Mathematisches Forschungsinstitut Oberwolfach

Report No. 54/2025

DOI: 10.4171/OWR/2025/54

Functional Inequalities: Geometric Calculus meets Stochastic Analysis

Organized by
Masha Gordina, Storrs
Jessica Lin, Montreal
Emanuel Milman, Haifa
Karl-Theodor Sturm, Bonn

30 November – 5 December 2025

Abstract. Functional inequalities form a unifying theme across a wide spectrum of modern analysis, geometry, and probability. They encode deep geometric and analytic information – for instance through Poincaré, log-Sobolev, transportation, isoperimetric and curvature-dimension inequalities – and serve as crucial tools in the study of Markov semigroups, diffusion processes, metric measure spaces, and geometric flows. The workshop brought together researchers working in geometric analysis, stochastic analysis, and optimal transport in order to promote exchange of ideas and further strengthen the interaction between these rapidly developing fields. Substantial emphasis was placed on non-smooth or singular geometric structures, stochastic dynamics with degeneracies, and new bridges between discrete, fractal, and continuum settings.

Mathematics Subject Classification (2020): 26D10, 28A33, 35K08, 35R60, 39B72, 46B07, 49Q22, 52C07, 53C21, 58J65, 60D05, 60E15, 60H10, 60H30, 60G15, 60J46, 60J60.

Introduction by the Organizers

The 2025 workshop *Functional Inequalities: Geometric Calculus meets Stochastic Analysis* presented a broad picture of interconnected topics in analysis, geometry and probability. The uniting subject for many talks was functional inequalities in geometric settings (Riemannian and sub-Riemannian, metric measure spaces etc) and for various applications (Langevin dynamics, Dirichlet spaces and mean-field particle systems). The workshop provides a unique opportunity to encourage and foster interactions between mathematicians who share common research interests but often use different techniques or settings.

Organized by Masha Gordina (Storrs), Jessica Lin (Montreal), Emanuel Milman (Haifa), and Karl-Theodor Sturm (Bonn), the workshop was attended by 47 in person participants from Canada, China, Denmark, France, Germany, Hong Kong, Israel, Italy, Japan, Luxembourg, the United Kingdom, and the USA. The scientific program consisted of 27 invited talks and 6 short contributions, leaving time for informal discussions and collaborations. The topics included analysis on metric measure spaces, optimal transport, ergodicity of Langevin dynamics, random geometries etc. While the workshop gave an opportunity to present a number of remarkable recent breakthroughs, it also identified new directions for collaborations, bridging between discrete, fractal, and geodesic spaces, finding novel applications of optimal-transport based techniques and developing new analytic tools that link stochastic dynamics, geometric flows, and the structure of singular spaces.

One of the major themes of the workshop addressed *functional inequalities in high-dimensional convex geometry and related transport problems.* Joseph Lehec presented the recent resolution of the thin-shell concentration conjecture for isotropic log-concave measures. Pierre Bizeul described refined tools from convex geometry and stochastic localization leading to an essentially optimal bound on the diameter of the (non-symmetric) Banach-Mazur compactum. Boaz Klartag presented a breakthrough density estimate on high-dimensional lattice sphere packings using a stochastically evolving ellipsoid.

A complementary theme concerned *probabilistic and analytical perspectives on optimal transport.* Martin Huesmann analyzed random optimal transport cost fluctuations and related quadratic asymptotics, while Giuseppe Savaré presented recent progress on the geometry of spaces of random measures and corresponding optimal transport structures. Nathael Gozlan presented sharp regularity estimates for the optimal transport map between Euclidean densities and their functional-inequality consequences. Finally, Max Fathi spoke on cutoff phenomena for kinetic Langevin dynamics, illustrating how transport inequalities can yield sharp quantitative convergence estimates.

Another focal area involved *interacting particle systems, kinetic equations, and generalized gradient-flow structures.* Benoit Dagallier discussed large-scale convexity properties in mean-field particle systems and their implications for stability and hydrodynamic limits. Jan Maas presented transport metrics adapted to kinetic equations and the resulting hypocoercivity structure. Matthias Erbar described connections between interacting point processes, transport geometry, and variational calculus on spaces of measures. Continuing this theme, Max von Renesse reported on regularization by noise in McKean-Vlasov dynamics, identifying improved analytic behavior due to stochastic perturbations.

A substantial part of the workshop focused on *analysis on non-smooth spaces and Dirichlet forms.* Nageswari Shanmugalingam surveyed recent advances on Newtonian Sobolev spaces and nonlinear potential theory in general metric measure spaces. Jun Kigami described new classes of compact spaces supporting fractional energy forms and fractal-type differential calculus. Patricia Alonso Ruiz presented isoperimetric inequalities on fractal structures, revealing scale-dependent geometry through functional inequalities. Mathav Murugan's talk was on the resolution of the Bouleau-Hirsch energy image density conjecture for diffusions generated by Dirichlet forms, with applications to Malliavin-type regularity. Several talks addressed more geometric aspects of non-smooth spaces. Shin-ichi Ohta discussed discrete-time gradient flows and curvature bounds in metric spaces of negative curvature, highlighting new nonlinear contraction and comparison principles. Rotem Assouline presented recent work on curvature-dimension conditions for Lagrangian systems, developing generalized synthetic Ricci curvature bounds. In addition, two talks dealt with random geometry. Wei Qian discussed the fractal geometry and scaling structure of Brownian loop-soup clusters. Linan Chen discussed geometric features of high-dimensional Gaussian free fields.

Stochastic dynamics, ergodicity, and long-time behavior under degeneracy or randomness formed another important thread. Tai Melcher introduced modified Γ-calculus techniques for degenerate diffusions and their use in establishing ergodicity. Jian Wang presented exponential ergodicity results for stochastic Hamiltonian systems driven by Lévy noise. David Herzog developed quantitative positivity criteria for transition densities of randomly perturbed Hamiltonian flows. Long-range and random-media effects were addressed in talks by Takashi Kumagai and Laurent Saloff-Coste, who presented precise heat-kernel bounds and homogenization results for random walks with non-local or polynomial growth structures.

The geometric and analytic foundations underlying many of these results were further illustrated in talks on *sub-Riemannian and Riemannian analysis, spectral theory, and boundary phenomena.* Fabrice Baudoin presented comparison theorems and curvature–dimension inequalities for sub-Laplacians on Riemannian foliations with minimal leaves. Gilles Carron discussed branching phenomena in strong Kato limits and their spectral implications. Finally, Marie Bormann reported on gradient-flow structures and functional inequalities for heat flow with Wentzell boundary conditions.

The workshop was characterized by lively and productive discussions across traditionally separate fields, including geometric analysis, probability, PDE theory, and metric geometry. Early career researchers contributed actively through short talks and informal exchanges. Overall, the meeting demonstrated the vitality and coherence of the area, and it is expected that the new connections forged during the week at Oberwolfach will stimulate further progress in the coming years.

Acknowledgement: The MFO and the organizers gratefully acknowledge support from the National Science Foundation via grant DMS–2230648, "US Junior Oberwolfach Fellows," and from the Simons Foundation through the "Simons Visiting Professors" program.

Workshop: Functional Inequalities: Geometric Calculus meets Stochastic Analysis

Table of Contents

Abstracts

Asymptotics for Random Quadratic Transportation Cost

Martin Huesmann
(joint work with Michael Goldman, Dario Trevisan)

Denote by W_p the p-th Wasserstein distance on $\mathbb{R}^d$. For a given probability measure λ let $(X_i)_{i\geq n}$ be i.i.d. samples with distribution λ and empirical measure $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$. It is well known that $\mu_n \to \lambda$ and under mild conditions on λ also the rate of this convergence in W_p is well known. In fact combining [1],[6], and [5] the following can be shown:

Theorem 1. *For any $d \geq 3$ and $p \geq 1$, there exist constants $0 < \underline{\mathsf{c}}(p,d) \leq \overline{\mathsf{c}}(p,d) < \infty$ such that the following holds for every probability measure λ on $\mathbb{R}^d$. Assume that either*

(i) *$p < d/2$ and λ has finite q-th moment, for some $q > dp/(d-p)$;*
(ii) *or λ is absolutely continuous with respect to the Lebesgue measure on a bounded connected C^2-smooth or convex domain, with Hölder continuous density uniformly bounded from above and below by a strictly positive constant.*

Then denoting by λ_a the absolutely continuous part of λ with respect to the Lebesgue measure,

(1)
$$\underline{\mathsf{c}}(p,d)\int_{\mathbb{R}^d}\lambda_a^{1-p/d} \leq \liminf_{n\to\infty}\mathbb{E}\left[\mathsf{W}_p^p\left(\frac{1}{n}\sum_{i=1}^n\delta_{X_i},\lambda\right)\right]n^{p/d}$$
$$\leq \limsup_{n\to\infty}\mathbb{E}\left[\mathsf{W}_p^p\left(\frac{1}{n}\sum_{i=1}^n\delta_{X_i},\lambda\right)\right]n^{p/d} \leq \overline{\mathsf{c}}(p,d)\int_{\mathbb{R}^d}\lambda_a^{1-p/d},$$

where $(X_i)_{i=1}^n$ are i.i.d. random variables with common distribution λ. Moreover, the constants $0 < \underline{\mathsf{c}}(p,d) \leq \overline{\mathsf{c}}(p,d) < \infty$ are given by

$$\overline{\mathsf{c}}(p,d) = \lim_{L\to\infty}\frac{1}{L^d}\mathbb{E}\left[\mathsf{W}^p_{(0,L)^d}\left(\mu, \frac{\mu((0,L)^d)}{L^d}\right)\right]$$
$$\text{and}\quad \underline{\mathsf{c}}(p,d) = \lim_{L\to\infty}\frac{1}{L^d}\mathbb{E}\left[\mathsf{Wb}^p_{(0,L)^d}\left(\mu_{|(0,L)^d}, \frac{\mu((0,L)^d)}{L^d}\right)\right],$$

where μ is a Poisson point process with unit intensity on $\mathbb{R}^d$, $\mathsf{W}^p_{(0,L)^d}$ denotes the Wasserstein cost of order p between the measures restricted on the cube $(0,L)^d$, while $\mathsf{Wb}^p_{(0,L)^d}$ denotes instead its "boundary" variant defined for two probability *measures* α, β *on* $\Omega = (0,L)^d$ *by*

$$\mathsf{Wb}^p_{(0,L)^d}(\alpha,\beta) = \inf_{q\in \mathsf{Cpl}(\alpha,\beta)} \int \mathsf{b}(x,y) q(dx,dy),$$

$$\mathsf{b}(x,y) = \min\left\{ d(x,y), \left(d(x,\Omega^c)^p + d(y,\Omega^c)^p\right)^{1/p} \right\}$$

The main result obtained in [2] and presented in this talk is as follows:

Theorem 2. *For every* $d \geq 3$ *it holds*

$$\underline{\mathsf{c}}(2,d) = \overline{\mathsf{c}}(2,d).$$

Previously only results in dimension $d = 1,2$ were known. In fact, in [3] it was shown that $\underline{\mathsf{c}}(2,2) = \overline{\mathsf{c}}(2,2) = 1/4\pi$ and in [4] it was shown that for $0 < p < 1/2$ it holds that $\underline{\mathsf{c}}(p,1) = \overline{\mathsf{c}}(p,1)$. We also remark that if one is only interested in the equality of the lim inf and lim sup in (1) one can still conclude when λ_a is a constant, see e.g. [5], [1].

The proof combines functional analytic tools with novel techniques from the stability theory of optimal transportation and ideas from stochastic homogenization. In the following I will sketch the argument of the proof in the simplified linearized PDE setup that inspired much of the recent progress on the classical matching problem.

The starting point is the heuristic approximation of the Wasserstein distance by the Dirichlet energy of solutions to the continuity equation. Put $\Omega = (0,L)^d$ and for two probability measures μ and λ let f (resp. u) be the solutions to the Poisson problem

$$\Delta g = \mu - \lambda$$

with Neumann (resp. Dirichlet) boundary conditions on Ω. Then, heuristically (e.g. see [3]) we have $\mathsf{W}^2_\Omega(\mu,\lambda) \approx \int |\nabla f|^2$ and $\mathsf{Wb}^2_\Omega(\mu,\lambda) \approx \int |\nabla u|^2$. We always have

$$\int |\nabla u|^2 \leq \int |\nabla f|^2$$

and modifying u within a boundary layer of size $r = \delta L$ to become a competitor for the Neumann problem we can show that for any $\varepsilon \in (0,1)$,

(2)
$$\int_\Omega |\nabla f|^2 - \int_\Omega |\nabla u|^2 \lesssim \varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\left[\|\eta(\mu-\lambda)\|^2_{H^{-1,2}(\Omega)} + \left(r^{-2} + \operatorname{diam}(\Omega)^2 r^{-4}\right) \int_\Omega |u|^2 \right].$$

where $0 \leq \eta \leq 1$ is a cutoff function equal to 1 on $\{x : d(x,\Omega^c) \leq r\}$ and equal to 0 on $\{x : d(x,\Omega^c) \geq 2r\}$. We expect:

$$\frac{1}{L^d}\mathbb{E}\left[\int_{(0,L)^d} |\nabla u|^2\right] \approx \frac{1}{L^d} Wb_2^2(\mu,\lambda) \approx 1$$

$$\frac{1}{L^d}\|\eta(\mu-\lambda)\|^2_{H^{-1,2}((0,L)^d)} \approx \frac{1}{L^d} W_2^2(\eta\mu,\eta\lambda) \approx \delta$$

However, with $r = \delta L$ we have

$$\frac{1}{L^d}(r^{-2} + L^2 r^{-4}) \int |u|^2 \approx (\delta^{-2} + \delta^{-4}) \frac{1}{L^{d+2}} \int |u|^2$$

By the Poincaré inequality we would obtain

$$\frac{1}{L^{d+2}} \int |u|^2 \lesssim \frac{L^2}{L^{d+2}} \int |\nabla u|^2 \approx 1$$

such that the right hand side of (2) would blow up for $\delta \to 0$. Hence, we would need

$$\frac{1}{L^{d+2}} \int |u|^2 \overset{!}{\ll} 1 \tag{3}$$

to conclude by letting first $L \to \infty$ and then $\delta \to 0$. In fact, we are able to show (3) by introducing an intermediate length scale $L_0 = L^\gamma$. Subdividing $\Omega = (0, L)^d$ into $(L/L_0)^d$ many subcubes of sidelength L_0 we can define another competitor for the Dirichlet energy by solving the Poisson equation with Dirichlet boundary conditions on each subcube and gluing them together. By stationarity properties of the Poisson point process the Dirichlet energy of the competitor coincides in expectation with the Dirichlet energy on a single cube $(0, L_0)^d$. This Dirichlet energy can be shown to converge for L large to the same limit as the original energy with an error that can be made arbitrarily small in L. Using subadditivity this allows us to show (3).

The main part of our work is to translate this ansatz to the non-linear world of transportation distances. Notably, all terms in (2) have natural correspondences on the transport side. For instance, the control of $\int |u|^2$ corresponds to the control of the variance of the Kantorovich potential which we obtain by new quantitative stability estimates of optimal transport.

References

[1] F. Barthe and C. Bordenave, *Combinatorial optimization over two random point sets*, Séminaire de probabilités XLV, 483–535, 2013

[2] M. Goldman, M. Huesmann and D. Trevisan, *Asymptotics for Random Quadratic Transportation Costs*, Preprint, arXiv:2409.08612 [math.PR] (2024)

[3] L. Ambrosio, M. Goldman and D. Trevisan, *On the quadratic random matching problem in two-dimensional domains*, Electron. J. Probab., **27**, 2022

[4] M. Goldman and D. Trevisan, *On the concave one-dimensional random assignment problem and Young integration theory*, Preprint, arXiv:2305.09234 [math.PR] (2023)

[5] M. Goldman and D. Trevisan, *Optimal transport methods for combinatorial optimization over two random point sets*, Probab. Theory Relat. Fields, **188**, No. 3–4, 1315–1384 (2024; Zbl 1542.90200)

[6] S. Dereich, M. Scheutzow, and R. Schottstedt, *Constructive quantization: approximation by empirical measures*, Ann. Inst. Henri Poincaré, Probab. Stat., **49**, 4, 1183–1203, 2013

Global Regularity Estimates for Optimal Transport via Entropic Regularization

Nathael Gozlan
(joint work with Maxime Sylvestre)

This work develops a general methodology for establishing global regularity estimates for the Brenier optimal transport map between two probability measures on $\mathbb{R}^n$. The approach leverages the entropic regularization of the quadratic optimal transport problem and the Prékopa-Leindler inequality, offering a flexible alternative to classical methods based on the Monge-Ampère equation. The main results recover and extend several key theorems in the field, including Caffarelli's celebrated contraction theorem and Hölder regularity estimates by Kolesnikov, while also enabling the treatment of anisotropic settings and measures with singular supports.

According to Brenier's Theorem [2], if μ, ν are two probability measures on $\mathbb{R}^n$ with finite second moments, and if μ is absolutely continuous, then there exists a convex function φ such that the map $T = \nabla\varphi$ transports μ onto ν, in the sense that the image of μ under the map T is equal to ν. Moreover this map T achieves the minimal value in the quadratic transport problem. More precisely, if

$$W_2^2(\mu,\nu) = \inf_{X\sim\mu, Y\sim\nu} \mathbb{E}[|Y-X|^2]$$

denotes the classical squared Wasserstein distance between μ and ν on the Euclidean space $(\mathbb{R}^n, |\cdot|)$, then it holds

$$W_2^2(\mu,\nu) = \mathbb{E}[|T(X)-X|^2].$$

The regularity of the optimal transport map $T = \nabla\varphi$ is a fundamental question with deep implications in analysis, geometry, and functional inequalities. A landmark result is Caffarelli's contraction theorem :

Theorem 1 (Caffarelli [4]). *Let $\mu(dx) = e^{-V(x)}dx$, $\nu(dy) = e^{-W(y)}dy$ be two probability measures on $\mathbb{R}^n$ such that* $\mathrm{dom}V = \mathbb{R}^n$ *and* $\mathrm{dom}W$ *is convex with nonempty interior. Further assume that V, W are twice continuously differentiable on the interior of their domains and satisfy $\nabla^2 V \leq \alpha_V I_n$ and $\nabla^2 W \geq \beta_W I_n$ with $\alpha_V, \beta_W > 0$. Then the optimal transport map T for the quadratic transport problem from μ to ν is $\sqrt{\alpha_V/\beta_W}$-Lipschitz.*

This theorem provides a simple, dimension-free condition ensuring Lipschitz regularity and has been instrumental in transferring functional inequalities (e.g., Poincaré, log-Sobolev) from Gaussian to log-concave measures.

The original proof of this result relies on the sophisticated regularity theory for the Monge-Ampère equation [3]. More recently, works by Fathi, Gozlan and Prod'homme [7] and Chewi and Pooladian [5], introduced new proofs using entropic regularization, circumventing the Monge-Ampère equation and instead employing functional inequalities related to the Brunn-Minkowski theory.

Using this entropic regularization approach, the following generalization of Theorem 1 is obtained in [8]:

Theorem 2. *Let $\mu = e^{-V(x)}\,dx$ and $\nu(dy) = e^{-W(y)}\,dy$ be two probability measures such that V is S_V-smooth and W is R_W-convex. Then the Brenier potential φ for the transport of μ onto ν is $\bar{S}$-smooth, with $\bar{S}$ given by*

$$\bar{S}(d) = \int_0^1 \sup_{R_W^{**}(p) \le S_V(td)} \langle p, d\rangle\,dt, \qquad \forall d \in \mathbb{R}^n.$$

In particular,

$$\bar{S}^*(\nabla\varphi(y) - \nabla\varphi(x)) \le \bar{S}(y-x), \qquad \forall x, y \in \mathbb{R}^n.$$

The notions of S-smoothness and R-convexity have been introduced by Vladimirov, Nesterov and Chekanov [14] and Zalinescu [15] and further studied by Azé and Penot [1]. A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is S-smooth if for any $x_0, x_1 \in \mathbb{R}^n$ and $t \in [0,1]$,

$$\varphi((1-t)x_0 + tx_1) + t(1-t)S(x_1 - x_0) \ge (1-t)\varphi(x_0) + t\varphi(x_1).$$

A function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is R-convex if for any $x_0, x_1 \in \mathbb{R}^n$ and $t \in [0,1]$,

$$\psi((1-t)x_0 + tx_1) + t(1-t)R(x_1 - x_0) \le (1-t)\psi(x_0) + t\psi(x_1).$$

When $S = \alpha|\cdot|^2/2$ and φ is twice continuously differentiable, then φ is S-smooth if and only if $\nabla^2\varphi \le \alpha$. Similarly, a twice continuously differentiable function ψ is R-convex with $R = \beta|\cdot|^2/2$ if and only if $\nabla^2\psi \ge \beta$. Thus, Theorem 2 implies back Theorem 1. It also recovers Kolesnikov's Hölder estimate [9] when $S_V \le \alpha|\cdot|^p$ and $R_W \ge \beta|\cdot|^q$ with $p \le 2 \le q$, giving $|T(x) - T(y)| \le C|x-y|^{p/q}$, $x, y \in \mathbb{R}^n$, for some C depending on α, β, p and q.

A classical duality property states that the Fenchel-Legendre transform interchanges S-smooth and R-convex functions: if f is S-smooth, then f^* is S^*-convex; if g is R-convex, then g^* is R^*-smooth. The main tool used in the proof of Theorem 2 is a similar duality result for the so-called *entropic Legendre transform*:

$$\mathcal{L}_\varepsilon(\psi)(x) = \varepsilon \log\left(\int \exp\left(\frac{\langle x, y\rangle - \psi(y)}{\varepsilon}\right) dy\right).$$

This functional interpolates between the log-Laplace transform ($\varepsilon = 1$) and the Fenchel-Legendre transform ($\varepsilon \to 0$). As shown in [8], this transform puts is duality the classes of S-smooth and R-convex functions in the following sense:

- If ψ is R-convex, then $\mathcal{L}_\varepsilon(\psi)$ is R^*-smooth.
- If φ is S-smooth, then $\mathcal{L}_\varepsilon(\varphi)$ is S^*-convex.

The proof of the first implication relies crucially on the Prékopa-Leindler inequality [11, 12, 10]. This duality result is crucially used in the proof of Theorem 2 to control, independently of ε, the smoothness and convexity moduli of the so-called entropic potentials $(\varphi_\varepsilon, \psi_\varepsilon)$ arising in the resolution of the entropy regularized transport problem between μ and ν.

The framework described above is very flexible and allows several variants and various applications. For instance, the following anisotropic version of Theorem 1 is obtained in [8] via the entropic method: If $\nabla^2 V \leq A^{-1}$ and $\nabla^2 W \geq B^{-1}$ for symmetric positive definite matrices A, B, then

$$\nabla^2 \varphi \leq B^{1/2}(B^{-1/2}A^{-1}B^{-1/2})^{1/2}B^{1/2}.$$

This generalizes results by Valdimarsson [13] and Chewi and Pooladian [5], removing the assumption that A and B commute.

Several other regularity results are obtained in [8] :

- Theorem 2 can be generalized to measures supported on affine subspaces. This provides anisotropic Lipschitz estimates depending on the alignment of the subspaces.
- A version of Caffarelli's theorem is obtained for perturbations of product measures with densities proportional to $e^{-\|\cdot\|_p^p/p}$ and $e^{-\|\cdot\|_{p^*}^{p^*}/p^*}$, yielding regularity estimates in p-norms.
- For two log-Lipschitz perturbations of the Gaussian measure, the Brenier map is shown to be an approximate isometry. This leads to sharp concentration inequalities for such measures.
- Using a modified Prékopa-Leindler inequality, a recent result of De Philippis and Shenfeld [6] on the Laplacian of the Kantorovich potential is recovered and improved.

References

[1] D. Azé and J.-P. Penot. Uniformly convex and uniformly smooth convex functions. *Ann. Fac. Sci. Toulouse, Math. (6)*, 4(4):705–730, 1995.

[2] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44(4):375–417, 1991.

[3] L. A. Caffarelli. The regularity of mappings with a convex potential. *J. Am. Math. Soc.*, 5(1):99–104, 1992.

[4] L. A. Caffarelli. Monotonicity properties of optimal transportation and the FKG and related inequalities. *Communications in Mathematical Physics*, 214(3):547–563, November 2000.

[5] S. Chewi and A.-A. Pooladian. An entropic generalization of caffarelli's contraction theorem via covariance inequalities. *Comptes Rendus. Mathématique*, 361(G9):1471–1482, November 2023.

[6] G. De Philippis and Y. Shenfeld. Optimal transport maps, majorization, and log-subharmonic measures. Preprint, arXiv:2411.12109, 2024.

[7] M. Fathi, N. Gozlan, and M. Prod'homme. A proof of the Caffarelli contraction theorem via entropic regularization. *Calc. Var. Partial Differ. Equ.*, 59(3):18, 2020. Id/No 96.

[8] N. Gozlan, M. Sylvestre. Global Regularity Estimates for Optimal Transport via Entropic Regularisation. Preprint, arXiv:2501.11382v5, 2025.

[9] A. V. Kolesnikov. Global Hölder estimates for optimal transportation. *Mat. Zametki*, 88(5):708–728, 2010.

[10] L. Leindler. On a certain converse of Hölder's inequality. II. *Acta Sci. Math. (Szeged)*, 33(3-4):217–223, 1972.

[11] A. Prékopa. Logarithmic concave measures with application to stochastic programming. *Acta Sci. Math. (Szeged)*, 32:301–316, 1971.

[12] A. Prékopa. On logarithmic concave measures and functions. *Acta Sci. Math. (Szeged)*, 34:335–343, 1973.

[13] S. I. Valdimarsson. On the Hessian of the optimal transport potential. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 6(3):441–456, 2007.

[14] A. A. Vladimirov, Yu. E. Nesterov, and Yu. N. Chekanov. On uniformly convex functionals. *Mosc. Univ. Comput. Math. Cybern.*, 1978(3):10–21, 1978.
[15] C. Zalinescu. On uniformly convex functions. *J. Math. Anal. Appl.*, 95:344–374, 1983.

Large scale convexity in mean field particle models

Benoit Dagallier
(joint work with Roland Bauerschmidt and Thierry Bodineau)

We consider N particles in $\mathbb{R}^d$ with mean field interaction as described by the following probability measure:

$$m_T^N(dx) = \frac{1}{Z_T^N} e^{-H_T^N(x)}\, dx,$$

where for $x \in (\mathbb{R}^d)^N$ the energy $H_T^N(x)$ is given by:

$$H_T^N(x) = \frac{1}{2NT} \sum_{i,j=1}^{N} W(x_i, x_j) + \sum_{i=1}^{N} V(x_i).$$

Above, $V : \mathbb{R}^d \to \mathbb{R}$ is a confinement potential, $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is an interaction term, with strength parametrised by the temperature $T > 0$. The constant Z_T^N is a normalisation making m_T^N a probability measure. The typical example we have in mind is the so-called Curie-Weiss model where $W(x_1, x_2) = -(x_1, x_2)$ $(x_1, x_2 \in \mathbb{R}^d)$, and $V(x_1) = \lambda |x_1|^4/4 - |x_1|^2/2$ $(x_1 \in \mathbb{R}^d, \lambda > 0)$.

The aim of the talk is to bound the speed of convergence of the following Langevin dynamics, known to converge to m_T^N in long time:

$$\mathrm{d}X_t^N = -\nabla H_T^N(X_t^N)\, \mathrm{d}t + \sqrt{2}\, \mathrm{d}B_t^N,$$

with $B_\cdot^N$ a standard Brownian motion in $(\mathbb{R}^d)^N$. This is done by bounding the log-Sobolev constant of the dynamics, that is the best constant $\gamma > 0$ such that, for all smooth compactly supported test functions $F : (\mathbb{R}^d)^N \to \mathbb{R}$:

$$\mathrm{Ent}_{m_T^N}(F^2) \le \frac{2}{\gamma} \mathbb{E}_{m_T^N}\big[|\nabla F|^2\big],$$

where $\mathrm{Ent}_{m_T^N}(F^2) = \mathbb{E}_{m_T^N}[F^2 \log(F^2)] - \mathbb{E}_{m_T^N}[F^2] \log \mathbb{E}_{m_T^N}[F^2]$.

If H_T^N is uniformly convex with constant $c > 0$, the celebrated Bakry-Émery argument says that the log-Sobolev constant is at least $c > 0$ (in particular independent of the number N of particles if c is). Without uniform convexity a log-Sobolev inequality may not hold, or at least not with a uniform in N constant. Mean-field models provide a nice class of examples which are typically not convex, yet when the parameter T is sufficiently large the log-Sobolev constant can often be shown to be independent of N [2]. On the contrary, for T small it vanishes with N.

