

The wandering domain problem for attracting polynomial skew products

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Abstract. Wandering Fatou components were recently constructed by Astorg et al. (2016) for higher-dimensional holomorphic maps on projective spaces. Their examples are polynomial skew products with a parabolic invariant line. In this paper we study this wandering domain problem for polynomial skew product f with an attracting invariant line L (which is the more common case). We show that if f is unicritical (in the sense that the critical curve has a unique transversal intersection with L), then every Fatou component of f in the basin of L is an extension of a one-dimensional Fatou component of $f|_L$. As a corollary there is no wandering Fatou component. We will also discuss the multicritical case under additional assumptions.

1. Introduction

1.1. Backgrounds

In one-dimensional complex dynamics, a celebrated theorem of Sullivan [24] asserts that there is no wandering Fatou component for rational maps on the Riemann sphere \mathbb{P}^1 . This leads to a complete classification of Fatou components for rational maps: a Fatou component is preperiodic to an attracting basin, a parabolic basin, or a rotation domain.

In higher dimensions, let f be a *holomorphic endomorphisms* on the projective space \mathbb{P}^k , $k \geq 2$. The Fatou set is classically defined as the maximal locus such that $\{f^n\}_{n \geq 1}$ form a normal family. However, contrary to the one-dimensional case, wandering Fatou components were recently constructed by Astorg–Buff–Dujardin–Peters–Raissy [4] for holomorphic endomorphisms on the projective plane \mathbb{P}^2 .

The counter examples constructed in [4] are *polynomial skew products*. By definition a polynomial skew product is a self map on \mathbb{C}^2 of the following form:

$$f(z, w) = (p(z), q(z, w)),$$

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where p, q are polynomials of degree at least 2. A polynomial skew product f is called *regular* if it can be extended to a holomorphic endomorphism on \mathbb{P}^2 , which is equivalent to the condition that

$$\deg(p) = \deg(q) = d,$$

and $q(z, w) = cw^d + \text{lower degree terms}$ for some $c \neq 0$. For a regular polynomial skew product, the line at infinity is an attracting set, and the Fatou components in the attracting basin of the line at infinity was classified by Lilov [16]. (In this case the line at infinity is *super-attracting* by the terminology we defined below.)

Now assume that Ω is a Fatou component of f with bounded orbit, let π be the projection map to the z -coordinate, then $\pi(\Omega)$ is contained in the Fatou set of p . Recall that the polynomial map p does not have Herman rings. Since the classification of Fatou components of p is known (by Sullivan’s theorem), to investigate the dynamics of f on Ω , by passing to an iteration of f and a coordinate change, we may assume that the line $L = \{z = 0\}$ is invariant (i.e. $p(0) = 0$), and Ω is contained in a small neighborhood of L . The invariant line L is called *super-attracting, attracting, parabolic* or *elliptic* if 0 is a super-attracting, attracting, parabolic or elliptic fixed point of p , respectively. The counter examples constructed in [4] are regular polynomial skew products with a parabolic invariant line, and the *parabolic implosion technique* is crucial in their construction.

It remains an open problem whether there are wandering Fatou components for regular polynomial skew products with an attracting invariant line. Note that a generic (i.e. open and dense in the parameter space of polynomials of fixed degree) polynomial map on \mathbb{P}^1 does *not* have parabolic or elliptic periodic points. Since the construction of wandering Fatou components in the parabolic case relies heavily on the parabolic implosion technique, one might expect that in the attracting case there is actually no wandering Fatou component. This would imply that a *generic* regular polynomial skew product does not have wandering Fatou components, which is satisfactory for the purpose of understanding dynamics of generic regular polynomial skew products.

In this paper we, solve this problem when f is *unicritical*, i.e. the critical curve of f has a unique *transversal intersection* with L . Here the critical curve of f is by definition the critical locus of the map $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$. Equivalently this means that f can be conjugated to the following form in a neighborhood of L :

$$f(z, w) = (\lambda z, w^d + c(z)), \tag{1.1}$$

where $0 < |\lambda| < 1$, $d \geq 2$ and $c(z)$ is a (non-constant) holomorphic function in a neighborhood 0. Let $k \geq 1$ be the order of c at 0, i.e. $c(z)$ has the local expression $c(z) = c(0) + a_k z^k + \text{higher order term}$, with $a_k \neq 0$. By passing to a coordinate

change $z \rightarrow \alpha z$, we can further assume that

$$c(z) = c(0) + z^k + \text{higher order term}, \tag{1.2}$$

and we choose $0 < r_0 < 1$ such that, for every $z \in B(0, r_0)$,

$$|z^k| \geq |c(z) - c(0) - z^k|. \tag{1.3}$$

In the rest of the paper we always assume that f has this form unless otherwise stated. Let f_0 be the one-dimensional map $f_0(w) = w^d + c(0)$, which is the restriction of f on the invariant line $\{z = 0\}$.

1.2. Main results

The following is our main theorem.

Theorem 1.1. *Let $f(z, w) = (\lambda z, w^d + c(z))$ as in (1.1). Then every Fatou component of f is an extension of a Fatou component of f_0 . In particular, there is no wandering Fatou component.*

Note that the escaping Fatou component

$$\Omega_\infty := \{x \in B(0, r_0) \times \mathbb{C} : |f^n(x)| \rightarrow +\infty\}$$

is clearly non-wandering. Any other Fatou component of f is contained in the complement of Ω_∞ and hence is uniformly bounded.

We will discuss a partial generalization of Theorem 1.1 in Section 6 without assuming the unicritical condition.

Applying Theorem 1.1 to the setting when f is a globally defined holomorphic endomorphism on \mathbb{P}^2 , we get the following.

Theorem 1.2. *Let $f(z, w) = (p(z), q(z, w))$ be a regular polynomial skew product of degree at least 2. Assume that p does not have parabolic periodic points nor Siegel periodic points, and that for every attracting p -periodic point z_0 of periods s , the critical curve of f^s in \mathbb{C}^2 has a unique transversal intersection with the vertical line $L := \{z = z_0\} \subset \mathbb{C}^2$, then f has no wandering Fatou component in \mathbb{P}^2 .*

Here is a specific example that Theorem 1.2 can apply.

Theorem 1.3. *Let $f(z, w) = (z^2 + \lambda z, w^2 + az^2 + bz + c)$, where $\lambda, a, b, c \in \mathbb{C}$. Then f can be extended holomorphically to \mathbb{P}^2 . Assume that $|\lambda| < 1$. Then f has no wandering Fatou component in \mathbb{P}^2 .*

It is not hard to show that every quadratic regular polynomial skew product can be conjugated to the form $f(z, w) = (z^2 + \lambda z, w^2 + az^2 + bz + c)$; see [1, Lemma 2.9].

In the setting of Theorem 1.3, let $f_0(w) := w^2 + c$, which is the one-dimensional map acting on the invariant attracting vertical line $\{z = 0\}$. As far as we know, previously the results about no wandering Fatou components in \mathbb{P}^2 in the setting of Theorem 1.3, were only established in the following two special cases:

- (1) For a fixed triple $(a, b, c) \in \mathbb{C}^3$, when $|\lambda|$ is sufficiently small, [12]. See also [16].
- (2) When $|\lambda| < 1$ and f_0 is Collet–Eckmann and Weakly Regular, [13]. See also [18].

In order to prove Theorem 1.1, we prove two intermediate results which are of independent interests. The following is the first one concerning the lower bounds of the derivative along the orbit.

Let $f(z, w) = (\lambda z, w^d + c(z))$ as in (1.1). An orbit $\{x_i = (z_i, w_i)\}_{0 \leq i \leq n}$ of f is called *tame* if $|z_i|^k \leq |w_i|^d$ for every $0 \leq i < n$ and $|z_0| < r_0$. Here the integer k comes from (1.2) and the constant r_0 comes from (1.3).

Theorem 1.4 (Expansion along tame orbits). *Let $f(z, w) = (\lambda z, w^d + c(z))$ as in (1.1). Assume that f_0 has no attracting nor super-attracting cycle in \mathbb{C} . Given $0 < \lambda_0 < 1$, there exists $C = C(\lambda_0) > 0$ such that if $\{x_i = (z_i, w_i)\}_{0 \leq i \leq n}$ is a tame orbit of f , then*

$$|Df^n(x_0)(v)| \geq C \lambda_0^n \min_{i=0}^{n-1} |w_i|^{d-1},$$

where $v = (0, 1)$ is the unit vertical vector. Moreover, if $|w_n| \leq |w_j|$ for all $0 \leq j < n$, then

$$|Df^n(x_0)(v)| \geq C \lambda_0^n.$$

Theorem 1.4 is a generalization of the corresponding one-dimensional estimate obtained by Levin–Przytycki–Shen in [15].

Our next intermediate result is about the abundance of points satisfying the slow approach condition. For $\alpha > 0$, a point $x_0 \in B(0, r_0) \times \mathbb{C}$ is called α -slow approach (to the critical point),

$$|w_n| \geq e^{-\alpha n}$$

for every n large, where w_n is the projection of $f^n(x_0)$ to the w -coordinate.

Theorem 1.5. *Let $f(z, w) = (\lambda z, w^d + c(z))$ as in (1.1). Assume $(0, 0)$ is not a periodic point. Then for every $\alpha > 0$, Lebesgue a.e. point in $B(0, r_0) \times \mathbb{C}$ is α -slow approach.*

We note that under additional hyperbolicity assumptions on f_0 , the above theorem was proved in [13]. Here we do not make any hyperbolicity assumptions on f_0 . Theorem 1.5 is also a generalization of the corresponding one-dimensional result obtained by Levin–Przytycki–Shen in [15].

Throughout the paper, the vector v will denote the unit vertical tangent vector $v = (0, 1)$.

1.3. Previous results

The problem of classification of Fatou components for polynomial skew products gets much attention in recent years. We refer the readers to the survey paper by Dujardin [8].

As we have mentioned, wandering Fatou components for parabolic polynomial skew products were constructed in [4], see also Astorg–Boc Thaler–Peters [3], Hahn–Peters [10] and Astorg–Boc Thaler [2] for constructions of wandering domains using similar technique.

In the case that f is an attracting polynomial skew product, previously there are two kinds of results concerning non-existence of wandering Fatou components, both making additional assumptions on f . The first type of results assume the smallness of the multiplier λ . It was first showed by Lilov [16] that there is no wandering Fatou component if $p'(0) = 0$. Later this was generalized by the first named author [12] that there is no wandering Fatou component if λ is sufficiently small. The second type of results make hyperbolicity conditions on f_0 , see for instance Peters–Vivas [19], Peters–Smit [18] and the first named author [13]. For example the result in [13] asserts that if f_0 satisfies Collet–Eckmann condition and a Weakly Regular condition, then there is no wandering Fatou component and the Julia set has zero volume. Compared with Theorem 1.1, here we do not make any assumptions on λ nor on the hyperbolicity of f_0 .

The elliptic polynomial skew products were studied by Peters–Raissy [17]. Finally, we mention that in the context of complex Hénon maps, wandering Fatou components were constructed by Berger–Biebler [6].

The topology of Fatou components of polynomial skew products on \mathbb{C}^2 was studied by Roeder [23].

1.4. On methods of the proof

We introduce new methods in order to prove Theorem 1.1. Our methods are based on the *parameter exclusion technique* and *binding argument* initiated by Jakobson [11] and Benedicks–Carleson [5]. They successfully used these two techniques to show the existence of absolutely continuous invariant measures for a positive volume parameter set of quadratic interval maps.

As far as we know this is the first time that these two techniques are used to solve the problems about wandering domains. This is actually built on the following interesting analogue between families of one-dimensional dynamical systems and

polynomial skew products. Let $\{q_z\}_{z \in \mathbb{D}}$ be a family of polynomials of same degree, parametrized by the unit disk. We can associate this family to a polynomial skew product as

$$f: \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{D} \times \mathbb{C}, \quad f(z, w) := (z, q_z(w)).$$

Recall that a polynomial skew product with an attracting invariant line L has the normal form

$$f: U \times \mathbb{C} \rightarrow U \times \mathbb{C}, \quad f(z, w) = (\lambda z, q_z(w))$$

in a neighborhood $U \times \mathbb{C}$ of L , where $|\lambda| < 1$. Now it is obvious to observe the similarity of these two objects. Unlike excluding bad parameters as Jakobson and Benedicks–Carleson did, we exclude bad vertical lines instead for attracting polynomial skew products. The one-dimensional estimates in Levin–Przytycki–Shen [15] are also involved, to ensure the binding argument work. To prove Theorem 1.4 we use the binding argument with the help of the one-dimensional estimates in [15]. To prove Theorem 1.5 we use the binding argument and the parameter exclusion technique.

To complete the proof of Theorem 1.1, we also adapt some arguments in [16] and [12]: assume by contradiction that there is a wandering Fatou component Ω . Then by Theorems 1.4 and 1.5, we can construct a vertical disk (i.e. a disk contained in a vertical line) $D \subset \Omega$ such that the forward image $f^n(D)$ contains a vertical disk of radius at least λ_0^n when n large enough, where $\lambda_0 > 0$ is a constant smaller but close to 1. Since D is contained in a wandering Fatou component, the orbit of D cluster only on the Julia set of f_0 . Again by using the binding argument, we show that this vertical disk actually cannot exist. This gives a contradiction, and hence finishes the proof.

1.5. Organization of the paper

The organization of the paper is as follows. In Section 2, we do some preliminaries, we recall some one-dimensional results and we introduce the notion of binding time. In Section 3, we prove Theorem 1.4. In Section 4, we prove Theorem 1.5. In Section 5, we complete the proof of Theorem 1.1, and we prove Theorems 1.2 and 1.3. Finally, in Section 6, we discuss a partial generalization of our results when f is multicritical.

2. Preliminaries

2.1. Expansion of one-dimensional unicritical polynomials

In the following, f_0 will denote a unicritical polynomial

$$f_0(w) = w^d + c, \quad \text{where } c \in \mathbb{C}, d \geq 2. \tag{2.1}$$

The following proposition summarizes results on lower bounds of derivatives in [15].

