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Number Theory and Functional Equations. – *Automorphism groups of commuting polynomial maps of the affine plane*, by Joseph H. Silverman, accepted on 1 July 2025.

Dedicated to Enrico Bombieri on the occasion of his 85th birthday.

Abstract. – Let $\mathcal{L}$ be a finite-dimensional semisimple Lie algebra of rank N over an algebraically closed field of characteristic 0. Associated with $\mathcal{L}$ is a family of polynomial folding maps

$$\mathsf{F}_n : \mathbb{A}^N \to \mathbb{A}^N \quad \text{for } n \geq 1$$

having the property that F_n has topological degree n^N and

$$\mathsf{F}_m \circ \mathsf{F}_n = \mathsf{F}_n \circ \mathsf{F}_m \quad \text{for all } m, n \geq 1.$$

We derive formulas for the leading terms of the folding maps on $\mathbb{A}^2$ associated with the Lie algebras $\mathcal{A}_2$, $\mathcal{B}_2$, and $\mathcal{G}_2$, and we use these formulas to compute the affine automorphism group of each folding map.

Keywords. – algebraic dynamics, commuting maps, folding polynomial, Lie algebra.

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1. Introduction

The subject of commuting maps has been much studied in dynamics; see for example [1, 3, 7, 8, 10, 15–19]. We start with a general definition although this paper will concentrate on the case of maps of the affine plane.

Definition 1.1. Let K be an algebraically closed field of characteristic 0, let X/K be an algebraic variety, and let

$$f : X \longrightarrow X$$

be a finite morphism. We say that f is *permutable*[1] if there exists a finite morphism $g : X \to X$ such that

- f and g are not invertible, i.e., not automorphisms of X.
- $f \circ g = g \circ f$.
- f and g do not have a common iterate.

One problem that we will study in this paper is the computation of the group of automorphisms, also sometimes called the group of self-similarities, of permutable maps.

Definition 1.2. Let K be an algebraically closed field of characteristic 0, let X/K be a variety, and let

$$f : X \longrightarrow X$$

be a finite morphism. The *group of automorphisms of f* is the group

$$\operatorname{Aut}(f) = \{\alpha \in \operatorname{Aut}(X) : \alpha^{-1} \circ f \circ \alpha = f\}.$$

Example 1.3. A classical theorem of Julia [8] and Ritt [12, 13] (see also Eremenko [6]) classifies permutable maps on $\mathbb{A}^1$, and more generally on $\mathbb{P}^1$. In particular, a permutable map $f : \mathbb{A}^1 \to \mathbb{A}^1$ is conjugate to either a power map x^n or a Chebyshev polynomial T_n, whose automorphism groups are given by

$$\operatorname{Aut}(x^n) \cong \boldsymbol{\mu}_{n-1} \quad \text{and} \quad \operatorname{Aut}(T_n) = \begin{cases} \boldsymbol{\mu}_1 & \text{if } n \text{ is even,} \\ \boldsymbol{\mu}_2 & \text{if } n \text{ is odd.} \end{cases}$$

We note that one construction of the Chebyshev polynomial T_n is to start with the n-power map on $\mathbb{G}_m$ and take the quotient by the automorphism $z \to z^{-1}$. There is an analogous construction in higher dimensions, where $\mathbb{G}_m$ is replaced by a higher-dimensional torus associated with a Lie algebra and the quotient is via an action of an affine Weyl group. The details of the construction will not concern us, so we give only a brief sketch (at the end of this introduction) and provide some references.

Theorem 1.4 ([14, 15]). *Let $\mathfrak{L}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0, and let N be the rank of $\mathfrak{L}$, i.e., the dimension of a Cartan subalgebra. Then for each $n \ge 0$, there is an associated polynomial map*

$$\mathsf{F}_n[\mathfrak{L}] : \mathbb{A}^N \longrightarrow \mathbb{A}^N,$$

(1) These maps have various names in the literature; for example they are called *integrable* in [14–16].

called the nth folding map[2] associated with the Lie algebra $\mathcal{L}$*, having the following properties:*

(1) $\mathsf{F}_n[\mathcal{L}]$ *has topological degree* n^N.

(2) *For all* $m, n \geq 0$*, we have*

$$\mathsf{F}_m[\mathcal{L}] \circ \mathsf{F}_n[\mathcal{L}] = \mathsf{F}_{mn}[\mathcal{L}] = \mathsf{F}_n[\mathcal{L}] \circ \mathsf{F}_m[\mathcal{L}]. \tag{1}$$

Definition 1.5. The folding map $\mathsf{F}_n[\mathcal{L}]$ described in Theorem 1.4 is so named because it may be constructed geometrically in terms of folding the chambers of $\mathcal{L}$ associated with a system of roots using the reflection maps that generate the affine Weyl group of $\mathcal{L}$. See [18] for a nice description of this construction in dimension 2.

Remark 1.6. We observe that $\mathsf{F}_n[\mathcal{L}]$ for $n \geq 2$ is a permutable map on $\mathbb{A}^N$ since the consideration of topological degrees shows that $\mathsf{F}_m[\mathcal{L}]$ and $\mathsf{F}_n[\mathcal{L}]$ cannot have a common iterate if $\gcd(m, n) = 1$. More precisely, it follows from (1) that $\mathsf{F}_m[\mathcal{L}]$ and $\mathsf{F}_n[\mathcal{L}]$ have a common iterate if and only if $m^i = n^j$ for some $i, j \geq 1$. Further, it is known that if two simple Lie algebras $\mathcal{L}_1$ and $\mathcal{L}_2$ have non-isomorphic Weyl groups, then their folding maps are inequivalent; see [15, Corollary to Proposition 3].

It is possible to construct higher-dimensional permutable maps from lower-dimensional maps. The following definition characterizes one way that this can be done.

Definition 1.7. Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a polynomial map. We say that f is *triangular*[3] if there is an affine automorphism[4]

$$\ell : \mathbb{A}^N \longrightarrow \mathbb{A}^N$$

so that

$$\ell^{-1} \circ f \circ \ell(\boldsymbol{x}) = \big[\underbrace{\varphi_1(x_1)}_{\text{just } x_1}, \underbrace{\varphi_2(x_1, x_2)}_{\text{just } x_1, x_2}, \underbrace{\varphi_3(x_1, x_2, x_3)}_{\text{just } x_1, x_2, x_3}, \ldots\big].$$

A bold conjecture of Veselov[5] suggested that every non-triangular permutable map on affine space comes from the Lie algebra construction described in Theorem 1.4. This conjecture is not entirely correct since for example there are non-triangular permutable

(2) For $\mathcal{B}_2$, $\mathcal{G}_2$, and $\mathcal{F}_4$, there are additional folding maps $\mathsf{F}_{n\sqrt{2}}[\mathcal{B}_2]$, $\mathsf{F}_{n\sqrt{3}}[\mathcal{G}_2]$, and $\mathsf{F}_{n\sqrt{2}}[\mathcal{F}_4]$ that satisfy the commutativity property $\mathsf{F}_\alpha \circ \mathsf{F}_\beta = \mathsf{F}_{\alpha\beta} = \mathsf{F}_\beta \circ \mathsf{F}_\alpha$.

(3) These maps are called *reducible* in [15].

(4) An *affine automorphism of* $\mathbb{A}^N$ is an element $\ell \in \mathrm{GL}_N \rtimes \mathbb{G}_a^N$, where $\ell = (M, \boldsymbol{b})$ acts on $\mathbb{A}^N$ via matrix multiplication and translation, $(M, \boldsymbol{b}) \cdot \boldsymbol{v} = M\boldsymbol{v} + \boldsymbol{b}$.

(5) Quoting from [15]: "The following construction [14] gives a series of integrable mappings for arbitrary n which, the author supposes, exhausts all integrable polynomial mappings."

maps of $\mathbb{A}^2$ in Dinh's classification [4] that do not come from the Lie algebra construction. But Lie algebra folding maps do form very interesting families of permutable maps.

In particular, the classification of simple Lie algebras implies that the folding maps of $\mathbb{A}^2$ are associated with one of the Lie algebras $\mathcal{A}_2$, $\mathcal{B}_2$, or $\mathcal{G}_2$.[6] Withers [18] gives recursion formulas for these 2-dimensional folding maps. In this paper, we use Withers' recursions to find the leading terms of the folding map coordinate functions, which in turn we use to compute the automorphism groups of the 2-dimensional folding maps. We state the latter result here. For the former, see Proposition 2.5 for $\mathcal{A}_n$, Proposition 3.1 for $\mathcal{B}_n$, and Proposition 4.1 for $\mathcal{G}_n$.