In this presentation I explained how to find, in the case of the Curie-Weiss model, an appropriate functional and an appropriate notion of convexity, different from H_T^N and the Bakry-Émery one respectively, such that uniform convexity implies uniform in N bound on the log-Sobolev constant. The appropriate functional is obtained in terms of the so-called free energy $\mathfrak{F}_T$, the real-valued functional acting on probability measures $\rho = \rho(x)\,dx$ on $\mathbb{R}^d$ according to:

$$\mathfrak{F}_T(\rho) = \int \rho(x)\log\rho(x)\,dx + \int V(x)\,\rho(dx) - \frac{1}{2T}\Big(\int x\,\rho(dx)\Big)^2,$$

and $\mathfrak{F}_T(\rho) = \infty$ if ρ is not absolutely continuous.

Specifically, we prove in [1] that the log-Sobolev constant γ of m_T^N in the Curie-Weiss case is indeed bounded below uniformly in N for any $T > 0$ such that the free energy has the following convexity property:

$$\exists\lambda_T > 0, \forall m \in \mathbb{R}^d, \qquad \nabla^2\hat{\mathfrak{F}}_T(m) \geq \lambda_T\,\mathrm{id},$$

where $\hat{\mathfrak{F}}_T : \mathbb{R}^d \to \mathbb{R}$ is the following projection of the free energy on the mean:

$$\hat{\mathfrak{F}}_T(m) = \inf\Big\{\mathfrak{F}_T(\rho) : \int x\,\rho(dx) = m\Big\}, \qquad m \in \mathbb{R}^d.$$

The proof relies on a one-step renormalisation argument, which concretely corresponds to using the following Gaussian identity:

$$\exp\Big[\frac{1}{2NT}\Big|\sum_{i=1}^N x_i\Big|^2\Big] \propto \int_{\mathbb{R}^d} \exp\Big[-\frac{N|\varphi|^2}{2T} + \frac{1}{T}\Big(\varphi, \sum_{i=1}^N x_i\Big)\Big]\,d\varphi.$$

This splits the measure m_T^N in two probability measures: an infinite-temperature part (i.e. a product measure) driven by an external field $\varphi \in \mathbb{R}^d$, and a low-dimensional part $\nu_r(d\varphi) \propto e^{-NV_T(\varphi)}\,d\varphi$, where the so-called renormalised potential V_T reads:

$$\begin{aligned}V_T(\varphi) &= \frac{|\varphi|^2}{2T} - \log\int_{\mathbb{R}^d} e^{-V(x_1)+(\varphi,x_1)/T}\,dx_1 \\ &= \frac{|\varphi|^2}{2T} + \inf_\rho\Big\{\int\rho(x)\log\rho(x)\,dx + \int V(x)\,\rho(dx) - \frac{1}{T}\Big(\varphi, \int x\,\rho(dx)\Big)\Big\}.\end{aligned}$$

All information on temperature-dependence of m_T^N is encoded in V_T, in particular in its convexity. This convexity is equivalent to convexity of $\hat{\mathfrak{F}}_T$ as the two can be shown to more or less be Legendre transforms of one another.

References

[1] R. Bauerschmidt, T. Bodineau and B. Dagallier, *A criterion on the free energy for log-Sobolev inequalities in mean-field particle systems*, arXiv preprint arXiv:2503.24372 [math.PR].

[2] A. Guillin, W. Liu, L. Wu and C. Zhang, *Uniform Poincaré and logarithmic Sobolev inequalities for mean field particles systems*, Annals of Applied Probability **32** (3) (2022), 1590–1614.

Resolution of the thin-shell conjecture

Joseph Lehec
(joint work with Boaz Klartag)

A probability measure on $\mathbb{R}^n$ is called log-concave if it is absolutely continuous with respect to the Lebesgue measure on the affine subspace spanned by its support, and if its log-density is a concave function. For instance uniform measures on compact convex sets are log-concave, as well as Gaussian measures. The class of log-concave probability measures is closed under convolutions, weak limits and push-forwards under linear maps, as follows from the Prékopa-Leindler inequality. A log-concave probability measure have moments of all orders, by Borell's lemma, and we can consider its barycenter and covariance matrix. The measure is then called isotropic if the former is zero, and the latter is the identity. This condition is merely a matter of normalization, any probability measure having second moments and not supported on a hyperplane admits an affine image that is isotropic. If X is a random vector with values in $\mathbb{R}^n$, we say that it is log-concave if its law is log-concave, and similarly for the isotropy condition. The main result discussed in this talk is the following.

Theorem 1. *Let X be an isotropic, log-concave random vector in $\mathbb{R}^n$. Then,*

$$\operatorname{Var}|X|^2 \leq Cn, \tag{1}$$

where $C > 0$ is a universal constant.

Theorem 1 is tight, up to the value of the universal constant. Indeed, if X is a standard Gaussian random vector in $\mathbb{R}^n$ then $\operatorname{Var}|X|^2 = 2n$, whereas if X is distributed uniformly in the cube $[-\sqrt{3}, \sqrt{3}]^n$ then X is isotropic and log-concave, and $\operatorname{Var}|X|^2 = 4n/5$. For any isotropic random vector X we have the inequality

$$\mathbb{E}\left(|X| - \sqrt{n}\right)^2 \leq \frac{1}{n}\mathbb{E}(|X|^2 - n)^2 = \frac{1}{n}\operatorname{Var}|X|^2.$$

Therefore Theorem 1 implies the bound

$$\mathbb{E}(|X| - \sqrt{n})^2 \leq C \tag{2}$$

for any log-concave andistropic X on $\mathbb{R}^n$, and where C is a universal constant. Inequality (2) is a *thin-shell bound*, it implies that with high probability, the random vector X belongs to a thin spherical shell $\{\sqrt{n} - O(1) \leq |x| \leq \sqrt{n} + O(1)\}$ whose width is much smaller than its radius $\sqrt{n}$.

Thin-shell bounds in the spirit of (2) were conjectured by Anttila, Ball and Perissinaki [1] in the context of the central limit problem for convex bodies. Very roughly speaking, the latter asks whether this is the case that most marginals of high dimensional convex sets are approximately Gaussian. In the case where X is distributed uniformly in a convex body, the precise form of Theorem 1 was posed as an open problem by Bobkov and Koldobsky [2], who also observed that an affirmative answer would follow from the Kannan-Lovász-Simonovits (KLS) conjecture (which is not discussed in this talk).

The thin-shell conjecture (i.e., the statement of Theorem 1) is also related to Bourgain's slicing problem. In fact, Eldan and Klartag [5] proved that the thin-shell conjecture implies an affirmative answer to Bourgain's slicing problem. The latter was resolved in the affirmative in [13] by using a recent bound by Guan [7]. Guan's technique is also a crucial ingredient in our proof of Theorem 1.

The first non-trivial upper bound for the left-hand side of (2) was given in the proof of the central limit theorem for convex sets in [9]. The bound obtained was that for an isotropic, log-concave random vector X in $\mathbb{R}^n$,

$$\mathbb{E}\left(|X|-\sqrt{n}\right)^2 \leq \sigma_n^2$$

with $\sigma_n = O(\sqrt{n}/\log n)$. This bound was improved to $\sigma_n = O(n^{3/8})$ in Fleury [6] and to $\sigma_n = O(n^{1/3})$ in Guédon and Milman [8]. Roughly speaking, the proofs of these bounds relied on concentration of measure on the high-dimensional sphere. Eldan's stochastic localization was then used by Lee and Vempala [14] in order to show that in fact $\sigma_n = O(n^{1/4})$. Thanks to Eldan and Klartag [5], this yielded another proof of the $n^{1/4}$-bound for Bourgain's slicing problem, which was the state of the art at the time. The methods of Lee and Vempala were extended in a breakthrough work by Chen [3] who came up with a clever growth regularity estimate and proved the bound $\sigma_n = n^{o(1)}$. This was improved to $\sigma_n = O(\log^4 n)$ in [12] by combining Chen's work with spectral analysis. The bound $\sigma_n = O(\sqrt{\log n})$ was then obtained in [11] by replacing the use of growth regularity estimates with an improved Lichnerowicz inequality. This inequality was then used in an extremely intricate bootstrap analysis in Guan [7], for proving $\sigma_n = O(\log\log n)$. Our main theorem provides the definitive answer $\sigma_n = O(1)$.

Our proof of Theorem 1 employs an idea from the proof of the thin-shell conjecture in the special case where the vector X is uniformly distributed on a convex set having coordinate symmetries, from [10]. If μ is a log-concave probability measure in $\mathbb{R}^n$, we define the associated H^{-1} norm as follows:

$$\|f\|_{H^{-1}(\mu)} = \sup\left\{\int_{\mathbb{R}^n} fg\,d\mu \colon \int_{\mathbb{R}^n} |\nabla g|^2\,d\mu \leq 1\right\}.$$

It was shown in [10] by using the *Bochner formula* that in the log-concave case, for any smooth function f such that $\int \nabla f d\mu = 0$,

$$\mathrm{Var}_\mu(f) \leq \|\nabla f\|^2_{H^{-1}(\mu)} := \sum_{i=1}^n \|\partial_i f\|^2_{H^{-1}(\mu)}.$$

Specifying to the case where μ is isotropic (in addition to being log-concave) and $f(x) = |x|^2$ we get

$$\mathrm{Var}_\mu(|x|^2) \leq 4\sum_{i=1}^n \|x_i\|^2_{H^{-1}(\mu)}. \tag{3}$$

Consequently, Theorem 1 would follow from (3) once we prove the following:

Theorem 2. *Let μ be an isotropic, log-concave probability measure in $\mathbb{R}^n$. Then,*

$$\sum_{i=1}^{n} \|x_i\|_{H^{-1}(\mu)}^2 \leq Cn,$$

where $C > 0$ is a universal constant.

The proof of Theorem 2 relies on the well-known fact that the $H^{-1}(\mu)$-norm is related to an infinitesimal optimal transport distance. More specifically, we reduce the problem to estimating the transport distance between μ and log-affine perturbations of μ, which we also call exponential tilts. We then construct a certain coupling between those exponential tilts related to Eldan's stochastic localization and to the theory of non-linear filtering. Theorem 2 then boils down to some growth estimate for the covariance process of the stochastic localization, which we prove using a variant of Guan's technique.

References

[1] M. Anttila, K. Ball, and I. Perissinaki. The central limit problem for convex bodies. *Trans. Amer. Math. Soc.*, 355(12):4723–4735, 2003.

[2] S. G. Bobkov and A. Koldobsky. On the central limit property of convex bodies. In *Geometric aspects of functional analysis (2001–02)*, volume 1807 of *Lecture Notes in Math.*, pages 44–52. Springer, 2003.

[3] Y. Chen. An almost constant lower bound of the isoperimetric coefficient in the KLS conjecture. *Geom. Funct. Anal. (GAFA)*, 31(1):34–61, 2021.

[4] R. Eldan. Thin shell implies spectral gap up to polylog via a stochastic localization scheme. *Geom. Funct. Anal. (GAFA)*, 23(2):532–569, 2013.

[5] R. Eldan and B. Klartag. Approximately Gaussian marginals and the hyperplane conjecture. In *Concentration, functional inequalities and isoperimetry*, volume 545 of *Contemp. Math.*, pages 55–68. Amer. Math. Soc., 2011.

[6] B. Fleury. Concentration in a thin Euclidean shell for log-concave measures. *J. Funct. Anal.*, 259(4):832–841, 2010.

[7] Q. Guan. A note on Bourgain's slicing problem. Preprint, arXiv:2412.09075, 2024.

[8] O. Guédon and E. Milman. Interpolating thin-shell and sharp large-deviation estimates for isotropic log-concave measures. *Geom. Funct. Anal. (GAFA)*, 21(5):1043–1068, 2011.

[9] B. Klartag. A central limit theorem for convex sets. *Invent. Math.*, 168(1):91–131, 2007.

[10] B. Klartag. A Berry-Esseen type inequality for convex bodies with an unconditional basis. *Probab. Theory Related Fields*, 145(1-2):1–33, 2009.

[11] B. Klartag. Logarithmic bounds for isoperimetry and slices of convex sets. *Ars Inveniendi Analytica*, Paper No. 4, 17pp, 2023.

[12] B. Klartag and J. Lehec. Bourgain's slicing problem and KLS isoperimetry up to polylog. *Geom. Funct. Anal. (GAFA)*, 32(5):1134–1159, 2022.

[13] B. Klartag and J. Lehec. Affirmative resolution of Bourgain's slicing problem using Guan's bound. Preprint, arXiv:2412.15044, 2024.

[14] Y. T. Lee and S. Vempala. Eldan's stochastic localization and the KLS hyperplane conjecture: an improved lower bound for expansion. In *58th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2017*, pages 998–1007.

An isoperimetric inequality for (fractal) spaces with different micro- and macrostructure

Patricia Alonso Ruiz
(joint work with Fabrice Baudoin)

Imagine a material whose structure significantly differs depending on whether you work with it at a small scale or at a large scale, and that shows fractal-like features at least on one scale. A company would like to test this material and has a fixed amount of "perimeter points" they can use. To estimate the cost of the amount (volume/area) of material these points would cover, they would like to know how large that size may be. How to "detect" the correct scale when performing measurements?

The present talk addresses this question by presenting an *isoperimetric inequality* in spaces that feature such different scales. To mathematically describe the different scales, we will make use of the function

$$\sigma_{\nu_1,\nu_2}(r) := r^{\nu_1}1_{[0,1]}(r) + r^{\nu_2}1_{(1,+\infty)}(r), \quad r > 0,$$

for given $\nu_1, \nu_2 > 0$. The guiding example during the talk is the Sierpinski carpet cable system, see an approximation in Figure 1.

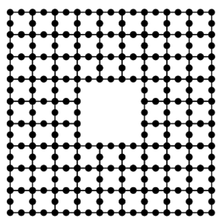

Figure 1. Approximation of a Sierpinski gasket cable system

As a metric measure space (X, d, μ), it satisfies the following key properties: For fixed parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ with $\alpha_i < \beta_i$ and $2 \leq 1 + \alpha_i$,

A1. for any ball $B(x, r)$ with $x \in X$ and $r > 0$,

$$\mu(B(x, r)) \asymp \sigma_{\alpha_1,\alpha_2}(r);$$

A2. there is an intrinsic Brownian motion $\{W_i\}_{t\geq 0}$ in X that admits a heat kernel $p_t(x, y)$ with estimates

$$p_t(x,y) \asymp \frac{1}{\sigma_{\alpha_1/\beta_1,\alpha_2/\beta_2}(t)} \exp\left(-ct\sigma_{\beta_2/(\beta_2-1),\beta_1/(\beta_1-1)}\left(\frac{d(x,y)}{t}\right)\right)$$

for μ-a.e. $x, y \in X$ and all $t > 0$;

A3. the associated heat semigroup $\{P_t\}_{t\geq 0}$ fulfills the so-called *weak Bakry-Émery* condition

$$|P_t u(x) - P_t u(y)| \leq C_{\mathrm{wBE}} \frac{\sigma_{\kappa_1,\kappa_2}\big(d(x,y)\big)}{\sigma_{\frac{\kappa_1}{\beta_1},\frac{\kappa_2}{\beta_2}}(t)} \|u\|_{L^\infty(X,\mu))}$$

for some $\kappa_1, \kappa_2 > 0$ and any $u \in L^\infty(X,\mu)$.

The main result discussed in this talk is an analogue to the classical isoperimetric inequality in $\mathbb{R}^n$, which for sets $E \subset \mathbb{R}^n$ with smooth boundary ∂E reads

$$\big(\mathcal{H}^n(E)\big)^{\frac{n-1}{n}} \leq c_n \mathrm{Per}(E), \tag{1}$$

where $\mathcal{H}^n(\cdot)$ denotes the n-dimensional Hausdorff measure on $\mathbb{R}^n$ and $\mathrm{Per}(E) := \mathcal{H}^{n-1}(\partial E)$ is the perimeter of E. In the metric measure space setting, the measure μ is the natural replacement for $\mathcal{H}^n$. Less clear at first sight is a suitable analogue/substitute for the perimeter measure.

In that regard, a first important observation towards stating an analogue of (1) is the connection between the isoperimetric inequality, the Sobolev embedding and functions of bounded variation (BV). That connection allowed de Giorgi to restate (1) as

$$\big(\mathcal{H}^n(E)\big)^{\frac{n-1}{n}} = \|\mathbf{1}_E\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq c_n \mathrm{Var}(\mathbf{1}_E), \tag{2}$$

where $\mathrm{Var}(\cdot)$ denotes the total variation measure given in $\mathbb{R}^n$ by

$$\mathrm{Var}(u) := \sup\Big\{ \int_{\mathbb{R}^n} u \,\mathrm{div}\varphi \, dx \colon \varphi \in C_c^1(\mathbb{R}^n),\ \|\varphi\|_{L^\infty} \leq 1\Big\}$$

for any BV function u, see e.g. [3]. The second key observation is a characterization of $\mathrm{Var}(u)$ in terms of the heat semigroup $\{P_t\}_{t\geq 0}$ proved in [4] in the Euclidean setting and reads

$$\mathrm{Var}(u) = \lim_{t\to 0^+} \sqrt{\frac{2}{\pi t}} \int_{\mathbb{R}^n} P_t(|u - u(x))(x)\, dx.$$

With these two observations in mind, our original question translates into proving a suitable Sobolev embedding in the metric measure space setting under the conditions **A1-A3** listed above. As a replacement for the $L^{\frac{n}{n-1}}$-norm in (2), a suitable Orlicz norm $\|\cdot\|_{\varphi_1}$ enters into play, and the corresponding variance becomes

$$\mathrm{Var}_{\Psi_1}(u) := \liminf_{t\to 0^+} \frac{1}{\Psi_1(t)} \int_X P_t(|u - u(x))(x)\, d\mu(x),$$

which is a generalization of the variance studied in [1] in the context of fractals with homogeneous scaling. The associated space of bounded variation functions is thus defined as

$$\mathbf{B}^{1,\Psi_1}(X) := \Big\{ u \in L^1(X,\mu) \colon \limsup_{t\to 0^+} \frac{1}{\Psi_1(t)} \int_X P_t(|u - u(x))(x)\, d\mu(x)\Big\}.$$

Theorem 1. *Under the Assumptions* **A1**-**A3***, let*

$$\Psi_1(s) := \sigma_{1-\frac{\kappa_1}{\beta_1},1-\frac{\kappa_2}{\beta_2}}(s).$$

There exists $C > 0$ such that

$$\|u\|_{\varphi_1} \leq C\mathrm{Var}_{\Psi_1}(u)$$

for any $u \in \mathbf{B}^{1,\Psi_1}(X)$. In particular, for any set $E \subset X$ of finite Ψ_1 perimeter,

$$\Phi_1(\mu(E)) \leq C\mathrm{Per}_{\Psi_1}(E),$$

where

$$\varphi_1(s) := \sigma_{\frac{\alpha_1}{\alpha_1-\beta_1+\kappa_1},\frac{\alpha_2}{\alpha_2-\beta_2+\kappa_2}}(s) \quad \text{and} \quad \Phi_1(s) := \sigma_{1+\frac{\kappa_1-\beta_1}{\alpha_1},1+\frac{\kappa_2-\beta_2}{\alpha_2}}(s).$$

The proof is based on an application of several results concerning pseudo-Poincaré inequalities that were obtained in [2].

References

[1] P. Alonso-Ruiz, F. Baudoin, L. Chen. L. Rogers, A. Teplyaev, N. Shanmugalingam, *Besov class via heat semigroup on Dirichlet spaces III: BV functions and sub-Gaussian heat kernel estimates*, Calc. Var. Partial Differential Equations **60** (2021), no. 5, Paper No. 170, 38 pp.;

[2] D. Bakry, T. Coulhon, M. Ledoux, L. Saloff-Coste, *Sobolev inequalities in disguise*, Indiana Univ. Math. J. **44** (1995), no. 4, 1033–1074;

[3] E. De Giorgi, *Sur una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni*, Ann. Mat. Pura Appl. (4) **36** (1954), 191–213;

[4] M. Miranda Jr., D. Pallara, F. Paronetto, M. Preunkert, *Short-time heat flow and functions of bounded variation in* $\mathbf{R}^{\mathbf{N}}$, Ann. Fac. Sci. Toulouse Math. (6) **16** (2007), no. 1, 125–145;

Sub-Laplacian comparison theorems on Riemannian foliations with minimal leaves and applications

Fabrice Baudoin

Let (M, g) be a complete $(n+m)$-dimensional Riemannian manifold endowed with a Riemannian foliation $\mathcal{F}$ with m-dimensional *minimal* leaves. The tangent bundle splits orthogonally as

$$TM = \mathcal{H} \oplus \mathcal{V},$$

where $\mathcal{H}$ is bracket-generating. The horizontal Laplacian acting on smooth functions is

$$\Delta_{\mathcal{H}} f = \mathrm{Tr}_{\mathcal{H}}(\mathrm{Hess}_D f),$$

which coincides with the generator of horizontal Brownian motion thanks to the minimality assumption.

The Levi–Civita connection does not preserve the horizontal distribution. Instead, the analysis relies on the *fundamental connection* ∇, a metric connection preserving $\mathcal{H}$ and $\mathcal{V}$ whose torsion measures the non-integrability of $\mathcal{H}$ and encodes the second fundamental form of the leaves. From ∇ and its torsion, we define a natural symmetric tensor $\mathbf{R}$ playing the role of a Ricci curvature; it combines: (i) curvature of ∇, (ii) torsion divergence, (iii) quadratic torsion terms, and (iv) a J-term induced by torsion.

Let $r_p(x) = d(p, x)$ denote the *Riemannian* distance from p. Assume the curvature lower bound

$$\mathbf{R}(U, U) \geq K|U|^2 \qquad \text{for all } U \in TM.$$

Then, outside the cut locus of p, the horizontal Laplacian of r_p satisfies the comparison estimate

$$\Delta_{\mathcal{H}} r_p(x) \leq \begin{cases} \sqrt{nK}\,\cot\big(\sqrt{K/n}\, r_p(x)\big), & K > 0, \\ \dfrac{n}{r_p(x)}, & K = 0, \\ \sqrt{n|K|}\,\coth\big(\sqrt{|K|/n}\, r_p(x)\big), & K < 0. \end{cases}$$

A related two-point inequality is obtained for a coupled horizontal Laplacian on $M \times M$, using a skewed parallel transport along minimizing geodesics.

The comparison results yield several global consequences:

- **Bonnet–Myers type theorem:** if $K > 0$, then M is compact and
$$\operatorname{diam}(M) \leq \pi\sqrt{\frac{n}{K}}.$$
- **Stochastic completeness:** the semigroup generated by $\Delta_{\mathcal{H}}$ is conservative.
- **Gradient and Lipschitz bounds:** for bounded Lipschitz functions,
$$\operatorname{Lip}(P_t f) \leq e^{-Kt}\operatorname{Lip}(f),$$
proved via coupling by (skewed) parallel transport.

Carnot groups of arbitrary step fit naturally into this framework, providing sub-Laplacian comparison theorems for the Riemannian distance and heat semigroup regularization results in a broad sub-Riemannian setting.

References

[1] F. Baudoin, *Sub-Laplacian comparison theorems on Riemannian foliations with minimal leaves and applications*, Preprint (2026).

Lattice packing of spheres in high dimensions using a stochastically evolving ellipsoid

Boaz Klartag

A lattice L in $\mathbb{R}^n$ is the image of $\mathbb{Z}^n$ under an invertible linear map, and the absolute value of the determinant of this linear map is the covolume of the lattice, denoted by $\mathrm{Vol}_n(\mathbb{R}^n/L)$. A *lattice sphere packing* is a family of disjoint Euclidean balls of equal radius whose centers form a lattice. The density of such a packing with radius r and lattice L is

$$\frac{\mathrm{Vol}_n(rB^n)}{\mathrm{Vol}_n(\mathbb{R}^n/L)}.$$

We denote by δ_n the supremum of all densities of lattice sphere packings in $\mathbb{R}^n$. The classical Minkowski-Hlawka theorem (see, e.g., [8]) implies

$$\delta_n \geq 2\zeta(n)\, 2^{-n},$$

and Rogers showed in [14] that

$$\delta_n \geq c\, n\, 2^{-n} \tag{1}$$

for some universal constant $c > 0$. This bound (1) was sharpened in various ways: Davenport and Rogers improved the constant in [7], Ball used Bang's solution of Tarski's plank problem to obtain $c = 2 - o(1)$ in [2], and Vance exploited quaternionic symmetries to improve c in certain dimensions [20]. Venkatesh developed a powerful random-lattice method with additional algebraic symmetries, proving that

$$\limsup_{n\to\infty} \frac{\delta_n}{n \log\log n\, 2^{-n}} \geq \frac{1}{2},$$

see [21].

Non-lattice constructions have also been very successful: Krivelevich, Litsyn and Vardy used graph-theoretic ideas to obtain a non-lattice packing of density $cn2^{-n}$ [10], and very recently Campos, Jenssen, Michelen and Sahasrabudhe improved this to density

$$\Big(\tfrac{1}{2} - o(1)\Big) n \log n\, 2^{-n}$$

using random graphs and the hard-core model [4]. Up to logarithmic factors, these works all lead to bounds of order $n2^{-n}$.

The main result of my talk is a polynomial improvement of the best known lower bound for *lattice* packings in high dimension.

Theorem. *There exists a universal constant $c > 0$ such that, for all $n \geq 2$,*

$$\delta_n \geq c\, n^2\, 2^{-n}.$$

By applying a linear map, this is equivalent to the following geometric statement: let $K \subset \mathbb{R}^n$ be a Euclidean ball centered at the origin with

$$\mathrm{Vol}_n(K) = c\, n^2. \tag{2}$$

Then there exists a lattice $L \subset \mathbb{R}^n$ of covolume one such that

$$L \cap K = \{0\}.$$

Equivalently, there is an origin-symmetric ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ of volume cn^2 with $\mathcal{E} \cap \mathbb{Z}^n = \{0\}$. We conjecture that the conclusion remains true for any origin-symmetric convex body K satisfying (2), and not only for Euclidean balls or ellipsoids; compare with earlier work of Schmidt under the weaker assumption $\mathrm{Vol}_n(K) \leq cn$ [18, 19].

There is a geometric heuristic behind the exponent 2. An origin-symmetric ellipsoid in $\mathbb{R}^n$ is specified by a positive-definite symmetric $n \times n$ matrix, i.e. by about $n(n+1)/2$ real parameters. One may therefore expect a "generic" ellipsoid to interact with on the order of n^2 lattice points. Indeed, it is not hard to construct an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ with $\mathcal{E} \cap \mathbb{Z}^n = \{0\}$ such that

$$\#(\partial\mathcal{E} \cap \mathbb{Z}^n) \geq n(n+1),$$

where $\partial\mathcal{E}$ is the boundary of $\mathcal{E}$. This is in sharp contrast with the proof of Rogers' bound (1), which relies on successive minima and an ellipsoid that essentially interacts with only n carefully chosen lattice vectors; see [8, 14, 17].

The proof combines random lattices and a Brownian-type stochastic evolution on the space of ellipsoids. We start from a random lattice L of covolume $\mathrm{Vol}_n(B^n)$, distributed according to the natural $SL_n(\mathbb{R})$-invariant probability measure on unimodular lattices. This point of view goes back to Siegel's mean value theorem [12] and underlies much work on the probabilistic geometry of numbers; see also Schmidt's early probabilistic methods [17] and the recent perspectives in [11].

For a fixed lattice L, we construct a stochastic process of positive-definite symmetric matrices $(A_t)_{t\geq 0}$ and consider the associated evolving ellipsoids

$$\mathcal{E}_t = \{x \in \mathbb{R}^n\,;\, A_t x \cdot x < 1\}.$$

The key feature of our process is a *sticky boundary* rule: whenever $\mathcal{E}_t$ first hits a new non-zero lattice point $x_0 \in L$, we impose the constraint

$$A_t x_0 \cdot x_0 = 1 \qquad \text{for all later times } t \geq t_0,$$

where t_0 is the hitting time of x_0. Each such event freezes one linear degree of freedom of A_t, and the process is then continued inside the corresponding affine subspace of the space of symmetric matrices. Since this space has dimension $n(n+1)/2$, the evolution can absorb on the order of n^2 lattice points (counted with symmetry) before it becomes rigid.

Heuristically, a typical random lattice of covolume $\mathrm{Vol}_n(B^n)$ behaves like a Poisson point process of intensity $1/\mathrm{Vol}_n(B^n)$ in $\mathbb{R}^n$ (the "Poisson heuristic"); see, for instance, [15, 11]. If the evolving ellipsoid must collect roughly n^2 lattice points on its boundary before freezing, then one expects that it must sweep a region of total volume of order $cn^2\,\mathrm{Vol}_n(B^n)$. Perhaps, at some time, its volume is at least of this order while still remaining free of non-zero lattice points in its interior.