Proposition 2.1. *Let $f_0(w) = w^d + c$ as in (2.1) such that f_0 has no attracting nor superattracting cycle in \mathbb{C} . Then the following hold: Given any $0 < \lambda_0 < 1$, there exists $C = C(\lambda_0) > 0$ such that for any $w \in \mathbb{C}$ and any $n \in \mathbb{N}$,*

$$|Df_0^n(w)| \geq C \lambda_0^n \min_{j=0}^{n-1} |f_0^j(w)|^{d-1}. \tag{2.2}$$

More precisely,

- (i) *given $0 < \lambda_0 < 1$ and $\delta > 0$, there exists $\kappa = \kappa(\lambda_0, \delta) > 0$ such that for every $w \in \mathbb{C}$ satisfying $|f_0^j(w)| \geq \delta$ for each $0 \leq j < n$, we have*

$$|Df_0^n(w)| \geq \kappa \lambda_0^n;$$

- (ii) *given $0 < \lambda_0 < 1$, there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, if $|w| < \delta$, $|f_0^n(w)| \leq \delta$, and $|f_0^j(w)| > \delta$ for all $1 \leq j < n$, then*

$$|Df_0^n(w)| \geq \lambda_0^n \min\left(1, \left(\frac{|w|}{|f_0^n(w)|}\right)^{d-1}\right);$$

- (iii) *given $0 < \lambda_0 < 1$, there exists $\kappa_0 = \kappa_0(\lambda_0) > 0$ such that if $|f_0^j(w)| \geq |f_0^n(w)|$ for each $0 \leq j < n$, then*

$$|Df_0^n(w)| \geq \kappa_0 \lambda_0^n.$$

Proof. We first prove (i)–(iii).

- (i) This follows from [15, Lemma 2.2].

(ii) If $|w| \geq |f_0^n(w)|$, this follows from [15, Lemma 4.2]. If $|w| < |f_0^n(w)|$, this follows from [15, Lemma 2.1].

(iii) Fix $\lambda_0 \in (0, 1)$, let $\delta_0 > 0$ be given by (ii) and let $\kappa_0 = \kappa(\lambda_0, \delta_0)$ be given by (i). If $|f_0^n(w)| \geq \delta_0$, then the desired estimate follows from (i). Otherwise, define $\dots > n_2 > n_1 \geq 0$ inductively such that

- n_1 is minimal such that $|f_0^{n_1}(w_1)| < \delta_0$,
- for each $i \geq 1$, n_{i+1} is minimal such that $n_{i+1} > n_i$ and $|f_0^{n_{i+1}}(w)| \leq |f_0^{n_i}(w)|$.

Since $|f_0^n(w)| \leq |f_0^j(w)|$ for all $0 \leq j < n$, there exists k such that $n_k = n$. So by (i) and (ii),

$$|Df_0^n(w)| = |Df_0^{n_1}(w)| \prod_{i=1}^{k-1} |Df_0^{n_{i+1}-n_i}(f_0^{n_i}(w))| \geq \kappa_0 \lambda^n.$$

Let us finally prove (2.2). By (iii), we only need to consider the case $|w| < |f_0^j(w)|$ for each $0 < j \leq n$. (If $|f_0^s(w)| = \min_{j=0}^{n-1} |f_0^j(w)|$ for some $0 < s \leq n - 1$, estimate $|Df_0^s(w)|$ and $|Df_0^{n-s}(f^s(w))|$ separately.) By (i), we may assume $|w| < \delta_0$. Let s be maximal in $\{0, 1, \dots, n\}$ such that $|f_0^s(w)| \leq \delta_0$. By (i) again,

$$|Df_0^{n-s}(f_0^s(w))| \geq d|f_0^s(w)|^{d-1} \kappa(\lambda_0, \delta_0) \lambda_0^{n-s-1}.$$

Let $0 = s_0 < s_1 < \dots < s_p = s$ be such that s_{i+1} is minimal such that $s_{i+1} > s_i$ and $|f_0^{s_{i+1}}(w)| = \min_{j=s_i+1}^n |f_0^j(w)|$. Then by (ii),

$$|Df_0^s(w)| = \prod_{i=1}^p |Df_0^{s_{i+1}-s_i}(f_0^{s_i}(w))| \geq \lambda_0^s \left(\frac{|w|}{|f_0^s(w)|} \right)^{d-1}.$$

Thus,

$$|Df_0^n(w)| \geq \kappa(\lambda_0, \delta_0) \lambda_0^n |w|^{d-1}. \quad \blacksquare$$

The following proposition was also proved in [15].

Proposition 2.2. *Let $f_0(w) = w^d + c$ as in (2.1) such that f_0 has no attracting nor superattracting cycle in \mathbb{C} . Then*

(1) *the lower Lyapunov exponent of f_0 at c is non-negative:*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log |Df_0^n(c)| \geq 0;$$

(2) *the power series*

$$F(z) = 1 + \sum_{n=1}^{+\infty} \frac{z^n}{(f_0^n)'(c)}$$

has the radius of convergence at least 1, and $F(z) \neq 0$ for every $|z| < 1$.

Proof. The first statement is one of the main theorems in [15], and the second one is [15, Corollary 5.1], which follows from the first by a result of Levin in [14]. ■

2.2. The binding time

In this subsection we introduce a version of the notion of *binding time* which was first used by Benedicks–Carleson [5] to study perturbation of non-uniformly expanding interval maps. Throughout we fix a map $f(z, w) = (\lambda z, w^d + c(z))$ as in (1.1).

Definition 2.3. Let $x \in B(0, r_0) \times \mathbb{C}$ and $y \in B(0, r_0) \times \mathbb{C}$. Let $\mu > 0$ be a constant. For every integer $m \geq 0$, we define $\xi_m(x)$ to be the projection of $f^m(x)$ to the

w -coordinate. We define the μ -binding time of the pair (x, y) , denoted by $b_\mu(x, y)$, to be the infimum of the set of non-negative integer n such that

$$|\xi_n(x) - \xi_n(y)| \geq \frac{\mu \min(|\xi_n(x)|, |\xi_n(y)|)}{(n + 1)^2}. \tag{2.3}$$

Here, we use the convention that $\inf \emptyset = +\infty$.

We note that by our definition of binding time, for $m < b_\mu(x, y)$, $\xi_j(y)$ and $\xi_j(x)$ are both non-zero for $0 \leq j < m$, we have

$$\frac{|\xi_m(x) - \xi_m(y)|}{|\xi_m(y)|} < \frac{\mu}{(m + 1)^2}. \tag{2.4}$$

We fix a small constant $\mu_0 \in (0, 1)$ such that

$$\sum_{i=1}^{\infty} \frac{2\mu_0}{i^2} < \frac{1}{4d},$$

and such that for any $t \in \mathbb{C}$ with $|t| < \mu_0$, we have

$$|t^d - 1| \leq d|t - 1|(1 + d|t - 1|). \tag{2.5}$$

Let

$$\mu_1 = \frac{\mu_0}{4}. \tag{2.6}$$

Let us show that for every positive integer n ,

$$\begin{aligned} |w_1 - w_2| &\leq \frac{\mu_1 \min(|w_1|, |w_2|)}{n^2}, |w_2 - w_3| \leq \frac{\mu_1 \min(|w_2|, |w_3|)}{n^2} \\ \implies |w_1 - w_3| &\leq \frac{\mu_0 \min(|w_1|, |w_3|)}{n^2}. \end{aligned} \tag{2.7}$$

Indeed,

$$\begin{aligned} |w_1 - w_3| &\leq |w_1 - w_2| + |w_2 - w_3| \\ &\leq \frac{\mu_1 \min(|w_1|, |w_2|)}{n^2} + \frac{\mu_1 \min(|w_2|, |w_3|)}{n^2} \\ &\leq \frac{2\mu_1 |w_2|}{n^2}. \end{aligned}$$

On the other hand, we have

$$|w_2| \leq |w_1| + |w_1 - w_2| \leq |w_1| + \frac{\mu_1 \min(|w_1|, |w_2|)}{n^2} \leq 2|w_1|.$$

Similarly, $|w_2| \leq 2|w_3|$. Hence (2.7) holds.

By (2.7), we have

$$b_{\mu_0}(x_1, x_3) \geq \min(b_{\mu_1}(x_1, x_2), b_{\mu_1}(x_2, x_3)). \tag{2.8}$$

We note that (2.8) will only be used in the proof of Lemma 5.3.

In the rest of the paper, a μ_0 -binding time is called a binding time for simplicity. Note that if n is the μ -binding time of (x, y) , where $0 < \mu \leq \mu_0$, then

$$\frac{1}{2}|\xi_j(y)| \leq |\xi_j(x)| \leq 2|\xi_j(y)| \quad \text{when } 0 \leq j < n. \tag{2.9}$$

The following two lemmas will be used frequently in the rest of the paper.

Lemma 2.4. *Let $x \in B(0, r_0) \times \mathbb{C}$ and $y \in B(0, r_0) \times \mathbb{C}$. Let $n = b_\mu(x, y)$, where $0 < \mu \leq \mu_0$. Then for every positive integer $m \leq n$, we have*

$$\left| \frac{Df^m(x)(v)}{Df^m(y)(v)} - 1 \right| < \frac{1}{2}.$$

Proof. For each $0 \leq i < n$, by (2.4) and the discussion just above (2.4), we have $\xi_i(x)$ and $\xi_i(y)$ are both non-zero, and

$$\left| \log \frac{\xi_i(x)}{\xi_i(y)} \right| \leq 2 \frac{|\xi_i(x) - \xi_i(y)|}{|\xi_i(y)|} \leq \frac{2\mu}{(i + 1)^2}.$$

Thus for $1 \leq m \leq n$,

$$\left| \sum_{i=0}^{m-1} \log \frac{\xi_i(x)}{\xi_i(y)} \right| \leq \sum_{i=0}^{m-1} \frac{2\mu}{(i + 1)^2} < \frac{\mu_0\pi^2}{3} < \frac{1}{4d}.$$

Consequently,

$$\left| \frac{Df^m(x)(v)}{Df^m(y)(v)} - 1 \right| = \left| e^{(d-1)\sum_{i=0}^{m-1} \log \frac{\xi_i(x)}{\xi_i(y)}} - 1 \right| < 2(d - 1) \left| \sum_{i=0}^{m-1} \log \frac{\xi_i(x)}{\xi_i(y)} \right| < \frac{1}{2}. \blacksquare$$

Let $x = (z_0, w_0) \in B(0, r_0) \times \mathbb{C}$, $y = (z, w) \in B(0, r_0) \times \mathbb{C}$ and $n \geq 1$. We define the following quantity

$$W(x, y, n) := 2|w_0 - w| + \sum_{i=1}^n \frac{2|c(\lambda^{i-1}z_0) - c(\lambda^{i-1}z)|}{|Df^i(x)(v)|},$$

where $c(z)$ is the critical value curve of f as in (1.1). By our assumption (1.2), this implies

$$W((0, w), (z_0, w_0), n) \leq 2|w_0 - w| + \sum_{i=1}^n \frac{4|\lambda^{k(i-1)}z_0^k|}{|Df_0^i(w)|}. \tag{2.10}$$

Lemma 2.5. *Let $f(z, w) = (\lambda z, w^d + c(z))$ as in (1.1). Let*

$$x = (z_0, w_0) \in B(0, r_0) \times \mathbb{C} \quad \text{and} \quad y = (z, w) \in B(0, r_0) \times \mathbb{C}.$$

If n is a positive integer and $n \leq b_\mu(x, y)$, where $0 < \mu \leq \mu_0$, then

$$|Df^n(x)(v)| \geq \frac{|\xi_n(x) - \xi_n(y)|}{W(x, y, n)}.$$

In particular, if $n = b_\mu(x, y)$, then

$$|Df^n(x)(v)| \geq \frac{\mu \min(|\xi_n(x)|, |\xi_n(y)|)}{(n + 1)^2 W(x, y, n)}.$$

Proof. First, for every $1 \leq m \leq n$, we have

$$\begin{aligned} & |\xi_m(x) - \xi_m(y)| \\ &= |(\xi_{m-1}(x))^d - (\xi_{m-1}(y))^d + c(\lambda^{m-1}z_0) - c(\lambda^{m-1}z)| \\ &\leq |(\xi_{m-1}(x))^d - (\xi_{m-1}(y))^d| + |c(\lambda^{m-1}z_0) - c(\lambda^{m-1}z)| \end{aligned} \tag{2.11}$$

We now estimate the first term in (2.11). Since

$$\left| \frac{\xi_{m-1}(y)}{\xi_{m-1}(x)} - 1 \right| \leq \frac{\mu}{m^2} \leq \mu_0,$$

by (2.5), we have

$$\begin{aligned} \frac{|\xi_{m-1}(x)^d - \xi_{m-1}(y)^d|}{|\xi_{m-1}(x)^d|} &\leq d \frac{|\xi_{m-1}(x) - \xi_{m-1}(y)|}{|\xi_{m-1}(x)|} \left(1 + d \frac{|\xi_{m-1}(x) - \xi_{m-1}(y)|}{|\xi_{m-1}(x)|} \right) \\ &\leq d \frac{|\xi_{m-1}(x) - \xi_{m-1}(y)|}{|\xi_{m-1}(x)|} \left(1 + \frac{d\mu}{m^2} \right), \end{aligned}$$

and hence

$$|\xi_{m-1}(x)^d - \xi_{m-1}(y)^d| \leq d |\xi_{m-1}(x)|^{d-1} |\xi_{m-1}(x) - \xi_{m-1}(y)| \left(1 + \frac{d\mu}{m^2} \right).$$

Therefore, for every $1 \leq m \leq n$, we have

$$\frac{|\xi_m(x) - \xi_m(y)|}{|Df^m(x)(v)|} \leq \left(1 + \frac{d\mu}{m^2} \right) \frac{|\xi_{m-1}(x) - \xi_{m-1}(y)|}{|Df^{m-1}(x)(v)|} + \frac{|c(\lambda^{m-1}z_0) - c(\lambda^{m-1}z)|}{|Df^m(x)(v)|}.$$

Next, we estimate $\frac{|\xi_{m-1}(x) - \xi_{m-1}(y)|}{|Df^{m-1}(x)(v)|}$ by the same method. Continuing this process, we finally have

$$\frac{|\xi_n(x) - \xi_n(y)|}{|Df^n(x)(v)|} \leq 2|w_0 - w| + \sum_{i=1}^n \frac{2|c(\lambda^{i-1}z_0) - c(\lambda^{i-1}z)|}{|Df^i(x)(v)|} = W(x, y, n),$$

using the fact that $\prod_{m=1}^{\infty} (1 + \frac{d\mu}{m^2}) \leq \prod_{m=1}^{\infty} (1 + \frac{d\mu_0}{m^2}) \leq e^{1/8} < 2$. Hence, we have

$$|Df^n(x)(v)| \geq \frac{|\xi_n(x) - \xi_n(y)|}{W(x, y, n)}.$$

In particular, by (2.3), when $n = b_\mu(x, y)$, we have

$$|Df^n(x)(v)| \geq \frac{\mu \min(|\xi_n(x)|, |\xi_n(y)|)}{(n + 1)^2 W(x, y, n)}. \quad \blacksquare$$

The following lemma will be used to bound from below the vertical derivatives for an orbit starting close to the critical value line, and will be used in Section 4 and Section 5.

