}_{[s,T]}(u),\ u \in [0,T], \qquad (2)
\end{aligned}
$$

where $\tilde{N}(\omega,dt,dx) = N(\omega,dt,dx) - \nu(\omega,dt,dx)$ is a compensated random measure on the probability space and X a $\mathbb{R}^d$-valued adapted càdlàg process.

In a second step we consider X to be a solution of a forward SDE:

$$
\begin{aligned}
dX_{\gamma_t}(s) &= b(s, X_{\gamma_t,s-})ds + \sigma(s, X_{\gamma_t,s-})dW(s) \\
&\quad + \int_{\mathbb{R}^l} \beta(s, X_{\gamma_t,s-}, z)\tilde{N}(ds,dz), \quad s \in [t,T], \\
X_{\gamma_t,t}(s) &= \gamma_t(s), \quad s \in [0,t), \qquad (3)
\end{aligned}
$$

and a BSDE with a deterministic generator $f : [t,T] \times D([0,T],\mathbb{R}^d) \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{R} \to \mathbb{R}$,

$$
\begin{aligned}
dY_{\gamma_t}(s) =& - f\left(s, X_{\gamma_t,s-}, Y_{\gamma_t}(s-), Z_{\gamma_t}(s), \int_{\mathbb{R}^l} U_{\gamma_t}(s,z)\delta(s,z)\nu(dz)\right)ds \\
&+ Z_{\gamma_t}(s)dW(s) + \int_{\mathbb{R}^l} U_{\gamma_t}(s,z)\tilde{N}(ds,dz),\ s \in [t,T], \\
Y_{\gamma_t}(T) =& \Phi(X_{\gamma_t,T}), \qquad (4)
\end{aligned}
$$

where the random measure $N(ds,dz)$ is the integer-valued random measure and $\tilde{N}(ds,dz) = N(ds,dz) - ds\nu(dz)$ i.e. $\eta(s) = 1$ and $Q(s,dz) = \nu(dz)$. Notice, that the path of the process X_{γ_t} defined in (3) satisfies equation (2). We then consider the following operator:

For a function $\psi \in C^{1,2}([0,T] \times D([0,T], \mathbb{R}^d); \mathbb{R})$ we define the operator $\mathcal{A}$ by $(\mathcal{A}_s\psi)(s,x) :=$

$$\frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} a_{ik}(s,x,x) D^2_{ik}\psi(s,x;[s,T]) + \sum_{i=1}^{d} b_i(s,x) D^1_i\psi(s,x;[s,T])$$

$$+ \int_{\mathbb{R}^l}\left[\psi(s,x^{\beta(s,x,z)}) - \psi(s,x) - \sum_{i=1}^{d}\beta_i(s,x,z) D^1_i\psi(s,x;[s,T])\right]\nu(dz) \tag{5}$$

and we consider a path-dependent partial integro-differential equation (PPIDE) of with boundary $\psi(T,x) = \Phi(x)$, $x \in D([0,T], \mathbb{R}^d)$ and

$$\begin{aligned} -\frac{\partial}{\partial s}\psi(s,x) &= \mathcal{A}_s\psi(s,x) + \\ &+ f\Big(s, x, \psi(s,x), D^1\psi(s,x;[s,T])\sigma(s,x), \\ &\int_{\mathbb{R}^l}[\psi(s,x^{\beta(s,x,z)}) - \psi(s,x)]\delta(s,z)\nu(dz)\Big). \end{aligned} \tag{6}$$

Here the differentials D^1, D^2 for functions on path space are defined in the Dupire sense.

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The Matrix A_2 conjecture fails and $3/2 > 1$

Stefanie Petermichl

Motivated by mixing problems for multivariate stationary stochastic processes as well as questions regarding Toeplitz operators, the characterization of those self adjoint locally integrable matrix weights W so that the Hilbert transform H acts $H : L^2(W) \to L^2(W)$ boundedly, became relevant in analysis. In this context we mean the tensor extension of the Hilbert transform and the inequality becomes

$$\int_{\mathbb{R}} \langle W(t)Hf(t), Hf(t)\rangle dt \le C(W)^2 \int_{\mathbb{R}} \langle W(t)f(t), f(t)\rangle dt,$$

where f is vector valued and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{C}^d$.

