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Mini-Workshop: Hyperbolic meets Stochastic Geometry

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Abstract. The mini-workshop brought together researchers from hyperbolic geometry and stochastic geometry with the aim of advancing the emerging field of hyperbolic stochastic geometry. It focused on understanding how negative curvature fundamentally influences the behaviour of random geometric models. Particular emphasis was placed on limit theorems, phase transitions, and scaling phenomena that differ substantially from those observed in Euclidean settings. The program combined survey lectures, research presentations, and discussion sessions to link geometric methods with probabilistic techniques tailored to hyperbolic spaces. As a result, the workshop clarified central challenges in the field, identified key open problems, and initiated new collaborations spanning geometry, probability, and related areas.

Mathematics Subject Classification (2020): Primary: 51M10, 60D05; Secondary: 52A22, 57M50, 60F05.

Introduction by the Organizers

Hyperbolic stochastic geometry has emerged as a rapidly developing field that provides a unifying framework for the study of random spatial structures in non-Euclidean settings. In contrast to Euclidean or spherical geometries, hyperbolic space is characterized by negative curvature, leading for example to exponential volume growth. This feature fundamentally alter both geometric and probabilistic aspects of random structures, giving rise to phenomena that are not only mathematically rich but also relevant for applications ranging from network theory to statistical physics.

A central theme in hyperbolic stochastic geometry is the strong sensitivity of second-order characteristics, such as variances and fluctuation behaviour, to curvature and dimension. While first-order quantities, such as expected lengths, areas, or volumes, may exhibit similarities to their Euclidean counterparts, second-order properties often reveal substantial differences. In hyperbolic settings, classical central limit theorems typically hold only under restrictive assumptions, and in many cases the limiting distributions are non-Gaussian, reflecting the intrinsic geometry of the underlying space.

Despite significant recent progress, the field remains at an early stage of development. Many fundamental questions are still open, including the precise nature of fluctuation limits in high-dimensional hyperbolic spaces, the identification of robust analytical techniques capable of capturing curvature-dependent effects, and a deeper understanding of phase transitions in random geometric structures.

The mini-workshop brought together, for the first time, experts from two historically distinct communities – hyperbolic geometers and stochastic geometers – in a genuinely interdisciplinary forum. Specialists in hyperbolic geometry contributed structural and geometric insights into negatively curved spaces, while researchers in stochastic geometry provided probabilistic perspectives and discussed recent advances in random spatial models. This exchange fostered a lively and productive dialogue between the two communities. By bridging these areas, the workshop stimulated new research directions, addressed open challenges, and laid the foundation for lasting collaborations in this rapidly developing field.

The schedule was carefully designed to maximize interaction, exchange, and collaboration. On Monday, Tuesday, and Wednesday, the workshop featured overview lectures of 2×45 minutes covering both classical aspects of hyperbolic geometry (H) and recent developments in stochastic geometry (S):

- (H1) Classical hyperbolic geometry (Norbert Peyerimhoff)
- (H2) Hyperbolic volume (Ruth Kellerhals)
- (H3) Integral geometry in hyperbolic space (Florian Besau)
- (H4) Hyperbolic surfaces and manifolds (Bram Petri)
- (S1) Random tessellations in hyperbolic space (Anna Gusakova)
- (S2) Boolean models in hyperbolic space (Daniel Hug)
- (S3) Random graphs on the hyperbolic plane (Nikolaos Fontoulakis)
- (S4) Random hyperbolic surfaces (Jean Raimbault)

These lectures introduced core concepts to participants from both communities and established a common foundation for subsequent discussions. During the remainder of the week, the program focused on shorter, more specialized talks (T1–T4) presented by participants, with two such contributions scheduled each day:

- (T1) Volumes of regular hyperbolic simplices (Zakhar Kabluchko)
- (T2) Subgraphs of kernel-based geometric random graphs (Clara Stegehuis)
- (T3) Around the ideal Poisson–Voronoi tessellation (Nicolas Curien)
- (T4) Hyperbolic radial spanning tree (Vanessa Trapp)

In addition to the formal lectures, the workshop offered ample opportunities for informal scientific exchange. Evening discussions and dedicated discussion sessions enabled participants to form smaller working groups addressing specific open problems and methodological challenges. This format proved highly effective in fostering active engagement and initiating new collaborations, supported by the excellent working conditions provided by the Oberwolfach institute.

In summary, the mini-workshop provided not only a platform for presenting and discussing recent research results but also a unique forum for bridging distinct mathematical communities. Through interdisciplinary exchange, in-depth discussions, and the initiation of joint projects, the workshop strengthened connections between hyperbolic geometry and stochastic geometry and laid important groundwork for future advances in the study of random structures in hyperbolic spaces.

Acknowledgement: The organizers would like to express their sincere gratitude to the Mathematisches Forschungsinstitut Oberwolfach for providing the opportunity to hold this mini-workshop. The excellent working conditions and the inspiring, focused atmosphere at the institute were essential for the success of the meeting and greatly facilitated scientific exchange and collaboration among the participants.

Mini-Workshop: Hyperbolic meets Stochastic Geometry

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Abstracts

(H1) Classical Hyperbolic Geometry

Norbert Peyerimhoff

The n-dimensional (real) hyperbolic space is a connected, simply connected complete Riemannian manifold with constant curvature -1. A standard reference for hyperbolic manifolds is [Ratcliffe]. We focus on the following concrete models of this space:

- Hyperboloid Model $\mathcal{H}^n = \{Q \equiv -1, x_{n+1} > 0\}$ with $Q(x_1, \ldots, x_n, x_{n+1}) = \left(\sum_{j=1}^n x_j^2\right) - x_{n+1}^2$ with Q as the Riemannian metric on the tangent spaces of $\mathcal{H}^n$.
- Unit Ball Model $\mathbb{D}^n = \{x \in \mathbb{R}^n : \|x\|_{Eucl} < 1\}$. This is a conformal model with metric $\langle v, w\rangle_x = \frac{4\langle v,w\rangle_{Eucl}}{(1-\|x\|^2)^2}$, related to $\mathbb{H}^n$ via projection from the point $(0, \ldots, 0, -1)$.
- Klein Model $\mathbb{K}^n = \{x \in \mathbb{R}^n : \|x\|_{Eucl} < 1\}$, connected to the hyperboloid model by choosing the n-dimensional horizontal unit ball in $\mathbb{R}^{n+1}$ at height 1 and project from the origin. This model is not conformal and totally geodesic subspaces are intersections of $\mathbb{K}^n$ with affine Euclidean subspaces.
- Upper Half Space Model $\mathbb{H}^n = \{z = (x_1, \ldots, x_n, y) \in \mathbb{R}^n : y > 0\}$ with metric $\langle v, w\rangle_z = \frac{\langle v,w\rangle_{Eucl}}{y^2}$.

Models of (real) hyperbolic $2, 3, 4$-dimensional spaces and higher dimensional hyperbolic spaces can be found in, e.g., [Parker].

For $\mathbb{H}^2$ with isometry group $G = PSL(2, \mathbb{R})$, we discuss the Iwasawa Decomposition $G = NAK$, acting transitively on the unit tangent vectors. $\mathbb{H}^n$ is a rank-one symmetric space of non-compact type. Other rank-one symmetric spaces of non-compact type are the complex hyperbolic spaces, the quaternionic complex hyperbolic spaces and the Cayley plane.