Lemma 2.6. *Assume that f_0 has no attracting nor superattracting cycle in \mathbb{C} . Fix $0 < \mu \leq \mu_0$. For any positive integer s_0 and $\lambda_0 \in (0, 1)$, there exists $\delta_0 > 0$ such that the following holds. Let $(z_j, w_j)_{j=0}^\infty$ be an f -orbit that satisfies:*

$$\delta := \max(|w_1 - c(0)|, |z_0|^k)^{1/d} \in (0, \delta_0). \tag{2.12}$$

Then there exists a positive integer n such that the following hold:

- (1) $s_0 \leq n \leq b_\mu((z_1, w_1), (0, c(0)))$;
- (2) $|w_j| \geq \delta/2, 1 \leq j \leq n$,
- (3) $|Df^n(z_1, w_1)(v)| \geq \lambda_0^n \delta^{-(d-1)}$.

Proof. Since f is Lipschitz, we have $m = b_\mu((z_1, w_1), (0, c(0))) \rightarrow +\infty$ when $\delta \rightarrow 0$. This implies that $m = b_\mu((z_1, w_1), (0, c(0)))$ can be made arbitrarily large, if $\delta_0 > 0$ is sufficiently small.

Fix a constant $\lambda_1 \in (0, 1)$ such that $\lambda_1 \geq \max(\lambda_0, |\lambda|^k)$. Let s be a large positive integer such that $s \geq s_0, \lambda_0^{-s/2} > 2d$ and

$$\frac{\mu}{2C(p + 1)^2} \geq \lambda_0^p \tag{2.13}$$

for each $p \geq s$, where

$$C = 2 + 4 \sum_{i=1}^m \frac{|\lambda|^{ki}}{|Df^i(c_0)|} \tag{2.14}$$

is a positive constant. Let $\delta_0 > 0$ be a small constant such that the following hold:

- $|f_0^j(0)| > \delta_0$ holds for all $1 \leq j \leq s$;
- $b_\mu((z_1, w_1), (0, c(0))) \geq s$ holds when $\delta \in (0, \delta_0)$, where δ is as in (2.12).

Now assume $\delta \in (0, \delta_0)$ and let $m = b_\mu((z_1, w_1), (0, c(0))) \in \mathbb{N} \cup \{\infty\}$. We distinguish two cases.

Case 1. There exists a minimal integer $0 \leq n \leq m$ such that $|f_0^{n+1}(0)| \leq \delta$. Then $n \geq s \geq s_0$ and $\lambda_0^{-n/2} > 2d$. By (2.9), we have $|w_j| \geq |f^j(0)|/2 > \delta/2$ for all $1 \leq j \leq n$. By Lemma 2.4, $|Df^n(z_1, w_1)| \geq |Df_0^n(c(0))|/2$, so it suffices to show

$$|Df_0^n(c(0))| \geq \delta^{-(d-1)} d^{-1} \lambda_0^{n/2}. \tag{2.15}$$

To this end, note that for any $t \in \mathbb{C}$ with $|t|$ very close to 0, there is δ' slightly larger than δ , such that $|t| < \delta'$, $|f_0^j(t)| > \delta'$ for $1 \leq j \leq n$ and $|t| < |f_0^{n+1}(t)| < \delta'$. By Proposition 2.1 (ii), we have

$$|Df_0^{n+1}(t)| \geq \lambda_0^{n/2} \left(\frac{|t|}{|f_0^{n+1}(t)|} \right)^{d-1},$$

which implies that

$$|Df_0^n(f_0(t))| = \frac{|Df_0^{n+1}(t)|}{|Df_0(t)|} \geq \frac{1}{d} \lambda_0^{n/2} \frac{1}{|f_0^{n+1}(t)|^{d-1}}.$$

Letting $t \rightarrow 0$, we obtain (2.15).

Case 2. For each $0 \leq j \leq m$, we have $|f_0^{j+1}(0)| > \delta$. Then by (2.9),

$$|w_j| \geq |f_0^j(0)| > \frac{\delta}{2} \quad \text{for all } 1 \leq j \leq m.$$

By Proposition 2.2, there exists $C_1 > 0$ such that

$$|Df_0^i(c(0))| \geq C_1 \lambda_1^i \quad \text{for all } i \geq 1.$$

If $m = \infty$, then it suffices to take n to be a large integer such that $n > s_0$ and $C_1 \lambda_1^n > 2\delta^{-(d-1)} \lambda_0^n$, since $|Df^n(z_1, w_1)| \geq |Df_0^n(c(0))|/2$. In the following, we assume $m < \infty$ and take $n = m$. Then $n = m \geq s \geq s_0$, so property (1) holds. As we noted above, property (2) also holds. Let us prove property (3). By (2.10) and (2.12),

$$W := W((0, c(0)), (z_1, w_1), m) \leq 2|w_1 - c(0)| + \sum_{i=1}^m \frac{4|\lambda|^{ki} |z_0|^k}{|Df^i(c(0))|} \leq C\delta^d,$$

where C is as in (2.14). By Lemma 2.5,

$$|Df^m(z_1, w_1)(v)| \geq \frac{\mu}{2(m+1)^2} \frac{|f_0^{m+1}(0)|}{W} \geq \frac{\mu}{2(m+1)^2 C} \delta^{-(d-1)},$$

which implies property (3) by (2.13) since $m \geq s$. ■

2.3. A variant of Przytycki’s lemma

We shall also frequently use the following slight generalization of a lemma due to Przytycki [20, Lemma 1]. In the following, the Euclidean norm in \mathbb{C}^2 is denoted by $\|\cdot\|$.

Lemma 2.7. *Let $f(z, w) = (\lambda z, w^d + c(z))$ as in (1.1). Assume that f_0 has no attracting nor superattracting cycle in \mathbb{C} . Then there exist a constant $C > 0$ such that for every $\varepsilon > 0$ and $n \geq 1$, if $x = (z_0, w_0) \in B(0, r_0) \times \mathbb{C}$ satisfies $|z_0|^k \leq \varepsilon$, $|w_0| \leq \varepsilon$ and $|\xi_n(x)| \leq \varepsilon$, then $n \geq C \log(1/\varepsilon)$.*

Recall that the constant k was defined in (1.2).

Proof. It is enough to prove the result when ε is sufficiently small. By [20, Lemma 1], there are constants $\varepsilon_0 \in (0, 1)$ and $C_0 > 0$ such that if $w_0 \in \mathbb{C}$ and $n \geq 1$ satisfies $|w_0| < \varepsilon$ and $|f_0^n(w_0)| < \varepsilon$ for some $\varepsilon \in (0, \varepsilon_0)$, then $n \geq C_0 \log(1/\varepsilon)$.

Let $R > 0$ be a constant such that for any $y = (z, w)$ with $|z| < r_0$ and $|w| > R$, we have $|\xi_1(y)| > 2|w|$. Thus $|\xi_n(y)| \leq R$ implies $|\xi_m(y)| \leq R$ for every $0 \leq m \leq n$. Let $M := \max(dR^{d-1}, 2)$.

Now consider $x = (z_0, w_0)$ with $|z_0|^k \leq \varepsilon$, $|w_0| < \varepsilon$ and $|\xi_n(x)| \leq \varepsilon$ for $\varepsilon < \min(\varepsilon_0^2/9, R)$. by the choice of R , for each $0 \leq m < n$, $|\xi_m(x)| \leq R$. Let n_1 be the maximal positive integer such that $n_1 \leq n$ and such that $|f_0^m(w_0)| < R$ for all $0 \leq m < n_1$. Then, for each $0 \leq m < n_1$,

$$\begin{aligned} |\xi_{m+1}(x) - f_0^{m+1}(w_0)| &= |\xi_m(x)^d - f_0^m(w_0)^d| + |c(z_m) - c(0)| \\ &\leq M|\xi_m(x) - f_0^m(w_0)| + 2|z_m|^k \\ &\leq M|\xi_m(x) - f_0^m(w_0)| + 2\varepsilon, \end{aligned}$$

hence

$$|\xi_{n_1}(x) - f_0^{n_1}(w_0)| \leq 2M^{n_1}\varepsilon. \tag{2.16}$$

Now if $M^{n_1} \geq \sqrt{1/\varepsilon}$ we get the desired estimate for $C = 1/2 \log M$. If $M^{n_1} \leq \sqrt{1/\varepsilon}$, then by (2.16) we have

$$|f_0^{n_1}(w_0)| \leq |\xi_{n_1}(x)| + 2M^{n_1}\varepsilon \leq 3\sqrt{\varepsilon}.$$

Thus we have $|w_0| \leq \varepsilon \leq 3\sqrt{\varepsilon}$ and $|f_0^n(w_0)| \leq 3\sqrt{\varepsilon}$, so

$$n \geq n_1 \geq C_0 \log \frac{1}{3\sqrt{\varepsilon}},$$

which implies the desired estimate for a suitably chosen C . ■

3. Lower bounds of derivatives along the orbits

In this section, we prove Theorem 1.4 which is a perturbed version of Proposition 2.1. To obtain the desired lower bounds for the vertical derivatives $|Df^n(z_0, w_0)(0, 1)|$, we decompose the orbit (z_j, w_j) into sub-orbits falling into the following two cases, and do estimates separately:

- Orbits staying bounded away from the critical set, i.e. the case $\inf_{j=0}^{n-1} |w_j|$ is bounded away from zero.
- Orbits corresponds to returns to a small neighborhood of the critical set, i.e. for some small $\delta > 0$, $|w_0|, |w_n| \leq \delta$, but $|w_j| > |w_n|$ for $1 \leq j < n$.

The first case follows from the corresponding non-perturbed result (see Proposition 2.1 (i) and (iii)) by continuity argument, see Lemmas 3.1 and 3.2. The second case is more delicate, and is derived from Proposition 2.1 (ii), using binding argument, see Lemma 3.4. The proof of Theorem 1.4 is given at the end of this section.

We start with the lemmas dealing with orbits staying away from the critical sets which follow from Proposition 2.1 by a continuity argument.

Lemma 3.1. *Let $f(z, w) = (\lambda z, w^d + c(z))$ as in (1.1) such that f_0 has no attracting nor superattracting cycle in \mathbb{C} . Given $0 < \lambda_0 < 1$ and $0 < \delta < 1$, there exists $\kappa = \kappa(\lambda_0, \delta) > 0$ and $\eta_1 = \eta_1(\lambda_0, \delta) > 0$ such that for any f -orbit $\{x_j = (z_j, w_j)\}_{i=0}^n$, if $|w_j| \geq \delta$ hold for all $0 \leq j < n$ and $|z_0| < \eta_1$, then*

$$|Df^n(x_0)(v)| \geq \kappa \lambda_0^n.$$

Proof. By Proposition 2.1 (i), there exists $\kappa > 0$ depending on λ_0 and δ such that for any f_0 -orbit $(y_j)_{j=0}^n$ with $|y_j| \geq \delta/2$ for all $0 \leq j < n$, then

$$|Df_0^n(y_0)| \geq 2\kappa \lambda_0^{n/2}.$$

Choose a positive integer N such that $\kappa^2 > \lambda_0^N$.

By continuity, there exists $\eta_1 > 0$ such that for any orbit $\{x_i = (z_i, w_i)\}_{i=0}^{m-1}$, $m \leq N$, with $\min_{j=0}^{m-1} |w_j| \geq \delta$ and $|z_0| < \eta_1$, then $\min_{j=0}^{m-1} |f_0^j(w_0)| > \delta/2$ and

$$|Df^m(x_0)(v)| \geq \frac{1}{2} |Df_0^m(w_0)|.$$

Since $|Df_0^m(w_0)| > 2\kappa \lambda_0^{m/2}$, we conclude

$$|Df^m(x_0)(v)| \geq \kappa \lambda_0^{m/2}.$$

In particular, when $m = N$, this gives us

$$|Df^N(x_0)(v)| > \lambda_0^N.$$

Now let us consider an f -orbit $(x_i = (z_i, w_i))_{i=0}^n$ with $|w_i| \geq \delta$ for $0 \leq i < n$. Let $n = qN + r$, where q is a non-negative integer and $0 \leq r < N$. Then by decomposing this orbit into q pieces of length N together with one piece of length less than $r < N$, we obtain

$$|Df^n(x_0)(v)| \geq \kappa \lambda_0^{r/2} \lambda_0^{qN} \geq \kappa \lambda_0^n. \quad \blacksquare$$

A completely similar argument, using Proposition 2.1 (iii) instead of (i), shows the following stronger result in the case $|w_n|$ is small. The difference between Lemma 3.1 and Lemma 3.2 is that in Lemma 3.2, κ_0 does not depend on δ , but in Lemma 3.1, κ depends on δ .

Lemma 3.2. *Let $f(z, w) = (\lambda z, w^d + c(z))$ as in (1.1) such that f_0 has no attracting nor superattracting cycle in \mathbb{C} . Given $0 < \lambda_0 < 1$, there exists $\kappa_0 = \kappa_0(\lambda_0) > 0$ such that for any $\delta > 0$ there exists $\eta_2 = \eta_2(\lambda_0, \delta) > 0$ with the following property: If $\{x_j = (z_j, w_j)\}_{j=0}^n$ is an f -orbit with $|w_j| \geq \delta$ hold for $0 \leq j < n$, $|w_n| \leq \delta$ and $|z_0| < \eta_2$, then*

$$|Df^n(x_0)(v)| \geq \kappa_0 \lambda_0^n.$$

Proof. We just specify the choice of κ_0 . By Proposition 2.1 (iii), there exists $\kappa_0 \in (0, 1)$ depending on λ_0 such that for any f_0 -orbit $(\tilde{w}_j)_{j=0}^n$, with $|\tilde{w}_j| \geq |\tilde{w}_n|$ for all $0 \leq j < n$, then

$$|Df_0^n(\tilde{w}_0)| \geq 2\kappa_0 \lambda_0^{n/2}.$$

Arguing in the same way as the previous proof by a continuity argument, we show that for any n , there exists $\eta'_n = \eta'_n(\delta)$ such that if $|z_0| < \eta'_n$ and $|w_j| \geq \delta$ for $0 \leq j < n$ and $|w_n| < \delta$, then

$$|Df^n(x_0)(v)| \geq \kappa_0 \lambda_0^n. \tag{3.1}$$

On the other hand, by Lemma 3.1, there exists

$$\kappa = \kappa(\lambda_0, \delta) > 0 \quad \text{and} \quad \eta_1 = \eta_1(\lambda, \delta) > 0$$

such that

$$|Df^n(x_0)(v)| \geq \kappa \lambda_0^{n/2},$$

provided that $|z_0| < \eta_1$. So there exists $N_1 = N_1(\lambda_0, \delta) > 0$ such that

$$|Df^n(x_0)(v)| > \kappa_0 \lambda_0^n \tag{3.2}$$

provided that $n > N_1$ and $|z_0| < \eta_1$.