THEOREM 1.8. *The affine automorphism groups of the 2-dimensional folding maps are as follows:*

$$\operatorname{Aut}\bigl(\mathsf{F}_n[\mathcal{A}_2]\bigr) \cong \begin{cases} \mathcal{S}_3 & \text{if } n \equiv 1 \pmod{3}, \\ \boldsymbol{\mu}_2 & \text{if } n \not\equiv 1 \pmod{3}, \end{cases}$$

$$\operatorname{Aut}\bigl(\mathsf{F}_n[\mathcal{B}_2]\bigr) \cong \begin{cases} \boldsymbol{\mu}_2 & \text{if } n \equiv 1 \pmod{2}, \\ 1 & \text{if } n \equiv 0 \pmod{2}, \end{cases}$$

$$\operatorname{Aut}\bigl(\mathsf{F}_n[\mathcal{G}_2]\bigr) \cong 1.$$

PROOF. See Theorem 2.6 for $\mathcal{A}_n$, Theorem 3.2 for $\mathcal{B}_n$, and Theorem 4.2 for $\mathcal{G}_n$. ∎

As a further application of our leading-term formulas, in Theorem 5.1, we describe the maps on $\mathbb{P}^2$ induced by homogenizing the folding maps on $\mathbb{A}^2$. In particular, we prove that $\mathsf{F}_n[\mathcal{A}_2]$ and $\mathsf{F}_n[\mathcal{B}_2]$ extend to endomorphisms on $\mathbb{P}^2$, while $\mathsf{F}_n[\mathcal{G}_2]$ extends to a rational map of $\mathbb{P}^2$ having either one or two points of indeterminacy, depending on the parity of n.

REMARK 1.9. There has been a considerable amount of work aimed at classifying permutable maps of $\mathbb{P}^N$. We mention in particular the paper of Dinh [4] that classifies permutable maps of $\mathbb{A}^2$ that extend to morphisms of $\mathbb{P}^2$, the paper of Dinh and Sibony [5] that extends this work to endomorphisms of $\mathbb{P}^N$ whose degrees are multiplicatively independent, and the paper of Kaufmann [9] that covers the case that the degrees are multiplicatively dependent.

([6]) The classification of finite-dimensional simple Lie algebras consists of four infinite families and a handful of exceptional cases. However, in dimension 2, there are isomorphisms $\mathcal{D}_2 \cong \mathcal{A}_1 \times \mathcal{A}_1$ and $\mathcal{B}_2 \cong \mathcal{C}_2$, so there are only three distinct cases.

In terms of the classification in [4], the folding map $\mathsf{F}_n[\mathcal{B}_2]$, which extends to an endomorphism of $\mathbb{P}^2$, corresponds to a Case (4) map with associated map on the line at infinity being the homogenized Chebyshev polynomial T_n. Explicitly, the map $\mathsf{F}_n[\mathcal{B}_2]$ is characterized by the functional equation

$$\mathsf{F}_n[\mathcal{B}_2](u+v,uv) = \bigl(T_n(u)+T_n(v),\ T_n(u)\cdot T_n(v)\bigr);$$

cf. [7], remembering that $\mathcal{B}_2 \cong \mathcal{C}_2$. On the other hand, the folding map $\mathsf{F}_n[\mathcal{A}_2]$, which also extends to an endomorphism of $\mathbb{P}^2$, does not fit into any of the four cases described in [4] although presumably it is covered by the more general treatment in [5]. Finally, the folding map $\mathsf{F}_n[\mathcal{G}_2]$ does not extend to an endomorphism of $\mathbb{P}^2$, and thus it is not included in any of the current classifications.

PROOF SKETCH OF THEOREM 1.4 (APRÈS [15]). We take $K=\mathbb{C}$. Let $\mathcal{L}_0 \cong \mathbb{C}^N$ be a Cartan subalgebra of $\mathcal{L}$, let $\mathcal{L}_0^*$ be its dual, let $w_1,\dots,w_N$ be a system of fundamental weights in $\mathcal{L}_0^*$, and let Λ be the lattice in $\mathcal{L}_0$ that is dual to $\operatorname{Span}_{\mathbb{Z}}(w_1,\dots,w_N) \subset \mathcal{L}_0^*$. Further, let W be the Weyl group with its action on $\mathcal{L}_0^*$. For each $1 \le k \le N$, define an exponential map

$$\varphi_k : \mathcal{L}_0/\Lambda \longrightarrow \mathbb{C}, \quad \varphi_k(\boldsymbol{x}) = \sum_{\sigma\in\mathsf{W}} e^{2\pi i\sigma(w_k)(\boldsymbol{x})},$$

and fit these maps together to give a map

$$\Phi_{\mathcal{L}} : \mathcal{L}_0/\Lambda \longrightarrow \mathbb{C}^N, \quad \Phi_{\mathcal{L}} = (\varphi_1,\dots,\varphi_N).$$

A theorem of Chevalley says that the algebra of exponential invariants for the action of W is the polynomial ring generated by $\varphi_1,\dots,\varphi_N$. Hence, for every $n \ge 2$, there is a unique polynomial map F_n characterized by

$$\Phi_{\mathcal{L}}(n\boldsymbol{x}) = F_n\bigl(\Phi_{\mathcal{L}}(\boldsymbol{x})\bigr).$$

For further details, see [14, 15], as well as [3, 7, 16, 17]. ∎

2. FOLDING MAPS FOR THE LIE ALGEBRA $\mathcal{A}_2$

We consider the folding maps $\mathsf{F}_n[\mathcal{A}_2]$. To ease notation in this section, we let

$$z = x+iy \quad\text{and}\quad A_n(z) = A_n(x,y) = \mathsf{F}_n[\mathcal{A}_2](x,y).$$

Then [18] says that the folding maps A_n are characterized by the formulas

$$A_0(z)=3, \quad A_1(z)=z, \quad A_2(z)=z^2-2\bar{z}, \tag{2}$$

$$A_n(z) = zA_{n-1}(z) - \bar{z}A_{n-2}(z) + A_{n-3}(z). \tag{3}$$

REMARK 2.1. Although we will use the description of the A_n in terms of z and $\bar{z}$, we note that (2) and (3) may be rewritten in terms of xy-coordinates by letting

$$(X_n, Y_n) = A_n(x, y).$$

Then

$$A_0(x, y) = (3, 0), \quad A_1(x, y) = (x, y), \quad A_2(x, y) = (x^2 - y^2 - 2x,\ 2xy + 2y),$$

and

$$\begin{aligned} X_n &= x(X_{n-1} - X_{n-2}) - y(Y_{n-1} + Y_{n-2}) + X_{n-3}, \\ Y_n &= x(Y_{n-1} - Y_{n-2}) + y(X_{n-1} + X_{n-2}) + Y_{n-3}. \end{aligned}$$

REMARK 2.2. It is clear from the recursion for A_n that $A_n \in \mathbb{Z}[z, \bar{z}]$, so in particular, the effect of complex conjugation is

$$\overline{A_n(z)} = A_n(\bar{z}).$$

Or, if one wants to be more formal and write $A_n(z, \bar{z})$, then

$$\overline{A_n(z, \bar{z})} = A_n(\bar{z}, z).$$

DEFINITION 2.3. We write $O(d)$ for a polynomial in $K[z, \bar{z}]$ or in $K[x, y]$ whose total degree is at most d; i.e., writing $f = g + O(d)$ means that $f - g$ has the form

$$\sum_{i+j \le d} a_{ij} z^i \bar{z}^j, \quad \text{or equivalently,} \quad \sum_{i+j \le d} a_{ij} x^i y^j.$$

LEMMA 2.4. (a) *Let $\zeta \in \mathbb{C}$ satisfy $\zeta^3 = 1$. Then*

$$A_n(\zeta z) = \zeta^n A_n(z). \tag{4}$$

(b) *The polynomial A_n has the form*

$$A_n(z) = \sum_{k=0}^{n} \sum_{\substack{i+j=k \\ i-j \equiv n \pmod 3}} a_{ij} z^i \bar{z}^{k-i}.$$

PROOF. (a) The assumption that ζ is a cube root of unity implies in particular that $|\zeta| = 1$, so $\bar{\zeta} = \zeta^{-1}$. We prove Lemma 2.4 by induction on n. The equality (4) is clearly true for $A_0(z) = 3$ and $A_1(z) = z$. For $A_2(z) = z^2 - 2\bar{z}$, we compute

$$A_2(\zeta z) = \zeta^2 z^2 - 2\overline{\zeta z} = \zeta^2 z^2 - 2\zeta^{-1}\bar{z} = \zeta^2 A_2(z).$$

Now assume that (4) is true up to A_{n-1}. Then

$$\begin{aligned} A_n(\zeta z) &= \zeta z A_{n-1}(\zeta z) - \overline{\zeta z} A_{n-2}(\zeta z) + A_{n-3}(\zeta z) \\ &= \zeta^n z A_{n-1}(z) - \bar{\zeta}\zeta^{n-2}\bar{z} A_{n-2}(z) + \zeta^{n-3} A_{n-3}(z) \\ &\qquad\qquad \text{by the induction hypothesis,} \\ &= \zeta^n \big(z A_{n-1}(z) - \bar{z} A_{n-2}(z) + A_{n-3}(z) \big) \\ &\qquad\qquad \text{since } \bar{\zeta} = \zeta^{-1} = \zeta^2 \text{ for } \zeta \in \boldsymbol{\mu}_3, \\ &= \zeta^n A_n(z). \end{aligned}$$