We discuss trigonometric formulas in the hyperbolic plane, right angled hexagons and how they lead to Y-pieces or pairs of pants, and how gluing of $2g - 2$ pairs of pants with prescribed lengths and twists (Fenchel-Nielsen Coordinates) in accordance with a trivalent graph with $2g-2$ vertices yield closed oriented hyperbolic surfaces of genus $g \geq 2$. In fact, all closed oriented hyperbolic surfaces can be constructed in this way, and the moduli space of all such genus g-hyperbolic spaces is $6g - 6$ dimensional. This is in stark contrast to closed hyperbolic manifolds of higher dimensions due to Mostow Rigidity, which states that, for any fixed dimension $n \geq 3$, a closed oriented hyperbolic manifold is uniquely determined by its fundamental group. We also discuss the Collar Theorem. These topics are available for the participants via shared slides. A beautiful source about many of these geometric topics is [Buser, Chapters 2,3,4,6]. Due to time restriction, we skip the Introduction of Teichmüller Space, Weil-Petersson Metric and convexity of curve systems. A standard reference for Teichmüller Theory is [Imatoshi-Taniguchi].

Another approach to oriented hyperbolic surfaces is via Fuchsian Groups, which are discrete subgroups Γ of $PSU(1,1)$ (orientation preserving isometries of $\mathbb{D}^2$) or $PSL(2,\mathbb{R})$ (orientation preserving isometries of $\mathbb{H}^2$). $S = \mathbb{D}^2/\Gamma$ is a smooth surface if Γ does not have elliptic elements, and cusps of S correspond to conjugacy classes of maximal parabolic subgroups. The Dirichlet Domain (or Voronoi Cell) of a point $p \in \mathbb{D}^2$ is defined by $D_\Gamma(p) = \{q \in \mathbb{D}^2 : d(q,p) \leq (q, \gamma p) \text{ for all } \gamma \in \Gamma\}$ and represents a convex, connected fundamental domain of Γ. Non-trivial closed oriented geodesics in a closed surface $S = \mathbb{H}^2/\Gamma$ are in $(1:1)$-correspondence to non-trivial free homotopy classes of closed curves in S and also in in $(1:1)$-correspondence to non-trivial conjugacy classes $[\gamma] = \{\gamma'\gamma(\gamma')^{-1} : \gamma' \in \Gamma\}$ of elements $\gamma \in \Gamma$, and the length $\ell(c)$ of such a geodesic c is related to the trace of the corresponding conjugacy class $[\gamma]$ via

$$\cosh\left(\frac{\ell(c)}{2}\right) = \frac{|\text{trace}(\gamma)|}{2},$$

agreeing also with the translation length of an axis of this hyperbolic element. A standard reference about Fuchsian groups is [Katok]. A classical reference for hyperbolic geometry and Fuchsian groups is also [Beardon].

We introduce the Length Spectrum of a closed hyperbolic surface, discuss that it is of unbounded multiplicity [Randol] and that the Length Spectrum and the Laplace Spectrum of a closed surface determine each other (Huber's Theorem). We sketch the proof via Selberg's Trace Formula (following [McKean1] and also [Buser, Chapter 9]) and briefly discuss the analogy between the Selberg Zeta Function and the Riemann Zeta Function and the role of small eigenvalues. (The paper [McKean1] contains the wrong statement that closed hyperbolic surfaces don't admit eigenvalues below 1/4, which the author acknowledged as being not correct in [McKean2].)

Regarding the Laplace Spectrum, we recall Cheeger's Inequality, Variational Principles of the eigenvalues (Minimax Principles), and discuss the geometric proofs of $\lambda_{4g-2}(S) > 1/4$ for all closed surfaces, mention the optimal result $\lambda_{2g-2}(S) > 1/4$, and that there are surfaces for which λ_{2g-3} is arbitrarily small. Here, our presentation is based again on [Buser, Chapter 8] and [Buser2].

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(H2) Hyperbolic Volume

Ruth Kellerhals

We provide an overview of classical and more recent results about hyperbolic volume. In the focus are polyhedra and their volumes in terms of their dihedral angles in a space X^n_K , $n \geq 2$, of constant curvature $K \in \{\pm 1\}$. By dissection, it suffices to study volumes of so-called n-orthoschemes which are simplices in X^n_K characterized by certain orthogonality relations and a system of n independent dihedral angles; see for example [5]. We present some general results valid in all dimensions and discuss in more detail the cases $n = 3$ and $n = 5$. Below is a summary of a few but important aspects about non-Euclidean polyhedron volume with corresponding references.

- **Schläfli's volume differential formula.**

For a non-Euclidean n-simplex $S \subset X^n_K$ with dihedral angles α_F at $(n-2)$-dimensional faces $F \subset S$, one has

$$(*) \qquad d\,\mathrm{vol}_n(S) = \frac{K}{n-1} \sum_F \mathrm{vol}_{n-2}(F)\, d\alpha_F\,, \quad \text{where } \mathrm{vol}_0(S) := 1\,.$$

There are several proofs of the formula $(*)$. Two are due to Schläfli [12] for $K = 1$, only, while Kneser [8], Milnor [9] and Vinberg [13] treated both curvature cases. Observe that Schläfli's formula shows a qualitative difference when treating non-Euclidean volume problems with respect to the dimension parity.

- **Schläfli's volume reduction formula.**

As a consequence of $(*)$, Schläfli [12] proved a reduction formula expressing the volume of an even-dimensional spherical simplex in terms of the volumes of all of its lower- and odd-dimensional (spherical) vertex link volumes. An analogous result holds in the hyperbolic context; see also [5].

- **Dimension $n = 3$.**

In 1836, Lobatschevsky derived an explicit volume formula for an orthoscheme $R = [\alpha, \beta, \gamma] \subset \mathbb{H}^3$ in terms of a new function $Л_2(\omega)$ closely related to the classical dilogarithm function

$$\mathrm{Li}_2(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^2}$$

as follows (see for example [1, 4, 6]).

$$\begin{aligned}\mathrm{vol}_3(R) = \frac{1}{4}\Big\{ &\text{Л}_2(\alpha+\theta) - \text{Л}_2(\alpha-\theta) + \text{Л}_2(\frac{\pi}{2}+\beta-\theta) + \text{Л}_2(\frac{\pi}{2}-\beta-\theta)+ \\ &+ \text{Л}_2(\gamma+\theta) - \text{Л}_2(\gamma+\theta) + 2\,\text{Л}_2(\frac{\pi}{2}-\theta)\Big\}, \quad \text{where} \\ &0 \le \theta = \arctan \frac{\sqrt{\cos^2\beta - \sin^2\alpha\,\sin^2\gamma}}{\cos\alpha\cos\gamma} \le \frac{\pi}{2}, \quad \text{and} \\ &\text{Л}_2(x) = -\int_o^x \log|2\sin t|\,dt = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin(2kx)}{k^2} = \frac{1}{2}\Im\,\mathrm{Li}_2(e^{2ix})\,.\end{aligned}$$

As a consequence, and by appropriate dissections, many nice volume formulas could be derived, especially for ideal hyperbolic tetrahedra, and more generally, for pyramids with ideal apex over k-gons ($k \ge 3$); see [9] and [13].
In [3], Coxeter presented a unified picture for the volume of an orthoscheme $R[\alpha,\beta,\gamma] \subset X^3_K$, by looking at Schläfli's normalized volume functional f_3 when $K = 1$, and by considering Lobachevsky's volume formula for $K = -1$. To this end, he introduced a certain infinite series $S(\alpha,\beta,\gamma)$ establishing a correspondence between the two cases, up to a constant multiple of i.

