

PBW-deformations and deformations à la Gerstenhaber of N -Koszul algebras

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Abstract. In this article we establish an explicit link between the classical theory of deformations à la Gerstenhaber (and a fortiori with the Hochschild cohomology) and (weak) PBW-deformations of homogeneous algebras. Our point of view is of cohomological nature. As a consequence, we recover a theorem by R. Berger and V. Ginzburg, which gives a precise condition for a filtered algebra to satisfy the so-called *PBW property*, under certain assumptions.

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Introduction

Given a graded k -algebra $A = TV/\langle R \rangle$ with $R \subseteq V^{\otimes N}$ there are two notions of deformation of A available: PBW-deformations and weak PBW-deformations, as defined by R. Berger and V. Ginzburg in [1] and classical deformations (after M. Gerstenhaber). The main goal of this article is to construct explicit equivalences between these concepts, under suitable hypotheses. Our construction is strongly related to Hochschild cohomology theory. One of our main motivations is to study the deformation theory of several examples of (graded) N -Koszul algebras of interest, and since the Hochschild cohomology of many of these algebras is known, we believe that it is quite fruitful to have such a direct connection.

We shall briefly explain our results in more detail. We consider a semisimple ring k containing a field F of characteristic zero, such that $k^e = k \otimes_F k^{\text{op}}$ is semisimple. Let $A = TV/\langle R \rangle$ be an N -homogeneous k -algebra (i.e., $R \subseteq V^{\otimes N}$). We are interested in studying filtered algebras $U = TV/\langle P \rangle$ with $P \subseteq \bigoplus_{i=0}^N V^{\otimes i}$ such that $R = \pi_N(P)$, for $\pi_N: TV \rightarrow V^{\otimes N}$ the canonical projection, which satisfy an extra property: the surjective morphism of graded algebras $p: A \rightarrow \text{gr}(U)$ induced

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by the projection $TV \rightarrow U$ is an isomorphism. In this case U is called a *PBW-deformation* of A . The condition of p being injective is equivalent to an infinite number of equalities between certain k -bimodules given by intersections of tensor powers of V and R (see Section 2.1). The filtered algebra U is said to be a *weak PBW-deformation* of A if only two of this set of equalities are satisfied (see (2.1) and (2.2)).

We provide explicit constructions under the hypothesis that $\mathrm{Tor}_3^A(k, k)$ is concentrated in degree $N + 1$ as follows: given a graded deformation of A we construct a weak PBW-deformation of A , and, conversely, given a weak PBW-deformation of A we construct a graded deformation, such that both constructions are inverse up to equivalence. They are completely explicit (see Sections 2.3 and 2.4). On the other hand, it is well known, without the assumption on the torsion group of A , that for any filtered algebra U satisfying the PBW property one can naturally obtain a graded deformation A_t of A by considering the Rees algebra $R(U)$, and conversely, given a graded deformation A_t of A , one obtains a filtered algebra U satisfying the PBW property taking a generic fiber $A_t/\langle t - 1 \rangle$. These two constructions provide quasi-inverse functors for the respective categories. Finally, under the assumption that $\mathrm{Tor}_3^A(k, k)$ is concentrated in degree $N + 1$, we notice that the explicit constructions we defined in fact coincide with the previous general ones, implying Theorem 1.2 and Theorem 3.4 of [1], which state that the notion of a weak PBW-deformation and of a PBW-deformation coincide if $\mathrm{Tor}_3^A(k, k)$ is concentrated in degree $N + 1$. We would like to point out that the procedure exhibited here allows to find the deformed product of A_t explicitly from the filtered algebra, even though the computations are often very hard to perform in general.

We would also like to remark that a similar construction has been already considered in the (second part of the) proof of Theorem 1.1 of the article [5] of G. Fløystad and J. E. Vatne. However, the mentioned proof contains a mistake. More precisely, following the notation of that article, on p. 122, after the identity defining γ on line 22, it is stated that $\gamma \circ \sigma$ vanishes. This is not necessarily true, because there is in principle no identification of the Koszul resolution K_\bullet inside the bar resolution B_\bullet , which is compatible with taking brackets $[1, -]$. In fact, this can also be noticed from the fact that the recursion formulas for the cochains giving the deformed product corresponding to a weak PBW-deformation explained in Proposition 2.11 of Section 2.3 are indeed more complicated.

The article is organized as follows. In the first section we recall some generalities about the theory of N -homogeneous algebras over a (not necessarily commutative) semisimple ring k containing a field F of characteristic zero such that $k^e = k \otimes_F k^{\mathrm{op}}$ is semisimple, and make an intensive study of the (reduced) Hochschild resolution and a “Koszul-like” projective resolution of bimodules of an N -homogeneous algebra satisfying some vanishing condition on the torsion groups. In Section 2, we first recall general facts on PBW-deformations and weak PBW-deformations of N -homogeneous algebras and the classical graded deformation theory à la Gerstenhaber. Later, in Sections 2.3 and 2.4 we establish a link between these concepts. Finally, in

Section 3 we prove the main results of this paper, namely Theorems 3.1, 3.3 and 3.5 and we give several examples.

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1. Generalities

From now on, k will be a (not necessarily commutative) semisimple ring containing a field F of characteristic zero. We consider $k^e = k \otimes_F k^{op}$. We assume further that k^e is semisimple. By k -bimodule we will always mean a k -bimodule such that the action of F is symmetric, i.e., $a \cdot m = m \cdot a$ for all $a \in F$, and m in the k -bimodule M . Note that we do not assume the action of k on the bimodule to be symmetric. It is clear that this definition of k -bimodule is equivalent to the notion of a (say left) k^e -module. As a matter of notation, all unadorned tensor products \otimes are over k .

A k -algebra denotes a monoid object in the monoidal category of k -bimodules, i.e., it is a k -bimodule A provided with a morphism $\mu: A \otimes A \rightarrow A$ of k -bimodules, which will be denoted $\mu(a \otimes a') = a \cdot a'$, and an element $1 \in A$, such that μ is associative, $1 \cdot a = a \cdot 1 = a$ for all $a \in A$, and $c \cdot 1 = 1 \cdot c$ for all $c \in k$. Equivalently, a k -algebra is a ring A provided with a unitary ring homomorphism $i_A: k \rightarrow A$ where $\text{Im}(i_A)$ is not necessarily contained in $\mathcal{Z}(A)$. For $n \geq 2$, $\mu^{(n)}: A^{\otimes n} \rightarrow A$ will denote the morphism of k -bimodules defined recursively by $\mu^{(2)} = \mu$ and $\mu^{(n+1)} = \mu^{(n)} \circ (\mu \otimes 1_A^{\otimes(n-1)})$ for $n > 2$. A morphism from a k -algebra A to a k -algebra B is a ring homomorphism $f: A \rightarrow B$ such that $f \circ i_A = i_B$.

A graded k -bimodule will be a k -bimodule V together with a decomposition as a direct sum of k -bimodules $V = \bigoplus_{n \in \mathbb{Z}} V_n$. A morphism of graded k -bimodules is just a degree-preserving morphism of k -bimodules. The category of graded k -bimodules is monoidal in the obvious manner. We will denote the homomorphism group between two graded k -bimodules M and M' by $\text{hom}_{k^e}(M, M')$. The shift functor $(-)[1]$, together with its iterations, is defined in the usual way, and we recall that the internal group of homomorphisms between two graded k -bimodules M and M' is given by $\mathcal{H}\text{om}_{k^e}(M, M') = \bigoplus_{i \in \mathbb{Z}} \text{hom}_{k^e}(M, M'[i])$. Moreover, a graded k -algebra is a monoid object in the monoidal category of graded k -bimodules, i.e., a k -algebra provided with a decomposition of k -bimodules of the form $A = \bigoplus_{n \in \mathbb{Z}} A_n$ such that $1 \in A_0$ and $A_n \cdot A_m \subseteq A_{n+m}$ for all $n, m \in \mathbb{Z}$. In fact, we shall usually assume that $A = \bigoplus_{n \in \mathbb{N}_0} A_n$.

Given a graded k -algebra A , a graded left A -module M is given by a graded k -bimodule structure on M together with a morphism of graded k -bimodules $\rho: A \otimes M \rightarrow M$, which will be denoted by $\rho(a \otimes m) = a \cdot m$ for $a \in A$ and $m \in M$ satisfying the usual mixed associative axiom, i.e., $a \cdot (a' \cdot m) = (a \cdot a') \cdot m$ for all $a, a' \in A$ and $m \in M$, and that $1 \cdot m = m$ for all $m \in M$. As usual, a morphism

of graded left A -modules is just a degree-preserving morphism of A -modules, and the homomorphism group between two graded left A -modules M and M' will be denoted by $\text{hom}_A(M, M')$. We note that shift functor $(-)[1]$ may also be defined on a graded left A -module M , where the underlying structure of graded k -bimodule is the same as before, and the obvious action. As in the previous case, the internal group of homomorphisms between two graded left A -modules M and M' is given by $\mathcal{H}\text{om}_A(M, M') = \bigoplus_{i \in \mathbb{Z}} \text{hom}_A(M, M'[i])$. The usual definition of *graded right A -module* and *graded A -bimodule* are analogous, and using the obvious structure of graded k -algebra on $A^e = A \otimes A^{\text{op}}$ we also see that the notion graded A -bimodule and graded (say left) A^e -module coincide.

Let N be a positive integer, $N \geq 2$. By *complex* (resp. N -*complex*) we mean a nonnegatively graded module over a k -algebra A , provided with an endomorphism d of degree -1 such that $d^2 = 0$ (resp. $d^N = 0$).

We note that, since k is semisimple, it is von Neumann regular, so all the considerations in [1], Section 2, in order to properly consider the notion of Koszul algebra also apply to this case. Let $A = TV/\langle R \rangle$ be an N -homogeneous algebra, where V is a k -bimodule (considered to be concentrated in degree 1) and R is a k -subbimodule of $V^{\otimes N}$. In this situation, we shall identify A/k with the k -subbimodule I_+ of A spanned by elements of strictly positive degree. We will make use of the number function given by

$$\zeta: \mathbb{N}_0 \rightarrow \mathbb{N}_0, \quad \zeta(n) = \begin{cases} Nm & \text{if } n = 2m, \\ Nm + 1 & \text{if } n = 2m + 1. \end{cases}$$

For $n \in \mathbb{N}_0$, define W_n the k -subbimodule of $V^{\otimes n}$ given by $V^{\otimes n}$ if $n < N$, and by

$$\bigcap_{i=0}^{n-N} V^{\otimes i} \otimes R \otimes V^{\otimes(n-i-N)}$$

if $n \geq N$. Then $W_i = V^{\otimes i}$ for $i = 0, \dots, N - 1$, $W_N = R$, and $W_{N+1} = (R \otimes V) \cap (V \otimes R)$, which we shall also denote by R_{N+1} . Note that W_n may be regarded as a graded k -bimodule concentrated in degree n .

We recall the *bimodule Koszul complex* $(K_*(A), d_*)$ of A , defined in [2] for the case that $k = F$ is a field. First, we consider the graded A -bimodule given by $(A \otimes W_n \otimes A)_{n \in \mathbb{N}_0}$. Then, for each $n \in \mathbb{N}$, there are two A -bimodule maps

$$d_n^L, d_n^R: A \otimes W_n \otimes A \rightarrow A \otimes W_{n-1} \otimes A$$

given by

$$\begin{aligned} d_n^L(a_0 \otimes \cdots \otimes a_{n+1}) &= a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}, \\ d_n^R(a_0 \otimes \cdots \otimes a_{n+1}) &= a_0 \otimes \cdots \otimes a_{n-1} \otimes a_n a_{n+1}. \end{aligned}$$

We note that d^L and d^R commute. We shall denote an element $a \otimes \alpha \otimes b \in A \otimes W_n \otimes A$, where $a, b \in A$ and $\alpha \in W_n$, in the shorter form $a|\alpha|b$.

If q is a primitive N -th root of unity, we consider the map of A -bimodules

$$d_n^b : A \otimes W_n \otimes A \rightarrow A \otimes W_{n-1} \otimes A$$

given by $d_n^b = d_n^L - q^{n-1}d_n^R$. It is trivially verified that $((A \otimes W_\bullet \otimes A), d_\bullet^b)$ is an N -complex. The bimodule Koszul complex $(K_\bullet(A), d_\bullet)$ of A is the $(1, N - 1)$ -contraction of the previous N -complex, that is $K_n(A) = A \otimes W_{\zeta(n)} \otimes A$ with differential d_\bullet given by the corresponding successive composition of the differential of the previous N -complex. It is easy to see that

$$d_n = \begin{cases} d_{Nm+1}^L - d_{Nm+1}^R & \text{if } n = 2m + 1, \\ d_{N(m-1)+2}^L \cdots d_{Nm}^L + d_{N(m-1)+2}^R \cdots d_{Nm}^R & \\ + \sum_{i=0}^{N-3} d_{N(m-1)+2}^L \cdots d_{Nm-i-1}^L d_{Nm-i}^R \cdots d_{Nm}^R & \text{if } n = 2m. \end{cases}$$

We thus notice that the N -th root of unity is in some sense superfluous, since we may define the Koszul complex without invoking it. The algebra A is called *Koszul* if the bimodule Koszul complex $(K_\bullet(A), d_\bullet)$ is a resolution of A -bimodules of A for the map $d_0 = \mu : K_0(A) = A \otimes A \rightarrow A$ given by the product μ of A . We recall that the category of A -bimodules is equivalent to the category of (say left) modules over $A^e = A \otimes_F A^{\text{op}}$ (not over $A^e = A \otimes_k A^{\text{op}}$, because we are not considering symmetric k -bimodules!). Moreover, since $\mathcal{H}om_{A^e}(A \otimes W_n \otimes A, -) \simeq \mathcal{H}om_{k^e}(W_n, -)$, we see that the bimodule Koszul resolution consists of projective graded A -bimodules, i.e., projective graded left A^e -modules.

For our purposes, we will be interested in a weaker condition than that of being Koszul: we shall suppose that A is an N -homogeneous algebra satisfying that $\text{Tor}_3^A(k, k)$ is concentrated in degree $N + 1$. This is equivalent to say that there exists a projective resolution (of graded A -bimodules) of A that coincides with the one given by the Koszul bimodule complex for homological degrees less than or equal to three (see [1], Sec. 2). This assumption comes from the fact that all the computations we shall perform in this article are restricted to those homological degrees. In any case, we shall still denote by $K_\bullet(A)$ the former projective resolution (of graded A -bimodules) of A , and call it *minimal*.

The (graded) *Hochschild complex* $(C_\bullet(A), b_\bullet)$ of A is given by $C_n(A) = A \otimes A^{\otimes n} \otimes A$ for $n \in \mathbb{N}_0$, with differential

$$b_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for $n \in \mathbb{N}$. We will also write $a_0 | \dots | a_{n+1}$ instead of $a_0 \otimes \cdots \otimes a_{n+1}$. This complex gives a projective resolution $C_\bullet(A) \rightarrow A$ of A in the category of (graded) A -bimodules provided with morphisms of degree zero via $b_0 = \mu : A \otimes A \rightarrow A$, called the *Hochschild resolution*. Moreover, we may consider the *reduced (graded) Hochschild resolution* $(\bar{C}_\bullet(A), \bar{b}_\bullet)$ of A . The underlying graded A -bimodule is given

by $\bar{C}_n(A) = A \otimes (A/k)^{\otimes n} \otimes A$ for $n \in \mathbb{N}_0$, so there exists a canonical projection $p_n: C_n(A) \rightarrow \bar{C}_n(A)$, and the differential \bar{b}_n is induced by b_n for $n \in \mathbb{N}$ such that p_\bullet becomes a morphism of complexes. It gives a projective resolution of A in the category of (graded) A -bimodules provided with morphisms of degree zero, called the *reduced Hochschild resolution* $\bar{C}_\bullet(A) \rightarrow A$ of A . Using the identification between A/k and the k -subbimodule of A spanned by the strictly positive elements of A , we will usually denote an element of $\bar{C}_n(A)$ simply by $a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}$, or $a_0|a_1| \dots |a_n|a_{n+1}$, where $\deg(a_i) \geq 1$ for $i = 1, \dots, n$, instead of the more correct $a_0 \otimes [a_1] \otimes \cdots \otimes [a_n] \otimes a_{n+1}$ (or $a_0|[a_1]| \dots |[a_n]|a_{n+1}$), where $[a] \in A/k$ is the class of $a \in A$. Furthermore, if by chance there exists $i \in \{1, \dots, n\}$ such that $\deg(a_i) = 0$, we may also consider $a_0|a_1| \dots |a_n|a_{n+1}$ as the zero element of $\bar{C}_n(A)$. We remark that, under this identification, \bar{b}_n is the restriction of b_n to $A \otimes I_+^{\otimes n} \otimes A$.