(b) We initially write $A_n(z)$ as

$$A_n(z) = \sum_{k=0}^{n} \sum_{i+j=k} a_{ij} z^i \bar{z}^j$$

and use (a) to show that certain coefficients vanish. Let ζ be a primitive cube root of unity. Then (a) implies that

$$\zeta^n \sum_{k=0}^{n} \sum_{i+j=k} a_{ij} z^i \bar{z}^j = \sum_{k=0}^{n} \sum_{i+j=k} a_{ij} (\zeta z)^i (\overline{\zeta z})^j = \sum_{k=0}^{n} \sum_{i+j=k} \zeta^{i-j} a_{ij} z^i \bar{z}^j .$$

Hence,

$$a_{ij} \neq 0 \implies \zeta^n = \zeta^{i-j} \implies i - j \equiv n \pmod 3. \qquad \blacksquare$$

Proposition 2.5. *For all $n \geq 2$, the polynomial $A_n \in \mathbb{Z}[z, \bar{z}]$ satisfies*

$$A_n(z) = z^n - n z^{n-2} \bar{z} + \frac{n^2 - 3n}{2} z^{n-4} \bar{z}^2 + O(n-3). \tag{5}$$

Proof. Using (2) and (3), we compute

$$\begin{aligned} A_3(z) &= z^3 - 3 z \bar{z} + 3, \\ A_4(z) &= z^4 - 4 z^2 \bar{z} + 4z + 2\bar{z}^2, \\ A_5(z) &= z^5 - 5 z^3 \bar{z} + 5 z^2 + 5 z \bar{z}^2 - 5\bar{z}. \end{aligned}$$

Hence, (5) is true for $n = 3, 4, 5$.[7] Assume that (5) is true up to $n - 1$ for some $n \geq 5$, so our induction assumption says that

$$A_j(z) = z^j - j z^{j-2} \bar{z} + \frac{j^2 - 3j}{2} z^{j-4} \bar{z}^2 + O(j-3) \quad \text{for all } j < n. \tag{6}$$

(7) For $n = 3$, the formula (5) appears to have a term of the form $z^{-1}\bar{z}^2$, but the coefficient $\frac{n^2-3n}{2}$ vanishes, so that term does not appear in the formula.

Then

$$\begin{aligned}
A_n &= zA_{n-1} - \bar{z}A_{n-2} + A_{n-3} \quad \text{from (3)}\\
&= z\left(z^{n-1} - (n-1)z^{n-3}\bar{z} + \frac{n^2-5n+4}{2}z^{n-5}\bar{z}^2 + O(n-4)\right)\\
&\quad - \bar{z}\left(z^{n-2} - (n-2)z^{n-4}\bar{z} + \frac{n^2-7n+10}{2}z^{n-6}\bar{z}^2 + O(n-5)\right)\\
&\quad + \left(z^{n-3} - (n-3)z^{n-5}\bar{z} + \frac{n^2-9n+18}{2}z^{n-7}\bar{z}^2 + O(n-6)\right)\\
&\qquad\qquad \text{using the induction hypothesis (6),}\\
&= \left(z^{n} - (n-1)z^{n-2}\bar{z} + \frac{n^2-5n+4}{2}z^{n-4}\bar{z}^2 + O(n-3)\right)\\
&\quad - \left(z^{n-2}\bar{z} - (n-2)z^{n-4}\bar{z}^2 + O(n-3)\right) + O(n-3)\\
&= z^n - nz^{n-2}\bar{z} + \frac{n^2-3n}{2}z^{n-4}\bar{z}^2 + O(n-3). \qquad \blacksquare
\end{aligned}$$

We use Proposition 2.5 to compute the affine automorphism group of A_n.

THEOREM 2.6. *Let $n \geq 2$. Then*

$$\operatorname{Aut}(A_n) \cong \begin{cases} \mathcal{S}_3 & \textit{if } n \equiv 1 \pmod{3},\\ \boldsymbol{\mu}_2 & \textit{if } n \not\equiv 1 \pmod{3}. \end{cases}$$

Explicitly, if we let $\zeta \in \boldsymbol{\mu}_3$ be a primitive cube root of unity, then the maps in $\operatorname{Aut}(A_n)$ *are*

$$\operatorname{Aut}(A_n) \cong \begin{cases} \{z,\ \zeta z,\ \zeta^2 z,\ \bar{z},\ \zeta\bar{z},\ \zeta^2\bar{z}\} & \textit{if } n \equiv 1 \pmod{3},\\ \{z,\ \bar{z}\} & \textit{if } n \not\equiv 1 \pmod{3}. \end{cases}$$

PROOF. Writing everything in terms of $z = x + iy$ and $\bar{z} = x - iy$, the elements of $\operatorname{Aut}(\mathbb{A}^2)$ are identified with maps of the form

$$\varphi_{\alpha,\beta,\gamma}(z) = \alpha z + \beta\bar{z} + \gamma \quad \text{for } \alpha, \beta, \gamma \in \mathbb{C}.$$

(We note that $\varphi_{\alpha,\beta,\gamma}(z)$ is invertible if and only if $|\alpha|^2 \neq |\beta|^2$.)

Suppose that $\varphi_{\alpha,\beta,\gamma} \in \operatorname{Aut}(A_n)$ for some $n \geq 2$. Then

$$\begin{aligned}
\varphi_{\alpha,\beta,\gamma} \circ A_n(z) &= A_n \circ \varphi_{\alpha,\beta,\gamma}(z) \quad \text{by definition of } \operatorname{Aut}(A_n),\\
\alpha A_n(z) + \beta\overline{A_n(z)} + \gamma &= A_n(\alpha z + \beta\bar{z} + \gamma) \quad \text{by definition of } \varphi_{\alpha,\beta,\gamma},\\
\alpha\bigl(z^n + O(n-1)\bigr) + \beta\bigl(\bar{z}^n + O(n-1)\bigr) + \gamma &\\
&= (\alpha z + \beta\bar{z} + \gamma)^n + O(n-1) \quad \text{from Proposition 2.5},\\
\alpha z^n + \beta\bar{z}^n &= (\alpha z + \beta\bar{z})^n + O(n-1).
\end{aligned}$$

Looking at the coefficients of z^n and $\bar{z}^n$ gives

$$\alpha^n = \alpha \quad \text{and} \quad \beta^n = \beta,$$

and then the lack of any other monomials on the left-hand side forces the equality $\alpha\beta = 0$. (This is where we use $n \geq 2$ and $\operatorname{char}(K) = 0$.)

We consider first the case that $\alpha^n = \alpha \neq 0$ and $\beta = 0$, so in particular we see that

$$\alpha^{n-1} = 1 \implies |\alpha| = 1 \implies \bar{\alpha} = \alpha^{-1}. \tag{7}$$

Using $\varphi_{\alpha,0,\gamma}(z) = \alpha z + \gamma$, we compute

$$\begin{aligned}
\varphi_{\alpha,0,\gamma} \circ A_n(z) = A_n \circ \varphi_{\alpha,0,\gamma}(z) \quad & \text{by definition of } \operatorname{Aut}(A_n),\\
\alpha A_n(z) + \gamma = A_n(\alpha z + \gamma) \quad & \text{by definition of } \varphi_{\alpha,0,\gamma},\\
\alpha\big(z^n - nz^{n-2}\bar{z} + O(n-2)\big) + \gamma = (\alpha z+\gamma)^n - n(\alpha z+\gamma)^{n-2}(\bar{\alpha}\,\bar{z}+\bar{\gamma}) + O(n-2) & \\
& \text{from Proposition 2.5},\\
\alpha z^n - \alpha n z^{n-2}\bar{z} + O(n-2) = \alpha^n z^n + n\alpha^{n-1}\gamma z^{n-1} - n\bar{\alpha}\,\alpha^{n-2} z^{n-2}\bar{z} + O(n-2). &
\end{aligned}$$

This gives

$$n\alpha^{n-1}\gamma z^{n-1} = 0 \quad \text{and} \quad -\alpha n z^{n-2}\bar{z} = -n\bar{\alpha}\,\alpha^{n-2} z^{n-2}\bar{z}. \tag{8}$$

The first equality tells us that $\gamma = 0$, and the second equality yields

$$\begin{aligned}
\alpha &= \bar{\alpha} \cdot \alpha^{n-2} && \text{from (8)},\\
&= \alpha^{-1} \cdot \alpha^{-1} && \text{from (7) and using } \alpha^n = \alpha.
\end{aligned}$$

Hence, $\alpha^3 = 1$. This concludes the proof that

$$\varphi_{\alpha,\beta,\gamma} \in \operatorname{Aut}(A_n) \quad \text{and} \quad \alpha \neq 0 \implies \varphi_{\alpha,\beta,\gamma}(z) = \varphi_{\alpha,0,0}(z) = \alpha z \quad \text{and} \quad \alpha^3 = 1.$$