The main technical work consists in turning this heuristic into a rigorous argument. We construct the stochastic process (A_t) and obtain uniform control on quantities such as $\log\det(A_t)$, which governs the volume of $\mathcal{E}_t$. We analyze the rate at which new lattice points are absorbed, using quantitative incarnations of the Poisson heuristic and large-deviation estimates to rule out pathological scenarios. We integrate these estimates over the space of lattices using Siegel's mean value formula and related tools, in the spirit of [1, 14, 17]. This yields a positive probability that a random lattice admits an ellipsoid of volume at least $cn^2\,\mathrm{Vol}_n(B^n)$ with no non-zero lattice points inside, which is equivalent to the desired lower bound $\delta_n \geq cn^2 2^{-n}$.

From a broader perspective, our result fits with the conjectural picture, suggested for instance by Venkatesh [21], that $2^n\delta_n$ should grow at most polynomially in n. It suggests that $n^2 2^{-n}$ might be the correct order of magnitude (up to constants or logarithmic factors) for the optimal density of high-dimensional lattice packings. On the other hand, the best known upper bounds, due to Kabatjanskiĭ and Levenšteĭn [9] and their refinements by Cohn–Zhao and Sardari–Zargar [6, 16], show that $\delta_n \lesssim (0.66)^n$, and the precise asymptotic behavior of δ_n remains wide open; see also the survey [5] for the special low-dimensional cases where the optimal density is known.

References

[1] Ajtai, M., *Random lattices and a conjectured 0–1 law about their polynomial time computable properties.* Proceedings of the 43rd Annual IEEE Symposium on Foundations of Computer Science, (2002), 733–742.

[2] Ball, K., *A lower bound for the optimal density of lattice packings.* Internat. Math. Res. Notices (IMRN), no. 10, (1992), 217–221.

[3] Blichfeldt, H. F., *The minimum value of quadratic forms, and the closest packing of spheres.* Math. Ann., Vol. 101, no. 1, (1929), 605–608.

[4] Campos, M., Jenssen, M., Michelen, M., Sahasrabudhe, J., *A new lower bound for sphere packing.* Preprint, arXiv:2312.10026.

[5] Cohn, H., *A conceptual breakthrough in sphere packing.* Notices Amer. Math. Soc., Vol. 64, no. 2, (2017), 102–115.

[6] Cohn, H., Zhao, Y., *Sphere packing bounds via spherical codes.* Duke Math. J., Vol. 163, no. 10, (2014), 1965–2002.

[7] Davenport, H., Rogers, C. A., *Hlawka's theorem in the geometry of numbers.* Duke Math. J., Vol. 14, (1947), 367–375.

[8] Gruber, P. M., Lekkerkerker, C. G., *Geometry of numbers.* Second edition. North-Holland Publishing Co., 1987.

[9] Kabatjanskiĭ, G. A., Levenšteĭn, V. I., *Bounds for packings on the sphere and in space.* Problemy Peredači Informacii 14, no. 1, (1978), 3–25. In Russian. English translation in: Problems of Information Transmission, Vol. 14, no. 1, (1978), 1–17.

[10] Krivelevich, M., Litsyn, S., Vardy, A., *A lower bound on the density of sphere packings via graph theory.* Internat. Math. Res. Notices (IMRN), no. 43, (2004), 2271–2279.

[11] Marklof, J., *Random lattices in the wild: from Pólya's orchard to quantum oscillators.* Lond. Math. Soc. Newsl., No. 493, (2021), 42–49.

[12] Siegel, C. L., *A mean value theorem in geometry of numbers.* Ann. of Math., Vol. 46, (1945), 340–347.

[13] Rankin, R. A., *On the closest packing of spheres in n dimensions.* Ann. of Math., Vol. 48, (1947), 1062–1081.

[14] Rogers, C. A., *Existence Theorems in the Geometry of Numbers.* Ann. of Math., Vol. 48, (1947), 994–1002.

[15] Rogers, C. A., *The number of lattice points in a set.* Proc. London Math. Soc., Vol. s3-6, (1956), 305–320.

[16] Sardari, N. T., Zargar, M., *New upper bounds for spherical codes and packings.* Math. Ann., Vol. 389, no. 4, (2024), 3653–3703.

[17] Schmidt, W. M., *The measure of the set of admissible lattices.* Proc. Amer. Math. Soc., Vol. 9, No. 3, (1958), 390–403.

[18] Schmidt, W. M., *On the Minkowski-Hlawka theorem.* Illinois J. Math., Vol. 7, (1963), 18–23.

[19] Schmidt, W. M., *Correction to my paper, "On the Minkowski-Hlawka theorem".* Illinois J. Math., Vol. 7, (1963), 714.

[20] Vance, S., *Improved sphere packing lower bounds from Hurwitz lattices.* Adv. Math., Vol. 227, no. 5, (2011), 2144–2156.
[21] Venkatesh, A., *A note on sphere packings in high dimension.* Internat. Math. Res. Notices (IMRN), no. 7, (2013), 1628–1642.

Distance between non-symmetric bodies: optimal bounds up to polylog

Pierre Bizeul
(joint work with Boaz Klartag)

The Banach–Mazur distance provides a quantitative way to compare convex bodies in high dimension up to affine transformations. Given two convex bodies $K, L \subset \mathbb{R}^n$ with non-empty interior, their Banach–Mazur distance is defined as

$$d_{BM}(K,L) = \inf\Big\{\lambda > 0 \ : \ T_1 K \subset T_2 L \subset \lambda\, T_1 K\Big\},$$

where T_1, T_2 range over invertible affine maps of $\mathbb{R}^n$. Equivalently, after suitable affine changes of coordinates, the norms induced by the two bodies are comparable up to a multiplicative factor λ. The logarithm of d_{BM} defines a genuine metric, and understanding its extremal behavior is a central problem in asymptotic convex geometry.

A fundamental question is to determine the *diameter of the Banach–Mazur compactum*, that is, the maximal possible distance between two convex bodies in dimension n.

The symmetric case. When K and L are centrally symmetric, the picture is well understood. John's theorem [3] asserts that any symmetric convex body $K \subset \mathbb{R}^n$ admits an ellipsoid $\mathcal{E}$ such that

$$\mathcal{E} \subset K \subset \sqrt{n}\,\mathcal{E}.$$

This immediately implies

$$d_{BM}(K, B_2) \leq \sqrt{n} \qquad \text{and hence} \qquad d_{BM}(K,L) \leq n$$

for any symmetric convex bodies K, L.

This upper bound is essentially optimal. A probabilistic construction due to Gluskin [1] shows that for suitable random symmetric polytopes P_N, with $N = 6n$ vertices,

$$d_{BM}(P_N, P'_N) \geq c\, n$$

with high probability. As a consequence, the diameter of the symmetric Banach–Mazur compactum is of order n.

The non-symmetric case. Without symmetry, the situation is more delicate. John's theorem yields a much weaker inclusion

$$\mathcal{E} \subset K \subset n\,\mathcal{E},$$

which leads to the crude bound

$$d_{BM}(K,L) \leq n^2$$

for arbitrary convex bodies $K, L \subset \mathbb{R}^n$. The example of the simplex shows that John's estimate is tight. In fact, for any symmetric convex body K and any non-degenerate simplex Δ,

$$d_{BM}(K,\Delta) = n.$$

Despite many contributions, the gap between the lower and upper bounds persisted until a substantial improvement by Rudelson [5], who proved an upper bound of order $n^{4/3}$ up to logarithmic factors.

Main result. In joint work with Boaz Klartag, we prove the following result.

Theorem 1. *For any convex bodies* $K, L \subset \mathbb{R}^n$*,*

$$d_{BM}(K,L) \leq C\,n \log^4 n,$$

where $C > 0$ *is a universal constant.*

Up to the logarithmic factor this estimate is optimal and matches the diameter of the symmetric Banach–Mazur compactum.

The proof uses the probabilistic method : the bound is attained when both bodies are placed in a randomly rotated isotropic position. Recall that a convex body K is said to be isotropic if a random vector uniformly distributed on K has zero mean and identity covariance matrix. Isotropy is preserved under orthogonal transformations, making it natural to introduce randomness via rotations.

Mean norm bounds and the probabilistic method. The main ingredient needed in the proof is a quantitative control on certain classical geometric parameters of convex bodies. In particular, the argument reduces to bounding two fundamental quantities: the *mean norm* and the *mean width*.

For a convex body $K \subset \mathbb{R}^n$, these are defined by

$$M(K) = \int_{\mathbb{S}^{n-1}} \|x\|_K \, d\sigma(x), \qquad M^*(K) = M(K^\circ) = \int_{\mathbb{S}^{n-1}} h_K(x)\, d\sigma(x),$$

where K° denotes the polar body and h_K its support function. Both quantities are ubiquitous in convex geometry, especially in the study of sections and projections of convex bodies. Furthermore they are dual to one another : when the body is a euclidean ball, its mean width is its radius while its mean norm is the inverse of its radius. More generally, the product $MM^*(K)$ is always greater than 1 and is scaling invariant. It may be thought of as a measure of how round is K in some average sense.

A fundamental result of Pisier [7] asserts that if K is symmetric, there exists a linear transformation T such that $MM^*(TK)$ is at most logarithmic in the dimension. It was a long standing open problem to provide a similar result in the nonsymmetric case, where Pisier's argument is known to fail.

A result of E. Milman [4] asserts that if K is in isotropic position, then

$$M^*(K) \leq C\sqrt{n}\,\log^2 n,$$

which is optimal up to the power of the log. In our work, we complement this estimate with a new, almost optimal bound for the mean norm, improving on previous works [2, 8].

Theorem 2. *If K is a convex body in isotropic position, then*

$$M(K) \leq \frac{C\log n}{\sqrt{n}},$$

for some universal constant $C > 0$.

Furthermore, assuming the KLS conjecture, our estimate improves to

$$M(K) \leq \frac{C\sqrt{\log n}}{\sqrt{n}},$$

which is optimal up to the universal constant C, as witnessed by the cube.

Together, the bounds on $M(K)$ and $M^*(K)$ allow one to control the effect of random rotations on isotropic convex bodies and lead to the stated upper bound on the Banach–Mazur distance.

References

[1] Gluskin, E. D., *The diameter of the Minkowski compactum is roughly equal to n.* Funktsional. Anal. i Prilozhen., Vol. 15, no. 1, (1981), 72–73. In Russian. English translation: Functional Anal. Appl., Vol. 15, no. 1, (1981), 57–58.

[2] Giannopoulos, A., Milman, E., *M-estimates for isotropic convex bodies and their L_q-centroid bodies.* Geom. aspects of Funct. Anal., Lecture Notes in Math., Vol. 2116, Springer (2014), 159–182.

[3] John, F., *Extremum problems with inequalities as subsidiary conditions.* Interscience Publishers, New York, (1948), 187–204.

[4] Milman, E., *On the mean-width of isotropic convex bodies and their associated L^p-centroid bodies.* Int. Math. Res. Not. (IMRN), No. 11, (2015), 3408–3423.

[5] Rudelson, M., *Distances between non-symmetric convex bodies and the MM^*-estimate.* Positivity, Vol. 4, no. 2, (2000), 161–178.

[6] Gordon, Y., Litvak, A. E., Meyer, M., and Pajor, A., *John's decomposition in the general case and applications.* J. Differential Geom., Vol. 68, no. 1, (2004), 99–119.

[7] Pisier, G., *Holomorphic semi-groups and the geometry of Banach spaces.* Ann. Math., Vol. 115, no. 2, (1982), 375–392.

[8] Vritsiou, B.-H., *Regular ellipsoids and a Blaschke-Santaló-type inequality for projections of non-symmetric convex bodies.* J. Funct. Anal., Vol. 286, no. 11, (2024), Paper No. 110414, 40 pp.

Heat flow with Wentzell boundary condition: Functional inequalities and gradient flow structure

Marie Bormann

(joint work with Léonard Monsaingeon, Michiel Renger, Max von Renesse, Feng-Yu Wang)

Let Ω be a smooth compact connected Riemannian manifold of dimension $d \geq 2$ with smooth connected boundary $\partial\Omega$. We consider the semigroup on $C(\Omega)$ induced from the Feller generator $(\mathcal{D}(L), L)$ given by

$$\mathcal{D}(L) = \{f \in C(\Omega) \mid Lf \in C(\Omega)\},$$
$$Lf = (\Delta f)\,\mathbb{1}_{\Omega^\circ} + \left(-\beta\frac{\partial f}{\partial N} + \delta\Delta^\tau f\right)\mathbb{1}_{\partial\Omega},$$

where $\frac{\partial f}{\partial N}$ is the outer normal derivative, Δ^τ is the Laplace-Beltrami operator on $\partial\Omega$, $\delta \geq 0$ and $\beta > 0$. The induced Markov process is a diffusion on Ω which performs Brownian motion in the interior while its boundary behaviour consists of Brownian motion with speed according to δ along the boundary as well as sticky reflection back into the interior with intensity β. Stickiness refers to the fact that the process sojourns at the boundary. The case $\delta = 0$ corresponds to pure sticky reflection without diffusion along the boundary. This boundary behaviour is a special case of the setting in [8] and fundamentally differs from simple reflection or killing at the boundary. The unique symmetric and invariant measure μ is a convex combination of the normalised volume measure on Ω and the normalised Hausdorff measure on $\partial\Omega$ depending on the value of the parameter β.
In the first part of the talk we discuss upper bounds for the Poincaré and logarithmic Sobolev constants (C, D) for Brownian motion with sticky-reflecting boundary diffusion in terms of the geometry of the manifold and its boundary and the Neumann Poincaré and logarithmic Sobolev constants (C_Ω, D_Ω), the Sobolev-Poincaré constants on Ω $(C_{p,q})$ as well as the Poincaré and logarithmic Sobolev constants on $\partial\Omega$ $(C_{\partial\Omega}, D_{\partial\Omega})$ as shown in [1]:

Theorem 1. *Let $k_1, k_2, \gamma_1, \gamma_2 \in \mathbb{R}$ such that*

$$Ric \geq (d-1)k_1,\ \ sect \leq k_2 \ \text{and}\ \gamma_1 id \leq \mathbb{I} \leq \gamma_2 id.$$

Then for any $\beta > 0$ and $\delta \geq 0$

$$C \leq C(\beta, \delta, k_1, k_2, \gamma_1, \gamma_2, d, C_\Omega, C_{\partial\Omega}, |\Omega|, |\partial\Omega|),$$
$$D \leq D(\beta, \delta, k_1, k_2, \gamma_1, \gamma_2, d, C_\Omega, C_{\partial\Omega}, L_\Omega, L_{\partial\Omega}, C_{p,2}, |\Omega|, |\partial\Omega|).$$

where the expressions on the right-hand side can be written out explicitly.

To illustrate the results we consider unit balls in the Euclidean and hyperbolic plane. The proofs are based on an interpolation approach that has first been used in [7] for the Poincaré inequality and which we have additionally extended for the logarithmic Sobolev inequality here. In both cases the interpolation approach combines classical functional inequalities for the (Neumann) Laplacian on Ω or $\partial\Omega$ with energy interactions between the boundary and the interior of the manifold.

The geometry of the manifold enters quantitatively via the use of the Laplacian comparison theorem for the distance to the boundary function. The problem of bounding C and D is connected to finding a lower bound on the first non-trivial Steklov eigenvalue and an upper bound on the norm of the Sobolev trace operator.

The Poincaré inequality in this setting was previously studied in [7] for the special case of positive Ricci curvature and convex boundary. Extensions to the doubly weighted case and to super and weak Poincaré inequalities using similar techniques as mentioned above as well as Cheeger-type inequalities can be found in [3, 9] and [4].

In the second part of the talk we consider the Fokker-Planck equation associated with Brownian motion with sticky-reflecting boundary diffusion where for simplicity we set $\beta = 1$ and consider a subset Ω of Euclidean space with properties as before:

$$\begin{cases} \dot{\omega}_t = \Delta\omega & \text{in } \Omega^\circ, \\ \omega = \gamma & \text{on } \partial\Omega, \\ \dot{\gamma}_t = \delta\Delta^\tau\gamma - \partial_N\omega & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here we have decomposed measures on Ω in interior/boundary part

$$\rho = \omega + \gamma \in \mathcal{P}(\Omega) \text{ with } \begin{cases} \omega = \rho|_{\Omega^\circ}, \\ \gamma = \rho|_{\partial\Omega}. \end{cases}$$

We discuss here whether equation (1) may be interpreted as a gradient flow equation, where notions of gradient flow may include Otto's calculus, generalised minimising movement schemes/JKO gradient flows or curves of maximal slope.

In [5] equation (1) has been identified as a gradient flow wrt. the μ-relative entropy Ent_μ and the quadratic Wasserstein distance $\mathcal{W}$ for $\delta = 1$. Furthermore, it has been conjectured that an analogous result holds when the boundary diffusion is sped up ($\delta > 1$) with the same driving functional but using a Wasserstein metric wrt. a different cost function $\mathcal{W}_{c_\delta}$. In this talk we consider the case of slow boundary diffusion $\delta \in (0, 1)$. We introduce the notion of Cheeger regular space as metric measure spaces (Ω, d, μ) for which the Cheeger energy is a 'nice' Dirichlet form which fulfills an integrated Varadhan formula (see [6]) based on the metric d. Using this notion one may show [2]:

Theorem 2. *For any $\delta > 0$ (1) can be identified as a Wasserstein gradient flow via Otto's calculus. But for $\delta \in (0, 1)$ there is no metric d on Ω inducing the Euclidean topology such that*

- *(Ω, d, μ) is a Cheeger-regular space and*
- *equation (1) can be interpreted as the JKO gradient flow of Ent_μ wrt. $\mathcal{W}_d$.*

This result can be of interest as it indicates that the different notions of gradient flow might not be equivalent.

References

[1] M. Bormann, M. von Renesse and F.-Y. Wang, *Functional inequalities for Brownian motion on manifolds with sticky-reflecting boundary diffusion*, Probability Theory and Related Fields (2024).

[2] M. Bormann, L. Monsaingeon, D. R. M. Renger and M. von Renesse, *A gradient flow that is none: Heat flow with Wentzell boundary condition*, arXiv: 2506.22093 (2025).

[3] M. Bormann, *Functional Inequalities for doubly weighted Brownian Motion with Sticky-Reflecting Boundary Diffusion*, arXiv: 2409.19336 (2024).

[4] M. Bormann, *A Cheeger-type inequality for the drift Laplacian with Wentzell-type boundary condition*, arXiv: 2503.16093 (2025).

[5] J.-B. Casteras, L. Monsaingeon and F. Santambrogio, *Sticky-reflecting diffusion as a Wasserstein gradient flow*, Journal de Mathématiques Pures et Appliquées **199** (2025).

[6] M. Hino and J. A. Ramírez, *Small-time Gaussian behavior of symmetric diffusion semigroups*, The Annals of Probability **31 (3)** (2003) 1254–1295.

[7] V. Konarovskyi, V. Marx and M. von Renesse, *Spectral gap estimates for Brownian motion on domains with sticky-reflecting boundary diffusion*, arXiv: 2106.00080 (2021).

[8] A. D. Venttsel' *On boundary conditions for multi-dimensional diffusion processes*, Theory of Probability and its Applications **4 (2)** (1959) 164–177.

[9] F.-Y. Wang, *Super and Weak Poincaré Inequalities for Sticky-Reflected Diffusion Processes*, arXiv: 2508.18846 (2025).

Examples of Kato's limits

Gilles Carron

(joint work with Ilaria Mondello and David Tewodrose)

1. Kato's condition for the Ricci curvature

When (M^n, g) is a complete Riemannian manifold, we introduce the function

$$\mathrm{Ric}_-\colon M \to [0, \infty)$$

so that

$$\forall x \in M\colon \mathrm{Ricci}_x \geq -\mathrm{Ric}_-(x) g_x \text{ in the sense of quadratic form,}$$

and the **Kato constant** of (M^n, g) at times T defined as follow

$$\mathrm{K}_T(M, g) = \sup_{x\in M} \int_0^T \left(e^{-t\Delta_g}\mathrm{Ric}_-\right)(x) ds.$$

Where $\left\{e^{-t\Delta_g}\right\}_{t\geq 0}$ is the heat flow of (M^n, g), that is to say for every $u \in L^2(M, \mathrm{dv}_g)$ the curve of function $t \mapsto u_t = e^{-t\Delta_g}(u)$ solves the evolution equation:

$$\begin{cases} \frac{\partial}{\partial t} u_t + \Delta_g u_t = 0 & \text{on }]0, +\infty[\times M, \\ u_{t=0} = u & \text{on } M. \end{cases}$$

Noticed our convention for the Laplace operator is that in local coordinates $g = \sum_{i,j} g_{i,j} dx_i dx_j$,

$$\Delta_g = -\sum_{i,j} g^{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \left(\Delta_g x_i\right) \frac{\partial}{\partial x_i}.$$

The Kato constant enables to control solution or sub-solution of the Schrödinger equation

$$\frac{\partial v}{\partial t} + \Delta_g v - \mathrm{Ric}_- v = 0$$

and using the Bochner formula, we have the following result:
Assume that for some $T > 0$: $K_T(M,g) < 1$ and define $\lambda > 0$ by $K_T(M,g) = 1 - e^{-\lambda T}$, then

$$\forall f \in Lip(M)\colon \left\|de^{-t\Delta_g}(f)\right\|_\infty \leq e^{\lambda T + \lambda t} \left\|df\right\|_\infty .$$

Comments:

- As far as I know, the Kato condition for the Ricci curvature appears for the first times in a paper of B. Güneysu and D. Pallara [7].
- In the Euclidean setting, Kato potentials are non negative $\mathrm{V} \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\lim_{T\to 0} \sup_{x\in\mathbb{R}^n} \int_0^T \left(e^{-t\Delta}\mathrm{V}\right)(x)ds = 0$$

  and this condition yields Hölder regularity for solution of the equation

$$\frac{\partial v}{\partial t} + \Delta v - \mathrm{V}v = 0$$

  and also Harnack inequality for positive solution of this Schrödinger equation [1].

Some estimates of the Kato constant:

- If the Ricci curvature is uniformly bounded from below : Ricci $\geq -\kappa^2 g$ then for any $T > 0 : \mathrm{K}_T(M,g) \leq \kappa^2 T$.
- [11] If $\mathrm{Ric}_- \in L^p, p > n/2$ and if (M,g) satisfies the Sobolev inequality:

$$\forall u \in \mathcal{C}^1_c(M)\colon \|u\|_{\frac{2n}{n-2}} \leq A\left(\|du\|_2 + \|u\|_2\right)$$

  then for any $T \in (0,1) : \mathrm{K}_T(M,g) \leq C(p,n,A)T^{1-\frac{n}{2p}}$.
- [2] If the $\mathbf{Q}_g$-curvature[1] of (M,g) is uniformly bounded from below :$\mathbf{Q}_g \geq -\kappa^4$ and the scalar curvature is uniformly bounded $|\mathrm{Scal}_g| \leq \kappa^2$, then for any $T \in (0, \kappa^{-2}) : \mathrm{K}_T(M,g) \leq C(n,\kappa)\sqrt{T}$.

2. Links with the Bakry-Emery condition and RCD spaces

A general intuition is that when some $T > 0$ the Kato constant at times T is small then the geometry looks like a one with Ricci curvature uniformly bounded from below $\sim -1/T$. The following result makes this intuition precise:

Theorem: [4] *Assume that*

$$K_T(M,g) \leq \frac{1}{3(n-2)}$$

[1] it is a fourth order expression on the metric g:

$$\mathbf{Q}_g = a_n \Delta_g \mathrm{Scal}_g - b_n \left|\mathrm{Ricci}\right|^2 + c_n \left|\mathrm{Scal}_g\right|^2$$

where a_n, b_b, c_n are positive explicit constants depending only on the dimension.

then there is a bounded $\mathcal{C}^2$*-function* f *such that* $\|f\|_\infty \leq 4K_T(M,g)$ *and such that the weighted Riemannian manifold* $\left(M, e^{2f}g, e^{2f}dv_g\right)$ *satisfies the Bakry-Emery curvature condition* $BE(-4K_T(M,g)/T, N)$ *where* $N = n + 4(n-2)^2 K_T(M,g)$*. In particular* $\left(M, e^{2f}g, e^{2f}dv_g\right)$ *is a* $RCD(-4K_T(M,g)/T, N)$ *space.*

Remarks:

- The result holds in dimension 2 provided that the Kato constant at times T is finite.
- The transformation $(M, g, \mathrm{dv}_g) \to \left(M, e^{2f}g, e^{2f}\mathrm{dv}_g\right)$ is called time changed, in dimension 2 it is a conformal deformation of the metric. The Laplacian of the weighted Riemannian manifold $\left(M, e^{2f}g, e^{2f}\mathrm{dv}_g\right)$ is $e^{-2f}\Delta_g$ in particular this transformation preserves harmonic function. The name time changed comes from the fact that the Brownian motion associated to both spaces follows the same trajectories (hence are the same up to reparametrization). It turns out that under time changed the Bakry-Emery curvature behave nicely : it does not depend on the Hessian of f but only to the gradient and the Laplacian of f [12, 8].
- This result together with RCD theory implies a precompactness result in the pointed Gromov-Hausdorff topology and also provides some structure of limits of sequence of manifold with a uniform bound on the Kato constant.

3. non collapsed strong Kato limits

Recall that a sequence of pointed metric spaces (X_ℓ, d_ℓ, o_ℓ) is said to converge to a pointed metric space $(X_\infty, d_\infty, o_\infty)$ in the pointed Gromov-Hausdorff topology, if for any $R > 0$ there are maps $\Phi_\ell \colon B(o_\ell, R) \to B(o_\infty, R)$ and $\varepsilon_\ell(R) > 0$ with

$$\lim_{\ell \to \infty} \varepsilon_\ell(R) = 0$$

(but not necessary uniformly with R) such that

- $\forall x \in B(o_\infty, R),\ \exists x_\ell \in B(o_\ell, R) : d_\infty(x, x_\ell) \leq \varepsilon_\ell(R)$,
- $\forall x, y \in B(o_\ell, R)\ :\ |d_\ell(x,y) - d_\infty(\Phi_\ell(x), \Phi_\ell(y))| \leq \varepsilon_\ell(R)$.

Theorem: [3] *Let* $v > 0$ *and* $T > 0$ *and let* $f \colon (0,T] \to (0,\infty)$ *non decreasing such that* $\int_0^T f(t)\frac{dt}{t} < \infty$*. Assume that* $\{(M_\ell, g_\ell, o_\ell)\}_\ell$ *is a sequence of complete Riemannian manifold such that*

- $\forall \ell, \forall t \in (0,T] \colon K_t(M_\ell, g_\ell) \leq f(t)$,
- $\forall \ell \colon vol_{g_\ell} B\left(o_\ell, \sqrt{T}\right) \geq vT^{\frac{n}{2}}$.

Then up to extraction of subsequences, the sequence $\{(M_\ell, d_{g_\ell}, o_\ell)\}_\ell$ *converges to a pointed metric space* $(X_\infty, d_\infty, o_\infty)$ *the pointed Gromov-Hausdorff topology and moreover*

- *the Riemannian measure* dv_{g_ℓ} *converges to the* n*-dimensional Hausdorff measure of* (X_∞, d_∞)*.*
- *the Laplacian* Δ_{g_ℓ} *converges in the Mosco sense to the Laplacian associated to the Cheeger energy of* $(X_\infty, d_\infty, d\mathcal{H}^n)$*.*

Our results extends the one Cheeger and Colding [6] in the case where there is a uniform lower bound on the Ricci curvature or of Petersen and Wei [9, 10] when we have a suitable $L^p, p > n/2$ bound on Ric_-.

4. Construction of examples in 2D

The following result produce many examples of such Gromov-Hausdorff limits of smooth surface with uniform Kato bounds.

Theorem: [5] *Let (M^2, g_0) be a complete Riemannian surface with bounded curvature:*

$$|K_{g_0}| \leq \kappa^2.$$

And let $f \in Lip(M)$ such that

$$\|f\|_\infty \leq L \text{ and } Lipf \leq \Lambda \text{ and } \Delta_{g_0} f \leq M \text{ weakly.}$$

Then we can find $f_\ell \in \mathcal{C}^\infty(M)$ such that fixing $o \in M$ we have that the sequence $(M^2, g_\ell = e^{2f_\ell} g_0, o)$ converges to $(M^2, e^{2f} g_0, o)$ in the pointed Gromov-Hausdorff topology and such that for some positive $a > 0$

- $\forall \ell, \forall t \in (0, 1]$: $K_t(M_\ell, g_\ell) \leq C(\kappa, L, \lambda, M)\sqrt{t}$.
- $\forall \ell$: $Aera_{g_\ell}(B_{g_\ell}(o, 1)) \geq a$.

