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Martin compactifications of affine buildings

Dedicated to Yves Guivarc'h with admiration

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Abstract. We carry out an in-depth study of Martin compactifications of affine buildings, from the viewpoint of potential theory and random walks. This work does not use any group action on buildings, although all the results are also stated within the framework of the Bruhat–Tits theory of semisimple groups over non-Archimedean local fields. This choice should allow the use of these building compactifications in intriguing geometric group theory situations, where only lattice actions are available. The resulting compactified spaces use and, at the same time, make it possible to understand geometrically the descriptions of asymptotic behavior of kernels resulting from the non-Archimedean harmonic analysis on affine buildings. Along the paper, we make explicit the most substantial differences from the case of symmetric spaces, namely absence of a group action but existence of precise asymptotics of Green kernels and, of course, no possibility of relying on standard techniques from PDEs.

Keywords: affine buildings, Martin compactification, Furstenberg compactification, random walks, potential theory, Green functions, Martin kernels, Bruhat–Tits theory, non-Archimedean harmonic analysis.

Introduction

This paper deals with Martin compactifications of affine buildings. In other words, it makes a connection between two very different mathematical topics. On the one hand, affine buildings are relevant to algebra and geometry; on the other hand, Martin compactifications refer to analysis, more precisely potential theory and probability theory. Therefore, our first task in this introduction, before mentioning the new results, is to introduce these two fields independently but in a way making them compatible with one another. At this stage let us simply say that dealing with compactifications associated

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with potential theory allows us to construct, from a concrete viewpoint, compactifications that before this approach could only be obtained artificially. Conversely, these compactifications provide a geometric way of understanding the various factors in the asymptotic formula for the Green function obtained previously by Gelfand–Fourier analytic methods.

In what follows, the geometric objects on which the various analytic concepts are defined (such as random walks, or heat and Martin kernels) are affine buildings. In many situations, those spaces are non-Archimedean analogues of Riemannian symmetric spaces; they were indeed designed for this purpose by F. Bruhat and J. Tits (see [13] and [14] for the full theory, and [55] for an overview). Affine buildings thus provide a suitable geometry that enables one to understand semisimple algebraic groups over non-Archimedean local fields, such as classical matrix groups over finite extensions of the field $\mathbb{Q}_p$ of p-adic numbers (e.g. the group $\mathrm{SL}_n(\mathbb{Q}_p)$). However, some affine buildings of low rank do not come from algebraic groups, and thus have interesting features in geometric group theory. For this reason we avoid using group actions for the basic results in the paper, even though we are led by this analogy and even though we eventually provide group-theoretic statements.

This approach is completely parallel to the way a semisimple real Lie group G with no compact factor is understood, that is, via its action on the associated symmetric space $X = G/K$ where K is a maximal compact subgroup (see [26,39]). As a result, geometric proofs of crucial tools in Lie theory and in representation theory, such as the well-known Cartan and Iwasawa decompositions for non-Archimedean semisimple Lie groups, are obtained. The main difference from Lie theory over the reals is that maximal compact subgroups are now also open, which is consistent with the fact that affine buildings are discrete structures, namely products of simplicial complexes. For instance, affine buildings of rank 1, i.e., corresponding to hyperbolic spaces, are semihomogeneous trees.

Apart from the above well-known algebraic consequences, this led to the possibility of studying questions which are also relevant to spherical harmonic analysis as inspired by the work of Harish-Chandra. Indeed, I. Satake [52] showed that non-Archimedean semisimple Lie groups can be considered in the general framework of Gelfand pairs since some suitable Hecke algebras of bi-invariant functions were shown to be commutative for the convolution law. The corresponding spherical functions were computed by I. G. Macdonald [33]. This opened the way to beautiful combinatorial problems [34]. It also provided a Gelfand–Fourier transform allowing one both to attack more advanced questions and to get a deeper understanding of analytic objects such as heat kernels.

Among these more advanced analytic problems we naturally find the ones related to random walks and integral representations of the corresponding harmonic functions. The pioneering work in this field is due to H. Furstenberg [21] who developed the crucial notion of boundaries from a probabilistic viewpoint in the Lie-theoretic context. This notion had a strong impact on many questions in group theory, in particular in rigidity theory [37], and it is likely to be still useful in many situations in geometric group theory where rich structures from Lie theory are missing and have to be replaced by more flexible measure-theoretic ones [5]. The Martin compactification procedure is relevant to this context. It deals with positive harmonic functions on symmetric spaces with respect to the

Laplace–Beltrami operator, which is a bi-invariant second order differential operator on the automorphism group. Later, it was extended to more general situations, at least at the bottom of the spectrum; for instance to integral equations with respect to well-behaved probability measures in the terminology introduced by Guivarc'h–Ji–Taylor [24]. Our aim is to construct Martin compactifications for affine buildings. Note that in this case, the absence of differential structure so to speak leads us to unify the approach via the use of averaging operators (i.e., difference operators). Averaging operators naturally correspond to Markov chains. We also wish to cover situations where no sufficiently transitive group action is available, which is a way to include some intriguing lower-dimensional exotic affine buildings (for existence see e.g. [50]), which is a first deviation from [24].

To be more precise, let us recall that an affine building $\mathscr{X}$ is a simplicial complex covered by subcomplexes all modeled on a given affine tiling, called apartments (see [10, Chapter V]). The subcomplexes are required to satisfy natural incidence axioms: any two simplices must be contained in an apartment, and given any two apartments there must exist a simplicial isomorphism fixing their intersection (see Section 1.1 for definitions). These axioms are particularly well-adapted to the construction of a complete non-positively curved distance on $\mathscr{X}$, which makes buildings a beautiful source of examples of CAT(0)-spaces for geometric group theory [12]. In this paper we try to stick as much as possible to the discrete point of view, which enables us to use more easily many notions from probability theory. The approach to Martin compactification in the discrete setup was described by J. L. Doob [20]. We want to compactify the set of all the so-called special vertices [13, Section 1.3.7] of a given affine building $\mathscr{X}$. However, when the root system of $\mathscr{X}$ is non-reduced, Fourier-analytic tools lead to splitting the set of special vertices into two subsets V_g and V_g^ε where V_g consists of the special vertices having the same type as the origin o (we call them good vertices – see Section 1.3 and [45]). In the reduced cases, all special vertices are good, so to treat all the buildings in a uniform way, we prefer to use the term good vertices. In any case, each maximal simplex (called an alcove) contains at least one good vertex, hence V_g is sufficiently large to provide a satisfactory compactification of $\mathscr{X}$.

We study an averaging operator A acting on functions on good vertices that is related to the transition function $p(x, y)$ of a finite range random walk defined on V_g (see Section 7.1) as follows:

$$Af(x) = \sum_{y \in V_g} p(x, y) f(y), \quad x \in V_g.$$

Finite range of the random walk guarantees that the embedding ι_ζ defined in (0.1) below has discrete image. We also assume that the random walk is isotropic and irreducible, which are the natural conditions corresponding to well-behaved probability measures on symmetric spaces. Let us stress that in the building case, there is no choice of a specific averaging operator which would correspond to the Laplace–Beltrami operator on symmetric spaces; this explains why we work with this class of measures. Let ϱ be the spectral radius of the operator A acting on $\ell^2(V_g)$; it can be computed in purely Lie-theoretic terms even without any group action: see formula (7.6). Classically, for each $\zeta \geqslant \varrho$ we can define the ζ-*Green function*

$$G_\zeta(x,y) = \sum_{n\geqslant 0} \zeta^{-n} p(n;x,y) \quad \text{for } x, y \in V_g,$$

which leads to the Martin kernels

$$K_\zeta(x,y) = \frac{G_\zeta(x,y)}{G_\zeta(o,y)}.$$

We are now in a position to define the Martin embedding associated with the transition function p and the real parameter ζ. Let us denote by $\mathscr{B}_\zeta(V_g)$ the set of positive ζ-*superharmonic* functions on V_g (i.e., functions f on V_g such that $Af \leqslant \zeta f$), normalized to take value 1 at the origin o. The set $\mathscr{B}_\zeta(V_g)$ endowed with the topology of pointwise convergence is a compact second countable Hausdorff space, thus it is metrizable. The corresponding Martin embedding is the map

$$\iota_\zeta : V_g \to \mathscr{B}_\zeta(V_g), \quad y \mapsto K_\zeta(\cdot, y), \tag{0.1}$$

which can be shown to be injective with discrete image, and $\mathrm{Aut}(\mathscr{X})$-equivariant for a suitably defined projective action on $\mathscr{B}_\zeta(V_g)$ (see formula (7.7)). Such an embedding with these properties is a typical map we will use in this paper in order to define compactifications in the sense of Section 1.4. The closure of the image of ι_ζ, which we denote by $\overline{\mathscr{X}}_{M,\zeta}$, is called the *Martin compactification* of $\mathscr{X}$ (associated with p and with the parameter $\zeta \geqslant \varrho$). The following theorem collects the main results of our paper.