Combine the estimates (3.1) and (3.2), the lemma holds with

$$\eta_2 := \min(\eta_1, \eta'_1, \dots, \eta'_{N_1}),$$

where $(\eta'_n)_{1 \leq n \leq N_1}$ are constants appearing before (3.1). \blacksquare

We use the binding argument to prove a perturbed version of Proposition 2.1 (ii). Let us first state the following one-dimensional slightly generalized form of the statement.

Lemma 3.3. *Given $\lambda_0 \in (0, 1)$, there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, if $(w_i)_{i=0}^n$ is an f_0 -orbit with $|w_0| < 2\delta$, $|w_n| < 2\delta$ and $|w_j| > \delta/2$ for all $1 \leq j < n$, then*

$$|Df_0^n(w_0)| \geq \lambda_0^n \min\left(1, \left(\frac{|w_0|}{|w_n|}\right)^{d-1}\right).$$

We note that the condition on the intermediate points of the orbit $|w_j| > \delta/2$ is weaker than it would have been for Proposition 2.1 (ii) (where it would be $|w_j| > 2\delta$).

Proof. Fix $\lambda_0 \in (0, 1)$. By Proposition 2.1 (ii), there exists $\delta_0 > 0$ such that the following holds: If $\{w_j\}_{j=0}^n$ is an f_0 -orbit for which there exists $\rho \in (0, \delta_0)$ with $|w_0| < \rho$, $|w_n| \leq \rho$, and $\min_{j=1}^{n-1} |w_j| > \rho$, then

$$|Df_0^n(w_0)| \geq \lambda_0^{n/2} \min\left(1, \left(\frac{|w_0|}{|w_n|}\right)^{d-1}\right). \tag{3.3}$$

Let

$$N(\delta) = \inf\{s \geq 1 : f_0^s(\overline{B(0, 2\delta)}) \cap \overline{B(0, 2\delta)} \neq \emptyset\}.$$

Then $N(\delta) \rightarrow \infty$ as $\delta \rightarrow 0$.

Now let $(w_j)_{j=0}^n$ be as in the lemma with $\delta > 0$ small.

Case 1. $|w_0|, |w_n| < \delta/2$. In this case the desired estimate follows immediately from formula (3.3) by taking $\rho = \delta/2$.

Case 2. $|w_0| \geq \delta/2$. Let $0 = n_0 < n_1 < n_2 < \dots < n_p = n$ be all the positive integers in $\{0, 1, \dots, n\}$ with $|w_{n_j}| \leq 2\delta$. For each $0 \leq j < p$, we have $|w_{n_i}|/|w_{n_{i+1}}| \geq 1/4$, so taking $\rho = 2\delta$ in (3.3), we obtain

$$|Df_0^{n_{j+1}-n_j}(w_{n_j})| \geq \lambda_0^{n/2} 4^{-(d-1)}.$$

Thus,

$$|Df_0^n(w_0)| \geq \lambda_0^{n/2} 4^{-p(d-1)}.$$

Since $n_{j+1} - n_j \geq N(\delta)$, we have $p \leq n/N(\delta)$. Provided that δ is small enough, $p \ll n$, so that $4^{-p(d-1)} \geq \lambda_0^{n/2}$. Then

$$|Df_0^n(w_0)| \geq \lambda_0^n.$$

Case 3. $|w_0| < \delta/2$ and $|w_n| \geq \delta/2$. Define n_j as above. Then

$$|Df_0^{n_1}(w_0)| \geq \lambda_0^{n/2} \left(\frac{|w_0|}{2\delta}\right)^{d-1} \geq 4^{-(d-1)} \lambda_0^{n/2} \left(\frac{|w_0|}{|w_n|}\right)^{d-1},$$

and

$$|Df^{n_{i+1}-n_i}(w_{n_i})| \geq \lambda_0^{n/2} 4^{-(d-1)}.$$

The desired estimate follows similarly to Case 2. ■

Recall the notion of tame orbits was introduced in Section 1, before the statement of Theorem 1.4.

Lemma 3.4. *Given $0 < \lambda_0 < 1$, there exists $\delta_0 > 0$ and $\eta_0 > 0$ such that the following holds. Let $\{x_i = (z_i, w_i)\}_{i=0}^n$ be a tame f -orbit such that*

- $|w_0|, |w_n| \leq \delta_0$;
- $|z_0| < \eta_0$;
- $|w_j| \geq |w_n|$ for each $0 < j < n$.

Then

$$|Df^n(x_0)(v)| \geq \lambda_0^n \min\left(1, \left(\frac{|w_0|}{|w_n|}\right)^{d-1}\right).$$

Proof. Without loss of generality, we may assume that $1 > \lambda_0 > |\lambda| > 0$. Let κ_0 be the constant given by Lemma 3.2. Let N be a large positive integer such that

$$\lambda_0^{-N} > \max(4^d, 2\kappa_0^{-1}), \tag{3.4}$$

and such that

$$\frac{\mu_0}{2(s+1)^2 C_1} > \lambda_0^s \tag{3.5}$$

holds for all $s \geq N$, where $C_1 = C_1(\lambda_0) > 0$ is a constant as in (3.12).

Let $\delta_0 > 0$ be a small constant such that the following hold:

- for any orbit $(z'_j, w'_j)_{j=0}^n$ satisfying $|z'_0| \leq \delta_0$ and $|w'_0| \leq \delta_0$ and $|w'_n| \leq \delta_0$, we have $n \geq N$;
- if $|z'_0| \leq \delta_0, |w'_0| \leq \delta_0$, we have $b_{\mu_0}((z'_0, w'_0), (0, w'_0)) > N$.

Note that for small δ_0 , the first property is guaranteed by the variant of Przytycki’s lemma (Lemma 2.7), and the second property follows from definition of binding time by continuity since the critical point of f_0 is not periodic. Let $\eta_2 = \eta_2(\lambda_0^{1/2}, \delta_0)$ be given by Lemma 3.2 and let $\eta_0 = \min(\eta_2, \delta_0)$.

We shall prove this lemma by induction on n .

Starting step. We take the trivial case $n = 0$ as the starting step of the induction.

Inductive step. Now let n_0 be a positive integer and assume that the lemma holds under the additional assumption that $n < n_0$. Let us consider an f -orbit $(w_j)_{j=0}^n$ satisfying the assumption of the lemma with $n = n_0$. Note that existence of such an orbit implies that $n_0 \geq N$.

Let s be the μ_0 -binding time of (z_0, w_0) and $(0, w_0)$, then $s \geq N$. Let $w_{i,0} = f_0^i(w_0)$. By Lemma 2.4, for any integer $1 \leq i \leq s$,

$$|Df^i(x_0)(v)| \geq \frac{1}{2} |Df_0^i(w_0)|. \tag{3.6}$$

Case 1. $s > n$. Then by Lemma 3.3,

$$|Df_0^n(w_0)| \geq \lambda_0^{n/2} \min\left(1, \frac{|w_0|}{|w_n|}\right)^{d-1},$$

which together with (3.6) implies the desired estimate, since

$$\lambda_0^{-n/2} \geq \lambda_0^{-N/2} > 2.$$

Case 2. $s \leq n$ and there exists $t \in \{1, 2, \dots, s-1\}$ such that $|w_t| \leq |w_0|$. Let t be minimal with the last property. Then $|w_{j,0}| > |w_j|/2 \geq |w_0|/2$ for each $0 < j < t$ and $|w_{t,0}| < 2|w_t| \leq 2|w_0|$. Since $t \geq N$, we have $\lambda_0^{-t/2} > 2^d$. So by Lemma 3.3,

$$|Df_0^t(w_0)| \geq \lambda_0^{t/2} \min\left(1, \left(\frac{|w_0|}{|w_{t,0}|}\right)^{d-1}\right) \geq 2\lambda_0^t \min\left(1, \frac{|w_0|}{|w_t|}\right)^{d-1} \geq 2\lambda_0^t,$$

hence $|Df^t(x_0)(v)| \geq \lambda_0^t$ by (3.6). By induction hypothesis,

$$|Df^{n-t}(x_t)(v)| \geq \lambda_0^{n-t} \min\left(1, \frac{|w_t|}{|w_n|}\right)^{d-1} \geq \lambda_0^{n-t}.$$

Thus the desired estimate holds.

Case 3. $s \leq n$, and for each $1 \leq j < s$, $|w_j| > |w_0|$. In this case, we first prove the following inequality:

$$|Df^s(x_0)(v)| \geq \frac{\lambda_0^{s/2}}{2}. \tag{3.7}$$

If $|w_{s,0}| < |w_0|$, then by Lemma 3.3,

$$|Df_0^s(w_0)| \geq \lambda_0^{s/2},$$

so (3.7) holds by (3.6). From now on, we assume that

$$|w_{s,0}| \geq |w_0|. \tag{3.8}$$

Let

$$W = W((z_0, w_0), (0, w_0), s).$$

By Lemma 2.5, we have

$$|Df^s(x_0)(v)| \geq \frac{|w_s - w_{s,0}|}{W}. \tag{3.9}$$

Let us show

$$|w_s - w_{s,0}| \geq \frac{\mu_0}{(s + 1)^2} \min(|w_s|, |w_{s,0}|) \geq \frac{\mu_0}{2(s + 1)^2} |w_0|. \tag{3.10}$$

Indeed, if $|w_s| < |w_0|/2$, then by (3.8), $|w_s - w_{s,0}| \geq |w_{s,0}| - |w_s| \geq |w_0|/2$, so (3.10) holds; if $|w_s| \geq |w_0|/2$, then $\min(|w_s|, |w_{s,0}|) \geq |w_0|/2$ and (3.10) follows from (2.3). Now, let us provide an upper bound for W . By Proposition 2.1,

$$|Df_0^j(w_0)| \geq C|w_0|^{d-1}\lambda_0^j$$

for all $1 \leq j \leq s$, where $C > 0$ is a constant depending only on λ_0 . Therefore, by (2.10),

$$W \leq \sum_{i=1}^s \frac{4|\lambda^{k(i-1)}|z_0|^k|}{|Df_0^i(w_0)|} \leq \sum_{i=1}^{\infty} \frac{4\lambda_0^{k(i-1)}|z_0|^k}{C|w_0|^{d-1}\lambda_0^i} \leq C_1|w_0|, \tag{3.11}$$

where

$$C_1 = \frac{4}{C\lambda_0^k(1 - \lambda_0^{k-1})}, \tag{3.12}$$

and we have used the assumption $|w_0| \geq |z_0|^{k/d}$ for the last inequality. Substituting (3.10) and (3.11) into (3.9), we obtain

$$|Df^s(x_0)(v)| \geq \frac{1}{2(s + 1)^2 C_1},$$

which implies (3.7) by (3.5), since $s \geq N$.

To complete the proof in Case 3, let $n_1 \geq s$ be minimal such that $|w_{n_1}| < \delta_0$. Then by Lemma 3.2,

$$|Df^{n_1-s}(x_s)(v)| \geq \kappa_0\lambda_0^{n_1-s}.$$

By induction hypothesis,

$$|Df^{n-n_1}(x_{n_1})(v)| \geq \lambda_0^{n-n_1}.$$

Therefore,

$$|Df^n(x_0)(v)| \geq \frac{\kappa_0\lambda_0^{-s/2}\lambda_0^n}{2} > \lambda_0^n,$$

where we have used (3.4) and $s \geq N$ for the last inequality. ■

Now we can prove Theorem 1.4.

Proof of Theorem 1.4. Fix $\lambda_0 \in (0, 1)$ and let δ_0, η_0 be given by Lemma 3.4. Let $\eta_1 = \eta_1(\lambda_0, \delta_0)$ and $\eta_2 = \eta_2(\lambda_0, \delta_0)$ be given by Lemmas 3.1 and 3.2, respectively. Let $\eta := \min(\eta_0, \eta_1, \eta_2)$.

Now let us consider a tame orbit $(x_i = (z_i, w_i))_{i=0}^n$. We may assume that $|z_0| < \eta$. Indeed, if $|z_0| \geq \eta$, let $N \geq 0$ be the minimal integer such that $|z_j| \geq \eta$ for $0 \leq j \leq N$. Then $\eta \leq |z_0| |\lambda|^N < |\lambda|^N$, so N is bounded. By the tameness assumption, $\inf_{i=0}^N |w_i|$ is bounded away from zero, then $|Df^{N+1}(x_0)(v)|$ is bounded away from zero.

If $|w_j| \geq \delta_0$ holds for all $0 \leq j < n$, then the desired estimate follows from Lemma 3.1. So we may assume that there is a minimal $n_0 \in \{0, 1, \dots, n-1\}$ such that $|w_{n_0}| < \delta_0$. By Lemma 3.2,

$$|Df^{n_0}(x_0)(v)| \geq \kappa_0 \lambda_0^{n_0}.$$

Let $n_1 \in \{n_0, n_0 + 1, \dots, n\}$ be maximal such that $|w_{n_1}| = \inf_{j=n_0}^{n_1} |w_j|$. Then by Lemma 3.4,

$$|Df^{n_1-n_0}(x_{n_0})(v)| \geq \lambda_0^{n_1-n_0}.$$

If $|w_n| \leq |w_j|$ for all $0 \leq j < n$, then $n_1 = n$, so the proof is completed. To deal with the general case, let us define a sequence of integers $n_1 < n_2 < \dots < n_t \leq n$ such that

- for each $1 \leq i < t$, $|w_{n_{i+1}}| = \inf_{j=n_i}^{n_{i+1}} |w_j| < \delta_0$;
- for each $n_t + 1 < j < n$, $|w_j| \geq \delta_0$.

Then $|w_{n_1}| \leq |w_{n_2}| \leq \dots \leq |w_{n_t}|$. By Lemma 3.4, for each $1 \leq i < t$,

$$|Df^{n_{i+1}-n_i}(x_{n_i})(v)| \geq \lambda_0^{n_{i+1}-n_i} \left(\frac{|w_{n_i}|}{|w_{n_{i+1}}|} \right)^{d-1}.$$

By Lemma 3.1,

$$|Df^{n-n_t}(x_{n_t})(v)| \geq d |w_{n_t}|^{d-1} \kappa \lambda_0^{n-n_t-1}.$$

Combining all the displayed inequalities, we obtain the desired estimates. ■

4. Slow approach to the critical point

In this section we prove Theorem 1.5. This theorem will be deduced from the following Theorem 4.2, which asserts that for almost every z with respect to the Lebesgue measure in $B(0, r_0)$, pull-backs of the ball $B(0, e^{-\alpha n})$ along the vertical mappings $w \mapsto \xi_n(z, w)$ have bounded criticality, uniformly in n . Let $L_z = \{z\} \times \mathbb{C}$ denote the vertical line passing through $(z, 0)$.