The following two examples are produced by this theorem:

- Consider two Euclidean copies of $\mathbb{R}^2 \setminus \mathbb{D}^2$ glued along their boundaries, where $\mathbb{D}^2 \subset \mathbb{R}^2$, we obtain the cylinder $\mathbb{R} \times \mathbb{S}^1$ endowed with the metric $dr^2 + (1 + |r|)^2 \, d\theta^2$.
- Let two copies of $\mathbb{S}^2 \setminus \mathbb{D}^2_\alpha$ be glued together along their boundaries, where $\mathbb{D}^2_\alpha$ is a geodesic ball of radius $\alpha \in (0, \pi/2)$ for the rounded metric.

Remarks:

(1) It should be noticed that in the above theorem, at any point $p \in M^2$ the tangent cone at p of $(M^2, e^{2f} g_0)$ is the Euclidean plane, hence from the geometric measure point of view every point in $(M^2, e^{2f} g_0)$ is regular.

(2) We have shown that the above example are not Gromov-Hausdorff limits of smooth surface with uniform lower bound on the Gaussian curvature and that the second example is not a Gromov-Hausdorff limits of smooth surface with uniform $L^{p>1}$ bound on the negative part of the Gaussian curvature.

(3) From the PDE regularity theory point of view, our examples can be understood as follow : intuitively asking for the Ricci curvature to be uniformly bounded implies that the Laplacian of the metric is bounded, hence this yields $W^{2,p}$ estimate on the metric. But asking for the Ricci curvature to be uniformly bounded in some Kato class, implies that the Ricci curvature is the Laplacian of a continuous functions, hence that the Laplacian of the metric is the Laplacian of a continuous functions hence we could not hope to get better that $\mathcal{C}^0$ estimate on the metric for Kato limits.

References

[1] M. Aizenman and B. Simon, *Brownian motion and Harnack inequality for Schrödinger operators*, Comm. Pure Appl. Math. **35** (1982), no. 2, 209–273.

[2] G. Carron, *Geometric inequalities for manifolds with Ricci curvature in the Kato class*, Ann. Inst. Fourier (Grenoble) **69** (2019), no. 7, 3095–3167.

[3] G. Carron, I. Mondello, and D. Tewodrose. *Limits of manifolds with a Kato bound on the Ricci curvature.* Geometry and Topology, **28**(6):2635–2745, 2024.

[4] G. Carron, I. Mondello, and D. Tewodrose. *Kato meets Bakry-Émery.* Calculus of Variations and Partial Differential Equations, **64**(6):193, 2025.

[5] G. Carron, I. Mondello, and D. Tewodrose. *Strong Kato limit can be branching.* Preprint arXiv:2411.18097.

[6] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differ. Geom. **46** (1997), no. 3, p. 406-480.

[7] B. Güneysu and D. Pallara, *Functions with bounded variation on a class of Riemannian manifolds with Ricci curvature unbounded from below*,Math. Ann. **363** (2015), no. 3–4, 1307–1331.

[8] B-H. Han and K-T. Sturm. *Curvature-dimension conditions under time change.* Annali di Matematica Pura ed Applicata, **201**(2):801–822, 2021.

[9] P. Petersen and G. Wei. *Relative volume comparison with integral curvature bounds.* Geom. Funct. Anal., **7**(6):1031–1045, 1997.

[10] P. Petersen and G. Wei. *Analysis and geometry on manifolds with integral Ricci curvature bounds. II.* Trans. Amer. Math. Soc., **353** (2):457–478, 2001.

[11] C. Rose and P. Stollmann, *The Kato class on compact manifolds with integral bounds on the negative part of Ricci curvature*, Proc. Amer. Math. Soc. **145** (2017), no. 5, 2199–2210.

[12] K-T. Sturm. *Ricci tensor for diffusion operators and curvature-dimension inequalities under conformal transformations and time changes.* J. Funct. Anal., **275**(4):793–829, 2018

Periodic and Stochastic Quantitative Homogenization for Non-Local Operators

Takashi Kumagai

(joint work with Xin Chen, Zhen-Qing Chen and Jian Wang)

We summarize our recent results on periodic and stochastic quantitative homogenization for non-local operators.

Periodic case. Let us first discuss the periodic case. In this case, qualitative homogenization for rather general non-symmetric Lévy-type processes has been obtained in [2] (see the references in the paper for related works). In the following, we discuss quantitative homogenization based on [6]. Consider

$$\mathcal{L}f(x) := \int_{\mathbb{R}^d} (f(y) - f(x)) \frac{K(x,y)}{|x-y|^{d+\alpha}}\, dy, \quad f \in C_c^2(\mathbb{R}^d),$$

where $\alpha \in (0,2)$, and $K \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ is a multivariate 1-periodic function such that

$$K(x,y) = K(y,x) \quad \text{and} \quad c_1 \le K(x,y) \le c_2 \quad \forall x, y \in \mathbb{R}^d.$$

Then, there exists unique strong Feller process $X := \{X_t\}_{t\ge 0}$ on $\mathbb{R}^d$ (in fact on $\mathbb{T}^d$) associated with $\mathcal{L}$. Next, for each $\varepsilon > 0$, define a scaled process by $X_t^\varepsilon = \varepsilon X_{\varepsilon^{-\alpha} t}$.

The associated generator can be written as

$$\mathcal{L}_\varepsilon f(x) := \int_{\mathbb{R}^d} (f(y) - f(x)) \frac{K(x/\varepsilon, y/\varepsilon)}{|x-y|^{d+\alpha}}\, dy.$$

In this case, one can prove (which is also a very special case of the results in [2]) that $\{X^\varepsilon\}$ converges weakly to $\bar{X} := (\bar{X}_t)_{t\geq 0}$ that has the generator as follows,

$$\bar{\mathcal{L}} f(x) := \int_{\mathbb{R}^d} (f(y) - f(x)) \frac{\bar{K}}{|y-x|^{d+\alpha}}\, dy, \quad \bar{K} := \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K(x,y)\, dx\, dy.$$

We consider quantitative homogenization for this model where the process is on a bounded domain D with a $C^{1,1}$ boundary. Let $h \in C(\bar{D})$ be in some function space whose elements are suitably smooth inside D (see [6] for precise definition of the function space), and let u_ε and $\bar{u}$ be solutions of

$$\begin{cases} \mathcal{L}_\varepsilon u_\varepsilon(x) = h(x), & x \in D, \\ u_\varepsilon(x) = 0, & x \in D^c, \end{cases}$$

$$\begin{cases} \bar{\mathcal{L}} \bar{u}(x) = h(x), & x \in D, \\ \bar{u}(x) = 0, & x \in D^c. \end{cases}$$

Namely,

$$u_\varepsilon(x) = \mathbb{E}_x \int_0^{\tau_D} h(X_s^\varepsilon) ds, \quad u(x) = \mathbb{E}_x \int_0^{\tau_D} h(\bar{X}_s) ds.$$

The following are our results on the quantitative homogenization.

Theorem 1. ([6]) (i) *There exists $C_0 > 0$ such that the following holds for all $\varepsilon \in (0,1)$;*

$$\|u_\varepsilon - \bar{u}\|_{L^1(D;dx)} \leq C_0 \begin{cases} \varepsilon^{\alpha/2}, & \alpha \in (0,1), \\ \varepsilon^{1/2}(1 + |\log\varepsilon|^2), & \alpha = 1, \\ \varepsilon^{(2-\alpha)/2}, & \alpha \in (1,2). \end{cases}$$

(ii) *If $\bar{u} \in C_c^2(D)$, then there exists $C_0 > 0$ such that the following holds for all $\varepsilon \in (0,1)$;*

$$\|u_\varepsilon - \bar{u}\|_{L^2(D;dx)} \leq C_0 \begin{cases} \varepsilon^{\alpha}, & \alpha \in (0,1), \\ \varepsilon(1 + |\log\varepsilon|^2), & \alpha = 1, \\ \varepsilon^{2-\alpha}, & \alpha \in (1,2). \end{cases}$$

We note that for the case of *i.i.d.* random variables in the domain of attraction for stable distributions, Banis [1] obtained the convergence rate of the generalized CLT with the rate $n^{-1/\alpha}$ when $\alpha \in (0,1)$ and $n^{1-2/\alpha}$ when $\alpha \in (1,2)$. Since $\varepsilon = n^{-1/\alpha}$, our homogenization rate in (ii) for $\alpha \in (1,2)$ is optimal. While we do not know the optimality of the case (i), our result indicates that the boundary decay behaviors of the solution to the equation in the limit affects the convergence rate in the homogenization.

Stochastic case. We next discuss stochastic homogenization. In this case, qualitative homogenization has been obtained in [9, 3] (also in [4] for a class of balanced random environments). In the following, we discuss quantitative homogenization based on [5].

Assumption (H) Let $\{w_{x,y} : (x,y) \in E\}$ be *i.i.d.* with $\mathbb{E}[w_{x,y}] = 1$, and there exists a non-random constant $C_1 > 0$ such that $w_{x,y} \le C_1$, $\mathbb{P}$-a.s.

We note that under the above assumption, $w_{x,y}$ may be degenerate. Also, note that we can relax the upper bound of (H) to a moment condition, although this may result in a slower rate of convergence.

Now, for fixed $f \in C_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; dx)$, let $\bar{u} \in C_c^\infty(\mathbb{R}^d)$ be the solution of $\lambda \bar{u}(x) + (-\Delta)^{\alpha/2}\bar{u}(x) = f(x)$, $x \in \mathbb{R}^d$, and $u_k : k^{-1}\mathbb{Z}^d \to \mathbb{R}$ be the solution of

$$\lambda u_k^\omega(x) - \mathcal{L}^{(k),\omega} u_k^\omega(x) = f(x), x \in k^{-1}\mathbb{Z}^d,$$

where

$$\mathcal{L}^{(k),\omega} u_k^\omega(x) := \sum_{z \in k^{-1}\mathbb{Z}^d} (u_k(x+z) - u_k(x)) \frac{w_{kx,k(x+z)}}{|z|^{d+\alpha}} k^{-d}.$$

Note that we can extend u_k to $u_k : \mathbb{R}^d \to \mathbb{R}$ naturally. Our results for quantitative stochastic homogenization are as follows.

Theorem 2. ([5]) *Suppose $d > \alpha$. Under* **Assumption (H)***, let u_k, $\bar{u}$ be the solutions above. Then, there exists $C_0 > 0$ (depends on f and λ) and $k_0(\omega) \ge 1$ (which only depends on λ) such that the following holds for all $k > k_0(\omega)$;*

$$\|u_k^\omega - \bar{u}\|_{L^2(\mathbb{R}^d;dx)} \le C_0 \begin{cases} \min\left\{k^{-\alpha/2}\log^\delta k,\ k^{-((1-\alpha)\wedge(d/2))}\log^{1/2} k\right\}, & \alpha \in (0,1), \\ k^{-1/2}\log k \, \log^\delta k, & \alpha = 1, \\ k^{-(2-\alpha)/2}\log^\delta k, & \alpha \in (1,2), \end{cases}$$

where δ depends on d and α.

We note that unlike the periodic case, in this case there is no global corrector, due to the lack of L^2-integrability. While we do not know the optimality of the estimates, one reason that the convergence rate $k^{-(2-\alpha)/2}$ for $\alpha \in (1,2)$ might be reasonable is the following. Consider the $H^{-\alpha/2}$ norm of the truncated potential $V_k^\omega(x) := \sum_{z\in\mathbb{Z}^d:|z|\le k} w_{x,x+z}(\omega) z|z|^{-d-\alpha}$; then it satisfies

$$\|V_k^\omega\|_{H^{-\alpha/2}}^2 \simeq \sum_{z\in\mathbb{Z}^d:|z|\le k} |z|^2|z|^{-d-\alpha} \simeq k^{2-\alpha}.$$

This blow up rate of $\|V_k^\omega\|_{H^{-\alpha/2}}$ would slow down the convergence rate from $k^{-(2-\alpha)}$ into $k^{-(2-\alpha)/2}$. (Note that for the nearest neighbour case, the limit of $\|V_k^\omega\|_{H^{-\alpha/2}}$ exists and is finite.)

Finally, we announce that quite recently quantitative homogenization on time-dependent long-range random conductance models has been established ([7]).

References

[1] I.I. Banis: Estimation of the rate of convergence in the metric L_p in the case of a limiting stable law. *Lith. Math.*, **13** (1974), 379–384.

[2] X. Chen, Z.-Q. Chen, T. Kumagai and J. Wang: Periodic homogenization of non-symmetric Lévy-type processes. *Ann. Probab.* **49** (2021), 2874–2921.

[3] X. Chen, Z.-Q. Chen, T. Kumagai and J. Wang: Homogenization of symmetric stable-like processes in stationary ergodic media. *SIAM J. Math. Anal.* **53** (2021), 2957–3001.

[4] X. Chen, Z.-Q. Chen, T. Kumagai and J. Wang: Quenched invariance principle for long range random walks in balanced random environments. *Ann. Inst. H. Poincaré Probab. Statist.* **57** (2021), 2243–2267.

[5] X. Chen, Z.-Q. Chen, T. Kumagai and J. Wang: Quantitative stochastic homogenization for random conductance models with stable-like jumps. *Probab. Theory Relat. Fields* **191** (2025), 627–669.

[6] X. Chen, Z.-Q. Chen, T. Kumagai and J. Wang: Quantitative periodic homogenization for symmetric non-local stable-like operators. `arXiv:2409.08120`.

[7] X. Chen, Z.-Q. Chen, T. Kumagai and J. Wang: Quantitative homogenization on time-dependent random conductance models with stable-like jumps. `arXiv:2511.22792`.

[8] Z.-Q. Chen, P. Kim and T. Kumagai: Discrete Approximation of Symmetric Jump Processes on Metric Measure Spaces. Probab. Theory Relat. Fields 155 (2013), 703–749.

[9] X. Chen, T. Kumagai and J. Wang: Random conductance models with stable-like jumps: quenched invariance principle. *Ann. Appl. Probab.*, **31** (2021), 1180–1231.

Long range random walks on groups with polynomial volume growth

Laurent Saloff-Coste

(joint work with Z-Q. Chen, T. Kumagai, J. Wang, R. Zhang, T. Zheng)

1. Stability

1.1. **Stability in $\mathbb{R}^n$.** In $\mathbb{R}$, a stable process is a Lévy process (independent, stationary increments) that is also self-similar (has a time-space scaling invariance property). These processes are attractors (potential limits) for proper rescaling schemes. In $\mathbb{R}^n$, the same concepts involve operator rescaling and operator stability: Instead of dilations families of the basic type $\delta_t : x \mapsto t^{1/\alpha}x$, operator stability involves groups of linear diffeomorphisms of the form $\delta_t = t^B = e^{\log(t)B}$ where the exponent B is a linear invertible map with some additional properties. We refer the reader to [5] which contains a vector space case overview and extensions to the case of non-commutative groups together with a thorough bibliography.

1.2. **Stability on groups.** G. Hunt and others established the theory of Lévy processes on Lie groups around 1960. The extension of operator stability for Lévy processes on groups is much more recent. see the references in [5]. The existence of contractive (or expanding) group isomorphisms is a very restrictive condition. Among all simply connected real Lie groups only certain nilpotent groups admit such group isomorphisms and the nature of such contractive isomorphisms, when they exist, is restricted by the bracket structure of the Lie algebra. On the group of matrices $g = (x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, $\delta_t(g) = \begin{pmatrix} 1 & t^{1/\alpha_1}x & t^{1/\alpha_3}z \\ 0 & 1 & t^{1/\alpha_2}y \\ 0 & 0 & 1 \end{pmatrix}$ is a group of group isomorphism if and only if $1/\alpha_1 + 1/\alpha_2 = 1/\alpha_3$.

1.3. **Group dilations and approximate group dilations.** On a simply connected nilpotent Lie group G, call group dilations any family of group isomorphisms $\{\delta_t, t > 0\}$ such that $\delta_t \delta_s = \delta_{ts}$ and $\lim_{t \to 0} \delta_t(g) = e$ uniformly over any compact set. Call *approximate group dilations* on G any family of diffeomorphisms $\{\delta_t, t > 0\}$ such that $\delta_t \delta_s = \delta_{ts}$ and $\lim_{t \to 0} \delta_t(g) = e$ uniformly over any compact set with the property that for any $g_1, g_2 \in G$, $\delta_t^{-1}(\delta_t(g_1)\delta_t(g_2))$ and $\delta_t^{-1}\left[(\delta_t(g_1))^{-1}\right]$ have uniform limits on compact sets when t tends to infinity. When this is the case, these limits define a **new** Lie group structure G_∞ on the set G.

2. Examples of stable like random walks

In this section, we simply describe different ways to define random walks on finitely generated nilpotent groups that one would expect to have a stable-like behaviors. For details see [3, 4, 6, 7].

2.1. **Word-length stable like measures.** Given a symmetric set of generators of a group H, consider the associated word length $|\cdot|$ and growth function $V(r) = \#\{g \in H : |g| \leq r\}$. If $V(r) \asymp r^d$, $r \geq 1$, then any symmetric probability measure ν such that $\nu(g) \asymp (1 + |g|)^{-\alpha - d}$, $\alpha \in (0, 2)$, is a stable stable-like candidate.

2.2. **1d-singular measures.** On the same group H as in the previous example, given a tuple of generators $S = (s_1, \ldots, s_k)$ and a tuple of exponents $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ consider the symmetric probability measure supported on the union of the one-parameter groups $< s_i >$ given by

$$\nu_{S,\boldsymbol{\alpha}}(g) = \frac{1}{k} \sum_{i=1}^{k} \sum_{m \in \mathbb{Z}} \frac{c_\alpha}{(1 + |m|)^{1+\alpha_i}} \mathbf{1}_{s_i^m}(g).$$

2.3. **Multidimensional coordinate-wise stable-like measures.** On the nilpotent group H, fix a generating tuple $S = (s_1, \ldots, s_k)$ and the associated map $\pi_S : \mathbb{Z}^k \to H$, $x = (x_1, \ldots, x_k) \mapsto s_1^{x_1} \ldots s_k^{x_k} \in H$. This map need not be injective or surjective (despite the moniker "coordinate-wise" we use). Now, given any probability measure ψ on $\mathbb{Z}^k$, we consider its symmetric push-forward

$$\nu_{\pi_S, \psi}(g) = \frac{1}{2} \sum_{x : \pi_S(x) \in \{g, g^{-1}\}} \psi(x).$$

More specifically, we are interested in the case when

$$\psi_{\boldsymbol{\alpha}}(x) = \frac{c}{(1 + \sum_{i=1}^{k} |x_i|^{\alpha_i})^{1 + \sum_1^k 1/\alpha_i}}, \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in (0, 2)^k.$$

In this case there exists $\gamma \geq 0$ such that $\nu_{\pi_S, \psi_{\boldsymbol{\alpha}}}^{(n)}(e) \asymp n^{-\gamma}$. See [6].

Finally, on a nilpotent group H, consider subgroups H_i, $1 \le i \le \ell$ with $H =< H_1, \dots, H_\ell >$. On each H_i, pick a measure μ_i of one of the type described above. Consider the set of all measures of the type $\mu = \sum_i^\ell c_i \mu_i$, $c_i > 0$, $\sum_1^\ell c_1 = 1$.

2.4. **Associated geometries.** On the nilpotent group H equipped with a generating tuple $S = (s_1, \dots, s_k)$ and a weight tuple $\mathbf{w} = (w_1, \dots, w_k)$, set

$$\|g\|_{S,\mathbf{w}} = \min \left\{ \max_{1 \le i \le k} \{[\deg_i(\omega)]^{1/w_i}\} : \boldsymbol{\omega} = g \right\}$$

where $\omega = \omega_1 \dots \omega_m$ is a word over the formal alphabet $\Sigma = \{s_1^{\pm 1}, \dots, s_k^{\pm 1}\}$ and $\deg_i(\omega) = \#\{j \in \{1, \dots, m\} : \omega_j \in \{s_i, s_i^{-1}\}$. It is proved in [7] that, given any choice of $S, \mathbf{w}$ on H, there exists a $\gamma \ge 0$ such that $\#\{g : \|g\|_{S,\mathbf{w}} \le r\} \asymp r^\gamma$, where γ is a function of the data $S, \mathbf{w}$ and the algebraic structure of H.

In each of the examples of classes of measures discussed above, there is some rather natural way to associate a pair $(S, \mathbf{w})$ to the given measure. For example, for measures of type 3.2 (1d-singular), $S = (s_1, \dots, s_k)$ is given to us and the weight given to s_i is $w_i = 1/\alpha_i$. The same applies to measures of type 3.3 (coordinate-wise). In each case, we get a quasi-norm $\| \cdot \|_\mu$ associated with the given symmetric probability measure μ.

2.5. **Near diagonal estimates.** Referring to the classes of measures discussed above, let μ and its associate quasi norm $\| \cdot \|_\mu$ be given. Then for any fixed $A > 0$ and all g such that $\|g\|_\mu \le An$,

$$\mu^{(n)}(g) \asymp n^{-\gamma} \tag{1}$$

where $\#\{g : \|g\|_\mu \le r\} \asymp r^\gamma$. For details see [3, 4, 6, 7].

In work in progress with Ruoqi Zhang, it is proved that two measures μ_1, μ_2 of our collection have equivalent Dirichlet forms if and only if their associated quasi-norms satisfy $\| \cdot \|_{\mu_1} \asymp \| \cdot \|_{\mu_2}$.

Extensions of these results to finitely generated groups of polynomial volume growth exists but require quite a bit of extra work and use in crucial ways the fact that any such group contains a nilpotent subgroup of finite index. See [3].

2.6. **Limit theorems.** Assume the finitely generated group H is torsion free and a co-compact lattice in the simply connected Lie group G. Under mild additional assumptions on the measure μ in any of the family described above, there exists a approximate group dilation family $\{\delta_t, t > 0\}$ defined on G such that, after rescaling using the family δ_t, the random walk associated to μ converges in the sense of functional limit theorems and local limit theorems to a δ_t-stable Lévy process on the limit group G_∞. See [4] for the treatment of some of our examples. Other examples are treated by the same method. In other words, any of the examples described above belong to the domain of attraction of a true stable Lévy process, but one must allow the group structure to change along the rescaling process. Extensions of such limit theorems to group of polynomial growths should be possible but require non-trivial additional ideas.

Limit theorems of this sort have been obtained earlier for diffusions in the work of Crépel and Raugi, Alexoploulos, Hough and Bénard and Breuillard. See [1, 2] for complete lists of references. In that case, the choice of approximate group dilations and the limit group G_∞ are somewhat canonical whereas in the stable like case the approximate group dilations and the limit group G_∞ have to be discovered based on the properties of the random walk measure μ, see [4].

References

[1] Timothée Bénard and Emmanuel Breuillard, *The central limit theorem on nilpotent Lie groups*, Ann. Probab. 53 (2025), no. 2, 668–719. MR4888142

[2] Timothée Bénard and Emmanuel Breuillard, *Local limit theorems for random walks on nilpotent Lie groups*, Probab. Theory Related Fields **192** (2025), no. 3-4, 875–947. MR4951764

[3] Zhen-Qing Chen, Takashi Kumagai, Laurent Saloff-Coste, Jian Wang and Tianyi Zheng, *Long range random walks and associated geometries on groups of polynomial growth,*, Ann. Inst. Fourier (Grenoble) **72** (2022), no. 3, 1249–1304. MR 4485825

[4] Zhen-Qing Chen, Takashi Kumagai, Laurent Saloff-Coste, Jian Wang and Tianyi Zheng, *Limit theorems for some long range random walks on torsion-free nilpotent groups*, SpringerBriefs in Mathematics, Springer, Cham, [2023] ©2023.

[5] W. Hazod and E. Siebert, *Stable Probability Measures on Euclidean Spaces and on Locally Compact Groups: Structural Properties and Limit Theorems*, Math. Appl., **531** Kluwer Academic Publishers, Dordrecht, 2001, xviii+612 pp.

[6] Laurent Saloff-Coste and Ruoqi Zhang, *Examples of stable-like random walks on groups of polynomial growth*, Arxiv:2509.00970.

[7] Laurent Saloff-Coste and Tianyi Zheng, *Random walks on nilpotent groups driven by measures supported on powers of generators*, Groups Geom. Dyn. **9** (2015), no. 4, 1047–1129. MR 3428409

Strong Feller Regularisation of One-Dimensional Nonlinear Transport by Reflected Ornstein–Uhlenbeck Noise

Max v. Renesse

(joint work with Feng-Yu Wang, Alexander Weiss)

1. Deterministic nonlinear transport

Let $\mathcal{P}_2(\mathbb{R})$ denote the space of Borel probability measures on $\mathbb{R}$ with finite second moment. We consider the nonlinear transport equation

$$\partial_t \mu_t = -\mathrm{div}\big(\mu_t\, b(\cdot,\mu_t)\big), \qquad \mu_{t=0} = \mu, \tag{1}$$

where the drift $b\colon \mathbb{R}\times\mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ describes a smooth self-interaction. Typical examples are given by McKean–Vlasov-type interactions

$$b(u,\mu) = \int_{\mathbb{R}} h(u-v)\,\mu(dv), \qquad h \in C_c^\infty(\mathbb{R}).$$

In one spatial dimension, the equation admits a natural Lagrangian formulation in terms of inverse cumulative distribution functions. Writing F^μ for the inverse cdf of μ, the measure-valued flow $(\mu_t)_{t\geq 0}$ can be equivalently described by the $L^2([0,1])$-valued evolution

$$\partial_t F_t = b(F_t, \mu_t), \qquad \mu_t = \lambda \circ F_t^{-1}. \tag{2}$$

The monotonicity of F_t encodes positivity of the density of μ_t and must be preserved under any admissible perturbation.

2. Regular measures and derivative formulation

Assuming sufficient regularity, we represent $\mu \in \mathcal{P}_2(\mathbb{R})$ by the pair

$$g^\mu = (F^\mu)', \qquad M^\mu = \int_{\mathbb{R}} x\,\mu(dx),$$

and introduce the space of regular measures

$$\mathcal{P}_2^1(\mathbb{R}) := \{\mu \in \mathcal{P}_2(\mathbb{R}) \,:\, F^\mu \in H^1((0,1))\}.$$

On $\mathcal{P}_2^1(\mathbb{R})$ we use the metric

$$\rho^2(\mu,\nu) = |M^\mu - M^\nu|^2 + \|g^\mu - g^\nu\|^2_{L^2([0,1])},$$

which dominates the quadratic Wasserstein distance locally and yields a convenient linear structure.

Differentiating (2) formally leads to a closed system for (g_t, M_t),

$$\begin{aligned}\partial_t g_t &= b'(A[g_t, M_t], \mu_t)\, g_t,\\ \partial_t M_t &= \int_0^1 b(A[g_t, M_t](x), \mu_t)\, g_t(x)\,dx,\end{aligned}$$

where $A[g,M]$ reconstructs the quantile function from (g,M) via

$$A[(g,M)](u) := \int_0^1 \int_v^u g(r)\,dr\,dv + M, \qquad u \in [0,1].$$

Correspondingly, the measure associated to (g,M) is

$$\mu_t := \lambda \circ A[(g_t, M_t)]^{-1},$$

where λ denote the Lebesgue measure on $[0,1]$.

3. Stochastic perturbation by reflected SPDEs

To regularise the dynamics while preserving monotonicity, we perturb the system at the level of g_t by an infinite-dimensional Ornstein–Uhlenbeck noise with reflection at the boundary $\{g=0\}$. This yields the reflected SPDE–SDE system

$$\begin{cases} dg_t = \Delta g_t\,dt + b'(A[g_t, M_t], \mu_t)\, g_t\,dt + dW_t + \eta_t,\\ dM_t = \displaystyle\int_0^1 b(A[g_t, M_t](x), \mu_t)\, g_t(x)\,dx\,dt + dB_t,\\ g_t \geq 0, \qquad \langle \eta, g\rangle = 0,\end{cases} \tag{3}$$

where Δ is the Dirichlet-Laplacian on $(0,1)$, W is a cylindrical Wiener process on $L^2([0,1])$, B an independent real Brownian motion, and η is the reflection measure enforcing positivity.

Under global Lipschitz and boundedness assumptions on b and b', we prove well-posedness of (3) and obtain a Markov diffusion $(\mu_t^\mu)_{t\geq 0}$ on $\mathcal{P}_2^1(\mathbb{R})$.

4. Quantitative strong Feller estimate

Our main result establishes a quantitative regularisation effect.

