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Rigidity of totally geodesic hypersurfaces in negative curvature

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Abstract. Let M be a closed hyperbolic manifold containing a totally geodesic hypersurface S, and let N be a closed Riemannian manifold homotopy equivalent to M with sectional curvature bounded above by -1. We study the following question: if $\pi_1(S)$ can be represented by a totally geodesic hyperbolic hypersurface in N, then must N be isometric to M? We show that many such S are rigid in the sense that the answer to this question is positive. On the other hand, we construct examples of S for which the answer is negative.

1. Introduction

A lattice Γ in a nonpositively curved symmetric space X is a discrete subgroup of the isometry group of X with the property that the quotient $\Gamma\backslash X$ has finite volume. The *rank* of a nonpositively curved symmetric space is the maximum dimension of a totally geodesic submanifold isometric to flat $\mathbb{R}^k$ that it contains. In groundbreaking work, Margulis showed that every lattice in a higher rank (rank greater than one) symmetric space is arithmetic [26]. This means that it can be constructed by a procedure analogous to taking the integer points of a group of matrices. Later, Corlette proved an arithmeticity theorem for lattices in quaternionic and octonionic hyperbolic spaces [11]. There are, however, examples of non-arithmetic lattices in rank one negatively curved symmetric spaces [19], and it was not until quite recently that an analog of Margulis's arithmeticity theorem was found in the setting of real and complex hyperbolic manifolds.

A closed, immersed, totally geodesic submanifold of a Riemannian manifold is called *maximal* if it is not contained in another closed, immersed, totally geodesic submanifold. It has recently been shown that the presence of infinitely many maximal totally geodesic submanifolds of dimension at least two in the quotient actually forces a real or complex hyperbolic lattice to be arithmetic.

This was proved by Margulis–Mohamaddi [32] for lattices in three-dimensional real hyperbolic space, Bader–Fisher–Miller–Stover [2, 3] for lattices in real and complex

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hyperbolic spaces of any dimension, and Baldi–Ulmmo [4] in the complex hyperbolic case, assuming that the totally geodesic submanifolds are subvarieties (see [5, 20, 29] for results in infinite covolume and [38] for a result in moduli space). This answered a question asked independently by McMullen, Reid for the hyperbolic 3-manifolds (see [12, Question 7.6], [30, Question 8.2].)

The attempt to find analogs outside the world of locally symmetric spaces for these kinds of rigidity statements has proven to be a fruitful line of inquiry [16,17,37]. With this in mind, we state the following question that was asked by David Fisher[1].

Question 1.1. Must a closed, negatively curved Riemannian manifold N that contains infinitely many maximal totally geodesic hypersurfaces be real hyperbolic (and thus arithmetic)?

One goal of this paper is to give an affirmative answer to this question under additional assumptions on the topology and curvature of N and the totally geodesic hypersurfaces that it contains (see Theorem 1.2). Although the setting in which we work is more restricted, it has the merit that our main theorems can also be viewed as rigidity statements for minimal hypersurfaces. This was in fact the original motivation for the paper, and it is explained more fully in Section 3.

We say that a hypersurface Σ in a Riemannian manifold N is *well distributed* in N if every point in the universal cover $\widetilde{N}$ of N is contained in an embedded solid hypercube, all of whose hyperfaces lie in lifts of Σ to $\widetilde{N}$. The following is our first main theorem.

Theorem 1.1. *Let N be a Riemannian manifold with sectional curvature at most -1 that contains a totally geodesic hyperbolic hypersurface Σ. Then, if Σ is well distributed in N, N must be hyperbolic.*

Let M be a closed hyperbolic manifold that contains infinitely many totally geodesic hypersurfaces Σ_k (see [25,31], for examples). Then, using Theorem 1.1, we can prove the following theorem.

Theorem 1.2. *Fix a closed hyperbolic manifold M containing infinitely many totally geodesic hypersurfaces. Then, there exists $K \in \mathbb{N}$ (depending on M and an ordering $\Sigma_1, \Sigma_2, \ldots$ of the totally geodesic hypersurfaces in M) so that the following holds. Fix Σ_k for $k > K$. Let N be a Riemannian manifold homotopy equivalent to M with sectional curvature at most -1. Assume that there is a totally geodesic hyperbolic hypersurface in N that represents $\pi_1(\Sigma_k)$ in $\pi_1(N)$, where we have used the homotopy equivalence to identify $\pi_1(M)$ and $\pi_1(N)$. Then, N is isometric to M in its hyperbolic metric.*

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[1] Personal communication.

A subgroup H of $\pi_1(N)$ is said to be *represented* by a submanifold Σ of N if $\pi_1(\Sigma)$ injectively includes to a subgroup conjugate to H under the map on fundamental groups induced by inclusion. In Section 3, we explain how the corollary below follows from Theorem 1.2.

Corollary 1.1. *Fix M as in the previous theorem. Then, there exists $K \in \mathbb{N}$ (depending on M and the ordering of the totally geodesic hypersurfaces $\Sigma_1, \Sigma_2, \ldots$) so that the following holds. Fix Σ_k for $k > K$. Let N be a Riemannian manifold homotopy equivalent to M with sectional curvature at most -1. Assume that $\pi_1(\Sigma_k)$ cannot be represented by an immersed hypersurface with volume smaller than the volume of Σ_k in its hyperbolic metric. Then, N is isometric to M in its hyperbolic metric.*

Although it is unclear how sharp Theorem 1.2 is certainly, some assumption on the hypersurface Σ_k in the model case is needed by the construction in Section 6. To the author's knowledge, Corollary 1.1 and the construction in Section 6 give the first example of a rigidity phenomenon for minimal hypersurfaces that is sensitive to how the minimal hypersurface is distributed in the ambient space. We comment that Farrell–Jones proved that if N is a closed manifold homotopy equivalent to a closed hyperbolic manifold M of dimension above four, then N and M must be homeomorphic [14] but not necessarily diffeomorphic [15]. The sectional curvature of such an N can, moreover, be taken to be pinched between $-1-\varepsilon$ and -1 for any $\varepsilon > 0$. On the other hand, all closed negatively curved 3-manifolds have hyperbolic metrics by geometrization.

The results of this paper are evidence for the following conjecture.

Conjecture 1.1. For every $D > 0$ (resp., $V > 0$) and $k \in \mathbb{N}$, there exists $A = A(D, k)$ (resp., $A = A(V, k)$) so that the following holds. Let N be a closed Riemannian manifold of dimension k, sectional curvature bounded above by -1, and diameter at most D (resp., volume at most V.) Then, if N contains a totally geodesic hyperbolic hypersurface with volume greater than A, N must be isometric to a hyperbolic manifold.

One could formulate a similar conjecture for totally geodesic submanifolds Σ of higher codimension. In this case, one would need to modify the statement by requiring that Σ is not contained in some higher-dimensional totally geodesic submanifold with small volume, and one would need to allow for the possibility that the universal cover of N is some other negatively curved symmetric space. It might also be interesting to drop the compactness assumption on N.

1.1. Related work

In contrast to the examples described in Section 6, there is a general rigidity statement for totally geodesic surfaces in the setting of positive curvature lower bounds in three dimensions, due to Mazet–Rosenberg [28]. Their work was inspired by a similar statement for geodesics in Riemannian 2-spheres, due to Calabi (see [1]). Mazet recently proved a version for totally geodesic spheres in higher codimension [27]. We also mention that Espinar–Rosenberg obtained rigidity statements for totally umbilic surfaces in negatively curved 3-manifolds [13].

There are many examples of closed hyperbolic manifolds that contain closed totally geodesic hypersurfaces (see [25] for the three-dimensional case, and [31], for examples, coming from so-called arithmetic hyperbolic manifolds of simplest type.) An arithmetic hyperbolic manifold that contains one totally geodesic hypersurface must contain infinitely many maximal totally geodesic hypersurfaces [24, 34]. Every even-dimensional closed arithmetic hyperbolic manifold contains infinitely many maximal totally geodesic hypersurfaces. In the other direction and as mentioned above, Bader–Fisher–Miller–Stover [2] and Margulis–Mohamaddi (the latter in dimension 3) [32] proved that a finite volume hyperbolic manifold with infinitely many maximal totally geodesic hypersurfaces must be arithmetic. There are examples of closed hyperbolic 3-manifolds that contain no totally geodesic surfaces [25]. It is an open problem whether there are closed hyperbolic manifolds in dimension above three that contain no totally geodesic hypersurfaces.

Let M be a closed hyperbolizable 3-manifold. Calegari–Marques–Neves introduced an entropy functional on metrics g defined by asymptotic counts of minimal surfaces in (M, g), and proved it to be uniquely minimized at the hyperbolic metric over all metrics with sectional curvature at most -1 [9]. The way we move information obtained via homogenous dynamics from constant curvature to variable curvature in the proof of Theorem 1.2 is inspired by their ideas.