If M is a graded A -bimodule, the *graded Hochschild cohomology groups* of A with coefficients in M , which will be also denoted by $H^\bullet(A, M)$, are given by the cohomology of the cochain complex

$$\begin{aligned} \mathcal{H}om_{A^e}(C_\bullet(A), M) &= \bigoplus_{j \in \mathbb{Z}} \text{hom}_{A^e}(C_\bullet(A), M[j]) \\ &\simeq \mathcal{H}om_{k^e}(A^{\otimes \bullet}, M) \\ &= \bigoplus_{j \in \mathbb{Z}} \text{hom}_{k^e}(A^{\otimes \bullet}, M[j]) \end{aligned}$$

with the induced differential, where $\text{hom}(-, -)$ is the set of degree-preserving homomorphisms. Again, these cohomology groups can be computed using either the complex $\mathcal{H}om_{A^e}(C_\bullet(A), M)$ or $\mathcal{H}om_{A^e}(\bar{C}_\bullet(A), M)$. If $M = A$, we shall write $HH^\bullet(A)$ instead of $H^\bullet(A, A)$. In this case, we know that $H^\bullet(A, M) \simeq \mathcal{E}xt_{A^e/k^e}^\bullet(A, M)$, the relative derived functors of $\mathcal{H}om_{A^e}(A, -)$. We see that $H^\bullet(A, M)$ has an internal grading or weight that comes from the gradings of A and M , which we shall denote by $H^\bullet(A, M)_\bullet$.

Before proceeding further, we shall state some notation. Let us consider a positive integer $p \leq n$ and a subset $I = \{i_1 < \cdots < i_m\}$ of $\{1, \dots, p\}$. Given a decomposition $n = n_1 + \cdots + n_p$ of n ($n_j \in \mathbb{N}_0$) satisfying that $n_{i_j} = 1$ for $j = 1, \dots, m$, we define the collection of k^e -linear maps $q_i^{(n_1, \dots, n_p)}$ homogeneous of degree zero given by the canonical map $q_i^{(n_1, \dots, n_p)}: V^{\otimes n_i} \rightarrow A$, if $i \notin I$, and $q_i^{(n_1, \dots, n_p)} = 1_V$, if $i \in I$. Then, consider the map $\text{par}_{(n_1, \dots, n_p), I} = q_1^{(n_1, \dots, n_p)} \otimes \cdots \otimes q_p^{(n_1, \dots, n_p)}$, so it is a map from $V^{\otimes n}$ to

$$A^{\otimes(i_1-1)} \otimes V \otimes A^{\otimes(i_2-i_1-1)} \otimes V \otimes \cdots \otimes V \otimes A^{\otimes(i_m-i_{m-1}-1)} \otimes V \otimes A^{\otimes(p-i_m)}.$$

We now define the maps $\text{par}'_{(n_1, \dots, n_p), I}$ and $\text{Par}_{p, I}$ from $V^{\otimes n}$ to $A \otimes V \otimes A \otimes V \otimes \cdots \otimes V \otimes A \otimes V \otimes A$, where the k -bimodule V appears m times in the last tensor product. The former is given by the composition of

$$(\mu^{(i_1-1)} \otimes 1_V \otimes \mu^{\otimes(i_2-i_1-1)} \otimes 1_V \otimes \cdots \otimes 1_V \otimes \mu^{(i_m-i_{m-1}-1)} \otimes 1_V \otimes \mu^{(p-i_m)})$$

and $\text{par}_{(n_1, \dots, n_p), I}$. The latter is defined as $\text{Par}_{p, I} = \sum \text{par}'_{(n_1, \dots, n_p), I}$, where the last sum is indexed over all decompositions (n_1, \dots, n_p) of n such that $n_{i_j} = 1$ for $j = 1, \dots, m$. In general, we shall write

$$\text{Par}_{p, I}(\alpha) = \alpha_{(1)} \otimes \bar{\alpha}_{(2)} \otimes \alpha_{(3)} \otimes \cdots \otimes \alpha_{(2m-1)} \otimes \bar{\alpha}_{(2m)} \otimes \alpha_{(2m+1)},$$

where the bars are used for the elements with indices in I . We remark that in the previous expression a sum over all decompositions of α in p terms such that the ones with bars belong to V is implicit. For instance, if $\alpha = v_1 \dots v_n \in V^{\otimes n}$, we denote

$$\text{Par}_{3, \{2\}}(\alpha) = \alpha_{(1)} | \bar{\alpha}_{(2)} | \alpha_{(3)} \in A \otimes V \otimes A$$

the sum over all decompositions of α such that the term with the bar $\bar{\alpha}_{(2)}$ belongs to V , i.e., it is given by

$$1 | v_1 | v_2 \dots v_n + \sum_{i=1}^{n-2} v_1 \dots v_i | v_{i+1} | v_{i+2} \dots v_n + v_1 \dots v_{n-1} | v_n | 1.$$

This should be seen as a similar notation to Sweedler’s one for coproducts. We emphasize that each term $\alpha_{(j)}$ is homogeneous and may be (and shall be) taken equal to 1 if it has degree 0.

The following remarks are easy consequences of the definitions:

Fact 1.1. If $\alpha = v_1 \dots v_n \in V^{\otimes n}$ and $\beta = w_1 \dots w_m \in V^{\otimes m}$, then

$$(\alpha\beta)_{(1)} | (\overline{\alpha\beta})_{(2)} | (\alpha\beta)_{(3)} = \alpha_{(1)} | \bar{\alpha}_{(2)} | \alpha_{(3)} \beta + \alpha \beta_{(1)} | \bar{\beta}_{(2)} | \beta_{(3)}.$$

Fact 1.2. If $\alpha = v_1 \dots v_n \in V^{\otimes n}$, let us consider $\alpha_{(1)} | \bar{\alpha}_{(2)} | \alpha_{(3)} \in A \otimes V \otimes A \subseteq A \otimes A \otimes A$. Then,

$$b_1(\alpha_{(1)} | \bar{\alpha}_{(2)} | \alpha_{(3)}) = \alpha_{(1)} \bar{\alpha}_{(2)} | \alpha_{(3)} - \alpha_{(1)} | \bar{\alpha}_{(2)} \alpha_{(3)} = \alpha | 1 - 1 | \alpha.$$

We see that

$$d_2(1 | r | 1) = r_{(1)} | \bar{r}_{(2)} | r_{(3)}, \tag{1.1}$$

$$d_3(1 | w | 1) = v_i | s_i | 1 - 1 | r_i | u_i = \bar{w}_{(1)} | w_{(2)} | 1 - 1 | w_{(1)} | \bar{w}_{(2)}, \tag{1.2}$$

where $r \in R$, and $w = \sum_{i \in I} r_i u_i = \sum_{i \in I} v_i s_i \in (V \otimes R) \cap (R \otimes V) = R_{N+1}$ for $u_i, v_i \in V, r_i, s_i \in R$ and a finite set I of indices. We have omitted the sum in (1.2) and we shall do so for the typical elements of R_{N+1} : we shall simply write $w = r_i u_i = v_i s_i$. Moreover, the previous choice of letters will be the usual one to denote elements of R and R_{N+1} , unless we say the contrary.

Since both the Hochschild and the minimal bimodule resolutions are projective resolutions of graded A -bimodules of A , there exist unique (up to homotopy) comparison morphisms of graded A -bimodules $\sigma_\bullet: K_\bullet(A) \rightarrow C_\bullet(A)$ and $\tau_\bullet: C_\bullet(A) \rightarrow$

$K_\bullet(A)$. We define the morphisms of graded A -bimodules given by the extension of the following expressions

$$\sigma_0 = 1_A \otimes 1_A, \quad (\text{SIGMA}_0)$$

$$\sigma_1 = 1_A \otimes \text{inc}_{V \subset A} \otimes 1_A, \quad (\text{SIGMA}_1)$$

$$\sigma_2(1|r|1) = 1|r_{(1)}|\bar{r}_{(2)}|r_{(3)}, \quad (\text{SIGMA}_2)$$

$$\sigma_3(1|w|1) = 1|v_i|s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)} = 1|\bar{w}_{(1)}|w_{(2)}|\bar{w}_{(3)}|w_{(4)}. \quad (\text{SIGMA}_3)$$

We define $\bar{\sigma}_\bullet = p_\bullet \circ \sigma_\bullet$. It is easy to see that $\sigma_0 d_1 = b_1 \sigma_1$ and $\sigma_1 d_2 = b_2 \sigma_2$. These identities immediately imply that $\bar{\sigma}_0 d_1 = \bar{b}_1 \bar{\sigma}_1$ and $\bar{\sigma}_1 d_2 = \bar{b}_2 \bar{\sigma}_2$ hold. We shall check that $\sigma_2 d_3 = b_3 \sigma_3$, which also yields $\bar{\sigma}_2 d_3 = \bar{b}_3 \bar{\sigma}_3$. On the one hand, we see that

$$\begin{aligned} b_3 \sigma_3(1|w|1) &= b_3(1|v_i|s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)}) \\ &= v_i|s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)} - 1|v_i s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)} + 1|v_i|s_{i,(1)}\bar{s}_{i,(2)}|s_{i,(3)} \\ &\quad - 1|v_i|s_{i,(1)}|\bar{s}_{i,(2)}s_{i,(3)} \\ &= v_i|s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)} - 1|v_i s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)} + 1|v_i|s_i|1 - 1|v_i|1|s_i \\ &= v_i|s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)} - 1|v_i s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)}, \end{aligned}$$

where we have used Fact 1.2 and that $1|v_i|s_i|1 = 1|v_i|1|s_i = 0$ since each s_i vanishes in A . On the other hand,

$$\begin{aligned} \sigma_2 d_3(1|w|1) &= \sigma_2(v_i|s_i|1 - 1|r_i|u_i) \\ &= v_i|s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)} - 1|r_{i,(1)}|\bar{r}_{i,(2)}|r_{i,(3)}u_i. \end{aligned}$$

The equality $\sigma_2 d_3 = b_3 \sigma_3$ then follows from the simple result:

Fact 1.3. If $w = r_i u_i = v_i s_i \in R_{N+1}$ for $u_i, v_i \in V$, $r_i, s_i \in R$ (summation understood), then

$$1|v_i s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)} = 1|r_{i,(1)}|\bar{r}_{i,(2)}|r_{i,(3)}u_i.$$

Proof. We point out that in the first member of the previous identity we are considering the decompositions of w in three terms where the first one has degree greater than or equal to 1 and the second one has degree 1, whereas in the second member we consider the decompositions of w in three terms where the second one has degree 1 but the third one has degree greater than or equal to 1.

We may decompose the sum $1|v_i s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)}$ in two separate cases: when $s_{i,(3)} \in k$ (in which case, it is equal to 1) and when $s_{i,(3)}$ has degree greater than or equal to 1. We may write this as

$$1|v_i s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)} = \underbrace{1|v_i s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)}}_{\deg(s_{i,(3)}) > 0} + 1|v_i s_{i,(1)}|\bar{s}_{i,(2)}|1.$$

The same reasoning applies to $1|r_{i,(1)}|\bar{r}_{i,(2)}|r_{i,(3)}u_i$, to give

$$1|r_{i,(1)}|\bar{r}_{i,(2)}|r_{i,(3)}u_i = \underbrace{1|r_{i,(1)}|\bar{r}_{i,(2)}|r_{i,(3)}u_i}_{\deg(r_{i,(1)})>0} + 1|1|\bar{r}_{i,(1)}|r_{i,(2)}u_i.$$

However, since the terms $1|v_i s_{i,(1)}|\bar{s}_{i,(2)}|1$ indicate all the decompositions of w in two terms such that the second one has degree 1, we see that $1|v_i s_{i,(1)}|\bar{s}_{i,(2)}|1 = 1|r_i|u_i|1$, which vanishes, since $1|r_i|u_i|1 \in C_2(A)$ and $r_i = 0$ in A . A similar argument implies that $1|1|\bar{r}_{i,(1)}|r_{i,(2)}u_i = 1|1|v_i|s_i \in C_2(A)$ vanishes.

Hence, we only have to prove that

$$\underbrace{1|v_i s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)}}_{\deg(s_{i,(3)})>0} = \underbrace{1|r_{i,(1)}|\bar{r}_{i,(2)}|r_{i,(3)}u_i}_{\deg(r_{i,(1)})>0}.$$

This identity holds, since both members are built from the decompositions of w in three terms, where the first and third ones have degree greater than or equal to 1, and the second one has degree 1. □

Definition 1.4. Given an homogeneous element $1|a_1| \dots |a_n|1 \in C_n(A)$ (resp. $a_1 \otimes \dots \otimes a_n \in A^{\otimes n}$), we shall say that it is N -normalized if $\deg(a_1) + \dots + \deg(a_n) < N$. Since N is fixed throughout this work, in both cases we will more simply say that it is *normalized*.

If $\sum_{i \in I} 1|a_1^i| \dots |a_n^i|1 \in C_n(A)$ (resp. $\sum_{i \in I} a_1^i \otimes \dots \otimes a_n^i \in A^{\otimes n}$) satisfies that $\deg(a_1^i) + \dots + \deg(a_n^i) = N$, $\deg(a_1^i), \deg(a_n^i) > 0$ and $\sum_{i \in I} a_1^i \dots a_n^i = 0$ in A (i.e., it belongs to R as an element of the tensor algebra TV), we shall say that $\sum_{i \in I} 1|a_1^i| \dots |a_n^i|1$ (resp. $\sum_{i \in I} a_1^i \otimes \dots \otimes a_n^i \in A^{\otimes n}$) is a *relation decomposition*. This could be also abbreviated by *rel. decomp.*

Finally, if $\sum_{i \in I} 1|a_1^i|a_2^i|a_3^i|1 \in C_3(A)$ (resp. $\sum_{i \in I} a_1^i \otimes a_2^i \otimes a_3^i \in A^{\otimes 3}$) satisfies that $\deg(a_1^i) + \deg(a_2^i) + \deg(a_3^i) = N + 1$, $\deg(a_1^i), \deg(a_3^i) = 1$ and $\sum_{i \in I} a_1^i a_2^i a_3^i \in R_{N+1}$ as an element of the tensor algebra TV , we shall say that $\sum_{i \in I} 1|a_1^i|a_2^i|a_3^i|1$ (resp. $\sum_{i \in I} a_1^i \otimes a_2^i \otimes a_3^i \in A^{\otimes 3}$) is a *double relation decomposition*. It will be occasionally abbreviated by *double rel. decomp.*

Using the identification $A/k \simeq I_+$, we have the analogous versions of the previous three definitions for elements of $\bar{C}_n(A)$ or $(A/k)^{\otimes n}$, which will be called in the same way. We would like to note that the set of homogeneous elements in $C_n(A)$ (resp. $A^{\otimes n}$, $\bar{C}_n(A)$, $(A/k)^{\otimes n}$) which are normalized form a k -subbimodule. The same applies to the set of homogeneous elements which are relation decompositions, or double relation decompositions.