Main result. *For every $T>0$ and $\theta \in (0,1)$, the Markov semigroup $(P_T)_{T>0}$ associated with (3) satisfies the entropy*

$$\mathrm{Ent}\big(P_T(\cdot,\nu)|P_T(\cdot,\mu)\big) \leq H_{T,\theta}(\rho_{\mu,\nu},\rho_\mu)$$

where $\rho_{\mu,\nu} = \rho(\mu,\nu)$, $\rho_\mu = \rho(\delta_{M^\mu},\mu)$, and for some constant $C>0$

$$\begin{aligned} &H_{T,\theta}(\rho_{\mu,\nu},\rho_\mu) \\ &\quad = C\Bigg(\frac{1}{T\wedge 1}\big(\rho_{\mu,\nu}^2 + \rho_{\mu,\nu}^\theta + \rho_{\mu,\nu}\big) + \big(\rho_{\mu,\nu}^{1+\theta} + \rho_{\mu,\nu}^2\big) \\ &\qquad + (1+\rho_\mu^2)^{1+\theta}\Big(\frac{T\wedge 1}{1-\theta}\Big)^\theta \big(\log(1+\rho_{\mu,\nu}^{-1})\big)^{-\theta}\Bigg). \end{aligned}$$

By Pinsker's inequality this implies the strong Feller property in quantitative form, i.e. continuity of $\mu \mapsto P_T F(\mu)$ for bounded measurable F. The estimate highlights a Gaussian-type short-time behaviour together with logarithmic corrections inherent to the infinite-dimensional reflected setting.

5. Relation to previous work

The construction is inspired by recent breakthroughs on regularisation by noise for conservative measure-valued dynamics, notably the rearrangement-based approach of Delarue and Hammersley and its applications to mean-field games. Compared to these works, the explicit reflected SPDE formulation allows for a direct coupling argument and a transparent entropy estimate. Our results contribute to the emerging theory of stochastic regularisation on nonlinear state spaces of probability measures.

References

[1] F. Delarue and S. Hammersley. Strong Feller diffusions on the space of probability measures. *Ann. Probab.*, 2024.

[2] F. Delarue and S. Hammersley. Intrinsic regularisation by noise in one-dimensional mean-field games. *Probab. Theory Relat. Fields*, 2024.

[3] F. Flandoli. *Random Perturbation of PDEs and Fluid Dynamic Models*. Springer, 2011.

[4] F.-Y. Wang. Harnack inequalities and applications for stochastic partial differential equations. *Springer*, 2013.

Quantitative positivity of transition densities for random perturbations of Hamiltonian systems

David P. Herzog
(joint work with Shimaa Elesealy, Kyle Liss)

We present results from [2] concerning a class of diffusion processes arising from random perturbations of conservative Hamiltonian systems. Under a set of abstract hypotheses—including basic structural assumptions on the Hamiltonian, a weak Lyapunov structure, and a quantitative notion of hypoellipticity—we prove that transition densities satisfy a sharp, uniform pointwise lower bound over Hamiltonian sublevel sets in the small noise limit $\varepsilon \to 0$. By applying our general theorem, we obtain quantitative minorization estimates for a variety of models including Langevin dynamics, chains of oscillators coupled to heat bathes at different temperatures, and finite-dimensional fluid models such as stochastically forced Galerkin truncations of the Navier-Stokes equations and the Lorenz '96 system. As a corollary, assuming a stronger Lyapunov structure, our main result yields a sharp exponential rate of convergence to equilibrium for $0 < \varepsilon \ll 1$ in a weighted total variation norm. A central feature of our approach is that it does not require knowledge of the explicit form of the invariant measure, nor even its existence, and hence is broadly applicable to deduce minorization for physically relevant systems where invariant measures are inaccessible. The work combines the analytical ideas of Bedrossian and Liss [1] with the probabilistic ideas of Meyn and Tweedie [3].

References

[1] J. Bedrossian and K. Liss, *Quantitative spectral gaps for hypoelliptic stochastic differential equations with small noise*, Probability and Mathematical Physics, Vol. 2 (2021), No. 3, 477–532.

[2] S. Elesaely, D.P. Herzog, K. Liss, *Quantitative positivity of transition densities for random perturbations of Hamiltonian systems* , arXiv:2509.02448 (2025), 61 pp.

[3] S.P. Meyn and R.L. Tweedie, *Markov chains and stochastic stability*, Springer Science & Business Media (2012).

Modified gamma calculus for degenerate diffusions

Tai Melcher
(joint work with Fabrice Baudoin, Maria Gordina, David Herzog, Jina Kim, Donnelly Phillips)

The gamma calculus is a powerful tool for the analysis of semigroups and their associated diffusions. In particular, the celebrated Bakry-Émery inequality is known to imply a host of functional inequalities, including log Sobolev, Poincaré, and their "reverse" inequalities, whose optimal coefficients give geometric data about the space in which the diffusion lives. (Non-elliptic) Diffusions generated by collections of Hörmander vector fields are degenerate, in the sense that they can't satisfy the classical Bakry-Émery inequality, but they are still expected to have nice properties as they have a strictly positive smooth density with respect to a given reference measure. In this talk we discuss some recent results for modified gamma operators and inequalities for two classes of degenerate diffusions which represent model cases under strong and weak Hörmander conditions.

For the strong Hörmander condition, modified Bakry-Émery inequalities were first proposed by Baudoin, Bonnefont, and Garofalo in a series of work including [1]. These were applied in [3] to produce reverse log-Sobolev inequalities with dimension independent coefficients on infinite-dimensional Heisenberg groups with finite-dimensional centers, and we proved a Cameron-Martin-type quasi-invariance result for heat kernel measures on these spaces. More recently in [4] and forthcoming preprints, we give geometric interpretations of the coefficients appearing in these inequalities, one of which is an analytic quantification of the strong Hörmander condition. The strong Hörmander condition induces a natural geometry via the horizontal distance which is tied to the analysis of the diffusion. For these Heisenberg groups with infinite-dimensional centers, we prove quasi-invariance for the heat kernel measure under translation by elements which are finite horizontal distance from the identity. We show that these elements form a topological group, but in contrast to the finite-dimensional setting may no longer include the whole space.

The weak Hörmander setting has previously lacked a natural notion of distance, and the missing intrinsic geometry has been a complicating factor in its general analysis. In [2] we consider gamma operators modified by natural time-dependent weights for a class of degenerate diffusions with linear drift. Using these operators, we prove reverse Poincaré and log Sobolev inequalities. In particular, the choice of gamma operator appears to be canonical, as these estimates resemble the classical ones for the zero curvature case, although now the time dependency of the coefficients is incorporated into the time-dependent gradient. These modified gamma operators also induce a control distance that preserves scaling properties when the diffusion satisfies one and that can be used to prove Wang-type Harnack inequalities and other transportation cost type inequalities. We again discuss applications to infinite-dimensional diffusions, including quasi-invariance and also uniqueness of stationary distributions.

References

[1] F. Baudoin, M. Bonnefont, N. Garofalo, *A sub-Riemannian curvature-dimension inequality, volume doubling property and the Poincare inequality*, Mathematische Annalen **358** (2014).

[2] F. Baudoin, M. Gordina, D. Herzog, J. Kim, T. Melcher, *Functional inequalities for a family of infinite-dimensional diffusions with degenerate noise*, Journal of Functional Analysis **288** (2025).

[3] F. Baudoin, M. Gordina, T. Melcher, *Quasi-invariance for heat kernel measures on sub-Riemannian infinite-dimensional Heisenberg groups*, Transactions of the AMS **365** (2013).

[4] D. Phillips, *The Hypoelliptic Heat Kernel Measure on Infinite-Dimensional Lie Groups: Heisenberg-Like Quasi-Invariance and the Taylor Isomorphism*, Ph.D dissertation, Department of Mathematics, University of Virginia (2024).

L^2-exponential ergodicity of stochastic Hamiltonian systems with α-stable Lévy noises

Jian Wang

(joint work with Jianhai Bao)

In physics, the Hamiltonian system, as a mathematical formalism due to W.R. Hamilton, describes the evolution of particles in physical systems. From the perspective of practical applications, the deterministic Hamiltonian systems are often subject to environmental noises. Then, the environmentally perturbed system, named as the stochastic Hamiltonian system in literature, is brought into being. So far, stochastic Hamiltonian systems have been applied ubiquitously (see e.g. [2]) in finance describing some risky assets, in physics portraying the synchrotron oscillations of particles in storage rings due to the impact of external fluctuating electromagnetic fields, and in stochastic optimal control serving as a stochastic version of the maximum principle of Pontryagin's type, to name just a few.

With regard to the mathematical formulation, the stochastic Hamiltonian system is described by the following degenerate stochastic differential equation (SDE for short) on $\mathbb{R}^{2d} := \mathbb{R}^d \times \mathbb{R}^d$:

$$(1)\qquad \begin{cases} \mathrm{d}X_t = \nabla_v H(X_t, V_t)\,\mathrm{d}t, \\ \mathrm{d}V_t = -\big(\nabla_x H(X_t, V_t) + F(X_t, V_t)\nabla_v H(X_t, V_t)\big)\,\mathrm{d}t + \mathrm{d}Z_t, \end{cases}$$

where H is the Hamiltonian function, $\nabla_x H$ and $\nabla_v H$ stand for the gradient operators with respect to the position variable x and the velocity variable v, respectively, F means the damping coefficient, and $(Z_t)_{t\geq 0}$ is a d-dimensional stochastic noise.

In this talk, the d-dimensional noise process $(Z_t)_{t\geq 0}$ can be modelled naturally by a pure jump Lévy process. For simplicity, we consider

$$(2)\qquad \begin{cases} \mathrm{d}X_t = \nabla\Phi(V_t)\,\mathrm{d}t, \\ \mathrm{d}V_t = -\nabla U(X_t)\,\mathrm{d}t - \nabla\Phi(V_t)\,\mathrm{d}t + \mathrm{d}L_t, \end{cases}$$

where $(L_t)_{t\geq 0}$ is a symmetric α-stable process with $\alpha \in (1, 2)$. Concerning the SDEs (2), the problem on addressing explicit expressions of invariant probability measures is a tough task and is impossible for almost all of scenarios. This is the prime issue we must be confronted with when we explore the L^2-exponential ergodicity for stochastic Hamiltonian systems with Lévy noises. To make sure that the probability measure

$$\mu(\mathrm{d}x, \mathrm{d}v) := C_{U,\Phi}^{-1} \mathrm{e}^{-(U(x)+\Phi(v))}\,\mathrm{d}x\,\mathrm{d}v,$$

with $C_{U,\Phi} := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{e}^{-(U(x)+\Phi(v))}\,\mathrm{d}x\,\mathrm{d}v < \infty$, is an invariant probability measure, we need to alter the drift term of (2) in a suitable manner. One of the potential ways is that the drift term $\nabla\Phi$ in the position component is untouched while the drift part $-\nabla\Phi$ in the velocity component is substituted by the following one:

$$(3)\qquad b_\Phi(v) := \mathrm{e}^{\Phi(v)}\nabla\big((-\Delta)^{\alpha/2-1}\mathrm{e}^{-\Phi(v)}\big), \quad v \in \mathbb{R}^d,$$

when $d > 2-\alpha$. Herein, $(-\Delta)^{\alpha/2-1}$ is the fractional Laplacian operator defined via the inverse of the Riesz potential. Subsequently, (2) can be rewritten as

$$\begin{cases} \mathrm{d}X_t = \nabla\Phi(V_t)\mathrm{d}t, \\ \mathrm{d}V_t = \big(-\nabla U(X_t) + b_\Phi(V_t)\big)\mathrm{d}t + \mathrm{d}L_t. \end{cases} \tag{4}$$

To proceed, we interpret more backgrounds on stochastic Hamiltonian systems driven by symmetric α-stable noise. The SDE (4), with the driven noise $(L_t)_{t\ge 0}$ being a symmetric α-stable process, is named as a fractional underdamped (or kinetic) Langevin dynamic in [3] and a fractional stochastic Hamiltonian Monte Carlo in [4].

Before proceeding to state our main result, we present assumptions on the coefficients U and Φ in (4). Firstly, concerning the potential U, we assume that

($\mathbf{A}_U$) *The term* $U:\mathbb{R}^d\to\mathbb{R}_+:=[0,\infty)$ *satisfies the following two assumptions*:

($\mathbf{A}_{U,1}$) $U\in C^\infty(\mathbb{R}^d;\mathbb{R}_+)$ *is a compact function* (*i.e., for any* $r>0$, $\{x\in\mathbb{R}^d: U(x)\le r\}$ *has a compact closure*) *such that* $\mathbb{R}^d\ni x\mapsto \mathrm{e}^{-U(x)}$ *is integrable and* $\mathbb{R}^d\ni x\mapsto|\nabla U(x)|$ *is a compact function* (*i.e., for any* $r>0$, $\{x\in\mathbb{R}^d:|\nabla U(x)|\le r\}$ *has a compact closure*); *moreover, there exist constants* $c_1,c_2>0$ *such that for all* $x\in\mathbb{R}^d$,

$$\|\nabla^2 U(x)\| \le c_1|\nabla U(x)| + c_2, \tag{5}$$

where ∇^2 *stands for the second-order gradient operator* (*i.e., the Hessian operator*) *and* $\|\cdot\|$ *denotes the operator norm.*

($\mathbf{A}_{U,2}$) $\liminf_{|x|\to\infty}\frac{\langle\nabla U(x),x\rangle}{|x|}>0$.

With regarding to Φ, we suppose that

($\mathbf{A}_\Phi$) *The function* $\Phi:\mathbb{R}^d\to\mathbb{R}_+$ *fulfills the following assumptions*:

($\mathbf{A}_{\Phi,1}$) $\Phi\in C^3(\mathbb{R}^d;\mathbb{R}_+)$ *is radial such that* $\Phi(v)=\psi(|v|^2)$ *for all* $v\in\mathbb{R}^d$ *and some* $\psi\in C^3(\mathbb{R}_+;\mathbb{R}_+)$; *moreover,* $|v|\mapsto\Phi(|v|)$ *is non-decreasing, and* $\mathbb{R}^d\ni v\mapsto \mathrm{e}^{-\Phi(v)}$ *is integrable.*

($\mathbf{A}_{\Phi,2}$) $\|\nabla\Phi\|_\infty+\|\nabla^3\Phi\|_\infty<\infty$ *and*

$$\sup_{v\in\mathbb{R}^d}\big(\|\nabla^2\Phi(v)\|\cdot|v|\big)<\infty,\qquad \sup_{v\in\mathbb{R}^d}|\Phi(v)-\Phi(v/2)|<\infty,$$

where ∇^3 *indicates the third-order gradient operator; moreover, there exist constants* $c^*,v^*>0$ *such that for all* $v\in\mathbb{R}^d$ *with* $|v|\ge v^*$,

$$\sup_{u\in B_1(v)}\|\nabla^i\Psi(u)\|\le c^*\|\nabla^i\Psi(v)\|,\quad i=1,2,3,$$

where $B_1(v)$ *denotes the unite ball with center* v *and radius* 1.

($\mathbf{A}_{\Phi,3}$) $\mathbb{R}^d\ni v\mapsto\frac{\mathrm{e}^{\Phi(v)}}{(1+|v|)^{2(d+\alpha)}}$ *is integrable.*

($\mathbf{A}_{\Phi,4}$) $\liminf_{|v|\to\infty}\frac{\mathrm{e}^{\Phi(v)}}{|v|^{d+\alpha}}>0$.

The main result of this talk is presented as below.

Theorem 3. *Assume $d > 2 - \alpha$, and suppose further that both $(\mathbf{A}_U)$ and $(\mathbf{A}_\Phi)$ are satisfied. Then, the process $(X_t, V_t)_{t\geq 0}$ solving (4) is L^2-exponentially ergodic, i.e., there exist constants $c, \lambda > 0$ such that for all $f \in L^2(\mu)$ and $t > 0$,*

$$\mathrm{Var}_\mu(P_t f) \leq c\,\mathrm{e}^{-\lambda t}\mathrm{Var}_\mu(f), \tag{6}$$

where $(P_t)_{t\geq 0}$ is the Markov semigroup generated by $(X_t, V_t)_{t\geq 0}$ and μ is an invariant probability measure of $(P_t)_{t\geq 0}$.

As a direct consequence of Theorem 3, we can obtain the following statement. Assume $d > 2 - \alpha$ and $(\mathbf{A}_U)$, and suppose for some $\beta \in [\alpha, 2\alpha)$,

$$\Phi(v) = \frac{1}{2}(d + \beta)\log(1 + |v|^2), \quad x \in \mathbb{R}^d,$$

Then, the assertion (6) holds true, i.e., the process $(X_t, V_t)_{t\geq 0}$ solving (4) is L^2-exponentially ergodic.

For the details of results above, the readers can be referred to [1].

References

[1] J. Bao and J. Wang, *L^2-exponential ergodicity of stochastic Hamiltonian systems with α-stable Lévy noises*, to appear in *Forum Mathematicum*, https://doi.org/10.1515/forum-2024-0047

[2] J. Hong and L. Sun, *Symplectic Integration of Stochastic Hamiltonian Systems*, Lecture Notes in Mathematics, Vol. 2314, Springer, 2022.

[3] U. Şimşekli, L. Zhu, Y.W. Teh, and M. Gürbüzbalaban, Fractional underdamped Langevin dynamics: retargeting SGD with momentum under heavy-tailed gradient noise, *Proceedings of the 37th International Conference on Machine Learning, PMLR*, **119**, 2020, 8970–8980.

[4] N. Ye and Z. Zhu, Stochastic fractional Hamiltonian Monte Carlo, *Proceedings of the 27th International Joint Conference on Artificial Intelligence* (IJCAI-18), 2018, 3019–3025.

L^2-optimal transport for laws of random measures in Euclidean spaces

Giuseppe Savaré

(joint work with Alessandro Pinzi)

A classical and deep feature of quadratic optimal transport in $\mathbb{R}^d$ [3, 7] is its tight link with convex analysis. For $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, the quadratic Wasserstein cost

$$\mathsf{w}_2^2(\mu, \nu) := \min_{\boldsymbol{\gamma}\in\Gamma(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x - y|^2 \,\mathrm{d}\boldsymbol{\gamma}$$

admits an equivalent maximization formulation

$$\mathsf{w}_2^2(\mu, \nu) = \mathsf{m}_2^2(\mu) + \mathsf{m}_2^2(\nu) - 2\,[\mu, \nu], \qquad [\mu, \nu] := \max_{\boldsymbol{\gamma}\in\Gamma(\mu,\nu)} \int x \cdot y \,\mathrm{d}\boldsymbol{\gamma},$$

so that minimizers of w_2^2 coincide with maximizers of $[\mu, \nu]$. Combined with Kantorovich duality and cyclical monotonicity, this yields the classical characterization of optimal plans by convex subdifferentials and, under mild regularity on μ, existence and uniqueness of optimal transport maps solving the Monge problem.

Random measures and the lifted quadratic transport problem. Since $(\mathcal{P}_2(\mathbb{R}^d), \mathsf{w}_2)$ is itself a Polish metric space, one can study optimal transport between *laws of random measures*, namely elements of

$$\mathscr{P}_2(\mathbb{R}^d) := \mathcal{P}_2(\mathcal{P}_2(\mathbb{R}^d)),$$

endowed with the quadratic cost induced by w_2:

$$\begin{aligned}\mathsf{W}_2^2(\mathtt{M}, \mathtt{N}) := & \min_{\Pi \in \Gamma(\mathtt{M},\mathtt{N})} \int_{\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{P}_2(\mathbb{R}^d)} \mathsf{w}_2^2(\mu,\nu)\, \mathrm{d}\Pi(\mu,\nu) \\ = & \min \mathbb{E}\big[\mathsf{w}_2^2(M,N)\big], \quad M \sim \mathtt{M},\ N \sim \mathtt{N}.\end{aligned}$$

A priori, the classical convex-analytic landscape is expected to fail on general metric spaces and, in particular, on $(\mathcal{P}_2(\mathbb{R}^d), \mathsf{w}_2)$, which is positively curved in the Alexandrov sense. However, in the lifted Euclidean setting, a substantial part of that landscape can be recovered by replacing linear convexity with an appropriate notion of convexity along *arbitrary* couplings (not only Wasserstein geodesics).

Total displacement convexity and a Kantorovich–Legendre transform. Given $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$ and any coupling $\boldsymbol{\mu} \in \Gamma(\mu_0, \mu_1)$, define the associated displacement interpolation

$$\mu_t := (\pi_t)_\# \boldsymbol{\mu}, \qquad \pi_t(x_0, x_1) := (1-t)x_0 + t x_1, \quad t \in [0,1].$$

A functional $\varphi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ is *totally displacement convex* if

$$\varphi(\mu_t) \le (1-t)\varphi(\mu_0) + t\varphi(\mu_1) \quad \text{for every } \mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d) \text{ and every } \boldsymbol{\mu} \in \Gamma(\mu_0, \mu_1).$$

This notion (systematically developed in [4]) is stronger than McCann displacement convexity [7] and enjoys significantly better stability and duality properties. A key input is that, for fixed $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, the map $\mu \mapsto [\mu, \nu]$ is totally displacement convex. This enables the definition of a Kantorovich–Legendre–Fenchel transform

$$\varphi^\star(\nu) := \sup_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} \Big\{ [\nu, \mu] - \varphi(\mu) \Big\},$$

and a biconjugation principle $\varphi = \varphi^{\star\star}$ for proper, lower semicontinuous, totally displacement convex functionals.

Duality and optimality conditions for W_2. The previous structure yields an intrinsic characterization of optimal Kantorovich potentials for W_2: they coincide with totally displacement convex functionals. More precisely, for every $\mathtt{M}, \mathtt{N} \in \mathscr{P}_2(\mathbb{R}^d)$ there exist conjugate totally convex potentials $\varphi, \varphi^\star$ such that

$$\int \varphi(\mu)\, \mathrm{d}\mathtt{M}(\mu) + \int \varphi^\star(\nu)\, \mathrm{d}\mathtt{N}(\nu) = [\![\mathtt{M}, \mathtt{N}]\!],$$

where $[\![\mathtt{M}, \mathtt{N}]\!]$ is the maximization functional associated with the identity

$$[\![\mathtt{M}_1, \mathtt{M}_2]\!] = \mathsf{W}_2^2(\mathtt{M}, \mathtt{N}) = \mathsf{M}_2^2(\mathtt{M}) + \mathsf{M}_2^2(\mathtt{N}) - 2\,[\![\mathtt{M}, \mathtt{N}]\!], \qquad \mathsf{M}_2^2(\mathtt{M}) := \int \mathsf{m}_2^2(\mu)\, \mathrm{d}\mathtt{M}(\mu)$$

$$[\![\mathtt{M}, \mathtt{N}]\!] := \max_{\Pi \in \Gamma(\mathtt{M},\mathtt{N})} \int [\mu, \nu]\, \mathrm{d}\Pi(\mu, \nu).$$

Optimal couplings can be equivalently described via *random coupling laws* on $\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$; their support is characterized by a *total subdifferential*, a closed subset of $\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ playing the role of a multivalued "probability vector field". Optimality admits a formulation in terms of total cyclical monotonicity.

Monge and strict Monge formulations. A central application concerns the Monge problem for W_2. If the total subdifferential of an optimal potential is single-valued for $\mathtt{M}$-a.e. μ, then the transport problem between $\mathtt{M}$ and $\mathtt{N}$ admits a *unique* optimal coupling, concentrated on a Monge map $\mathbf{F} : \mathcal{P}_2(\mathbb{R}^d) \to \mathcal{P}_2(\mathbb{R}^d)$, i.e. $\Pi = (\mathrm{id} \times \mathbf{F})_\# \mathtt{M}$, and

$$\mathsf{W}_2^2(\mathtt{M}, \mathtt{N}) = \int \mathsf{w}_2^2(\mu, \mathbf{F}(\mu)) \, \mathrm{d}\mathtt{M}(\mu).$$

Moreover, the map $\mathbf{F}$ is induced by the distinguished minimal selection of the total subdifferential of an optimal Kantorovich potential φ, represented by a deterministic, nonlocal field $f^\circ : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ through $\mathbf{F}(\mu) = f^\circ(\cdot, \mu)_\# \mu$; in particular f° solves a strict Monge formulation where one optimizes directly over such fields.

Lagrangian (Hilbertian) lifting and super-regularity. The crucial issue is to provide conditions on $\mathtt{M}$ ensuring $\mathtt{M}$-a.e. single-valuedness (see [5] for a different type of conditions). Total displacement convexity is precisely the condition allowing a Lions-type Lagrangian lifting to a Hilbert space. Fix a nonatomic standard Borel probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and set $\mathcal{H} := L^2(\Omega, \mathbb{P}; \mathbb{R}^d)$. Every $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ can be represented as $\mu = X_\# \mathbb{P}$ for some $X \in \mathcal{H}$. The law map $\iota : X \mapsto X_\# \mathbb{P}$ is 1-Lipschitz and surjective, and total displacement convexity of φ is equivalent to linear convexity of $\widehat{\varphi} := \varphi \circ \iota$ on $\mathcal{H}$. This correspondence extends to subdifferentials, so that points where the total subdifferential is single-valued correspond to points of Gâteaux differentiability of $\widehat{\varphi}$.

We introduce a natural class of *super-regular* measures $\mathtt{M}$ on $\mathcal{P}_2(\mathbb{R}^d)$ obtained by combining: (R1) concentration on the usual class of regular measures (giving null mass to d.c. hypersurfaces in $\mathbb{R}^d$), and (R2) a lifted negligibility condition excluding sets whose preimage under ι lies in a d.c. hypersurface of $\mathcal{H}$. These conditions are stable under absolute continuity. Since d.c. hypersurfaces in Hilbert spaces are Gaussian-null, a canonical source of examples is provided by *laws of Gaussian-generated random measures* (LGGRM), obtained as $\mathtt{G} = \iota_\# \gamma$ where γ is a nondegenerate Gaussian measure on $\mathcal{H}$. Such measures have full support in $\mathcal{P}_2(\mathbb{R}^d)$ and satisfy (R2); in particular, in dimension $d = 1$ every LGGRM is super-regular, and broad super-regular families can be constructed in all dimensions. As a byproduct, super-regular measures are dense in $\mathscr{P}_2(\mathbb{R}^d)$.

References

[1] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed., Birkhäuser, 2008.

[2] N. Aronszajn, *Differentiability of Lipschitzian mappings between Banach spaces*, Studia Math. **57** (1976), 147–190.

[3] Y. Brenier, *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math. **44** (1991), no. 4, 375–417.

[4] G. Cavagnari, G. Savaré, and G. Sodini, *A Lagrangian approach to totally dissipative evolutions in Wasserstein spaces*, arXiv.2305.05211 (2023).

[5] P. Emami and B. Pass, *Optimal transport with optimal transport cost: the Monge–Kantorovich problem on Wasserstein space*, Calc. Var. Partial Differential Equations, Calculus of Variations and Partial Differential Equations, **64** (2025).

[6] R. R. Phelps, *Gaussian null sets and differentiability of Lipschitz maps on Banach spaces*, Pacific J. Math. **77** (1978), no. 2, 523–531.

[7] C. Villani, *Optimal Transport: Old and New*, Grundlehren der Mathematischen Wissenschaften, Vol. 338, Springer, 2009.

Optimal transport and gradient flows for point processes

Matthias Erbar

(joint work with Martin Huesmann, Jonas Jalowy, Bastian Müller, Hanna Stange)

We develop a theory of optimal transport for stationary random measures with a focus on stationary point processes. We construct two types of distances on stationary probability measures on the configuration space geared towards different dynamics of point configurations. The first one is based on continuous displacement of points and leads to a two layer optimal transport problem with a probabilitstic constraint. It gives rise to a natural notion of displacement interpolation between point processes and provides e.g. a characterisation of the evolution of infinitely many Brownian motions as the gradient flow of the specific relative entropy w.r.t. the Poisson point process. The second notion is based on a birth-death dynamic of points and modelled on the Benamou-Brenier formula. In a perturbative regime around the Poisson process we obtain an EVI gradient flow characterisation of the associated Glauber dynamics.