- **Thurston's problem no. 23.**

There are several important conjectures involving hyperbolic volume in three dimensions. Let us just mention Thurston's problem no. 23 which raises the question whether there are two hyperbolic 3-orbifolds whose volume quotient is irrational; see [10]. As an example, look at the cusped manifolds given by the Figure Eight knot complement resp. the Whitehead link complement which are commensurable with the Coxeter orthoschemes $[\frac{\pi}{3},\frac{\pi}{6},\frac{\pi}{3}]$ resp. $[\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{4}]$ in an explicit way. The computation of their volumes according to Lobachevsky's formula leads to the question whether the quotient $\text{Л}_2(\frac{\pi}{3})/\,\text{Л}_2(\frac{\pi}{4})$ is an irrational number.

- **Dimension $n = 5$.**

For *2-asymptotic* orthoschemes $[\alpha_1,\dots,\alpha_5] \subset \mathbb{H}^5$, which generate the relevant scissors congruence group, an explicit, but complicated formula in terms of trilogarithm functions could be derived in [7]. In the case of the infinite subfamily consisting of elements of the form $R = [\alpha,\beta,\gamma,\alpha,\beta] \subset \mathbb{H}^5$, that is, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, the volume can be expressed according to

$$\begin{aligned}\mathrm{vol}_5(R) = &\frac{3}{64}\zeta(3) + \frac{1}{4}\{\text{Л}_3(\alpha) + \text{Л}_3(\beta) - \frac{1}{2}\text{Л}_3(\frac{\pi}{2}-\gamma)\} - \\ &- \frac{1}{16}\{\text{Л}_3(\frac{\pi}{2}+\alpha+\beta) + \text{Л}_3(\frac{\pi}{2}-\alpha+\beta)\}, \quad \text{where}\end{aligned}$$

$$\text{Л}_3(\alpha) = \frac{1}{4}\zeta(3) - \int_0^\alpha \text{Л}_2(t)\,dt = \frac{1}{4}\sum_{r=1}^{\infty}\frac{\cos(2r\alpha)}{r^3} = \frac{1}{4}\Re\big(\mathrm{Li}_3(e^{2i\alpha})\big)$$

denotes the *Trilobachevsky function.* This function is π-periodic, even and satisfies the distribution law

$$\frac{1}{m^2}\,\text{Л}_3(m\alpha) = \sum_{r=0}^{m-1} \text{Л}_3\left(\alpha + \frac{r\pi}{m}\right).$$

In particular, the values $\text{Л}_3(0) = \frac{1}{4}\,\zeta(3)$, $\text{Л}_3(\frac{\pi}{6}) = \frac{1}{12}\,\zeta(3)$ and $\text{Л}_3(\frac{\pi}{2}) = -\frac{3}{16}\,\zeta(3)$ are irrational.

As an application, the volumes of the smallest cusped hyperbolic 5-orbifold and the two known small cusped hyperbolic 5-manifolds could be derived and turn out to be explicit rational multiples of $\zeta(3)$; see for example [2].

- **Rudenko's work for arbitrary dimension n.**

In the recent work [11], Rudenko considers orthoschemes $R \subset \mathbb{H}^n$ parametrized not by dihedral angles but by configurations of points in $\mathbb{P}^1$. This point of view allows him to transfer the methods which he developed to prove the depth conjecture of Goncharov about multiple polylogarithms. In order to build the bridge to the theory of hyperbolic orthoscheme volume in terms of dihedral angles, Rudenko uses the structural results of Böhm [1] expressing $\text{vol}_n(R)$ by means of multiple polylogarithms. In this way, he provides the *quadrangulation formula* for the hyperbolic orthoscheme volume in terms of *alternating polylogarithms* and projective configurations; see [11, Theorem 1.5].

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(H3) Integral Geometry in Hyperbolic Spaces

Florian Besau

In this extended abstract we survey the hyperbolic Steiner formula, intrinsic and intersection volumes, and classical kinematic formulas in d-dimensional hyperbolic space. Emphasis is placed on the structural relations between these notions and on their interpretation via integral geometry.

1. Steiner formula and intrinsic volumes in hyperbolic space

Let $\mathbb{H}^d$ denote the d-dimensional hyperbolic space, and let $d(\mathbf{x}, \mathbf{y})$ be the geodesic distance between $\mathbf{x}, \mathbf{y} \in \mathbb{H}^d$. The group of isometries of $\mathbb{H}^d$ is denoted by $\mathcal{I}_d$. In the hyperboloid model, $\mathcal{I}_d$ can be identified with the group $O^+(d,1)$ of time-orientation preserving Lorentz-orthogonal transformations of $\mathbb{R}^{d+1}$ endowed with the indefinite inner product

$$\mathbf{z} \circ \mathbf{z} = z_1^2 + \cdots + z_d^2 - z_{d+1}^2, \qquad \mathbf{z} \in \mathbb{R}^{d+1}.$$

Let $\mathcal{K}(\mathbb{H}^d)$ be the space of hyperbolic convex bodies, i.e., compact and geodesically convex subsets of $\mathbb{H}^d$. For $K \in \mathcal{K}(\mathbb{H}^d)$ and $\varepsilon > 0$ we define the ε-parallel set of K by

$$K_\varepsilon := \{\mathbf{x} \in \mathbb{H}^d : d(\mathbf{x}, K) \le \varepsilon\},$$

where $d(\mathbf{x}, K) := \inf_{\mathbf{y} \in K} d(\mathbf{x}, \mathbf{y})$.

The hyperbolic Steiner formula asserts the existence of functionals $V_i : \mathcal{K}(\mathbb{H}^d) \to [0, \infty)$, $i = 0, \dots, d-1$, satisfying the following properties:

(i) *Valuation property*: for $K, L \in \mathcal{K}(\mathbb{H}^d)$ with $K \cup L \in \mathcal{K}(\mathbb{H}^d)$,

$$V_i(K) + V_i(L) = V_i(K \cap L) + V_i(K \cup L).$$

(ii) *Continuity and invariance*: V_i is continuous with respect to the Hausdorff metric and invariant under the action of $\mathcal{I}_d$.

(iii) *Hyperbolic Steiner formula*: for all $K \in \mathcal{K}(\mathbb{H}^d)$ and $\varepsilon > 0$,

$$\mathcal{H}^d(K_\varepsilon) = \mathcal{H}^d(K) + \sum_{i=0}^{d-1} V_i(K)\, \ell_i^d(\varepsilon),$$

where $\mathcal{H}^d$ denotes d-dimensional Hausdorff measure on $\mathbb{H}^d$ and

$$\ell_i^d(\varepsilon) := \int_0^\varepsilon (\sinh t)^{d-1-i} (\cosh t)^i \, \mathrm{d}t.$$

The functionals V_i are called *intrinsic volumes*. Up to normalization, they coincide with the i-dimensional Hausdorff measure of i-dimensional convex bodies contained in totally geodesic i-subspaces of $\mathbb{H}^d$. We also set $V_d := \mathcal{H}^d$.

If K has smooth boundary ∂K, then

$$V_i(K) = \binom{d-1}{i} M_{d-1-i}(\partial K), \qquad i = 0, \dots, d-1,$$

where $M_k(\partial K)$ denotes the k-th total normalized mean curvature of ∂K. In this case the Steiner formula reduces to Weyl's classical tube formula for smooth submanifolds of $\mathbb{H}^d$ [3].