In the late 90s Treil and Volberg resolved this question [15], providing the matrix A_2 characteristic as necessary and sufficient condition for the boundedness in the above inequality. W is in the A_2 class if

$$[W]_{A_2} = \sup_I \|\langle W\rangle_I^{1/2}\langle W^{-1}\rangle_I^{1/2}\| < \infty,$$

where the supremum runs over all intervals and $\langle\cdot\rangle_I$ denotes the average (of a function) over I.

One might ask why this question is not raised for operators instead of matrices, but most of the basic tools do not have a suitable extension to the infinite dimensional case (with the exact growth known in some cases) [14]. But the weighted Hilbert transform itself was also shown to have a blow up with dimension [4].

The work of Treil and Volberg on the Hilbert transform in matrix weighted spaces sparked interest in the scalar theory, where a weight w is a locally integrable positive function. The equivalent to the Matrix A_2 condition had long been known through the work of Hunt, Muckenhaupt and Wheeden [5]. The condition

$$[w]_{A_2} = \sup_I \langle w \rangle_I \langle w^{-1} \rangle_I < \infty$$

determines the boundedness of the Hilbert transform. In part due to interest in different sides of PDE, the sharp weighted theory became of importance, where the exact growth of an operator norm can have a decisive role. Through a use of Jensen's inequality one can see that

$$[w]_{A_2} \geq 1 \text{ and } [W]_{A_2} \geq 1,$$

so that estimates with lower powers are better than those with higher powers.

While some leading experts thought that the norm of the Hilbert transform had an estimate of the form $[w]_{A_2}^{3/2}$, the dyadic methods developed by Nazarov, Treil and Volberg, notably the Bellman method based on the Haar system in combination with a now well known formula for the Hilbert transform by the author [9], the author showed that

$$\|H\|_{L^2(w)\to L^2(w)} \leq C[w]_{A_2},$$

which is the sharp estimate for the Hilbert transform [10].

The theory of sharp weighted estimates became very popular and methods developed through the effort of many [2, 7, 8]. But the estimates with matrix weights appeared to be much more difficult. With modern methods, Nazarov, Treil, Volberg and the author [13] proved

$$\|H\|_{L^2(W)\to L^2(W)} \leq C[W]_{A_2}^{3/2},$$

thus quantifying and improving the original estimate of Treil and Volberg as much as we could. We, as well as many, thought that this estimate was not optimal. Recently Domelevo, Treil, Volberg and the author [3] finally constructed an example showing that the 3/2 exponent cannot be improved. This is the negative answer to the matrix A_2 conjecture.

While the scalar conjecture was 3/2 even before this estimate was established and turned out to be improvable to 1, the conjecture in the matrix case stood for years and was believed to be linear. However, we showed that the best growth possible is the 3/2 power.

Our example uses the entire landscape of developments in the sharp weighted theory. We make us of dyadic models based on the Haar system. These reduce the statement to that of a paraproduct. From here, we try to lean as much as possible on the scalar extremizers to make use of the freedom of non commutativity [12] and create a matrix valued martingale that is a dyadic A_2 weight, which shows the

3/2 blow up for these dyadic models. As expected, the matrix weight constructed has infinitely many singularities and is not a classical A_2 weight. In a second step we show that an appropriate, quite complicated modification of the weight is in the A_2 class and provides a 3/2 blow up for the Hilbert transform. These ideas resemble the work of Bourgain [1] and their extensions to weighted spaces by Nazarov [11] and, more recently, Kakaroumpas and Treil [6].

The example ends a long line of investigation and highlights the decisive differences in the vector world.