Definition 4.1. For every $\alpha > 0$, let Λ_α be the subset of $B(0, r_0)$ characterized by the following property: for every $z \in \Lambda_\alpha$, there is a constant $N = N(z) > 0$ such that for every integer $n \geq 1$, for every connected component V of $\xi_n^{-1}(B(0, e^{-\alpha n})) \cap L_z$, there are at most N integers $0 \leq m \leq n$ satisfying $0 \in f^m(V)$.

The following theorem implies Theorem 1.5.

Theorem 4.2. Let $f(z, w) = (\lambda z, w^d + c(z))$ as in (1.1). Assume $(0, 0)$ is not a periodic point. Then for any $\alpha > 0$, Λ_α is a full Lebesgue measure subset of $B(0, r_0)$.

Proof of Theorem 1.5 assuming Theorem 4.2. By our assumption, there is $R > 1$ such that for any (z, w) in the domain of f with $|w| > R$, we have $|\xi_1(z, w)| \geq |w| > R$.

Since Λ_α has full area, it is sufficient to prove for every $z \in \Lambda_\alpha$, a.e. $x \in L_z$ (with respect to the Lebesgue measure) is 2α -slow approach. Let

$$E_{n,2\alpha} = \{x \in L_z : |\xi_n(x)| < e^{-2\alpha n}\}$$

and

$$E_{n,\alpha} = \{x \in L_z : |\xi_n(x)| < e^{-\alpha n}\}.$$

Let V be a connected component of $E_{n,\alpha}$. By our assumption on z , the map

$$\xi_n: V \rightarrow B(0, e^{-\alpha n})$$

has degree at most d^N . Since $\xi_n(V \cap E_{n,2\alpha}) \subset B(0, e^{-2\alpha n})$, it follows from a version of the Koebe distortion theorem for multivalent maps, see for instance [21, Lemma 2.1], that there exists $\alpha' = \alpha'(\alpha, N) > 0$ such that

$$\frac{\text{Vol}(V \cap E_{n,2\alpha})}{\text{Vol}(V)} \leq e^{-\alpha' n}.$$

Since $\xi_n(V \cap E_{n,2\alpha}) \subset B(0, e^{-2\alpha n}) \subset B(0, R)$, we have $V \subset \{z\} \times B(0, R)$. Thus, $\text{Vol}(E_{n,2\alpha})$ is exponentially small with respect to n , and thus

$$\sum_{n=1}^{+\infty} \text{Vol}(E_{n,2\alpha}) < +\infty.$$

By the Borel–Cantelli lemma, Lebesgue a.e. $x \in L_z$ is contained in only finitely many $E_{n,2\alpha}$. This is equivalent to saying that Lebesgue a.e. $x \in L_z$ is 2α -slow approach. ■

The rest of this section is devoted to prove Theorem 4.2. We shall use the following result which is a special case of [13, Lemma 4.5].

Lemma 4.3. *Let $f(z, w) = (\lambda z, w^d + c(z))$ be as in (1.1). Assume $(0, 0)$ is not a periodic point. Then for every $\alpha > 0$, $0 < \beta < 1$, there is a constant $N = N(\alpha, \beta) > 0$ such that for every integer $n \geq 1$ and for every $z \in B(0, r_0)$, for every connected component V of $\xi_n^{-1}(B(0, e^{-\alpha n})) \cap L_z$, there are at most N integers $\beta n \leq m \leq n$ satisfying $0 \in \xi_m(V)$.*

To deduce Theorem 4.2 from Lemma 4.3, we shall show that for almost every z , $\xi_n(\lambda^m z, 0)$ cannot be too close to zero. More precisely, we shall show that for almost every z , $\lambda^m z$ belongs to the set $\Omega_{\alpha, m}$ defined below.

Definition 4.4. For each $\alpha > 0$ and each integer $m \geq 0$, define

$$\Omega_{\alpha, m} = \{z \in B(0, |\lambda|^m r_0) : |\xi_n(z, 0)| > |z|^{k/d} e^{-\alpha n} \text{ for every } n \geq 1\}.$$

For each integer $l \geq 1$, define $K_{\alpha, m, l}$ to be the following set:

$$\{z \in B(0, |\lambda|^m r_0) : l \geq 1 \text{ is minimal such that } |\xi_l(z, 0)| \leq |z|^{k/d} e^{-\alpha l}\}.$$

We have the following relations between Λ_α , $\Omega_{\alpha, m}$ and $K_{\alpha, m, l}$.

Lemma 4.5. *Assume $(0, 0)$ is not a periodic point of f . Then for every $\alpha > 0$, $m \geq 0$ and $k \geq 1$, the following hold:*

- (1) *For every integer $N \geq 0$, $\bigcap_{m=N}^\infty \lambda^{-m} \Omega_{\alpha, m} \subset \Lambda_{2\alpha} \cup \{0\}$.*
- (2) *$B(0, |\lambda|^m r_0) \setminus \Omega_{\alpha, m} = \bigcup_{l=1}^\infty K_{\alpha, m, l}$.*

Here for $\theta \in \mathbb{C}$ and $X \subset \mathbb{C}$, $\theta X := \{\theta x : x \in X\}$.

Proof. The second statement is obvious by the definition of $\Omega_{\alpha, m}$ and $K_{\alpha, m, l}$. We only prove (1). Let $0 \neq z \in \bigcap_{m=N}^\infty \lambda^{-m} \Omega_{\alpha, m}$. Then for every $m \geq N$, $n > m$, we have

$$|\xi_{n-m}(\lambda^m z, 0)| > |\lambda|^{km/d} |z|^{k/d} e^{-\alpha(n-m)}.$$

Thus when $|\lambda|^{km/d} \geq e^{-\alpha n/3}$ and $|z|^{k/d} \geq e^{-\alpha n/3}$, i.e. when $m \leq \alpha dn / (-3k \log |\lambda|)$, and n is large enough, we have that

$$|\xi_{n-m}(\lambda^m z, 0)| > e^{-2\alpha n}.$$

This in particular implies that for n large enough, for every connected component V of $\xi_n^{-1}(B(0, e^{-2\alpha n})) \cap L_z$, if $m \leq \alpha dn / (-3k \log |\lambda|)$, then $0 \notin f^m(V)$.

On the other hand, by Lemma 4.3, we know that for every connected component V of $\xi_n^{-1}(B(0, e^{-2\alpha n})) \cap L_z$, there are at most $N(2\alpha, d\alpha / (-3k \log |\lambda|))$ integers m such that $0 \in f^m(V)$ for $m > \alpha dn / (-3k \log |\lambda|)$. To summarize, for every connected component V of $\xi_n^{-1}(B(0, e^{-2\alpha n})) \cap L_z$, there are at most $N(2\alpha, d\alpha / (-3k \log |\lambda|))$ integers m such that $0 \in f^m(V)$. Hence $z \in \Lambda_{2\alpha}$. This implies (1). ■

To complete the proof, we shall show that $K_{\alpha,m,l}$ has exponentially small area relative to $B(0, |\lambda|^m r_0)$. To this end, we shall analyze the property of the map

$$z \mapsto \xi_l(z, 0).$$

We shall show that for each $z_0 \in K_{\alpha,m,l}$, $\xi_l(z, 0)$ maps a neighborhood of z_0 conformally onto its image which contains a ball much larger than $B(0, |z_0|^{k/d} e^{-\alpha l})$, see Lemma 4.7. The strategy bears strong analogue with the parameter exclusion technique introduced by Benedicks–Carleson in [5]. Indeed, the vertical fibres

$$w \mapsto \xi_l(z, w)$$

is a composition of maps close to f_0 , parametrized in z . Provided that $\xi_j(z_0, 0)$ is not too close to 0 for $1 \leq j < l$, we can relate the derivative $\frac{\partial \xi_l(z, 0)}{\partial z}$ with $Df_0^{l-1}(c(0))$ in a very precise manner, see Lemma 4.6 below. Let

$$\mathcal{X}_l(z) = \frac{d}{dz} \xi_l(z, 0).$$

Let

$$\widehat{K}_{\alpha,m,l} = \{z \in B(0, 2|\lambda|^m r_0) : 4|\xi_j(z, 0)| \geq |z|^{k/d} e^{-\alpha j}, 1 \leq j < l\}.$$

Recall that by Proposition 2.2 (2),

$$X_0 := \sum_{i=0}^{\infty} \frac{\lambda^{ik}}{(f_0^i)'(c(0))} \neq 0.$$

Lemma 4.6. *Let $f(z, w) = (\lambda z, w^d + c(z))$ be as in (1.1). Assume $(0, 0)$ is not a periodic point and let $\alpha > 0$ be such that $e^{d\alpha} |\lambda|^k < 1$. Then for m and l large and for every $z_0 \in \widehat{K}_{\alpha,m,l}$, we have*

$$\left| \frac{\mathcal{X}_l(z_0)}{Df^{l-1}(f(z_0, 0))(v)} - kX_0 z_0^{k-1} \right| \leq \frac{k|X_0||z_0|^{k-1}}{2}.$$

Proof. Fix $1 > \lambda_0 > |\lambda|$. For $z_0 \in \widehat{K}_{\alpha,m,l}$, write $x_i = (z_i, w_i) = f^i(z_0, 0)$. Let us first relate the derivative $\mathcal{X}_l(z_0)$ to the derivative $|Df^{l-1}(x_1)(v)|$. For each positive integer j ,

$$\xi_j(z, 0) = (\xi_{j-1}(z, 0))^d + c(\lambda^{j-1} z).$$

Taking derivatives on both side, we have

$$\mathcal{X}_j(z) = d(\xi_{j-1}(z, 0))^{d-1} \mathcal{X}_{j-1}(z) + \lambda^{j-1} c'(\lambda^{j-1} z).$$

Thus

$$\frac{\mathcal{X}_j(z_0)}{Df^{j-1}(x_1)(v)} = \frac{\mathcal{X}_{j-1}(z_0)}{Df^{j-2}(x_1)(v)} + \frac{\lambda^{j-1}c'(\lambda^{j-1}z_0)}{Df^{j-1}(x_1)(v)}.$$

As $\mathcal{X}_0 = 0$, by induction, we obtain

$$\frac{\mathcal{X}_l(z_0)}{Df^{l-1}(x_1)(v)} = \sum_{i=0}^{l-1} \frac{\lambda^i c'(\lambda^i z_0)}{Df^i(x_1)(v)}.$$

By the definition of k (see (1.3)), there exists a constant $A > 0$ such that

$$|c'(z) - kz^{k-1}| \leq A|z|^k$$

for all $|z| < |\lambda|r_0$. Let

$$X := \sum_{i=0}^{l-1} \frac{\lambda^{ik}}{Df^i(x_1)(v)}.$$

Then we have

$$\begin{aligned} \left| \frac{\mathcal{X}_l(z_0)}{Df^{l-1}(x_1)(v)} - kz_0^{k-1}X \right| &\leq \sum_{i=0}^{l-1} \frac{|\lambda^i| |c(\lambda^i z_0) - k(\lambda^i z_0)^{k-1}|}{|Df^i(x_1)(v)|} \\ &\leq A \sum_{i=0}^{l-1} \frac{|\lambda|^{(k+1)i}}{|Df^i(x_1)(v)|} |z_0|^k. \end{aligned} \tag{4.1}$$

Let us now provide lower bounds for $|Df^i(x_1)(v)|$. Let s_0 be a large integer such that $4^d (e^{\alpha d} |\lambda|^k)^{s_0} < 1$. Let s be the μ_0 -binding time of the pair $(x_1, (z_1, c(0)))$. Put

$$\delta = \max(|w_1 - c(0)|, |z_0|^k)^{1/d} = \max(|c(z_0) - c(0)|, |z_0|^k)^{1/d} = |z_0|^{k/d}.$$

Provided that m is large enough, δ is small. So by Lemma 2.6, there exists $s_0 \leq n \leq s$, such that

$$|Df^n(x_1)(v)| \geq |z_0|^{-k(d-1)/d} e^{-\alpha n}. \tag{4.2}$$

Since $z_0 \in \widehat{K}_{\alpha, m, l}$, for each $n \leq i < l$, we have

$$|w_i|^d \geq |z|^k e^{-\alpha i d} 4^{-d} > |z|^k |\lambda|^{ik} = |z_i|^k,$$

so $(x_i)_{i=n}^l$ is tame. By Theorem 1.4, there exists $C = C(\alpha) > 0$ such that, for each $n < i \leq l$,

$$|Df^{i-n}(x_n)(v)| \geq C e^{-\alpha(i-n)} \min_{j=n}^i |w_j|^{d-1}.$$

As $|w_j| \geq |z_0|^{k/d} e^{-\alpha j} / 4$ for each $1 \leq j < l$, we obtain

$$|Df^{i-n}(x_n)(v)| \geq C' |z_0|^{k(d-1)/d} e^{-\alpha(di-n)},$$

where C' is a constant.

Together with (4.2), this implies

$$|Df^i(x_1)(v)| = |Df^n(x_0)(v)||Df^{i-n}(x_n)(v)| \geq C'e^{-d\alpha i} \tag{4.3}$$

for each $n \leq i < l$. On the other hand, by Lemma 2.4, for each $0 \leq i \leq s$,

$$|Df^i(x_1)(v)| \geq \frac{1}{2}|Df_0^i(c(0))|.$$

Combining with Proposition 2.2, this implies that (4.3) remains true for $1 \leq i < n$ (replacing C' by a larger constant if necessary). It follows that

$$\sum_{i=0}^{l-1} \frac{|\lambda|^{(k+1)i}}{|Df^i(x_1)(v)|}$$

is bounded from above by a constant. These lower bounds also imply that

$$|X - X_0| \leq \frac{1}{4},$$

provided that m, l are large enough. By (4.1), we obtain the desired estimate. ■

Consider two sets of positive real numbers $\{a_i\}_{i \in I}, \{b_i\}_{i \in I}$, where I is an index set. In the following lemma we use the asymptotic notions \asymp and \succeq . We say $a_i \asymp b_i$ if there exists a constant $C > 0$ such that $b_i/C \leq a_i \leq Cb_i$ for every $i \in I$. We say $a_i \succeq b_i$ if there exists a constant $C > 0$ such that $a_i \geq Cb_i$ for every $i \in I$.