A far-reaching extension of the Steiner formula to sets of positive reach in hyperbolic space was established by Kohlmann [4], who also proved a hyperbolic version of the *Gauss–Bonnet theorem*:

$$\frac{\omega_{d+1}}{2}\,\chi(K) = \sum_{i=0}^{\lfloor d/2\rfloor} (-1)^i V_{2i}^0(K),$$

where $\omega_d := \frac{2\pi^{d/2}}{\Gamma(d/2)}$ and

$$V_i^0(K) := \frac{\omega_{d+1}}{\omega_{i+1}\,\omega_{d-i}}\,V_i(K), \qquad i = 0,\dots,d.$$

Here $\chi(K)$ denotes the Euler characteristic, which equals 1 for convex bodies in $\mathbb{H}^d$.

For $d = 2$ and a hyperbolic triangle T with interior angles α,β,γ, one recovers the classical area formula $\mathcal{H}^2(T) = \pi - (\alpha+\beta+\gamma)$.

2. Classical kinematic formulas in hyperbolic space

The isometry group $\mathcal{I}_d$ carries a locally finite Haar measure λ, called the *kinematic measure*, unique up to scale. We normalize λ by requiring that for any Borel set $B \subset \mathbb{H}^d$,

$$\mathcal{H}^d(B) = \lambda\big(\{\rho \in \mathcal{I}_d : \rho(\mathbf{x}_0) \in B\}\big),$$

for a fixed $\mathbf{x}_0 \in \mathbb{H}^d$.

Similarly, the Grassmannian $\overline{\mathrm{Gr}}_k(\mathbb{H}^d)$ of totally geodesic k-planes admits an invariant probability measure μ_k, normalized such that

$$\int_{\overline{\mathrm{Gr}}_k(\mathbb{H}^d)} \mathcal{H}^k(K\cap L^k)\,\mu_k(\mathrm{d}L^k) = \mathcal{H}^d(K)$$

for all $K \in \mathcal{K}(\mathbb{H}^d)$.

The *general Cauchy–Crofton formula* states that for $r \le k \le d$,

$$\frac{\omega_{d+1-r}}{\omega_{d+1}} \int_{\overline{\mathrm{Gr}}_{d-r}(\mathbb{H}^d)} V_{k-r}^0(K\cap L^{d-r})\,\mu_{d-r}(\mathrm{d}L^{d-r}) = V_k^0(K),$$

see [1, 5, 6].

The *intersection volumes* U_i^0 are defined by

$$U_i^0(K) := \frac{\omega_{d+1-i}}{2} \int_{\overline{\mathrm{Gr}}_{d-i}(\mathbb{H}^d)} \chi(K\cap L^{d-i})\,\mu_{d-i}(\mathrm{d}L^{d-i}),$$

for $i = 0,\dots,d-1$, with $U_d^0(K) := V_d^0(K) = \mathcal{H}^d(K)$. One has $U_0^0(K) = \omega_{d+1}/2$.

Combining Gauss–Bonnet and Crofton formulas yields the *hyperbolic Cauchy–Crofton relations*

$$U_{d-k}^0(K) = \sum_{i=0}^{\lfloor k/2\rfloor} (-1)^i V_{d-k+2i}^0(K), \qquad k = 0,\dots,d.$$

Thus intrinsic volumes and intersection volumes are linearly related. In particular,

$$\frac{\omega_{d+1}}{2}\chi(K) = V_0^0(K) - U_2^0(K),$$

see [6].

The *generalized kinematic formula* asserts that for $K, L \in \mathcal{K}(\mathbb{H}^d)$,

$$\int_{\mathcal{I}_d} V_k^0(K \cap \rho L)\,\lambda(\mathrm{d}\rho) = \sum_{a+b=d+k} V_a^0(K)V_b^0(L).$$

Applying Gauss–Bonnet yields the *principal kinematic formula*

$$\frac{\omega_{d+1}}{2}\int_{\mathcal{I}_d} \chi(K \cap \rho L)\,\lambda(\mathrm{d}\rho) = \sum_{i+j=d} V_i^0(K)U_j^0(L).$$

Using Alesker's theory of smooth valuations one can reproduce the generalized kinematic formula from the principal kinematic formula, see, for example, [1]. Furthermore, the principal kinematic formula also easily reproduces the Steiner formula:

$$\begin{aligned}\mathcal{H}^d(K_\varepsilon) &= \int_{\mathcal{I}_d} \mathbf{1}\{\rho(\mathbf{x}_0) \in K_\varepsilon\}\,\lambda(\mathrm{d}\rho) = \int_{\mathcal{I}_d} \chi(K \cap \rho B_\varepsilon(\mathbf{x}_0))\,\lambda(\mathrm{d}\rho)\\ &= \frac{2}{\omega_{d+1}}\left(V_d^0(K)U_0^0(B_\varepsilon(\mathbf{x}_0)) + \sum_{i=0}^{d-1} V_i^0(K)U_{d-i}^0(B_\varepsilon(\mathbf{x}_0))\right)\\ &= \mathcal{H}^d(K) + \sum_{i=0}^{d-1} V_i(K)\ell_i^d(\varepsilon),\end{aligned}$$

where $B_\varepsilon(\mathbf{x}_0)$ is the closed ball of radius ε around $\mathbf{x}_0$ and we used the fact that $2\,U_k^0(B_\varepsilon(\mathbf{x}_0)) = \omega_k\,\omega_{d+1-k}\,\ell_{d-k}^d(\varepsilon)$ which can be verified by directly by calculation.

Recent work of Bernig, Faifman, and Solanes [2] extends Crofton-type formulas to pseudo-Riemannian space forms, encompassing and generalizing the hyperbolic setting.

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(H4) Hyperbolic surfaces and manifolds

Bram Petri

(joint work with Maxime Fortier Bourque, Will Hide, Mingkun Liu, Anna Roig Sanchis, Joe Thomas)

We will discuss hyperbolic surfaces and hyperbolic 3-manifolds. We will do so through examples and extremal problems.

The first lecture will be focused on surfaces and the problem of the maximal possible **systole** - the length of the shortest closed geodesic - of a closed and orientable hyperbolic surface of a given genus. We will start with a brief history of the problem. We will also mention our results with Maxime Fortier Bourque [FBP23] on linear programming bounds for this problem. Moreover, we will mention work with Mingkun Liu on random constructions of hyperbolic surfaces with large systoles [LP23]. The largest part of the talk will be spent on Jenni's theorem [Jen81]: the fact that the Bolza surface uniquely maximizes the systole among surfaces of genus two. This is still the only genus in which a complete answer is known.

In the second lecture, we will discuss 3-manifolds (and orbifolds). We will start with an example: the **Apollonian orbifold**. This is a geometrically finite orbifold of infinite volume that naturally appears when studying the Apollonian gasket. We will also discuss the super Apollonian group: the right-angled Coxeter group whose defining graph is the 1-skeleton of the 3-cube. This group acts on hyperbolic 3-space as a lattice and contains the Apollonian group – the Kleinian group corresponding to the Apollonian orbifold – as a subgroup of infinite index. The extremal problem we will discuss in this lecture is that of the **spectral gap**: the bottom of the spectrum of the Laplacian. The question is how large this can be as the volume of the underlying manifold grows. In particular, Magee and Thomas conjectured that there are sequences of closed hyperbolic 3-manifolds whose volume tends to infinity and whose spectral gap tends to that of hyperbolic 3-space. We will end this lecture by describing joint work with Will Hide, Anna Roig Sanchis and Joe Thomas [HPRST25] on a random construction of hyperbolic 3-manifolds with an explicit spectral gap (that is however below that of hyperbolic 3-space).