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Sharp bounds for two-weighted commutators

Sandra Pott

In [4], the authors showed that Bloom-type commutators can be realized as off-diagonal corners of matrix-weighted operators, thereby providing relatively simple proofs for Bloom-type estimates in a variety of settings. In this talk, we choose the converse approach and show that each matrix-weighted linear operator can be realized as a combination of lower-dimensional matrix-weighted operators and a Bloom-type commutator, effectively providing an induction over dimension for matrix-weighted estimates. As a result, we obtain new extrapolation results and weak-type estimates for Bloom-type commutators, using the results of [1] and [2]. By means of the recent results in [3], we provide sharp bounds for Bloom-type commutators. Moreover, our method offers an alternative approach to the disproof of the A_2-conjecture for matrix weights in [3].

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Reflected BSDEs in non-convex domains and martingales on Euclidean manifolds with boundary

Adrien Richou

(joint work with Marc Arnaudon, Jean-François Chassagneux, Sergey Nadtochiy)

Given a Brownian motion W, $(\mathcal{F}_t)$ its augmented natural filtration and $\mathcal{D}$ a bounded domain of $\mathbb{R}^d$, we are looking for (Y, Z, K) a triple of progressively measurable processes that satisfy the following reflected backward stochastic differential equation (BSDE)

$$\begin{cases} (i)\ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T dK_s - \int_t^T Z_s dW_s,\ 0 \leq t \leq T, \\ (ii)\ Y_. \in \bar{\mathcal{D}} \text{ a.s.}, \quad \dot{K}_. \in \mathfrak{n}(Y_.)\ dt \otimes d\mathbb{P} - a.e., \quad \int_0^T \mathbb{1}_{\{Y_s \notin \partial \mathcal{D}\}} d\mathrm{Var}_s(K) = 0, \end{cases}$$

where ξ is a bounded $\mathcal{F}_T$-measurable random variable, the generator $f(., y, z)$ is a progressively measurable process for all (y, z) and $\mathfrak{n}(y)$ denotes the normal exterior cone of $\bar{\mathcal{D}}$ in $y \in \bar{\mathcal{D}}$ with $\mathfrak{n}(y) = \{0\}$ when $y \in \mathcal{D}$. If we look at the equation (i) without the K term, we have a classical BSDE (i.e. non reflected) and if moreover we consider the zero-generator case $f = 0$, then Y is just the conditional expectation of ξ. So the reflecting term K is here in order to force the process Y to stay in $\bar{\mathcal{D}}$ and acts only when it is necessary.

The theory of reflected BSDEs is well understood in the one-dimensional case. However, the multidimensional case poses serious additional challenges, notably due to the lack of a comparison principle. Well-posedness results in higher dimensions have primarily been established under convexity assumptions in [3]. A key observation is that, when the generator is zero and $\mathcal{D}$ is convex, the conditional expectation remains in $\bar{\mathcal{D}}$ which is no longer guaranteed in the non-convex case. In other words, for convex domains, the reflection only needs to counter the drift term arising from the generator, whereas in the non-convex case it may also need to take into account the Brownian term.

The first result in the non-convex setting was obtained recently in [2]. This work provides existence and uniqueness results under several restrictive conditions, such as a weak star-shapedness assumption on $\mathcal{D}$ and a smooth boundary, which notably does not contain [3]. A smallness condition on the terminal condition is also required, though it is not necessary in a Markovian framework. A particularly insightful remark in [2] connects the zero-generator case with the theory of Γ-martingales on manifolds. Specifically, when the terminal condition lies in a sufficiently "concave" part of $\partial\mathcal{D}$, the solution remains on the boundary and becomes a Γ-martingale on the manifold $\partial\mathcal{D}$. On the other hand, if the terminal condition lies within a convex subset of $\mathcal{D}$, the solution is a classical martingale in $\mathbb{R}^d$. These observations naturally lead to two questions: can we define a notion of Γ-martingale on a manifold with boundary (here, $\bar{\mathcal{D}}$ is viewed as a Euclidean manifold with boundary) and is the solution of a reflected BSDE with zero generator a Γ-martingale? The first goal of [1] is to provide positive answer to both questions. In particular, we are able to characterize reflected zero-generator BSDEs as processes Y such that for any Γ-convex function Ψ, i.e. a function $\mathbb{R}^d \to \mathbb{R}$ that is convex along geodesics of $\bar{\mathcal{D}}$ (for the intrinsic distance), $\Psi(Y)$ is (locally) a sub-martingale.