We partially define the morphisms of A -bimodules τ_\bullet as follows:

$$\tau_0 = 1_A \otimes 1_A, \quad (\text{TAU}_0)$$

$$\tau_1(1|a|1) = a_{(1)}|\bar{a}_{(2)}|a_{(3)} \quad \text{if } \deg(a) < N, \quad (\text{TAU}_1)$$

$$\tau_2(1|a|b|1) = 0, \quad \text{if } 1|a|b|1 \text{ is normalized,} \quad (\text{TAU}_{2,1})$$

$$\tau_2(1|a_i|b_i|1) = 1|r|1 \quad \text{if } r = \sum_{i \in I} a_i \otimes b_i \text{ is a rel. decomp.,} \quad (\text{TAU}_{2,2})$$

$$\tau_3(1|a|b|c|1) = 0 \quad \text{if } 1|a|b|c|1 \text{ is normalized,} \quad (\text{TAU}_{3,1})$$

$$\tau_3(1|a_i|b_i|c_i|1) = 0 \quad \text{if } r = \sum_{i \in I} a_i \otimes b_i \otimes c_i \text{ is a rel. decomp.,} \quad (\text{TAU}_{3,2})$$

$$\tau_3(1|a_i|b_i|c_i|1) = 1|w|1 \quad \text{if } w = \sum_{i \in I} a_i \otimes b_i \otimes c_i \text{ is a double rel. decomp.} \quad (\text{TAU}_{3,3})$$

We remark that the previous identities induce maps $\bar{\tau}_\bullet$ from $\bar{C}_\bullet(A)$ to $K_\bullet(A)$ (partially defined on the image under p_\bullet of the domain of definition of τ_\bullet). It is trivial to verify that $\tau_0 b_1 = d_1 \tau_1$ holds, wherever τ_1 is defined. We shall check that $\tau_1 b_2 = d_2 \tau_2$ and $\tau_2 b_3 = d_3 \tau_3$ are verified where we have defined them. These identities would imply that $\bar{\tau}_0 \bar{b}_1 = d_1 \bar{\tau}_1$, $\bar{\tau}_1 \bar{b}_2 = d_2 \bar{\tau}_2$ and $\bar{\tau}_2 \bar{b}_3 = d_3 \bar{\tau}_3$ hold where they are defined. These maps can be extended to complete morphisms of complexes of A^e -modules $\tau_\bullet: C_\bullet(A) \rightarrow K_\bullet(A)$ and $\bar{\tau}_\bullet: \bar{C}_\bullet(A) \rightarrow K_\bullet(A)$, giving quasi-isomorphisms, by the semisimplicity assumption on k^e .

Let us start with $\tau_1 b_2 = d_2 \tau_2$.

- If we apply $d_2 \tau_2$ to a normalized element of the form $1|a|b|1$, we see that $d_2 \tau_2(1|a|b|1) = 0$. On the other hand,

$$\begin{aligned} \tau_1 b_2(1|a|b|1) &= \tau_1(a|b|1 - 1|ab|1 + 1|a|b) \\ &= ab_{(1)}|\bar{b}_{(2)}|b_{(3)} - (ab)_{(1)}|(\overline{ab})_{(2)}|(ab)_{(3)} + a_{(1)}|\bar{a}_{(2)}|a_{(3)}b, \end{aligned}$$

which trivially vanishes by Fact 1.1, so $\tau_1 b_2 = d_2 \tau_2$ for the elements $1|a|b|1$, with $\deg(a) + \deg(b) < N$.

- If we apply $d_2 \tau_2$ to a relation decomposition of the form $1|a_i|b_i|1$ (with $r = \sum_{i \in I} a_i \otimes b_i$), we see that $d_2 \tau_2(1|a_i|b_i|1) = d_2(1|r|1) = r_{(1)}|\bar{r}_{(2)}|r_{(3)}$. Analogously,

$$\begin{aligned} \tau_1 b_2(1|a_i|b_i|1) &= \tau_1(a_i|b_i|1 - 1|a_i b_i|1 + 1|a_i|b_i) \\ &= a_i b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)} + a_{i,(1)}|\bar{a}_{i,(2)}|a_{i,(3)}b_i, \end{aligned}$$

since $1|a_i b_i|1 = 1|r|1 \in C_1(A)$, which vanishes. Now, using that

$$r_{(1)}|\bar{r}_{(2)}|r_{(3)} = a_i b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)} + a_{i,(1)}|\bar{a}_{i,(2)}|a_{i,(3)}b_i,$$

we conclude that $\tau_1 b_2 = d_2 \tau_2$ where we have defined it.

Let us now prove that $\tau_2 b_3 = d_3 \tau_3$.

- If we apply $d_3\tau_3$ to a normalized element of the form $1|a|b|c|1$, we see that $d_3\tau_3(1|a|b|c|1) = 0$. On the other hand,

$$\tau_2b_3(1|a|b|c|1) = \tau_2(a|b|c|1 - 1|ab|c|1 + 1|a|bc|1 - 1|a|b|c),$$

which trivially vanishes by (TAU_{2,1}), so $\tau_2b_3 = d_3\tau_3$ for the elements of the form $1|a|b|c|1$, with $\deg(a) + \deg(b) + \deg(c) < N$.

- If we apply $d_3\tau_3$ to a relation decomposition of the form $1|a_i|b_i|c_i|1$, we see that $d_3\tau_3(1|a_i|b_i|c_i|1) = 0$. Also,

$$\tau_2b_3(1|a_i|b_i|c_i|1) = \tau_2(a_i|b_i|c_i|1 - 1|a_i b_i|c_i|1 + 1|a_i|b_i c_i|1 - 1|a_i|b_i|c_i) = 0,$$

where we have used (TAU_{2,1}), since $\deg(a_i), \deg(c_i) > 0$.

- If we apply $d_3\tau_3$ to a double relation decomposition of the form $1|a_i|b_i|c_i|1$ (with $w = \sum_{i \in I} a_i \otimes b_i \otimes c_i$), we see that $d_3\tau_3(1|a_i|b_i|c_i|1) = d_3(1|w|1) = a_i|b_i c_i|1 - 1|a_i b_i|c_i$. We note that, if $w = r_i u_i = v_i s_i$, then $a_i|b_i c_i|1 = v_i|s_i|1$ and $1|a_i b_i|c_i = 1|r_i|u_i$, since, for the first identity, each member is a decomposition of w in two terms such that first one has degree 1, and the argument for the second identity is analogous. On the other hand,

$$\begin{aligned} \tau_2b_3(1|a_i|b_i|c_i|1) &= \tau_2(a_i|b_i|c_i|1 - 1|a_i b_i|c_i|1 + 1|a_i|b_i c_i|1 - 1|a_i|b_i|c_i) \\ &= a_i|b_i c_i|1 - 1|a_i b_i|c_i, \end{aligned}$$

where we have used that $1|a_i b_i|c_i|1 = 1|r_i|u_i|1$ and $1|a_i|b_i c_i|1 = 1|v_i|s_i|1$ vanish in $C_2(A)$, and (TAU_{2,2}) for the other two terms. We conclude that $\tau_2b_3 = d_3\tau_3$, where we have defined it.

Finally, there exist homotopies s_\bullet and t_\bullet for $\bullet \in \mathbb{N}_0$, which are morphisms of A -bimodules, on the complexes $\bar{C}_\bullet(A)$ and $K_\bullet(A)$, respectively, such that

$$\bar{b}_{n+1}s_n + s_{n-1}\bar{b}_n = 1_{\bar{C}_n(A)} - \bar{\sigma}_n\bar{\tau}_n, \tag{1.14}$$

and

$$d_{n+1}t_n + t_{n-1}d_n = 1_{K_n(A)} - \bar{\tau}_n\bar{\sigma}_n, \tag{1.15}$$

hold for $n \in \mathbb{N}_0$, respectively. We set

$$s_{-1} = 0, \tag{S_{-1}}$$

$$s_0 = 0, \tag{S_0}$$

$$s_1(1|a|1) = -1|a_{(1)}|\bar{a}_{(2)}|a_{(3)}, \text{ if } \deg(a) < N, \tag{S_1}$$

$$s_2(1|a_i|b_i|1) = 1|a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)}, \tag{S_2}$$

if $1|a_i|b_i|1$ is normalized or a relation decomposition,

$$s_3(1|a_i|b_i|c_i|1) = -1|a_i|b_i|c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)}, \tag{S_3}$$

if $1|a_i|b_i|c_i|1$ is normalized, a rel. decomp. or a double rel. decomp.

Note that in fact $s_3(1|a_i|b_i|c_i|1)$ vanishes if $1|a_i|b_i|c_i|1$ is a double relation decomposition.

As previously indicated, the semisimplicity hypothesis on k^e tells us that these maps can be extended to complete morphisms of A^e -modules $s_\bullet: \bar{C}_\bullet(A) \rightarrow \bar{C}_{\bullet+1}(A)$ satisfying the identity (1.14). We see clearly that $1_{\bar{C}_0(A)} - \bar{\sigma}_0 \bar{\tau}_0 = \bar{b}_1 s_0 + s_{-1} \bar{b}_0$ holds. Moreover, (1.14) for $n = 1$ is also verified, since, for $\deg(a) < N$,

$$(1_{\bar{C}_1(A)} - \bar{\sigma}_1 \bar{\tau}_1)(1|a|1) = 1|a|1 - a_{(1)}|\bar{a}_{(2)}|a_{(3)},$$

and

$$\begin{aligned} (\bar{b}_2 s_1 + s_0 \bar{b}_1)(1|a|1) &= \bar{b}_2(-1|a_{(1)}|\bar{a}_{(2)}|a_{(3)}) \\ &= -a_{(1)}|\bar{a}_{(2)}|a_{(3)} + 1|a_{(1)}\bar{a}_{(2)}|a_{(3)} - 1|a_{(1)}|\bar{a}_{(2)}a_{(3)} \\ &= -a_{(1)}|\bar{a}_{(2)}|a_{(3)} + 1|a|1 = 1|a|1 - a_{(1)}|\bar{a}_{(2)}|a_{(3)}, \end{aligned}$$

where we have used Fact 1.2 in the penultimate equality.

We shall now check that (1.14) holds for $n = 2$ when applied to a normalized $1|a|b|1$, or to a relation decomposition $1|a_i|b_i|1$.

- If we apply (1.14) for $n = 2$ to a normalized element of the form $1|a|b|1$, we see that $(1_{\bar{C}_2(A)} - \bar{\sigma}_2 \bar{\tau}_2)(1|a|b|1) = 1|a|b|1$, since $\bar{\tau}_2(1|a|b|1) = 0$. On the other hand,

$$\begin{aligned} s_1 \bar{b}_2(1|a|b|1) &= s_1(a|b|1 - 1|ab|1 + 1|a|b) \\ &= -a|b_{(1)}|\bar{b}_{(2)}|b_{(3)} + 1|(ab)_{(1)}|(\overline{ab})_{(2)}|(ab)_{(3)} \\ &\quad - 1|a_{(1)}|\bar{a}_{(2)}|a_{(3)}b \\ &= -a|b_{(1)}|\bar{b}_{(2)}|b_{(3)} + 1|ab_{(1)}|\bar{b}_{(2)}|b_{(3)}, \end{aligned}$$

where we have used Fact 1.1. Besides,

$$\begin{aligned} \bar{b}_3 s_2(1|a|b|1) &= \bar{b}_3(1|a|b_{(1)}|\bar{b}_{(2)}|b_{(3)}) \\ &= a|b_{(1)}|\bar{b}_{(2)}|b_{(3)} - 1|ab_{(1)}|\bar{b}_{(2)}|b_{(3)} + 1|a|b_{(1)}\bar{b}_{(2)}|b_{(3)} \\ &\quad - 1|a|b_{(1)}|\bar{b}_{(2)}b_{(3)} \\ &= a|b_{(1)}|\bar{b}_{(2)}|b_{(3)} - 1|ab_{(1)}|\bar{b}_{(2)}|b_{(3)} + 1|a|b|1, \end{aligned}$$

where we have used Fact 1.2 in the last equality. By adding the previous computations, we see that (1.14) holds for $n = 2$ when applied to $1|a|b|1$ if $\deg(a) + \deg(b) < N$.

- If we apply (1.14) for $n = 2$ to a relation decomposition $1|a_i|b_i|1$ (with $r = \sum_{i \in I} a_i \otimes b_i$), we see that

$$\begin{aligned} (1_{\bar{C}_2(A)} - \bar{\sigma}_2 \bar{\tau}_2)(1|a_i|b_i|1) &= 1|a_i|b_i|1 - \bar{\sigma}_2(1|r|1) \\ &= 1|a_i|b_i|1 - 1|a_i b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)} \\ &\quad - 1|a_{i,(1)}|\bar{a}_{i,(2)}|a_{i,(3)}b_i. \end{aligned}$$

On the other hand,

$$\begin{aligned} s_1 \bar{b}_2(1|a_i|b_i|1) &= s_1(a_i|b_i|1 + 1|a_i|b_i) \\ &= -a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)} - 1|a_{i,(1)}|\bar{a}_{i,(2)}|a_{i,(3)}b_i, \end{aligned}$$

where we have used that the elements $1|a_i|b_i|1 = 1|r|1$ in $\bar{C}_1(A)$ vanish. Also,

$$\begin{aligned} \bar{b}_3 s_2(1|a_i|b_i|1) &= \bar{b}_3(1|a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)}) \\ &= a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)} - 1|a_i b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)} \\ &\quad + 1|a_i|b_{i,(1)}\bar{b}_{i,(2)}|b_{i,(3)} - 1|a_i|b_{i,(1)}|\bar{b}_{i,(2)}b_{i,(3)} \\ &= a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)} - 1|a_i b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)} + 1|a_i|b_i|1, \end{aligned}$$

where we have used Fact 1.2 in the last equality. Adding both computations, we see that (1.14) holds for $n = 2$ when applied to $1|a_i|b_i|1$, with $r = \sum_{i \in I} a_i \otimes b_i \in R$, $\deg(a_i), \deg(b_i) > 0$.

We shall now check that (1.14) holds for $n = 3$ when applied to a normalized $1|a|b|c|1$, to a relation decomposition or to a double relation decomposition $1|a_i|b_i|c_i|1$.

- If we apply (1.14) for $n = 3$ to a normalized element or to a relation decomposition of the form $1|a_i|b_i|c_i|1$, we see that $(1_{\bar{C}_3(A)} - \bar{\sigma}_3 \bar{\tau}_3)(1|a_i|b_i|c_i|1) = 1|a_i|b_i|c_i|1$, since $\bar{\tau}_3(1|a_i|b_i|c_i|1) = 0$. On the other hand,

$$\begin{aligned} s_2 \bar{b}_3(1|a_i|b_i|c_i|1) &= s_2(a_i|b_i|c_i|1 - 1|a_i b_i|c_i|1 + 1|a_i|b_i c_i|1 - 1|a_i|b_i|c_i) \\ &= a_i|b_i|c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)} - 1|a_i b_i|c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)} \\ &\quad + 1|a_i|(b_i c_i)_{(1)}|\overline{(b_i c_i)}_{(2)}|(b_i c_i)_{(3)} \\ &\quad - 1|a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)}c_i \\ &= a_i|b_i|c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)} - 1|a_i b_i|c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)} \\ &\quad + 1|a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)}c_i + 1|a_i|b_i c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)} \\ &\quad - 1|a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)}c_i \\ &= a_i|b_i|c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)} - 1|a_i b_i|c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)} \\ &\quad + 1|a_i|b_i c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)}, \end{aligned}$$

where we have used Fact 1.1. Besides,

$$\begin{aligned} \bar{b}_4 s_3(1|a_i|b_i|c_i|1) &= \bar{b}_4(-1|a_i|b_i|c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)}) \\ &= -a_i|b_i|c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)} + 1|a_i b_i|c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)} \\ &\quad - 1|a_i|b_i c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)} + 1|a_i|b_i|c_i|1, \end{aligned}$$

where we have used Fact 1.2 in the last equality. The addition of these computations tells us that (1.14) holds for $n = 2$ when applied to a normalized element or a relation decomposition.

- If we apply (1.14) for $n = 3$ to a double relation decomposition $1|a_i|b_i|c_i|1$ (with $w = \sum_{i \in I} a_i \otimes b_i \otimes c_i$, $w = r_i u_i = v_i s_i$), we see that

$$\begin{aligned} & (1\bar{c}_3(A) - \bar{\sigma}_3\bar{\tau}_3)(1|a_i|b_i|c_i|1) \\ &= 1|a_i|b_i|c_i|1 - \bar{\sigma}_3(1|w|1) \\ &= 1|a_i|b_i|c_i|1 - 1|a_i|(b_i c_i)_{(1)}|\overline{(b_i c_i)}_{(2)}|(b_i c_i)_{(3)} \\ &= 1|a_i|b_i|c_i|1 - 1|a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)}c_i \\ &\quad - 1|a_i|b_i c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)} \\ &= -1|a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)}c_i, \end{aligned}$$

where we have used that $1|a_i|b_i c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)} = 1|a_i|b_i|c_i|1$, for the degree of c_i is one. On the other hand,

$$\begin{aligned} s_2\bar{b}_3(1|a_i|b_i|c_i|1) &= s_2(a_i|b_i|c_i|1 - 1|a_i|b_i|c_i) \\ &= -1|a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)}c_i, \end{aligned}$$

where we have used that $1|a_i|b_i|c_i|1 = 1|r_i|u_i|1$ and $1|a_i|b_i c_i|1 = 1|v_i|s_i|1$ vanish in $\bar{C}_2(A)$, and $a_i|b_i|c_{i,(1)}|\bar{c}_{i,(2)}|c_{i,(3)}$ vanish in $\bar{C}_3(A)$. Also,

$$\bar{b}_4 s_3(1|a_i|b_i|c_i|1) = 0,$$

using that s_3 vanishes on a double relation decomposition. We conclude thus that (1.14) holds for $n = 3$ when applied to a double relation decomposition.