Lemma 4.7. *There exist constants $C > 0$ and $\omega \in (0, 1)$ such that for any $\alpha > 0$ with $e^{d\alpha}|\lambda|^k < 1$, the following holds provided that m, l are sufficiently large. For each $z_0 \in K_{\alpha, m, l}$, there exists a neighborhood V' of z_0 such that $\varphi_l(z) = \xi_l(z, 0)$ maps V' conformally onto $B(0, C|z_0|^{k/d}e^{-\alpha\omega l})$ and such that $V' \subset B(0, 2|z_0|)$.*

Proof. We shall prove that there is $r_1 \in (0, |z_0|)$ such that $\varphi_l(z) := \xi_l(z, 0)$ is univalent on $B(z_0, r_1)$ and $\varphi_l(B(z_0, r_1))$ contains a ball centered at $\xi_l(z_0, 0)$ with radius at least $C|z_0|^{k/d}e^{-\alpha\omega l}$, where $C > 0$ and $\omega \in (0, 1)$ are constants.

Note that $0 \notin K_{\alpha, m, l}$, so $z_0 \neq 0$. Let r be the maximal radius satisfying the following: $r \leq 2|z_0|$ and for every $z \in B(z_0, r)$, if s denotes the μ_0 -binding time of the pair $((\lambda z_0, \xi_1(z_0, 0)), (\lambda z, \xi_1(z, 0)))$, then $s \geq l - 1$. By Lemma 2.5 for each $1 \leq j < l$,

$$2|\xi_j(z, 0)| \geq |\xi_j(z_0, 0)| \geq |z_0|^{k/d}e^{-\alpha j}.$$

Let $\varepsilon > 0$ be a small constant such that $2(1 - \varepsilon)^{k/d} > 1$. Let $r_1 = \min(r, \varepsilon|z_0|)$. Then for $z \in B(z_0, r_1)$, we have

$$4|\xi_j(z, 0)| \geq |z|^{k/d}e^{-\alpha j}$$

for all $1 \leq j < l$. Thus $B(z_0, r_1) \subset \widehat{K}_{\alpha, m, l}$. By Lemma 4.6, for each $z \in B(z_0, r_1)$,

$$\left| \frac{\varphi'_l(z)}{kz^{k-1}X_0D_{l-1}(z)} - 1 \right| < \frac{1}{2} \quad \text{for all } z \in B(z_0, r),$$

where $D_i(z) = Df^i(f(z, 0))(v)$. Then φ_l is univalent in $B(z_0, r_1)$ and it suffices to show that $|\varphi'(z_0)|r_1 \geq C|z_0|^{k/d}e^{-\alpha\omega l}$ provided that m, l are large enough. By the Koebe distortion theorem, we only need to show that

$$|z_0^{k-1}r_1D_{l-1}(z_0)| \geq C|z_0|^{k/d}e^{-\alpha\omega l}. \tag{4.4}$$

Case 1. $r > |z_0|$. Then $r_1 = \varepsilon|z_0|$. In this case $0 \in B(z_0, r)$, so by the definition of r ,

$$|Df_0^{l-1}(c(0))| \asymp |D_{l-1}(z_0)|. \tag{4.5}$$

Subcase 1.1. Assume that $|f_0^l(0)| \geq 2|z_0|^{k/d}e^{-\alpha l/2}$. Since $|\xi_l(z_0, 0)| \leq |z_0|^{k/d}e^{-\alpha l}$, we have

$$|\xi_l(0, 0) - \xi_l(z_0, 0)| \geq |z_0|^{k/d}e^{-\alpha l/2}.$$

Then by (4.5) and Proposition 2.2 (1), we have

$$W := 2|\xi_1(z_0, 0) - \xi_1(0, 0)| + \sum_{i=1}^{l-1} \frac{2|c(\lambda^{i-1}z_0) - c(0)|}{|D_i(z_0)|} \leq C_0|z_0|^k,$$

where $C_0 > 0$ is a constant. Then, by Lemma 2.5, we have

$$D_{l-1}(z_0) \geq \frac{|\xi_l(0, 0) - \xi_l(z_0, 0)|}{W} \geq \frac{|z_0|^{k/d}e^{-\alpha l/2}}{C_0|z_0|^k}.$$

Hence,

$$|z_0^{k-1}r_1D_{l-1}(z_0)| \geq |z_0|^{k/d} \frac{e^{-\alpha l/2}\varepsilon}{C_0},$$

and hence (4.4) holds.

Subcase 1.2. Assume that $|f_0^l(0)| < 2|z_0|^{k/d}e^{-\alpha l/2}$. Then by Proposition 2.2 (1),

$$e^{-\alpha l/2} \leq |Df_0^l(c(0))| = |Df_0^{l-1}(c(0))|d|f_0^l(0)|^{d-1}$$

and

$$|Df_0^{l-1}(c(0))|d|f_0^l(0)|^{d-1} \leq |Df_0^{l-1}(c(0))|2^d d|z_0|^{k(d-1)/d}e^{-\alpha l(d-1)/2},$$

which, by (4.5), again implies that

$$|z_0^{k-1}r_1D_{l-1}(z_0)| \geq |z_0|^{k/d}e^{-\alpha l/2},$$

hence (4.4) holds.

Case 2. $r \leq |z_0|$. Then $r \asymp r_1$. By the maximality of r and the definition of the binding time, there is a minimal integer $2 \leq n \leq l - 1$ such that, for $z \in \overline{B(z_0, r)}$,

$$|\xi_n(z_0, 0) - \xi_n(z, 0)| = \frac{\mu_0 \min(|\xi_n(z, 0)|, |\xi_n(z_0, 0)|)}{n^2} \geq \frac{\mu_0 |\xi_n(z_0, 0)|}{2n^2}.$$

It follows that

$$|D_{n-1}(z_0)|r_1 \asymp |D_{n-1}(z_0)|r \geq n^{-2}|\xi_n(z_0, 0)| \geq l^{-2}|\xi_n(z_0, 0)|.$$

Note that $\{f^j(z_0, 0)\}_{j=n}^l$ is tame, and $|\xi_l(z_0, 0)| < |\xi_j(z_0, 0)|$ for all $n \leq j < l$. By the last statement of Theorem 1.4,

$$\frac{|D_{l-1}(z_0)|}{|D_{n-1}(z_0)|} \geq e^{-\alpha(l-n)/2},$$

hence

$$D_{l-1}(z_0)r_1 \geq l^{-2}|\xi_n(z_0, 0)|e^{-\alpha(l-n)/2}. \tag{4.6}$$

Case 2.1. $|\xi_n(z_0, 0)| \geq |z_0|^{k/d} e^{-\alpha l/4}$. Then

$$D_{l-1}(z_0)r \geq l^{-2}|z_0|^{k/d} e^{-3\alpha l/4} \geq |z_0|^{k/d} e^{-4\alpha l/5},$$

provided that l is large enough.

Case 2.2. $|\xi_n(z_0, 0)| < |z_0|^{k/d} e^{-\alpha l/4}$. By Lemma 2.7, $l - n \asymp l$. Since $z_0 \in K_{\alpha, m, l}$, we have $|\xi_n(z_0, 0)| \geq |z_0|^{k/d} e^{-\alpha n}$. By (4.6),

$$|D_{l-1}(z_0)r_1| \geq l^{-1}|z_0|^{k/d} e^{-\alpha(l+n)/2} \geq e^{-\alpha\omega l}$$

for a suitably chosen ω , provided that l is large enough. ■

For each integer $m_1 \geq m$, let

$$K_{\alpha, m, l}^{m_1} = \{z \in K_{\alpha, m, l} : |\lambda|^{m_1+1}r_0 \leq |z| < |\lambda|^{m_1}r_0\}.$$

The following estimate of the volume of $K_{\alpha, m, l}^{m_1}$ is crucial in the proof of Theorem 4.2.

Lemma 4.8. *Let $f(z, w) = (\lambda z, w^d + c(z))$ as in (1.1). Assume $(0, 0)$ is not a periodic point. Then for every $\alpha > 0$ satisfying $e^{-d\alpha} > |\lambda|^k$, there exists $\gamma = \gamma(\alpha) > 0$ such that, for m and l large, the following holds for all $m_1 \geq m$:*

$$\frac{\text{Vol}(K_{\alpha, m, l}^{m_1})}{\text{Vol}(B(0, |\lambda|^{m_1}r_0))} \leq e^{-\gamma l}.$$

Proof. Fix $m_1 \geq m$. Let V'_j , $j = 1, 2, \dots$, be the connected components of the set $\varphi_l^{-1}(B(0, |\lambda|^{km_1/d} e^{-\alpha\omega l}))$ such that $V'_j \subset B(0, 2|\lambda|^{m_1} r_0)$ and such that

$$\varphi_l: V'_j \rightarrow B(0, |\lambda|^{km_1/d} e^{-\alpha\omega l})$$

is a conformal map. Let

$$V_j = \{z \in V'_j : |\varphi_l(z)| \leq |\lambda|^{km_1/d} e^{-\alpha l}\}.$$

By the Koebe distortion theorem,

$$\frac{\text{area}(V_j)}{\text{area}(V'_j)} \leq C e^{-2\alpha(1-\omega)l}.$$

The previous lemma implies that $K_{\alpha, m, l}^{m_1} \subset \bigcup_j V_j$. Since V'_j are pairwise disjoint, we obtain the desired estimate. ■

Now we can prove Theorem 4.2.

Proof of Theorem 4.2. It is sufficient to show that for every $\alpha > 0$ small, and for every $\varepsilon > 0$, we have $\text{Vol}(B(0, r_0) \setminus \Lambda_{2\alpha}) < \varepsilon$. We shall use Lemma 4.5 to estimate $\text{Vol}(B(0, r_0) \setminus \Lambda_{2\alpha})$. By Lemma 2.7, there exists a constant $\theta = \theta(\alpha) > 0$ such that for every $m \geq 0$ and every $z \in B(0, |\lambda|^m r_0)$, we have $|\xi_l(z)| > e^{-\alpha l} |\lambda|^{km/d}$ provided that $l \leq \theta m$. In other words, $K_{\alpha, m, l} = \emptyset$ provided that $l \leq \theta m$. By Lemma 4.8,

$$\text{area}(K_{\alpha, m, l}) \leq \sum_{m_1=m}^{\infty} e^{-\alpha\gamma l} \text{area}(B(0, |\lambda|^{m_1} r_0)) \leq C e^{-\alpha\gamma l} \text{area}(B(0, |\lambda|^m r_0)).$$

Thus by Lemma 4.5 (2), for m large enough, we have

$$\text{Vol}(B(0, |\lambda|^m r_0) \setminus \Omega_{\alpha, m}) \leq \sum_{l=\theta m}^{+\infty} \text{Vol}(K_{\alpha, m, l}) \leq Q e^{-\gamma\theta m} \text{Vol}(B(0, |\lambda|^m r_0)),$$

where $Q := C \sum_{l=1}^{+\infty} e^{-\gamma l}$.

Thus by Lemma 4.5 (1), for N large enough, we have

$$\begin{aligned} \text{Vol}(B(0, r_0) \setminus \Lambda_{2\alpha}) &\leq \sum_{m=N}^{+\infty} |\lambda|^{-2m} \text{Vol}(B(0, |\lambda|^m r_0) \setminus \Omega_{\alpha, m}) \\ &\leq \sum_{m=N}^{+\infty} |\lambda|^{-2m} Q e^{-\gamma\theta m} \text{Vol}(B(0, |\lambda|^m r_0)) \\ &= \sum_{m=N}^{+\infty} Q \pi |r_0|^2 e^{-\gamma\theta m} \leq \varepsilon. \end{aligned}$$

The conclusion follows. ■

5. Non-wandering Fatou components

In this section, we prove Theorem 1.1. The strategy is the following. We will first show that if a vertical disk $D \subset L_{z_0}$ centered at $x_0 = (z_0, w_0)$ is very close to the invariant line L (which means that the radius of D is much larger than $|z_0|$), then for every $n \geq 0$, we have $f_0^n(w_0) \in \xi_n(D)$. The proof of this proposition is again by using the binding argument. Now assume by contradiction there is a wandering Fatou component, by Theorems 1.4 and 1.5, we can select a vertical disk D contained in a wandering Fatou component, which is very close to the invariant line L . Finally, we will get a contradiction.

More precisely, we first prove the following result.

Proposition 5.1. *Let $f(z, w) = (\lambda z, w^d + c(z))$ be as in (1.1) such that f_0 has no attracting nor superattracting cycle in \mathbb{C} . There exists $\rho > 0$ such that the following hold. Let $D \subset L_{z_0}$ be a vertical disk centered at $x = (z_0, w_0)$ of radius $\delta \in (0, \rho)$ with $|z_0| < \delta^{2d}$. Then for every $0 < \lambda_0 < 1$, there is a constant $C = C(\lambda_0) > 0$ such that for each $n \geq 1$,*

$$B(f_0^n(w_0), C\lambda_0^n\delta) \subset \xi_n(D).$$

The proof of this proposition will be given after we prove two lemmas.

Lemma 5.2. *For any $\delta_0 > 0$ and $\lambda_0 \in (0, 1)$, there exist positive integers $N = N(\delta_0, \lambda_0)$, $\rho = \rho(\delta_0, \lambda_0) > 0$ and $C = C(\delta_0, \lambda_0) > 0$ such that, for every $w_0 \in \mathbb{C}$, the f_0 -orbit of w_0 is bounded, the following hold:*

- (1) *If $|f_0^i(w_0)| > \delta_0/2$ for every $0 \leq i < N$ and $|z_0|^k \leq \delta^2$ for some $\delta \in (0, \rho)$, then*

$$B(f_0^N(w_0), \lambda_0^N\delta) \subset \xi_N(\{z_0\} \times B(w_0, \delta)),$$

and

$$B(f_0^j(w_0), C\lambda_0^j\delta) \subset \xi_j(\{z_0\} \times B(w_0, \delta)), \quad 1 \leq j < N.$$

- (2) *If $n \leq N$ is a positive integer such that $|f_0^i(w_0)| > \delta_0/2$ for all $0 \leq i < n$ and $|f_0^n(w_0)| \leq \delta_0/2$, and $|z_0|^k \leq \delta^2$ for some $\delta \in (0, \rho)$, then*

$$B(f^n(w_0), \frac{\kappa_0\lambda_0^n\delta}{4}) \subset \xi_n(\{z_0\} \times B(w_0, \delta)),$$

where $\kappa_0 = \kappa_0(\lambda_0) > 0$ is a constant, and

$$B(f_0^j(w_0), C\lambda_0^j\delta) \subset \xi_j(\{z_0\} \times B(w_0, \delta)), \quad 1 \leq j < n.$$

Proof. Without loss of generality, we may assume $\lambda_0 \in (|\lambda|, 1)$. Let $\kappa = \kappa(\lambda_0^{1/2}, \delta_0/2)$ and $\kappa_0 = \kappa_0(\lambda_0^{1/2})$ be given by Proposition 2.1. Choose N such that if $\kappa\lambda_0^{N/2} > 4\lambda_0^N$.