Results that assume a negative sectional curvature upper bound often have analogs that instead assume a negative Ricci or scalar curvature lower bound. For example, in joint work with Neves [23], we establish the weaker analog of Corollary 1.1 and inequality (3.1) below. Let (M, g_{hyp}) be a closed hyperbolic 3-manifold containing a sequence of closed totally geodesic surfaces Σ_n with areas tending to infinity. Then, for any metric g on M with scalar curvature at least -6, the liminf of the ratio between the minimal area of a surface in (M, g) homotopic to Σ_n and

$$\text{Area}_{g_{\text{hyp}}}(\Sigma_n) = -2\pi\chi(\Sigma_n) \tag{1.1}$$

must be at least 1, and rigidity in the case of equality holds in that if it is equal to 1 then g is isometric to g_{hyp}. It is an interesting problem to determine whether this result could be strengthened to results similar to those of this paper.

1.2. Outline

This paper has four main parts. The reader is encouraged to assume that M and N are three dimensional on first reading.

In the first part, we cover some preliminaries (Section 2) and explain how Corollary 1.1 follows from Theorem 1.2 (Section 3).

In the second part, we prove Theorem 1.1. Let N be a negatively curved manifold containing a totally geodesic hyperbolic hypersurface Σ that satisfies the well-distribution condition. Every point in the universal cover of N is then contained in some solid embedded hypercube $\square$ whose hyperfaces lie in lifts $\tilde{\Sigma}$ of Σ to the universal cover. The crucial point for our argument is that there is an isometric embedding Φ of the boundary of $\square$ in $\mathbb{H}^n$. After constructing Φ, we define a new metric on $\mathbb{H}^n$ by gluing the unbounded connected component of $\mathbb{H}^n - \Phi(\partial\square)$ to $\square$. Using the Rauch comparison theorem, we are able to argue that this metric must have been isometric to $\mathbb{H}^n$. We thus conclude that the metric on the interior of $\square$ has constant curvature -1. Since the point we chose was arbitrary, this shows that N has constant curvature -1.

In the third part, we find examples of totally geodesic hypersurfaces Σ in closed hyperbolic manifolds M that satisfy a slightly stronger version of the well-distribution condition. Here, we rely on the work by Mozes–Shah that implies uniform distribution statements for totally geodesic hypersurfaces in closed hyperbolic manifolds. The strong well-distribution condition will imply that for any Riemannian manifold N homotopy equivalent to M with sectional curvature at most -1, any totally geodesic hypersurface in N corresponding to Σ will satisfy the well-distribution condition. This allows us to apply Theorem 1.1 to prove Theorem 1.2.

In the fourth part, we construct examples of closed Riemannian manifolds M with sectional curvature at most -1 that contain totally geodesic hyperbolic hypersurfaces without themselves being hyperbolic.

2. Background

In this section, we collect some facts that we will need in the paper.

2.1. Comparison geometry

Informally speaking, the Rauch comparison theorem states that if one Riemannian manifold is more negatively curved than the other, then its Jacobi fields will grow faster than those of the other. We will need the following corollary of that theorem (see [10, Corollary 1.35 and Remark 1.37]).

Corollary 2.1 (Corollary of Rauch comparison theorem). *Let (M, g) be a Riemannian manifold of dimension n and let $m \in M$. Assume the sectional curvature K_g of (M, g) satisfies $K_g \leq -1$. Choose a point $m_0 \in \mathbb{H}^n$, and let $I : TM_m \to T\mathbb{H}^n_{m_0}$ be a linear isometry between the tangent spaces. Let $c : [0, 1] \to M$ be a geodesic segment* *in* (M, g) *so that* $\exp_m$ *is a nonsingular embedding on* $s \cdot \exp_m^{-1}(c(t))$ *for* $0 \le s \le 1$ *and* $0 \le t \le 1$. *Then, we have*

$$L[c] \ge L[\exp_{m_0} \circ I \circ \exp_m^{-1}(c)]. \tag{2.1}$$

In other words, the distance between $c(0)$ *and* $c(1)$ *in* M *is at least the distance between* $\exp_{m_0} \circ I \circ \exp_m^{-1}(c(0))$ *and* $\exp_{m_0} \circ I \circ \exp_m^{-1}(c(1))$ *in* $\mathbb{H}^n$. *Moreover, in the case of equality, the image of the map*

$$(s,t) \mapsto \exp_m(s \cdot \exp_m^{-1}(c(t))), \quad 0 \le s \le 1,\ 0 \le t \le 1,$$

is a solid totally geodesic triangle, and every 2-plane tangent to the image has constant sectional curvature -1.

2.2. Negatively curved manifolds

Let M be a closed hyperbolic n-manifold. Then, M is the quotient of $\mathbb{H}^n$ by a discrete group isomorphic to $\pi_1(M)$ acting properly discontinuously and by isometries. If N is closed negatively curved Riemannian manifold, then M is homotopy equivalent to N if and only if $\pi_1(M)$ is isomorphic to $\pi_1(N)$. Assume that this is the case, and fix a homotopy equivalence $F : M \to N$. We can lift F to a map $\widetilde{F} : \widetilde{M} \to \widetilde{N}$ that is equivariant for the actions of $\pi_1(M)$ and $\pi_1(N)$ by deck transformations.

A map $Q : (X, d_X) \to (Y, d_Y)$ between two metric spaces (X, d_X) and (Y, d_Y) is a *quasi-isometry* if there exist constants C_1, C_2 such that for all x_1 and x_2 in X,

$$\frac{1}{C_1} d_X(x_1, x_2) - C_2 \le d_Y(Q(x_1), Q(x_2)) \le C_1 d_X(x_1, x_2) + C_2.$$

The map $\widetilde{F}$ defines a quasi-isometry between $\widetilde{M}$ and $\widetilde{N}$.

The boundary at infinity $\partial_\infty(X)$ of a simply connected negatively curved manifold X is the set of geodesic rays in X up to the equivalence relation of remaining at uniformly bounded distance for all time when parametrized by arc-length. For any $x \in X$, the exponential map defines a bijection between the unit sphere in the tangent space to X and $\partial_\infty(X)$. The sphere at infinity $\partial_\infty(X)$ is topologized so that this map is a homeomorphism for all x. Every geodesic in X is uniquely determined by its two endpoints in $\partial_\infty(X)$, and conversely any two distinct points in $\partial_\infty(X)$ determine a geodesic.

The fundamental groups $\pi_1(M)$ and $\pi_1(N)$ act by homeomorphisms on $\partial_\infty(\widetilde{M})$ and $\partial_\infty(\widetilde{N})$. Because M and N are closed, the map $\widetilde{F}$ induces a homeomorphism $\widetilde{F}_\infty : \partial_\infty \widetilde{M} \to \partial_\infty \widetilde{N}$ between the boundaries at infinity of $\widetilde{M}$ and $\widetilde{N}$, that is equivariant for the actions of $\pi_1(M)$ and $\pi_1(N)$.

Suppose that Σ and Σ_N are closed totally geodesic hyperbolic hypersurfaces of M and N so that $F_*\pi_1(\Sigma)$ is conjugate to $\pi_1(\Sigma_N)$ in $\pi_1(N)$. Let $\widetilde{\Sigma}$ be a lift of Σ to $\widetilde{M}$, and suppose that the intersection

$$G = \bigcap_{i=1}^{n-k} \gamma_i \widetilde{\Sigma}$$

is transverse and nonempty for some $\gamma_1, \dots, \gamma_{n-k} \in \pi_1(M)$. Then G is a k-dimensional totally geodesic subspace in $\widetilde{M} \cong \mathbb{H}^n$. The embedded $n-2$-sphere $\widetilde{F}_\infty(\partial_\infty(\gamma_i \widetilde{\Sigma}))$ in $\partial_\infty(N)$ bounds a unique totally geodesic hyperplane that lifts Σ_N, which by an abuse of notation we call $\gamma_i \widetilde{\Sigma}_N$. It will be important for us that

$$\bigcap_{i=1}^{n-k} \gamma_i \widetilde{\Sigma}_N$$

is a totally geodesic k-plane with boundary at infinity $\widetilde{F}_\infty(\partial_\infty G)$. This implies, for example, that if $\square$ is an embedded hypercube in $\widetilde{M}$ whose hyperfaces are totally geodesic and extend to totally geodesic hyperplanes containing lifts $\gamma\widetilde{\Sigma}$ of Σ to $\widetilde{M}$, then the region $\square_N$ bounded by the corresponding lifts of Σ_N in $\widetilde{N}$ will also be a hypercube.

Since this last fact—that the lifts of Σ_N to $\widetilde{N}$ corresponding to $\square$ bound a hypercube $\square_N$—is important in the paper, we give a detailed justification. It is a direct consequence of the following lemma.