The second main contribution of this paper is to improve existence and uniqueness results of [2] in dimension $d = 2$. More precisely, we are able to drastically reduce assumptions asked on $\bar{\mathcal{D}}$ in order to obtain an existence and uniqueness result: we just need to assume that $\bar{\mathcal{D}}$ is simply connected and is locally C^2-diffeomorphic to a convex set. In particular, this theorem generalizes results from [2] and strictly contains [3]. Moreover, the simply connected assumption is very natural: if $\mathcal{D}$ is not simply connected, we can construct easily two different geodesics with same ending points and then two Γ-martingales with same terminal condition which invalidates the uniqueness results for reflected BSDEs with zero-generator.

The approach used in [1] for the existence result differs substantially from that of [2] which consists, as is usual for this type of problem, in considering a penalized version of the non reflected BSDE and then making the penalty tend to infinity. In this article, we rather adapt the approach used for martingales in manifolds by Kendall [4]: we look at our domain as a metric space (for the intrinsic distance) which has the property to be a CAT(0) space (a.k.a. Hadamard space), we consider a classical notion of mean in metric spaces, called Fréchet mean, and we take some backward infinitesimal iterations of Fréchet means. Nevertheless, in order to tackle reflected BSDEs with non null exogenous generator (i.e. a generator that does not depend on the solution), we have to add some extra steps: between two Fréchet means steps, we add a transport step in the generator direction. Finally, the general case is treated by using some Picard iteration schemes and BMO tools. Let us emphasize that in all these steps, but also for uniqueness results, a key tool is the replacement of the usual use of the squared Euclidean distance between two processes by the squared intrinsic distance of the domain. So, we had to prove an Itô formula for this intrinsic distance.

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Functional calculus for the Laplacian on weighted spaces and applications to (S)PDEs

Floris Roodenburg

(joint work with Nick Lindemulder, Emiel Lorist, Mark Veraar)

The basis for analysing a nonlinear parabolic (S)PDE is understanding the corresponding linearised equation which takes the form

$$\begin{cases} \mathrm{d}u + Au\mathrm{d}t = f\mathrm{d}t + g\mathrm{d}W_t & \text{for } t \in \mathbb{R}_+, \\ u|_{t=0} = 0, \end{cases} \tag{1}$$

where A is a differential operator on a Banach space X and W is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which models noise in space and time. The inhomogeneities $f : \mathbb{R}_+ \to X$ and $g : \mathbb{R}_+ \times \Omega \to X$ arise from both forcing terms in the physical model and the linearisation procedure. If one has sharp regularity estimates for (1), then combining this with a Banach fixed point argument yields existence, uniqueness and regularity results for solutions to the nonlinear (S)PDE, see for instance [1].

A fruitful approach to study (1) is via abstract functional and harmonic analytical tools for the differential operator A such as maximal regularity and holomorphic H^∞-functional calculus. In particular, the boundedness of the H^∞-functional calculus for an operator A on a Banach space X guarantees boundedness of certain singular integral operators. This can immediately be used as a black box to get well-posedness for the stochastic equation in (1), see [8]. The standard example is the Dirichlet Laplacian $-\Delta$ on $X = L^p(\mathcal{O})$ and domain $D(\Delta) = W^{2,p}(\mathcal{O}) \cap W^{1,p}_0(\mathcal{O})$ with $p \in (1, \infty)$ and $\mathcal{O}$ a bounded C^2-domain. This operator admits a bounded H^∞-functional calculus and therefore we obtain well-posedness of (1), which, in this case, is the (stochastic) heat equation.