Theorem A. *Let $\mathscr{X}$ be a thick regular locally finite affine building.*

(i) *For any isotropic irreducible finite range random walk on $\mathscr{X}$, the following dichotomy holds:*
   - [At the bottom of the spectrum] *If $\zeta = \varrho$, then $\overline{\mathscr{X}}_{M,\varrho}$ is* $\mathrm{Aut}(\mathscr{X})$*-equivariantly isomorphic to any of the Furstenberg (measure-theoretic) or the Caprace–Lécureux (combinatorial) compactifications of the set V_g of good vertices.*
   - [Above the bottom of the spectrum] *If $\zeta > \varrho$, then $\overline{\mathscr{X}}_{M,\zeta}$ is* $\mathrm{Aut}(\mathscr{X})$*-equivariantly isomorphic to the join of any of the previously mentioned compactifications with the Gromov (horofunction) compactification.*

(ii) *If the root system of the building is non-reduced, there exists an isotropic irreducible finite range random walk on $\mathscr{X}$ providing Martin compactifications of special vertices satisfying the same dichotomy.*

Note that when the root system of the building is reduced, the notions of special and good vertices coincide. In Figure 1 we present an example of the closure of an apartment in a Martin compactification when the parameter ζ is above the bottom ϱ of the spectrum. As the picture suggests, and as stated in the theorem, the Martin compactification dominates both the Gromov compactification and the (maximal) Furstenberg compactification. This is illustrated in Figure 2 at the level of closures of apartments.

The idea of the proof is (as in [24]) to use a family of remarkable unbounded sequences – called *core sequences* (1.4) – such that

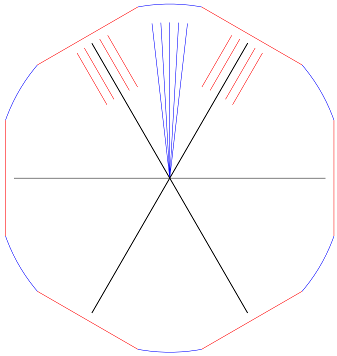

Fig. 1. Closure of an apartment in the Martin compactification above the bottom of the spectrum ($\widetilde{A}_2$ case).

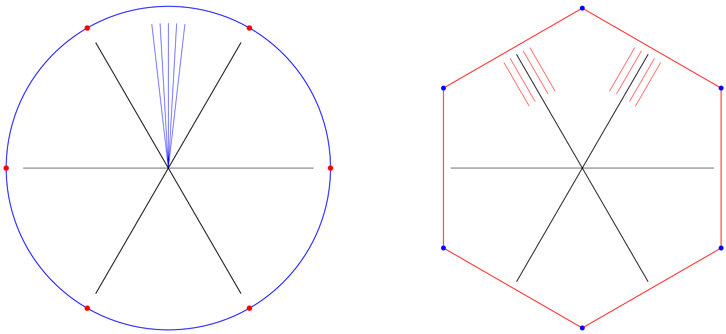

Fig. 2. Left: the closure of an apartment in the Gromov compactification. Right: the closure of an apartment in the (maximal) Furstenberg compactification ($\widetilde{A}_2$ case).

- any unbounded sequence of good vertices admits a core subsequence;
- any core sequence converges in most of our compactifications, the limit being precisely localized thanks to the parameters describing the core sequence.

In this context, identifying two compactifications amounts to showing that the exact localization of the limit of a core sequence in the boundaries is obtained via the same process from the parameters of the core sequence for both compactifications. This explains why the heart of the proof of the identification theorems as above is the combination of a convergence statement and a uniqueness statement. Consequently, Theorem A (i) in the case $\zeta = \varrho$ basically follows from Theorem 7.5 for convergence and Theorem 7.3 for uniqueness (see Theorem 7.6), and Theorem A (i) for $\zeta > \varrho$ follows from Theorem 7.7 for convergence and Theorem 7.4 for uniqueness (see Theorem 7.9); finally, Theorem A (ii) is proved in the Appendix (see Theorem A.2).

Let us now discuss some known compactifications. Many of them were initially defined when the affine building comes from a semisimple algebraic group **G** over a locally compact non-Archimedean valued field k, the first ones being due to E. Landvogt thanks to a gluing procedure [31]. In this context, by Bruhat–Tits theory [13], the group $\mathbf{G}(k)$ acts on an affine building $\mathscr{X}$ and the action is strongly transitive in the sense that $\mathbf{G}(k)$ acts transitively on the inclusions of an alcove (i.e., a maximal simplex) in an apartment. To our knowledge, the richest situation from the perspective of algebraic structures, where full Bruhat–Tits buildings are compactified (not only their vertices) and where integral models of $\mathbf{G}$ (as defined in [14]) are taken into account, is treated in [46]. This requires using Berkovich analytic geometry over non-Archimedean fields, and ultimately leads to connections with representation theory [47] and algebraic geometry [48]. The outcome is a finite family of compactifications, indexed (as for symmetric spaces) by the conjugacy classes of parabolic subgroups. In the biggest class, corresponding to the choice of a minimal parabolic subgroup, the closure of the vertices is equivariantly isomorphic to the group-theoretic compactification, hence to the Martin compactification at the bottom of the spectrum (i.e., when $\zeta = \rho$). Therefore, the case of a parameter above the bottom of the spectrum, $\zeta > \rho$, provides new compactifications of vertices which take into account additional radial parameters for the convergence of suitable unbounded sequences (in addition to distances with respect to some faces of the Weyl sectors in the building). The following theorem combines Theorems 2.7 and 7.10 (more precisely, the identification of the Martin compactification at the bottom of the spectrum with the maximal Satake–Berkovich, polyhedral or group-theoretic compactification is provided by Theorem 2.7, and the Martin compactification above the bottom of the spectrum is described in Theorem 7.10).

Theorem B. *Let $\mathbf{G}$ be a semisimple algebraic group defined over a locally compact non-Archimedean valued field k, and let $\mathscr{X}$ be its Bruhat–Tits building. Choose a good vertex in $\mathscr{X}$ and denote by K its stabilizer. Let p be a compactly supported bi-K-invariant well-behaved probability measure on $\mathbf{G}(k)$. Then the Martin compactification $\bar{\mathscr{X}}_{M,\varrho}$ can be equivariantly identified with the maximal Satake–Berkovich compactification of $\mathscr{X}$ (from analytic geometry); hence for $\zeta > \varrho$, the Martin compactification $\bar{\mathscr{X}}_{M,\zeta}$ is the join of $\bar{\mathscr{X}}_{M,\varrho}$ and the Gromov compactification.*

In other words, in Theorems A and B we provide a concrete potential-theoretic way to construct compactifications that could only be obtained artificially by means of joining two previously known compactifications (i.e., by means of embedding diagonally the building in their product and taking the closure of the image).

In this paper, we also choose to work in a situation which is as discrete as possible. This is consistent with standard references on random walks on graphs [59] and with some recent works dealing with spherical harmonic analysis on buildings, in particular those due to A. M. Mantero and A. Zappa (e.g. [35, 36]) and J. Parkinson (e.g. [44, 45]). This also leads to significant differences from [24]. Moreover, the statements of Theorem A are valid even with small, possibly trivial, automorphism groups for $\mathscr{X}$: according to J. Tits' classification [56], there exist affine buildings that are not relevant to any algebraic group situation only when the dimension is 1 (trees) or 2. Nevertheless, the excluded cases are very interesting because in dimension 2 they lead to situations in which natural questions such as the linearity of some automorphism groups, (super)rigidity, arithmeticity or, on the contrary, simplicity properties of lattices, and also property (T) and its strengthenings, make sense and are strong motivations to develop more tools of geometric and analytic nature (see for instance [4, 32] for striking recent results in this field).

Another purely geometric approach was developed by C. Charignon [17] and G. Rousseau [51]. For our purposes this is an interesting viewpoint since, without any group action, it sticks as much as possible to Bruhat–Tits' original approach governed by Lie combinatorics. By this we mean that the compactifications obtained, isomorphic to the previously mentioned ones, are those which make most easily appear the modular structure of the compact space: the boundary can be seen as a disjoint union of affine buildings at infinity of smaller rank, called façades ([1]) in loc. cit. In the Bruhat–Tits case, these façades are proven to be the affine buildings attached to the parabolic subgroups of the initial non-Archimedean semisimple group [46]. We rely on this known structure in order to study the convergence of sequences of harmonic measures on affine buildings in the spirit of Furstenberg compactifications. Each of these harmonic measures is defined on the maximal boundary Ω of the affine building $\mathscr{X}$, which is the set of parallelism classes of sectors endowed with a natural totally disconnected topology (Ω, as a set, consists of the chambers of the spherical building at infinity $\mathscr{X}^\infty$; see [1, Section 11.8]).

Each harmonic measure is attached to a well-defined special vertex in the building and, roughly speaking, is characterized by the fact that it is the most symmetric probability measure on Ω with respect to that vertex (see [43, Chapter 7] and Section 5.1). Noting that each stratum at infinity, being an affine building, can carry its own harmonic measures on its own maximal boundary, the following result, which we prove in Theorem 6.4, makes sense.

Theorem C. *Let $\mathscr{X}$ be a thick regular locally finite affine building. The closure of the collection of harmonic measures on $\mathscr{X}$ in the space $\mathcal{P}(\Omega)$ of probability measures on the maximal boundary Ω endowed with the weak-$*$ topology is* $\mathrm{Aut}(\mathscr{X})$*-equivariantly isomorphic to the polyhedral and to the combinatorial compactification of $\mathscr{X}$. More precisely, the maximal boundary of each affine building at infinity, or stratum, can be seen as a residue in Ω and any cluster value of any unbounded sequence of harmonic measures in $\mathscr{X}$ is a harmonic measure on a well-defined stratum.*

Let us roughly sum up this part of the paper. Affine buildings provide a suitable framework to generalize to higher dimensions the classical study of harmonic measures on infinite graphs (since the set of initial harmonic measures has a strong geometric structure related to Lie theory). The resulting measures at infinity are again harmonic measures for smaller affine buildings in the boundary, and therefore provide an analogous geometric structure at infinity. We emphasize that in our purely geometric context, we cannot use the interpretation of maximal boundaries in terms of maximal flag varieties, and the related fibrations from the maximal flag variety to a flag variety associated with a non-minimal parabolic subgroup. Instead, we have to geometrically investigate sets of residues of a given type and interpret them as maximal boundaries of smaller affine buildings

[1] The term façade comes from a French word meaning the front face of a building; here a façade is an affine building at infinity.