Lemma 2.1. *Suppose that X is a simply connected negatively curved manifold, and that there is a quasi-isometric diffeomorphism $Q : \mathbb{H}^n \to X$. Suppose that the totally geodesic hyperplanes $H_1, \dots, H_{2n+2}$ bound a compact region $\square$ in $\mathbb{H}^n$ homeomorphic to a solid hypercube, and that each image $Q(H_i)$ of one of the H_i is at a uniformly bounded distance from a totally geodesic hyperplane H_i' in X. Then, there is a connected component of the complement of the union of the H_i' whose closure is homeomorphic to a solid hypercube.*

Proof. We argue by induction on the dimension n of X. In the case that $n=2$ and the Σ_i are geodesics, the statement is clear. Assume that it holds in all dimensions less than or equal to $n-1$. Then, note that, for each fixed H_i, $Q|_{H_i}$ composed with the nearest distance projection to H_i' defines a quasi-isometry between H_i, which is an isometric copy of $\mathbb{H}^{n-1}$, and H_i'. The nearest-distance projection to H_i' is well-defined because X is negatively curved and H_i' is totally geodesic. Note also that the $H_i \cap H_j$ are totally geodesic hypersurfaces in H_i that bound a solid hypercube in H_i, and they map under Q composed with the nearest distance projection to H_i' to hypersurfaces of H_i' at uniformly bounded distance from the $H_i' \cap H_j'$. Therefore, the hypotheses of the theorem are met with $X = H'_i$, and so, by the induction hypothesis the union of the intersections of the H'_j with a fixed H'_i bound a region in H'_i homeomorphic to a solid hypercube $\square_i$ in H'_i.

By applying the inductive hypothesis to a fixed k-dimensional intersection $H_{i_1} \cap \cdots \cap H_{i_{n-k}}$ in a similar way, we can conclude that the compact region in $H'_{i_1} \cap \cdots \cap H'_{i_{n-k}}$ bounded by intersections with the H'_j is homeomorphic to a solid hypercube in $H'_{i_1} \cap \cdots \cap H'_{i_{n-k}}$.

This shows that the cube complex $\square^\partial$ obtained by taking the union of the $\square_i$ is specified by the same combinatorial data as the boundary of the standard n-dimensional hypercube, and therefore must be homeomorphic to the boundary of the standard n-dimensional hypercube.

By Alexander duality, the complement of $\square^\partial$ in X (or equivalently the one-point compactification of X) has two connected components. Call the connected component with compact closure K. Then, K is convex as the intersection of convex regions in X (namely, connected components of complements of totally geodesic hypersurfaces). The fact that K is star-shaped with respect to any point in its interior and that $\partial K = \square^\partial$ is homeomorphic to the boundary of a solid hypercube implies that the closure of K is homeomorphic to a solid hypercube. ∎

2.3. Homogenous dynamics

We state the result from homogenous dynamics that we will need. Let Σ_k be a sequence of distinct totally geodesic hypersurfaces in a closed hyperbolic n-manifold M. Denote by $\widehat{\Sigma}_k$ their lifts to the Grassmann bundle $\mathrm{Gr}_{n-1}(M)$ of unoriented tangent $n-1$ planes to M. The $\widehat{\Sigma}_k$ define probability measures μ_k on $\mathrm{Gr}_{n-1}(M)$ by

$$\mu_k(f) := \frac{1}{\mathrm{Vol}(\widehat{\Sigma}_k)} \int_{\widehat{\Sigma}_k} f \, d\widehat{\Sigma}_k$$

for f a continuous function on $\mathrm{Gr}_{n-1}(M)$, and where $d\widehat{\Sigma}_k$ is the volume form for the hyperbolic metric on Σ_k. The hyperbolic metric induces a metric on $\mathrm{Gr}_{n-1}(M)$ on which we denote the volume measure, normalized to have unit volume, by μ_{Leb}. It follows from Ratner's measure classification theorem that any weak-$*$ limit of the μ_k is equal to a convex combination of measures supported on totally geodesic submanifolds and μ_{Leb}. That we can rule out ergodic components that are supported on totally geodesic submanifolds is a consequence of work by Mozes–Shah [33, Theorem 1.1].

Theorem 2.1. *The μ_k weak-$*$ converge to μ_{Leb}.*

In fact, we will only require a weaker statement. We say that a surface Σ is *ε-dense* if every tangent plane in $\mathrm{Gr}_{n-1}(M)$ is at a distance of at most ε from some tangent plane to Σ, where the distance is measured in the natural metric on $\mathrm{Gr}_{n-1}(M)$. The corollary below follows from Theorem 2.1.

Corollary 2.2. *For every $\varepsilon > 0$, there is K so that Σ_k is ε-dense if $k > K$.*

3. Area-minimizing hypersurfaces

In this section, we explain how Corollary 1.1 follows from Theorem 1.2 and provide some context.

There is a sharp upper bound for the area of a minimal surface in a Riemannian 3-manifold with a negative upper bound on its sectional curvature. Namely, if (M, g) is a Riemannian 3-manifold containing a minimal surface Σ, and if the sectional curvature K_g through every tangent 2-plane satisfies $K_g \leq -1$, then

$$\mathrm{Area}_g(\Sigma) \leq 2\pi|\chi(\Sigma)|. \tag{3.1}$$

It is simple to check this inequality using the Gauss equation and the Gauss–Bonnet formula, but the question of what happens in the case of equality is more interesting. Equality in (3.1) implies that Σ is totally geodesic and hyperbolic (constant curvature -1) in its induced metric. The straightforward way for this to happen is if g itself is hyperbolic, but is this the only way?

Lima [21] showed that the answer to this question is negative by constructing non-hyperbolic examples of metrics on $\Sigma \times \mathbb{R}$ with sectional curvature at most -1 for which equality in (3.1) is attained [21]. In Section 6, we construct closed non-hyperbolic Riemannian 3-manifolds with sectional curvature at most -1 that contain totally geodesic hyperbolic surfaces. These give examples of metrics for which rigidity in the case of equality in the area inequality (3.1) fails, even in the case when the ambient manifold is closed. One way of describing the goal of this paper is to give conditions under which equality in (3.1) forces the ambient metric to be hyperbolic.

Our arguments will work in any dimension. For a closed hyperbolic n-manifold Σ and a closed $n+1$-manifold N with sectional curvature less than or equal to -1 the following holds. Suppose we have a map $F : \Sigma \to N$ so that the induced map $\pi_1(\Sigma) \to \pi_1(N)$ is injective. Assume also that there is a closed hyperbolic manifold homotopy equivalent to N via a homotopy equivalence that sends $F(\Sigma)$ to the homotopy class of a totally geodesic hypersurface. Then, as we explain shortly, the next theorem is a consequence of the work of Besson–Courtois–Gallot. Together with Theorem 1.2, it directly implies Corollary 1.1.

Theorem 3.1. *There exists a smooth map $\overline{F}$ homotopic to F so that the n-volume of the image $\overline{F}(\Sigma)$ of Σ in N is less than or equal to the volume of Σ in its hyperbolic* *metric. The map $\overline{F}$ has the property that in the case that the volume of $\overline{F}(\Sigma)$ is equal to the volume of Σ in its hyperbolic metric. Then, the following statements hold.*

(1) *The image $\overline{F}(\Sigma)$ of Σ under $\overline{F}$ is a totally geodesic hypersurface in N with induced metric the hyperbolic metric on Σ.*

(2) *The volume of any immersed hypersurface in N homotopic to and distinct from $\overline{F}(\Sigma)$ up to reparametrization is greater than that of $\overline{F}(\Sigma)$.*

Remark 3.1. If N is three dimensional, the previous theorem is essentially a consequence of the fact that $\overline{\Sigma}$ is homotopic to a least area minimal immersion [35] and inequality (3.1).

We explain how Theorem 3.1 follows from the work of Besson–Courtois–Gallot. That $\overline{F}$ is homotopic to a smooth map with Jacobian pointwise smaller than 1 is a direct consequence of [6, Theorem 1.10], taking the manifold X in the statement of that theorem to be the universal cover $\widetilde{N}$ of N, and the representation to correspond to the action of $\pi_1(\Sigma) \subset \pi_1(N)$ on $\widetilde{N}$ by deck transformations. That the action of $\pi_1(\Sigma)$ is convex cocompact follows from the fact that N is homotopy equivalent to a closed hyperbolic manifold via a homotopy equivalence that sends $F(\Sigma)$ to a totally geodesic hypersurface. Lifting the homotopy equivalence to a quasi-isometry $\tilde{h} : \mathbb{H}^n \to \widetilde{N}$ between universal covers, [8, Proposition 2.5.4] implies that there is a convex set in $\widetilde{N}$, contained for some R in the R-neighborhood of the image under $\tilde{h}$ of a totally geodesic hyperplane in $\mathbb{H}^n$ preserved by $\pi_1(\Sigma) \subset \pi_1(M)$. This convex set is then contained in the orbit of a compact set under the action of $\pi_1(\Sigma) \subset \pi_1(N)$. Therefore, $\pi_1(\Sigma) \subset \pi_1(N)$ acts cocompactly on the convex hull of its limit set, and is convex cocompact.

For the equality case, if no immersion homotopic to $\overline{F}$ has smaller volume than the volume of Σ in its hyperbolic metric, then the volume of $\overline{F}(\Sigma)$ is equal to the volume of Σ in the hyperbolic metric, and the Jacobian of $\overline{F}(\Sigma)$ must be everywhere equal to 1. In that case [6, Theorem 1.2] implies that the differential of $\overline{F}$ is at every point an isometry onto its image. The image $\overline{F}(\Sigma)$ is therefore isometric to the hyperbolic metric on Σ in its induced metric.