The comparison morphism $\bar{\sigma}_\bullet: K_\bullet(A) \rightarrow \bar{C}_\bullet(A)$ is an injection for $\bullet = 0, \dots, 3$. Furthermore, the morphisms $\bar{\sigma}_\bullet$ and $\bar{\tau}_\bullet$ satisfy the following result.

Lemma 1.5. *According to the previous definition of the comparison morphisms $\bar{\sigma}_\bullet: K_\bullet(A) \rightarrow \bar{C}_\bullet(A)$ and $\bar{\tau}_\bullet: \bar{C}_\bullet(A) \rightarrow K_\bullet(A)$ for $\bullet = 0, \dots, 3$, we see that $\bar{\tau}_\bullet \bar{\sigma}_\bullet = 1_{K_\bullet(A)}$.*

Proof. It is clear that $\bar{\tau}_0 \bar{\sigma}_0 = 1_A \otimes 1_A$ and $\bar{\tau}_1 \bar{\sigma}_1 = 1_A \otimes 1_V \otimes 1_A$. Moreover,

$$\begin{aligned} \bar{\tau}_2 \bar{\sigma}_2(1|r|1) &= \bar{\tau}_2(1|r_{(1)}|\bar{r}_{(2)}|r_{(3)}) \\ &= \bar{\tau}_2 \underbrace{(1|r_{(1)}|\bar{r}_{(2)}|r_{(3)})}_{\deg(r_{(3)}) > 0} + 1|r_{(1)}|\bar{r}_{(2)}|1 \\ &= 0 + \bar{\tau}_2(1|r_{(1)}|\bar{r}_{(2)}|1) = 1|r|1. \end{aligned}$$

Finally,

$$\begin{aligned} \bar{\tau}_3 \bar{\sigma}_3(1|w|1) &= \bar{\tau}_3(1|v_i|s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)}) \\ &= \bar{\tau}_3 \underbrace{(1|v_i|s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)})}_{\deg(s_{i,(3)}) > 1} + \underbrace{1|v_i|s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)}}_{\deg(s_{i,(3)}) = 1} \\ &\quad + 1|v_i|s_{i,(1)}|\bar{s}_{i,(2)}|1 \\ &= 0 + \bar{\tau}_3(1|\bar{r}_{i,(1)}|r_{i,(2)}|\bar{r}_{i,(3)}|u_i) + \bar{\tau}_3(1|v_i|s_{i,(1)}|\bar{s}_{i,(2)}|1) = 1|w|1, \end{aligned}$$

where $w = v_i s_i = r_i u_i \in R_{N+1}$ and we have used that $1|v_i|s_{i,(1)}|\bar{s}_{i,(2)}|\bar{s}_{i,(3)} = 1|\bar{r}_{i,(1)}|r_{i,(2)}|\bar{r}_{i,(3)}|u_i$. \square

Remark 1.6. The Lemma implies that we may choose $t_\bullet = 0$ for $\bullet = -1, \dots, 2$.

We have thus obtained a partial description of both projective resolutions and their comparison in lower degrees that can be depicted as follows:

$$\begin{array}{cccccccccccc}
 \cdots & \xleftarrow{t_3} & A \otimes R_{N+1} \otimes A & \xrightarrow{d_3} & A \otimes R \otimes A & \xleftarrow{t_2} & A \otimes V \otimes A & \xrightarrow{d_1} & A \otimes A & \xrightarrow{d_0} & A \rightarrow 0 \\
 & & \bar{\tau}_3 \updownarrow \bar{\sigma}_3 & & \bar{\tau}_2 \updownarrow \bar{\sigma}_2 & & \bar{\tau}_1 \updownarrow \bar{\sigma}_1 & & \bar{\tau}_0 \updownarrow \bar{\sigma}_0 & & \parallel \\
 \cdots & \xleftarrow{s_3} & A \otimes (A/k)^{\otimes 3} \otimes A & \xrightarrow{\bar{b}_3} & A \otimes (A/k)^{\otimes 2} \otimes A & \xleftarrow{\bar{b}_2} & A \otimes (A/k) \otimes A & \xrightarrow{\bar{b}_1} & A \otimes A & \xrightarrow{b_0} & A \rightarrow 0 \\
 & & \bar{s}_3 \updownarrow \bar{s}_2 & & \bar{s}_2 \updownarrow \bar{s}_1 & & \bar{s}_1 \updownarrow \bar{s}_0 & & \bar{s}_0 \updownarrow \bar{s}_{-1} & & \parallel
 \end{array}$$

2. PBW-deformations and deformations à la Gerstenhaber of homogeneous algebras

In this section we shall briefly recall the definitions of PBW-deformations and of the (graded) deformations à la Gerstenhaber, which we will usually just call deformations. We shall also establish a link between both concepts.

2.1. PBW-deformations of homogeneous algebras. We start recalling the definition of a PBW-deformation, and we mainly follow [1]. We first recall that a *filtered k -algebra* B is a k -algebra provided with an increasing sequence $\{F^\bullet B\}_{\bullet \in \mathbb{N}_0}$ of k -subbimodules of B such that $F^m B \cdot F^n B \subseteq F^{m+n} B$, for all $m, n \in \mathbb{N}_0$, and $1_B \in F^0 B$. As usual, such filtrations may also be seen to be indexed over \mathbb{Z} , where the negatively indexed terms vanish. Given a k -bimodule V , the tensor algebra TV has a filtration $\{F^\bullet\}_{\bullet \in \mathbb{N}_0}$ defined by $F^i = \bigoplus_{j=0}^i V^{\otimes j}$. Now, given $P \subset F^N$, we shall consider the algebra $U = TV/\langle P \rangle$, with the filtration $\{F^\bullet U\}_{\bullet \in \mathbb{N}_0}$ induced by the filtration of the tensor algebra, i.e., $F^\bullet U = \pi(F^\bullet)$, where π denotes the canonical projection from TV to U . Of course, π is a morphism of filtered algebras. The filtration can be described more concretely as follows: if $\langle P \rangle^i = F^i \cap \langle P \rangle$, then $F^i U = F^i / \langle P \rangle^i$ for $i \in \mathbb{N}_0$. If $\pi_i : TV \rightarrow V^{\otimes i}$ is the canonical projection, let us denote $R = \pi_N(P)$ and define the N -homogeneous algebra $A = TV/\langle R \rangle$.

Remark 2.1. We recall here the standard fact that, even though the ideal $\langle P \rangle$ coincides with $\sum_{i,j \geq 0} V^{\otimes i} P V^{\otimes j}$, $\langle P \rangle^n$ may be strictly bigger than the sum $\sum_{i+j \leq n-N} V^{\otimes i} P V^{\otimes j}$, which in particular vanishes if $n < N$. As we shall see below, the PBW property will be the exact condition in order to avoid this phenomenon.

We shall now consider the associated graded algebra $\text{gr}(U)$ to the previous filtration. First, we state the following direct results.

Lemma 2.2. *If $S = \bigoplus_{i \in \mathbb{N}_0} S_i$ is an \mathbb{N}_0 -graded k -algebra and $F^\bullet S$ is the filtration induced by the grading of S , i.e., $F^i S = \bigoplus_{j=0}^i S_j$, then there exists a canonical isomorphism $\iota: S \rightarrow \text{gr}(S)$ of \mathbb{N}_0 -graded k -algebras, such that the restriction $\iota|_{S_i}: S_i \rightarrow F^i S / F^{i-1} S$ is the canonical k^e -linear isomorphism.*

Proof. Easy. □

Since $\pi: TV \rightarrow U$ is a morphism of filtered algebras, it induces a morphism of graded algebras $\text{gr}(\pi): \text{gr}(TV) \rightarrow \text{gr}(U)$. Moreover, the filtration of U is induced by the filtration of TV , so $\text{gr}(\pi)$ is surjective. On the other hand, since the filtration of TV comes from a grading on the tensor algebra, we see that there exists a canonical isomorphism $\iota: TV \simeq \text{gr}(TV)$, by Lemma 2.2. So we may consider the surjective morphism of graded k -algebras given by the composition $\text{gr}(\pi) \circ \iota: TV \rightarrow \text{gr}(U)$, which we shall call Π . It is easy to see that $\Pi(R) = 0$, since $\iota(R) = P/F^{N-1}$. Hence Π induces a surjective morphism of graded k -algebras $p: A \rightarrow \text{gr}(U)$. We say that U satisfies the PBW property or that U is a PBW-deformation of A if p is an isomorphism.

Remark 2.3. If $k = F$ is a field and V is a finite dimensional vector space over k , p is an isomorphism if and only if there exists an isomorphism of graded k -algebras $A \simeq \text{gr}(U)$. This is proved as follows. One direction is obvious. Let us assume that there exists an isomorphism of graded k -algebras $A \simeq \text{gr}(U)$. Since V is finite dimensional, A (and a fortiori $\text{gr}(U)$) is evidently locally finite dimensional, i.e., each homogeneous component is finite dimensional. Hence, since p is surjective, each restriction of p to a homogeneous component is surjective, so bijective. Therefore, p is an isomorphism.

The morphism p can be more concretely described as follows. We just need to consider its restriction to A_i ($i \in \mathbb{N}_0$). First, we see that

$$\text{gr}(U)_i = F^i U / F^{i-1} U = \frac{(F^i / (F^i \cap \langle P \rangle))}{(F^{i-1} / (F^{i-1} \cap \langle P \rangle))} \simeq F^i / ((F^i \cap \langle P \rangle) + F^{i-1}),$$

where we have used that $F^{i-1} \cap \langle P \rangle = (F^i \cap \langle P \rangle) \cap F^{i-1}$ and the Second and Third Isomorphism Theorems. Then, $p|_{A_i}$ is induced by the k^e -linear morphism $V^{\otimes i} \rightarrow F^i / ((F^i \cap \langle P \rangle) + F^{i-1}) \simeq \text{gr}(U)_i$ given by the composition of the canonical injection $V^{\otimes i} \rightarrow F^i$ and the canonical projection $F^i \rightarrow F^i / ((F^i \cap \langle P \rangle) + F^{i-1})$. So it is easy to see that U satisfies the PBW property if and only if

$$(\langle P \rangle \cap F^n) + F^{n-1} = (\langle R \rangle \cap F^n) + F^{n-1},$$

for all $n \in \mathbb{N}_0$, which is equivalent to

$$\langle P \rangle \cap F^n \subset (\langle R \rangle \cap F^n) + F^{n-1},$$

for all $n \in \mathbb{N}_0$.

As noticed by R. Berger and V. Ginzburg (see [1], Prop. 3.2), the filtered algebra U satisfies the PBW property if and only if $\langle P \rangle^n = \sum_{i+j \leq n-N} V^{\otimes i} P V^{\otimes j}$ for all $n \in \mathbb{N}_0$ (in fact, it is sufficient to prove the equality for $n \geq N - 1$). Moreover, if we denote $J_n = \sum_{i+j \leq n-N} V^{\otimes i} P V^{\otimes j}$ for $n \in \mathbb{N}_0$, Prop. 3.3. in [1] states that U satisfies the PBW property if and only if $J_n \cap F^{n-1} = J_{n-1}$ for all $n \in \mathbb{N}_0$ (or just $n \geq N$). The identity $J_N \cap F^{N-1} = J_{N-1}$ is simply

$$P \cap F^{N-1} = 0, \tag{2.1}$$

whereas $J_{N+1} \cap F^N = J_N$ is easily seen to be equivalent to

$$(V \otimes P + P \otimes V) \cap F^N \subset P. \tag{2.2}$$

From now on we shall suppose that identity (2.1) holds, which implies that the map $\pi_N: F^N \rightarrow V^{\otimes N}$ gives an isomorphism between P and $R = \pi_N(P)$. Then there exists a k^e -linear map $\varphi: R \rightarrow F^{N-1}$ such that $\text{id} - \varphi$ is the inverse of $\pi_N|_P$, i.e., $P = \{r - \varphi(r) : r \in R\}$. We further write, $\varphi = \sum_{j=0}^{N-1} \varphi_j$, where $\varphi_j: R \rightarrow V^{\otimes j}$ is the composition of φ with the canonical morphism $F^{N-1} \rightarrow V^{\otimes j}$. Then it is easy to see that identity (2.2) is equivalent to (see [1], Prop. 3.5)

$$(\varphi \otimes 1_V - 1_V \otimes \varphi)(R_{N+1}) \subset P,$$

or equivalently (see [1], Prop. 3.6)

$$(\varphi_{N-1} \otimes 1_V - 1_V \otimes \varphi_{N-1})(R_{N+1}) \subset R, \tag{2.3}$$

$$\varphi_0 \circ (\varphi_{N-1} \otimes 1_V - 1_V \otimes \varphi_{N-1})(R_{N+1}) = 0, \tag{2.4}$$

$$(\varphi_j \circ (\varphi_{N-1} \otimes 1_V - 1_V \otimes \varphi_{N-1}) + (\varphi_{j-1} \otimes 1_V - 1_V \otimes \varphi_{j-1}))(R_{N+1}) = 0, \tag{2.5}$$

for $0 < j < N$.

Definition 2.4. Given a filtered algebra $U = TV/\langle P \rangle$, where $P \subset F^N$, such that (2.1), (2.3), (2.5) and (2.4) hold, we say that U is a *weak PBW-deformation* of $A = TV/\langle R \rangle$, where $R = \pi_N(P)$. We remark that each weak PBW-deformation U of A is provided with a surjective morphism of graded algebras $p: A \rightarrow \text{gr}(U)$. Given two weak deformations U and U' of A (with induced morphisms p and p' , respectively), they are called *equivalent* if there exists an isomorphism of filtered algebras $g: U \rightarrow U'$ such that $\text{gr}(g) \circ p = p'$.

It is immediate to see that a PBW-deformation is a weak PBW-deformation. Using considerations of deformation theory à la Gerstenhaber we shall provide another proof of the converse as a consequence of Theorem 3.1 when A is an N -homogeneous algebra satisfying that $\text{Tor}_3^A(k, k)$ is concentrated in degree $N + 1$ (cf. [1], Thm. 3.4).

2.2. Graded deformations in the sense of Gerstenhaber. Let us now briefly recall the definition of a graded deformation and some results that we shall use in the sequel. Most of what we will present is implicit in the work of M. Gerstenhaber (see [6], Sec. 1.2-1.5), and it is explained in more detail by A. Braverman and D. Gaitsgory in [3]. We would like to remark, however, that we are working over a not necessarily commutative ring k and this situation needs more sophisticated tools (cf. [7], Sec. 2, where the author is dealing with the case $k = \mathbb{C}[G]$ or more generally k a separable \mathbb{C} -algebra, in order to assure that the Hochschild cohomology over k coincides with that over \mathbb{C}).

In what follows, we consider $k[t]$ as an \mathbb{N}_0 -graded k -algebra such that $\deg(t) = 1$ and t is central. Given a k -bimodule V , we shall denote by $V[t]$ the $k[t]$ -bimodule with elements $\sum_{i \in I} v_i t^i$ for $v_i \in V$ and finite subsets $I \subseteq \mathbb{N}_0$, provided with the action $ct^m (\sum_{i \in I} v_i t^i) c' t^n = \sum_{i \in I} c.v_i.c' t^{i+m+n}$ for $c, c' \in k$.