(at infinity). We also need to prove a disintegration formula for harmonic measures in this context which leads to measure-theoretic substitutes for maps between flag varieties.

The reader who knows about compactifications of non-compact Riemannian symmetric spaces has already understood that our results perfectly parallel the latter situation, at least at the level of the statements obtained. From this perspective, we owe a lot to the book [24] by Y. Guivarc'h, L. Ji and J.-C. Taylor (see also [22, 23]) where, among many other things, the precise descriptions of the Martin compactifications of symmetric spaces, both at the bottom and above the bottom of the spectrum, are given. We have taken from that book the idea of exhibiting well-chosen classes of unbounded sequences that become convergent after applying a suitable embedding in a compact metrizable target space (e.g. a space of probability measures on a flag manifold, the Chabauty space of closed subgroups of the isometry group etc.); this is a good tool to compare the various compactifications, including the most algebraic ones [46]. The book and Y. Guivarc'h's cited articles are the first places where most Bruhat–Tits analogues were conjectured. Those references mainly study Riemannian symmetric spaces, for which the most important ingredients are sufficiently precise Green kernel estimates [3] and some uniqueness results for solutions of well-chosen PDEs taking into account invariance under suitable group actions.

We decided to push the logic of using sequences to its fullest application. This means that we have chosen to study the compactifications, possibly by adapting the class of sequences according to the finally expected boundary, through the parametrization of the points at infinity provided by the initial data characterizing the unbounded sequence used. Both for symmetric spaces and for Bruhat–Tits buildings, we know that the closure of a Weyl sector completely describes the compactification (this is a consequence of the Cartan decomposition). For a given compactification, the question is then to know which geometric parameters related to these simplicial cones are to be taken into account. For the Gromov compactification, whose boundary is in all cases (Archimedean or not) a single spherical building, we know that the parameters are radial and provide a direction of escape in the cone. In the case of all other compactifications considered, with the exception of the Martin compactification above the bottom of the spectrum, the correct parameters are a partition of codimension 1 faces of Weyl sectors into two subsets: one for which the distances to the corresponding walls explode and the other for which the distances to the walls converge. In our approach, we obtain an identification between compactifications by showing that the redundancies of parametrization of the limit points according to the sequence parameters are exactly the same on both sides. The beauty of the Martin compactification above the bottom of the spectrum is that we have to use a similar partition to those before, but we use an additional radial (partial) parameter for the subset of walls with exploding distances (see Theorem 7.7); this explains why this compactification is obtained by joining the visual compactification and any of the other previously mentioned compactifications.

Apart from the fact that we avoid using group actions in order to make our results available for the study of exotic situations, another significant difference from the book [24] is that one key ingredient there was provided by estimates of the heat kernel and of Green functions due to J.-Ph. Anker and L. Ji [3], while here we use the asymptotics for the Green kernels, obtained by the second author [57]. Those asymptotics are exact, so they can be directly used for our purposes. In particular, we cannot (and need not) use uniqueness arguments for solutions of partial differential equations, assumed in addition to be invariant under some unipotent subgroup of the full isometry group (as in [24, Theorem 7.22]). This more direct approach can be seen as a further manifestation of the fact that some formulas in spherical harmonic analysis can be simplified more efficiently in the non-Archimedean case: Harish-Chandra's integral formula for spherical functions remains what it is on $\mathbb{R}$, while it was algebraized by I. G. Macdonald already in the 1970s [33].

Choices and conventions

Let us quickly repeat some choices. We are generically dealing with affine buildings without assuming the existence of any sufficiently transitive group action. When dealing with affine buildings arising from semisimple groups over local fields, we will explicitly call the corresponding spaces *Bruhat–Tits buildings*; in other words, no Bruhat–Tits building if no group of rational points $\mathbf{G}(k)$. Also, we use the notation $\mathrm{Stab}_G(x)$ for the stabilizer of a point $x \in X$ in a group G acting on X.

In order to optimally use analytic formulas (e.g. Green kernel asymptotics), we mainly see buildings as sets of (special or good) vertices; the only exception is Section 2 introducing affine buildings at infinity according to G. Rousseau's approach (façades in his terminology). Moreover, for a given affine building, the only apartment system we use on it is the complete one (in other words, in metric language, any subset isometric to a Euclidean space and maximal for this property is an apartment).

Finally, we will use subroot systems corresponding to subsets of simple roots; if I is such a subset, then the index $_I$ attached to the standard notation for a given notion will mean that the object under consideration is defined with respect to the subroot system generated by I. This convention applies to root systems: Φ_I is the subroot system generated by I, but also to analytic notions: for instance, c_I will be the Harish-Chandra function associated with Φ_I etc.

Structure of the paper

Section 1 recalls as quickly as possible the useful notions from building theory; it introduces the relevant classes of unbounded sequences and the definition of a compactification in our context. Section 2 is where we use different, more metric, definitions of buildings, in order to recall the purely geometric construction of affine buildings at infinity. Section 3 contains a discrete variation on the theme of visual compactifications; this adaptation is useful to describe Martin compactifications above the bottom of the spectrum when seeing a building as a set of vertices. Section 4 deals with combinatorial compactifications of buildings which can be introduced in a remarkably elementary way in the affine case and which will mainly be used as a tool in the paper. Section 5 deals with harmonic measures on the maximal boundary of affine buildings; it contains results preparing the study of the

Furstenberg compactification which may be useful in its own right. The goal of Section 6 is precisely to describe compactifications of affine buildings obtained by suitably embedding them into the spaces of probability measures on maximal boundaries; the point is to show that cluster values of unbounded sequences of harmonic measures are still harmonic measures for affine buildings at infinity. Section 7 is the main part of the paper: it studies the Martin compactifications of affine buildings and proves the main Theorem A; this requires recalling some notions from potential and probability theory. In Appendix A, we construct a distinguished random walk when the root system of the affine building is non-reduced; this random walk provides the desired Martin compactifications on the set of all special vertices (not only the good ones). Note that at the end of each relevant section, we illustrate our geometric results by providing their Bruhat–Tits consequences.

1. Affine buildings and compactifications

1.1. Buildings

A family $\mathscr{X}$ of non-empty finite subsets of some set V is an *abstract simplicial complex* if for all $\sigma \in \mathscr{X}$, each subset $\gamma \subseteq \sigma$ also belongs to $\mathscr{X}$. The elements of $\mathscr{X}$ are called *simplices*. The *dimension* of a simplex σ is $\#\sigma - 1$. Zero-dimensional simplices are called *vertices*. The set $V(\mathscr{X}) = \bigcup_{\sigma \in \mathscr{X}} \sigma$ is the *vertex set* of $\mathscr{X}$. The *dimension* of the complex $\mathscr{X}$ is the maximal dimension of its simplices. A *face* of a simplex σ is a non-empty subset $\gamma \subseteq \sigma$. For a simplex σ we denote by $\mathrm{St}(\sigma)$ the collection of simplices containing σ; in particular, $\mathrm{St}(\sigma)$ is a simplicial complex. Two abstract simplicial complexes $\mathscr{X}$ and $\mathscr{X}'$ are *isomorphic* if there is a bijection $\psi : V(\mathscr{X}) \to V(\mathscr{X}')$ such that for all $\sigma = \{x_1, \ldots, x_k\} \in \mathscr{X}$ we have $\psi(\sigma) = \{\psi(x_1), \ldots, \psi(x_k)\} \in \mathscr{X}'$. With every abstract simplicial complex $\mathscr{X}$ one can associate its *geometric realization* $|\mathscr{X}|$ in the vector space of functions $V \to \mathbb{R}$ with finite support (see e.g. [42, Section 2]).

A set $\mathscr{C}$ equipped with a collection $\{\sim_i : i \in I\}$, where $I = \{0, \ldots, r\}$, of equivalence relations is called a *chamber system*. The elements of $\mathscr{C}$ are called *chambers* and the relation $\sim_i$ is called *i-adjacency*. A *gallery* of type $f = i_1 \cdots i_k$ in $\mathscr{C}$ is a sequence $(c_1, \ldots, c_k)$ of chambers such that for all $j \in \{2, \ldots, k\}$, we have $c_{j-1} \sim_{i_j} c_j$ and $c_{j-1} \neq c_j$. If $J \subseteq I$, a *residue* of type J is a subset of $\mathscr{C}$ such that any two chambers can be joined by a gallery of type $f = i_1 \cdots i_k$ with $i_1, \ldots, i_k \in J$. From a chamber system $\mathscr{C}$ we can construct an abstract simplicial complex where each residue of type J corresponds to a simplex of dimension $r - \#J$. Then, for a given vertex x, we denote by $\mathcal{C}(x)$ the set of chambers containing x.

A *Coxeter group* is a group W given by a presentation

$$\langle r_i : (r_i r_j)^{m_{i,j}} = 1 \text{ for all } i, j \in I \rangle$$

where $M = (m_{i,j})_{I \times I}$ is a symmetric matrix with entries in $\mathbb{Z} \cup \{\infty\}$ such that for all $i, j \in I$,

$$m_{i,j} = \begin{cases} \geqslant 2 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

For a word $f = i_1 \cdots i_k$ in the free monoid I we denote by r_f the element of W given by $r_f = r_{i_1} \cdots r_{i_k}$. The *length* of $w \in W$, denoted $\ell(w)$, is the smallest integer k such that there is a word $f = i_1 \cdots i_k$ and $w = r_f$. We say that f is *reduced* if $\ell(r_f) = k$. A Coxeter group W may be turned into a chamber system by introducing in W the following collection of equivalence relations: $w \sim_i w'$ if and only if $w = w'$ or $w = w' r_i$. The corresponding simplicial complex Σ is called a *Coxeter complex*.