We leave to the reader the similar calculation for $\alpha = 0$ and $\beta^n = \beta \neq 0$, which leads to the similar conclusion

$$\varphi_{\alpha,\beta,\gamma} \in \operatorname{Aut}(A_n) \quad \text{and} \quad \beta \neq 0 \implies \varphi_{\alpha,\beta,\gamma}(z) = \varphi_{0,\beta,0}(z) = \beta\bar{z} \quad \text{and} \quad \beta^3 = 1.$$

Let $\zeta \in \boldsymbol{\mu}_3$ be a cube root of unity. It remains to check under what circumstances the maps

$$\varphi_{\zeta,0,0}(z) = \zeta z \quad \text{and/or} \quad \varphi_{0,\zeta,0}(z) = \zeta\,\bar{z}$$

are in $\operatorname{Aut}(A_n)$. Lemma 2.4 (a) tells us that

$$A_n \circ \varphi_{\zeta,0,0}(z) = A_n(\zeta z) = \zeta^n A_n(z) = \zeta^{n-1} \cdot \varphi_{\zeta,0,0} \circ A_n(z). \tag{9}$$

Hence, for $\zeta \in \boldsymbol{\mu}_3$, we have

$$\varphi_{\zeta,0,0} \in \operatorname{Aut}(A_n) \iff \zeta = 1 \text{ or } n \equiv 1 \pmod{3}.$$

We next observe that

$$\begin{aligned}
&A_n \circ \varphi_{0,\zeta,0}(z) \\
&\quad = A_n(\zeta\,\bar z) \quad \text{by definition of } \varphi_{0,\zeta,0}(z) = \zeta\,\bar z, \\
&\quad = \zeta^n A_n(\bar z) \quad \text{since the formula in Lemma 2.4 is a formal identity in the variable } z = x + iy \text{, so it remains true if we replace } z \text{ with } \bar z = x - iy, \\
&\quad = \zeta^{n-1} \cdot \zeta \cdot \overline{A_n(z)} \quad \text{since } A_n(z) \in \mathbb{R}[z,\bar z], \text{ so } \overline{A_n(z)} = A_n(\bar z), \\
&\quad = \zeta^{n-1} \cdot \varphi_{0,\zeta,0} \circ A_n(z) \quad \text{by definition of } \varphi_{0,\zeta,0}(z) = \zeta\,\bar z.
\end{aligned}$$

So for $\zeta \in \boldsymbol{\mu}_3$, we have

$$\varphi_{0,\zeta,0} \in \operatorname{Aut}(A_n) \iff \zeta = 1 \text{ or } n \equiv 1 \pmod{3}.$$

This completes the proof that

$$\operatorname{Aut}(A_n) = \begin{cases} \{\varphi_{\zeta,0,0} : \zeta \in \boldsymbol{\mu}_3\} \cup \{\varphi_{0,\zeta,0} : \zeta \in \boldsymbol{\mu}_3\} & \text{if } n \equiv 1 \pmod{3}, \\ \{\varphi_{1,0,0},\ \varphi_{0,1,0}\} & \text{if } n \not\equiv 1 \pmod{3}. \end{cases}$$

In the case that $n \equiv 1 \pmod{3}$, we may identify $\operatorname{Aut}(A_n)$ with $\mathcal{S}_3$ by noting that the listed set of maps is the group of $\mathbb{R}$-linear transformations of $\mathbb{C}$ preserving the equilateral triangle with vertices $\{1, \zeta, \zeta^2\}$. We also note that the map $\varphi_{0,1,0}(z) = \bar z$ is simply the map $(x, y) \to (x, -y)$, so in the case that $n \not\equiv 1 \pmod{3}$, the group $\operatorname{Aut}(A_n)$ is naturally identified with $\boldsymbol{\mu}_2$. ■

3. Folding maps for the Lie algebra $\mathcal{B}_2$

We consider the folding maps $\mathsf{F}_n[\mathcal{B}_2]$. To ease notation in this section, we let

$$(X_n, Y_n) = B_n(x, y) = \mathsf{F}_n[\mathcal{B}_2](x, y), \quad \text{so } X_n, Y_n \in \mathbb{Z}[x, y]. \tag{10}$$

According to [18], the first few B_n are

$$\begin{aligned}
B_0(x, y) &= (4, 4), \\
B_1(x, y) &= (x, y), \\
B_2(x, y) &= (x^2 - 2y - 4,\ y^2 - 2x^2 + 4y + 4), \\
B_3(x, y) &= (x^3 - 3xy - 3x,\ y^3 - 3x^2y + 6y^2 + 9y),
\end{aligned}$$

and then writing $B_n(x,y) = (X_n, Y_n)$, the subsequent B_n are given by the recursion

$$X_{n+4} = x(X_{n+3} + X_{n+1}) - (2+y)X_{n+2} - X_n, \tag{11}$$

$$Y_{n+4} = y(Y_{n+3} + Y_{n+1}) - (x^2 - 2y - 2)Y_{n+2} - Y_n. \tag{12}$$

PROPOSITION 3.1. *With notation $(X_n, Y_n) = B_n(x,y)$ as in (10) and with big-O notation as described in (2.3), we have*

$$X_n = x^n - nx^{n-2}y - nx^{n-2} + \frac{n(n-3)}{2}x^{n-4}y^2 + O(n-3),$$
$$Y_n = y^n - nx^2y^{n-2} + x^3 \cdot O(n-3) + O(n-1).$$

PROOF. The stated formulas are true for $n \geq 3$ from the explicit formulas for B_n in these cases. So we assume that they are true up to $n+3$ and prove that they are true for $n+4$ by induction.

We use the recursion (11) for X_n and the induction hypothesis to compute

$$\begin{aligned} X_{n+4} &= xX_{n+3} + xX_{n+1} - (2+y)X_{n+2} - X_n \\ &= x\Big(x^{n+3} - (n+3)x^{n+1}y - (n+3)x^{n+1} + \frac{n(n+3)}{2}x^{n-1}y^2 + O(n)\Big) \\ &\quad + x\big(x^{n+1} + O(n)\big) - (2+y)\big(x^{n+2} - (n+2)x^n y + O(n)\big) + O(n) \\ &= \Big(x^{n+4} - (n+3)x^{n+2}y - (n+3)x^{n+1} + \frac{n(n+3)}{2}x^n y^2 + O(n+1)\Big) \\ &\quad + x^{n+2} - 2x^{n+2} - x^{n+2}y + (n+2)x^n y^2 + O(n+1) \\ &= x^{n+4} - (n+4)x^{n+2}y - (n+4)x^{n+2} + \frac{(n+4)(n+1)}{2}x^n y^2 + O(n+1). \end{aligned}$$

We next use the recursion (12) for Y_n and the induction hypothesis to compute

$$\begin{aligned} Y_{n+4} &= yY_{n+3} + yY_{n+1} - (x^2 - 2y - 2)Y_{n+2} - Y_n \\ &= y\big(\underbrace{y^{n+3} - (n+3)x^2y^{n+1}}_{\boxed{\uparrow\uparrow}} + x^3 \cdot O(n) + O(n+2)\big) \\ &\quad + y\big(y^{n+1} - (n+1)x^2y^{n-1} + x^3 \cdot O(n-2) + O(n)\big) \\ &\quad - \big(\underbrace{x^2}_{\boxed{\uparrow\uparrow}} - 2y - 2\big)\big(\underbrace{y^{n+2}}_{\boxed{\uparrow\uparrow}} - (n+2)x^2y^n + x^3 \cdot O(n-1) + O(n+1)\big) \\ &\quad - \big(y^n - nx^2y^{n-2} + x^3 \cdot O(n-3) + O(n-1)\big) \\ &= y^{n+4} - (n+4)x^2y^{n+2} + x^3 \cdot O(n+1) + O(n+3), \end{aligned}$$

where we have marked with $\boxed{\uparrow\uparrow}$ the quantities that form the terms that are not absorbed into the big-O error terms. ∎

THEOREM 3.2. *For all $n \geq 2$, we have*

$$\mathrm{Aut}(B_n) = \begin{cases} \boldsymbol{\mu}_2 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

More precisely, when n is odd, the non-trivial element of $\mathrm{Aut}(B_n)$ *is* $\varphi(x, y) = (-x, y)$.

PROOF. We write

$$(X_n, Y_n) = B_n(x, y) \quad \text{with } X_n, Y_n \in \mathbb{Z}[x, y],$$

and we suppose that

$$\varphi(x, y) = (ax + by + c, dx + ey + f) \in \mathrm{Aut}(\mathbb{A}^2)$$

commutes with B_n.