We consider random measures $\xi^\bullet$, i.e. random variables with values in the space $\mathcal{M}(\mathbb{R}^d)$ of locally finite measures on $\mathbb{R}^d$. $\xi^\bullet$ is a point process if it almost surely takes values in the $\mathbb{N}$. The distribution of a random measure is an element of $\mathcal{P}\big(\mathcal{M}(\mathbb{R}^d)\big)$ the set of probability measures over $\mathcal{M}(\mathbb{R}^d)$. $\mathbb{R}^d$ naturally acts on $\mathcal{M}(\mathbb{R}^d)$ by shift of the support and we say that $\mathsf{P} \in \mathcal{P}\big(\mathcal{M}(\mathbb{R}^d)\big)$ is *stationary* if it conicides with its image measure under the shift operation by any vector $x \in \mathbb{R}^d$. Stationarity of the distribution of a random measure is implied by the following stronger property. A random measure $\xi^\bullet : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathcal{M}(\mathbb{R}^d)$ is called *invariant* if the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ admits a measurable flow, i.e. a family of measurable mappings $\mathbb{R}^d \times \Omega \ni (x, \omega) \mapsto \theta_x \omega \in \Omega$ with $\theta_0 = \mathsf{id}$ and $\theta_x \circ \theta_y = \theta_{x+y}$ for all $x, y \in \mathbb{R}^d$, such that $\mathbb{P}$ is invariant under θ and for all $x \in \mathbb{R}^d$, $\omega \in \Omega$, we have $\xi^\omega(\cdot) = \xi^{\theta_x \omega}(\cdot - x)$.

On the space $\mathcal{M}(\mathbb{R}^d)$ we introduce a cost function that measures the asymptotic transport cost per volume:

$$(1) \qquad c(\xi,\eta) = \inf_{\mathsf{q}\in\mathsf{cpl}(\xi,\eta)} \limsup_{n\to\infty} \frac{1}{n^d}\int_{\Lambda_n\times\mathbb{R}^d} |x-y|^p \mathsf{q}(dx,dy) ,$$

where $\mathsf{cpl}(\xi,\eta)$ denotes the set of all couplings $\mathsf{q}\in\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)$ of ξ and η and $\Lambda_n = [-n/2,n/2]^d$ denotes the box of side length n centered at the origin. Given $\mathsf{P}_0,\mathsf{P}_1 \in \mathsf{P}\big(\mathcal{M}(\mathbb{R}^d)\big)$ two distributions of stationary random measures with unit intensity we consider the transport problem

$$(2) \qquad \mathsf{C}(\mathsf{P}_0,\mathsf{P}_1) = \inf_{\mathsf{Q}\in\mathsf{Cpl}_s(\mathsf{P}_0,\mathsf{P}_1)} \int c(\xi,\eta)\mathsf{Q}(d\xi,d\eta) = \inf_{(\xi^\bullet,\eta^\bullet)} \mathbb{E}\big[c(\xi^\bullet,\eta^\bullet)\big] ,$$

where the first infimum is taken over the set of stationary couplings of P_0 and P_1 and the second infimum is taken over all pairs of jointly invariant random measures $(\xi^\bullet,\eta^\bullet)$ with distributions P_0 and P_1 respectively.

Note that (2) is a two layer optimisation problem. The first layer of optimisation is on the level of the coupling of the distributions, the second layer is on the level of the spatial coupling of the realisations of the random measures in the transport problem defining c. Moreover, it is an optimal transport problem with an additional probabilistic constraint, namely stationarity. Alternatively, the value of the transport problem can be represented as

$$(3) \qquad \mathsf{C}(\mathsf{P}_0,\mathsf{P}_1) = \inf_{(\xi^\bullet,\eta^\bullet)} \mathsf{cost}(\xi^\bullet,\eta^\bullet).$$

where the infimum runs over all jointly invariant random measures $(\xi^\bullet,\eta^\bullet)$ such that $\xi^\bullet \sim \mathsf{P}_0, \eta^\bullet\sim\mathsf{P}_1$. Moreover, if $\mathsf{C}(\mathsf{P}_0,\mathsf{P}_1)<\infty$, there exists an optimal pair $(\mathsf{Q},\mathsf{q}^\bullet)$ of a coupling Q of $\mathsf{P}_0,\mathsf{P}_1$ and $\mathsf{q}^\bullet$ achieving the infimum. We fix $p\geq 1$ in (1) and put $\mathsf{W}_p := \mathsf{C}^{\frac{1}{p}}$.

Theorem 1. W_p *defines a geodesic extended distance on the space of stationary distributions* $\mathcal{P}_s(\mathcal{M}(\mathbb{R}^d))$ *with unit intensity.*

For P_0 and P_1 with $\mathsf{W}_p(\mathsf{P}_0,\mathsf{P}_1)<\infty$ and an optimal pair $(\mathsf{Q},\mathsf{q}^\bullet)$ the geodesic $(\mathsf{P}_a)_{a\in[0,1]}$ is given as follows. Let $\xi_a^\bullet := (\mathsf{geo}_a)_\#\mathsf{q}^\bullet$ be the random measure interpolating the points of $\xi_0^\bullet\sim\mathsf{P}_0$ and $\xi_1^\bullet\sim\mathsf{P}_1$ along straight lines, i.e. $\mathsf{geo}_a(x,y) = x+a(y-x)$, and put $\mathsf{P}_a = \mathsf{law}(\xi_a^\bullet)$.

The novel geometry induced by the transport distance W_2 can be used to study functionals on stationary point processes and infinite particle dynamics. We consider the *specific relative entropy* of a stationary point process P with respect to the Poisson point process Poi given by

$$\mathcal{E}(\mathsf{P}) := \limsup_{n\to\infty}\frac{1}{n^d}\mathsf{Ent}(\mathsf{P}_{\Lambda_n}|\mathsf{Poi}_{\Lambda_n}) ,$$

where P_{Λ_n} denotes the restriction of P to Λ_n. For a stationary point process P, let P_t be the point process obtained by evolving each point of P by an independent Brownian motion for time t. Our second main result is the following

Theorem 2. *$(\mathsf{P}_t)_t$ is the gradient flow of $\mathcal{E}$ wr.t. W_2 in the sense of the following* Evolution Variational Inequality*: For any stationary point processes* R *with* $\mathsf{W}_2(\mathsf{P},\mathsf{R})<\infty$ *we have*

$$\mathsf{W}_2^2(\mathsf{P}_t,\mathsf{R})-\mathsf{W}_2^2(\mathsf{P},\mathsf{R})\le 2t\big[\mathcal{E}(\mathsf{R})-\mathcal{E}(\mathsf{P}_t)\big]\ .$$

In particular, $\mathcal{E}$ is convex along W_2-geodesics.

This result can be seen as the analogue for stationary point processes of the celebrated results by Jordan, Kinderlehrer and Otto that the heat flow is the Wasserstein gradient flow of the Boltzmann entropy as well as McCann that the Boltzmann entropy is displacement convex. —n dimension $d=1$, similar characterisations of dynamics with interaction have been obtained by Huesmann and Müller in [2] using a variant of the transport problem.

In [3] we consider birth-death dynamics on the configuration space described by the semigroup (S_t) with generator

$$L_\kappa F(\xi)=\int_{\mathbb{R}^d}\kappa(\xi,x)\big[F(\xi+\delta_x)-F(\xi)\big]dx+\sum_{x\in\xi}\big[F(\xi-\delta_x)-F(\xi)\big]\ ,$$

where $\kappa:\Gamma(\mathbb{R}^d)\times\mathbb{R}^d\to[0,1]$ is a Papangelou intensity. A particular case are Gibbs measures with Hamiltonian H, activity z and inverse temperature β where, with $D_x^+F(\xi):=F(\xi+\delta_x)-F(\xi)$, we have $\kappa(\xi,x)=z\exp\big(-\beta D_x^+H(\xi)\big)$. We assume that κ satisfies monotonicity $\kappa(\xi+\delta_x,y)\le\kappa(\xi,y)$ and the cocycle condition $\kappa(\xi,x)\kappa(\xi+\delta_x,y)=\kappa(\xi,y)\kappa(\xi,\delta_y,x)$ for all $\xi\in\Gamma,x,y\in\mathbb{R}^d$, and admits a invariant Papangelou measure $\mathsf{R}\in\mathcal{P}\big(\Gamma(\mathbb{R}^d)\big)$, i.e. $\iint F(\xi-\delta_x)\xi(dx)\mathsf{R}(d\xi)=\iint F(\xi,x)\kappa(\xi,x)dx\mathsf{R}(d\xi)$ for all $F\ge 0$. We further assume that κ is $\mathsf{R}\otimes\mathcal{L}$-a.e. continuous.

For point processes $\mathsf{P}_0,\mathsf{P}_1$ We construct the non-local transport distance

$$\mathcal{W}_\kappa^2(\mathsf{P}_0,\mathsf{P}_1)=\inf\left\{\int_0^1\int_{\Gamma\times\mathbb{R}^d}\frac{|V_t|^2}{\theta\big(\kappa(\mathsf{P}_t\otimes\mathcal{L},\mathsf{C}_{\mathsf{P}_t}\big)}dt:(\mathsf{P},V)\in\mathsf{C}E(\mathsf{P}_0,\mathsf{P}_1)\right\}\ ,$$

where $\theta(a,b)=(a-b)/(\log a-\log b)$ is the logarithmic mean, C_t is the reduced Campbell measure of P_t and the infimum is taken over all solutions $(\mathsf{P}_t,V_t)_{t\in[0,1]}$ to the continuity equation

$$\frac{d}{dt}\int Fd\mathsf{P}_t+\int D_x^+F(\xi,x)dV_t(\xi,x)=0\qquad\forall F\ ,$$

with curves (P_t) interpolating $\mathsf{P}_0,\mathsf{P}_1$ and families of measure V_t on $\Gamma\times\mathbb{R}^d$.

We then have the following result.

Theorem 3. *Assume that*

$$\varepsilon:=\sup_{\xi,x}\int_{\mathbb{E}=R^d}\kappa(\xi,y)-\kappa(\xi+\delta_x,y)dy<1\ .$$

Then the Glauber dynamis $\mathsf{P}_t=S_t\mathsf{P}_0$ is the gradient flow of the relative entropy $\mathsf{E}nt(\cdot|\mathsf{R})$ w.r.t. the Papangelou measure wr.t. $\mathcal{W}_\kappa$ in the sense of the following EVI*:*

For any Q *with* $\mathcal{W}_2(\mathsf{Q},\mathsf{R})<\infty$ *we have*

$$\frac{d}{dt}\frac{1}{2}\mathcal{W}_\kappa^2(\mathsf{P}_t,\mathsf{R})+(1-\varepsilon)\frac{1}{2}\mathcal{W}_\kappa^2(\mathsf{P}_t,\mathsf{Q})\le \mathsf{Ent}(Q|\mathsf{R})-\mathsf{Ent}(\mathsf{P}_t|\mathsf{R})\ .$$

References

[1] M. Erbar, M. Huesmann, J. Jalowy, B. Müller, *Optimal transport of stationary point processes: Metric structure, gradient flow and convexity of the specific entropy*, J. Funct. Anal. 289 (2025).

[2] M. Huesmann, B. Müller, *Gap metrics for stationary point processes and quantitative convexity of the free energy*, arXiv:2509.08659 (2025).

[3] M. Erbar, M. Huesmann, H. Stange, *Wasserstein geometry for Papangelou spaces*, in preparation.

Discrete-time gradient flows in Gromov hyperbolic spaces

Shin-ichi Ohta

The theory of gradient flows for convex functions on metric spaces has made impressive progress since 1990s, including those on CAT(0)-spaces [7], CAT(1)-spaces [13], Alexandrov spaces and the Wasserstein spaces over them [6, 8, 15, 16], and RCD(K,∞)-spaces [17]. These spaces are, however, all "Riemannian" in the sense that they exclude non-Riemannian Finsler manifolds (in particular, non-inner product normed spaces). In fact, despite great success for Riemannian spaces, much less is known for non-Riemannian spaces (even for normed spaces); we refer to [11, 14] for the failure of the contraction property. Motivated by this large gap and a fact that some non-Riemannian Finsler manifolds can be Gromov hyperbolic, we shall consider convex optimization on Gromov hyperbolic spaces (see also [9] for a related study on barycenters in Gromov hyperbolic spaces).

The *Gromov hyperbolicity*, introduced in a seminal work [2] of Gromov, is a notion of negative curvature of large scale. A metric space (X,d) is said to be *Gromov hyperbolic* if it is δ*-hyperbolic* for some $\delta\ge 0$ in the sense that

$$(x|z)_p\ge\min\{(x|y)_p,(y|z)_p\}-\delta \tag{1}$$

holds for all $p,x,y,z\in X$, where

$$(x|y)_p:=\frac{1}{2}\{d(p,x)+d(p,y)-d(x,y)\}$$

is the *Gromov product*. If (1) holds with $\delta=0$, then the quadruple p,x,y,z is isometrically embedded into a tree.

In the investigation of gradient flows in Gromov hyperbolic spaces, we employ discrete-time gradient flows of large time step τ ("giant steps"), because of the inevitable local perturbations. Precisely, for a convex function $f\colon X\longrightarrow\mathbb{R}$ and an arbitrary initial point $x_0\in X$, we study the behavior of recursive applications of the *proximal* (or *resolvent*) *operator*:

$$x_k\in\mathsf{J}^f_\tau(x_{k-1}):=\operatorname*{arg\,min}_{y\in X}\left\{f(y)+\frac{d^2(x_{k-1},y)}{2\tau}\right\},\quad k\in\mathbb{N}. \tag{2}$$

The resulting sequence $x_0, x_1, x_2, \ldots$ can be regarded as a discrete approximation of a (continuous-time) gradient curve for f starting from x_0. In [10], we established the following.

Theorem 1 (Theorem 1.1 in [10] with $K = 0$)**.** *Let (X, d) be a proper, geodesic δ-hyperbolic space and $f\colon X \longrightarrow \mathbb{R}$ be an L-Lipschitz convex function such that $f(p) = \inf_X f$ for some $p \in X$. Then, for any $x \in X$ and $y \in \mathsf{J}^f_\tau(x)$ with $\tau > 0$, we have*

$$d(p, y) \le d(p, x) - d(x, y) + 4\sqrt{2\tau L\delta}.$$

Theorem 2 (Theorem 1.3 in [10] with $K = 0$)**.** *Let (X, d) and f be as in Theorem 1. Take any $x_1, x_2 \in X$, $\tau > 0$, and $y_i \in \mathsf{J}^f_\tau(x_i)$ for $i = 1, 2$, and assume $d(p, y_1) \le d(p, y_2)$.*

(i) *If $d(p, y_1) \ge (x_1|x_2)_p$, then we have*

$$d(y_1, y_2) \le d(x_1, x_2) - d(x_1, y_1) - d(x_2, y_2) + 20\sqrt{2\tau L\delta} + 24\delta.$$

(ii) *If $d(p, y_1) < (x_1|x_2)_p$, then we have*

$$d(y_1, y_2) \le d(x_1, x_2) - (p|x_2)_{x_1} + C(L, D, \tau, \delta),$$

where $D := \max\{d(p, x_1), d(p, x_2)\}$, $C(L, D, \tau, \delta) = O_{L,D,\tau}(\delta^{1/4})$ as $\delta \to 0$.

In Theorem 1, it is in fact sufficient to assume $f(p) \le f(y)$. The estimates in Theorem 2 can be thought of as contraction properties akin to tress.

Next, inspired by Hirai–Sakabe's recent work [3], we consider the case where the target convex function is unbounded below, and study divergence to the steepest direction in the boundary at infinity ∂X. This kind of phenomenon was established by Karlsson–Margulis [5] for *semi-contractions* $\varphi\colon Y \longrightarrow Y$ (i.e., $d_Y(\varphi(x), \varphi(y)) \le d_Y(x, y)$) on nonpositively curved metric spaces (Y, d_Y). Their *multiplicative ergodic theorem* [5, Theorem 2.1] asserts that, for a complete, uniformly convex metric space (Y, d_Y) of nonpositive curvature in the sense of Busemann and almost every $y_0 \in Y$, there exists a ray $\xi\colon [0, \infty) \longrightarrow Y$ satisfying

$$\lim_{k \to \infty} \frac{d(\xi(\alpha k), y_k)}{k} = 0, \quad \text{where } y_k := \varphi(y_{k-1}),$$

provided that $\alpha := \lim_{k\to\infty} d(y_0, y_k)/k > 0$. We refer to [4, Proposition 5.1] for a generalization to semi-contractions on Gromov hyperbolic spaces.

The results in [3] extract and discretize convex optimization ingredients of the *moment-weight inequality* for reductive group actions by Georgoulas–Robbin–Salamon [1] in geometric invariant theory, and have applications to operator scaling problems. In Gromov hyperbolic spaces, although the proximal operator is not a semi-contraction, we obtain the following by using Theorem 1.

Theorem 3 (Theorem 1.1 in [12])**.** *Let (X, d) be a proper, geodesic δ-hyperbolic space with $\partial X \neq \emptyset$, and $f\colon X \longrightarrow \mathbb{R}$ be an L-Lipschitz convex function. Assume $\alpha := -\inf_{v\in\partial X} \partial_\infty f(v) > 0$. Then,*

(i) *there exists a unique element $v_* \in \partial X$ satisfying $\partial_\infty f(v_*) < 0$;*

(ii) *for any* $x_0 \in X$*, the discrete-time gradient curve* $(x_k)_{k\in\mathbb{N}}$ *as in* (2) *tends to* $v_* \in \partial X$*;*

(iii) *letting* $\xi\colon [0,\infty) \longrightarrow X$ *be a ray representing* v_* *with* $\xi(0) = x_0$*, we have*

$$(\xi(t)|x_k)_{x_0} \geq k\sqrt{\tau}\left\{\left(\frac{\alpha^2}{4L} + \frac{\alpha^4}{16L^3}\right)\sqrt{\tau} - 2\sqrt{2L\delta}\right\} \tag{3}$$

for all $k \in \mathbb{N}$ *and* $t > 0$ *satisfying* $f(\xi(t)) \leq f(x_k)$*.*

Here, $\partial_\infty f(v)$ is the *asymptotic slope* defined by

$$\partial_\infty f(v) := \lim_{t\to\infty} \frac{f(\xi(t))}{t}$$

for a ray $\xi\colon [0,\infty) \longrightarrow X$ representing $v \in \partial X$. We remark that the uniqueness in (i) is a specific feature of the negative curvature; it is not the case for CAT(0)-spaces. The last assertion (3) can be regarded as an estimate of the rate of convergence of x_k to v_*.

Let us close with two further problems.

(A) Gromov hyperbolicity makes sense for discrete spaces as well. Therefore, it is interesting to explore some generalizations of the results in [10, 12] to discrete (non-geodesic) Gromov hyperbolic spaces. Then, it is a challenging problem to formulate and analyze convex functions on discrete Gromov hyperbolic spaces (possibly for some special classes such as hyperbolic groups).

(B) Even in geodesic Gromov hyperbolic spaces, it is worth considering a certain "large-scale convexity" of functions, preserved by *quasi-isometries*, since Gromov hyperbolicity is preserved by quasi-isometries between geodesic spaces.

References

[1] V. Georgoulas, J. W. Robbin and D. A. Salamon, *The moment-weight inequality and the Hilbert–Mumford criterion—GIT from the differential geometric viewpoint*, Lecture Notes in Math., **2297**, Springer, Cham, 2021.

[2] M. Gromov, *Hyperbolic groups*, Essays in group theory, 75–263, Math. Sci. Res. Inst. Publ., **8**, Springer, New York, 1987.

[3] H. Hirai and K. Sakabe, *Gradient descent for unbounded convex functions on Hadamard manifolds and its applications to scaling problems*, 65th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2024, pages 2387–2402. IEEE, 2024.

[4] A. Karlsson, *Non-expanding maps and Busemann functions*, Ergodic Theory Dynam. Systems **21** (2001), 1447–1457.

[5] A. Karlsson and G. Margulis, *A multiplicative ergodic theorem and nonpositively curved spaces*, Comm. Math. Phys. **208** (1999), 107–123.

[6] A. Lytchak, *Open map theorem for metric spaces*, St. Petersburg Math. J. **17** (2006), 477–491.

[7] U. F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Comm. Anal. Geom. **6** (1998), 199–253.

[8] S. Ohta, *Gradient flows on Wasserstein spaces over compact Alexandrov spaces*, Amer. J. Math. **131** (2009), 475–516.

[9] S. Ohta, *Barycenters and a law of large numbers in Gromov hyperbolic spaces*, Rev. Mat. Iberoam. **40** (2024), 1185–1206.

[10] S. Ohta, *Discrete-time gradient flows in Gromov hyperbolic spaces*, Israel J. Math. (to appear), available at arXiv:2205.03156

[11] S. Ohta, *How does the contraction property fail for convex functions on normed spaces?*, Tohoku Series in Mathematical Sciences (to appear), available at arXiv:2311.15152
[12] S. Ohta, *Discrete-time gradient flows for unbounded convex functions on Gromov hyperbolic spaces*, Commun. Contemp. Math. (to appear), available at arXiv:2503.19652
[13] S. Ohta and M. Pálfia, *Gradient flows and a Trotter–Kato formula of semi-convex functions on CAT(1)-spaces*, Amer. J. Math. **139** (2017), 937–965.
[14] S. Ohta and K.-T. Sturm, *Non-contraction of heat flow on Minkowski spaces*, Arch. Ration. Mech. Anal. **204** (2012), 917–944.
[15] A. Petrunin, *Semiconcave functions in Alexandrov's geometry*, Surveys in differential geometry. Vol. XI, 137–201, Surv. Differ. Geom., **11**, Int. Press, Somerville, MA, 2007.
[16] G. Savaré, *Gradient flows and diffusion semigroups in metric spaces under lower curvature bounds*, C. R. Math. Acad. Sci. Paris **345** (2007), 151–154.
[17] K.-T. Sturm, *Gradient flows for semiconvex functions on metric measure spaces—existence, uniqueness, and Lipschitz continuity*, Proc. Amer. Math. Soc. **146** (2018), 3985–3994.

Curvature-Dimension for Autonomous Lagrangians

Rotem Assouline

The characterization of lower Ricci curvature and upper dimension bounds using optimal transport, which finds its roots in the works of Cordero-Erausquin, McCann and Schmuckenschläger [11], Sturm [24] and Lott-Villani [16], served as a point of departure for a rich theory of synthetic curvature-dimension bounds on metric measure spaces. A prototypical instance of this equivalence is the following: a complete n-dimensional Riemannian manifold (M,g) has nonnegative Ricci curvature if and only if for all $N \in [n,\infty]$, the function $t \mapsto S_N[\mu_t|\mathrm{Vol}_g]$ is convex for every 2-Wasserstein geodesic μ_t, where $S_N[\nu|\mu]$ is the N-entropy

$$S_N[\nu|\mu] := \begin{cases} \int_M (d\nu/d\mu)^{-1/N}\, d\nu & N<\infty, \\ \int_M \log(d\nu/d\mu)\, d\nu, & N=\infty. \end{cases}$$

More generally, if μ is a measure on M with a smooth density, then $S_N[\mu_t\,|\,\mu]$ is convex for every 2-Wasserstein geodesic μ_t if and only if $\mathrm{Ric}_{g,\mu,N} \geq 0$, where $\mathrm{Ric}_{g,\mu,N} = \mathrm{Ric}_g + \mathrm{Hess}\psi - (N-n)^{-1}(d\psi)^2$ is the weighted (Bakry-Émery) Ricci tensor. The connection between curvature-dimension bounds and the aforementioned *displacement convexity of entropy* has since been established in other settings, including sub-Riemannian [3, 4], Finslerian [19] and Lorentzian [18, 9] manifolds. We present here a new extension, to weighted manifolds endowed with a Lagrangian. Displacement convexity in this setting was studied in Lee [15] Ohta [20], Schachter [23] and Yang [25]. However, a complete characterization such as the one stated below did not exist previously.

Let M be a smooth manifold, and let $L : TM \to \mathbb{R}$ be a function which is smooth away from the zero section $\mathbf{0} \subseteq TM$ and strictly convex and superlinear on each fiber (a *Tonelli Lagrangian*). Thus we consider Lagrangians that are time-independent, i.e. *autonomous*. The cost associated with the Lagrangian L is

$$\mathrm{c} : M \times M \to \mathbb{R}, \qquad \mathrm{c}(x,x') := \inf_\gamma \int L(\dot\gamma(t))dt,$$

where the infimum is over all piecewise-C^1 curves γ joining x to x'. A curve achieving the above infimum is called a *minimizing extremal.*

For a pair of Borel probability measures μ_0, μ_1 on M, we say that a measure κ on $M \times M$ is an *optimal coupling* between μ_0 and μ_1 if it minimizes the expected cost $\int \mathrm{c}\, d\kappa$ among all measures on $M \times M$ whose first and second marginals are μ_0 and μ_1, respectively. The existence and fundamental properties of optimal couplings for costs arising from Tonelli Lagrangians were established by Bernard-Buffoni [5] and Fathi-Figalli [12].

We impose the following rather natural global assumptions on L: (i) every pair of points is joined by at least one minimizing extremal, (ii) for every compact set A there exists a compact set $\widetilde{A} \supseteq A$ containing all minimizing extremals with endpoints in A, and (iii) for every closed curve γ we have $\int L(\dot\gamma) > 0$.

Write $\mathcal{P}_1(L)$ for the collection of absolutely continuous probability measures μ on M for which the function $|\mathrm{c}(x_0, \cdot)| + |\mathrm{c}(\cdot, x_0)|$ lies in $L^1(\mu)$ for some $x_0 \in M$.

Theorem ([5, 12, 1])**.** *Under the above assumptions, for every pair $\mu_0, \mu_1 \in \mathcal{P}_1(L)$ there exists a family $\{\mu_\lambda\}_{0<\lambda<1}$ of absolutely continuous probability measures with the following property: there exists a random minimizing extremal $\gamma : [0,T] \to M$ such that for every $0 \le \lambda \le \lambda' \le 1$, the joint law of $\gamma(\lambda T)$ and $\gamma(\lambda' T)$ is an optimal coupling between μ_λ and $\mu_{\lambda'}$.*

The family $\{\mu_\lambda\}$ in the above theorem is called a *displacement interpolation* between μ_0 and μ_1.

In [1], given a Lagrangian L as above, a measure μ on M with a smooth density, and $N \in [n, \infty]$, we construct a function $\mathrm{Ric}_{L,\mu,N} : TM \setminus \mathbf{0} \to \mathbb{R}$, generalizing the Riemannian and Finslerian weighted Ricci curvature, such that the following holds.

Theorem (A. '25 [1])**.** *Let M be a smooth manifold, let μ be a measure on M with a smooth density, let $L : TM \to \mathbb{R}$ be a Lagrangian satisfying the above assumptions, and let $N \in [n, \infty]$. Then the following conditions are equivalent:*

(1) $\mathrm{Ric}_{\mu,L,N} \ge 0$ *on the energy level* $E^{-1}(0)$.

(2) For every pair $\mu_0, \mu_1 \in \mathcal{P}_1(L)$ there exists a displacement interpolation $\{\mu_\lambda\}_{0<\lambda<1}$ such that the function $\lambda \mapsto S_N[\mu_\lambda|\mu]$ is convex.

Here the *energy* $E : TM \setminus \mathbf{0} \to \mathbb{R}$ is defined by $E = H \circ \mathcal{L}^{-1}$, where H is the Hamiltonian associated to the Lagrangian L and $\mathcal{L} : T^*M \to TM$ is the Legendre transform. For instance, for a Riemannian metric g we take $L = (g+1)/2$, and then $E^{-1}(0)$ is the unit tangent bundle. Any bound $\mathrm{Ric}_{\mu,L,N} \ge K \in \mathbb{R}$ can be similarly characterized by displacement convexity, with a slightly more complicated statement than the one above.

It is important to remark that the optimal transport problem we consider is analogous to L^1, rather than L^2, optimal transport in the metric setting. For this reason, the theorem asserts the *existence* of a displacement interpolation along which entropy is convex. Indeed, the L^1 displacement interpolation may not be unique. L^1-Displacement convexity in the metric case was established in [7, 10].

Our main technical tool is the *needle decomposition technique*. This technique was introduced by Klartag [14], who employed L^1 mass transport to construct a Riemannian counterpart to a powerful localization method originating in the works of Payne-Weinberger [22], Gromov-Milman [13] and Lovasz-Simonovitz [17]. The needle decomposition technique has since been extended to various other settings [21, 8, 6].