A simplicial complex $\mathscr{X}$ is called a *building of type* Σ if it contains a family of subcomplexes called *apartments* such that

(B0) each apartment is isomorphic to Σ,

(B1) any two simplices of $\mathscr{X}$ lie in a common apartment,

(B2) for any two apartments $\mathscr{A}$ and $\mathscr{A}'$ having a chamber in common there is an isomorphism $\psi : \mathscr{A} \to \mathscr{A}'$ fixing $\mathscr{A} \cap \mathscr{A}'$ pointwise.

The *rank* of the building is the cardinality of the set I. We always assume that $\mathscr{X}$ is irreducible. A simplex c is a chamber in $\mathscr{X}$ if it is a chamber in any of its apartments. We denote by $C(\mathscr{X})$ the set of chambers in $\mathscr{X}$. Using the building axioms we see that $C(\mathscr{X})$ has a chamber system structure. However, it is not unique. A geometric realization of the building $\mathscr{X}$ is its geometric realization as an abstract simplicial complex. In this article we assume that the system of apartments in $\mathscr{X}$ is *complete*, meaning that any subcomplex of $\mathscr{X}$ isomorphic to Σ is an apartment. We denote by $\mathrm{Aut}(\mathscr{X})$ the group of automorphisms of the building $\mathscr{X}$.

1.2. Affine Coxeter complexes

In this section we recall basic facts about root systems and Coxeter groups. A general reference is [10], which deals with Coxeter systems attached to reduced root systems. Since from the outset we use possibly non-reduced root systems, we will also refer to [36, 43].

Let Φ be an irreducible, but not necessarily reduced, finite root system in Euclidean space $\mathfrak{a}$ with associated norm denoted by $|\cdot|$. Select a fixed base $\{\alpha_i : i \in I_0\}$ of Φ, where $I_0 = \{1, \ldots, r\}$, and let Φ^+ be the corresponding set of all positive roots. Since Φ is irreducible, there is a unique highest root $\alpha_0 = \sum_{i \in I_0} m_i \alpha_i$, $m_i \in \mathbb{N}_0$. We set

$$I_g = \{0\} \cup \{i \in I_0 : m_i = 1\}.$$

For each $\alpha \in \Phi$, we define the dual root

$$\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha.$$

Let $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ be the dual root system. Then the *coroot lattice* Q is the $\mathbb{Z}$-span of $\Phi^\vee$. Let $Q^+ = \sum_{\alpha \in \Phi^+} \mathbb{N}_0 \alpha^\vee$. The dual basis to $\{\alpha_i : i \in I_0\}$ is formed by the fundamental coweights $\{\lambda_i : i \in I_0\}$. The *coweight lattice* P is the $\mathbb{Z}$-span of $\{\lambda_i : i \in I_0\}$. A coweight $\lambda \in P$ is called *dominant* if $\lambda = \sum_{i \in I_0} x_i \lambda_i$ with $x_i \geqslant 0$ for all $i \in I_0$. Finally, the cone of all dominant coweights is denoted by P^+. If $x_i > 0$ for all $i \in I_0$, then λ is *strongly dominant*. We set

$$\tilde{\rho} = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

Let $\mathscr{H}$ be the family of affine hyperplanes, called *walls*, of the form

$$H_{\alpha;k} = \{x \in \mathfrak{a} : \langle x, \alpha \rangle = k\}$$

where $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$. Each wall determines two half-apartments

$$H^-_{\alpha;k} = \{x \in \mathfrak{a} : \langle x, \alpha \rangle \leqslant k\} \quad \text{and} \quad H^+_{\alpha;k} = \{x \in \mathfrak{a} : \langle x, \alpha \rangle \geqslant k\}.$$

Note that for a given α, the family $H^-_{\alpha;k}$ is increasing in k while $H^+_{\alpha;k}$ is decreasing. To each wall we associate $r_{\alpha;k}$, the orthogonal reflection in $\mathfrak{a}$, defined by

$$r_{\alpha;k}(x) = x - (\langle x, \alpha \rangle - k)\alpha^\vee.$$

Set $r_0 = r_{\alpha_0;1}$, and $r_i = r_{\alpha_i;0}$ for each $i \in I_0$.

The *finite Weyl group* W is the subgroup of $\mathrm{GL}(\mathfrak{a})$ generated by $\{r_i : i \in I_0\}$. Let us denote by w_0 the longest element in W. The *fundamental sector* in $\mathfrak{a}$ defined by

$$S_0 = \{x \in \mathfrak{a} : \langle x, \alpha_i \rangle \geqslant 0 \text{ for all } i \in I_0\} = \bigoplus_{i \in I_0} \mathbb{R}_+ \lambda_i = \bigcap_{i \in I_0} H^+_{\alpha_i;0}$$

is the fundamental domain for the action of W on $\mathfrak{a}$.

The *affine Weyl group* W^a is the subgroup of $\mathrm{Aff}(\mathfrak{a})$ generated by $\{r_i : i \in I\}$. Observe that W^a is a Coxeter group. The hyperplanes $\mathscr{H}$ give the geometric realization of its Coxeter complex Σ_Φ. To see this, let $C(\Sigma_\Phi)$ be the family of closures of the connected components of $\mathfrak{a} \setminus \bigcup_{H \in \mathscr{H}} H$. We denote by C_0 the *fundamental chamber* (or *fundamental alcove*) defined by

$$C_0 = \{x \in \mathfrak{a} : \langle x, \alpha_0 \rangle \leqslant 1 \text{ and } \langle x, \alpha_i \rangle \geqslant 0 \text{ for all } i \in I_0\} = \bigcap_{i \in I_0} H^+_{\alpha_i;0} \cap H^-_{\alpha_0;1},$$

which is the fundamental domain for the action of W^a on $\mathfrak{a}$. Moreover, the group W^a acts simply transitively on $C(\Sigma_\Phi)$. This allows us to introduce a chamber system in $C(\Sigma_\Phi)$: For two chambers C and C' and $i \in I$, we set $C \sim_i C'$ if and only if $C = C'$ or there is $w \in W^a$ such that $C = w.C_0$ and $C' = wr_i.C_0$.

The vertices of C_0 are $\{0, \lambda_1/m_1, \ldots, \lambda_r/m_r\}$. Let us denote by $V(\Sigma_\Phi)$ the set of vertices of all $C \in C(\Sigma_\Phi)$. Under the action of W^a, the set $V(\Sigma_\Phi)$ is made up of the $r+1$ orbits $W^a.0$ and $W^a.(\lambda_i/m_i)$ for all $i \in I_0$. Thus setting $\tau_{\Sigma_\Phi}(0) = 0$, and $\tau_{\Sigma_\Phi}(\lambda_i/m_i) = i$ for $i \in I_0$, we obtain the unique labeling $\tau_{\Sigma_\Phi} : V(\Sigma_\Phi) \to I$ such that any chamber $C \in C(\Sigma_\Phi)$ has one vertex with each label.

For each simplicial automorphism $\varphi : \Sigma_\Phi \to \Sigma_\Phi$ there is a permutation π of the set I such that for all chambers C and C', we have $C \sim_i C'$ if and only if $\varphi(C) \sim_{\pi(i)} \varphi(C')$, and

$$\tau_{\Sigma_\Phi}(\varphi(v)) = \pi(\tau_{\Sigma_\Phi}(v)) \quad \text{for all } v \in V(\Sigma_\Phi).$$

A vertex v is called *special* if for each $\alpha \in \Phi^+$ there is k such that $v \in H_{\alpha;k}$. The set of all special vertices is denoted by $V_s(\Sigma_\Phi)$. A codimension 1 simplex whose vertices have their labels in $I \setminus \{i\}$ is called an i*-panel*.

Given $\lambda \in P$ and $w \in W^a$, the set $S = \lambda + w.S_0$ is called a *sector* in Σ_Φ with a *base vertex* λ. Its *sector* i*-panel* is $\lambda + w.(S_0 \cap H_{\alpha_i;0})$.

Moreover, by [1, Corollary 3.20], an affine Coxeter complex Σ_Φ uniquely determines the affine Weyl group W^a but not a finite root system Φ. In fact, the root systems C_r and BC_r have the same affine Weyl group.

1.3. Affine buildings

A building $\mathscr{X}$ of type Σ is called an *affine building* if Σ is a Coxeter complex corresponding to an affine Weyl group. Select a chamber c_0 in $C(\mathscr{X})$ and an apartment $\mathscr{A}_0$ containing c_0. Using an isomorphism $\psi_0 : \mathscr{A}_0 \to \Sigma$ such that $\psi_0(c_0) = C_0$, we define the labeling in $\mathscr{A}_0$ by

$$\tau_{\mathscr{A}_0}(v) = \tau_\Sigma(\psi_0(v)), \quad v \in V(\mathscr{A}_0).$$

Now, thanks to the building axioms the labeling can be uniquely extended to $\tau : V(\mathscr{X}) \to I$. We turn $C(\mathscr{X})$ into a chamber system over I by declaring two chambers c and c' to be i-adjacent if they share all vertices except the one of type i (equivalently, they intersect along an i-panel). For each $c \in C(\mathscr{X})$ and $i \in I$, we define

$$q_i(c) = \#\{c' \in C(\mathscr{X}) : c' \sim_i c\} - 1.$$

Throughout, we *assume* that $q_i(c)$ only depends on i, i.e., $q_i(c)$ is independent of c, and therefore the building $\mathscr{X}$ is *regular*; we henceforth write q_i instead of $q_i(c)$. We also assume that $1 < q_i < \infty$ and therefore the building $\mathscr{X}$ is *thick* and *locally finite*. Notice that for Bruhat–Tits buildings all the assumptions about thickness are automatic. A vertex of $\mathscr{X}$ is special if it is special in any of its apartments. The set of special vertices is denoted by V_s. We choose the finite root system Φ in such a way that Σ is its Coxeter complex. In all cases except when the affine group has type C_r or BC_r, the choice is unique. In the remaining cases we select C_r if $q_0 = q_r$, otherwise we take BC_r. This guarantees that $q_{\tau(\lambda)} = q_{\tau(\lambda+\lambda')}$ for all $\lambda, \lambda' \in P$; see the discussion in [36, Section 2.13].