If A denotes an \mathbb{N}_0 -graded associative k -algebra and $i \in \mathbb{N}$, an i -th level graded deformation of A means a graded $k[t]/(t^{i+1})$ -algebra structure on the $k[t]/(t^{i+1})$ -bimodule $A_i = A[t]/(t^{i+1})$ such that the identity $A_i/t.A_i \simeq A$ is an isomorphism of graded algebras. By a (polynomial) graded deformation of A we mean a graded $k[t]$ -algebra structure on the $k[t]$ -bimodule $A_t = A[t]$ such that the identity $A_t/t.A_t \simeq A$ is an isomorphism of graded algebras. In the previous definitions, we are always using the obvious graded k^e -linear map $A \rightarrow A_i = A[t]/(t^{i+1})$ (resp., $A \rightarrow A_t = A[t]$) given by $a \mapsto a$. We will usually denote the product of A_i (resp. A_t) by \times^i (resp. \times), which can be written as

$$a \times^i b = ab + \sum_{h=1}^i \psi_h(a, b)t^h \quad (a \times b = ab + \sum_{h \in \mathbb{N}} \psi_h(a, b)t^h).$$

Since ψ_h has degree $-h$, we must note that the sum for \times is finite for any pair of homogeneous elements a and b in A . We remark that ψ_h may be considered as an element of $\mathcal{H}om_{A^e}(C_2(A), A)$, and that the associativity of \times is equivalent to

$$d\psi_1 = 0, \tag{2.6}$$

$$-d\psi_{j+1}(a, b, c) = \sum_{i=1}^j (\psi_i(a, \psi_{j+1-i}(b, c)) - \psi_i(\psi_{j+1-i}(a, b), c)), \quad \text{for } j \in \mathbb{N}. \tag{2.7}$$

The right member of the last equation is usually denoted by $\text{sq}(\psi_1, \dots, \psi_j)(a, b, c)$. We note that there exists a trivial polynomial deformation of A given by the trivial product on $A[t]$, i.e., such that $(at^m) \times_0 (a't^n) = (a.a')t^{m+n}$ for $a, a' \in A$ and $m, n \in \mathbb{N}_0$.

Given a filtered algebra B with filtration $\{F^\bullet B\}_{\bullet \in \mathbb{N}_0}$, we recall that the Rees algebra associated to it is the graded $k[t]$ -algebra

$$R(B) = \left\{ \sum_{i \in I} b_i t^i : I \text{ is finite and } b_i \in F^i B \right\},$$

which is considered as a subalgebra of $B[t]$ provided with the trivial product \times_0 (in this case B is concentrated in degree zero). We remark that the underlying graded algebra structure of $R(B)$ is $\bigoplus_{\bullet \in \mathbb{N}_0} F^\bullet B$ with the product induced by that of B . It is easy to see that $R(B)/\langle t - \lambda \rangle \simeq B$ for $\lambda \in k^\times \cap \mathcal{Z}(k)$ such that its action on B is central (i.e., $\lambda b = b\lambda$ for all $b \in B$) and $R(B)/\langle t \rangle \simeq \text{gr}(B)$ (cf. [4], Cor. 2.3.8, whose proof can be applied also to this case). Moreover, $R(-)$ defines a functor from the category of filtered k -algebras to the category of graded $k[t]$ -algebras.

We would like to make some remarks about the algebra $A_t/\langle t - 1 \rangle$ (or about the algebra $A_t/\langle t - \lambda \rangle$, with $\lambda \in k^\times \cap \mathcal{Z}(k)$ such that its action on A is central, to which the following arguments also apply). There exists a k^e -linear map $A \rightarrow A_t/\langle t - 1 \rangle$ given by the composition of the canonical injection $A \rightarrow A_t$ and the projection $A_t \rightarrow A_t/\langle t - 1 \rangle$. We consider the filtration on $A_t/\langle t - 1 \rangle$ induced by the filtration of A under the previous map. We remark that the filtration of $A_t/\langle t - 1 \rangle$ induced by the filtration of A_t that comes from the grading is trivial. We shall see that the associated graded algebra of $A_t/\langle t - 1 \rangle$ is isomorphic to A as graded algebras. This is proved as follows. We consider the k^e -linear map $\rho' : A_t \rightarrow A$ (not an algebra map) given by $\sum_{j=0}^m a_j t^j \mapsto \sum_{j=0}^m a_j$. It is trivially verified that ρ' respects the filtrations coming from the gradings, and that $\rho'((t - 1)b) = 0$ for any $b \in A_t$, so it induces a morphism of filtered k -bimodules $\rho^* : A_t/\langle t - 1 \rangle \rightarrow A$, which is obviously surjective and injective. Its inverse is just the aforementioned map $A \rightarrow A_t/\langle t - 1 \rangle$. Hence it induces an isomorphism of graded k -bimodules $\rho : \text{gr}(A_t/\langle t - 1 \rangle) \rightarrow \text{gr}(A) \simeq A$ (the last isomorphism by Lemma 2.2). Now, if we denote the product of $A_t/\langle t - 1 \rangle$ by \times_1 and taking into account that the product of two elements $a, b \in A$ in $A_t/\langle t - 1 \rangle$ is given by

$$a \times_1 b = a \cdot b + \sum_{j \in \mathbb{N}} \psi_j(a, b),$$

and the degree of ψ_j is $-j$, we see that ρ is an algebra morphism, so an isomorphism of graded algebras. This further implies that, if A is generated by the image of the k^e -linear map $V \rightarrow A$, then $A_t/\langle t - 1 \rangle$ is generated by the image of the composition of $V \rightarrow A$ and the k^e -linear map $A \rightarrow A_t/\langle t - 1 \rangle$.

The importance of the algebra $A_t/\langle t - 1 \rangle$ is explained in the following proposition.

Proposition 2.5. *Let A be a graded algebra and let A_t be a graded deformation of A . Then, there exists a canonical isomorphism of graded $k[t]$ -algebras $R(A_t/\langle t - 1 \rangle) \rightarrow A_t$, such that the induced morphism $A \simeq R(A_t/\langle t - 1 \rangle)/\langle t \rangle \rightarrow A_t/\langle t \rangle \simeq A$ is the identity, where $A \simeq R(A_t/\langle t - 1 \rangle)/\langle t \rangle$ is given by the composition of the inverse of ρ and the canonical isomorphism of graded algebras $\text{gr}(A_t/\langle t - 1 \rangle) \simeq R(A_t/\langle t - 1 \rangle)/\langle t \rangle$.*

Proof. We first remark that we are going to identify $A_t/\langle t - 1 \rangle$ with A as k^e -bimodules under the previous filtered k^e -linear isomorphism $A \rightarrow A_t/\langle t - 1 \rangle$, so

we will denote the elements of this last algebra by elements a of A . We will continue denoting the product of $A_t/\langle t-1 \rangle$ by \times_1 and we remark that, if $a, b \in A$ are of degree i and j respectively, then

$$a \times_1 b = ab + \sum_{h=1}^{i+j} \psi_h(a, b),$$

since ψ_h has degree $-h$. Under this identification, the homogeneous i -th component of $R(A_t/\langle t-1 \rangle)$ is just $\bigoplus_{j=0}^i A_j$. So the elements of $R(A_t/\langle t-1 \rangle)$ are sums of elements of the form $a_i t^j$ for $j \geq i \geq 0$, with $a_i \in A_i$.

We recall that the elements of A_t are sums of elements of the form $a_i t^j$ for $i, j \geq 0$ for $a_i \in A_i$.

We now define the map $\text{com} : R(A_t/\langle t-1 \rangle) \rightarrow A_t$ given by the linear extension of

$$a_i t^j \mapsto a_i t^{j-i}.$$

It is trivially verified that com is an isomorphism of graded $k[t]$ -bimodules. In order to prove that it is a morphism of algebras, we only need to show that

$$\text{com}((a_i t^j) \times_1 (b_l t^m)) = \text{com}(a_i t^j) \times \text{com}(b_l t^m),$$

for $a_i, b_l \in A$ of degree i and l respectively, and $i \leq j$ and $l \leq m$. The left member is given by

$$\begin{aligned} \text{com}((a_i t^j) \times_1 (b_l t^m)) &= \text{com}((a_i \times_1 b_l) t^{j+m}) \\ &= \text{com}((a_i b_l + \sum_{h=1}^{i+l} \psi_h(a_i, b_l)) t^{j+m}) \\ &= \text{com}(a_i b_l t^{j+m}) + \sum_{h=1}^{i+l} \text{com}(\psi_h(a_i, b_l) t^{j+m}) \\ &= a_i b_l t^{j+m-i-l} + \sum_{h=1}^{i+l} \psi_h(a_i, b_l) t^{j+m+h-i-l}, \end{aligned}$$

where we have used that ψ_h has degree $-h$. On the other hand, the right member is given by

$$\begin{aligned} \text{com}(a_i t^j) \times \text{com}(b_l t^m) &= (a_i t^{j-i}) \times (b_l t^{m-l}) \\ &= (a_i \times b_l) t^{j+m-i-l} \\ &= ((a_i b_l + \sum_{h=1}^{i+l} \psi_h(a_i, b_l) t^h) t^{j+m-i-l}) \\ &= a_i b_l t^{j+m-i-l} + \sum_{h=1}^{i+l} \psi_h(a_i, b_l) t^{j+m+h-i-l}, \end{aligned}$$

where we have again used that ψ_h has degree $-h$. Hence, com is an isomorphism of graded $k[t]$ -algebras. It is clear that the induced map $A \simeq R(A_t/\langle t-1 \rangle)/\langle t \rangle \rightarrow A_t/\langle t \rangle \simeq A$ is the identity. \square

Let $E(A)$ denote the groupoid of all graded deformations of A where the morphisms are by definition isomorphisms of graded $k[t]$ -algebras, such that the induced morphism of $A \simeq A_t/t.A_t$ is the identity. Analogously, let $E_i(A)$ denote the groupoid of all i -th level graded deformations of A . Given $i \in \mathbb{N}$, we denote by F_i the functor from $E(A)$ to $E_i(A)$ given by reduction modulo t^{i+1} , i.e., if A_t is a graded deformation of A , then $F_i(A_t) = A_t/t^{i+1}A_t$ and the definition on morphisms is the obvious one. Moreover, given $i < j$ natural numbers, we denote by $F_{i < j}$ the functor from $E_j(A)$ to $E_i(A)$ given by reduction modulo t^{i+1} , so if A_j denotes a j -th level deformation, $F_{i < j}(A_j) = A_j/t^{i+1}A_j$ and for the morphisms it is obvious.

The following lemma is trivial (cf. [3], Lemma 1.3).

Lemma 2.6. *The collection of functors F_i define an equivalence between the category $E(A)$ and the inverse limit of the categories $E_i(A)$ with respect to the functors $F_{i < j}$.*

Given an i -th level deformation A_i of A , a *continuation to an $(i + 1)$ -th level deformation of A_i* is an $(i + 1)$ -th level deformation A_{i+1} of A such that $F_{i < i+1}(A_{i+1}) = A_i$. Given two continuations A_{i+1} and A'_{i+1} of A_i to an $(i + 1)$ -th level deformation, a morphism f from A_{i+1} to A'_{i+1} is a morphism in $E_{i+1}(A)$ such that $F_{i < i+1}(f) = 1_{A_i}$. The following proposition is also immediate (cf. [3], Prop. 1.5, or [7], 2.6).

Proposition 2.7. (a) *The set of isomorphism classes of objects of $E_1(A)$ can be canonically identified with $HH^2(A)_{-1}$.*

(b) *Let A_i be an object of $E_i(A)$. Then the obstruction for its continuation to an $(i + 1)$ -th level deformation lies in $HH^3(A)_{-i-1}$.*

(c) *Let A_i be as in (b). Then the set of isomorphism classes of continuations of A_i to an $(i + 1)$ -th level deformation has a natural structure of an $HH^2(A)_{-i-1}$ -homogeneous space.*

Finally we state the following proposition, which is analogous to Prop. 3.7 of [3].

Proposition 2.8. *Let A be an N -homogeneous algebra that is Koszul. Then,*

- (i) *the functors $F_{i-1 < i} : E_i(A) \rightarrow E_{i-1}(A)$ are injective on isomorphism classes of objects for $i > N$,*
- (ii) *the functors $F_{i-1 < i}$ are surjective on isomorphism classes of objects for $i > N + 1$.*

Proof. It is easy to see from the bimodule Koszul complex that $HH^2_{-i}(A)$ vanishes for $i > N$. Hence Proposition 2.7 (c) implies (i). Analogously, $HH^3_{-i}(A)$ vanishes for $i > N + 1$, so Proposition 2.7 (b) implies (ii). \square

In what follows, we shall only consider graded deformations such that the unit of the original k -algebra is also a unit of the deformed algebra. This is equivalent to ask that the 2-cochains ψ_j actually belong to $\mathcal{H}om_{A^e}(\bar{C}_\bullet(A), A)$. We shall say that such graded deformations *preserve the unit*. Since any graded deformation is equivalent to another one preserving the unit (because the complexes $\mathcal{H}om_{A^e}(\bar{C}_\bullet(A), A)$ and $\mathcal{H}om_{A^e}(C_\bullet(A), A)$ are quasi-isomorphic, and the equations (2.6) and (2.7) for both complexes are preserved under the corresponding quasi-isomorphism), there would be no loss of generality in restricting to such a situation.

2.3. From deformations to weak PBW-deformations. After having recalled the basic definitions and results we shall provide a link between both concepts. From now on, we stress the fact that we assume that the graded k -algebra A is of the form $TV/\langle R \rangle$, where V is considered to be concentrated in degree 1 and $R \subseteq V^{\otimes N}$ for $N \geq 2$ satisfying the assumption that $\text{Tor}_3^A(k, k)$ is concentrated in degree $N + 1$, so we may use the results of Section 1. First, we set some notation: if $\psi : W \rightarrow A$ denotes a k^e -linear map, we shall denote $\psi^\sim : A \otimes W \otimes A \rightarrow A$, its unique A^e -linear extension. A k^e -linear map $\psi : (A/k)^{\otimes 2} \rightarrow A$ is called *normalized* if it vanishes on normalized elements, i.e., if $\psi(a \otimes b) = 0$, whenever $a, b \in I_+ \simeq A/k$ are homogeneous elements satisfying that $\text{deg}(a) + \text{deg}(b) < N$. Further, we say that ψ is *extranormalized* if it is normalized and if it vanishes on relation decompositions. We shall also say in this case that ψ^\sim is normalized or extranormalized.

Let us now consider a graded deformation of A given by

$$a \times b = a.b + \sum_{i \geq 1} \psi_i(a, b)t^i,$$

where $\psi_i : (A/k)^{\otimes 2} \rightarrow A$ are normalized k^e -linear maps (see Lemma 2.12). We remark that ψ_i is a morphism of degree $-i$, and we have that the associativity of \times is equivalent to (2.6) and (2.7).

We set $\varphi_{N-j}^\sim = \psi_j^\sim \circ \bar{\sigma}_2$ for $j = 1, \dots, N$. We point out that the grading implies that $\psi_j^\sim \circ \bar{\sigma}_2$ vanishes for $j > N$. We shall see that, once we assume that (2.6) and (2.7) hold, the induced morphisms $\varphi_{N-j} : R \rightarrow F^{N-1}$ satisfy the weak PBW property expressed in (2.3), (2.5) and (2.4).

Let us first prove (2.3). We shall state a simple fact that we shall use in the sequel.

Fact 2.9. Given $\varphi : R \rightarrow A$ a k^e -linear map, then $d\varphi : R_{N+1} \rightarrow A$ satisfies that

$$d\varphi = (1_V \otimes \varphi - \varphi \otimes 1_V),$$

where we remark that the map $(1_V \otimes \varphi - \varphi \otimes 1_V)$ is defined from R_{N+1} to A .

Proof. If $w = r_i u_i = v_i s_i \in R_{N+1}$, then

$$\begin{aligned} d\varphi(w) &= d\varphi^\sim(1|w|1) = (\varphi^\sim \circ d_3)(1|w|1) = \varphi^\sim(v_i|r_i|1 - 1|s_i|u_i) \\ &= v_i\varphi(r_i) - \varphi(s_i)u_i = (1_V \otimes \varphi - \varphi \otimes 1_V)(w). \end{aligned} \quad \square$$

We note that (2.6) implies the equality

$$d\varphi_{N-1}^{\sim} = d(\psi_1^{\sim} \circ \bar{\sigma}_2) = \psi_1^{\sim} \circ \bar{\sigma}_2 \circ d_3 = \psi_1^{\sim} \circ \bar{b}_3 \circ \bar{\sigma}_3 = d\psi_1^{\sim} \circ \bar{\sigma}_3 = 0.$$

Hence, Fact 2.9 implies that $(1_V \otimes \varphi_{N-1} - \varphi_{N-1} \otimes 1_V)(w) = 0$. So, when we consider $(1_V \otimes \varphi_{N-1} - \varphi_{N-1} \otimes 1_V)$ as a map from R_{N+1} to $V^{\otimes N}$, we get (2.3). More precisely, we see that equation (2.6) composed with $\bar{\sigma}_3$ is equivalent to equation (2.3).