The quantity $\mathrm{Ric}_{L,\mu,N}$ can be computed explicitly in several cases, notably that of *magnetic* Lagrangians, in which the Lagrangian is the sum of a Riemannian metric and a 1-form. By analyzing the weighted Ricci curvature of a particular magnetic Lagrangian on the *complex hyperbolic space* $\mathbb{C}\mathbf{H}^d$, and using our adaptation of the needle decomposition technique to the Lagrangian setting, we were able to prove the following higher-dimensional generalization of the horocyclic Brunn-Minkowski inequality on the hyperbolic plane [2]:

Theorem (Horocyclic Brunn-Minkowski inequality in complex hyperbolic space)**.** *Let $A_0, A_1 \subseteq \mathbb{C}\mathbf{H}^d$ be Borel sets of positive measure and let $0 \le \lambda \le 1$. Denote by A_λ the set of points of the form $\gamma(\lambda)$, where $\gamma : [0,1] \to \mathbb{C}\mathbf{H}^d$ is a constant-speed horocycle contained in a single complex geodesic and satisfying $\gamma(0) \in A_0$ and $\gamma(\ell) \in A_1$. Then*

$$\mathrm{Vol}(A_\lambda)^{1/n} \ge (1-\lambda) \cdot \mathrm{Vol}(A_0)^{1/n} + \lambda \cdot \mathrm{Vol}(A_1)^{1/n},$$

where Vol *denotes the hyperbolic volume measure and $n = 2d$.*

The interest in the above theorem lies in the fact that the Brunn-Minkowski inequality has the same form as the classical one in Euclidean space, even though complex hyperbolic space has negative Ricci curvature. Thus, loosely speaking, interpolation with horocycles rather than geodesics compensates for the negative Ricci curvature of complex hyperbolic space. In [1] we also obtain a Brunn-Minkowski inequality for contact magnetic geodesics on odd-dimensional spheres.

References

[1] Assouline. arXiv:2409.08001.
[2] Assouline, Klartag. Adv. Math. 436 (2024), Paper No. 109381, 39 pp.
[3] Balogh, Kristály, Sipos. C. R. Math. Acad. Sci. Paris 354 (2016), no. 9, 916–919.
[4] Barilari, Mondino, Rizzi. arXiv:2211.07762.
[5] Bernard, Buffoni. Adv. Math. 207 (2006), no.2, 691–706.
[6] Braun, McCann. arXiv:2312.17158.
[7] Cavalletti, Milman. Invent. Math. 226 (2021), no. 1, 1–137.
[8] Cavalletti, Mondino. Geom. Topol. 21(2017), no.1, 603–645.
[9] Cavalletti, Mondino. Camb. J. Math. 12 (2024), no.2, 417–534.
[10] Cavalletti, Gigli, Santarcangelo. Ann. Fac. Sci. Toulouse Math. (6) 30 (2021), no. 2, 411–427.
[11] Cordero-Erausquin, McCann, Schmuckenschläger. Invent. Math. 146 (2001), no.2, 219–257.
[12] Fathi, Figalli. Israel J. Math. 175 (2010), 1–59.
[13] Gromov, Milman. Compositio Math., 62, 3, (1987), 263–282.
[14] Klartag. Mem. Amer. Math. Soc. 249 (2017), no.1180, v+77 pp.

[15] Lee. Funct. Anal. 265 (2013), no.12, 3163–3203.
[16] Lott, Villani. Ann. of Math. (2) 169 (2009), no.3, 903–991.
[17] Lovász, Simonovits. Random Structures Algorithms, 4, 4, (1993), 359–412.
[18] McCann. Camb. J. Math. 8 (2020), no. 3, 609–681.
[19] Ohta. Calc. Var. Partial Differential Equations 36 (2009), no. 2, 211–249.
[20] Ohta. Anal. Geom. Metr. Spaces 2 (2014), no.1, 81–114.
[21] Ohta. J. Math. Soc. Japan 70 (2018), no.2, 651–693.
[22] Payne, Weinberger. Arch. Rational Mech. Anal. 5 (1960), 286–292.
[23] Schachter (PhD Thesis). University of Toronto (Canada), 2017.
[24] Sturm. Acta Math. 196 (2006), no.1, 133–177.
[25] Yang. Calc. Var. Partial Differential Equations 62 (2023), no. 7, Paper No. 206, 20 pp.

On the geometry of Gaussian free fields in dimension $d > 2$

Linan Chen
(joint work with Louis Meunier)

Gaussian free fields (GFFs), as natural analogs of Brownian motion with multi-dimensional time parameters, are sample points under a centered Gaussian measure on a space of functions (or generalized functions). In the 2D case, due to its origin in quantum mechanics and statistical physics, the *log-correlated* GFF (GFF) in 2D has been extensively studied in recent years. In particular, as a promising model of random surface, the study of GFF has lead to rich developments in the realm of 2D random geometry. However, much less is known in higher dimensions. As an attempt to explore random geometry in dim $d > 2$, we consider a class of GFFs and discuss some problems arising from the study of geometric characteristics of these models.

Formally speaking, an *instance* of the d-dimensional GFF, denoted by θ, is a random "function" on $\mathbb{R}^d$ such that $\{\theta(x) : x \in \mathbb{R}^d\}$ forms a centered Gaussian family with the following covariance:

$$\mathbb{E}[\theta(x)\theta(y)] = K_{I-\Delta}(x, y) \quad \text{for } x, y \in \mathbb{R}^d,$$

where $K_{I-\Delta}(\cdot, \cdot)$ is the kernel of $(I - \Delta)^{-1}$ on $\mathbb{R}^d$. Due to the singularity of $K_{I-\Delta}(\cdot, \cdot)$ along the diagonal, the instance θ is a.s. **not** a point-wisely defined function, but a generalized function instead. When $d \geq 3$, since the kernel has polynomial singularity ($K_{I-\Delta}(x, y) \sim |x - y|^{2-d}$ near the diagonal), the GFF is said to be *polynomial-correlated*; moreover, the higher d is, the more singular θ becomes. Therefore, in order to treat such singular GFFs analytically, it is necessary to apply a regularization procedure. The specific regularizations we adopt are all based on the "averages" of θ: for $x \in \mathbb{R}^d$ and $t > 0$,

(1) spherical average: $\theta^\circ_t(x)$:=average of θ over $\partial B(x, t)$,
(2) ball average: $\theta^\bullet_t(x)$:=average of θ over $B(x, t)$,
(3) bump-function average: $\theta^f_t(x) := \theta * f_t(x)$ with $f_t := \frac{1}{t^d} f(\frac{\cdot}{t})$, where $f \in C^\infty_c(B(0, 1))$ is a radially symm. mollifier.

Aiming at studying the behavior of θ "at" x, we need to carefully examine the asymptotics of these *regularized* GFFs as $t \searrow 0$. In particular, the family of spherical averages $\{\theta_t^\circ(x) : x \in \mathbb{R}^d, t > 0\}$ satisfies technically favorable properties called the *harmonic conditions* [1], but the same cannot be said about the families of $\theta_t^\bullet(x)$ or $\theta_t^f(x)$.

Our work is most inspired by [3], in which the authors developed a **multifractality** result for the 2D log-correlated GFF[1] via studying the fractal dimensions of the *thick point* sets defined by θ_t°, and further proved the **conform invariance** for the thick point sets corresponding to $\theta_t^\bullet$.

Our work on the geometry of polynomial-correlated GFFs has two components: first, we extend the notion of "thick point" to the higher-dimensional models, for which we establish analogous results of multifractality and conformal invariance; second, we investigate the dependence of these results on the choice of regularization, which, in particular, leads to an analysis of the difference between θ_t° and $\theta_t^\bullet$. In both fronts, we observe both similarities and distinctions between the polynomial-correlated GFFs and the log-correlated counterparts. Heuristically speaking, due to higher level of singularity compared with the 2D case, the geometry of polynomial-correlated GFFs is more complex and untractable, with features that are not fully captured by fractal dimensions. Below we give a brief overview of the results we have obtained, with part of the work still ongoing.

For θ being an instance of the polynomial-correlated GFF on $\mathbb{R}^d$ for $d \geq 3$, given $\gamma \geq 0$, we define the γ-*thick point* set of θ to be the following subset of the unit cube $\overline{S(O,1)}$:

$$T_\gamma^\circ(\theta) := \left\{ x \in \overline{S(O,1)} : \limsup_{t \searrow 0} \frac{\theta_t^\circ(x)}{\sqrt{-\mathrm{Var}(\theta_t^\circ(x)) \ln t}} \geq \sqrt{2d\gamma} \right\}. \tag{1}$$

The harmonic conditions of θ_t° implies that, for every $\gamma > 0$, T_γ° is an exceptional set (i.e., T_γ° is a Lebesgue null set a.s.) and, as an analog of "extrema" for the generalized function θ, $T_\gamma^\circ(\theta)$ consists of locations where θ achieves "unusually" large values. We derive in [1] the explicit Hausdorff dimension of these thick point sets as: for $\gamma \in [0,1]$,

$$\dim_{\mathcal{H}}(T_\gamma^\circ) = d(1-\gamma) \text{ a.s.},$$

which confirms that the geometry of polynomial-correlated GFF is *multifractal*, i.e., possessing a continuous spectrum of fractal dimensions (instead of a single dimension). In an ongoing work [4], switching to the regularization based on the bump-function average θ_t^f and defining the thick point set $T_\gamma^f(t)$ as in (1) with "θ_t°" replaced by "θ_t^f", we have also obtained the *conformal invariance* of T_γ^f in the sense that, if $\Psi : U \to V$ is a conformal diffeomorphism between the two bounded domains $U, V \subseteq \mathbb{R}^d$ (assuming $U, V \supseteq \overline{S(O,1)}$), then a.s.,

$$x \in T_\gamma^f(\theta) \Leftrightarrow \Psi(x) \in T_\gamma^f(\theta \circ \Psi^{-1}) \text{ for all } \gamma \in [0,1].$$

These results extend the literature on thick point sets (and exceptional sets in general) from the 2D GFF case to higher dimensional settings.

[1] In [3], the 2D GFF is constructed using the Dirichlet Laplacian Δ on the unit disk $\mathbb{D}$, but this model of GFF has the same local behaviors as our model.

We have also obtained comparison results on the thick point sets defined via different regularizations, which shed light on a novel aspect in the study of random geometry associated with GFFs. We discover in [4] that, to the contrary of the 2D log-correlated case, the thick point sets of polynomial-correlated GFF vary "drastically" with the change of regularizations: since a.s. $(\theta_t^\circ(x) - \theta_t^\bullet(x))/\sqrt{-t^{2-d}\ln t}$ is non-trivial asymptotically as $t \searrow 0$, even the difference between T_γ° and $T_\gamma^\bullet$, where the latter is defined by (1) with "θ_t°" replaced by "$\theta_t^\bullet$", possesses non-trivial multifractal structures. This result is proven with the framework and the techniques we developed for another type of exceptional sets known as *steep point set* [2]. Similar phenomenon occurs to thick point sets under "spatial" transformation. Namely, in the aforementioned setup of T_γ^f under a diffeomorphism Ψ, if Ψ is not conformal, then a.s., T_γ^f and $\Psi(T_\gamma^f(\theta \circ \Psi^{-1}))$ contain vastly different elements.

These discoveries highlight the sensitivity (and hence the importance) of the choice of regularization in the study of singular GFFs, especially in the polynomial-correlated setting. Ultimately, we hope to develop understanding on the "intrinsic" properties of the geometry of GFFs, and one would expect that such properties can be formulated or described in a way that is universal for all choices of regularizations. However, the current finding suggests that, to identify these properties, we may need to develop novel tools that can capture geometric structures of GFFs at a more refined level.

References

[1] L. Chen, *Thick points of high-dimensional Gaussian free fields*, Annales. Institut Henri Poincaré **53** (2018), 1492–1526.

[2] L. Chen, *Steep Points of Gaussian free fields in any dimension*, Journal of Theoretical Probability Theory **34** (2021), 1959–2004.

[3] X. Hu, J. Miller, Y. Peres *Thick points of the Gaussian free field*, Annals of Probability **38** (2010), 896–926.

[4] L. Meunier, L. Chen, *Thick point sets of Gaussian free fields under transformations*, ongoing (2025+).

Multiple points on the boundaries of Brownian loop-soup clusters

Wei Qian

(joint work with Yifan Gao, Xinyi Li, Runsheng Liu)

1. Background: Multiple points in the planar Brownian motion

Lévy [16] pointed out, already in 1948, that the double points on the frontier of a planar Brownian motion are dense. Later, we got to know that the planar Brownian motion contains points of infinite multiplicity on its trace (Dvoretzky, Erdös and Kakutani [5] and Le Gall [15]). It was proved in 1996 by Burdzy and Werner [2] that there do not exist triple points on the frontier of a planar Brownian motion. In 2001, Lawler, Schramm and Werner [9] further computed the Hausdorff dimension of the Brownian frontier, which is 4/3, confirming the famous Mandelbrot's conjecture. Finally, it was proved by Kiefer and Mörters [8] that the dimension of double points on the Brownian frontier is $(\sqrt{97}+1)/24$.

The computation of the exact Hausdorff dimensions of the Brownian frontier and double points on the Brownian frontier relied on a so-called *Brownian disconnection exponent* $\eta(n)$. The value of $\eta(n)$ was first conjectured by Duplantier and Kwon [4] to be

$$\eta(n) = \frac{1}{48}\left(\left(\sqrt{24n+1}-1\right)^2 - 4\right). \tag{1}$$

The rigorous proof of (1) was achieved by Lawler, Schramm and Werner [10, 11, 13, 12] using the Schramm-Loewner evolutions (SLE).

2. Brownian loop soup and generalized disconnection exponent

The Brownian loop soup, introduced by Lawler and Werner [14], plays an important role in random conformal geometry, due to its close connection to the conformal loop ensembles (CLE), the Gaussian free fields (GFF), etc. The Brownian loop soup is a Poisson point process which depends on an intensity parameter $c > 0$. It was proved in [20] that for $c \in (0,1]$, the Brownian loop soup in the unit disk a.s. exhibits infinitely many clusters, while for $c > 1$ it contains a single cluster. Moreover, for $c \in (0,1]$, the outer boundaries of the outermost clusters are distributed as a CLE_κ, where

$$c(\kappa) = (6-\kappa)(3\kappa-8)/(2\kappa). \tag{2}$$

For $c \in (0,1]$, since the outer boundary of a Brownian loop has dimension $4/3$ and the CLE_κ loop has dimension $1+\kappa/8 > 4/3$ [1], we can deduce that most points on the outer boundary of a cluster are not visited by any Brownian loop in the loop soup. On the other hand, [19, Lemma 4] implies that there do exist some Brownian loops that touch the outer boundary of a cluster.

Whether there exist double points on the outer boundary of a cluster for $c \in (0,1]$ is far from obvious. We are particularly interested to answer this question for the critical intensity $c=1$, due to the following question raised in a joint work with Werner [19]. There, we prove that at $c=1$, the set of Brownian loops that touch the outer boundary of a cluster can be decomposed into a Poisson point process of Brownian excursions away from this boundary. It naturally leads to the question of how to reconnect the excursions back into loops. In particular, we would like to know if there is any additional randomness in the reconnection procedure. It was indicated in [21] that loops in the critical loop soup may exchange their trajectories at double points, without changing the global distribution of the loop soup. The double points on the boundary of a cluster in a critical loop soup would therefore be a potential source of randomness in this reconnection procedure.

Motivated by the aforementioned questions, we have defined in [18] the *generalized disconnection exponents* inside a loop soup with intensity $c \in (0,1]$, and computed their values

$$\eta_c(\beta) = \frac{1}{48}\left(\left(\sqrt{24\beta+1-c}-\sqrt{1-c}\right)^2 - 4\,(1-c)\right). \tag{3}$$

The computation of η_c relies on a family of radial hypergeometric SLE (hSLE) introduced in the same paper, which are the natural counterpart of the chordal hSLE's introduced in [17].

3. Multiple points on the boundary of the Brownian loop-soup clusters

In a joint work with Gao and Li [7], we prove that the dimensions of simple and double points on the boundary of a cluster in a loop soup with intensity $c \in (0, 1]$ are respectively equal to $2 - \eta_c(2)$ and $2 - \eta_c(4)$, also relying on [3] as an input. Note that $\eta_1(4) = 2$ by (3), so the dimension of double points on the boundary of a cluster in the critical loop soup is exactly zero. The computation of this dimension is therefore not sufficient to answer whether there exist double points on the boundary of a critical loop-soup cluster.

In a joint work with Gao, Li and Liu [6], we develop a unified approach to prove the non-existence of several random fractals related to the Brownian motion which have dimension exactly zero. In particular, we prove the non-existence of the pioneer triple points of the planar Brownian motion, the pioneer double cut points of the planar and three-dimensional Brownian motions, and the double points on the boundaries of the clusters of the planar Brownian loop soup at the critical intensity.

To illustrate our proof in the pioneer triple points (PTP) case, we consider the set $\mathfrak{S}_n$ of good boxes with side-length 2^{-n} in the unit disk which contain a δ-approximate PTP. It suffices to prove that for some $c > 0$,

$$\mathbb{P}(\#\mathfrak{S}_n \geq 1) = O(n^{-c}). \tag{4}$$

Our proof of (4) is based on the intuition that it is very likely to find many good boxes near a typical good box. More concretely, we will prove the following conditional negative first moment estimate for any given box S

$$\mathbb{E}\left[(\#\mathfrak{S}_n)^{-1} \mid S \in \mathfrak{S}_n\right] = O(n^{-c}),$$

which implies (4) through

$$\mathbb{P}(\#\mathfrak{S}_n \geq 1) = \sum_S \mathbb{E}\left[(\#\mathfrak{S}_n)^{-1} \mid S \in \mathfrak{S}_n\right] \mathbb{P}(S \in \mathfrak{S}_n).$$

References

[1] V. Beffara. The dimension of the SLE curves. *Ann. Probab.*, 36(4):1421–1452, 2008.

[2] K. Burdzy and W. Werner. No triple point of planar Brownian motion is accessible. *Ann. Probab.*, 24(1):125–147, 1996.

[3] G. Cai, Y. Gao. Uniqueness of generalized conformal restriction measures and Malliavin-Kontsevich-Suhov measures for $c \in (0, 1]$. *Preprint*, available at arXiv:2502.05890.

[4] B. Duplantier and K.-H. Kwon. Conformal invariance and intersections of random walks. *Phys. Rev. Lett.*, 61:2514–2517, Nov 1988.

[5] A. Dvoretzky, P. Erd″ os, and S. Kakutani. Points of multiplicity *germc* of plane Brownian paths. *Bull. Res. Council Israel Sect. F*, 7F:175–180, 1958.

[6] Y. Gao, X. Li, R. Liu, and W. Qian. Non-existence of several random fractals in the Brownian motion and the Brownian loop soup. *ArXiv e-prints*, 2025.

[7] Y. Gao, X. Li, and W. Qian. Multiple points on the boundaries of Brownian loop-soup clusters. *Annals of Probability, to appear*, 2022.
[8] R. Kiefer and P. Mörters. The Hausdorff dimension of the double points on the Brownian frontier. *J. Theoret. Probab.*, 23(2):605–623, 2010.
[9] G. F. Lawler, O. Schramm, and W. Werner. The dimension of the planar Brownian frontier is 4/3. *Math. Res. Lett.*, 8(4):401–411, 2001.
[10] G. F. Lawler, O. Schramm, and W. Werner. Values of Brownian intersection exponents. I. Half-plane exponents. *Acta Math.*, 187(2):237–273, 2001.
[11] G. F. Lawler, O. Schramm, and W. Werner. Values of Brownian intersection exponents. II. Plane exponents. *Acta Math.*, 187(2):275–308, 2001.
[12] G. F. Lawler, O. Schramm, and W. Werner. Analyticity of intersection exponents for planar Brownian motion. *Acta Math.*, 189(2):179–201, 2002.
[13] G. F. Lawler, O. Schramm, and W. Werner. Values of Brownian intersection exponents. III. Two-sided exponents. *Ann. Inst. H. Poincaré Probab. Statist.*, 38(1):109–123, 2002.
[14] G. F. Lawler and W. Werner. The Brownian loop soup. *Probab. Theory Related Fields*, 128(4):565–588, 2004.
[15] J.-F. c. Le Gall. Le comportement du mouvement brownien entre les deux instants où il passe par un point double. *J. Funct. Anal.*, 71(2):246–262, 1987.
[16] P. Lévy. *Processus Stochastiques et Mouvement Brownien. Suivi d'une note de M. Loève.* Gauthier-Villars, Paris, 1948.
[17] Wei Qian. Conformal restriction: the trichordal case. *Probab. Theory Related Fields*, 171(3-4):709–774, 2018.
[18] W. Qian. Generalized disconnection exponents. *Probab. Theory Related Fields*, 179(1-2):117–164, 2021.
[19] W. Qian and W. Werner. Decomposition of Brownian loop-soup clusters. *J. Eur. Math. Soc.*, 21(10):3225–3253, 2019.
[20] S. Sheffield and W. Werner. Conformal loop ensembles: the Markovian characterization and the loop-soup construction. *Ann. of Math. (2)*, 176(3):1827–1917, 2012.
[21] W. Werner. On the spatial Markov property of soups of unoriented and oriented loops. In *Séminaire de Probabilités XLVIII*, volume 2168 of *Lecture Notes in Math.*, pages 481–503. Springer, Cham, 2016.

"Sobolev spaces" on compact metric spaces and a new class of self-similar sets as examples

Jun Kigami
(joint work with Yuka Ota)

A wide variety of "Analysis" has been developed and applied to every area of science since the introduction of differentiation by Newton and Leibniz. However, the emergence of "fractals" as models of natural objects and phenomena has raised the question of how we can tackle analysis on spaces that are nowhere smooth. For example, typical self-similar sets, such as the Sierpinski gasket and carpet, do not possess a differential structure; hence, it is difficult to apply analyses based on differentiation.

To be more exact, let us recall the definition of $W^{1,p}$-Sobolev spaces on $\mathbb{R}^n$. Define

$$\mathcal{E}_p(u,v) = \int (\nabla u, \nabla v)|\nabla u|^{p-2}dx = -\int \Delta_p u {\cdot} v dx,$$

for $u, v \in L^p(\mathbb{R}^n)$, where ∇u and ∇v are the gradient in the sense of distribution and $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}u)$, where Δ_p is called the p-Laplacian.

Note that

$$\mathcal{E}_p(u,u) = \int |\nabla u|^p dx.$$

Then $(1,p)$-Sobolev space $W^{1,p}(\mathbb{R}^n)$ is defined as

$$W^{1,p}(\mathbb{R}^n) = \{f | f \in L^p(\mathbb{R}^n), \int |\nabla f|^p dx < \infty\}.$$

In this definition, the gradient ∇u is indispensable. So how can we extend this in the case of "non-smooth metric spaces".

One of the answers is to use the local Lipschitz constant of Lipschitz functions as a substitute for derivative. Let (X,d) be a metric space. Then the collection of Lipschitz continuous functions are given by

$$LIP(X,d) = \{u | u : X \to \mathbb{R}, \sup_{x,y \in X, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} < \infty\}$$

For $u \in LIP(X,d)$, define the local Lipschitz constant $\overline{\nabla} u(x)$ at $x \in X$ by

$$\overline{\nabla} u(x) = \limsup_{r \downarrow 0} \sup_{y \in B_d(x,r)} \frac{|u(x) - u(y)|}{r}.$$

The rough idea is to use $\overline{\nabla} u(x)$ as a substitute of $|\nabla u|$. Since the middle 1990's, several authors have pursued this idea and constructed counterparts of $(1,p)$-Sobolev spaces on metric spaces. The pioneering works in this direction are Hajłasz[3], Cheeger[2] and Shanmugalingam [10]. In fact, they use a more sophisticated notion called "upper gradient", which is a generalization of $\overline{\nabla} u$, in their work. This direction has been explored intensively over the years, and one can see the landscape of this theory in the book [4].

However, recently, in [6, 5], Kajino and Murugan have shown that this theory does not work for some of the well-known self-similar sets like the Sierpinski carpet and the Vicsek set. More precisely, Barlow and Bass constructed the Brownian motion on the Sierpinski gasket in [1]. They have shown that the domain of the Dirichlet form associated with the Brownian motion can not be any of $(1,2)$-Sobolev spaces constructed by the upper gradient theory. Recall that in the case of the Brownian motion on $\mathbb{R}^n$, the domain of the associated Dirichlet form is $W^{1,2}(\mathbb{R}^n)$.

So what is the proper definition of "Sobolev" spaces on such spaces as the Sierpinski carpet and the Vicsek set? Our naive idea comes from the following observation. For $n \in \mathbb{N}$, and $f : [0,1] \to \mathbb{R}$, define

$$\mathcal{E}_p^n(f) = \sum_{i=1}^{2^n} \left| f\Big(\frac{i-1}{2^n}\Big) - f\Big(\frac{i}{2^n}\Big) \right|^p.$$

Then we have the following elementary fact: iff $f \in C^1([0,1])$ for $p \geq 1$, then

$$(2^{p-1})^n \mathcal{E}_p^n(f) \underset{n \to \infty}{\longrightarrow} \int_0^1 |\nabla f|^p dx.$$

In fact, we can show the above fact even if $p > 1$ and $f \in W^{1,p}([0,1])$. In light of the above equation, we have an alternative definition of $W^{1,p}([0,1])$ for $p > 1$ as follows;

$$W^{1,p}(\mathbb{R}^n) = \{f | f \in C([0,1]), (2^{p-1})^n \mathcal{E}_p^n(f) \text{ is convergent as } n \to \infty\}.$$

This gives a definition of the Sobolev space without using the derivation ∇f.

From the above observation, our idea to define "Sobolev spaces" on a general compact metric space (X, d) is as follows:

Approximate the compact metric space (X, d) be a sequence of discrete graphs $\{(T_m, E_m)\}_{m \geq 0}$, where T_m and E_m are collections of vertices and edges respectively. Then define a discrete p-energy of a function $f : T_m \to \mathbb{R}$ by

$$\mathcal{E}_p^n(f) = \frac{1}{2} \sum_{(x,y) \in E_n} |f(x) - f(y)|^p.$$

If there exists a scaling constant σ_p such that

$$\{f | f : X \to \mathbb{R},\ (\sigma_p)^n \mathcal{E}_p^n(f) \text{ is "convergent" in some sense.}\}$$

is a nice function space possessing the properties of the ordinary $(1, p)$-Sobolev spaces on $\mathbb{R}^n$, then the above function space is a $(1, p)$-Sobolev space on the metric space (X, d).

The first part, which is the construction of a sequence of graphs, can be done by the theory of partitions of metric spaces studied in [7]. In [8], we pursued the above idea to define a counterpart of the Sobolev space and obtained the notion of "**p-conductive homogeneity**" as the condition ensuring the realization of the construction of $(1, p)$-Sobolev space on a metric space (X, d) through the above idea. The key notions are the conductance constant and the neighbor disparity constant. In particular, if a metric space has 2-conductive homogeneity and $\dim_{ARC}(X, d) > 2$, where $\dim_{ARC}(X, d)$ is the Ahlfors regular conformal dimension of (X, d), the constructed $(1, 2)$-Sobolev space is a domain of a non-trivial diffusion process, which may be called the Brownian motion. (The Ahlfors regular conformal dimension is the quasi-symmetric invariant of metric spaces. For its definition, see [7] for example.)

Once we have established the notion of the conductive homogeneity, it is important to have a variety of new examples having the property. Without them, the theory is like a gorgeous apartment with no residents. In this aspect of the study, we established a new class of planar self-similar sets, called locally symmetric polygon-based self-similar set, in the joint paper [9] with Y. Ota. Typical examples of locally symmetric polygon-based self-similar sets are shown in Figure 1. Those examples have no global symmetry, but they are p-conductively homogeneous for $p > \dim_{ARC}(X, d)$, where $\dim_{ARC}(X, d)$ is the Hausdorff dimension of (X, d). In particular, since $\dim_{ARC}(X, d) < 2$, we have constructed "Brownian motions" on them.

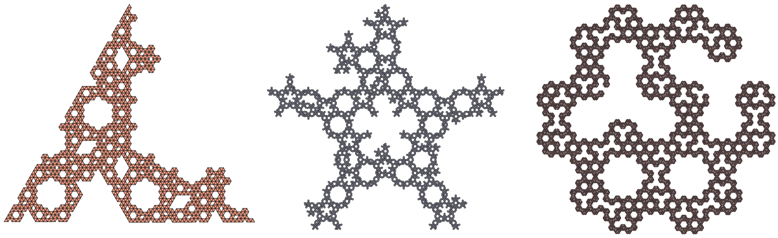

FIGURE 1. Locally symmetric polygon-based self-similar sets

References

[1] M. T. Barlow and R. F. Bass, *The construction of Brownian motion on the Sierpinski carpet*, Ann. Inst. Henri Poincaré **25** (1989), 225–257.

[2] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **3** (1999), 428–517.

[3] P. Hajłasz, *Sobolev spaces on an arbitrary metric spaces*, Pontential Analysis **5** (1996), 403–415.

[4] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson, *Sobolev Spaces on Metric Measure Spaces - An Approach Based on Upper Gradients*, New Mathematical Monographs, Cambridge University Press, 2015.

[5] N. Kajino and M. Murugan, *On the conformal walk dimension II: Non-attainment for some Sierpiński carpets*, in preparation.

[6] ———, *On the conformal walk dimension: Quasisymmetric uniformization for symmetric diffusions*, Invent. Math. **231** (2023), 263–405.