If the regularity of the domain $\mathcal{O}$ is too low, then derivatives of the solution to (1) may drastically blow up near the boundary $\partial\mathcal{O}$. As a consequence, the canonical domain of the Dirichlet Laplacian on $L^p(\mathcal{O})$ is no longer a closed subspace of $W^{2,p}(\mathcal{O})$. Moreover, if one is interested in higher-order Sobolev regularity of the solution u, then more smoothness of the domain and compatibility conditions for the data are needed. To set up a satisfying well-posedness and regularity theory for (S)PDE *without* such additional regularity and compatibility assumptions, one can use weighted function spaces for the solution u. In particular, we consider spatial weights of the form $w_\kappa^{\partial\mathcal{O}}(x) := \operatorname{dist}(x, \partial\mathcal{O})^\kappa$ for some $\kappa \in \mathbb{R}$, which may compensate the blow-up of the derivatives of the solution near $\partial\mathcal{O}$ and relax compatibility conditions. The literature for theory for (S)PDEs on weighted spaces is well-established, see for instance [2, 3].

An alternative way to study (S)PDEs on weighted spaces is via functional calculus as explained above. The application of powerful tools from different fields such as function space theory, functional and harmonic analysis leads to several improvements of certain results in the aforementioned literature. In [4, 5, 6] we have obtained boundedness of the H^∞-functional calculus for the Laplace operator with Dirichlet boundary conditions on weighted Sobolev spaces of the form

$$W^{k,p}(\mathcal{O}, w_\kappa^{\partial\mathcal{O}}) := \Big\{ f : \textstyle\sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{L^p(\mathcal{O}, w_\kappa^{\partial\mathcal{O}}(x)\mathrm{d}x)} < \infty \Big\}, \quad p \in (1,\infty),\ k \in \mathbb{N}.$$

One of the main results in [6] reads as follows.

Theorem. Let $p \in (1,\infty)$, $k \in \mathbb{N}$, $\lambda \in [0,1]$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. Furthermore, suppose that

$$\lambda > 1 - \tfrac{\gamma+1}{p} \qquad \text{or, equivalently} \qquad \gamma > (1-\lambda)p - 1 \tag{2}$$

and $\mathcal{O}$ is a bounded $C^{1,\lambda}$-domain. Then the operator $-\Delta_{\mathrm{Dir}}$ on $W^{k,p}(\mathcal{O}, w_{\gamma+kp}^{\partial\mathcal{O}})$ with

$$D(-\Delta_{\mathrm{Dir}}) = \{ f \in W^{k+2,p}(\mathcal{O}, w_{\gamma+kp}^{\partial\mathcal{O}}) : \mathrm{Tr} f = 0 \}$$

has a bounded H^∞-calculus of angle zero.

The main feature is that there is a trade-off between the regularity of the domain and the weight exponent. Moreover, if the weight exponent is large enough (i.e., $\gamma > p-1$), then we can allow for C^1-domains *independently* of the smoothness of the Sobolev space. Furthermore, w_γ belongs to the class of Muckenhoupt weights (the class of weights for which, e.g., Mihlin's multiplier theorem in harmonic analysis holds) if and only if $\gamma \in (-1, p-1)$. However, due to the additional decay of the kernel of the Dirichlet semigroup close to the boundary, the range of weight exponents can be extended to $\gamma \in (p-1, 2p-1)$, see [4]. Finally, we note that the condition (2) is almost optimal, see [7, Section 15.6].

As a direct consequence, we obtain (stochastic) maximal L^q-regularity with time integrability $q \in (1,\infty)$ for the (stochastic) heat equation together with quantitative estimates of the blow-up of the solution near the boundary of the domain.

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Schauder Estimates and a Diagram-Free Approach to Regularity Structures

Jonas Sauer
(joint work with Felix Otto, Scott A. Smith, Hendrik Weber)

In this talk, I will present a diagram-free approach to the theory of regularity structures. In particular, our method can be applied to show a priori estimates in Hölder spaces for renormalized, classically ill-defined quasilinear SPDEs in the subcritical regime. We first discuss a novel and efficient method to obtain (linear) Schauder estimates for germs which correspond to solutions of elliptic equations in anisotropic settings. The notion of a germ in regularity structures is a generalization of the standard Taylor polynomials. This method does not use kernel estimates, but is based on a scaling argument originally introduced by Simon in the classical case. I will then show how these linear estimates can be applied to derive estimates for the nonlinear problem via our diagram-free approach.