Let us now prove (2.5) and (2.4). We shall need the following simple fact.

Fact 2.10. If $\psi : (A/k)^{\otimes 2} \rightarrow A$ is a normalized cocycle and $\gamma \in V^{\otimes N}$, then it holds that $\psi(\bar{\gamma}_{(1)}, \gamma_{(2)}) = \psi(\gamma_{(1)}, \bar{\gamma}_{(2)})$. More generally, let us suppose that $\psi_1, \dots, \psi_j : (A/k)^{\otimes 2} \rightarrow A$ are normalized k^e -linear maps such that $-d\psi_j = \text{sq}(\psi_1, \dots, \psi_{j-1})$. Hence, if $\gamma \in V^{\otimes N}$, we see that $\psi_j(\bar{\gamma}_{(1)}, \gamma_{(2)}) = \psi_j(\gamma_{(1)}, \bar{\gamma}_{(2)})$.

Proof. It is obvious that the second statement generalizes the first one, but we give a detailed proof of both. In the first case we note that, since ψ is a cocycle,

$$\begin{aligned} 0 &= d\psi(\bar{\gamma}_{(1)}, \gamma_{(2)}, \bar{\gamma}_{(3)}) \\ &= \bar{\gamma}_{(1)}\psi(\gamma_{(2)}, \bar{\gamma}_{(3)}) - \psi(\bar{\gamma}_{(1)}\gamma_{(2)}, \bar{\gamma}_{(3)}) \\ &\quad + \psi(\bar{\gamma}_{(1)}, \gamma_{(2)}\bar{\gamma}_{(3)}) - \psi(\bar{\gamma}_{(1)}, \gamma_{(2)})\bar{\gamma}_{(3)} \\ &= -\psi(\gamma_{(1)}, \bar{\gamma}_{(2)}) + \psi(\bar{\gamma}_{(1)}, \gamma_{(2)}), \end{aligned}$$

where we have used the normalization of ψ in the third equality.

Now we prove the second statement. On the one hand, just as before we have that

$$\begin{aligned} d\psi_j(\bar{\gamma}_{(1)}, \gamma_{(2)}, \bar{\gamma}_{(3)}) &= \bar{\gamma}_{(1)}\psi_j(\gamma_{(2)}, \bar{\gamma}_{(3)}) - \psi_j(\bar{\gamma}_{(1)}\gamma_{(2)}, \bar{\gamma}_{(3)}) \\ &\quad + \psi_j(\bar{\gamma}_{(1)}, \gamma_{(2)}\bar{\gamma}_{(3)}) - \psi_j(\bar{\gamma}_{(1)}, \gamma_{(2)})\bar{\gamma}_{(3)} \\ &= -\psi_j(\gamma_{(1)}, \bar{\gamma}_{(2)}) + \psi_j(\bar{\gamma}_{(1)}, \gamma_{(2)}), \end{aligned}$$

where we have used the normalization of ψ_j . On the other hand,

$$\begin{aligned} \text{sq}(\psi_1, \dots, \psi_{j-1})(\bar{\gamma}_{(1)}, \gamma_{(2)}, \bar{\gamma}_{(3)}) \\ = \sum_{i=1}^{j-1} (\psi_i(\bar{\gamma}_{(1)}, \psi_{j-i}(\gamma_{(2)}, \bar{\gamma}_{(3)})) - \psi_i(\psi_{j-i}(\bar{\gamma}_{(1)}, \gamma_{(2)}), \bar{\gamma}_{(3)})) = 0, \end{aligned}$$

since $\psi_1, \dots, \psi_{j-1}$ are normalized. Hence our statement follows. □

Using Fact 2.9 for φ_{N-j} ($0 < j < N$) we see that

$$(d\psi_{j+1}^{\sim} \circ \bar{\sigma}_3)(1|w|1) = (1_V \otimes \varphi_{N-j-1} - \varphi_{N-j-1} \otimes 1_V)(w). \tag{2.8}$$

Besides, $(\text{sq}(\psi_1, \dots, \psi_j) \sim \circ \bar{\sigma}_3)(1|w|1)$ is equal to

$$\begin{aligned} & \text{sq}(\psi_1, \dots, \psi_j) \sim (1|v_i|s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)}) \\ &= \sum_{i=1}^j (\psi_i(v_i, \psi_{j+1-i}(s_{i,(1)}, \bar{s}_{i,(2)})) - \psi_i(\psi_{j+1-i}(v_i, s_{i,(1)}, \bar{s}_{i,(2)}))s_{i,(3)}) \\ &= \sum_{i=1}^j (\psi_i(v_i, \psi_{j+1-i}(s_{i,(1)}, \bar{s}_{i,(2)})) - \psi_i(\psi_{j+1-i}(v_i, s_{i,(1)}, \bar{s}_{i,(2)}))) \\ &= \sum_{i=1}^j (\psi_i(v_i, \psi_{j+1-i}(s_{i,(1)}, \bar{s}_{i,(2)})) - \psi_i(\psi_{j+1-i}(\bar{r}_{i,(1)}, r_{i,(2)}, u_i))) \\ &= \psi_j(v_i, \psi_1(s_{i,(1)}, \bar{s}_{i,(2)})) - \psi_j(\psi_1(\bar{r}_{i,(1)}, r_{i,(2)}, u_i)), \end{aligned}$$

where we have used in the third equality the normalization of ψ_1, \dots, ψ_j . In the last equality we have used the following simple fact: ψ_{j+1-i} has degree $-(j + 1 - i) = i - j - 1$, so ψ_{j+1-i} applied to an element of degree N gives an element of degree $N + i - j - 1$. In consequence, the elements $v_i \otimes \psi_{j+1-i}(s_{i,(1)}, \bar{s}_{i,(2)})$ and $\psi_{j+1-i}(\bar{r}_{i,(1)}, r_{i,(2)}) \otimes u_i$ have degree $N + j - i \leq N$ (for $i \leq j$) and the degree is exactly N if and only if $i = j$. The normalization of ψ_j forces only to consider the terms with degree N and the last equality follows.

Now, using Fact 2.10 we see that $\psi_1(\bar{r}_{i,(1)}, r_{i,(2)}) = \psi_1(r_{i,(1)}, \bar{r}_{i,(2)})$. Furthermore, by its very definition, $\varphi_{N-1}(r) = \psi_1(r_{(1)}, \bar{r}_{(2)})$ for any $r \in R$. Therefore,

$$(\text{sq}(\psi_1, \dots, \psi_j) \sim \circ \bar{\sigma}_3)(1|w|1) = \psi_j(v_i, \varphi_{N-1}(s_i)) - \psi_j(\varphi_{N-1}(r_i), u_i).$$

We need to compare the previous expression with $\varphi_{N-j} \circ (1_V \circ \varphi_{N-1} - \varphi_{N-1} \otimes 1_V)(w)$, as we shall proceed to do. We first note that $\varphi_{N-j} \circ (1_V \otimes \varphi_{N-1} - \varphi_{N-1} \otimes 1_V)(w)$ is given by

$$\begin{aligned} & \varphi_{N-j}(v_i \otimes \varphi_{N-1}(s_i)) - \varphi_{N-j}(\varphi_{N-1}(r_i) \otimes u_i) \\ &= (\psi_j \sim \circ \bar{\sigma}_2)(1|\overbrace{(v_i \otimes \underbrace{\varphi_{N-1}(s_i)}_{\alpha_i} - \underbrace{\varphi_{N-1}(r_i)}_{\beta_i} \otimes u_i)}^{\in R})|1) \\ &= \psi_j \sim (1|v_i \alpha_{i,(1)}|\bar{\alpha}_{i,(2)}|\alpha_{i,(3)} - 1|\beta_{i,(1)}|\bar{\beta}_{i,(2)}|\beta_{i,(3)}u_i - 1|\beta_i|u_i|1) \\ &= \psi_j(v_i \alpha_{i,(1)}, \bar{\alpha}_{i,(2)})\alpha_{i,(3)} - \psi_j(\beta_{i,(1)}, \bar{\beta}_{i,(2)})\beta_{i,(3)}u_i - \psi_j(\beta_i, u_i) \\ &= \psi_j(v_i \alpha_{i,(1)}, \bar{\alpha}_{i,(2)}) - \psi_j(\beta_i, u_i) = \psi_j(v_i, \alpha_i) - \psi_j(\beta_i, u_i) \\ &= \psi_j(v_i, \varphi_{N-1}(s_i)) - \psi_j(\varphi_{N-1}(r_i), u_i), \end{aligned}$$

where we have considered in the third member that $v_i \otimes \varphi_{N-1}(s_i) - \varphi_{N-1}(r_i) \otimes u_i$ is an element of R , we have used the normalization of ψ_j in the fourth equality, and Fact 2.10 in the first term of the penultimate member. This implies that

$$(\text{sq}(\psi_1, \dots, \psi_j) \sim \circ \bar{\sigma}_3)(1|w|1) = \varphi_{N-j} \circ (1_V \otimes \varphi_{N-1} - \varphi_{N-1} \otimes 1_V)(w). \quad (2.9)$$

Now, from equations (2.8) and (2.9) we see that equation (2.7) composed with $\bar{\sigma}_3$ (for $j = 1, \dots, N - 1$) is exactly (2.5). The case of identity (2.7) for $j = N$ leads exactly to equation (2.4), since $d\psi_{N+1}^\sim \circ \bar{\sigma}_3 = 0$ by degree reasons.

We have thus proved the following result.

Proposition 2.11. *Let \times be a graded deformation of A given by*

$$a \times b = a.b + \sum_{i \geq 1} \psi_i(a, b)t^i,$$

where $\psi_i: (A/k)^{\otimes 2} \rightarrow A$ (resp. $\psi_i^\sim: \bar{C}_2(A) \rightarrow A$) are normalized k^e -linear maps (resp. A^e -linear maps). We define $\varphi_{N-j}^\sim = \psi_j^\sim \circ \bar{\sigma}_2$ for $j = 1, \dots, N$. Then, the induced morphisms $\varphi_{N-j}: R \rightarrow F^{N-1}$ satisfy the weak PBW property expressed in (2.3), (2.5) and (2.4).

Even though we have considered graded deformations given by normalized maps in the sense of Definition 1.4, the following lemma shows that the assumption is in fact unnecessary.

Lemma 2.12. *Let A be an N -homogeneous algebra such that $\text{Tor}_3^A(k, k)$ is concentrated in degree $N + 1$. Then, any graded deformation of A is equivalent to a deformation preserving the unit given by normalized maps.*

Proof. Let A_t be a graded deformation of A given by a collection of maps

$$\{\psi_j: (A/k)^{\otimes 2} \rightarrow A\}_{j \in \mathbb{N}}$$

(each of degree $-j$) which define a product \times . We only need to show that there exists another graded deformation A'_t given by maps $\{\psi'_j: (A/k)^{\otimes 2} \rightarrow A\}_{j \in \mathbb{N}}$ such that the first N maps ψ'_1, \dots, ψ'_N are normalized, for the $\{\psi'_j\}_{j > N}$ are always automatically normalized by degree reasons.

Using the main property (1.14) of the homotopy s_* , we see that $\psi_1^\sim - \psi_1^\sim \circ \bar{\sigma}_2 \circ \bar{\tau}_2 = d(\psi_1^\sim \circ s_1)$. It is clear that $\psi_1^\sim \circ \bar{\sigma}_2 \circ \bar{\tau}_2$ is normalized. Let us define $\alpha_1: \bar{A} \rightarrow A$ the map induced by $\psi_1^\sim \circ s_1$. So we see that $\exp(t\alpha_1)$ gives an equivalence from the algebra (A_t, \times) to another deformation (A_t^1, \times_1) of A given by the maps $\{\psi_\bullet^1: A^{\otimes 2} \rightarrow A\}_{\bullet \in \mathbb{N}}$ such that $(\psi_1^1)^\sim = \psi_1^\sim \circ \bar{\sigma}_2 \circ \bar{\tau}_2$ is normalized.

We proceed now by (finite) induction. Let $i \leq N$ and let us suppose that A_t is equivalent to a deformation A_t^i given by a collection of maps $\{\psi_\bullet^i: (A/k)^{\otimes 2} \rightarrow A\}_{\bullet \in \mathbb{N}}$ such that ψ_\bullet^i are normalized for $\bullet \leq i$. Using again (1.14), we conclude that

$$(\psi_{i+1}^i)^\sim - ((\psi_{i+1}^i)^\sim \circ \bar{\sigma}_2 \circ \bar{\tau}_2 - \text{sq}(\psi_1^i, \dots, \psi_i^i)^\sim \circ s_2) = d((\psi_{i+1}^i)^\sim \circ s_1).$$

Also, it is clear that $(\psi_{i+1}^i)^\sim \circ \bar{\sigma}_2 \circ \bar{\tau}_2 - \text{sq}(\psi_1^i, \dots, \psi_i^i)^\sim \circ s_2$ is normalized (each summand is obviously so). Let us define $\alpha_{i+1}: \bar{A} \rightarrow A$ the map induced by $(\psi_{i+1}^i)^\sim \circ s_1$. This tells us that $\exp(t^{i+1}\alpha_{i+1})$ gives an equivalence from the algebra (A_t^i, \times_i)

to another deformation $(A_t^{i+1}, \times_{i+1})$ of A given by the maps $\{\psi_j^{i+1}: (A/k)^{\otimes 2} \rightarrow A\}_{j \in \mathbb{N}}$ such that $\psi_j^{i+1} = \psi_j^i$ for $j \leq i$ and the map $(\psi_{i+1}^{i+1})^\sim = (\psi_{i+1}^i)^\sim \circ \bar{\sigma}_2 \circ \bar{\tau}_2 - \text{sq}(\psi_1^i, \dots, \psi_i^i)^\sim \circ s_2$ is normalized. \square

2.4. From weak PBW-deformations to deformations. Now, we shall give an inverse construction, which is a little more complicated. Let us suppose that $\varphi = \sum_{j=0}^{N-1} \varphi_j, \varphi_j: R \rightarrow V^{\otimes j}$, gives a weak PBW-deformation of an N -homogeneous algebra A satisfying that $\text{Tor}_3^A(k, k)$ is concentrated in degree $N + 1$. We shall construct a (possible infinite) sequence of normalized maps $\{\psi_j\}_{j \in \mathbb{N}}$, where $\psi_j: (A/k)^{\otimes 2} \rightarrow A$, such that

$$a \times b = a.b + \sum_{i \geq 1} \psi_i(a, b)t^i$$

is a graded deformation of A .

First, we define $(\psi_j')^\sim = \varphi_{N-j}^\sim \circ \bar{\tau}_2$ for $j = 1, \dots, N$, and zero otherwise. We note that ψ_j' is a normalized homogeneous morphism of degree $-j$ for $j = 1, \dots, N$.

We note that ψ_1' is a cocycle since

$$d(\psi_1')^\sim = (\psi_1')^\sim \circ \bar{b}_3 = \varphi_{N-1}^\sim \circ \bar{\tau}_2 \circ \bar{b}_3 = \varphi_{N-1}^\sim \circ d_3 \circ \bar{\tau}_3 = d\varphi_{N-1}^\sim \circ \bar{\tau}_3,$$

and Fact 2.9 tells us that the evaluation of the last expression at $1|w|1$, for $w \in R_{N+1}$, is equal to the element of A given by $(1_V \otimes \varphi_{N-1} - \varphi_{N-1} \otimes 1_V)(w)$. The identity (2.3) says that this element vanishes, so $d(\psi_1')^\sim = 0$. We define $\psi_1 = \psi_1'$.

We will proceed recursively on $j \in \mathbb{N}$. Let us suppose that we have defined ψ_1, \dots, ψ_j such that $\psi_i^\sim - (\psi_i')^\sim$ is an extranormalized k^e -linear map of degree $-i$ for all $i = 1, \dots, j$, and that

$$-d\psi_{i+1} = \text{sq}(\psi_1, \dots, \psi_i)$$

holds for all $i = 0, \dots, j - 1$. We will denote $\eta_i^\sim = \psi_i^\sim - (\psi_i')^\sim$.