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(S1) Random tessellations in hyperbolic space

Anna Gusakova

Random tessellations are among the classical models of stochastic geometry. In Euclidean space $\mathbb{R}^d$, the study of random tessellations goes back to the 1960s and remains an active research direction (see [9] for a survey). The idea of extending the Euclidean setting to non-Euclidean ones was proposed at roughly the same time, but it has not attracted much attention and only a few results were obtained in this direction. Recently, the study of stochastic geometry models, and in particular random tessellations, in hyperbolic space $\mathbb{H}^d$ has become a topic of interest. A number of very recent results have revealed interesting new phenomena in the behaviour of random geometric models in $\mathbb{H}^d$ that are not observed in $\mathbb{R}^d$. In this overview talk we discuss the concept of random tessellations in hyperbolic space and consider some models, such as Poisson hyperplane and Poisson–Voronoi tessellations, in more detail.

Random tessellations in $\mathbb{H}^d$ and their characteristics: A tessellation T in $\mathbb{H}^d$ is a locally finite countable system of convex, closed subsets of $\mathbb{H}^d$ (cells) such that each cell has interior points, the interiors of distinct cells do not intersect, and the cells cover the space. In contrast to the Euclidean case, we do not require the cells to be bounded. A random tessellation $\mathcal{T}$ is defined as a random collection of cells that is almost surely a tessellation.

Given a random tessellation $\mathcal{T}$, let $C_o(\mathcal{T})$ be the almost surely unique cell of $\mathcal{T}$ containing the fixed origin o of $\mathbb{H}^d$. In the case when all cells of $\mathcal{T}$ are almost surely bounded and the law of $\mathcal{T}$ is invariant under the isometry group $I(\mathbb{H}^d)$ of $\mathbb{H}^d$, we may also consider the cell intensity and the typical cell of $\mathcal{T}$, defined as follows. Let $\mathcal{C}^d$ be the space of compact subsets of $\mathbb{H}^d$ and let $z : \mathcal{C}^d \mapsto \mathbb{H}^d$ be a measurable function satisfying $z(\rho K) = \rho z(K)$ for any $K \in \mathcal{C}^d$ and $\rho \in I(\mathbb{H}^d)$. Let $B \subset \mathbb{H}^d$ be a subset of hyperbolic volume 1. Then the cell intensity $\gamma_d(\mathcal{T})$ is given by

$$\gamma_d(\mathcal{T}) := \mathbb{E}\Big[\sum_{C\in\mathcal{T}} 1\{z(C) \in B\}\Big].$$

Further, given $x \in \mathbb{H}^d$, let $I(\mathbb{H}^d, o, x) = \{\rho \in I(\mathbb{H}^d)\colon \rho o = x\}$ and let $\kappa(x, \cdot)$ be the unique invariant probability measure on $I(\mathbb{H}^d, o, x)$. Then the typical cell $C_{\rm typ}(\mathcal{T})$ of $\mathcal{T}$ is a random polytope with distribution

$$\mathbb{P}(C_{\rm typ}(\mathcal{T}) \in \cdot) = \gamma_d(\mathcal{T})^{-1}\mathbb{E}\Big[\sum_{C\in\mathcal{T}} \int_{I(\mathbb{H}^d)} 1\{\rho^{-1}C \in \cdot, z(C) \in B\}\kappa(z(C), \mathrm{d}\rho)\Big].$$

We note that the definition of cell intensity and, hence, the typical cell relies on the choice of the centre function z. Using the mass transport principle, one can show [3] that there is no such choice of centre function in the case when $\mathcal{T}$ contains cells of infinite volume. One open problem is to find a natural extension of the notions of cell intensity and the typical cell to the case when $\mathcal{T}$ contains unbounded cells.

Poisson hyperplane tessellation: One possibility to generate a random tessellation is to consider the dissection of space by a set of hyperplanes. Let η_t be an isometry-invariant Poisson process on the space of hyperplanes (totally geodesic subspaces of dimension $(d-1)$) of $\mathbb{H}^d$ with intensity measure $t \cdot \mu_{d-1}$, where μ_{d-1} is an invariant measure on the space of hyperplanes and $t > 0$ is a parameter, called the intensity. The Poisson hyperplane tessellation $\mathcal{H}_t$ is defined as the collection of closures of the connected components of $\mathbb{H}^d \setminus \bigcup_{H \in \eta_t} H$. An interesting phenomenon (not observed in $\mathbb{R}^d$) appears for the zero cell $C_o(\mathcal{H}_t)$: namely, there exists $t_c \in (0, \infty)$ such that $C_o(\mathcal{H}_t)$ is almost surely bounded if and only if $t \geq t_c$ [1, 3, 5, 8]. Further results include some exact mean value formulas for the zero cell [5, 2], but no exact formulas for the cell intensity or results for the typical cell are known.

Poisson–Voronoi tessellation: Let ξ_t be a Poisson point process with intensity measure $t \cdot \mathrm{vol}_{\mathbb{H}^d}$. Given $x \in \mathbb{H}^d$, we define the Voronoi cell of x with respect to ξ_t as

$$V(x, \xi_t) := \{y \in \mathbb{H}^d \colon d_{\mathbb{H}^d}(y, x) \leq d_{\mathbb{H}^d}(y, x') \, \forall x' \in \xi_t\}.$$

The Poisson–Voronoi tessellation $\mathcal{V}_t := \{V(x, \xi_t) \colon x \in \xi_t\}$ is the collection of all Voronoi cells. The cell intensity and the typical cell are easy to describe in this case, since $\gamma_d(\mathcal{V}_t) = t$ and $C_{\mathrm{typ}}(\mathcal{V}_t)$ has the same distribution as $V(o, \xi_t)$. Some mean value characteristics for the typical cell have been derived in [5, 6, 7], but no results for the zero cell are available in the literature. A new phenomenon related to the hyperbolic nature of the model arises when we consider the low-intensity limit $t \to 0$. The random tessellation $\mathcal{V}_t$ converges to a non-trivial limit, called the ideal Poisson–Voronoi tessellation [4].

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(S2) Boolean Models in Hyperbolic Space

Daniel Hug

The union of the particles of a stationary Poisson process of compact (convex) sets in Euclidean space is called Boolean model and is a classical topic of stochastic geometry. Alternatively, the Boolean model is obtained by independent marking of the points of a Poisson point process by compact particles. The Boolean model is a benchmark model of stochastic geometry and the subject of numerous investigations and applications. We first introduce the model in Euclidean space and derive several of its first and second order properties in an exemplary way (see [6] for recent results and references).

To describe the Boolean model Z in d-dimensional hyperbolic space and to contrast properties and special features that are specific for the hyperbolic setting, we start from a new general decomposition (and characterization) result for isometry invariant measures on compact particles. This result applies in particular to an isometry invariant Poisson particle process η, which underlies the Boolean model Z. Geometric functionals f such as the volume, the surface area or the Euler characteristic of the intersection of the Boolean model with a compact convex observation window W are studied. In particular, the asymptotic behavior of $f(Z \cap W)$, for balls $\mathbb{B}_R$ with increasing radii R as observation windows W, is investigated. Exact and asymptotic formulas for expectations, variances, and covariances are shown for $f(Z \cap W)$. In addition, qualitative and quantitative central limit theorems are derived. Compared to the Euclidean framework, several new phenomena can be observed. Basically, due to the non-vanishing of boundary terms, additional contributions in limit results show up. Moreover, several terms arising in the limit involve integration with respect to horoballs or the restriction of points to horoballs. The results presented are mainly based on [7]. We indicate how several new results from integral geometry in hyperbolic geometry enter the arguments.