Let us define the set V_g of *good* vertices to consist of those $x \in V(\mathscr{X})$ having the type $\tau(x) \in I_g$. If Φ is reduced, then all special vertices are good. However, we do not have this property in the case of BC_r. Indeed, there are two types of special vertices, 0 and r, but only those of type 0 are mapped to P. To deal with the vertices of type r we modify the isomorphisms ψ_0 that we have used to define τ. For this purpose we use the non-trivial automorphism φ_ε of the Coxeter complex which permutes the vertices of the base chamber C_0. Let ε be the corresponding permutation of the set I. We compose the isomorphism ψ_0 with the automorphism φ_ε getting a new isomorphism ψ_ε which we use to get the new labeling $\tau_\varepsilon = \varepsilon \circ \tau$ on $V(\mathscr{X})$, which we call ε*-type*. In particular, the vertices of type r have ε-type 0 and vice versa. We set $V_g^\varepsilon = \{x \in V(\mathscr{X}) : \tau(x) = r\}$. Hence, $V_s = V_g \sqcup V_g^\varepsilon$. Now, all the statements for good vertices hold true for V_g^ε after application of the permutation ε. In general, all notations with superscript ε will correspond to the standard notions after applying this twist of types.

A *half-apartment* in $\mathscr{X}$ is a half-apartment in any of its apartments. For a subset X of $V(\mathscr{X})$, the *convex hull* of X is the set of all vertices of $\mathscr{X}$ that belong to every half-apartment containing X. The convex hull of two vertices x and y is denoted by $[x, y]$. A subcomplex $\mathscr{S}$ is called a *sector* of $\mathscr{X}$ if it is a sector in any apartment. We say that π is the *sector i-panel* of $\mathscr{S}$ if there are an apartment $\mathscr{A}$ containing $\mathscr{S}$ and a type-preserving isomorphism $\psi : \mathscr{A} \to \Sigma$ such that $\psi(\pi)$ is the sector i-panel of the sector $\psi(\mathscr{S})$. Two sectors are *equivalent* (often called *parallel*) if they contain a common subsector. The set of equivalence classes of sectors is denoted by Ω and is called the *maximal boundary* of $\mathscr{X}$. For any special vertex $x \in V_s$ and $\omega \in \Omega$, there is a unique sector denoted by $[x, \omega]$ which has base vertex x and represents ω. For any $\omega_1, \omega_2 \in \Omega$ there is an apartment of $\mathscr{X}$ containing the sectors $[x, \omega_1]$ and $[x, \omega_2]$ for a certain special vertex x. If the apartment is unique we say that ω_1 and ω_2 are *opposite* each other and the apartment is denoted by $[\omega_1, \omega_2]$. For $x \in V_s$ and $y \in V(\mathscr{X})$, we set

$$\Omega(x, y) = \{\omega \in \Omega : y \in [x, \omega]\}.$$

Then for each $x \in V_s$, the collection $\{\Omega(x, y) : y \in V_s\}$ generates a totally disconnected compact Hausdorff topology in Ω. In fact, the topology is independent of the choice of the reference vertex x (see e.g. [36, Proposition 3.15]). Let us observe that each $\Omega(x, y)$ is clopen in Ω. The action of the automorphism group $\mathrm{Aut}(\mathscr{X})$ can be extended to a continuous action on Ω. We fix once and for all the origin o which is a good vertex of the chamber c_0.

The maximal boundary Ω can be equipped with adjacency relations to turn it into a chamber system. To be more precise, we say that two chambers ω and ω' are i-adjacent if there are sectors $\mathscr{S}$ and $\mathscr{S}'$ representing ω and ω' respectively, and such that $\mathscr{S} \cap \mathscr{S}'$ contains an i-panel. In fact, the resulting structure is a spherical building called the *spherical building at infinity* $\mathscr{X}^\infty$. For details see e.g. [49, Theorem 9.6].

Given two special vertices $x, y \in V_s$, let $\mathscr{A}$ be an apartment containing x and y and let $\psi : \mathscr{A} \to \Sigma$ be a type-rotating isomorphism such that $\psi(x) = 0$ and $\psi(y) \in S_0$ (see [43, Definition 4.1.1]). We set $\sigma(x, y) = \psi(y) \in \frac{1}{2}P^+$. If x and y are good vertices then $\sigma(x, y) \in P^+$. For $\lambda \in P^+$ and $x \in V_g$, we denote by $V_\lambda(x)$ the set of all good vertices $y \in V_g$ such that $\sigma(x, y) = \lambda$. The building axioms entail that the cardinality of $V_\lambda(x)$ depends only on λ (see [45, Proposition 1.5]). Let N_λ be that common value.

Let Φ^{++} be the set of roots $\alpha \in \Phi^+$ such that $\frac{1}{2}\alpha \notin \Phi^+$. If $\alpha \in \Phi^{++}$ then $q_\alpha = q_i$ provided that $\alpha \in W.\alpha_i$ for $i \in I$. We define

$$\tau_\alpha = \begin{cases} 1 & \text{if } \alpha \notin \Phi, \\ q_\alpha & \text{if } \alpha \in \Phi, \text{ but } \frac{1}{2}\alpha, 2\alpha \notin \Phi, \\ q_{\alpha_0} & \text{if } \alpha, \frac{1}{2}\alpha \in \Phi, \text{ and therefore } 2\alpha \notin \Phi, \\ q_\alpha q_{\alpha_0}^{-1} & \text{if } \alpha, 2\alpha \in \Phi, \text{ and therefore } \frac{1}{2}\alpha \notin \Phi. \end{cases}$$

We set

$$\eta = \frac{1}{2} \sum_{\alpha \in \Phi^+} (\log \tau_\alpha) \alpha = \frac{1}{2} \sum_{\alpha \in \Phi^{++}} (\log \tau_\alpha \tau_{2\alpha}^2) \alpha.$$

Since r_i sends α_i to $-\alpha_i$ and permutes other indivisible positive roots, we have

$$r_i.\eta = \eta - (\log \tau_{\alpha_i} \tau_{2\alpha_i}^2) \alpha_i.$$

and thus

$$\eta = \frac{1}{2} \sum_{i=1}^{r} (\log \tau_{\alpha_i} \tau_{2\alpha_i}^2) \langle \alpha_i, \alpha_i \rangle \lambda_i. \tag{1.1}$$

Let us define a multiplicative function on $\mathfrak{a}$ by

$$\chi(\lambda) = \prod_{\alpha \in \Phi^+} \tau_\alpha^{\langle \lambda, \alpha \rangle}, \quad \lambda \in \mathfrak{a}.$$

For $w \in W^a$ having the reduced expression $w = r_{i_1} \cdots r_{i_k}$, we set $q_w = q_{i_1} \cdots q_{i_k}$. Then by [45, Proposition 1.5 & Appendix A],

$$N_\lambda = \frac{W(q^{-1})}{W_\lambda(q^{-1})} \chi(\lambda) \tag{1.2}$$

where $W_\lambda = \{w \in W : w.\lambda = \lambda\}$, and where for any subset $U \subseteq W$ we set

$$U(q^{-1}) = \sum_{w \in U} q_w^{-1}.$$

We also have

$$N_{\varepsilon;\lambda} = \frac{W(q_\varepsilon^{-1})}{W_\lambda(q_\varepsilon^{-1})} \chi(\lambda). \tag{1.3}$$

Lastly, let us define the *horocycle* (or Busemann) *function* $h : V_s \times V_s \times \Omega \to \frac{1}{2}P$: for two special vertices x and y, and $\omega \in \Omega$, we set

$$h(x, y; \omega) = \sigma(x, z) - \sigma(y, z) \tag{1.4}$$

where z is any special vertex belonging to $[x, \omega] \cap [y, \omega]$. In fact, the value $h(x, y; \omega)$ is independent of z (see e.g. [36, Proposition 3.3]). Figure 3 gives the geometric interpretation of $h(x, y; \omega)$.