For any monomial $M \in \mathbb{Z}[x, y]$ and any polynomial $P \in \mathbb{Z}[x, y]$, we let

$$\mathsf{Coef}(M, P) = \text{coefficient of the monomial } M(x, y) \text{ in } P(x, y),$$

and if $V = (v_1, \ldots, v_N)$ is any list, we write

$$V[i] = v_i = i\text{th entry of the list.}$$

We use the formulas for X_n and Y_n in Proposition 3.1 to compute

$$\begin{aligned} \mathsf{Coef}\big(xy^{n-1}, (\varphi \circ B_n)(x, y)[1]\big) &= \mathsf{Coef}\big(xy^{n-1}, aX_n + bY_n + c\big) \\ &= 0, \end{aligned} \tag{13}$$

$$\begin{aligned} \mathsf{Coef}\big(xy^{n-1}, (B_n \circ \varphi)(x, y)[1]\big) &= B_n(ax + by + c, dx + ey + f)[1] \\ &= \mathsf{Coef}\big(xy^{n-1}, (ax + by + c)^n\big) \\ &= nax(by)^{n-1}. \end{aligned} \tag{14}$$

Equating (13) and (14) yields $ab = 0$.

We next look at the coefficient of x^2y^{n-2}:

$$\begin{aligned} \mathsf{Coef}\big(x^2y^{n-2}, (\varphi \circ B_n)(x, y)[1]\big) &= \mathsf{Coef}\big(x^2y^{n-2}, aX_n + bY_n + c\big) \\ &= -bn, \end{aligned} \tag{15}$$

$$\begin{aligned} \mathsf{Coef}\big(x^2y^{n-2}, (B_n \circ \varphi)(x, y)[1]\big) &= B_n(ax + by + c, dx + ey + f)[1] \\ &= \mathsf{Coef}\big(x^2y^{n-2}, (ax + by + c)^n\big) \\ &= \binom{n}{2}(ax)^2(by)^{n-2}. \end{aligned} \tag{16}$$

Equating (15) and (16) gives

$$-bn = \frac{n(n-1)}{2}a^2b^{n-2}, \quad \text{which gives} \quad b = 0 \text{ or } \frac{n-1}{2}a^2b^{n-3} = -1.$$

Since we know from earlier that $ab = 0$, and since the invertibility of φ implies that a and b cannot both vanish, this proves that $b = 0$.

Using the fact that $b = 0$, we look at the first few terms in the expansion of the x-coordinates of $\varphi \circ B_n$ and $B_n \circ \varphi$. Thus,

$$\begin{aligned}
&(\varphi \circ B_n)(x,y)[1] = aX_n + c \\
&\quad = ax^n - anx^{n-2}y - anx^{n-2} + a\frac{n(n-3)}{2}x^{n-4}y^2 + O(n-3), \\
&(B_n \circ \varphi)(x,y)[1] = B_n(ax+c, dx+ey+f)[1] \\
&\quad = (ax+c)^n - n(ax+c)^{n-2}(dx+ey+f) - n(ax+c)^{n-2} \\
&\qquad + \frac{n(n-3)}{2}(ax+c)^{n-4}(dx+ey+f)^2 + O(n-3) \\
&\quad = \Big(a^nx^n + na^{n-1}cx^{n-1} + \tbinom{n}{2}a^{n-2}c^2x^{n-2}\Big) \\
&\qquad - \Big(na^{n-2}dx^{n-1} + na^{n-2}ex^{n-2}y + na^{n-2}fx^{n-2} \\
&\qquad\qquad + n(n-2)a^{n-3}cdx^{n-2} + n(n-2)a^{n-3}cex^{n-3}y\Big) \\
&\qquad - na^{n-2}x^{n-2} + \Big(\tfrac{n(n-3)}{2}a^{n-4}d^2x^{n-2} + \tfrac{n(n-3)}{2}a^{n-4}e^2x^{n-4}y^2\Big) + O(n-3) \\
&\quad = a^nx^n + \big(na^{n-1}c - na^{n-2}d\big)x^{n-1} - na^{n-2}ex^{n-2}y \\
&\qquad + \Big(\tbinom{n}{2}a^{n-2}c^2 - na^{n-2}f - n(n-2)a^{n-3}cd - na^{n-2} + \tfrac{n(n-3)}{2}a^{n-4}d^2\Big)x^{n-2} \\
&\qquad - n(n-2)a^{n-3}cex^{n-3}y + \tfrac{n(n-3)}{2}a^{n-4}e^2x^{n-4}y^2 + O(n-3).
\end{aligned}$$

Equating the x^n coefficients gives

$$a = a^n, \quad \text{so} \quad a^{n-1} = 1.$$

Then equating the x^{n-1} coefficients gives

$$na^{n-2}(ac-d) = 0, \quad \text{so } d = ac.$$

Then equating the $x^{n-2}y$ coefficients and using $a^{n-1} = 1$ gives

$$na^{n-2}(a^2-e) = 0, \quad \text{so } e = a^2.$$

In particular, we have $e \neq 0$. Then equating the $x^{n-3}y$ coefficients and using $a^{n-1} = 1$ and $e \neq 0$ gives

$$n(n-2)a^{n-3}ce = 0, \quad \text{so } c = 0.$$

We note that this also implies that $d = ac = 0$. To recapitulate, we have shown that

$$a^{n-1} = 1, \quad b = 0, \quad c = 0, \quad d = 0, \quad e = a^2.$$

Equating the x^{n-2} coefficients and using these values gives

$$f = a^2 - 1.$$

Hence,

$$\begin{aligned} \varphi(x,y) &= (ax, ey + f) \\ &= (ax, a^2 y + a^2 - 1) \quad \text{for some } a \text{ satisfying } a^{n-1} = 1. \end{aligned} \tag{17}$$

We next look at the $x^2 y^{n-2}$ terms of the y-coordinate Y_n of $B_n(x,y)$. Thus,

$$\begin{aligned} \mathsf{Coef}\big(x^2y^{n-2}, (\varphi\circ B_n)(x,y)[2]\big) &= \mathsf{Coef}\big(x^2y^{n-2}, a^2Y_n + f\big) \quad \text{from (17)}, \\ &= -na^2 \quad \text{from Proposition 3.1}, \\ \mathsf{Coef}\big(x^2y^{n-2}, (B_n\circ\varphi)(x,y)[2]\big) &= B_n(ax, a^2y + f)[2] \quad \text{from (17)}, \\ &= \mathsf{Coef}\big(x^2y^{n-2}, -n(ax)^2(a^2y+f)^{n-2}\big) \\ &= -na^2(a^2)^{n-2} \quad \text{from Proposition 3.1}, \\ &= -n \quad \text{since } a^{n-1} = 1. \end{aligned}$$

Hence,

$$a^2 = 1, \quad \text{which implies that} \quad e = a^2 = 1 \quad \text{and} \quad f = a^2 - 1 = 0.$$

We have thus shown that

$$\varphi(x,y) = (ax, y) \quad \text{with } a = \pm 1.$$

It remains to check when $\varphi(x,y) = (-x,y)$ commutes with B_n. More generally, we will prove by induction that for all $n \ge 0$, we have

$$X_n(-x,y) = (-1)^n X_n(x,y) \quad \text{and} \quad Y_n(-x,y) = Y_n(x,y). \tag{18}$$

The explicit formulas for $B_0, \dots, B_3$ show that (18) is true for $0 \le n \le 3$. Assume now that (18) is true up to $n+3$. Then

$$\begin{aligned} X_{n+4}(-x,y) &= x\big(X_{n+3}(-x,y) + X_{n+1}(-x,y)\big) \\ &\qquad - (2+y)X_{n+2}(-x,y) - X_n(-x,y), \\ &= (-x)\big((-1)^{n+3}X_{n+3}(x,y) + (-1)^{n+1}X_{n+1}(x,y)\big) \\ &\qquad - (-1)^{n+2}(2+y)X_{n+2}(x,y) - (-1)^n X_n(x,y) \\ &= (-1)^{n+4}X_{n+4}(x,y), \end{aligned}$$

$$\begin{aligned} Y_{n+4}(-x,y) &= y\big(Y_{n+3}(-x,y)+Y_{n+1}(-x,y)\big) \\ &\quad -\big((-x)^2-2y-2\big)Y_{n+2}(-x,y)-Y_n(-x,y) \\ &= y\big(Y_{n+3}(x,y)+Y_{n+1}(x,y)\big) \\ &\quad -(x^2-2y-2)Y_{n+2}(x,y)-Y_n(x,y) \\ &= Y_{n+4}(x,y). \end{aligned}$$

It follows from (18) that

$$\varphi(x,y)=(-x,y)\in \operatorname{Aut}(B_n) \iff n\equiv 1 \pmod 2. \qquad \blacksquare$$

4. Folding maps for the Lie algebra $\mathcal{G}_2$

We consider the folding maps $\mathsf{F}_n[\mathcal{G}_2]$. To ease notation in this section, we let

$$(X_n,Y_n)=G_n(x,y)=\mathsf{F}_n[\mathcal{G}_2](x,y), \quad \text{so } X_n,Y_n\in\mathbb{Z}[x,y]. \tag{19}$$