Since $\overline{F}(\Sigma)$ achieves the minimal volume over all hypersurfaces homotopic to it, we know that $\overline{F}(\Sigma)$ is a minimal hypersurface. Thus, at every point the principal curvatures $\lambda_1, \ldots, \lambda_n$ sum to zero. If any were nonzero, then we could find $\lambda_i > 0$ and $\lambda_j < 0$. The tangent plane to $\overline{F}(\Sigma)$ spanned by the corresponding principal directions would then have sectional curvature strictly less than one by the Gauss equation and the fact that the sectional curvature of N is less than or equal to -1, which contradicts the fact that $\overline{F}(\Sigma)$ is hyperbolic in its induced metric. Therefore, $\overline{F}(\Sigma)$ is totally geodesic. If there were some other immersed hypersurface in the homotopy class of

$\overline{F}(\Sigma)$ with the same volume, then one could argue as in [7, Proof of Theorem 1.2] to show that it is equal to $\overline{F}(\Sigma)$.

4. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. Again, we recommend that the reader assume N is three dimensional on a first reading.

Assume that Σ is a closed totally geodesic hypersurface in N that satisfies the well-distribution condition. This means that every point $p \in \widetilde{N}$ is contained in an embedded solid hypercube $\square$ whose hyperfaces are contained in lifts of Σ to $\widetilde{N}$. We claim that there is an isometric embedding Φ of the boundary $\partial\square$ of $\square$ in $\mathbb{H}^n$.

4.1. Warm-up: Construction of an isometric embedding for tetrahedra

To illustrate some of the ideas, we first give a proof of a similar statement with the boundary of the hypercube $\partial\square$ replaced by the boundary Δ of a solid embedded tetrahedron $\Delta \subset \widetilde{N}$ whose faces are contained in lifts of Σ. We assume also that N is three dimensional. We originally tried to work with tetrahedra rather than hypercubes, but it does not seem possible to follow the approach of Section 5 using tetrahedra.

We claim that Δ isometrically embeds in $\mathbb{H}^3$. Label the vertices of Δ as A, B, C, and D. Note that the faces of Δ meet at constant angles, since the faces are totally geodesic, i.e., for any edge of Δ, the angle that the two faces meeting at that edge make at any point on the edge is the same. Then, since the face ABC is contained in a totally geodesic plane isometric to $\mathbb{H}^2$, we can map ABC isometrically into some copy of $\mathbb{H}^2$ contained in $\mathbb{H}^3$. Call this map Φ.

Note also that the choice of Φ together with a choice of orientation for $\mathbb{H}^3$ and for the triangle ABC uniquely determines linear isometries between the tangent spaces to A, B, and C and, respectively, $\Phi(A)$, $\Phi(B)$, and $\Phi(C)$—fix such a linear isometry

$$d\Phi_A : T_A(M) \to T_{\Phi(A)}(\mathbb{H}^3),$$

and similarly for B and C. We can then extend Φ to the line segment AD by sending it to the geodesic ray beginning at $\Phi(A)$ with tangent vector at $\Phi(A)$ equal to the image under $d\Phi_A$ of the tangent vector to AD at A, and similarly for B and C.

By the fact that the faces of Δ meet along constant angles, we know that $\Phi(AD)$ and $\Phi(BD)$ intersect, and that triangle $\Phi(ABD)$ is isometric to triangle ABD, and similarly for triangles ACD and BCD. We can thus extend Φ to an isometry on all of Δ as desired.

4.2. Construction of an isometric embedding for hypercubes

In the hypercube case, it is possible by a similar but more complicated argument to define an isometric embedding Φ of $\partial\Box$ in $\mathbb{H}^n$ "by hand". Instead, we give a short proof that this is possible using an analytic-continuation-type argument.

Note that every point p on a hyperface of $\Box$ has a neighborhood in $\partial\Box$ that isometrically embeds in $\mathbb{H}^n$. If p is contained in the interior of a hyperface then this is immediate. Otherwise, we can first define an isometric embedding Φ' on the intersection of a neighborhood of p with a hyperface H containing p. Then, the fact that hyperfaces of $\Box$ meet along constant angles, since they are contained in totally geodesic hypersurfaces, allows us to extend Φ' to a neighborhood of p in $\partial\Box$.

To see that any two hyperfaces of $\Box$ meet along a constant angle, suppose that two incident hyperfaces of $\Box$ are contained in totally geodesic hypersurfaces S_1 and S_2. Take two points p and q on $S_1 \cap S_2$, and join them by a geodesic γ parametrized by arc-length. Let $\eta_1(0)$ and $\eta_2(0)$ be normal vectors to $S_1 \cap S_2$ inside, respectively, S_1 and S_2. Parallel transport $\eta_1(0)$ and $\eta_2(0)$ in S_1 and S_2 along γ to obtain normal vector fields $\eta_1(t)$ and $\eta_2(t)$ defined along $\gamma(t)$. Then, if we denote the Levi-Civita connection on S_i by $\nabla_{g_{S_i}}$ and in the ambient space by ∇_g, we have

$$\frac{d}{dt}\langle \eta_1(t), \eta_2(t)\rangle_g = \langle \nabla_g \eta_1(t), \eta_2(t)\rangle_g + \langle \eta_1(t), \nabla_g \eta_2(t)\rangle_g.$$

Since S_1 and S_2 are totally geodesic, this is equal to

$$\langle \nabla_{g_{S_1}} \eta_1(t), \eta_2(t)\rangle_g + \langle \eta_1(t), \nabla_{g_{S_2}} \eta_2(t)\rangle_g.$$

Both terms in the previous sum vanish because $\eta_i(t)$ is the parallel transport of $\eta_i(0)$ in along γ in S_i. Therefore, the inner product of $\eta_1(t)$ and $\eta_2(t)$ is constant in t, and S_1 and S_2 meet at the same angle at p and q.

To define Φ, we fix a point p in $\partial\Box$, and define Φ on a neighborhood of p in $\partial\Box$ to be some isometric embedding of that neighborhood in $\mathbb{H}^n$. Then, for any $p' \in \partial\Box$ and any path γ in $\partial\Box$ joining p to p', we can use the fact that each point has a neighborhood in $\partial\Box$ that isometrically embeds in $\mathbb{H}^n$ to extend Φ along the path γ and define $\Phi(p')$. Since $\partial\Box$ is simply connected, $\Phi(p')$ is well defined independent of the choice of γ. Therefore, the map Φ is a well-defined local isometry onto its image from $\partial\Box$ into $\mathbb{H}^n$. Note that Φ is an isometry restricted to each hyperface of $\partial\Box$ and that the images of opposite hyperfaces under Φ are disjoint. Therefore, Φ is an isometric embedding onto its image. We comment that this argument would fail in the case that the dimension n of the ambient space were equal to 2, since $\partial\Box$ would not be simply connected.

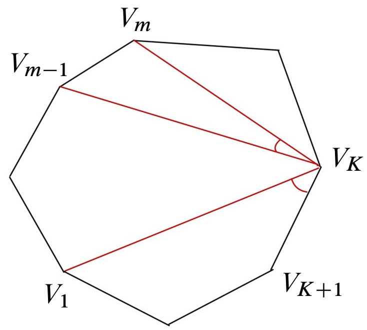

Figure 1. The polygon P in the totally geodesic plane S.

4.3. Definition of the new Riemannian manifold H and rigidity via comparison geometry

We define a new Riemannian manifold H as follows. First, we take the closure of the non-compact connected component of the complement of $\Phi(\square)$ in $\mathbb{H}^n$ equipped with the hyperbolic metric. We then "fill in" the boundary $\Phi(\square)$ by gluing in the region $R_\square$ bounded by $\square$ in $\widetilde{N}$ to obtain H. Since Φ is an isometry, this defines a Riemannian manifold which we denote by H. Note that a priori the metric tensor of H is only continuous.

We claim that H must be isometric to $\mathbb{H}^n$. Our proof is similar to some of the arguments in [36] (see also the recent paper [18] and references therein).

Every totally geodesic two-dimensional plane S in $\mathbb{H}^n$ that intersects $\Phi(\square)$ transversely corresponds to a totally geodesic surface with boundary S' in H, whose polygonal boundary is contained in $\Phi(\square)$. The surface S' is isometric to S with the compact region bounded by some m-sided polygon P' removed. Note that P' is isometric to a polygon P in $S \cong \mathbb{H}^2$, and label the vertices of P (resp., P') as $v_1, \ldots, v_\ell$ (resp., $v'_1, \ldots, v'_\ell$.)

First, consider the case that P is a triangle. Note that P has the same angles and side-lengths as a geodesic triangle in $\mathbb{H}^2$. Then, since $R_\square$ is geodesically convex and has curvature bounded above by -1, the equality case of Corollary 2.1 implies that P bounds a totally geodesic hyperbolic solid triangle in $R_\square$, and S', thus, extends to an embedded hyperbolic plane in H. See Figure 1.