Let $D = \{z_0\} \times B(w_0, \delta)$. For every $x \in D$, let s_x be the μ_0 -binding time of the pair $(x, (0, w_0))$. Let $s = \min_{x \in D} s_x$. Provided that δ is small enough, we have $s > N$.

Lemma 2.4 implies that $w \mapsto \xi_m(z_0, w)$ is univalent in $B(w_0, \delta)$ and hence $\xi_m(D)$ contains a disk centered at $\xi_m(x_0)$ with radius at least $R_m := 1/2|(f_0^m)'(w_0)|\delta$. Let $s_1 = N$ in case (1) and $s_1 = n$ in case (2). Let $x_0 = (z_0, w_0)$ and $y_0 = (0, w_0)$. By (2.10) and Proposition 2.1, we have

$$W(x_0, y_0, s_1) \leq \sum_{i=1}^{s_1} \frac{4|\lambda^{k(i-1)}z_0^k|}{|(f_0^i)'(w_0)|} \leq C|z_0|^k,$$

where C is a constant depending on λ_0 and δ_0 . By Lemma 2.5,

$$|\xi_{s_1}(x_0) - f_0^{s_1}(w_0)| \leq |(f_0^{s_1})'(w_0)|W(x_0, y_0, s_1) \leq C|(f_0^{s_1})'(w_0)||z_0|^k < \frac{R_{s_1}}{2},$$

provided that δ is small enough. Therefore,

$$\xi_{s_1}(D) \supset B\left(f_0^{s_1}(w_0), \frac{R_{s_1}}{2}\right).$$

The lemma follows since in case (1) we have $|Df_0^{s_1}(w_0)| > \lambda_0^N$ and in case (2) we have $|Df_0^{s_1}(w_0)| \geq \kappa_0\lambda_0^n$, and in both cases, $|Df_0^j(w_0)| \geq C\lambda_0^j$ for $1 \leq j < s_1$. ■

Lemma 5.3. *For each $\lambda_0 \in (0, 1)$ and each $K > 0$, there exist constants $C = C(\lambda_0, K) > 0$ and $\delta_0 > 0$ such that the following holds. Assume $|w_0| \leq \delta_0$ and $|z_0|^k < \delta^{d+1}$ for some $0 < \delta < \delta_0$. Then there exists a positive integer n such that*

$$B(f_0^{n+1}(w_0), K\lambda_0^{n+1}\delta) \subset \xi_{n+1}(\{z_0\} \times B(w_0, \delta)),$$

and

$$B(f_0^j(w_0), C\lambda_0^j\delta^d) \subset \xi_j(\{z_0\} \times B(w_0, \delta)), \quad 1 \leq j \leq n.$$

Proof. Fix constants $\lambda_0 \in (0, 1)$, $K > 0$ and a large positive integer s_0 such that $\lambda_0^{-s_0/2} \geq C_3^{-1}K$, where C_3 is a constant to be determined below. We may and will assume $\lambda_0 > |\lambda|$.

For each $j \geq 0$, write $x_j = (z_j, w_j) = f^j(z_0, w_0)$ and

$$D_j = f^j(\{z_0\} \times B(w_0, \delta)) \subset \{z_j\} \times \mathbb{C}.$$

Then $D_1 = \{z_1\} \times \xi_1(D_0) \supset \{z_1\} \times B(w_1, \delta_1)$, where

$$\delta_1 = \theta\delta \max(|w_0|, \delta)^{d-1} \tag{5.1}$$

for some uniform constant $\theta > 0$. Let μ_1 be as in (2.6) and let

$$s = \inf\{b_{\mu_1}((z, w), (0, c(0))) : |z| \leq |\lambda||z_0|, w \in \xi_1(D_0)\}.$$

Claim 1. Provided that δ_0 is small enough, for each $1 \leq j \leq s$,

$$\xi_{j+1}(D_0) \supset B\left(f_0^{j+1}(w_0), \frac{R_{j+1}}{4}\right),$$

where

$$R_{j+1} = |Df^j(z_1, w_1)(v)|\delta_1.$$

Proof of Claim 1. Indeed, as in the proof of the previous lemma, for each integer $0 \leq j \leq s$, we have $\xi_{j+1}(D_0) \supset \xi_j(D_1) \supset B(\xi_{j+1}(x_0), R_{j+1}/2)$. So it suffices to prove

$$|\xi_{j+1}(x_0) - f_0^{j+1}(w_0)| \leq \frac{R_{j+1}}{4}.$$

To prove this, we first apply Proposition 2.2 (1) to obtain a constant $C_1 > 0$ such that

$$|Df_0^i(c(0))| \geq 2C_1\lambda_0^i \quad \text{for all } i \geq 0. \tag{5.2}$$

By (2.10),

$$W((z_1, w_1), (0, w_1), j) \leq 4 \sum_{i=1}^j \frac{|\lambda|^{(i-1)k} |z_1|^k}{|Df_0^i(c_0)|} \ll \delta_1.$$

By (2.8),

$$b_{\mu_0}((z_1, w_1), (0, w_1)) \geq \min(b_{\mu_1}((z_1, w_1), (0, c(0))), b_{\mu_1}((0, w_1), (0, c(0)))) \geq s.$$

By Lemma 2.5, it follows that

$$|\xi_j(z_1, w_1) - f_0^{j+1}(w_0)| \leq |Df^j(z_1, w_1)(v)|W((z_1, w_1), (0, w_1), j) \leq \frac{R_{j+1}}{4}.$$

The claim is proved. ■

By Lemma 2.4, for each $1 \leq j \leq s$, we have $|Df^j(z_1, w_1)| \geq |Df_0^j(c(0))|/2$. So by (5.2) and (5.1), we obtain

$$\xi_{j+1}(D_0) \supset B(f_0^{j+1}(w_0), C_1\theta\lambda_0^i\delta^d). \tag{5.3}$$

Claim 2. Provided that $\delta_0 > 0$ is small enough, there exists a positive integer $n \leq s$ such that $R_{n+1} > 2K\lambda_0^n\delta$.

Proof of Claim 2. To prove the claim, let (z'_1, w'_1) be such that $|z'_1| \leq |\lambda z_0|$ and $w'_1 \in \xi_1(D_0)$ with

$$b_{\mu_1}((z'_1, w'_1), (0, c(0))) = s.$$

By Lemma 2.6, there exists a positive integer $s_0 \leq n \leq s$ such that

$$|Df^n(z'_1, w'_1)(v)| \geq \lambda_0^{n/2}\delta^{-(d-1)},$$

where

$$\delta'^d = \min(|w'_1 - c(0)|, |z'_1|^k).$$

Let $w'_0 \in B(w_0, \delta)$ be such that $w'_1 = \xi_1(z_0, w'_0)$. Then

$$\begin{aligned} |w'_1 - c(0)| &= |w_0'^d + c(z_0) - c(0)| \\ &\leq (|w_0| + \delta)^d + 2|z_0|^k \leq (|w_0| + \delta)^d + 2\delta^{d+1}. \end{aligned}$$

Since $|z_1|^k \leq |z_0|^k < \delta^{d+1}$, there exists a constant $C_2 > 0$ such that

$$\delta' \leq C_2 \max(|w_0|, \delta).$$

It follows, by the definition of δ_1 , that

$$|Df^n(z'_1, w'_1)(v)|\delta_1 \geq \lambda_0^{n/2} C_2^{-(d-1)} \theta \delta.$$

By Lemma 2.4,

$$|Df^n(z'_1, w'_1)(v)| \leq 2|Df_0^n(c(0))| \leq 4|Df^n(z_1, w_1)(v)|.$$

Therefore,

$$R_{n+1} = |Df^n(z_1, w_1)(v)|\delta_1 \geq \frac{1}{4}|Df^n(z'_1, w'_1)(v)|\delta_1 \geq C_3 \lambda_0^{n/2} \delta,$$

where C_3 is a constant. By our choice of s_0 (at the beginning of the proof), this implies the claim. ■

By Claims 1 and 2 and (5.3), the lemma follows. ■

Now we can prove Proposition 5.1.

Proof of Proposition 5.1. We may assume that $\lambda_0^{d+1} > |\lambda|$. Fix such $\lambda_0 \in (|\lambda|, 1)$ and let $\kappa_0 = \kappa_0(\lambda_0) > 0$ be given by Lemma 5.2 (2). Choose $K = 4/\kappa_0$. Let $\delta_0 = \delta_0(\lambda_0, K) > 0$ be given by Lemma 5.3. Let $\delta > 0$ be small. Suppose that we have found a non-negative integer m such that

$$\xi_m(D) \supset B(f_0^m(w_0), \lambda_0^m \delta)$$

and

$$\xi_j(D) \supset B(f_0^j(w_0), C \lambda_0^j \delta^d) \quad \text{for } 1 \leq j < m.$$

Note that $m = 0$ satisfies these properties. For $\delta(m) = \lambda_0^m \delta$, we have

$$|z_m| = |\lambda|^m |z_0| \leq |\lambda|^m \delta^{2d} < \delta(m)^2,$$

so that we may apply Lemma 5.2 to the f -orbit of $(z_m, f_0^m(w_0))$. If $|f_0^j(w_0)| \geq \delta_0/2$ for all $j \geq m$, then by Lemma 5.2 (1) we obtain

$$\xi_j(D) \supset \xi_{j-m}(\{z_m\} \times \xi_m(D)) \supset B(f_0^j(w_0), C\lambda_0^j\delta) \quad \text{for all } j,$$

and hence we are done. Otherwise, let $t \geq m$ be minimal such that $|f_0^t(w_0)| < \delta_0/2$. Then, by Lemma 5.2 (2), we obtain

$$\xi_t(D) \supset B\left(f_0^t(w_0), \frac{\kappa_0\lambda_0^t\delta}{4}\right)$$

together with

$$\xi_j(D) \supset B(f_0^j(w_0), C\lambda_0^j\delta) \quad \text{for } m \leq j < t.$$

For $\delta(t) = \kappa_0\lambda_0^t\delta$, we have

$$|z_t| = |\lambda|^t|z_0| \leq |\lambda|^t\delta^{2d} < \delta(t)^{d+1}$$

holds, so applying Lemma 5.3 to the f -orbit of $(z_t, f_0^t(w_0))$, we obtain a positive integer $m' > t$ such that

$$\xi_{m'}(D) \supset B\left(f_0^{m'}(w_0), \frac{K\kappa_0\lambda_0^{m'}\delta}{4}\right) \supset B(f_0^{m'}(w'), \lambda_0^{m'}),$$

and

$$\xi_j(D) \supset B(f_0^j(w_0), C\lambda_0^j\delta^d) \quad \text{for } t < j < m'.$$

Repeat the argument for m' instead of m and continue, the conclusion follows. ■

Remark 5.4. Proposition 5.1 is even new in dimension one. Taking $D \subset L$, we get the following one-dimensional result: let $f_0(w) = w^d + c$ as in (2.1), then for every $0 < \lambda_0 < 1$, there are constants $C = C(\lambda_0) > 0$ and $\delta_0 = \delta_0(\lambda_0) > 0$ such that for every w not be contained in an attracting basins of f_0 , $\delta < \delta_0$ and $n \geq 0$, we have

$$B(f_0^n(w), C\lambda_0^n\delta^d) \subset f_0^n(B(w, \delta)).$$

This improves a result of Denker–Przytycki–Urbanski [7, Lemma 3.4] for uni-critical polynomials. In [7], it is proved that for rational map f_0 , there are constants $0 < L < 1$, $\rho > 0$ and $\delta_0 > 0$ such that $B(f_0^n(w), L^n\delta^\rho) \subset f_0^n(B(w, \delta))$ for w in the Julia set of f_0 and $\delta < \delta_0$. But the constants L and ρ are not controlled.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. First it is proved in [12] that every Fatou component of f_0 can be extended to a two-dimensional Fatou component of f . These kind of Fatou components are clearly non-wandering, since their restrictions on L is non-wandering,

by Sullivan’s theorem [24]. Next we show that every Fatou component of f in a small neighborhood of L is an extension of a Fatou component of f_0 .

Since f_0 is unicritical, if f_0 has an attracting nor superattracting cycle in \mathbb{C} , then f_0 is hyperbolic. In this case it is not hard to show, by the shadowing lemma, that the Fatou set of f is the union of basins of attracting cycles. See also [18] or [13]. In the following we assume that f_0 has no attracting nor superattracting cycle in \mathbb{C} .

We argue by contradiction. Assume there is a Fatou component Ω such that Ω is not an extension of a one-dimensional Fatou component, i.e. Ω is wandering. Clearly, the f -orbit of Ω is uniformly bounded. Fix a small constant $\alpha > 0$ such that

$$|\lambda|^k < e^{-2d^2\alpha} \tag{5.4}$$

and fix λ_0 such that $|\lambda|^k e^{2d^2\alpha} < \lambda_0 < 1$. Since Ω is open, it has positive volume. By Theorem 1.5, there exists $x' = (z'_0, w'_0) \in \Omega$ such that for n large enough, $|\xi_n(x')| \geq e^{-n\alpha}$. Then there exists an integer $N > 0$ such that the orbit of $x'_N := (z'_N, w'_N)$ is tame. (See the definition of tame orbits in Section 1.) By Theorem 1.4, for every $n \geq 1$

$$|Df^n(x'_N)(v)| \geq C \lambda_0^n e^{-(d-1)(n+N)\alpha},$$

where $C = C(\lambda_0) > 0$.

Claim. There exists a constant $\kappa > 0$ and an arbitrarily large n such that

$$\{z'_n\} \times B(w'_n, \kappa \lambda_0^n e^{-(d+1)\alpha n}) \subset \Omega_n =: f^n(\Omega).$$

Proof of the claim. To prove this claim, let $\varphi_m(w) = \xi_m(z'_N, w)$ for each $m \geq 0$ and let $\varepsilon_0 > 0$ be so small that $\{z'_N\} \times B(w'_N, \varepsilon_0) \subset \Omega_N$. We distinguish two cases.