The paper [18] gives explicit formulas for $G_0,\dots,G_5$, which for the convenience of the reader we have reproduced in Figure 1. The subsequent values of $G_n(x,y)$ are then determined by the following recursions:

$$\begin{aligned} X_{n+6} &= x(X_{n+5}+X_{n+1})-(x+y+3)(X_{n+4}+X_{n+2}) \\ &\quad +(x^2-2y-4)X_{n+3}-X_n, \end{aligned} \tag{20}$$

$$\begin{aligned} Y_{n+6} &= y(Y_{n+5}+Y_{n+1})-(x^3-3xy-9x-5y-9)(Y_{n+4}+Y_{n+2}) \\ &\quad +(y^2-2x^3+6xy+18x+12y+8)Y_{n+3}-Y_n. \end{aligned} \tag{21}$$

Proposition 4.1. *We write $(X_n,Y_n)=G_n(x,y)$ as in (19) and we use big-O notation as described in (2.3).*

(a) *The values of $X_0,\dots,X_5$ are given in Figure 1. For $n\ge 5$, the top-order terms of X_n satisfy*

$$X_n = \underbrace{x^n}_{\deg n} \underbrace{-nx^{n-2}y}_{\deg n-1} + \underbrace{\frac{n^2-3n}{2}x^{n-4}y^2-nx^{n-3}y-3nx^{n-2}}_{\deg n-2}+O(n-3). \tag{22}$$

(b) *We have $Y_0=6$, and for $n\ge 1$, the top-order term of Y_n is given by*

$$Y_n=\begin{cases} (-1)^{n/2}2x^{3n/2}+O\left(\frac{3n-2}{2}\right) & \text{if } n \text{ is even,} \\ (-1)^{(n-1)/2}nx^{(3n-3)/2}y+O\left(\frac{3n-3}{2}\right) & \text{if } n \text{ is odd.} \end{cases} \tag{23}$$

n	x-coordinate of G_n	y-coordinate of G_n
0	6	6
1	x	y
2	$x^2 - 2x - 2y - 6$	$-2x^3 + 6xy + y^2 + 18x + 10y + 18$
3	$x^3 - 3xy - 9x - 6y - 12$	$-3x^3y - 6x^3 + 9xy^2 + y^3 + 45xy$ $+18y^2 + 54x + 63y + 60$
4	$x^4 - 4x^2y - 10x^2 - 4xy$ $+2y^2 - 8x + 8y + 6$	$2x^6 - 12x^4y - 4x^3y^2 - 36x^4 - 28x^3y$ $+18x^2y^2 + 12xy^3 + y^4 - 40x^3$ $+108x^2y + 120xy^2 + 24y^3 + 162x^2$ $+372xy + 134y^2 + 360x + 280y + 198$
5	$x^5 - 5x^3y - 15x^3 - 5x^2y$ $+5xy^2 - 10x^2 + 35xy$ $+10y^2 + 55x + 50y + 60$	$5x^6y + 10x^6 - 30x^4y^2 - 5x^3y^3$ $-150x^4y - 65x^3y^2 + 45x^2y^3 + 15xy^4$ $+y^5 - 180x^4 - 205x^3y + 360x^2y^2$ $+240xy^3 + 30y^4 - 190x^3 + 945x^2y$ $+1200xy^2 + 255y^3 + 810x^2 + 2415xy$ $+920y^2 + 1710x + 1495y + 900$

FIGURE 1. The first few G_n polynomials.

PROOF. (a) The formula (22) for X_n is true for $n = 5$ by inspection from Figure 1. Assume now that (22) is true up to $n + 5$ for some $n \geq 0$. The recursion (20) says that

$$X_{n+6} = x(X_{n+5} + X_{n+1}) - (x + y + 3)(X_{n+4} + X_{n+2}) + (x^2 - 2y - 4)X_{n+3} - X_n. \tag{24}$$

We use the induction hypothesis to compute the various pieces of X_{n+6} up to $O(n+3)$:

$$\begin{aligned} &x(X_{n+5} + X_{n+1}) \\ &\quad = x^{n+6} - (n+5)x^{n+4}y + \frac{(n+5)(n+2)}{2}x^{n+2}y^2 \\ &\qquad - (n+5)x^{n+3}y - 3(n+5)x^{n+4} + O(n+3) \quad \text{from (22)}, \end{aligned} \tag{25}$$

$$\begin{aligned} &(x + y + 3)(X_{n+4} + X_{n+2}) \\ &\quad = (x + y + 3)X_{n+4} + O(n+3) \\ &\quad = (x + y + 3)x^{n+4} - (x + y)(n+4)x^{n+2}y + O(n+3) \\ &\quad = x^{n+5} + x^{n+4}y + 3x^{n+4} - (n+4)x^{n+3}y - (n+4)x^{n+2}y^2 \\ &\qquad + O(n+3), \end{aligned} \tag{26}$$

$$
\begin{aligned}
(27)\qquad & (x^2-2y-4)X_{n+3}-X_n\\
&= (x^2-2y)X_{n+3}+O(n+3)\\
&= (x^2-2y)\big(x^{n+3}-(n+3)x^{n+1}y+O(n+1)\big)\\
&= x^{n+5}-2x^{n+3}y-(n+3)x^{n+3}y+O(n+3)\\
&= x^{n+5}-(n+5)x^{n+3}y+O(n+3).
\end{aligned}
$$

Substituting (25), (26), and (27) into (24) yields

$$
\begin{aligned}
X_{n+6} = \Big(& x^{n+6}-(n+5)x^{n+4}y+\frac{(n+5)(n+2)}{2}x^{n+2}y^2\\
&\qquad -(n+5)x^{n+3}y-3(n+5)x^{n+4}\Big)\\
&-\big(x^{n+5}+x^{n+4}y+3x^{n+4}-(n+4)x^{n+3}y-(n+4)x^{n+2}y^2\big)\\
&+\big(x^{n+5}-(n+5)x^{n+3}y\big)+O(n+3)\\
= {}& x^{n+6}-(n+6)x^{n+4}y+\frac{(n+6)^2-3(n+6)}{2}x^{n+2}y^2\\
&-(n+6)x^{n+3}y-3(n+6)x^{n+4}+O(n+3).
\end{aligned}
$$

This completes the induction proof that (22) holds for all $n \geq 0$.

(b) To ease notation, we let

$$
\lambda_n = \begin{cases} (-1)^{n/2}2 & \text{if } n \text{ is even,}\\ (-1)^{(n-1)/2}n & \text{if } n \text{ is odd.} \end{cases} \tag{28}
$$

An easy computation shows that the sequence λ_n described by (28) satisfies the recursion

$$
\lambda_{n+2} = \begin{cases} -\lambda_n & \text{if } n \text{ is even,}\\ \lambda_{n+1}-\lambda_n & \text{if } n \text{ is odd.} \end{cases} \tag{29}
$$

The formula (23) for Y_n is true for $0 \leq n \leq 5$ by inspection from Figure 1. Assume now that (23) is true up to $n+5$.

We consider first the case that $n = 2k$ is even. Then the induction hypothesis and the recursion (21) give

$$
\begin{aligned}
Y_{n+6} = {}& \underbrace{y(Y_{2k+5}+Y_{2k+1})}_{\boxed{y\cdot O(3k+6)}} - \underbrace{(x^3-3xy-9x-5y-9)(Y_{2k+4}+Y_{2k+2})}_{\boxed{(x^3+O(2))\cdot(\lambda_{2k+4}x^{3k+6}+O(3k+5))}}\\
&+\underbrace{(y^2-2x^3+6xy+18x+12y+8)Y_{2k+3}}_{\boxed{O(3)\cdot O(3k+3)}} - \underbrace{Y_{2k}}_{\boxed{O(3k)}}
\end{aligned}
$$

$$
\begin{aligned}
&= -\lambda_{2k+4}x^{3k+9} + O(3k+8) \\
&= -\lambda_{n+4}x^{3(n+6)/2} + O\left(\frac{3(n+6)}{2} - 1\right) \\
&= \lambda_{n+6}x^{3(n+6)/2} + O\left(\frac{3(n+6)}{2} - 1\right) \quad \text{from (29).}
\end{aligned}
$$

Similarly, if $n = 2k+1$ is even, then

$$
\begin{aligned}
Y_{n+6} = {} & \underbrace{y(Y_{2k+6} + Y_{2k+2})}_{\boxed{y\cdot(\lambda_{2k+6}x^{3k+9}+O(3k+8))}} - \underbrace{(x^3 - 3xy - 9x - 5y - 9)(Y_{2k+5} + Y_{2k+3})}_{\boxed{(x^3+O(2))\cdot(\lambda_{2k+5}x^{3k+6}y+O(3k+6))}} \\
& + \underbrace{(y^2 - 2x^3 + 6xy + 18x + 12y + 8)Y_{2k+4}}_{\boxed{O(3)\cdot O(3k+6)}} - \underbrace{Y_{2k+1}}_{\boxed{O(3k+1)}} \\
= {} & (\lambda_{2k+6} - \lambda_{2k+5})x^{3k+9}y + O(3k+9) \\
= {} & (\lambda_{n+5} - \lambda_{n+4})x^{(3n+15)/2}y + O\left(\frac{3n+15}{2}\right) \\
= {} & \lambda_{n+6}x^{(3(n+6)-3)/2}y + O\left(\frac{3(n-6)-3}{2}\right) \quad \text{from (29).}
\end{aligned}
$$

This completes the induction proof that (23) holds for all $n \ge 1$. ∎

We use Proposition 4.1 to compute the automorphism group of G_n.