To prove the general case, we will show that for any i, j with $1 \leq i, j \leq \ell$ and $\ell - |i-j|$ or $|i-j|$ less than k, the length of $v'_i v'_j$ is greater than or equal to that of $v_i v_j$. Take this as the inductive hypothesis, assume that it holds for k, and choose i and j with $|i-j| = k-1$. Relabeling if necessary we can assume $i = 1$ and $j = k$.

First, we claim that for any m such that $1 < m < k$, the angle $< v'_{m-1} v'_k v'_m$ is less than or equal to $< v_{m-1} v_k v_m$. To see this, by the inductive hypothesis $v'_{m-1} v'_k$

and $v'_m v'_k$ have length greater than or equal to, respectively, $v_{m-1}v_k$ and $v_m v_k$, and the length of $v'_{m-1}v'_m$ is the same as that of $v_{m-1}v_m$. The claim then follows from Corollary 2.1.

Second, we claim angle $> v'_1 v'_k v'_{k+1}$ is greater than or equal to angle $> v_1 v_k v_{k+1}$. Note that the sum of $> v'_1 v'_k v'_{k+1}$ and

$$\sum_{p=2}^{k-1} > v'_{p-1} v'_k v'_p \tag{4.1}$$

is equal to $> v'_{k-1} v'_k v'_{k+1}$, which is equal to $> v_{k-1} v_k v_{k+1}$. Therefore, since (4.1) is smaller than or equal to the same sum with the v'_i replaced by v_i, this implies that $> v'_1 v'_k v'_{k+1}$ is greater than or equal to $> v_1 v_k v_{k+1}$. Using Corollary 2.1, we can then conclude that the length of $v'_1 v'_{k+1}$ is greater than or equal to the length of $v_1 v_{k+1}$. In the case of equality, we could then use the equality case of Corollary 2.1 to conclude that $v'_1 v'_k$ has length equal to $v_1 v_k$ and $v'_1 v'_{k+1}$ has length equal to $v_1 v_{k+1}$, and that $\Delta v'_1 v'_{k+1} v'_k$ can be filled in by a totally geodesic hyperbolic triangle. Continuing in this way we see that we must have had equality at every previous stage of the induction, and we, finally, get that the length of $v'_1 v'_3$ equals the length of $v_1 v_3$, and so, $\Delta v'_1 v'_2 v'_3$ can be filled in by a hyperbolic triangle. Removing $\Delta v'_1 v'_2 v'_3$ we obtain a new polygon with one fewer side on which we can repeat the same argument. If we never have equality, then we can conclude the finite induction to get that $v'_1 v'_\ell$ has length strictly greater than $v_1 v_\ell$, which is a contradiction since the two have the same length. We have, thus, shown that S' extends to an embedded totally geodesic hyperbolic plane.

4.4. Conclusion: All tangent planes in H are tangent to totally geodesic hyperbolic planes

We explain why the previous subsection implies that H has constant curvature -1. Take a large metric sphere B in $\mathbb{H}^n$ that contains $\Phi(\square)$, and let $\mathcal{C}$ be the set of round (but not necessarily great) circles contained in the metric sphere ∂B. There is a natural disk bundle $\mathcal{D}$ over $\mathcal{C}$ whose fiber over $C \in \mathcal{C}$ is equal to the totally geodesic disk in B bounded by C. We define a map f from $\mathcal{D}$ to the Grassmann bundle $\mathrm{Gr}_2(\overline{R_\square})$ of unoriented tangent 2-planes to the closure of $R_\square$ as follows. Here, $\mathrm{Gr}_2(\overline{R_\square})$ is defined by viewing $\overline{R_\square}$ as a subset of N and pulling back $\mathrm{Gr}_2(N)$ to $\overline{R_\square}$.

Suppose $d \in \mathcal{D}$ corresponds to a circle $C \in \mathcal{C}$ that bounds a totally geodesic disk D containing d. In other words, d belongs to the fiber of C in the disk bundle introduced in the previous paragraph. We take the unique totally geodesic tangent plane S that intersects ∂B in D, which by what we have shown above corresponds to a totally geodesic plane S' in H. This S' contains an isometric copy D' of the hyperbolic disk $S \cap B$. We define $f(d)$ to be the tangent plane Π' to the point d' on D' corresponding to d if it is contained in $\overline{R_\square}$, or else the parallel transport of Π' along the geodesic joining d' to its nearest point projection to $\partial\overline{R_\square}$.

Note that f is continuous, and f is injective near every point that maps into the interior of $R_\square$. Since $\mathcal{D}$ and the interior of $\mathrm{Gr}_2(\overline{R_\square})$ both have dimension $3n-4$, the invariance of domain theorem implies that f is a local homeomorphism onto its image near any such point d. Since the image of f is both open and closed, the fact that $\mathrm{Gr}_2(\overline{R_\square})$ is connected implies that f is surjective. It follows that $R_\square$ has constant curvature -1, which implies that H is isometric to $\mathbb{H}^n$.

5. Proof of Theorem 1.2

Let M be a closed hyperbolic manifold, and let N be a Riemannian manifold homotopy equivalent to N with sectional curvature at most -1. Fix a homotopy equivalence $F : M \to N$. Let Σ be a totally geodesic hypersurface in M, and assume that N contains a totally geodesic hypersurface Σ_N whose fundamental group includes to $\pi_1(\Sigma)$ up to conjugacy, where we have used F to identify $\pi_1(M)$ and $\pi_1(N)$. To show that Σ_N satisfies the well-distribution property, we will actually need Σ to satisfy a stronger condition, which we introduce now.

5.1. Strong well-distribution in M

Definition 5.1. We say that Σ satisfies the *strong well-distribution property* if the following holds. Let $\mathcal{B}_\Sigma$ be the set of solid embedded hypercubes in $\mathbb{H}^n = \widetilde{M}$ each of whose boundary hyperfaces is contained in a lift of Σ to the universal cover $\mathbb{H}^n$. Then, for each geodesic γ in the universal cover $\widetilde{M} \cong \mathbb{H}^n$ of M, we can find hypercubes $\square_i \in \mathcal{B}_\Sigma$, $i \in \mathbb{Z}$, so that the following holds. This sequence of hypercubes is then said to *enclose* γ. See Figure 2.

(1) Exactly two hyperfaces $H_{i,\mathrm{top}}$ and $H_{i,\mathrm{bottom}}$ of $\square_i$ intersect γ.

(2) The totally geodesic hyperplanes that contain the hyperfaces of $\square_i$, excluding $H_{i,\mathrm{top}}$ and $H_{i,\mathrm{bottom}}$, do not intersect γ.

(3) $\square_{i-1}$ and $\square_i$ satisfy the *interlocking property*: the totally geodesic hyperplane that contains $H_{i-1,\mathrm{top}}$ is contained in the region between those of $H_{i,\mathrm{top}}$ and $H_{i,\mathrm{bottom}}$.

(4) Each point on γ is contained in some $\square_i$.

Now, let Σ_k be an infinite sequence of distinct closed totally geodesic hypersurfaces in M. We claim that for k large enough, Σ_k satisfies the strong well-distribution

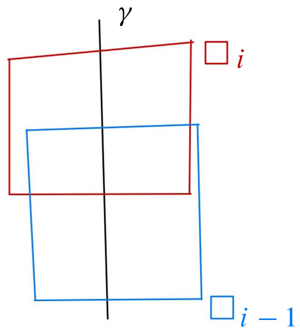

Figure 2. Hypercubes enclosing γ.

property. By Corollary 2.2, we know that for every $\varepsilon > 0$ the Σ_k are ε-dense in $\mathrm{Gr}_{n-1}(M)$ for k large enough.

For $\varepsilon > 0$ sufficiently small, we define the following hypercube $\square_{\text{model}}(\varepsilon)$ with totally geodesic faces in $\mathbb{H}^n$. Take some point P in $\mathbb{H}^n$, and let $E_1, \ldots, E_n$ be an orthonormal basis for the tangent space at that point. For each i, let H_i^+ and H_i^- be the totally geodesic hyperplanes orthogonal to the geodesic through P in the direction of E_i, and at signed distance of, respectively, ε and $-\varepsilon$ from P. Let U_i be the connected component of the complement of $H_i^+ \cup H_i^-$ in $\mathbb{H}^n$ containing P. We define $\square_{\text{model}}(\varepsilon)$ to be the closure of the intersection $\bigcap_{i=1}^n U_i$, where we have taken ε small enough that this intersection is a solid embedded hypercube.

We say that two hypercubes $\square$ and $\square'$ with totally geodesic hyperbolic faces are δ-close if there is a labeling of the vertices $V_1, \ldots, V_\ell$ of $\square$ and the vertices $V_1', \ldots, V_\ell'$ of $\square'$ such that

(1) for all i, j such that V_i and V_j are adjacent the length of $V_i V_j$ and the length of $V_i' V_j'$ differ by less than δ,

(2) for all i, j, p such that V_i and V_p are adjacent to V_j the angle $> V_i V_j V_p$ differs from the angle $> V_i' V_j' V_p'$ by less than δ.