We shall now prove that

$$-d(\psi_{j+1}')^\sim \circ \bar{\sigma}_3 = \text{sq}(\psi_1, \dots, \psi_j)^\sim \circ \bar{\sigma}_3. \tag{2.10}$$

Let us as usual consider $w = r_i u_i = v_i s_i \in R_{N+1}$. On the one side, $(d(\psi_{j+1}')^\sim \circ \bar{\sigma}_3)(1|w|1)$ is equal to

$$\begin{aligned} & d(\psi_{j+1}')^\sim(1|v_i|s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)}) \\ &= (\psi_{j+1}')^\sim(v_i|s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)}) \\ & \quad - (\psi_{j+1}')^\sim(1|v_i s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)}) + (\psi_{j+1}')^\sim(1|v_i|s_i|1) \end{aligned}$$

$$\begin{aligned}
 &= v_i \psi'_{j+1}(s_{i,(1)}, \bar{s}_{i,(2)})s_{i,(3)} - \psi'_{j+1}(v_i s_{i,(1)}, \bar{s}_{i,(2)})s_{i,(3)} \\
 &= v_i \psi'_{j+1}(s_{i,(1)}, \bar{s}_{i,(2)}) - \psi'_{j+1}(v_i s_{i,(1)}, \bar{s}_{i,(2)})s_{i,(3)} \\
 &= v_i \psi'_{j+1}(s_{i,(1)}, \bar{s}_{i,(2)}) - \underbrace{\psi'_{j+1}(v_i s_{i,(1)}, \bar{s}_{i,(2)})}_{\deg(s_{i,(3)})=0} - \underbrace{\psi'_{j+1}(v_i s_{i,(1)}, \bar{s}_{i,(2)})s_{i,(3)}}_{\deg(s_{i,(3)})>0} \\
 &= v_i \psi'_{j+1}(s_{i,(1)}, \bar{s}_{i,(2)}) - \underbrace{\psi'_{j+1}(v_i s_{i,(1)}, \bar{s}_{i,(2)})}_{\deg(s_{i,(3)})=0} - \underbrace{\psi'_{j+1}(v_i s_{i,(1)}, \bar{s}_{i,(2)})s_{i,(3)}}_{\deg(s_{i,(3)})=1},
 \end{aligned}$$

where we have used Fact 1.2 in the second equality, that s_i vanishes in A in the third equality, and that ψ'_{j+1} is normalized in the fourth and sixth ones. Moreover, since $v_i s_{i,(1)} \otimes \bar{s}_{i,(2)} = r_i \otimes u_i$, we see that

$$\psi'_{j+1}(v_i s_{i,(1)}, \bar{s}_{i,(2)}) = \psi'_{j+1}(r_i, u_i) = 0.$$

Also, taking into account that if $\deg(s_{i,(3)}) = 1$, then

$$v_i s_{i,(1)} \otimes \bar{s}_{i,(2)} \otimes s_{i,(3)} = r_{i,(1)} \otimes \bar{r}_{i,(2)} \otimes u_i,$$

we obtain that $\psi'_{j+1}(v_i s_{i,(1)}, \bar{s}_{i,(2)})s_{i,(3)} = \psi'_{j+1}(r_{i,(1)}, \bar{r}_{i,(2)})u_i$, so

$$(d(\psi'_{j+1}) \sim \circ \bar{\sigma}_3)(1|w|1) = v_i \psi'_{j+1}(s_{i,(1)}, \bar{s}_{i,(2)}) - \psi'_{j+1}(r_{i,(1)}, \bar{r}_{i,(2)})u_i.$$

By its very definition, $\psi'_j(r_{(1)}, \bar{r}_{(2)}) = \varphi_{N-j}(r)$ for all $r \in R$, $j = 1, \dots, N$. Therefore,

$$\begin{aligned}
 (d(\psi'_{j+1}) \sim \circ \bar{\sigma}_3)(1|w|1) &= v_i \varphi_{N-j-1}(s_i) - \varphi_{N-j-1}(r_i)u_i \\
 &= (1_V \otimes \varphi_{N-j-1} - \varphi_{N-j-1} \otimes 1_V)(w).
 \end{aligned} \tag{2.11}$$

On the other side, $(\text{sq}(\psi_1, \dots, \psi_j) \sim \circ \bar{\sigma}_3)(1|w|1)$ is equal to

$$\begin{aligned}
 &\text{sq}(\psi_1, \dots, \psi_j) \sim (1|v_i|s_{i,(1)}|\bar{s}_{i,(2)}|s_{i,(3)}) \\
 &= \sum_{i=1}^j (\psi_i(v_i, \psi_{j+1-i}(s_{i,(1)}, \bar{s}_{i,(2)})) - \psi_i(\psi_{j+1-i}(v_i, s_{i,(1)}, \bar{s}_{i,(2)}))s_{i,(3)}) \\
 &= \sum_{i=1}^j (\psi_i(v_i, \psi_{j+1-i}(s_{i,(1)}, \bar{s}_{i,(2)})) - \psi_i(\psi_{j+1-i}(v_i, s_{i,(1)}, \bar{s}_{i,(2)})) \\
 &= \sum_{i=1}^j (\psi_i(v_i, \psi_{j+1-i}(s_{i,(1)}, \bar{s}_{i,(2)})) - \psi_i(\psi_{j+1-i}(\bar{r}_{i,(1)}, r_{i,(2)}, u_i)) \\
 &= \psi_j(v_i, \psi_1(s_{i,(1)}, \bar{s}_{i,(2)})) - \psi_j(\psi_1(\bar{r}_{i,(1)}, r_{i,(2)}, u_i)),
 \end{aligned}$$

where we have used in the third equality that ψ_i is normalized. In the last one we have used the following simple fact which we have already explained: ψ_{j+1-i} has degree $-(j+1-i) = i-j-1$, so ψ_{j+1-i} applied to an element of degree N gives an element

of degree $N + i - j - 1$. In consequence, the elements $v_i \otimes \psi_{j+1-i}(s_{i,(1)}, \bar{s}_{i,(2)})$ and $\psi_{j+1-i}(\bar{r}_{i,(1)}, r_{i,(2)}) \otimes u_i$ have degree $N + j - i \leq N$ (for $i \leq j$) and the degree is exactly N if and only if $i = j$. The normalization of ψ_j forces only to consider the terms with degree N and the last equality follows.

By its very definition, $\psi_1(r_{(1)}, \bar{r}_{(2)}) = \psi'_1(r_{(1)}, \bar{r}_{(2)}) = \varphi_{N-1}(r)$ for $r \in R$, so

$$(\text{sq}(\psi_1, \dots, \psi_j) \sim \circ \bar{\sigma}_3)(1|w|1) = \psi_j(v_i, \varphi_{N-1}(s_i)) - \psi_j(\varphi_{N-1}(r_i), u_i).$$

Thus, $(\text{sq}(\psi_1, \dots, \psi_j) \sim \circ \bar{\sigma}_3)(1|w|1)$ is given by

$$\begin{aligned} & \psi_j(v_i \otimes \varphi_{N-1}(s_i) - \varphi_{N-1}(r_i) \otimes u_i) \\ &= (\varphi_{N-j} \sim \circ \bar{\tau}_2 + \eta_j \sim)(1|\underbrace{v_i \otimes \varphi_{N-1}(s_i) - \varphi_{N-1}(r_i) \otimes u_i}_{\in R}|1) \\ &= (\varphi_{N-j} \sim \circ \bar{\tau}_2)(1|\underbrace{v_i \otimes \varphi_{N-1}(s_i) - \varphi_{N-1}(r_i) \otimes u_i}_{\in A^{\otimes 2}}|1) \\ &= (\varphi_{N-j} \sim (1|\underbrace{v_i \otimes \varphi_{N-1}(s_i) - \varphi_{N-1}(r_i) \otimes u_i}_{\in R}|1) \\ &= \varphi_{N-j} \circ (1_V \otimes \varphi_{N-1} - \varphi_{N-1} \otimes 1_V)(w), \end{aligned}$$

where we have used that, by identity (2.3), $v_i \otimes \varphi_{N-1}(s_i) - \varphi_{N-1}(r_i) \otimes u_i$ can be seen as a relation decomposition of an element r of R in the second equality, and that $\eta_j \sim$ vanishes over it in the third equality. Finally, in the penultimate equality we have used that

$$\bar{\tau}_2(1|v_i \otimes \varphi_{N-1}(s_i) - \varphi_{N-1}(r_i) \otimes u_i|1) = 1|(v_i \otimes \varphi_{N-1}(s_i) - \varphi_{N-1}(r_i) \otimes u_i)|1,$$

where in the first member $v_i \otimes \varphi_{N-1}(s_i) - \varphi_{N-1}(r_i) \otimes u_i$ is seen as an element of $A^{\otimes 2}$, whereas in the second one it is regarded as an element of R . This thus implies that

$$(\text{sq}(\psi_1, \dots, \psi_j) \sim \circ \bar{\sigma}_3)(1|w|1) = \varphi_{N-j} \circ (1_V \circ \varphi_{N-1} - \varphi_{N-1} \otimes 1_V)(w). \tag{2.12}$$

Now, since (2.3), (2.5) and (2.4) hold, and putting together (2.11) and (2.12), we see that equation (2.10) holds. The standard identity (1.14) for the homotopy s_\bullet tells us that

$$\begin{aligned} d(\psi'_{j+1}) \sim \circ (1_{\bar{C}_3(A)} - \bar{\sigma}_3 \circ \bar{\tau}_3) &= d(\psi'_{j+1}) \sim \circ (\bar{b}_4 \circ s_3 + s_2 \circ \bar{b}_3) \\ &= d(\psi'_{j+1}) \sim \circ s_2 \circ \bar{b}_3 = d(d(\psi'_{j+1}) \sim \circ s_2), \end{aligned}$$

so

$$d(\psi'_{j+1}) \sim \circ \bar{\sigma}_3 \circ \bar{\tau}_3 = d(\psi'_{j+1}) \sim - d(d(\psi'_{j+1}) \sim \circ s_2).$$

Also, we see that

$$\begin{aligned} \text{sq}(\psi_1, \dots, \psi_j)^\sim \circ (1_{\bar{C}_3(A)} - \bar{\sigma}_3 \circ \bar{\tau}_3) &= \text{sq}(\psi_1, \dots, \psi_j)^\sim \circ (\bar{b}_4 \circ s_3 + s_2 \circ \bar{b}_3) \\ &= \text{sq}(\psi_1, \dots, \psi_j)^\sim \circ s_2 \circ \bar{b}_3 \\ &= d(\text{sq}(\psi_1, \dots, \psi_j)^\sim \circ s_2), \end{aligned}$$

i.e.,

$$\text{sq}(\psi_1, \dots, \psi_j)^\sim \circ \bar{\sigma}_3 \circ \bar{\tau}_3 = \text{sq}(\psi_1, \dots, \psi_j)^\sim - d(\text{sq}(\psi_1, \dots, \psi_j)^\sim \circ s_2).$$

Now, equation (2.10) yields that

$$-d((\psi'_{j+1})^\sim - (d(\psi'_{j+1})^\sim \circ s_2 + \text{sq}(\psi_1, \dots, \psi_j)^\sim \circ s_2)) = \text{sq}(\psi_1, \dots, \psi_j)^\sim.$$

We define $\eta_{j+1}^\sim = d(\psi'_{j+1})^\sim \circ s_2 + \text{sq}(\psi_1, \dots, \psi_j)^\sim \circ s_2$ and $\psi_{j+1}^\sim = (\psi'_{j+1})^\sim - \eta_{j+1}^\sim$. It is easy to see that η_{j+1}^\sim has degree $-j - 1$. We just need to prove that it is extranormalized in order to end this recursive process, since ψ_{j+1}^\sim defines a continuation of the j -th level deformation defined by $\psi_1^\sim, \dots, \psi_j^\sim$.

Let us prove that η_{j+1}^\sim is extranormalized. For this, consider $1|a_i|b_i|1 \in \bar{C}_2(A)$ to be normalized or a relation decomposition. Then, $(d(\psi'_{j+1})^\sim \circ s_2)(1|a_i|b_i|1)$ is equal to

$$\begin{aligned} &d(\psi'_{j+1})^\sim(1|a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)}) \\ &= (\psi'_{j+1})^\sim(a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)} - 1|a_i b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)} + 1|a_i|b_i|1) \\ &= a_i \psi'_{j+1}(b_{i,(1)}, \bar{b}_{i,(2)})b_{i,(3)} - \psi'_{j+1}(a_i b_{i,(1)}, \bar{b}_{i,(2)})b_{i,(3)} + \psi'_{j+1}(a_i, b_i) \\ &= -\psi'_{j+1}(a_i b_{i,(1)}, \bar{b}_{i,(2)}) + \psi'_{j+1}(a_i, b_i) \\ &= -\varphi_{N-j-1}(\bar{\tau}_2(1|a_i b_{i,(1)}|\bar{b}_{i,(2)}|1) - \bar{\tau}_2(1|a_i|b_i|1)), \end{aligned}$$

where we have used Fact 1.2 in the first equality and the normalization of ψ'_{j+1} on the fourth equality. By definition, we see that $\bar{\tau}_2(1|a_i b_{i,(1)}|\bar{b}_{i,(2)}|1) = \bar{\tau}_2(1|a_i|b_i|1) = 0$ if $1|a_i|b_i|1$ is normalized and that $\bar{\tau}_2(1|a_i b_{i,(1)}|\bar{b}_{i,(2)}|1) = \bar{\tau}_2(1|a_i|b_i|1)$ if $1|a_i|b_i|1$ is a relation decomposition. In any case, we conclude that $d(\psi'_{j+1})^\sim \circ s_2(1|a_i|b_i|1)$ vanishes.

On the other hand, we get

$$\begin{aligned} &(\text{sq}(\psi_1, \dots, \psi_j)^\sim \circ s_2)(1|a_i|b_i|1) \\ &= \text{sq}(\psi_1, \dots, \psi_j)^\sim(1|a_i|b_{i,(1)}|\bar{b}_{i,(2)}|b_{i,(3)}) \\ &= \sum_{i=1}^j ((\psi_i(a_i, \psi_{j+1-i}(b_{i,(1)}, \bar{b}_{i,(2)})) - \psi_i(\psi_{j+1-i}(a_i, b_{i,(1)}, \bar{b}_{i,(2)}))b_{i,(3)}), \end{aligned}$$

which vanishes by the normalization of ψ_{j+1-i} . Hence, η_{j+1}^\sim is extranormalized.

We may summarize the previous results as follows.

Proposition 2.13. *Let $U = TV/\langle P \rangle$ be a filtered algebra, with $P \subset F^N$, and let $A = TV/\langle R \rangle$, with $R = \pi_N(P) \subset V^{\otimes N}$, be the corresponding N -homogeneous algebra, which we assume to satisfy that $\text{Tor}_3^A(k, k)$ is concentrated in degree $N + 1$. We assume that U is a weak PBW-deformation of A , i.e., that (2.1) holds and that the k^e -linear maps $\varphi_j : R \rightarrow V^{\otimes j}$ (for $j = 0, \dots, N - 1$) which describe P out from R satisfy (2.3), (2.5) and (2.4). Set $(\psi'_j)^\sim = \varphi_{N-j} \circ \bar{\tau}_2$ for $j = 1, \dots, N$ and zero otherwise. We define $\psi_j^\sim : \bar{C}_2(A) \rightarrow A$ for $j \in \mathbb{N}$ recursively. First, $\psi_1^\sim = (\psi'_1)^\sim$. For $j \in \mathbb{N}$, after having defined ψ_1, \dots, ψ_j , we set $\eta_{j+1}^\sim = d(\psi'_{j+1})^\sim \circ s_2 + \text{sq}(\psi_1, \dots, \psi_j)^\sim \circ s_2$ and $\psi_{j+1}^\sim = (\psi'_{j+1})^\sim - \eta_{j+1}^\sim$. Then the ψ_\bullet are normalized morphisms that define a graded deformation of A .*

3. Main theorems

The following theorems provide a description of the previous constructions at the level of algebras, which contains the one given in Sec. 4.6 in [3], where the authors explored only one direction under the assumption of $k = F$ a field. Moreover, we also prove that conditions (2.1) and (2.2) are equivalent to the fact that U satisfies the PBW property (cf. Thm. 4.1 of [3] and Thm. 3.4 of [1]).