THEOREM 4.2. *For all $n \ge 2$, we have*

$$
\operatorname{Aut}(G_n) = 1.
$$

PROOF. We write

$$
(X_n, Y_n) = G_n(x,y) \quad \text{with } X_n, Y_n \in \mathbb{Z}[x,y],
$$

and we suppose that

$$
\varphi(x,y) = (ax + by + c, dx + ey + f) \in \operatorname{Aut}(\mathbb{A}^2)
$$

commutes with G_n.

Our assumption that $n \ge 2$ implies that

$$
n \text{ odd} \implies \frac{3n-3}{2} \ge n \quad \text{and} \quad n \text{ even} \implies \frac{3n-2}{2} \ge n.
$$

It follows from Proposition 4.1 (a) that $X_n = O(n)$, so the formulas for X_n and Y_n in Proposition 4.1 give

$$\begin{aligned}(\varphi \circ G_n)(x, y)[1] &= aX_n + bY_n + c \\ &= \begin{cases} bx^{3n/2} + O\left(\frac{3n-2}{2}\right) & \text{if } n \text{ is even,} \\ bx^{(3n-3)/2}y + O\left(\frac{3n-3}{2}\right) & \text{if } n \text{ is odd,} \end{cases} \\ (G_n \circ \varphi)(x, y)[1] &= G_n(ax + by + c, dx + ey + f)[1] \\ &= (ax + by + c)^n + O(n-1) \\ &= O(n).\end{aligned}$$

We observe that if $b \neq 0$, then the highest degree term of the composition $(\varphi \circ G_n)(x, y)[1]$ is strictly larger than the highest degree term of $(G_n \circ \varphi)(x, y)[1]$, so it would not be possible for them to be equal. Therefore,

$$b = 0 \quad \text{and} \quad \varphi(x, y) = (ax + c, dx + ey + f).$$

We now use Proposition 4.1 (a) to compare the next few terms in the expansions of the first coordinates of $G_n \circ \varphi$ and $\varphi \circ G_n$, keeping in mind that $b = 0$. Thus,

$$\begin{aligned}(G_n \circ \varphi)(x, y)[1] &= G_n(ax + c, dx + ey + f)[1] \\ &= (ax + c)^n - n(ax + c)^{n-2}(dx + ey + f) \\ &\quad + \frac{n^2 - 3n}{2}(ax + c)^{n-4}(dx + ey + f)^2 \\ &\quad - n(ax + c)^{n-3}(dx + ey + f) \\ &\quad - 3n(ax + c)^{n-2} + O(n-3),\end{aligned} \tag{30}$$

$$\begin{aligned}(\varphi \circ G_n)(x, y)[1] &= aX_n + c \\ &= ax^n - anx^{n-2}y + \frac{n^2 - 3n}{2}ax^{n-4}y^2 \\ &\quad - nax^{n-3}y - 3nax^{n-2} + O(n-3).\end{aligned} \tag{31}$$

We equate the degree n terms of (30) and (31). This yields

$$(ax)^n = \mathsf{Deg}_n\big((G_n \circ \varphi)(x, y)[1]\big) = \mathsf{Deg}_n\big((\varphi \circ G_n)(x, y)[1]\big) = ax^n,$$

and hence

$$a^{n-1} = 1.$$

(We cannot have $a = 0$ since we already know that $b = 0$, and if $a = b = 0$, then φ would not be invertible.)

We next pull out the degree $n-1$ terms of (30) and (31), dividing by n to simplify the expressions. Thus,

$$\frac{1}{n}\operatorname{Deg}_{n-1}\bigl((G_n\circ\varphi)(x,y)[1]\bigr) = (ax)^{n-1}c - (ax)^{n-2}(dx+ey), \tag{32}$$

$$\frac{1}{n}\operatorname{Deg}_{n-1}\bigl((\varphi\circ G_n)(x,y)[1]\bigr) = -ax^{n-2}y. \tag{33}$$

Setting (32) equal to (33), multiplying by a, and using $a^{n-1}=1$, we find that

$$(ac-d)x^{n-1} + (a^2-e)x^{n-2}y = 0.$$

Hence,

$$d = ac \quad\text{and}\quad e = a^2. \tag{34}$$

We next pull out the degree $n-2$ terms of (30) and (31), using $a^n = a$ and dividing by n to simplify the expressions. We find, after some algebra, that

$$\begin{aligned}
&\frac{1}{n}\operatorname{Deg}_{n-2}\bigl((G_n\circ\varphi)(x,y)[1]\bigr)\\
&= \left(\frac{n-1}{2}a^{-1}c^2 - a^{-1}f - (n-2)a^{-2}cd + \frac{n-3}{2}a^{-3}d^2 - a^{-2}d - 3a^{-1}\right)x^{n-2}\\
&\quad + \left(2\frac{n-3}{2}a^{-3}de - (n-2)a^{-2}ce - a^{-2}e\right)x^{n-3}y + \left(\frac{n-3}{2}a^{-3}e^2\right)x^{n-4}y^2.
\end{aligned}$$

Similarly, the degree $n-2$ terms of $(\varphi\circ G_n)(x,y)[1]$ are

$$\frac{1}{n}\operatorname{Deg}_{n-2}\bigl((\varphi\circ G_n)(x,y)[1]\bigr) = -3ax^{n-2} - ax^{n-3}y + \frac{n-3}{2}ax^{n-4}y^2.$$

Setting

$$\operatorname{Deg}_{n-2}\bigl((G_n\circ\varphi)(x,y)[1]\bigr) = \operatorname{Deg}_{n-2}\bigl((\varphi\circ G_n)(x,y)[1]\bigr),$$

we see that the coefficients of $x^{n-4}y^2$ are equal since we already know from (34) that $e=a^2$.

Next equating the coefficients of $x^{n-3}y$, we find that

$$(n-3)a^{-3}de - (n-2)a^{-2}ce - a^{-2}e = -a.$$

Using $d=ac$ and $e=a^2$ from (34), after some algebra, this becomes

$$c = 1-a.$$

So we now know that

$$a^{n-1}=1,\quad b=0,\quad c=1-a,\quad d=ac=a-a^2,\quad e=a^2. \tag{35}$$

Finally, equating the coefficients of x^{n-2}, we find that

$$\frac{n-1}{2}a^{-1}c^2 - a^{-1}f - (n-2)a^{-2}cd + \frac{n-3}{2}a^{-3}d^2 - a^{-2}d - 3a^{-1} + 3a = 0. \tag{36}$$

Substituting (35) into (36) yields

$$f = (a-1)(3a+4).$$

Hence,

$$\varphi(x, y) = \big(ax + (1-a),\ (a-a^2)x + a^2y + (a-1)(3a+4)\big). \tag{37}$$

In particular, we see that $a = 1$ implies that $\varphi(x, y) = (x, y)$ is the identity map.

We now look at the y-coordinates, where it is simpler to consider n even and odd separately. We start with the case that n is even, say

$$n = 2k.$$

Then

$$\begin{aligned} (G_n \circ \varphi)(x, y)[2] &= G_n(ax + c, acx + a^2y + f)[2] \quad \text{from (35)} \\ &= (-1)^k 2(ax + c)^{3k} + O(3k-1) \quad \text{from Proposition 4.1 (b)} \\ &= (-1)^k 2a^{3k}x^{3k} + O(3k-1), \\ (\varphi \circ G_n)(x, y)[2] &= acX_n + a^2Y_n + f \quad \text{from (35)} \\ &= (-1)^k 2a^2x^{3k} + O(3k-1) \quad \text{from Proposition 4.1 (b).} \end{aligned}$$

Using $G_n \circ \varphi = \varphi \circ G_n$, we find that

$$(-1)^k 2a^{3k}x^{3k} = (-1)^k 2a^2x^{3k} \quad \text{and hence that} \quad a^{3k-2} = 1.$$

But we also know from (35) that $a^{n-1} = a^{2k-1} = 1$, and hence

$$a^{\gcd(3k-2,2k-1)} = 1.$$

We have

$$2(3k-2) - 3(2k-1) = -1, \quad \text{so} \quad \gcd(3k-2, 2k-1) = 1.$$

Hence, if n is even, then $a = 1$.