Denote by $\mathcal{B}_{\Sigma_k}^{\varepsilon}$ the set of hypercubes $\square$ in $\mathcal{B}_{\Sigma_k}$ that are δ-close to $\square_{\text{model}}(\varepsilon)$, for some small δ be specified later. The reason for defining $\mathcal{B}_{\Sigma_k}^{\varepsilon}$ is that having lots of elements of $\mathcal{B}_{\Sigma_k}$ of controlled shape will be useful for enclosing γ.

Fix $\varepsilon > 0$ small enough that $\square_{\text{model}}(\varepsilon)$ is defined. For every geodesic γ in the universal cover, we claim that we can find a collection of solid hypercubes $\square$ contained in $\mathcal{B}_{\Sigma_k}^{\varepsilon}$ that enclose γ, provided that k was chosen sufficiently large and δ was chosen sufficiently small at the start. Let $\mathcal{B}_{\Sigma_k}^{\varepsilon}(\gamma)$ be the set of hypercubes $\square$ in the universal cover so that the following holds:

(1) $\square$ is the lift of some element of $\mathcal{B}_{\Sigma_k}^{\varepsilon}$ to the universal cover,

(2) γ intersects the boundary of $\square$ in exactly two points p_1 and p_2 on opposite hyperfaces H_{top} and H_{bottom} of $\square$,

(3) p_1 and p_2 are at a distance of less than δ from the centers of H_{top} and H_{bottom}, respectively. Here, the centers of H_{top} and H_{bottom} are the endpoints of the geodesic segment of shortest length beginning at H_{bottom} and ending at H_{top},

(4) γ makes an angle between $\pi/2 - \delta$ and $\pi/2 + \delta$ with H_{top} and H_{bottom} at, respectively, p_1 and p_2.

Here, δ is chosen small enough that for every γ and $\square$ satisfying the above four conditions, γ will be disjoint from all of the totally geodesic hyperplanes that contain hyperfaces of $\square$ different from H_{top} and H_{bottom}. We emphasize that all of the requirements we impose on δ and k will be independent of γ. That they can be satisfied follows from the fact that given any δ', any embedded copy of $\square_{\text{model}}(\varepsilon)$ in $\mathbb{H}^n$ is δ'-close to some element of $\mathcal{B}^{\varepsilon}_{\Sigma_k}(\gamma)$, provided δ was chosen small enough and k was chosen large enough at the start.

If k was chosen large enough, then every point on γ will be contained in some element of $\mathcal{B}^{\varepsilon}_{\Sigma_k}(\gamma)$. Suppose that $p \in \gamma$ is contained in some $\square_p \in \mathcal{B}^{\varepsilon}_{\Sigma_k}(\gamma)$, whose boundary intersects γ at p_1 and p_2. Then, provided δ was chosen small enough and k was chosen large enough, we can find some $\square_{p_1} \in \mathcal{B}^{\varepsilon}_{\Sigma_k}(\gamma)$ that contains p_1, and whose boundary intersects γ at points p' and p'' so that

$$\min(d(p', p_1), d(p'', p_1)) > \varepsilon/4.$$

In this case, again provided δ was chosen small enough, $\square_{p_1}$ and $\square_p$ satisfy the interlocking property, and in a similar way we can find a $\square_{p_2}$ containing p_2 in its interior so that $\square_{p_2}$ and $\square_p$ satisfy the interlocking property. Continuing in this way, we can find a collection of $\square \in \mathcal{B}^{\varepsilon}_{\Sigma_k}(\gamma)$ that enclose γ. This shows that Σ_k satisfies the strong well-distribution property for large enough k.

5.2. Well-distribution in N

Suppose that Σ satisfies the strong well-distribution property. Then, we claim that Σ_N is well distributed in N. First, note that each hypercube $\square$ bounded by lifts of Σ to the universal cover corresponds to a hypercube $\square'$ in N bounded by lifts of Σ_N. This can be seen as follows. Suppose that $\square$ is bounded by $2n + 2$ totally geodesic hyperplanes $H_1, \ldots, H_{2n+2}$ that are lifts of Σ.

Recall the discussion in Section 2.2. The map F defines a homeomorphism between the boundary at infinity of $\mathbb{H}^n$ and the boundary at infinity of the universal cover $\widetilde{N}$ of N. Taking the images of the boundaries at infinity of the H_i we obtain $2n + 2$, $n - 2$-disks in $\partial_\infty \widetilde{N}$. These bound totally geodesic hyperplanes that project down to

Σ_N. By Lemma 2.1, we obtain a hypercube $\square'$ in $\widetilde{N}$ whose sides are contained in lifts of Σ_N.

We claim that Σ_N satisfies the well-distribution property in N. Let γ_N be a geodesic in the universal cover $\widetilde{N}$. Let γ be a geodesic in $\widetilde{M} = \mathbb{H}^n$ with the same endpoints at infinity as γ_N, identifying the two boundaries at infinity as above. Since Σ satisfies the strong well-distribution property, we can find a sequence of hypercubes $\square_i : i \in \mathbb{Z}$ in $\mathbb{H}^n$ whose hyperfaces are contained in lifts of Σ and that enclose γ. In particular, the following properties hold.

(1) Exactly two hyperfaces $H_{i,\text{top}}$ and $H_{i,\text{bottom}}$ of $\square_i$ intersect γ.

(2) The totally geodesic hyperplane that contains $H_{i-1,\text{top}}$ is contained in the region between the totally geodesic hyperplanes containing $H_{i,\text{top}}$, $H_{i,\text{bottom}}$.

(3) Each point on γ is contained in some $\square_i$.

The hyperplanes corresponding to the faces of $\square_i$ excluding $H_{i,\text{top}}$ and $H_{i,\text{bottom}}$ bound an infinite solid rectangular “pillar” that contains γ.

The hypercubes $\square_i'$ corresponding to the $\square_i$ satisfy the same three properties in $\widetilde{N}$, but with γ replaced by γ_N. Item (1) follows from the fact that a totally geodesic hyperplane H in $\widetilde{N}$ will intersect a given geodesic γ exactly if the two endpoints of γ in $\partial_\infty \widetilde{N}$ are in separate components of the complement of $\partial_\infty H$ in $\partial_\infty \widetilde{N}$. This implies that whether γ intersects H is determined by what happens in $\mathbb{H}^n$ for the corresponding geodesic and lift of Σ. The second property follows in a similar way.

The first two properties imply that the subset of γ of points contained in some $\square_i'$ is both open and closed, and so, must be all of γ.

Therefore, since every point in N is contained in some geodesic, every point p of $\widetilde{N}$ is contained in an embedded hypercube $\square_p'$ as above. By Theorem 1.1, this implies that N has constant curvature -1. Since N is homotopy equivalent to M, N must be isometric to M by Mostow's rigidity theorem.

6. Examples where rigidity fails

In this section we prove the following theorem. We expect that a similar construction works in higher dimensions, but for simplicity we focus on the three-dimensional case.

Theorem 6.1. *There exists a closed hyperbolic 3-manifold (M, g_{hyp}) and a smooth Riemannian metric g on M that is not isometric to g_{hyp} such that the following hold:*

(1) *M contains a closed surface S that is totally geodesic in g_{hyp},*

(2) *S is a totally geodesic hyperbolic surface in the metric (M, g),*

(3) *the sectional curvature of g is less than or equal to -1.*

Our construction will be based on the following proposition.

Proposition 6.1. *Let $\ell > 1$ and let g_ℓ be a metric on $S \times \mathbb{R}$ with constant curvature $-\ell$ so that $S \times \{0\}$ is a totally geodesic surface in the metric g_ℓ. There exists a smooth metric g on $S \times \mathbb{R}$ satisfying the following conditions:*

(1) *$g = g_\ell$ outside of a compact set,*

(2) *g has sectional curvature $K_g \leq -1$,*

(3) *$S \times \{0\}$ is totally geodesic in g with constant sectional curvature -1.*

Lima [21] constructed non-hyperbolic examples as in the proposition, but with point (1) excluded. The examples we give were explained to us by Laurent Mazet, and we are grateful to him for allowing us to include them. Our verifications closely follow [21, Section 3].

Proof. Let (S, g_{hyp}) be a hyperbolic surface, and let f be a function defined by

$$f(t) := \frac{1}{\ell}\cosh(\ell t) + \chi(t)\Big(1 - \frac{1}{\ell}\Big)$$

for $\chi(t)$ a smooth non-negative function satisfying, for $\varepsilon > 0$ and M, respectively, sufficiently small and large to be specified later,

(1) $\chi(t) = 1$ near $t = 0$, $\chi(t) = 0$ for $|t| > M$,

(2) $\max(|\chi'(t)|, |\chi''(t)|) < \varepsilon$.