In view of Kostant's convexity theorem (see [30] or [45, Lemma 3.19]), if $\sigma(x, y) = \lambda$ then for each $\omega \in \Omega$ and $w \in W$, we have $\lambda - w.h(x, y; \omega) \in Q^+$. In particular,

$$|h(x, y; \omega)| \leqslant |\sigma(x, y)| \quad \text{and} \quad \langle h(x, y; \omega), \tilde{\rho} \rangle \leqslant \langle \sigma(x, y), \tilde{\rho} \rangle. \tag{1.5}$$

Moreover, h satisfies the *cocycle relation*, that is, for all $x, y, z \in V_s$ and $\omega \in \Omega$,

$$h(x, y; \omega) = h(x, z; \omega) + h(z, y; \omega). \tag{1.6}$$

Fix $x \in V_s$ and $\omega \in \Omega$, and let $\mathscr{A}$ be an apartment containing the sector $[x, \omega]$. The *retraction* on $\mathscr{A}$ with respect to ω and centered at x is the mapping $\rho_{\mathscr{A}}^{x,\omega} : \mathscr{X} \to \mathscr{A}$

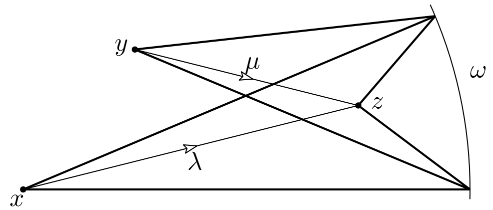

Fig. 3. Geometric interpretation of the quantity $h(x, y; \omega) = \lambda - \mu$.

defined as follows: If γ is any simplex in $\mathscr{X}$, there is a subsector $\mathscr{S}$ of $[x, \omega]$ such that $\mathscr{S}$ and γ are in one apartment $\mathscr{A}'$. By the building axiom (B2), there is an isomorphism $\phi : \mathscr{A}' \to \mathscr{A}$ fixing $\mathscr{S}$ pointwise. Then $\rho_{\mathscr{A}}^{x,\omega}(\gamma) = \phi(\gamma)$. This definition is independent of the choice of $\mathscr{A}'$. If x is a special vertex and $\psi : \mathscr{A} \to \Sigma$ is a type-rotating isomorphism such that $\psi([x, \omega]) = S_0$, then

$$h(x, y; \omega) = \psi(\rho_{\mathscr{A}}^{x,\omega}(y))$$

for all $y \in V_s$.

Given $\omega \in \Omega$, let $\Omega'(\omega)$ denote the set of all $\omega' \in \Omega$ which are opposite ω. It is an open subset of Ω. Indeed, let $\omega' \in \Omega(\omega)$. Set $\mathscr{A} = [\omega', \omega]$. We select any special vertex $x \in \mathscr{A}$ and take $y \neq x$, $y \in [x, \omega']$. Then $\rho_{\mathscr{A}}^{y,\omega}$ restricted to $[y, \omega] \cap [y, \omega'']$, for any $\omega'' \in \Omega(x, y)$, is an isomorphism onto $[y, \omega] \cap [y, \omega']$. Hence, there is a unique apartment containing $[x, \omega]$ and $[x, \omega']$, thus $\Omega(x, y) \subset \Omega'(\omega)$. The subset $\Omega'(\omega)$ is called the *big cell* associated with ω.

Lemma 1.1. *Let F be a facet in the spherical building at infinity $\mathscr{X}^\infty$. Then the corresponding residue $R(F)$ is closed as a subset of the maximal boundary Ω.*

Proof. Let R be the residue attached to F; it is a proper subset of Ω. Let us show that $\Omega \setminus R$ is open. We pick $\omega \notin R$. There exists an apartment $\mathscr{A}$ whose boundary $\mathscr{A}^\infty$ contains F and ω. Let $-\omega$ be the chamber in $\mathscr{A}^\infty$ which is opposite ω. Since the big cell $\Omega'(-\omega)$ is an open subset containing ω, it is enough to show that $\Omega'(-\omega) \cap R = \varnothing$. Let ρ denote the retraction of the spherical building $\mathscr{X}^\infty$ onto its apartment $\mathscr{A}^\infty$ centered at $-\omega$. The map ρ preserves the Weyl distance from $-\omega$ and preserves each adjacency relation in $\mathscr{X}^\infty$; here the first statement implies that $\rho(\Omega'(-\omega)) = \{\omega\}$ and the second implies that $\rho(R)$ is the residue of F in $\mathscr{A}^\infty$. As a consequence, since $\omega \notin R$ we obtain $\Omega'(-\omega) \cap R = \varnothing$, as required. ∎

1.4. Compactifications

A *compactification* of the building $\mathscr{X}$ is a pair (ι, H) where H is a compact second countable Hausdorff space and ι is an embedding of all special vertices of $\mathscr{X}$ into H such that $\iota(V_s)$ is a discrete set. Then the closure of the image $\iota(V_s)$ equipped with the induced topology is a compact Hausdorff space. Since H is metrizable, to describe this closure it is sufficient to consider sequences (x_n) of special vertices such that $(\iota(x_n))$ converges in H. If G is a subgroup of the automorphism group of $\mathscr{X}$ that acts on H, the compactification is called a G-*compactification* if ι is G-equivariant, that is, $\iota(g.x) = g.\iota(x)$ for all $x \in V_s$ and $g \in G$.

We say that (ι_1, H_1) *dominates* (ι_2, H_2) if there is a continuous mapping $\eta : H_1 \to H_2$ such that $\iota_2 = \eta \circ \iota_1$. If η happens to be a homeomorphism then (ι_1, H_1) is said to be *isomorphic* to (ι_2, H_2). Lastly, if both are G-compactifications then (ι_1, H_1) is G-*isomorphic* to (ι_2, H_2) provided η is G-equivariant.

Given two compactifications (ι_1, H_1) and (ι_2, H_2) of the building $\mathscr{X}$, we can produce another compactification $(\iota, H_1 \times H_2)$ by setting $\iota(x) = (\iota_1(x), \iota_2(x))$ for $x \in V_s$. It is denoted by $(\iota_1, H_1) \vee (\iota_2, H_2)$. Notice that $(\iota_1, H_1) \vee (\iota_2, H_2)$ is the smallest compactification that dominates both (ι_1, H_1) and (ι_2, H_2).

To describe compactifications, we study sequences of good vertices of $\mathscr{X}$ approaching infinity. To be precise, a sequence $(x_n : n \in \mathbb{N})$ of vertices of $\mathscr{X}$ *approaches infinity* if for any finite subset $F \subset V_s$ there is $N \in \mathbb{N}$ such that $x_n \notin F$ for all $n \geqslant N$. In fact, the way the sequence approaches infinity can be specified. A sequence $(x_n : n \in \mathbb{N})$ of good vertices of $\mathscr{X}$ is a *core sequence* if all x_n have the same type and there are $\omega \in \Omega$ and a sequence $(u_n : n \in \mathbb{N}_0)$ of good vertices, a subset $J \subsetneq I_0$, possibly empty, and numbers $(c_j : j \in J) \in \frac{1}{2}\mathbb{N}_0^J$, such that

(1) $u_n \in [o, \omega]$;

(2) for all $m \geqslant n$,

$$\Omega(o, x_m) \subseteq \Omega(o, u_m) \subsetneq \Omega(o, u_n), \quad \text{but} \quad \Omega(o, x_m) \nsubseteq \Omega(o, u_{m+1});$$

(3) for all $j \in J$ and $n \in \mathbb{N}$,

$$\langle \sigma(o, u_n), \alpha_j \rangle = \langle \sigma(o, x_n), \alpha_j \rangle = c_j;$$

(4) for all $i \in I_0 \setminus J$,

$$\lim_{n\to\infty} \langle \sigma(o, u_n), \alpha_i \rangle = \infty.$$

In this case (x_n) is often called an (ω, J, c)-*core sequence* (with respect to the origin o) and (u_n) is called an *auxiliary sequence* for (x_n). Let us observe that if all x_n are special vertices then $(c_j : j \in J) \in \mathbb{N}_0^J$.

Lemma 1.2. *Any unbounded sequence of good vertices contains a core subsequence.*

Proof. Let $(x_n : n \in \mathbb{N})$ be a sequence leaving any compact subset. We set

$$J = \Big\{ i \in I_0 : \limsup_{n\to\infty} \langle \sigma(o, x_n), \alpha_i \rangle < +\infty \Big\}.$$

Since x_n approaches infinity, $J \subsetneq I_0$ and, up to extracting a subsequence, we can find $(c_j : j \in J) \in \frac{1}{2}\mathbb{N}_0^J$ such that

- for all $j \in J$ and all $n \in \mathbb{N}$, we have $\langle \sigma(o, x_n), \alpha_j \rangle = c_j$,
- for all $i \in I_0 \setminus J$, we have $\liminf_{n\to\infty} \langle \sigma(o, x_n), \alpha_i \rangle = \infty$.

Let $K = \sum_{j \in J} c_j$. For each $m \geqslant 1$, we define the finite set B_m to consist of all special vertices in the vectorial spheres $V_\mu(o)$ for any $\mu \in P^+ \cap (Km\rho - Q^+)$. There is $n_0 \geqslant 1$ such that $x_n \notin B_1$ for all $n \geqslant n_0$. For every $n \geqslant n_0$, there is $v_n^{(1)} \in [o, x_n] \cap B_1$. Since B_1 is finite, there is $u_1 \in \{v_n^{(1)} : n \geqslant n_0\}$ such that $\{n \geqslant n_0 : v_n^{(1)} = u_1\}$ is infinite. Let $\varphi_1 : \mathbb{N} \to \mathbb{N}$ be a strictly increasing map such that $u_1 = v^{(1)}_{\varphi_1(n)}$ for all $n \geqslant 1$. We repeat the above procedure for the sequence $(x_{\varphi_1(n)})$, replacing B_1 by $B_2 \setminus B_1$. Now, by the diagonal process, we obtain a strictly increasing map $\varphi : \mathbb{N} \to \mathbb{N}$ and a sequence $(u_m : m \in \mathbb{N})$ such that $\langle \sigma(o, u_m), \alpha_j \rangle = c_j$ for all $j \in J$ and $m \geqslant 1$, and $u_m = v^{(m)}_{\varphi(n)} \in B_m \setminus B_{m-1}$ for all $n \geqslant 1$.