Recent work on pathwise approaches to singular parabolic SPDEs have two central analytic ingredients, known as 'reconstruction' and 'integration' in the terminology of rough paths, regularity structures, and paracontrolled calculus ([5], [6], [7]). These tools can be viewed as methods for estimating the nonlinearity in the SPDE and the solution itself, respectively. For example, in pathwise approaches to the multiplicative stochastic heat equation

$$(\partial_t - \Delta)u = \sigma(u)\xi, \tag{1}$$

cf. [8], [11], [3], the reconstruction theorem is used to estimate the product $\sigma(u)\xi$, while the integration theorem exploits the regularizing properties of $(\partial_t - \Delta)^{-1}$ to estimate u. In fact, typically the strategy is not to estimate either of these quantities directly, but rather the remainder that arises from re-centering them around one or more explicit functions (or distributions) which are amenable to direct calculations, such as multi-linear functionals of the noise ξ. For an equation with additive noise like

$$(\partial_t - \Delta)u = -u^3 + \xi, \tag{2}$$

the Da-Prato/Debussche [4] trick amounts to considering a remainder of the form $u - v$, where $(\partial_t - \Delta)v = \xi$. Even with ξ being a space-time white noise in dimension 1, this yields a nice insight that despite both u and v individually belonging to $C^{1/2-}$, the difference $u - v$ belongs to $C^{5/2-}$, both Hölder spaces being understood with respect to the parabolic metric. It is not immediately apparent what the analogous statement would be for (1), as a simple subtraction of v amounts only to a change of the non-linearity to $(\sigma(u) - 1)\xi$, which does not generally provide a cancellation. The theory of regularity structures resolved this question as well as a closely related one, the generalization of the classical Wong-Zakai theorem from ODE's to (1), cf. [8].

Writing down a single explicit equation for the remainder is often not possible or natural for many SPDEs of interest. The basic idea in the rough path/regularity structures approach to singular SPDEs is to instead consider a remainder depending on a base-point, so there is actually a 'family of remainders' each satisfying a slightly different equation. For example, freezing in the diffusion coefficient $\sigma(u)$ near a space-time point $z = (x, s)$ leads to

$$(\partial_t - \Delta)\big(u - \sigma(u)(z)v - P_z\big) = \big(\sigma(u) - \sigma(u)(z)\big)\xi,$$

where P_z is a polynomial in the kernel of $(\partial_t - \Delta)$. One can then try to analyze the family of functions $(U_z)_z$, where $U_z := u - \sigma(u)(z)v - P_z$, which we refer to in this talk as a germ, borrowing the terminology of [1, 2]. A typical goal is to show that despite u and v being individually only α-Hölder continuous, for a suitable choice of P_z, it holds that $U_z(w)$ vanishes to order $\eta > \alpha$ as the parabolic distance between z and w approaches zero. This is accomplished with the Schauder estimate for germs.

The 'integration' theorem of [9], inspired by the integration theorem for modelled distributions [7] and proved based on the Safonov approach as in [11], is a generalized Schauder estimate which applies naturally to such germs. In the talk, I will show that various Schauder type estimates similar to the ones obtained in [9] and [10] can alternatively be proved by the indirect blow-up arguments of [13].

The talk is based on joint work with Scott A. Smith [12], and on joint work with Felix Otto, Scott A. Smith, and Hendrik Weber [10].