We take $\ell > 1$. Define g to be the warped product metric

$$g := f^2(t) g_{\text{hyp}}^2 + dt^2.$$

Writing g_ℓ as

$$g_\ell = \frac{1}{\ell^2}\cosh^2(\ell t) g_{\text{hyp}}^2 + dt^2,$$

only the second item in the statement of the proposition requires an argument. In computing the curvature of g, we follow [21]. For a vector field V on $(S \times \mathbb{R}, g)$, define the projection to S by

$$V^S := V - g(V, \partial_t)\partial_t.$$

The curvature tensor R of g can be written in terms of the curvature tensor R_S of (S, g_{hyp}) as

$$\begin{aligned} R(X, Y, Z) = R_S(X^S, Y^S)Z^S &- \Big(\frac{f'}{f}\Big)^2 (g(X, Z)Y - g(Y, Z)X) \\ &+ (\log f)'' g(Z, \partial_t)(g(Y, \partial_t)X - g(X, \partial_t)Y) \\ &- (\log f)''(g(X, Z)g(Y, \partial_t) - g(Y, Z)g(X, \partial_t)\partial_t) \end{aligned}$$

for X, Y, and Z smooth vector fields. Given a plane Π spanned by unit length orthogonal vector fields X, Y, we have that the sectional curvature of Π is given by

$$K_\Pi = \frac{K_S - (f')^2}{f^2} + \left(\frac{-K_S + (f')^2 - f'' f}{f^2}\right)(g(X, \partial_t)^2 + g(Y, \partial_t)^2). \quad (6.1)$$

For $|t|$ smaller than some t_0, we have that $\chi(t)$ is equal to 1. The numerator of the coefficent of $(g(X, \partial_t)^2 + g(Y, \partial_t)^2)$ is then equal to

$$= 1 + \sinh^2(\ell t) - \ell \cosh \ell t \left(\frac{1}{\ell}\cosh(\ell t) + \chi(t)\left(1 - \frac{1}{\ell}\right)\right) = (1 - \ell)\cosh \ell t,$$

which is less than zero, so we have that (6.1) is maximized when $(g(X, \partial_t)^2 + g(Y, \partial_t)^2) = 0$. In this case, we just need to show that the initial term

$$\frac{K_S - (f')^2}{f^2} = \frac{-1 - (\sinh(\ell t) + \chi'(t)(1 - \frac{1}{\ell}))^2}{(\frac{1}{\ell}\cosh(\ell t) + \chi(t)(1 - \frac{1}{\ell}))^2} \quad (6.2)$$

is at most -1. This will actually be true for all t, not just $|t| < t_0$, which we verify now. Note that this holds at $t = 0$. It is then enough to show that the quantity (6.2) is decreasing in t for $t > 0$ and increasing in t for $t < 0$. For the derivative of the quantity (6.2) to be negative, we need

$$\begin{aligned} 0 > -2&\left(\sinh(\ell t) + \chi'(t)\left(1 - \frac{1}{\ell}\right)\right)\left(\ell\cosh(\ell t) + \chi''(t)\left(1 - \frac{1}{\ell}\right)\right) \\ &\times \left(\frac{1}{\ell}\cosh(\ell t) + \chi(t)\left(1 - \frac{1}{\ell}\right)\right) \\ &+ \left(1 + \left(\sinh(\ell t) + \chi'(t)\left(1 - \frac{1}{\ell}\right)\right)^2\right)\left(\sinh(\ell t) + \chi'(t)\left(1 - \frac{1}{\ell}\right)\right). \end{aligned} \quad (6.3)$$

Choose t_0 so that $\chi(t) = 1$ for $|t| < t_0$. We first do the case $|t| < t_0$. Assume that $t > 0$. To verify (6.3), it is enough to show that

$$0 \geq \sinh(\ell t)\left(-2\cosh^2(\ell t) - 2\ell\cosh(\ell t)\chi(t)\left(1 - \frac{1}{\ell}\right) + 1 + \sinh^2(\ell t)\right), \quad (6.4)$$

where the right-hand side of (6.4) is all of the terms of the right-hand side of (6.3) that do not contain $\chi'(t)$ or $\chi''(t)$. Since $\sinh(\ell t)(-2\ell\cosh(\ell t)\chi(t)(1 - \frac{1}{\ell}))$ is always non-positive provided $t > 0$, for (6.4) to hold it is enough that

$$0 \geq -2\cosh^2(\ell t) + 1 + \sinh^2(\ell t),$$

which is true since the right-hand side equals $-\cosh^2(\ell t)$. Thus, the quantity (6.2) is negative if $t_0 > t > 0$, and the same reasoning shows it is positive if $-t_0 < t < 0$.

Now, assume that $t \geq t_0$ (the case $t \leq -t_0$ is similar) Then, by the previous computations, we will be able to conclude that (6.2) is smaller than -1 for all t as long as $-\sinh(\ell t)\cosh^2(\ell t)$ is larger in absolute value than all of the terms of the right-hand side of (6.3) that, when expanded out, contain $\chi'(t)$ or $\chi''(t)$. This will be true provided ε was taken small enough at the beginning, since $\max(|\chi'(t)|, |\chi''(t)|) < \varepsilon$ and no term in (6.3) has more than three factors equal to $\sinh(\ell t)$ or $\cosh(\ell t)$. We have, thus, shown that (6.2) is smaller than -1 for all t.

We now check the case that $|t| > t_0$. Because the initial term in (6.1) is always less than or equal to -1, if the second term in (6.1) is negative then we are done. If not, it is enough to check (6.1) in the case that $g(X, \partial_t)^2 + g(Y, \partial_t)^2$ assumes its maximum possible value

$$g(X, \partial_t)^2 + g(Y, \partial_t)^2 = 1.$$

The quantity (6.1) is then equal to $-f''/f$. Therefore, we need to show that

$$-1 \geq -f''/f = -\frac{\ell^2\cosh(\ell t) + \ell\chi''(t)(1 - \frac{1}{\ell})}{\cosh(\ell t) + \ell\chi(t)(1 - \frac{1}{\ell})}.$$

Since $\chi \leq 1$, it is enough to show that

$$1 \leq \frac{\ell^2\cosh(\ell t)}{\cosh(\ell t) + \ell - 1} + \frac{\chi''(t)(\ell - 1)}{\cosh(\ell t) + \ell - 1}. \tag{6.5}$$

Note that

$$\ell^2\cosh(\ell t) \geq \ell\cosh(\ell t) + \ell(\ell - 1),$$

so the first term on the right-hand side of (6.5) is greater than or equal to

$$\frac{\ell\cosh(\ell t) + \ell(\ell - 1)}{\cosh(\ell t) + \ell - 1} = \ell.$$

We can make the second term on the right-hand side of (6.5) as small as desired by choosing ε small enough because $|\chi''| < \varepsilon$. Therefore, inequality (6.5) holds, which completes the proof of Proposition 6.1. ∎

We also need the following lemma.

Lemma 6.1. *Let M be a closed hyperbolic 3-manifold containing an embedded two-sided totally geodesic surface S. Then, for any $L > 0$, there exists a finite cover M_0 of M which contains a totally geodesic surface S_0 isometric to S and with normal injectivity radius greater than L.*

Recall that S_0 has normal injectivity radius greater than L in M_0 if no two distinct geodesic segments of length less than L beginning normal to S_0 intersect. This means that the normal exponential map $S_0 \times (-L, L) \to M_0$ is a diffeomorphism onto its image, for $S_0 \times (-L, L)$ identified with a tubular neighborhood of the zero section of the normal bundle of S_0 in M_0 in the natural way.

Proof. This is a consequence of subgroup separability for Fuchsian subgroups

$$\pi_1(S, p) \subset \pi_1(M, p)$$

for some choice of basepoint p on S [25, Lemma 5.3.6]. This means that for any finite subset $\{g_1, \ldots, g_n\} \subset \pi_1(M, p)$ disjoint from $\pi_1(S, p)$, we can choose a finite index subgroup of $\pi_1(M, p)$ containing $\pi_1(S, p)$ but none of $g_1, \ldots, g_n$.

Let F be the Fuchsian cover of M corresponding to $\pi_1(S)$. Fix a connected polyhedral fundamental domain P for the action of $\pi_1(M)$ on the universal cover $\mathbb{H}^3$ of M. Then, we can choose a fundamental domain in $\mathbb{H}^3$ for the cover F that is tiled by copies of P, and any fixed normal neighborhood of S in F is contained in the projections of finitely many such copies. The images of the projections to F of the copies of P correspond to cosets of $\pi_1(S)$ in $\pi_1(M)$. Choose finitely many coset representatives $g_1, \ldots, g_n$ so that the corresponding projections of copies of P contain the L-neighborhood of S in F.

We can now use the Fuchsian subgroup separability result described in the first paragraph of the proof to choose a finite index subgroup of $\pi_1(M)$ containing $\pi_1(S)$ but none of $g_1, \ldots, g_n$. The finite cover of M corresponding to this finite index subgroup is then also covered by F, and the restriction of the covering map to the L-neighborhood of S in F is a diffeomorphism onto its image. This shows that we can make the normal injectivity radius of S as large as desired by passing to finite covers. ■

We can now prove Theorem 6.1. (Compare [22, Section 6], where a similar construction appeared.) Take a closed hyperbolic 3-manifold (M, g_{hyp}) containing an embedded two-sided totally geodesic surface S. See [25], for examples.

Let $F \cong S \times \mathbb{R}$ be the Fuchsian cover of (M, g_{hyp}) corresponding to S. Let g_ℓ be the homothetic scaling of the hyperbolic metric on F that has constant curvature $-\ell$ for some $\ell > 1$. We apply Proposition 6.1 to obtain a metric g_K on F that agrees with g_ℓ outside a compact set K, and so, that $S \times \{0\}$ is a totally geodesic hyperbolic surface.