At this stage, the last conditions (3) and (4) on core sequences (dealing with vectorial distances) are fulfilled. It remains to pass to subsequences of $(x_{\varphi(n)})$ and (u_m) to fulfill the first and last conditions on shadows in (3) of the definition of core sequences. Finally, the sets $\Omega(o, u_m)$ are clopen and decreasing, so by compactness of Ω, there is $\omega \in \Omega$ such that

$$\omega \in \bigcap_{m \geqslant 1} \Omega(o, u_m),$$

which provides (1), and the lemma follows. ■

To describe the Gromov compactification (Section 3) and the Martin compactification above the bottom of the spectrum (Section 7.5), we need a further refinement of the sequences approaching infinity. Let

$$\mathbb{S}^{r-1}_+ = \{u \in \mathbb{S}^{r-1} : \langle u, \alpha \rangle \geqslant 0 \text{ for all } \alpha \in \Phi^+\}$$

where $\mathbb{S}^{r-1}$ denotes the unit sphere in $\mathfrak{a}$. A sequence $(x_n : n \in \mathbb{N})$ of good vertices of $\mathscr{X}$ is an *angular sequence* if there are $\omega \in \Omega$, a sequence $(u_n : n \in \mathbb{N})$ of good vertices and $\theta \in \mathbb{S}^{r-1}_+$ such that

(1) $u_n \in [o, \omega]$;

(2) for all $m \geqslant n$,

$$\Omega(o, x_m) \subseteq \Omega(o, u_m) \subsetneq \Omega(o, u_n), \quad \text{but} \quad \Omega(o, x_m) \nsubseteq \Omega(o, u_{m+1});$$

(3) for all $i \in I_0$, if $\langle \theta, \alpha_i \rangle \neq 0$, then

$$\liminf_{n \to \infty} \langle \sigma(o, u_n), \alpha_i \rangle = +\infty;$$

(4)

$$\lim_{n \to \infty} \frac{\sigma(o, x_n)}{|\sigma(o, x_n)|} = \theta.$$

We often say that (x_n) is an (ω, θ)-*sequence*. The sequences that are at the same time core and angular are called *angular core sequences*. By compactness of $\mathbb{S}^{r-1}$ and Lemma 1.2 we have the following result.

Lemma 1.3. *Any unbounded sequence of good vertices contains an angular core subsequence.* ■

1.5. The algebraic group setting

In this section we reformulate the previous notions in the situation when the building $\mathscr{X}$ is Bruhat–Tits, i.e., when it is associated with a semisimple algebraic group over a (locally compact) non-Archimedean local field k. Let k° be its ring of integers; we denote by ϖ a uniformizer of k, by $\kappa = k^\circ/\varpi k^\circ$ the residue field of k and by $\mathrm{ord}_k : k^\times \to \mathbb{Z}$ the associated valuation. We let $\mathbf{G}$ be a connected semisimple algebraic group over k. Then it follows from Bruhat–Tits theory that the group $\mathbf{G}(k)$ acts on an affine building $\mathscr{X} = \mathscr{X}(\mathbf{G}, k)$ [55, Section 2]. The action is well-balanced in the sense that it is *strongly transitive* (i.e., transitive on the inclusions of an alcove in an apartment) and *proper* (i.e., the facet stabilizers are compact); the action is type-preserving as long as we assume that $\mathbf{G}$ is simply connected. In this context, apartments in the building are in bijective correspondence with maximal k-split tori. Let us fix such a maximal k-split torus $\mathbf{S}$ and let us denote by $\mathbf{Z} = \mathbf{Z}_{\mathbf{G}}(\mathbf{S})$ (resp. $\mathbf{N} = \mathbf{N}_{\mathbf{G}}(\mathbf{S})$) its centralizer (resp. its normalizer). The spherical Weyl group, classically defined as the quotient $W^{\mathrm{sph}} = \mathbf{N}(k)/\mathbf{Z}(k)$, is also the Weyl group associated with the spherical root system $\Phi^{\mathrm{sph}} = \Phi(\mathbf{G}, \mathbf{S})$, defined as generated by reflections [9, Théorème 5.3].

Let us now recall how the apartment $\mathcal{A} = \mathcal{A}(\mathbf{S}, k)$ associated with $\mathbf{S}$ is constructed [55, Section 1]. For any algebraic k-group $\mathbf{H}$, we denote by $X^*(\mathbf{H})$ (resp. $X_*(\mathbf{H})$) the group of characters $\mathbf{H} \to \mathrm{GL}_1$ (resp. the group of cocharacters $\mathrm{GL}_1 \to \mathbf{H}$) defined over k. The geometric realization $\mathcal{A}$ of the apartment $\mathscr{A}$ is a Euclidean affine space under the real vector space $V = X_*(\mathbf{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ admitting a suitable $\mathbf{N}(k)$-action $\xi : \mathbf{N}(k) \to \mathrm{Aff}(\mathcal{A})$. Roughly speaking, it is constructed as follows. We first define a map $\xi : \mathbf{N}(k) \to \mathrm{Aff}(\mathcal{A})$ by duality; namely, for any $z \in \mathbf{Z}(k)$ the image $\xi(z)$ is characterized by

$$\chi(\xi(z)) = -\mathrm{ord}_k(\chi(z))$$

for all $\chi \in X^*(\mathbf{Z})$ where χ on the left-hand side is seen as a linear form on V. The kernel of the map ξ is denoted by Z_c; it is the unique maximal compact subgroup of $\mathbf{N}(k)$ [31, Proposition 1.2] (it does not come from an algebraic subgroup). The quotient group $\Lambda = \mathbf{Z}(k)/Z_c$ is a free Abelian group of rank $\dim \mathbf{S} = \dim V$ [31, Lemma 1.3]. Setting $\widetilde{W} = \mathbf{N}(k)/Z_c$, we obtain an exact sequence

$$0 \to \Lambda \to \widetilde{W} \to W^{\mathrm{sph}} \to 1.$$

The desired $\mathbf{N}(k)$-action will be via $\widetilde{W}$. The provisional map ξ obtained so far corresponds to the action of the translation part Λ of $\widetilde{W}$. More precisely, by a standard pushforward argument [31, Proposition 1.6], we finally obtain

- an affine space $\mathcal{A}$ with underlying Euclidean vector space V;
- an affine action $\xi : \mathbf{N}(k) \to \mathrm{Aff}(\mathcal{A})$;
- a collection Φ^{aff} of affine linear forms on $\mathcal{A}$;
- a map $\alpha \mapsto X_\alpha$ attaching to each $\alpha \in \Phi^{\mathrm{aff}}$ a subgroup X_α of $\mathbf{G}(k)$

such that

(i) for any $n \in \mathbf{N}(k)$ we have $nX_\alpha n^{-1} = X_{\alpha\circ\xi(n)}$;

(ii) the set of vectorial parts of the affine linear forms in Φ^{aff} is equal to Φ^{sph};

(iii) for any $a \in \Phi^{\mathrm{sph}}$ the subgroups X_α, for α of vectorial part a, form a filtration of $\mathbf{U}_a(k)$.

Any affine linear form $\alpha \in \Phi^{\mathrm{aff}}$ is called an *affine root* of $\mathbf{G}$ over k. The zero set $\partial\alpha$ of an affine root α is called a *wall* and we denote by r_α the Euclidean reflection in $\partial\alpha$. The reflections r_α generate a Euclidean reflection group W^a called the *affine Weyl group* of $\mathcal{A}$; it is a finite index normal subgroup in $\widetilde{W}$. The root system Φ we introduce in Section 1.2 (and its Weyl group W) is related to Φ^{sph} by the fact that they both provide, up to proportionality, the vectorial parts of the affine roots in Φ^{aff}, in particular $W^{\mathrm{sph}} \simeq W$ (but these two finite root systems are not globally proportional in general [55, Section 1.7]).

It follows from Borel–Tits theory [8, Theorem 21.15] that if we choose a minimal parabolic k-subgroup $\mathbf{P}$ containing $\mathbf{S}$, then the couple $(\mathbf{P}(k), \mathbf{N}(k))$ is a BN-pair (or a Tits system in Bourbaki's terminology [10, IV.2]) for the group $\mathbf{G}(k)$; we will use this when going back to the maximal boundary Ω and its big cells. The spherical building at infinity $\mathscr{X}^\infty$, which is defined geometrically [56, Proposition 1], is the building that is naturally associated with this combinatorial structure. To each chamber at infinity ω of $\mathscr{X}^\infty$ is attached a minimal parabolic k-subgroup $\mathbf{P}_\omega$ such that $\mathbf{P}_\omega(k) = \mathrm{Stab}_{\mathbf{G}(k)}(\omega)$. More generally, if F is a facet in $\mathscr{X}^\infty$, there exists a parabolic k-subgroup $\mathbf{P}_F$ such that $\mathbf{P}_F(k) = \mathrm{Stab}_{\mathbf{G}(k)}(F)$, and all parabolic k-subgroups of $\mathbf{G}$ are obtained this way.

1.5.1. Cartan and Iwasawa decompositions, Busemann functions. Let $\mathcal{A}$ be the apartment associated with a maximal k-split torus $\mathbf{S}$ as before, and let c be an alcove in it whose closure contains a special vertex x. The cone with tip x and generated by c is an open Weyl sector in $\mathcal{A}$, which we denote $\mathcal{S}$; its closure $\overline{\mathcal{S}}$ is a fundamental domain for the action of $\mathrm{Stab}_{W^a}(x) \simeq W^{\mathrm{sph}}$ on $\mathcal{A}$. We denote by ω the chamber at infinity represented by $[x, \omega]$. In order to suitably formulate the Cartan and Iwasawa decompositions with respect to these geometric choices, we use the map $\xi : \mathbf{N}(k) \to \mathrm{Aff}(\mathcal{A})$ giving the affine action of the group $\mathbf{N}(k)$ on $\mathcal{A}$ and whose image is isomorphic to $\widetilde{W}$. We denote by Y the group of all translations contained in $\widetilde{W}$ and by Y^+ the translations of Y sending x to a vertex in $\overline{\mathcal{S}}$.