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Well-posedness of the stochastic thin-film equation with an interface potential

Max Sauerbrey
(joint work with Antonio Agresti)

The stochastic thin-film equations reads

$$\partial_t u \,=\, -\nabla\cdot\big(u^n\nabla(\Delta u-\phi'(u))\big)\,+\,\nabla\cdot(u^{n/2}\xi_\epsilon), \tag{1}$$

Here we take $\xi_\epsilon = \varphi_\epsilon * \xi$ for a spatial mollifier φ_ϵ as an approximation to the physically prescribed space-time white noise ξ in order to avoid issues related to renormalization. It serves as a model for the evolution of a free liquid-gas interface given by the graph of a function $u \geq 0$. The fluid layer is assumed to be thin in the sense that the vertical scales for distance and velocity are much smaller than the horizontal ones. Moreover, we assume it to be merely several nanometers high. Under these assumptions, one can reduce the full free boundary problem to the sole equation (1) for the film-height u, whose evolution is driven by surface tension, the molecular interaction forces between the liquid and the substrate as well as thermal fluctuations. The free energy of a configuration u is accordingly given by

$$\mathcal{E}(u)\,=\,\int\frac{1}{2}|\nabla u|^2\,+\,\phi(u)\,\mathrm{d}x,$$

where ϕ is the effective interface potential. A typical example of the latter is

$$\phi(u)\,=\,u^{-8}\,-\,u^{-2}\,+\,1, \tag{2}$$

which can be obtained from a 6–12–Lennard Jones interaction for each pair of molecules. The emerging parameter n reflects furthermore the boundary condition on the fluid velocity imposed near the fixed liquid–solid interface, for which physically relevant choices include $n \in [1, 3]$.

Concerning the mathematical solution theory for stochastic thin-film equations initiated in [5], previous research was mostly concerned with the construction of weak martingale solutions. Many follow-up works like [4] treat (1) with $\phi = 0$ and rely on tailored approximation schemes paired with a stochastic compactness argument, and therefore treat specific sub-ranges of $n \in [2, 4)$. The idea of our work [1] concerning (1) with $\phi \neq 0$ is to exploit positivity properties of u implied by the finiteness of $\mathcal{E}(u)$ for sufficiently repulsive interaction potentials like (2), to obtain the existence and uniqueness of probabilistically and analytically strong solutions. Indeed, in one (effective) spatial dimension $d = 1$ one can obtain the following Gagliardo–Nirenberg type estimates:

Lemma 1. *In $d = 1$, for ϕ as in (2), and $u \in L^1(\mathbb{T})$, we have the estimates*

$$\begin{aligned} \sup_{x\in\mathbb{T}} u^3(x) &\lesssim \Big(\int_{\mathbb{T}} u \,\mathrm{d}x\Big)\mathcal{E}(u) + \Big(\int_{\mathbb{T}} u \,\mathrm{d}x\Big)^3, \\ \Big(\inf_{x\in\mathbb{T}} u(x)\Big)^{-3} &\lesssim \mathcal{E}(u) + \Big(\int_{\mathbb{T}} u \,\mathrm{d}x\Big)^{-3}. \end{aligned} \tag{3}$$

We notice that by the divergence form of (1), the mass of a solution u is constant in time, thereby controlling the additional terms in (3). Moreover, by the Sobolev embedding $H^1(\mathbb{T}) \hookrightarrow C^{1/2}(\mathbb{T})$ a control on the energy of a solution would result in the bounds

$$\inf_{(t,x)\in[0,T]\times\mathbb{T}} u(t,x) > 0, \qquad \sup_{t\in[0,T]} \|u\|_{C^{1/2}(\mathbb{T})} < \infty. \tag{4}$$

In any spatial dimension, each process u satisfying the above $\mathbb{P}$-a.s. as well as the Itô interpretation of (1), solves at the same time a linear SPDE of the form

$$\mathrm{d}u + \nabla\cdot(v\nabla\Delta v)\,\mathrm{d}t = f\,\mathrm{d}t + \textstyle\sum_k g_k\,\mathrm{d}\beta^k. \tag{5}$$

Here, $(\beta^k)_{k\in\mathbb{N}}$ are i.i.d. Brownian motion, and we have set

$$v = u^n, \qquad f = \nabla\cdot(u^n\nabla(\phi'(u))), \qquad g_k = \nabla\cdot(u^{n/2}\varphi_\epsilon * e_k), \tag{6}$$

for an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{T}^d)$. Our first main result is the existence of unique, local, positive solutions to (1), which we obtain as follows:

- We prove $L^p(\Omega; L^p([0,T], t^\kappa; H^{s,q+2}(\mathbb{T}^d)) \cap C([0,T]; B^{s+2-4(1+\kappa)/p}_{q,p}(\mathbb{T}^d)))$-bounds on the solutions to the linear equations (5) for a spatially regular and strictly positive coefficient function v.
- Show compatible local Lipschitz estimates on the nonlinearities (6).