Next, we apply Lemma 6.1 to find a finite cover of M' so that F also covers M', and the restriction of the covering map $\rho : F \to M'$ to K is injective. We can then define a metric on M' to be the scaled hyperbolic metric with constant curvature $-\ell$ outside of $\rho(K)$, and to be the pushforward of g_K under ρ on $\rho(K)$. Since g_ℓ and g_K agree outside of the compact set K, this gives a smooth well-defined metric on M', which completes the proof.

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References

[1] L. Andersson and R. Howard, Comparison and rigidity theorems in semi-Riemannian geometry. *Comm. Anal. Geom.* **6** (1998), no. 4, 819–877 Zbl 0963.53038 MR 1664893

[2] U. Bader, D. Fisher, N. Miller, and M. Stover, Arithmeticity, superrigidity, and totally geodesic submanifolds. *Ann. of Math. (2)* **193** (2021), no. 3, 837–861 Zbl 07353243 MR 4250391

[3] U. Bader, D. Fisher, N. Miller, and M. Stover, Arithmeticity, superrigidity and totally geodesic submanifolds of complex hyperbolic manifolds. *Invent. Math.* **233** (2023), no. 1, 169–222 Zbl 1534.53056 MR 4601998

[4] G. Baldi and E. Ullmo, Special subvarieties of non-arithmetic ball quotients and Hodge theory. *Ann. of Math. (2)* **197** (2023), no. 1, 159–220 Zbl 1519.14025 MR 4513144

[5] Y. Benoist and H. Oh, Geodesic planes in geometrically finite acylindrical 3-manifolds. *Ergodic Theory Dynam. Systems* **42** (2022), no. 2, 514–553 Zbl 1485.37005 MR 4362901

[6] G. Besson, G. Courtois, and S. Gallot, Lemme de Schwarz réel et applications géométriques. *Acta Math.* **183** (1999), no. 2, 145–169 Zbl 1035.53038 MR 1738042

[7] G. Besson, G. Courtois, and S. Gallot, Rigidity of amalgamated products in negative curvature. *J. Differential Geom.* **79** (2008), no. 3, 335–387 Zbl 1206.53038 MR 2433927

[8] B. H. Bowditch, Geometrical finiteness with variable negative curvature. *Duke Math. J.* **77** (1995), no. 1, 229–274 Zbl 0877.57018 MR 1317633

[9] D. Calegari, F. C. Marques, and A. Neves, Counting minimal surfaces in negatively curved 3-manifolds. *Duke Math. J.* **171** (2022), no. 8, 1615–1648 Zbl 1502.53094 MR 4432012

[10] J. Cheeger and D. G. Ebin, *Comparison theorems in Riemannian geometry*. AMS Chelsea, Providence, RI, 2008 Zbl 1142.53003 MR 2394158

[11] K. Corlette, Archimedean superrigidity and hyperbolic geometry. *Ann. of Math. (2)* **135** (1992), no. 1, 165–182 Zbl 0768.53025 MR 1147961

[12] K. Delp, D. Hoffoss, and J. F. Manning, Problems in groups, geometry, and three-manifolds. 2015, arXiv:1512.04620v1

[13] J. M. Espinar and H. Rosenberg, Area estimates and rigidity of capillary H-surfaces in three-manifolds with boundary. *Math. Z.* **289** (2018), no. 3-4, 1261–1279 Zbl 1407.53008 MR 3830248

[14] F. T. Farrell and L. E. Jones, Negatively curved manifolds with exotic smooth structures. *J. Amer. Math. Soc.* **2** (1989), no. 4, 899–908 Zbl 0698.53027 MR 1002632

[15] F. T. Farrell and L. E. Jones, Rigidity and other topological aspects of compact nonpositively curved manifolds. *Bull. Amer. Math. Soc. (N.S.)* **22** (1990), no. 1, 59–64 Zbl 0703.57018 MR 1001606

[16] D. Fisher, Superrigidity, arithmeticity, normal subgroups: Results, ramifications, and directions. In *Dynamics, geometry, number theory—the impact of Margulis on modern mathematics*, pp. 9–46, The University of Chicago Press, Chicago, IL, 2022 Zbl 1508.37010 MR 4422050

[17] A. Furman, A survey of measured group theory. In *Geometry, rigidity, and group actions*, pp. 296–374, Chicago Lectures in Math., The University of Chicago Press, Chicago, IL, 2011 Zbl 1267.37004 MR 2807836

[18] M. Ghomi and J. Spruck, Rigidity of nonpositively curved manifolds with convex boundary. *Proc. Amer. Math. Soc.* **151** (2023), no. 11, 4935–4940 Zbl 1527.53041 MR 4634894

[19] M. Gromov and I. Piatetski-Shapiro, Nonarithmetic groups in Lobachevsky spaces. *Inst. Hautes Études Sci. Publ. Math.* (1988), no. 66, 93–103 Zbl 0649.22007 MR 0932135

[20] M. Lee and H. Oh, Orbit closures of unipotent flows for hyperbolic manifolds with Fuchsian ends. *Geom. Topol.* **28** (2024), no. 7, 3373–3473 Zbl 7953193 MR 4833684

[21] V. Lima, Area estimates and rigidity of non-compact H-surfaces in 3-manifolds. *Proc. Amer. Math. Soc.* **147** (2019), no. 10, 4499–4512 Zbl 1423.53072 MR 4002559

[22] B. Lowe, Deformations of totally geodesic foliations and minimal surfaces in negatively curved 3-manifolds. *Geom. Funct. Anal.* **31** (2021), no. 4, 895–929 Zbl 1487.53038 MR 4317506

[23] B. Lowe and A. Neves, Minimal surface entropy and average area ratio. [v1] 2021, [v2] 2024, arXiv:2110.09451v2

[24] C. Maclachlan and A. W. Reid, Commensurability classes of arithmetic Kleinian groups and their Fuchsian subgroups. *Math. Proc. Cambridge Philos. Soc.* **102** (1987), no. 2, 251–257 Zbl 0632.30043 MR 0898145

[25] C. Maclachlan and A. W. Reid, *The arithmetic of hyperbolic 3-manifolds*. Grad. Texts in Math. 219, Springer, New York, 2003 Zbl 1025.57001 MR 1937957

[26] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*. Ergeb. Math. Grenzgeb. (3) 17, Springer, Berlin, 1991 Zbl 0732.22008 MR 1090825

[27] L. Mazet, Rigidity of Riemannian manifolds containing an equator. *Comm. Anal. Geom.* **32** (2024), no. 1, 21–63 Zbl 07956679 MR 4806762

[28] L. Mazet and H. Rosenberg, On minimal spheres of area 4π and rigidity. *Comment. Math. Helv.* **89** (2014), no. 4, 921–928 Zbl 1307.53034 MR 3284299

[29] C. T. McMullen, A. Mohammadi, and H. Oh, Geodesic planes in hyperbolic 3-manifolds. *Invent. Math.* **209** (2017), no. 2, 425–461 Zbl 1383.53044 MR 3674219

[30] D. B. McReynolds and A. W. Reid, The genus spectrum of a hyperbolic 3-manifold. *Math. Res. Lett.* **21** (2014), no. 1, 169–185 Zbl 1301.53039 MR 3247048

[31] J. J. Millson, On the first Betti number of a constant negatively curved manifold. *Ann. of Math. (2)* **104** (1976), no. 2, 235–247 Zbl 0364.53020 MR 0422501

[32] A. Mohammadi and G. Margulis, Arithmeticity of hyperbolic 3-manifolds containing infinitely many totally geodesic surfaces. *Ergodic Theory Dynam. Systems* **42** (2022), no. 3, 1188–1219 Zbl 1483.57017 MR 4374970

[33] S. Mozes and N. Shah, On the space of ergodic invariant measures of unipotent flows. *Ergodic Theory Dynam. Systems* **15** (1995), no. 1, 149–159 Zbl 0818.58028 MR 1314973

[34] A. W. Reid, Totally geodesic surfaces in hyperbolic 3-manifolds. *Proc. Edinburgh Math. Soc. (2)* **34** (1991), no. 1, 77–88 Zbl 0714.57010 MR 1093177

[35] R. Schoen and S. T. Yau, Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature. *Ann. of Math. (2)* **110** (1979), no. 1, 127–142 Zbl 0431.53051 MR 0541332

[36] V. Schroeder and W. Ziller, Local rigidity of symmetric spaces. *Trans. Amer. Math. Soc.* **320** (1990), no. 1, 145–160 Zbl 0724.53033 MR 0958901

[37] Y. T. Siu, Geometric superrigidity. In *Algebraic geometry and related topics (Inchon, 1992)*, pp. 105–119, Conf. Proc. Lecture Notes Algebraic Geom. 1, International Press, Cambridge, MA, 1993 Zbl 0826.53034 MR 1285377

[38] A. Wright, Totally geodesic submanifolds of Teichmüller space. *J. Differential Geom.* **115** (2020), no. 3, 565–575 Zbl 1450.30065 MR 4120819

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