The central limit theorems for the Boolean model are contrasted with asymptotic fluctuation results for processes of k-planes in d-dimensional hyperbolic space, where also non-normal (but infinitely divisible) limit distributions arise, depending on the relation of the dimension parameters d and k. Starting with [5], recent work in this direction is covered by [1, 2, 4, 9], see also the monograph [8].

We conclude with a question concerning the visible volume in the complement of the Boolean model [3].

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(S3) Random graphs on the hyperbolic plane

Nikolaos Fountoulakis
(joint work with Michel Bode, Tobias Müller)

This mini-course comprised of two introductory lectures on the basic theory of random graphs on the hyperbolic plane. The focus of the lectures was the model of Krioukov, Papadopoulos, Kitsak,Vahdat and Boguñá [1]. This model was introduced in the context of modelling complex networks. In a nutshell, the model creates random geometric graphs on the hyperbolic plane that exhibit common features that are typically observed in a wide range of networks that emerge in a variety of contexts such as computer networks, social networks or biological networks. The idea behind this model is the explanation of these features as emerging from the underlying hyperbolic geometry.

The first lecture introduced the model and exhibited the basic features of the resulting random graphs as the basic parameters of the model vary.

The second lecture covered two central results related to the typical structure of the random graph. The first result is about the connectivity of the random graph. We gave an overview of a result by Bode, Fountoulakis and Müller [3] which gives the critical values of the parameters of the model around which the random graph transitions from connectivity to non-connectivity. The other result we covered is about the emergence of the giant component, that is, having a largest component of linear order. The presentation covered two results by Bode, Fountoulakis and Müller [2] and by Fountoulakis and Müller [4]. The first result provides a proof of the critical condition for the emergence of a giant component. The second result provides a more detailed proof of this result, relating the model with a continuum percolation model and the emergence of an infinite component therein.

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(S4) Random hyperbolic surfaces

Jean Raimbault

Introduction. In this talk we discuss models of random surfaces and their properties. In each case the model is supported on closed orientable surfaces of bounded genus and the properties we are interested in are asymptotically generic; for a family of random objects depending on an integer parameter n, we say as usual that a property holds *asymptotically almost surely* (abbreviated as a.a.s.) if the probability that it holds converges to 1 as n goes to infinity.

One of the models presented in the talk is essentially combinatorial in nature while the other is more analytical. There are various motivations behind the study of random surfaces. Some originate in theoretical physics, which we will not discuss here (see [PS06]). For us their use will be mostly as a tool to study extremal and typical properties of hyperbolic surfaces. That is, given a quantity defined over the moduli space $\mathcal{M}_g$ of hyperbolic surfaces of genus g, find its global extrema, and whether it admits a typical (with respect to a given random model) behaviour over $\mathcal{M}_g$ for large g. With regards to the latter we will be interested in seeing similar behaviour reproduce for seemingly unrelated models.

First part: Two random models and their properties. The first model is that of *random Belyí surfaces*, originally studied by Brooks–Makover [BM04]. It is constructed by taking a random triangulation coming from a random locally oriented (or ribbon) trivalent graph, the latter according to the configuration model with i.i.d. uniform local orientations. A geometric structure is then put on the exterior of the vertices by taking every triangle to be equal to a fixed given equilateral triangle, and then extending the conformal structure to the vertices. The resulting closed Riemann surface is a.a.s. connected and hyperbolic when the number of triangles goes to infinity. We will call $\mu_{\mathrm{bm}}^{(n)}$ the distribution of the resulting random surface (with $2n$ triangles).

The second model may be called *random Weil–Petersson surfaces*. The moduli space $\mathcal{M}_g$ has a Kähler structure of finite volume and the random surface is taken with respect to the probability normalisation of this measure. The measure can be fairly easily described locally: it is the Lebesgue measurein Fenchel–Nielsen coordinates. The main technical tools to study this model are due to Mirzakhani [Mir13] and we will not say anything about them. We will call $\mu_{\mathrm{wp}}^{(g)}$ the distribution of the resulting random genus g surface.

We discuss the asymptotic behaviour of a few geometric invariants of random surfaces. There are many other results that we do not have the time to present.

The first result on the random Belyí surfaces is their asymptotic genus: Brooks and Makover proved that it concentrates around $\frac{n}{2}$.

One of the main open problems about hyperbolic surfaces of large genus is whether it is possible to find $X \in \mathcal{M}_g$ for arbitrarily large with first positive eigenvalue of the Laplacian $\lambda_1(X) > \frac{1}{4}$ (which is asymptotically optimal). Both random models come close to this: we have $\lambda_1(X) > \frac{1-\varepsilon}{4}$ a.a.s. for any $\varepsilon > 0$, due to Anantharaman–Monk for $\mu_{\mathrm{wp}}^{(g)}$ and to Shen–Wu for $\mu_{\mathrm{bm}}^{(n)}$ [AM24, SW25]. It is unclear at present whether the probability that $\lambda_1 > \frac{1}{4}$ is asymptotically positive or not for either model.

Recently it was shown by Budzinski–Curien–Petri that the asymptotic minimum for the diameter of surfaces in $\mathcal{M}_g$ is $\log(g)$, using an ad hoc random model. The same team also established that for random Belyí surfaces the diameter concentrates around $2\log(n)$ for large n [BCP21]. For random Weil–Petersson surfaces there is an asymptotic upper bound of $2\log(g)$ but it is unknown if it is sharp [Mag].

Finally, a more subtle invariant is the length spectrum of a surface, the collection of all lengths of closed unoriented primitive geodesics. For a random surface this can be seen as a point process on the half-line $]0, +\infty[$. For both the Belyí surface and the Weil–Petersson random model it converges in distribution to a Poisson point process, due to Petri in the first case and Mirzakhani–Petri in the second [Pet17, MP19]. However the intensities of the limit processes are very different: for $\mu_{\mathrm{wp}}^{(g)}$ it has a density w.r.t Lebesgue measure, while in the second case it is supported on a uniformly discrete subset. This shows a difference between the two models: while the Weil–Petersson random surfaces occupy the whole $\mathcal{M}_g$, the typical Belyí surface lies in a relatively shallow part of it, even though Belyí surfaces themselves are dense in every moduli space.

Second part: how to study random Belyí surfaces. In the second part of the talk we explained some of the techniques used to study random Belyí surfaces. These blend combinatorics and hyperbolic geometry.

We only discussed three results : Gamburd's theorem that the asymptotic distribution for the degree of vertices of random triangulations follows the same law as the asymptotic distribution of cycles for random permutations (the so-called Poisson–Dirichlet distribution) [Gam06], Brooks' theorem comparing the geometry of ideal triangulations with those of their conformal fillings [Bro99], and (very briefly and sketchily) Budzinski–Curien–Petri's result on the typical diameter of random Belyí surfaces.