We next consider the case that x is odd, say

$$n = 2k + 1.$$

Then

$$\begin{aligned}(G_n \circ \varphi)(x,y)[2] &= G_n(ax+c, acx+a^2y+f)[2] \quad \text{from (35)}\\ &= (-1)^k n(ax+c)^{3k}(acx+a^2y+f) + O(3k)\\ &\qquad\qquad\qquad\qquad \text{from Proposition 4.1 (b)}\\ &= (-1)^k n a^{3k} x^k (acx+a^2y) + O(3k),\\ (\varphi \circ G_n)(x,y)[2] &= acX_n + a^2Y_n + f \quad \text{from (35)}\\ &= (-1)^k n a^2 x^{3k} y + O(3k) \quad \text{from Proposition 4.1 (b).}\end{aligned}$$

Using $G_n \circ \varphi = \varphi \circ G_n$, we find that

$$(-1)^k n a^{3k+1} c x^{3k+1} + (-1)^k n a^{3k+2} x^{3k} y = (-1)^k n a^2 x^{3k} y.$$

The coefficient of x^{3k+1} must vanish, which shows that $c = 0$, and then (35) forces $a = 1$. So we have proven that if n is odd, then $a = 1$.

To recapitulate, we have shown that $a = 1$ for all $n \geq 2$. Then (37) tells us that $\varphi(x,y) = (x,y)$ is the identity map, which completes the proof that

$$\operatorname{Aut}(G_n) = 1 \quad \text{for all } n \geq 2. \qquad \blacksquare$$

5. Geometric properties of 2-dimensional folding maps

In this section, we study the maps on $\mathbb{P}^2$ induced by 2-dimensional folding maps.

Theorem 5.1. (a) *Let $\bar{A}_n(X,Y,Z)$ be the homogenization of $A_n = \mathsf{F}_n[\mathcal{A}_2]$. Then*

$$\bar{A}_n : \mathbb{P}^2 \longrightarrow \mathbb{P}^2 \text{ is a morphism of degree } n.$$

(b) *Let $\bar{B}_n(X,Y,Z)$ be the homogenization of $B_n = \mathsf{F}_n[\mathcal{B}_2]$. Then*

$$\bar{B}_n : \mathbb{P}^2 \longrightarrow \mathbb{P}^2 \text{ is a morphism of degree } n.$$

(c) *Let $\bar{G}_n(X,Y,Z)$ be the homogenization of $G_n = \mathsf{F}_n[\mathcal{G}_2]$. Then*

$$\bar{G}_n : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \text{ is a rational map of degree } \left\lfloor \frac{3n}{2} \right\rfloor.$$

If $n \geq 2$, then $\bar{G}_n$ is not a morphism, and its indeterminacy locus is given by

$$I(\bar{G}_n) = \begin{cases} \{[0,1,0]\} & \text{if } n \text{ is even,}\\ \{[0,1,0],[1,0,0]\} & \text{if } n \text{ is odd.}\end{cases} \tag{38}$$

The dynamical degree of $\bar{G}_n$ is $\operatorname{DynDeg}(\bar{G}_n) = n$.

PROOF. (a) Proposition 2.5 tells us that

$$A_n(x,y) = \bigl(\mathrm{Re}(x+iy)^n + O(n-1),\ \mathrm{Im}(x+iy)^n + O(n-1)\bigr),$$

so when we homogenize, we get

$$\bar{A}_n(X,Y,Z) = \bigl[\mathrm{Re}(X+iY)^n + Z\underbrace{u(X,Y,Z)}_{\boxed{\text{homog. deg } n-1}},\ \mathrm{Im}(X+iY)^n + Z\underbrace{v(X,Y,Z)}_{\boxed{\text{homog. deg } n-1}},\ Z^n\bigr].$$

Then

$$\begin{aligned}\bar{A}_n(X,Y,Z) = [0,0,0] &\iff \mathrm{Re}(X+iY)^n = \mathrm{Im}(X+iY)^n = Z = 0\\ &\iff (X+iY)^n = Z = 0\\ &\iff X = Y = Z = 0.\end{aligned}$$

Hence, $\bar{A}_n$ is a morphism of degree n.

(b) Proposition 3.1 tells us that

$$B_n(x,y) = \bigl(x^n + O(n-1),\ y^n + x^2\cdot O(n-2) + O(n-1)\bigr),$$

so when we homogenize, we get

$$\bar{B}_n(X,Y,Z) = \bigl[X^n + Z\underbrace{u(X,Y,Z)}_{\boxed{\text{homog. deg } n-1}},\ Y^n + X^2\underbrace{v(X,Y,Z)}_{\boxed{\text{homog. deg } n-2}} + Z\underbrace{w(X,Y,Z)}_{\boxed{\text{homog. deg } n-1}},\ Z^n\bigr].$$

Setting $\bar{B}_n(X,Y,Z) = [0,0,0]$, the third coordinate gives $Z=0$, then the first coordinate gives $X=0$, and then the second coordinate gives $Y=0$. Hence, $\bar{B}_n$ is a morphism of degree n.

(c) Suppose first that n is even, say $n=2k$. Then Proposition 4.1 tells us that

$$\begin{aligned}G_n(x,y) &= (x^{2k} + \cdots, (-1)^k 2x^{3k} + \cdots)\\ &= \bigl(x^{2k} + O(2k-1),\ (-1)^k 2x^{3k} + O(3k-1)\bigr).\end{aligned}$$

Hence,

$$\bar{G}_n(X,Y,Z) = \bigl[X^{2k}Z^k + Z\underbrace{u(X,Y,Z)}_{\boxed{\text{homog. deg } 3k-1}}, (-1)^k 2X^{3k} + Z\underbrace{v(X,Y,Z)}_{\boxed{\text{homog. deg } 3k-1}}, Z^{3k}\bigr].$$

From this we can read off

$$\deg \bar{G} = 3k = \frac{3n}{2} \quad\text{and}\quad I(\bar{G}_n) = \{X = Z = 0\}.$$

Next suppose that $n = 2k + 1$ is odd for some $k \ge 1$. Again using Proposition 4.1, we find that

$$G_n(x, y) = \bigl(x^{2k+1} + O(2k), (-1)^k(2k+1)x^{3k}y + O(3k-1)\bigr).$$

Hence,

$$\bar{G}_n(X, Y, Z) = \Bigl[X^{2k+1}Z^k + Z\,\overbrace{u(X, Y, Z)}^{\text{homog. deg } 3k},\; (-1)^k(2k+1)X^{3k}Y + Z\,\underbrace{v(X, Y, Z)}_{\text{homog. deg } 3k},\, Z^{3k+1}\Bigr].$$

From this and the assumption that $k \ge 1$, we can read off

$$\deg \bar{G} = 3k + 1 = \frac{3n-1}{2} = \left\lfloor \frac{3n}{2} \right\rfloor \quad \text{since } n = 2k + 1 \text{ is odd,}$$
$$I(\bar{G}_n) = \{X = Z = 0\} \cup \{Y = Z = 0\}.$$

This completes the proof of (c) except for the computation of the dynamical degree. Using $\deg \bar{G}_n = \lfloor 3n/2 \rfloor$, the limit formula definition of the dynamical degree yields

$$\operatorname{DynDeg}(\bar{G}_n) = \lim_{m\to\infty} (\deg \bar{G}_n^m)^{1/m} = \lim_{m\to\infty} (\deg \bar{G}_{n^m})^{1/m}$$
$$= \lim_{m\to\infty} \left\lfloor \frac{3n^m}{2} \right\rfloor^{1/m} = n. \qquad \blacksquare$$

REMARK 5.2. As noted in the footnote to Theorem 1.4, the Lie group $\mathcal{B}_2$ admits some additional folding maps arising from the fact that a right isosceles triangle can be folded in half to form two right isosceles triangles. More precisely, there is a $\mathcal{B}_2$ folding map

$$B_{\sqrt{2}}(x, y) = (y, x^2 - 2y - 4) \quad \text{satisfying} \quad B_{\sqrt{2}}^2 = B_2,$$

leading to additional folding maps via $B_{n\sqrt{2}} = B_n \circ B_{\sqrt{2}}$. The associated homogenized map on $\mathbb{P}^2$ is

$$\bar{B}_{\sqrt{2}}(X, Y, Z) = [YZ, X^2 - 2YZ - 4Z^2, Z^2],$$

which is not a morphism, since it is not defined at $[0, 1, 0]$. So $\bar{B}_{\sqrt{2}}$ is an example of a non-morphism whose second iterate $\bar{B}_2$ is a morphism.

REMARK 5.3. As noted in the footnote to Theorem 1.4, the Lie group $\mathcal{G}_2$ admits some additional folding maps arising from the fact that a 30-60-90 triangle can be folded onto itself 3-to-1. More precisely, there is a $\mathcal{G}_2$ folding map

$$G_{\sqrt{3}}(x, y) = (y, x^3 - 3xy - 9x - 6y - 12) \quad \text{satisfying} \quad G_{\sqrt{3}}^2 = G_3,$$

leading to additional folding maps via $G_{n\sqrt{3}} = G_n \circ G_{\sqrt{3}}$. The associated homogenized map on $\mathbb{P}^2$ is

$$\bar{G}_{\sqrt{3}}(X,Y,Z) = [YZ^2, X^3 - 3XYZ - 9XZ^2 - 6YZ^2 - 12Z^3, Z^3].$$

It is a rational map with indeterminacy locus $[0,1,0]$.

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