Thereby, following [2, 3], we deduce that positive solutions to (1) exist uniquely and regularize instantaneously to $C^{1/2-,\infty}_{\mathrm{loc}}((0,\tau)\times\mathbb{T}^d)$, where τ is the maximal time of existence. These solutions satisfy additionally a suitable blow-up alternative.

Our second main result is that $\tau = \infty$ in dimension $d = 1$, e.g., if $n \in [0, 6)$ and ϕ is as in (2), which we prove in the following steps:

- We show a priori estimates on the dissipation of the mathematical α-entropy $\frac{1}{\alpha(\alpha+1)}\int_{\mathbb{T}} u_t^{\alpha+1}\mathrm{d}x$ of the local positive solution.

- This allows us to estimate the energy production by the noise and show a bound on the energy $\mathcal{E}(u_t)$, uniformly in time.

Then, the result that (4) holds $\mathbb{P}$-a.s. suffices to verify the condition for non-blow-up of the solution. Next to its theoretical significance, the latter has also computational implications: By means of the Gyöngy–Krylov lemma one can deduce that the numerical scheme proposed in [5] converges in probability to the unique strong solution that we provide.

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Permeable Sets: From SDEs to Piecewise Lipschitz Functions

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(joint work with Zoltán Buczolich, Gunther Leobacher, Tapio Rajala, Jörg Thuswaldner)

Real world applications, especially energy market modelling or insurance dividend optimization, lead to stochastic differential equations (SDEs) with discontinuous drift terms and degenerate diffusion. The drift terms are usually multidimensional, with multidimensional discontinuities. With their 2017 article, G. Leobacher and M. Szölgyenyi [5] started off a research direction in SDEs focusing on numerical schemes and rates for such equations. They propose a transformation method requiring the points of discontinuity of the drift coefficient to be C^4-smooth. On each separate region created by such a discontinuity manifold, the SDE's drift coefficient has to be Lipschitz. An attempt to generalize the assumption of a smooth discontinuity manifold to 'discontinuous' or 'irregular discontinuities' led directly to the question what 'piecewise Lipschitz functions' in several dimensions actually should be.

A serviceable answer to this question is given by to the notion of *permeable subsets* of the $\mathbb{R}^d$. In this talk we explain the notion, present a number of (random and deterministic) examples, non-examples, properties and conditions, and we illustrate that the concept leads to a rich theory:
A key result we present is, that functions $f\colon \mathbb{R}^d \to \mathbb{R}$ that are continuous on $\mathbb{R}^d$ and Lipschitz w.r.t. the intrinsic metric on $\mathbb{R}^d \setminus \Theta$ are globally Lipschitz when Θ is

a permeable subset of $\mathbb{R}^d$. This ensures, that Lipschitz continuity with respect to the intrinsic metric outside a permeable exception set is a good way of generalizing 'piecewise Lipschitzness' into several dimensions [3]. Further, the connections between permeability and dimension- and measure theory are pointed out, see [4], and criteria for permeability of self-similar sets are shown, see also [4]. We then present conditions for real functions $g\colon [0,1] \to \mathbb{R}$ such that their graph is permeable in $\mathbb{R}^2$ and give an example of a function g that has an impermeable graph, available in [1]. Finally, we show upcoming results, e.g. the permeability of a fractional Brownian motion graph for Hurst index $H \geq \frac{1}{2}$ (in [2]) and certain grpahs of nowhere differentiable deterministic functions.

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