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(T1) Volumes of regular hyperbolic simplices

Zakhar Kabluchko
(joint work with Philipp Schange)

We present an explicit, dimension-uniform formula for the hyperbolic volume of a regular d-simplex of prescribed side length in hyperbolic space of curvature -1. The representation is given by a one-dimensional contour integral involving the standard normal distribution function analytically continued to complex arguments. As a limit case, the same formula yields the volume of the ideal regular simplex. The method extends further to orthocentric simplices and produces analogous formulas in both hyperbolic and spherical geometries.

Setting. We work in the Klein model of d-dimensional hyperbolic space of constant curvature $\kappa = -1$, realized as the open unit ball

$$\mathbb{B}^d := \{x \in \mathbb{R}^d : \|x\| < 1\}.$$

The hyperbolic volume of a Borel set $A \subseteq \mathbb{B}^d$ is

$$\mathrm{Vol}_{d,-1}(A) := \int_A \frac{\mathrm{d}y}{(1 - \|y\|^2)^{(d+1)/2}}.$$

In the Klein model, a *hyperbolic simplex* is simply a Euclidean simplex

$$[v_0, \dots, v_d] \subseteq \overline{\mathbb{B}}^d := \{x \in \mathbb{R}^d : \|x\| \le 1\}.$$

It is *ideal* if all vertices lie on the unit sphere $\mathbb{S}^{d-1} = \partial\mathbb{B}^d$. A simplex is *regular* if any permutation of its vertices is induced by a hyperbolic isometry; equivalently (for vertices in $\mathbb{B}^d$), all hyperbolic edge lengths are equal. We write Δ^d_ℓ for a regular hyperbolic d-simplex with hyperbolic side length $\ell > 0$, and Δ^d_∞ for an ideal regular hyperbolic simplex.

Our formulas for $\mathrm{Vol}_{d,-1}(\Delta_\ell^d)$ involve the standard normal distribution function extended to complex arguments,

$$\Phi : \mathbb{C} \to \mathbb{C}, \qquad \Phi(z) := \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^z \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x, \tag{1}$$

where the integral is taken along any contour in $\mathbb{C}$ connecting 0 and z. It is well known that Φ is an entire function.

Main theorem. The following statement provides an explicit contour-integral representation for the volume of Δ_ℓ^d in every dimension.

Theorem 1 (Volume of a regular hyperbolic simplex)**.** *Let $d \geq 2$. In the d-dimensional hyperbolic space of constant curvature $\kappa = -1$ consider a regular d-dimensional hyperbolic simplex Δ_ℓ^d with hyperbolic side length $\ell > 0$. Then, the hyperbolic volume of Δ_ℓ^d is given by*

$$\frac{\sqrt{2}\,\pi^{d/2}}{\mathrm{i}^d\,\Gamma\left(\frac{d+1}{2}\right)} \int_{-(1-\mathrm{i})\infty}^{+(1-\mathrm{i})\infty} \Phi^{d+1}\left(\mathrm{i}\sqrt{\frac{\cosh \ell}{1 + d\cosh \ell}}\,x\right) \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x.$$

The hyperbolic volume of the ideal regular d-dimensional hyperbolic simplex Δ_∞^d is given by

$$\frac{\sqrt{2}\,\pi^{d/2}}{\mathrm{i}^d\,\Gamma\left(\frac{d+1}{2}\right)} \int_{-(1-\mathrm{i})\infty}^{+(1-\mathrm{i})\infty} \Phi^{d+1}\left(\frac{\mathrm{i}x}{\sqrt{d}}\right) \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x.$$

Here $\Gamma(\cdot)$ denotes the Gamma function and $\mathrm{i} = \sqrt{-1}$. Integrals of the form $\int_0^{\omega\infty} f(x)\,\mathrm{d}x$, with $\omega \in \mathbb{C}\setminus\{0\}$ and entire f, are interpreted as improper integrals via

$$\int_0^{\omega\infty} f(x)\,\mathrm{d}x := \lim_{B\to\infty} \int_0^B f(\omega y)\,\omega\,\mathrm{d}y,$$

whenever the limit exists.

Relation to previous work. Volumes of hyperbolic (and spherical) simplices and polytopes have a long history going back to Lobachevsky, Bolyai, and Schläfli; see, for example, [1, 5, 6]. For ideal regular simplices, Haagerup and Munkholm [2] proved that Δ_∞^d uniquely maximizes volume among all hyperbolic d-simplices. On the spherical side, explicit formulas for volumes of regular spherical simplices appear in the work of Vershik and Sporyshev [7], motivated by asymptotic face numbers of random polytopes and neighborliness phenomena. In the present work, the special function governing volumes of regular spherical simplices and appearing in [7] is analytically continued to a larger domain; this continuation naturally extends the spherical volume formulas to cover regular hyperbolic simplices as well.

Beyond regular simplices. The theorem above is a special case of a more general framework: we derive corresponding hyperbolic and spherical volume formulas for *orthocentric* simplices. Regular simplices appear naturally within this class, but the orthocentric setting is substantially broader while still permitting explicit volume expressions in terms of Φ.

This talk is based on the paper [4]. The spherical case for orthocentric simplices was previously treated in [3].

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(T2) Subgraphs of Kernel-Based Geometric Random Graphs

Clara Stegehuis

(joint work with Riccardo Michielan, Matthias Walter)

The study of subgraphs in random networks has long been motivated by the analysis of *motifs*, statistically significant patterns that recur in real-world networks. Classical null models, such as the configuration model, often fail to capture clustering and geometric dependencies, leading to unrealistic comparisons. Geometric scale-free random graphs provide a more realistic framework, combining power-law degree distributions with spatial embedding. This interplay between geometry and degree heterogeneity strongly influences the occurrence of subgraphs and their scaling behavior.

1. Model

We consider kernel-based geometric random graphs, a broad class that includes hyperbolic random graphs, spatial preferential attachment models, and geometric inhomogeneous random graphs. Vertices are assigned random weights $(w_i)_{i\in[n]}$ following a power-law distribution, and positions x_i, uniformly sampled on the d-dimensional torus $\mathbb{T}^d$. Every pair of vertices is independently connected by an edge with probability $p(w_i, w_j, |x_i - x_j|)$. Interestingly, this model is equivalent to the Hyperbolic random graph [2].

2. Main Results

Our analysis shows that:

- The scaling of subgraph counts in the number of vertices n can be expressed through a *mixed-integer linear program* (MILP), which encodes the expected number of subgraph opies at sepcific weights and distances.
- A phase transition occurs for Hamiltonian subgraphs: when the degree-exponent of the power-law is large, the optimizer corresponds to a geometric regime (close vertices of low degree), while for small degree-exponent, they are in a non-geometric regime (distant hubs of high degree).
- For hyperbolic random graphs, this means that for small degree-exponent, almost all Hamiltonian subgraphs are formed within the disk of radius $R/2$, between vertices of any distances. For larger degree-exponent, asymptotically all subgraphs are formed at the boundary of the disk of radius R, between close-by vertices.
- These concentration results allow us to develop better tools to distinguish hyperbolic and non-geometric random graphs [3].

3. Methods

We employ a divide et impera strategy: first, we count subgraphs restricted to vertex classes with prescribed weights and distances; then, we optimize over these classes using MILP formulations. This approach allows us to overcome edge dependence in geometric models.

4. Conclusion

Our results highlight the interplay between geometry and degrees in random graphs. Subgraphs appear in highly structured and concentrated locations in hyperbolic random graphs, either near the center of the disk or in clustered regions at the edge of the disk. The MILP framework provides a powerful tool to analyze these phenomena systematically.