Theorem 3.1. *Let A be an N -homogeneous algebra satisfying that $\text{Tor}_3^A(k, k)$ is concentrated in degree $N + 1$. Let us consider a graded deformation A_t of a A , which we suppose to be given by normalized maps $\{\psi_j : (A/k)^{\otimes 2} \rightarrow A\}_{j \in \mathbb{N}}$. We apply Proposition 2.11 to produce maps $\{\varphi_j : R \rightarrow V^{\otimes j}\}_{0 \leq j < N}$, and to obtain thus a filtered algebra $U = TV/\langle P \rangle$, with $P = \{r - \sum_{j=0}^{N-1} \varphi_j(r) : r \in R\}$. Then, there exists an isomorphism of filtered k -algebras $U \rightarrow A_t/\langle t - 1 \rangle$.*

Conversely, let us consider a filtered algebra $U = TV/\langle P \rangle$, with $P \subset F^N$ and let $A = TV/\langle R \rangle$, with $R = \pi_N(P) \subset V^{\otimes N}$, be the corresponding N -homogeneous algebra, which we assume to satisfy that $\text{Tor}_3^A(k, k)$ is concentrated in degree $N + 1$. We assume that U is a weak PBW-deformation of A , and define a deformation A_t of A following Proposition 2.13. We again see that there exists an isomorphism of filtered k -algebras $U \rightarrow A_t/\langle t - 1 \rangle$.

In both cases, U is a PBW-deformation of A and the induced morphism

$$A \xrightarrow{p} \text{gr}(U) \rightarrow \text{gr}(A_t/\langle t - 1 \rangle) \xrightarrow{p} A$$

is the identity.

Proof. Since the proof is similar for both implications, except for some minor changes, we shall only treat each case separately when necessary.

We consider the k^e -linear map $\text{inc} : V \rightarrow A_t/\langle t - 1 \rangle$, given by the composition of the inclusion $V \rightarrow A$, the canonical map $A \rightarrow A_t$ and the projection $A_t \rightarrow$

$A_t/\langle t - 1 \rangle$. This induces a morphism of k -algebras $q': TV \rightarrow A_t/\langle t - 1 \rangle$. It is clear that q' respects the filtrations (where we recall that TV is filtered by F^\bullet).

We shall see that $q'(P) = 0$. In order to prove this statement, take $r \in R$ and let $r - \sum_{i=0}^{N-1} \varphi_i(r) \in P$ be a generic element. We will show that q' vanishes over it, using that the associated maps ψ_\bullet are normalized.

First, we remark the fact that, if $\alpha \in V^{\otimes j}$, with $j < N$, then $q'(\alpha) = \alpha$, which can be proved as follows. It suffices to treat the case $\alpha = v_1 \dots v_j$ for $v_1, \dots, v_j \in V$. By definition q' acts as the identity for $j = 0, 1$. Let us thus assume that $j \geq 2$ and prove the statement by induction on j . We assume that q' acts as the identity on F^{j-1} and we shall prove that it does the same on F^j ($j < N$). The inductive hypothesis implies that $q'(v_1 \dots v_{j-1}) = v_1 \times \dots \times v_{j-1} = v_1 \dots v_{j-1}$. Hence,

$$\begin{aligned} q'(v_1 \dots v_j) &= v_1 \times \dots \times v_j \\ &= (v_1 \times \dots \times v_{j-1}) \times v_j \\ &= (v_1 \dots v_{j-1}) \times v_j \\ &= v_1 \dots v_{j-1} v_j + \sum_{l \geq 1} \psi_l(v_1 \dots v_{j-1}, v_j) \\ &= v_1 \dots v_{j-1} v_j, \end{aligned}$$

since the maps ψ_\bullet are normalized.

Now, take $r = v_{1,i} \dots v_{N,i} \in R$ (summation understood). Then,

$$\begin{aligned} q'(r) &= v_{1,i} \times \dots \times v_{N,i} \\ &= (v_{1,i} \times \dots \times v_{N-1,i}) \times v_{N,i} \\ &= (v_{1,i} \dots v_{N-1,i}) \times v_{N,i} \\ &= v_{1,i} \dots v_{N-1,i} v_{N,i} + \sum_{l \in \mathbb{N}} \psi_l(v_{1,i} \dots v_{N-1,i}, v_{N,i}). \end{aligned}$$

If we are considering the first statement, using that, by definition, $\varphi_{N-l}(r) = \psi_l(r_1, \bar{r}_2)$ for $l = 1, \dots, N$, and by degree reasons $\psi_l(r_1, \bar{r}_2) = 0$ for $l > N$, then $q'(r) = r + \sum_{l=1}^N \varphi_{N-l}(r) = \sum_{l=0}^{N-1} \varphi_l(r)$, since r vanishes in A and $t = 1$ in $A_t/\langle t - 1 \rangle$. For the second statement, since $v_{1,i} \dots v_{N-1,i} \otimes v_{N,i} \in A^{\otimes 2}$ is a relation decomposition we see that $\eta_j(v_{1,i} \dots v_{N-1,i}, v_{N,i})$ vanishes, because it is extranormalized, and $\bar{\tau}_2(1|v_{1,i} \dots v_{N-1,i}|v_{N,i}|1) = 1|r|1$. Hence,

$$\begin{aligned} \psi_l(v_{1,i} \dots v_{N-1,i}, v_{N,i}) &= \psi'_l(v_{1,i} \dots v_{N-1,i}, v_{N,i}) \\ &= \begin{cases} \varphi_{N-l}(r) & \text{if } l = 1, \dots, N, \\ 0 & \text{else.} \end{cases} \end{aligned}$$

As a consequence, we again have that $q'(r) = r + \sum_{l=1}^N \varphi_{N-l}(r) = \sum_{l=0}^{N-1} \varphi_l(r)$, since r vanishes in A and $t = 1$ in $A_t/\langle t - 1 \rangle$. Therefore, $q'(r - \sum_{i=0}^{N-1} \varphi_i(r)) = 0$ and q' thus induces a morphism of filtered algebras $q: U \rightarrow A_t/\langle t - 1 \rangle$.

We will now prove that q is an isomorphism. Since

$$A \xrightarrow{p} \text{gr}(U) \xrightarrow{\text{gr}(q)} \text{gr}(A_t/\langle t-1 \rangle) \xrightarrow{\rho} A$$

is the identity, ρ is an isomorphism and p is surjective, we conclude that p is an isomorphism and $\text{gr}(q)$ is also an isomorphism. Hence, q is an isomorphism and U is a PBW-deformation of A . \square

Remark 3.2. The preceding theorem implies that $A_t/\langle t-1 \rangle$ may be seen as a PBW-deformation of A equivalent to U , where the morphism $A \rightarrow \text{gr}(A_t/\langle t-1 \rangle)$ is the inverse of ρ . Furthermore, by Lemma 2.12 we see that the procedure in Section 2.3 may be defined in the set of equivalence classes of deformations and it sends equivalent deformations to equivalent PBW-deformations. So it defines a map

$$\text{gp}: \{\text{eq. classes of deformations of } A\} \rightarrow \{\text{eq. classes of PBW-deformations of } A\}.$$

Theorem 3.3. *Let A be an N -homogeneous algebra satisfying that $\text{Tor}_3^A(k, k)$ is concentrated in degree $N + 1$. Let us consider a graded deformation A_t of a A by normalized maps $\{\psi_j: (A/k)^{\otimes 2} \rightarrow A_j\}_{j \in \mathbb{N}}$. We apply the construction of Proposition 2.11 to produce maps $\{\varphi_j: R \rightarrow V^{\otimes j}\}_{0 \leq j < N}$, and to obtain thus a filtered algebra $U = TV/\langle P \rangle$, with $P = \{r - \sum_{j=0}^N \varphi_j(r) : r \in R\}$. Then, there exists an isomorphism of graded $k[t]$ -algebras $R(U) \rightarrow A_t$ such that the induced morphism $A \simeq R(U)/\langle t \rangle \rightarrow A_t/\langle t \rangle \simeq A$ is the identity, where $A \simeq R(U)/\langle t \rangle$ is the map described in Section 2.2.*

Conversely, let us consider a filtered algebra $U = TV/\langle P \rangle$ for $P \subset F^N$ and let $A = TV/\langle R \rangle$ for $R = \pi_N(P) \subset V^{\otimes N}$, be the corresponding N -homogeneous algebra, which we assume to satisfy that $\text{Tor}_3^A(k, k)$ is concentrated in degree $N + 1$. We suppose that U is a weak PBW-deformation of A , and define a deformation A_t of A following Proposition 2.13. We again see that there exists an isomorphism of graded $k[t]$ -algebras $R(U) \rightarrow A_t$ such that the induced morphism $A \simeq R(U)/\langle t \rangle \rightarrow A_t/\langle t \rangle \simeq A$ is the identity, where $A \simeq R(U)/\langle t \rangle$ is the map described in Section 2.2.

Proof. From Proposition 2.5, we see that, given any graded deformation A_t of A , there exists an isomorphism of graded $k[t]$ -algebras $\text{com}: R(A_t/\langle t-1 \rangle) \rightarrow A_t$. Since $q: U \rightarrow A_t/\langle t-1 \rangle$ is an isomorphism of filtered algebras, $R(q): R(U) \rightarrow R(A_t/\langle t-1 \rangle)$ is an isomorphism of graded $k[t]$ -algebras. The composition $\text{com} \circ R(q)$ gives the desired isomorphism. Using the canonical isomorphism $R(-)/\langle t \rangle \simeq \text{gr}(-)$, we see that $A \simeq R(U)/\langle t \rangle \rightarrow R(A_t/\langle t-1 \rangle) \simeq A$ identifies with

$$A \xrightarrow{p} \text{gr}(U) \xrightarrow{\text{gr}(q)} \text{gr}(A_t/\langle t-1 \rangle) \xrightarrow{\rho} A,$$

which we have already seen to be the identity. \square

Remark 3.4. The preceding theorem implies that $R(U)$ may be seen as a deformation of A equivalent to A_t . Moreover, it also says that the procedure in Section 2.4 sends equivalence classes of PBW-deformations to equivalent deformations, i.e., it defines a map

$$\text{pg}: \{\text{eq. classes of PBW-deformations of } A\} \rightarrow \{\text{eq. classes of deformations of } A\}.$$

Theorem 3.5. *By the previous theorems we see that the procedure performed in Section 2.3 consists in $A_t \mapsto A_t/(t-1)$ whereas the one done in Section 2.4 is just $U \rightarrow R(U)$, both up to equivalence. Thus Proposition 2.5, Theorem 3.1 and Theorem 3.3 imply that pg and gp are mutually inverse.*

We finish by exhibiting two examples.

Example 3.6. Let $k = F$ be a field, $V = \text{span}_k \langle x, y \rangle$ a k -vector space of dimension 2, and $A = TV/\langle [x, y] \rangle$ a quadratic algebra, so $R \subset V^{\otimes 2}$ has a basis formed by a unique element $r = [x, y]$. We note that $R_3 = 0$.

We can provide in this case the maps $\bar{\tau}_\bullet$ for $\bullet = 1, 2$, in their complete domain of definition:

$$\begin{aligned} \bar{\tau}_1(1|y^n x^m|1) &= \sum_{\alpha_1+\alpha_2=n-1} y^{\alpha_1}|y|y^{\alpha_2}x^m + \sum_{\beta_1+\beta_2=m-1} y^n x^{\beta_1}|x|x^{\beta_2}, \\ \bar{\tau}_2(1|y^{n_1} x^{m_1}|y^{n_2} x^{m_2}|1) &= \sum_{\alpha_1+\alpha_2=n_2-1} \sum_{\beta_1+\beta_2=m_1-1} y^{n_1+\alpha_1} x^{\beta_1}|r|y^{\alpha_2} x^{m_2+\beta_2}. \end{aligned}$$

Furthermore, the homotopies s_\bullet for $\bullet = 1, 2$, are given as follows. The image of $1|y^n x^m|1$ under s_1 is the class in $\bar{C}_2(A)$ of

$$- \sum_{\alpha+\alpha_2=n-1} 1|y^{\alpha_1}|y|y^{\alpha_2}x^m - \sum_{\beta_1+\beta_2=m-1} 1|y^n x^{\beta_1}|x|x^{\beta_2},$$

and the image of $1|y^{n_1} x^{m_1}|y^{n_2} x^{m_2}|1$ under s_2 is the class in $\bar{C}_3(A)$ of

$$\begin{aligned} &\sum_{\substack{\alpha_1+\alpha_2=n_2-1 \\ \beta_1+\beta_2=m_1-1}} (1|y^{n_1+\alpha_1} x^{\beta_1}|y|x|y^{\alpha_2} x^{m_2+\beta_2} - 1|y^{n_1+\alpha_1} x^{\beta_1}|x|y|y^{\alpha_2} x^{m_2+\beta_2}) \\ &+ \sum_{\alpha_1+\alpha_2=n_2-1} 1|y^{n_1} x^{m_1}|y^{\alpha_1}|y|y^{\alpha_2} x^{m_2} + \sum_{\beta_1+\beta_2=m_2-1} 1|y^{n_1} x^{m_1}|y^{n_2} x^{\beta_1}|x|x^{\beta_2}. \end{aligned}$$

Suppose that $U = TV/\langle [x, y] - y \rangle$. Note that the maps defining the filtration are $\varphi_0 = 0$ and $\varphi_1(r) = y$, which obviously satisfy the weak PBW-property. A simple inductive argument then implies that the cochains defining the deformed product are

$$\psi_l(y^{n_1} x^{m_1}, y^{n_2} x^{m_2}) = \binom{m_1}{l} n_2^l y^{n_1+n_2} x^{m_1+m_2-l}.$$

On the other hand, if $U = TV/\langle [x, y] - 1 \rangle$, the so-called *Weyl algebra*, the maps defining the filtration are $\varphi_0(r) = 1$ and $\varphi_1 = 0$, which also satisfy the weak

PBW-property. It is direct to prove in this case that the cochains giving the deformed product are of the form

$$\psi_l(y^{n_1}x^{m_1}, y^{n_2}x^{m_2}) = \begin{cases} i! \binom{m_1}{i} \binom{n_2}{i} y^{n_1+n_2-i} x^{m_1+m_2-i} & \text{if } l = 2i, i \in \mathbb{N}, \\ 0 & \text{else.} \end{cases}$$

Example 3.7. Let $k = F$ be a field, $V = \text{span}_k \langle x \rangle$ a k -vector space of dimension 1, and $A = TV/\langle x^N \rangle$ an N -homogeneous algebra, so $R \subset V^{\otimes N}$ has a basis given by $r = x^N$. Notice that $R_3 = \text{span}_k \langle x^{N+1} \rangle$.

In this case, the comparison map $\bar{\tau}_1$ is actually given in (TAU₁) in its complete domain of definition $\bar{C}_1(A)$. As for $\bar{\tau}_2$, its full expression may be given by (see also [8]):

$$\bar{\tau}_2(1|x^{m_1}|x^{m_2}|1) = \begin{cases} 1|x^N|x^{m_2+m_1-N} & \text{if } m_1 + m_2 \geq N, \\ 0 & \text{else,} \end{cases}$$

where we consider $1 \leq m_1, m_2 \leq N - 1$. Furthermore, the homotopy s_1 is also given in (S₁) in its complete domain of definition. On the other hand, the homotopy s_2 is given by the obvious extension

$$s_2(1|x^{m_1}|x^{m_2}|1) = \sum_{i=1}^{m_2-i-1} 1|x^{m_1}|x^i|x|x^{m_2-i-1},$$

where $1 \leq m_1, m_2 \leq N - 1$, and, as previously stated, we suppose that the right member lies in $\bar{C}_3(A)$.

Let $f = \sum_{i=0}^{N-1} a_i x^i \in k[x]$ be a polynomial of degree less than or equal to $N - 1$, and suppose that $U = TV/\langle x^N - f \rangle$. As usual we assume that $a_i = 0$, if $i \notin \{0, \dots, N - 1\}$. Note that the maps defining the filtration are $\varphi_i(r) = a_i x^i$ for $i = 0, \dots, N - 1$, which obviously satisfy the weak PBW-property. A simple inductive argument then implies that the cochains defining the deformed product are

$$\psi_l(x^{m_1}, x^{m_2}) = \begin{cases} a_{N-l} x^{m_1+m_2-l} & \text{if } m_1 + m_2 \geq N, \\ 0 & \text{if } m_1 + m_2 < N, \end{cases}$$

as one could have expected.

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