Case 1. There exists $\varepsilon \in (0, \varepsilon_0)$ such that φ_m is univalent on $B(w'_N, \varepsilon e^{-\alpha m})$ for all $m \geq 1$. Then, by the Koebe 1/4 theorem, $\{z'_{N+m}\} \times B(w'_{N+m}, R_m) \subset \Omega_{N+m}$, where

$$R_m = |Df^m(z'_N, w'_N)(v)| \frac{e^{-\alpha m} \varepsilon}{4}.$$

Then the conclusion of the claim holds for all n large enough.

Case 2. For each $\varepsilon \in (0, \varepsilon_0)$, there exists a minimal $m = m(\varepsilon)$ such that φ_{m+1} is not univalent on $B(w'_N, \varepsilon e^{-\alpha(m+1)})$. Then $m \rightarrow \infty$ as $\varepsilon \rightarrow 0$. By the minimality of m , φ_m is univalent on $B(w'_N, \varepsilon e^{-\alpha m})$ and we have $\{w'_{m+N}, 0\} \subset \varphi_m(B(w'_N, \varepsilon e^{-\alpha(m+1)}))$, so that its diameter is at least $e^{-\alpha(m+N)}$. By the Koebe distortion theorem, it follows that the image $\varphi_m(B(w'_N, \varepsilon e^{-\alpha(m+1)}))$ contains a ball centered at w'_{m+N} and of radius at least $\rho(\alpha)e^{-\alpha(m+N)}$, where $\rho(\alpha) > 0$ is a constant. Then $n = m + N$ satisfies the requirement of the claim. The claim is proved. ■

Take n_0 large for which the property in the claim is satisfied. Rename z'_{n_0} as z_0 and choose $w_0 \in F(f_0)$ very close to w'_{n_0} such that

$$w_0 \in B\left(w'_{n_0}, \frac{1}{2}\kappa\lambda_0^{n_0}e^{-(d+1)\alpha n_0}\right).$$

(We can choose such w_0 since the Fatou set $F(f_0)$ is dense in \mathbb{C} .)

Let $\delta := \frac{1}{2}\kappa\lambda_0^{n_0}e^{-(d+1)\alpha n_0}$ and let $D := \{z_0\} \times B(w_0, \delta)$, which is a vertical disk. By the claim, D is contained in the Fatou component Ω_{n_0} . Moreover, by the choice of α in (5.4), the vertical disk D satisfies the condition in Proposition 5.1, i.e. we have $\delta \in (0, \rho)$ and $|z_0| < \delta^{2d}$, provided that n_0 was chosen large enough.

By Proposition 5.1, it follows, for every $n \geq 1$, that

$$f_0^n(w_0) \in \xi_n(D) \quad \text{and} \quad f^n(D) \subset \Omega_{n_0+n}. \tag{5.5}$$

By the classification of Fatou components in one dimension, $f_0^n(w_0)$ converges either to a parabolic cycle, a Siegel disk or to ∞ . In the latter two cases, by (5.5), for n large enough $f^n(D)$ must intersect with an extension of a one-dimensional Fatou component, a contradiction. In the parabolic cycle case, it is proved in [12, Theorem 3.3], which is a special case of Ueda [26, Section 7.2], that there exist attracting petals of the form

$$U_j = \{|z| < \varepsilon\} \times (U_j \cap \{z = 0\}),$$

such that $f^n(x)$ converges to the parabolic cycle p implies $f^N(x) \in U_j$ for some N and j . Moreover, these attracting petals are contained in the extension of the one-dimensional parabolic basin. Thus by (5.5), for n large enough $f^n(D)$ must intersect with an attracting petal, a contradiction. The proof is completed. ■

Proof of Theorem 1.2. By Lilov’s result [16], f does not have wandering Fatou components in the attracting basin of the line at infinity. Let Ω be a Fatou component with bounded orbit, we need to show that Ω is non-wandering. Let π be the projection to the z -coordinate, then $\pi(\Omega)$ is contained in a bounded Fatou component of p . Since $\pi(\Omega)$ is connected, $\pi(\Omega)$ is contained in a Fatou component of p . Since p is a polynomial without parabolic periodic points and Siegel periodic points, all Fatou components of p are attracting or superattracting basins. The superattracting case is again covered by [16]. We can assume that $\pi(\Omega)$ is contained in the basin of an attracting periodic point z_0 of p with periods s . Replacing f by f^s , z_0 will be a fixed point of p . By our assumption that the critical curve of f^s in \mathbb{C}^2 has a unique transversal intersection with the vertical line $L := \{z = z_0\} \subset \mathbb{C}^2$, f^s can be conjugated in a neighborhood U of the vertical line $\{z = z_0\}$, to the form as in (1.1). Clearly, there exists $N \geq 1$ such that $U \cap f^N(\Omega) \neq \emptyset$. Apply Theorem 1.1 we get that $U \cap f^N(\Omega)$ is contained in a non-wandering Fatou component, This implies that Ω itself is non-wandering. ■

Proof of Theorem 1.3. It is clear that f can be extended to a holomorphic endomorphism on \mathbb{P}^2 . We need to show that when $|\lambda| < 1$, f does not have wandering Fatou components. When $|\lambda| < 1$, p has no parabolic periodic points, nor Siegel periodic points. The only attracting periodic point of p is 0, which is fixed. The critical curve of f in \mathbb{C}^2 has two components $\{z = -\lambda/2\}$ and $\{w = 0\}$. So the critical curve has a unique transversal intersection with the invariant vertical line $\{z = 0\}$. (The intersection point is $(0, 0)$). So our map f satisfies all the assumptions in Theorem 1.2, and we conclude the result by applying Theorem 1.2. ■

6. The multicritical case

In this section, we discuss the situation when f is multicritical (i.e. when f is a regular polynomial skew product with an invariant attracting line). For a regular polynomial skew product with an invariant attracting line, we may locally conjugate f to the following form

$$f(z, w) = (\lambda z, F(z, w)), \tag{6.1}$$

where $|\lambda| < 1$ and $F(z, w) = w^d + \sum_{i=0}^{d-1} c_i(z)w^i$ is a polynomial in w with coefficients holomorphic in z , $d \geq 2$. Let f_0 be the one-dimensional map

$$f_0(w) = w^d + \sum_{i=0}^{d-1} c_i(0)w^i,$$

which is the restriction of f on the invariant line. It is natural to ask whether our Theorem 1.1 hold in this more general setting.

Problem 6.1. Let $f(z, w) = (\lambda z, F(z, w))$ as in (6.1). Is that true that every Fatou component of f is an extension of a Fatou component of f_0 ?

A positive answer will imply that there is no wandering Fatou component. Notice that most of our techniques have multicritical version (e.g. the binding argument and the parameter exclusion technique). However, there is a major difference between unicritical and multicritical polynomials: it was proved in [15] that for unicritical polynomial f_0 the lower Lyapunov exponent of the critical value c is always non-negative, provided c is not contained in an attracting basin of f_0 . But in the multicritical case, there are semi-hyperbolic polynomials carrying a critical value on the Julia set with $-\infty$ Lyapunov exponent, see Przytycki–Rohde [22].

However, assuming the following two conditions, we expect our methods can apply and the answer of Problem 6.1 is yes. Let

$$C(f) := \{x \in \mathbb{C}^2 : Df(x) \text{ is not invertible}\}$$

be the critical curve, and let $L := \{z = 0\}$.

(1) *Lyapunov exponent.* For every critical value c of f_0 such that c is contained in the Julia set of f_0 , its lower Lyapunov exponent satisfies

$$\chi_-(c) := \liminf_{n \rightarrow +\infty} \frac{1}{n} \log |Df_0^n(c)| > \log \lambda.$$

(2) *Non-degeneracy condition.* The critical curve $C(f)$ intersects L transversally, and the following power series converges to a non-zero constant for every critical value c of f_0 which is contained in the Julia set of f_0 ,

$$G(c) + \sum_{i=1}^{+\infty} \frac{\lambda^i G(f_0^i(c))}{(f_0^i)'(c)} \neq 0, \tag{6.2}$$

where G is a polynomial defined by $G(w) := \sum_{i=0}^{d-1} c'_i(0)w^i$.

We give some explanation of the second condition. By Proposition 2.2, for a unicritical polynomial $f_0(w) = w^d + c$ satisfying f_0 has no attracting cycle in \mathbb{C} , we have

$$1 + \sum_{i=1}^{+\infty} \frac{\lambda^i}{(f_0^i)'(c)} \neq 0$$

for every $|\lambda| < 1$. Thus if f is unicritical of the form (1.1) such that $c'(0) \neq 0$, the polynomial G in (6.2) is a non-zero constant, and the non-degeneracy condition (6.2) is automatically true. A similar non-degeneracy condition is raised in other contexts; see, for instance, Tsujii [25] and Gao–Shen [9].

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References

[1] M. Astorg and F. Bianchi, [Hyperbolicity and bifurcations in holomorphic families of polynomial skew products](#). *Amer. J. Math.* **145** (2023), no. 3, 861–898 Zbl 1527.37049 MR 4596179

[2] M. Astorg and L. Boc Thaler, [Dynamics of skew-products tangent to the identity](#). *J. Eur. Math. Soc.* (2024), published online first

[3] M. Astorg, L. Boc Thaler, and H. Peters, [Wandering domains arising from Lavaurs maps with Siegel disks](#). *Anal. PDE* **16** (2023), no. 1, 35–88 Zbl 1518.32004 MR 4578525

- [4] M. Astorg, X. Buff, R. Dujardin, H. Peters, and J. Raissy, [A two-dimensional polynomial mapping with a wandering Fatou component](#). *Ann. of Math. (2)* **184** (2016), no. 1, 263–313 Zbl [1368.37055](#) MR [3505180](#)
- [5] M. Benedicks and L. Carleson, [On iterations of \$1 - ax^2\$ on \$\(-1, 1\)\$](#) . *Ann. of Math. (2)* **122** (1985), no. 1, 1–25 Zbl [0597.58016](#) MR [0799250](#)
- [6] P. Berger and S. Biebler, [Emergence of wandering stable components](#). *J. Amer. Math. Soc.* **36** (2023), no. 2, 397–482 Zbl [1522.37056](#) MR [4536902](#)
- [7] M. Denker, F. Przytycki, and M. Urbański, [On the transfer operator for rational functions on the Riemann sphere](#). *Ergodic Theory Dynam. Systems* **16** (1996), no. 2, 255–266 Zbl [0852.46024](#) MR [1389624](#)
- [8] R. Dujardin, [Geometric methods in holomorphic dynamics](#). In *ICM 2022. International Congress of Mathematicians. Vol. 5. Sections 9–11*, pp. 3460–3482, EMS Press, Berlin, 2023 Zbl [1545.37001](#) MR [4680370](#)
- [9] B. Gao and W. Shen, [Summability implies Collet–Eckmann almost surely](#). *Ergodic Theory Dynam. Systems* **34** (2014), no. 4, 1184–1209 Zbl [1316.37023](#) MR [3227153](#)
- [10] D. Hahn and H. Peters, [A polynomial automorphism with a wandering Fatou component](#). *Adv. Math.* **382** (2021), article no. 107650 Zbl [1487.37065](#) MR [4224048](#)
- [11] M. V. Jakobson, [Absolutely continuous invariant measures for one-parameter families of one-dimensional maps](#). *Comm. Math. Phys.* **81** (1981), no. 1, 39–88 Zbl [0497.58017](#) MR [0630331](#)
- [12] Z. Ji, [Non-wandering Fatou components for strongly attracting polynomial skew products](#). *J. Geom. Anal.* **30** (2020), no. 1, 124–152 Zbl [1461.37050](#) MR [4058508](#)
- [13] Z. Ji, [Non-uniform hyperbolicity in polynomial skew products](#). *Int. Math. Res. Not. IMRN* (2023), no. 10, 8755–8799 Zbl [1547.37060](#) MR [4589086](#)
- [14] G. Levin, [On an analytic approach to the Fatou conjecture](#). *Fund. Math.* **171** (2002), no. 2, 177–196 Zbl [0984.37046](#) MR [1880383](#)
- [15] G. Levin, F. Przytycki, and W. Shen, [The Lyapunov exponent of holomorphic maps](#). *Invent. Math.* **205** (2016), no. 2, 363–382 Zbl [1352.37136](#) MR [3529117](#)
- [16] K. Lilov, [Fatou theory in two dimensions](#). PhD thesis, University of Michigan, 2004
- [17] H. Peters and J. Raissy, [Fatou components of elliptic polynomial skew products](#). *Ergodic Theory Dynam. Systems* **39** (2019), no. 8, 2235–2247 Zbl [1421.37022](#) MR [3977341](#)
- [18] H. Peters and I. M. Smit, [Fatou components of attracting skew-products](#). *J. Geom. Anal.* **28** (2018), no. 1, 84–110 Zbl [1397.32003](#) MR [3745850](#)
- [19] H. Peters and L. R. Vivas, [Polynomial skew-products with wandering Fatou-disks](#). *Math. Z.* **283** (2016), no. 1–2, 349–366 Zbl [1339.37039](#) MR [3489070](#)
- [20] F. Przytycki, [Lyapunov characteristic exponents are nonnegative](#). *Proc. Amer. Math. Soc.* **119** (1993), no. 1, 309–317 Zbl [0787.58037](#) MR [1186141](#)
- [21] F. Przytycki and S. Rohde, [Porosity of Collet–Eckmann Julia sets](#). *Fund. Math.* **155** (1998), no. 2, 189–199 Zbl [0908.58054](#) MR [1606527](#)
- [22] F. Przytycki and S. Rohde, [Rigidity of holomorphic Collet–Eckmann repellers](#). *Ark. Mat.* **37** (1999), no. 2, 357–371 Zbl [1034.37026](#) MR [1714763](#)
- [23] R. K. W. Roeder, [A dichotomy for Fatou components of polynomial skew products](#). *Conform. Geom. Dyn.* **15** (2011), 7–19 Zbl [1218.32009](#) MR [2769221](#)

- [24] D. Sullivan, [Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains](#). *Ann. of Math. (2)* **122** (1985), no. 3, 401–418
Zbl [0589.30022](#) MR [0819553](#)
- [25] M. Tsujii, [Positive Lyapunov exponents in families of one-dimensional dynamical systems](#). *Invent. Math.* **111** (1993), no. 1, 113–137 Zbl [0787.58029](#) MR [1193600](#)
- [26] T. Ueda, [Local structure of analytic transformations of two complex variables. I](#). *J. Math. Kyoto Univ.* **26** (1986), no. 2, 233–261 Zbl [0611.32001](#) MR [0849219](#)

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