[7] ———, *Geometry and Analysis of Metric Spaces via Weighted Partitions*, Lecture Notes in Math. vol. 2265, Springer, 2020.

[8] ———, *Conductive homogeneity of compact metric spaces and construction of p-energy*, Mem. European Math. Soc. **5** (2023).

[9] J. Kigami and Y. Ota, Conductive homogeneity of locally symmetric polygon-based self-similar sets, preprint 2025.

[10] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamer. **16** (2000), 243–279.

On the energy image density conjecture of Bouleau and Hirsch

MATHAV MURUGAN

(joint work with Sylvester Eriksson-Bique)

We report on the recent affirmative resolution of the energy image density conjecture of Bouleau and Hirsch (1986) [5, Theorem 1.7], [1, p. 251]. This conjecture generalizes a foundational result in Malliavin calculus: the non-degeneracy of the Malliavin matrix of a random variable implies absolute continuity of its law. This property is a key step in Malliavin's probabilistic proof of Hörmander's hypoellipticity theorem. Going beyond the original framework of Dirichlet structures, we establish the energy image density property in a unified setting that includes classical Dirichlet forms, Sobolev spaces defined via upper gradients, and self-similar energies on fractals. As an application of independent interest, we show that the martingale dimension of a diffusion satisfying a sub-Gaussian heat kernel estimate is finite, thereby verifying a recent conjecture [6, Conjecture 3.12]. As another application, we provide a new proof of Cheeger's conjecture on the Hausdorff dimension of images of differentiability charts in PI spaces [3, Conjecture 4.63].

For a precise statement of this conjecture, we recall the relevant definitions below.

Definition 1. *A* Dirichlet structure $(X, \mathcal{X}, \mu, \mathcal{E}, \mathcal{F})$ *is a probability space* $(X, \mathcal{X}, \mu)$ *along with a quadratic form* $(\mathcal{E}, \mathcal{F})$ *on* $L^2(X, \mu)$ *that satisfies the following properties.*

(i) (densely defined, non-negative definite, quadratic form) $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ *is bilinear, where* $\mathcal{F}$ *is a dense subspace of* $L^2(X, \mu)$ *and* $\mathcal{E}(f, f) \geq 0$ *for all* $f \in \mathcal{F}$.

(ii) (closed form) $(\mathcal{E}, \mathcal{F})$ *is a* closed *form; that is,* $\mathcal{F}$ *is a Hilbert space equipped with the inner product*

$$\mathcal{E}_1(f, g) := \mathcal{E}(f, g) + \langle f, g\rangle_{L^2(\mu)}, \quad \text{for all } f, g \in \mathcal{F}.$$

(iii) (Markovian property) $f \in \mathcal{F}$ *implies that* $\widetilde{f} := 0 \vee (f \wedge 1) \in \mathcal{F}$ *and* $\mathcal{E}(\widetilde{f}, \widetilde{f}) \leq \mathcal{E}(f, f)$.

(iv) $\mathbf{1} \in \mathcal{F}$ *and* $\mathcal{E}(\mathbf{1}, \mathbf{1}) = 0$, *where* $\mathbf{1}$ *is the constant function on* X *that is identically one.*

(v) (existence of carré du champ operator) For all $f \in \mathcal{F} \cap L^\infty(m)$, *there exists* $\gamma(f, f) \in L^1(m)$ *such that for all* $h \in \mathcal{F} \cap L^\infty(m)$, *we have*

$$\mathcal{E}(fh, f) - \frac{1}{2}\mathcal{E}(h, f^2) = \int h\gamma(f, f)\, d\mu.$$

(vi) (strong locality) For all $f, g \in \mathcal{F}$ *and* $a \in \mathbb{R}$ *such that* $(f + a)g = 0$ *implies* $\mathcal{E}(f, g) = 0$.

By [2, Proposition I.4.1.3], for any Dirichlet structure $(X, \mathcal{X}, \mu, \mathcal{E}, \mathcal{F})$ there exists a unique positive symmetric and continuous bilinear form $\gamma : \mathcal{F} \times \mathcal{F} \to L^1(\mu)$ (called the *carré du champ operator*) such that

$$\frac{1}{2}\mathcal{E}(fh, g) + \frac{1}{2}\mathcal{E}(gh, f) - \frac{1}{2}\mathcal{E}(h, fg) = \int h\gamma(f, g)\, d\mu, \quad \text{for all } f, g, \in \mathcal{F} \cap L^\infty. \tag{1}$$

For a function $f = (f_1, \ldots, f_n) \in \mathcal{F}^n$, we define the carré du champ matrix $\gamma(f)$ as the $n \times n$ matrix

$$\gamma(f) = \left[\gamma(f_i, f_j)\right]_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}. \tag{2}$$

The carré du champ matrix is the natural generalization of the Malliavin matrix to the setting of strongly local Dirichlet forms. In the special case of the Ornstein–Uhlenbeck Dirichlet form on Wiener space, this carré du champ matrix coincides with the classical Malliavin matrix. We refer to [2, Definition II.2.3.3] and [7, p. 92] for the definitions of the Ornstein–Uhlenbeck Dirichlet form on Wiener space and Malliavin matrix respectively. Hence the energy image density property in Definition 2 and Conjecture 3 below can be viewed as a broad generalization of Malliavin's criterion for absolute continuity of the law of a random variable on Wiener space using non-degeneracy of the Malliavin matrix under minimal regularity assumption.

Next, we recall the energy image density property introduced by Bouleau and Hirsch.

Definition 2. *Let $(X, \mathcal{X}, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet structure with the associated carré du champ operator $\gamma : \mathcal{F} \times \mathcal{F} \to L^1(\mu)$. We say that the Dirichlet structure $(X, \mathcal{X}, \mu, \mathcal{E}, \mathcal{F})$ satisfies the* energy image density property*, if for any $n \in \mathbb{N}$, $f \in \mathcal{F}^n$, we have*

$$f_*(\mathbf{1}_{\{\det(\gamma(f))>0\}} \cdot \mu) \ll \mathcal{L}_n, \tag{3}$$

where $\mathcal{L}_n$ is the Lebesgue measure on $\mathbb{R}^n$.

Let us briefly explain the condition $\det(\gamma(f)) > 0$ in (3). Consider the Dirichlet form for Brownian motion of $\mathbb{R}^n$ given by $\mathcal{E}(g, g) = \int |\nabla g|(x)^2\, dx$ for all $g \in \mathcal{F} = W^{1,2}(\mathbb{R}^n)$, where ∇g denotes the distributional gradient of g. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function whose components are in the Sobolev space $W^{1,2}(\mathbb{R}^n)$. In this case, $\gamma(f_i, f_j) = \nabla f_i \cdot \nabla f_j$ and hence the condition $\det(\gamma(f))(x) > 0$ is equivalent to the statement that the differential Df of f at x is surjective, or equivalently, f is a submersion at x. For an abstract Dirichlet form, this condition can be interpreted more generally as requiring that f 'varies in all infinitesimal directions' at x. Intuitively, the invertibility of the carré du champ matrix ensures that f has enough local variability so that its image is sufficiently 'spread out', a viewpoint that directly motivates the energy image density condition in more abstract settings.

The Bouleau–Hirsch conjecture [1, p. 251] is stated as follows.

Conjecture 3 (EID conjecture)**.** *Every Dirichlet structure satisfies the energy image density property.*

The main result of [5] is the positive answer to Bouleau–Hirsch conjecture.

Theorem 4. *Let $(X, \mathcal{X}, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet structure with the associated carré du champ operator $\gamma : \mathcal{F} \times \mathcal{F} \to L^1(\mu)$. For any $n \in \mathbb{N}$, $\varphi \in \mathcal{F}^n$, we have*

$$f_*(\mathbf{1}_{\{\det(\gamma(\varphi))>0\}} \cdot \mu) \ll \mathcal{L}_n,$$

where $\mathcal{L}_n$ is the Lebesgue measure on $\mathbb{R}^n$, and $\gamma(\varphi)$ is the carré du champ matrix as defined in (2)*.*

A key ingredient in the proof is a deep structure theorem of measures and normal currents in $\mathbb{R}^n$ developed by De Philippis and Rindler [4, Corollary 1.12] recalled below.

Theorem 5. [4, Corollary 1.12] *Let $T_1 = \vec{T_1} \|T_1\|, \dots, T_n = \vec{T_n} \|T_n\|$ be one-dimensional normal currents on $\mathbb{R}^n$ such that there exists a positive Radon measure ν on $\mathbb{R}^n$ with the following properties:*

(i) $\nu \ll \|T_i\|$ for $i = 1, \dots, n$;

(ii) For ν-a.e. $x \in \mathbb{R}^n$, $\operatorname{span}\{\vec{T_1}(x), \dots, \vec{T_n}(x)\} = \mathbb{R}^n$.

Then $\nu \ll \mathcal{L}_n$.

References

[1] Bouleau, N. and Hirsch, F. Formes de dirichlet générales et densité des variables aléatoires réelles sur l'espace de Wiener. *J. Funct. Anal.* **69** (1986), no. 2, pp. 229–259.
[2] Bouleau, N. and Hirsch, F. *Dirichlet forms and analysis on Wiener space.* De Gruyter Stud. Math., **14**, Walter de Gruyter & Co., Berlin, 1991. x+325 pp.
[3] Cheeger, J. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal. 9*, 3 (1999), pp. 428–517.
[4] De Philippis, G. and Rindler, F. On the structure of $\mathscr{A}$-free measures and applications. *Ann. of Math.* (2) **184** (2016), no. 3, pp. 1017–1039.
[5] Eriksson-Bique, M. and Murugan, M. On the energy image density conjecture of Bouleau and Hirsch, arXiv:2510.13659 (preprint) 2025
[6] Murugan, M. Martingale and analytic dimensions coincide under gaussian heat kernel bounds. *Bull. Sci. Math.* **204** (2025), Paper No. 103710.
[7] Nualart, D. The Malliavin calculus and related topics. 2nd ed., Probab. Appl. (N. Y.), Springer-Verlag, Berlin, 2006. xiv+382 pp.

Absolutely continuous curves in kinetic optimal transport

Jan Maas

(joint work with Giovanni Brigati, Filippo Quattrocchi)

The classical notion of absolutely continuity can be naturally generalised to curves in metric spaces: A curve $\gamma : (0,T) \to X$ with values in a metric space (X,d) is said to be 2-absolutely continuous if there exists $\alpha \in L^2(0,T)$ such that

$$d\big(\gamma(s), \gamma(t)\big) \le \int_s^t \alpha(r)\, dr$$

for all $0 < s < t < T$. A central result in the field of Optimal Transport [1] gives a characterisation of 2-absolutely continuous curves in the Wasserstein space $\mathcal{P}_2(\mathbf{R}^d)$, the space of probability measures on $\mathbf{R}^d$ having finite second moment endowed with the L^2-Wasserstein metric W_2. (Recall that

$$W_2(\mu,\nu)^2 := \inf \mathbf{E}|X-Y|^2 \,,$$

where the infimum runs over all random variables (X,Y) with $\mathrm{law}(X) = \mu$ and $\mathrm{law}(Y) = \nu$.) The result asserts that for a weakly continuous curve $t \mapsto \mu_t \in \mathcal{P}_2(\mathbf{R}^d)$ the following statements are equivalent:

(1) $t \mapsto \mu_t$ is 2-absolutely continuous in the Wasserstein space $(\mathcal{P}_2(\mathbf{R}^d), W_2)$;
(2) $t \mapsto \mu_t$ solves the continuity equation $\partial_t \mu + \nabla \cdot (v\mu) = 0$ for some vector field v satisfying $\displaystyle\int_0^T \int_{\mathbf{R}^d} |v(t,x)|^2 \, d\mu_t(x)\, dt < \infty$.

This result provides a bridge between the metric geometry of the Wasserstein space and a concrete class of partial differential equations.

It is natural to ask for an analogous result in kinetic theory, where the continuity equation is replaced by the Vlasov equation

$$\partial_t \mu + v \cdot \nabla_x \mu + \nabla_v \cdot (F\mu) = 0 \,.$$

This equation describes a time-evolving measure $\mu_t(dx, dv)$ on position-velocity space driven by a time-dependent accelaration field $F_t(x, v)$. Instead of the Wasserstein metric W_2, we consider a natural second-order discrepancy C_T between probability measures on position-velocity space, which depends nontrivially on the length of the chosen time horizon $T > 0$. More precisely, for $\mu, \nu \in \mathcal{P}_2(\mathbf{R}^{2d})$ we define

$$C_T(\mu, \nu)^2 := \inf \mathbf{E}\left[\frac{1}{12}\left|\frac{Y-X}{T} - \frac{V+W}{2}\right|^2 + |V-W|^2\right].$$

The infimum runs over all random variables $\big((X, V), (Y, W)\big)$ with $\mathrm{law}(X, Y) = \mu$ and $\mathrm{law}(Y, W) = \nu$. This object appears in earlier work on variational discretisations of kinetic Fokker–Planck equations [5, 3, 4]. The underlying cost function, which may look mysterious at first glance, arises naturally from the theory of large deviations: it appears in the (nonistropic Gaussian) heat kernel of the Kolmogorov diffusion $\partial_t\mu + v \cdot \nabla_x\mu = \Delta\mu$, just as the isotropic quadratic cost function $|x - y|^2$ defining the 2-Wasserstein metric features in the heat kernel of the usual diffusion equation $\partial_t\mu = \Delta\mu$.

While C_T is not a metric, it is natural to declare a curve of measures $t \mapsto \mu_t$ to be absolutely continuous with respect to $(C_T)_T$ if there exists $\alpha \in L^2(0, T)$ such that

$$C_{t-s}(\mu_s, \mu_t) \le \int_s^t \alpha(r)\, dr$$

for all $s < t$. One of our main results [2] asserts that, under appropriate technical assumptions, these absolutely continuous curves are exactly those that satisfy the Vlasov equation $\partial_t\mu + v \cdot \nabla_x\mu + \nabla_v \cdot (F\mu) = 0$ for some accelaration field F with $\displaystyle\int_0^T \int_{\mathbf{R}^{2d}} |F(t, x, v)|^2 \, d\mu_t(x, v)\, dt < \infty$. This result provides a natural kinetic counterpart to the characterisation of absolutely continuous curves in the Wasserstein space, potentially allowing for further applications of optimal transport in kinetic theory.

References

[1] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows in metric spaces and in the space of probability measures.* Birkhäuser Verlag, second edition (2008).

[2] Giovanni Brigati, Jan Maas, and Filippo Quattrocchi. Kinetic optimal transport (OTIKIN) – Part 1: Second-order discrepancies between probability measures, *arXiv:2502.15665* (2025).

[3] Manh Hong Duong, Mark A. Peletier, and Johannes Zimmer. GENERIC formalism of a Vlasov-Fokker-Planck equation and connection to large-deviation principles. *Nonlinearity* **26** (2013), 2951–2971.

[4] Manh Hong Duong, Mark A. Peletier, and Johannes Zimmer. Conservative-dissipative approximation schemes for a generalized Kramers equation. *Math. Methods Appl. Sci.* **37** (2014), 2517–2540.

[5] Chaocheng Huang. A variational principle for the Kramers equation with unbounded external forces. *Journal of Mathematical Analysis and Applications* **250** (2000), 333–367.

Cutoff for underdamped Langevin dynamics

Max Fathi

(joint work with Arnaud Guillin, Cyril Labbé, Justin Salez)

This talk presented some results on cutoff for underdamped (or kinetic) Langevin dynamics, building up on the recent work of Salez [3] on cutoff for nonnegatively curved diffusion processes.

The dynamics considered are of the form

$$dX_t = V_t dt; \quad dV_t = -\nabla W(X_t) - V_t + \sqrt{2} dB_t \tag{1}$$

where (X_t, V_t) is the position and velocity, and W a potential. Its invariant measure is of the form

$$\pi(dx, dv) = \frac{1}{Z}\exp(-W(x) - |v|^2/2)dxdv.$$

The cutoff phenomenon is an abrupt convergence to equilibrium for sequences of Markovian dynamics, based on the notion of (total variation) mixing time

$$t_{mix}^{(S)}(\varepsilon) := \inf\{t \geq 0; d_{TV}(\mathcal{L}(X_t, V_t), \pi) \leq \varepsilon \quad \forall (x_0, v_0) \in S\}$$

where S is a set of possible initial data, which in compact settings is often simply the whole state space, but here we consider non-compact spaces, so we must take its choice into account. Cutoff is said to occur when, for a sequence of Markovian dynamics, we have

$$\frac{t_{mix}^{(n,S_n)}(\varepsilon) - t_{mix}^{(n,S_n)}(1-\varepsilon)}{t_{mix}^{(n,S_n)}(1-\varepsilon)} \longrightarrow 0 \quad \forall \varepsilon \in (0, 1/2).$$

This property means that the first order behavior of the mixing time does not depend on ε, and that the time it takes to go from distance $1-\varepsilon$ to ε to equilibrium is much smaller than the time it takes for the distance to start being smaller than 1 (which is the maximum value). It depends on the sequence of dynamic, as well as the choice of sequence of initial data.

The varentropy method for cutoff of [3] is combined with certain estimates from hypocoercivity theory, such as Wasserstein contraction [1] and short-time entropic regularization [2], to obtain

Theorem 1. *Consider a sequence of dynamics of the form* (1)*, with varying dimension d_n, potential W_n and set of initial data $S_n \subset B_{\mathbb{R}^{2d_n}}(0, R_n)$. Assume that the equilibrium measures are centered, and that $m < \nabla^2 W_n < M$ for $0 < m < M$ such that $\sqrt{M} - \sqrt{m} < 1$. Then cutoff occurs if*

$$t_{mix}^{(n,S_n)}(1-\varepsilon) >> \log\log(d_n + R_n^2).$$

As an application, we obtain cutoff for the classical mean-field Langevin dynamics for potentials satisfying the assumptions, including for example

$$W(x_1, \ldots, x_N) := \frac{1}{2}\sum_{i=1}^{N} |x_i|^2 + \frac{1}{2N}\sum_{i \neq j} f(x_i - x_j)$$

as soon as $||\nabla^2 f||_\infty$ is not too large and the set of initial data is a ball of radius of order $\sqrt{N}$ (which is the natural scale in this problem). These dynamics have a mixing time of order at least $\log N$ (because of propagation of chaos), so the criterion for cutoff is satisfied.

References

[1] F. Baudoin, Bakry-Émery meet Villani. *J. Funct. Anal.* 273, No. 7, 2275-2291 (2017).

[2] A. Guillin and F.-Y. Wang, Degenerate Fokker-Planck equations: Bismut formula, gradient estimate and Harnack inequality. *J. Differ. Equations* 253, No. 1, 20-40 (2012).

[3] J. Salez, Cutoff for non-negatively curved diffusions. *Arxiv preprint* 2025.

Fine properties of upper gradient-based Sobolev spaces in metric measure spaces, with application to Dirichlet boundary value problems

Nageswari Shanmugalingam

The talk gives an overview of a version of Sobolev spaces of functions on non-smooth metric measure spaces using the notion of *p-weak upper gradients* - a notion first proposed by Heinonen and Koskela [1]. The context is that of a metric space (X, d), equipped with a Borel regular measure μ. A measurable function $u : X \to [-\infty, \infty]$ is said to have a non-negative Borel regular function $g : X \to [0, \infty]$ as an upper gradient if

$$|u(\gamma(b)) - u(\gamma(a))| \le \int_a^b g(\gamma(t))\, |\dot\gamma(t)|\, dt$$

whenever $\gamma : [a, b] \to X$ is a rectifiable curve (that is, a continuous map of finite length). In the above, we also require that if *at least* one of $u(\gamma(b))$ and $u(\gamma(a))$ is not a real number, then the right-hand side of the above integral must be infinite.

Unlike in the case of classical Sobolev spaces in Euclidean domains, and horizontal Sobolev spaces in domains in Carnot groups, we do not allow for perturbation of such a function u arbitrarily on a set of measure zero; to allow for arbitrary perturbation, the set must, in addition to being of measure zero, also have the property that the collection of all rectifiable curves in X that intersect the set must have zero p-modulus; with such a relaxation, we consider the Newton-Sobolev class $N^{1,p}(X)$ of equivalence classes of functions on X. In the Euclidean and Carnot setting as well as in Riemannian setting, these function classes consist of functions that belong to an equivalence class in the classical Sobolev space but with fine continuity property. More on this can be found in the book [2].

In the second part of the talk we discuss how this new way of looking at Sobolev spaces also allow us to consider solutions to Dirichlet problems in Euclidean domains whose boundary need not be Lipschitz regular. If Ω is a domain with the complement $X \setminus \Omega$ of the domain of positive measure (it suffices to have positive p-capacity), the key test-class in dealing with such a Dirichlet problem is to consider the class of functions u in $N^{1,p}(\Omega)$ for which the zero-extension of u gives a function in $N^{1,p}(X)$. What this means is that for p-modulus almost every rectifiable curve $\gamma : [a, b] \to X$ with $\gamma(a) \in \Omega$ and $\gamma(b) \in X \setminus \Omega$, and for each $t_0 \in \gamma^{-1}(\partial\Omega)$, we have $\lim_{[a,b]\ni t\to t_0} u(\gamma(t)) = 0$. Thus we can make sense of what it means for two Newton-Sobolev functions on Ω to have the *same boundary data* in formulating the Dirichlet problem. This approach was first explored in [3].

References

[1] J. Heinonen, P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), no. 1, 1–61.

[2] J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson, *spaces on metric measure spaces. An approach based on upper gradients.* New Mathematical Monographs, **27**. Cambridge University Press, Cambridge, 2015. xii+434 pp.

[3] N. Shanmugalingam, *Harmonic functions on metric spaces*, Illinois J. Math. **45** (2001), no. 3, 1021–1050.

Reporters: Marie Bormann, Erik Hupp

Participants

Prof. Dr. Patricia Alonso Ruiz
Fakultät für Mathematik und Informatik
Friedrich-Schiller-Universität
07743 Jena
GERMANY

Dr. Sebastian Andres
Institut für Mathematische Stochastik
Technische Universität Braunschweig
Universitätsplatz 2
38106 Braunschweig
GERMANY

Dr. Rotem Assouline
Institut de Mathématiques de Jussieu (IMJ-PRG)
Sorbonne Université
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Franck Barthe
Institut de Mathématiques de Toulouse
Université de Toulouse
118, Route de Narbonne
31062 Toulouse Cedex 9
FRANCE

Prof. Dr. Fabrice Baudoin
Matematisk Institut
Aarhus Universitet
Ny Munkegade 118
8000 Aarhus C
DENMARK

Pierre Bizeul
c/o Prof. Steven Gelbart
Dept. of Maths
Weizmann Institute of Sciences
Rehovot 76100
ISRAEL

Dr. Marie Bormann
Hausdorff Center for Mathematics
Universität Bonn
Endenicher Allee 62
53115 Bonn
GERMANY

Prof. Dr. Gilles Carron
Département de Mathématiques "Jean Leray"
UMR 6629
Nantes Université
2, rue de la Houssinière
44322 Nantes Cedex 03
FRANCE

Prof. Dr. Linan Chen
Dept. of Mathematics and Statistics
McGill University
805 Sherbrooke St West
Burnside Hall 1005
Montréal QC H3A 0B9
CANADA

Prof. Dr. Zhen-Qing Chen
Department of Mathematics
University of Washington
Seattle WA 98195
UNITED STATES

Prof. Dr. Dario Cordero-Erausquin
Institut de Mathématiques de Jussieu
Sorbonne Université
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Benoit Dagallier
Department of Mathematics
Imperial College of Science,
Technology and Medicine
180 Queen's Gate, Huxley Bldg.
London SW7 2BZ
UNITED KINGDOM

Prof. Dr. Lorenzo Dello Schiavo
Università degli Studi di Roma
"Tor Vergata"
Viale della Ricerca Scientifica 1
00133 Roma 00133
ITALY

Prof. Dr. Jean Dominique Deuschel
Institut für Mathematik
Sekr. MA 7-4
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin
GERMANY

Prof. Dr. Matthias Erbar
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Dr. Max Fathi
Laboratoire Jacques-Louis Lions &
Laboratoire de Probabilités, Statistique
et Modélisation
Batiment Sophie Germain
8, Place Aurélie Nemours
75013 Paris Cedex
FRANCE

Dr. Mira Gordin
Division of Applied Mathematics
Brown University
Box F
182 George Street
Providence, RI 02912
UNITED STATES

Prof. Dr. Masha Gordina
Department of Mathematics
University of Connecticut
341 Mansfield Road
Storrs CT 06269-3009
UNITED STATES

Dr. Nathael Gozlan
Université Paris Cité
MAP 5
45 rue des Saints Pères
75006 Paris
FRANCE

Assoc. Prof. Dr. Karen Habermann
Department of Statistics
University of Warwick
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. David P. Herzog
Dept. of Mathematics
Iowa State University
Carver Hall
411 Morrill Rd.
Ames IA, 50011
UNITED STATES

Prof. Dr. Martin Huesmann
Institut für Mathematische Stochastik
Universität Münster
Orléans-Ring 10
48149 Münster
GERMANY

Erik Hupp
Institut für Angewandte Mathematik
Universität Bonn
53115 Bonn
GERMANY

Dr. Naotaka Kajino
Research Institute for Mathematical Sciences
Kyoto University
Kitashirakawa-Oiwake-cho, Sakyo-ku
Kyoto 606-8502
JAPAN

Prof. Dr. Jun Kigami
Graduate School of Informatics
Kyoto University
Yoshida-honmachi, Sakyo-ku
Kyoto 606-8501
JAPAN

Prof. Dr. Bo'az Klartag
Department of Mathematics
The Weizmann Institute of Science
P. O. Box 26
76 100 Rehovot
ISRAEL

Dr. Eva Kopfer
Institut für angewandte Mathematik (IaM)
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Takashi Kumagai
Department of Mathematics
Waseda University
Ohkubo 3-4-1
Shinjuku-ku
Tokyo 169-8555
JAPAN

Dr. Joseph Lehec
Laboratoire Mathématiques et Applications
Université de Poitiers
BP 30179
Site du Futuroscope, Teleport 2
11, Boulevard Marie et Pierre Curie
86073 Poitiers Cedex 9
FRANCE

Zhenhao Li
Fakultät für Mathematik
Universität Bielefeld
Universitätsstr. 25
33615 Bielefeld
GERMANY

Dr. Jessica Lin
Department of Mathematics and Statistics
McGill University
Burnside Hall 1217
805 Sherbrooke Street West
Montréal QC H3A 0B9
CANADA

Prof. Dr. Jan Maas
IST Austria
Am Campus 1
3400 Klosterneuburg
AUSTRIA

Prof. Dr. Tai Alexis Melcher
Department of Mathematics
University of Virginia
141 Cabell Drive, Kerchof Hall
P.O. Box 400137
Charlottesville, VA 22904-4137
UNITED STATES

Prof. Dr. Emanuel Milman
Department of Mathematics
TECHNION
Israel Institute of Technology
Amado 630
Haifa 32000
ISRAEL

Dr. Ilaria Mondello
Faculté de Sciences et Technologie
Université Paris-Est, Créteil
Bâtiment P3
61, Avenue du General de Gaulle
94010 Créteil Cedex
FRANCE

Dr. Mathav Murugan
Department of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver BC V6T 1Z2
CANADA

Dr. Shin-ichi Ohta
Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-1, Toyonaka
560-0043 Osaka 560-0043
JAPAN

Dr. Wei Qian
Department of Mathematics
University of Hong Kong
Pokfulam Road
Hong Kong
CHINA

Prof. Dr. Laurent Saloff-Coste
Department of Mathematics
Cornell University
Malott Hall
Ithaca, NY 14853-7901
UNITED STATES

Prof. Dr. Giuseppe Savaré
Department of Decision Sciences,
Bocconi University
Via Roentgen 1
Milano 20136
ITALY

Dr. Nageswari Shanmugalingam
Dept. of Mathematical Sciences
University of Cincinnati
P.O. Box 210025
Cincinnati OH 45221-0025
UNITED STATES

Prof. Dr. Theo Sturm
Institut für angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Anton Thalmaier
Department of Mathematics
University of Luxembourg
Campus Belval, Maison du Nombre
6, Avenue de la Fonte
4364 Esch-sur-Alzette
LUXEMBOURG

Prof. Dr. Max von Renesse
Mathematisches Institut
Universität Leipzig
Augustusplatz 10
04109 Leipzig
GERMANY

Prof. Dr. Jian Wang
School of Mathematics and Statistics
Fujian Normal University
South 603, Technology Building
Qishan Campus
No. 1 Keji Road, Shangjie, Minhou
Fuzhou 350 117
CHINA

Dr. Kai-Hsiang Wang
Faculty of Mathematics
Technion – I.I.T.
Haifa 32000
ISRAEL