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(T3) Around the ideal Poisson-Voronoi tessellation

Nicolas Curien
(joint work with Matteo D'Achille, Nathanaël Enriquez, Russell Lyons and Meltem Ünel)

We investigate the low-intensity limit of Poisson–Voronoi tessellations in real hyperbolic spaces $\mathbb{H}_d$ for dimensions $d \geq 2$. In contrast to the Euclidean setting, where the Voronoi tessellation of a homogeneous Poisson point process with decreasing intensity becomes trivial, in hyperbolic geometry a rich and nontrivial limiting structure emerges as the point intensity tends to zero. We show that this limit defines a new isometry-invariant decomposition of $\mathbb{H}_d$, termed the *ideal Poisson–Voronoi tessellation* $\mathcal{V}_d$, consisting of countably many unbounded convex polytopes, each having a unique end and boundary at the ideal boundary of hyperbolic space. This ideal tessellation arises naturally from a Poisson process on the space of horospheres in $\mathbb{H}_d$, and its construction exploits the exponential volume growth and negative curvature inherent to hyperbolic geometry. We establish convergence of the finite-intensity Voronoi cells to $\mathcal{V}_d$ in an appropriate topology as the intensity parameter tends to zero, and analyze central geometric properties of the limiting cells, including distributional and invariance characteristics under the full group of hyperbolic isometries. The cells in $\mathcal{V}_d$ retain a statistical homogeneity and exhibit novel structural features tied to the geometry of the hyperbolic boundary, distinguishing them from their Euclidean counterparts. Our results provide the first rigorous description of a nontrivial ideal geometric tessellation emerging from a low-intensity limit of classical stochastic geometry models in non-Euclidean spaces, and lay groundwork for further probabilistic and geometric analysis of random partitions in curved spaces.

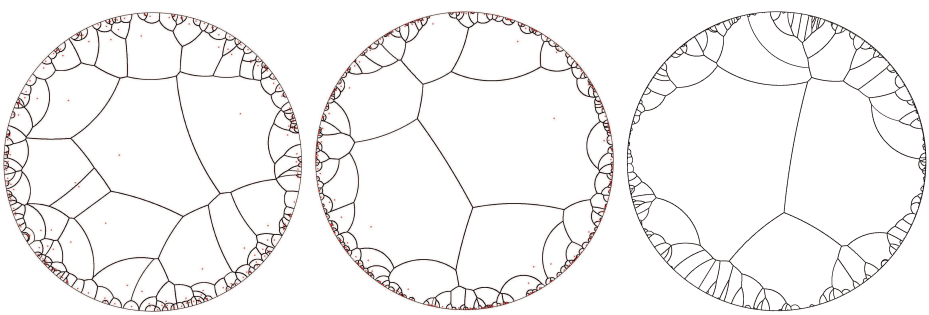

Figure 1. A simulation of three Poisson–Voronoi tessellations of $\mathbb{H}^2$ with intensity $1, 0.1$ and 0.001. The limiting tessellations has no nuclei but still reveals an interesting geometry.

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(T4) Hyperbolic radial spanning tree

Vanessa Trapp

(joint work with Daniel Rosen, Matthias Schulte and Christoph Thäle)

We consider the radial spanning tree in hyperbolic space generated by a unit intensity Poisson process, which was first introduced in [2]. A simulation of a radial spanning tree in the upper half space model can be seen in Figure 1.

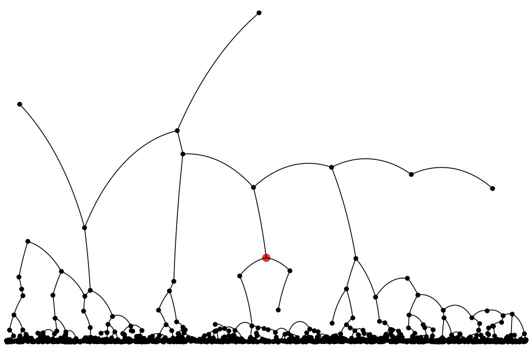

Figure 1. Simulation in the upper half-space model and added origin (red)

In particular, we focus on two characteristics of the model: the degree of the origin and the total edge-length.

For the degree of the origin o, the following explicit integral representation of its expectation can be shown.

$$\mathbb{E}[\deg(o)] = \omega_d \int_0^\infty \sinh^{d-1}(r) \exp[-2\kappa_{d-1} I(r)] \mathrm{d}r,$$

where ω_d is the surface content of the $(d-1)$-dimensional Euclidean unit sphere and κ_{d-1} denotes the volume of the $(d-1)$-dimensional Euclidean unit ball. The expression $I(r)$ is related to an intersection volume. More precisely,

$$2\kappa_{d-1} I(r) := 2\kappa_{d-1} \int_0^{r/2} \Big(\frac{\cosh^2 r}{\cosh^2(r-t)} - 1 \Big)^{\frac{d-1}{2}} \mathrm{d}t$$

is the volume of the intersection of two hyperbolic balls of radius r whose centres are of distance r. An approximate evaluation of this expression in dimension two indicates that the expected degree in hyperbolic space is larger than its Euclidean counterpart, which was computed in [1, Equation (7)].

In contrast to the Euclidean case, where the degree of the origin is almost surely bounded (see for example [1, Lemma 3.2] for dimension two), the hyperbolic setting exhibits a fundamentally different behaviour: all moments of the degree exist, yet the degree is not almost surely bounded. The proof of this unboundedness mainly relies on a geometric construction in hyperbolic space, showing that point configurations forcing an arbitrarily large degree of the origin can occur with positive probability.

We further investigate the total edge-length of the radial spanning tree restricted to balls of increasing radius around the origin, i.e. for a Poisson process η we consider

$$\mathcal{L}_R = \sum_{x \in \eta \cap B^d(o,R)} \ell(x, \eta),$$

where $B^d(o, R)$ denotes the ball around the origin o of radius R and $\ell(x, \eta)$ is the distance of x to its radial nearest neighbour in the radial spanning tree generated by the Poisson process η.

For this functional, we analyse the asymptotic behaviour of the expectation, derive lower and upper variance bounds and establish quantitative central limit theorems as $R \to \infty$. Let $V_d(R)$ denote the volume of $B^d(o, R)$. As $R \to \infty$, both the expectation and the variance are of order $V_d(R)$. The convergence rates in the quantitative central limit theorems - measured in Wasserstein or Kolmogorov distance - are of order $1/\sqrt{V_d(R)}$ as $R \to \infty$. Analogous results in the Euclidean setting were obtained in [5].

To prove lower variance bounds and central limit theorems, we employ tools originally developed in [3, Theorem 6.1] and [6, Theorem 1.1] for more general spaces, involving so-called difference operators. These methods can be applied to the total edge-length of the radial spanning tree in Euclidean space as well as in hyperbolic space.

Although the results for the total edge-length in Euclidean and hyperbolic space appear similar, there are two major differences. First, hyperbolic balls scale differently with their radius than Euclidean balls, leading to different orders of the asymptotic expectation and variance, as well as different convergence rates in the central limit theorems. Second, instead of half-spaces in Euclidean space, horoballs appear in the proofs and formulas in the hyperbolic setting. Horoballs are important objects of hyperbolic geometry without Euclidean counterparts. They are obtained as limits of increasing balls with a fixed point in their boundaries and centres tending to infinity along a geodesic ray. The volume of an intersection of a ball with a horoball, for example, is contained in the formula for the asymptotic expectation.

The results in hyperbolic space and the corresponding proof ideas presented here are based on [4].

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