The *Cartan decomposition* of $\mathbf{G}(k)$ with respect to the choices of $x \in \mathcal{A}$ in $\mathcal{X}$ is the partition

$$\mathbf{G}(k) = \bigsqcup_{t \in Y^+} K_x \xi^{-1}(t) K_x$$

where we write K_x for the maximal compact subgroup $\mathrm{Stab}_{\mathbf{G}(k)}(x)$ in $\mathbf{G}(k)$. The geometric interpretation of the Cartan decomposition is that a fundamental domain for the K_x-action on the building $\mathcal{X}$ is given by the closed Weyl sector $\overline{\mathcal{S}}$.

This can be seen by going back to the problem of describing (at least partially) root group actions on $\mathcal{X}$. More precisely, we denote by $(\mathbf{U}_a : a \in \Phi^{\mathrm{sph}})$ the collection of root groups in $\mathbf{G}$ with respect to the maximal k-split torus $\mathbf{S}$ defining $\mathcal{A}$ (these metabelian groups are denoted by $\mathbf{U}_{(a)}$ in [8, Proposition 21.9] and [9, Section 5.2] but we stick to the notation in [55]). By construction of the action ξ (see Section 1.5), the group $\mathbf{S}(k)$ acts by translations; the question here is to describe the action of a group $\mathbf{U}_a(k)$ and for this we use its filtration by the subgroups X_α where the α's have linear part a. If we fix such an affine root α, then the compact subgroup X_α of $\mathbf{U}_a(k)$ fixes the positive half-space $A_\alpha = \alpha^{-1}(\{0\})$ and folds the other half-apartment of $\mathcal{A}$ into another apartment in $\mathcal{X}$. This description is very useful for the interpretation of retractions in terms of actions of well-chosen unipotent subgroups; retractions centered inside the building are related to affine Bruhat–Tits decompositions and Cartan decomposition, while retractions centered at a chamber (or, more generally, at a spherical facet) at infinity are related to Iwasawa decompositions (or, more generally, to horospherical decompositions).

For the Cartan decomposition with respect to $x \in \mathcal{A}$ in $\mathcal{X}$, we choose the family of groups X_α geometrically characterized by the condition that $x \in A_\alpha$; in view of the previous description, all such X_α's are included in K_x. In a first step we can even impose the slightly stronger condition that $c \subset A_\alpha$. Then strong transitivity of the action is illustrated by the fact that any alcove or vertex y in $\mathcal{X}$ can be sent into $\mathcal{A}$ by the action of finitely many elements of well-chosen subgroups X_α with $c \subset A_\alpha$: this can be arranged by induction whose last step is illustrated in Figure 4.

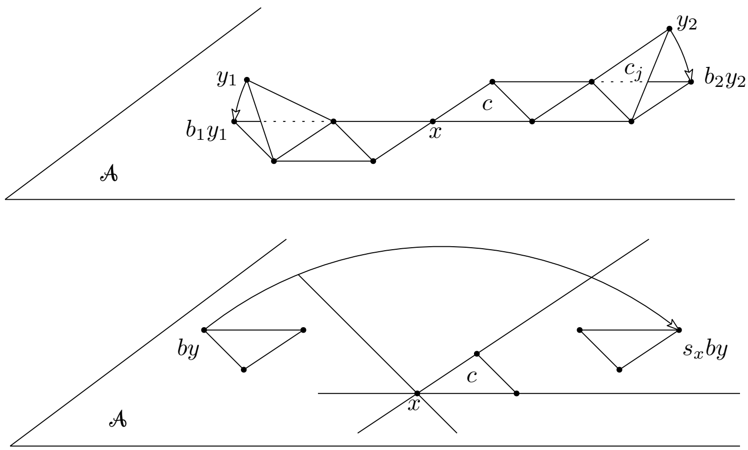

Fig. 4. Cartan decomposition: first, fold onto the chosen apartment by root group action (top); then use spherical Weyl group action to go to the chosen Weyl sector (bottom).

At this stage, we find b fixing c and sending an arbitrary vertex, y say, into $\mathcal{A}$. We can now work in the apartment $\mathcal{A}$ and make $W_x = \mathrm{Stab}_{W^a}(x)$ act: since x was chosen to be special, a fundamental domain for the W_x-action $\mathcal{A}$ is given by $\overline{\mathcal{S}}$ and it remains to note that if $n \in \mathbf{N}(k)$ lifts an element of W_x which sends $b.y$ into $\overline{\mathcal{S}}$, then $n \in K_x$.

Figure 5 illustrates in one stroke the two steps when the k-rank of $\mathbf{G}$ is equal to 1 (i.e., $\mathcal{X}$ is a tree): the effects of retracting with respect to an edge c in a given geodesic, and then of using (if needed) the symmetry with respect to the chosen vertex x.

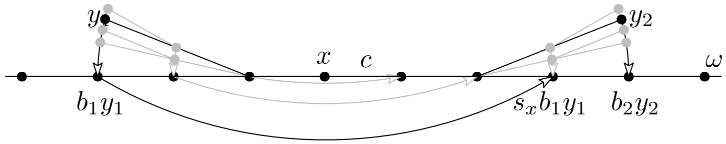

Fig. 5. Cartan decomposition in the rank 1 case.

The *Iwasawa decomposition* of $\mathbf{G}(k)$ with respect to the choices of $[x, \omega] \subset \mathscr{A}$ in $\mathscr{X}$ is the partition

$$\mathbf{G}(k) = \bigsqcup_{t \in Y} K_x \xi^{-1}(t) \mathbf{U}^{\omega}(k)$$

where $\mathbf{U}^{\omega}$ denotes the unipotent radical of the minimal parabolic k-subgroup $\mathbf{P}_{\omega}$.

For this decomposition we also have a geometric interpretation but now we have to use a retraction onto the apartment $\mathscr{A}$ and based at ω. The difference is that the collection of groups used to perform the foldings now consists of all the full root groups $\mathbf{U}_a(k)$ where a runs over the set of positive roots defined by the choice of the chamber ω in the spherical building at infinity $\mathscr{X}^{\infty}$. Concretely, for a positive root $a \in \Phi$ there is no restriction on the affine root α (with vectorial part a) used since the fixed half-apartment can be, so to speak, arbitrarily close to ω to fold by induction galleries from an arbitrary alcove into $\mathscr{A}$. Figure 6 illustrates an inductive step of such folding applied to a gallery given by the projection of an arbitrary alcove into $\mathscr{A}$.

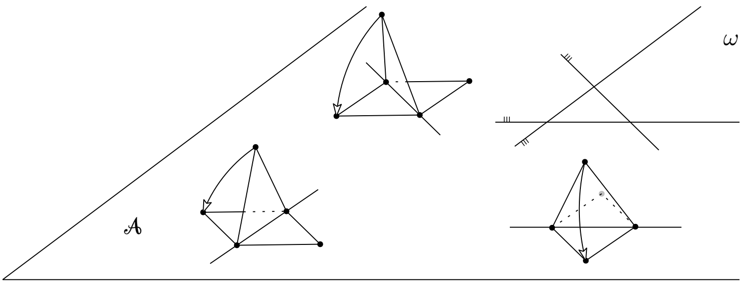

Fig. 6. Iwasawa decomposition in general.

Figure 7 illustrates the geometric interpretation of the Iwasawa decomposition in the tree case.

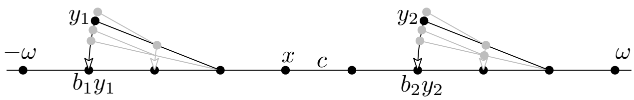

Fig. 7. Iwasawa decomposition in the rank 1 case.

For completeness, we shall explain the connection between these decompositions and the various distances and similar quantities we use to study the convergence of unbounded sequences of vertices. We keep the geometric choices made before. If y is a special vertex, then the Cartan decomposition with respect to $x \in \mathcal{A}$ says that there exists $k \in K_x$ such that $k.y \in \overline{\mathcal{S}}$ and this vertex is represented by a positive coweight in the identification between $\mathcal{S} \subset \mathcal{A}$ and $S_0 \subset \mathfrak{a}$: this coweight is $\sigma(x, y)$; it is the non-Archimedean version of the radial component in bi-invariant harmonic analysis. If y lies in the same $\mathbf{G}(k)$-orbit as x, then there exists $n \in \mathbf{Z}(k)$ such that $\xi(n)$ is a translation of Y^+ and $k.y = \xi(n).x$; if the $\mathbf{G}(k)$-action on $\mathcal{X}$ is type-preserving, then a vertex y is in the same $\mathbf{G}(k)$-orbit as x if and only if the vertices have the same type. Keeping the special vertex y, the Iwasawa decomposition of $\mathbf{G}(k)$ with respect to $[x,\omega] \subset \mathcal{A}$ implies that there exists $u \in \mathbf{U}^{\omega}(k)$ such that $u.y \in \mathcal{A}$. The vertex $u.y$ is represented by an arbitrary coweight in the identification between $[x,\omega] \subset \mathcal{A}$ and $S_0 \subset \mathfrak{a}$; this coweight is $h(x, y; \omega)$. If y lies in the same $\mathbf{G}(k)$-orbit as x, then there exists $n \in \mathbf{Z}(k)$ such that $\xi(n)$ is a translation of Y and $u.y = \xi(n).x$.

1.5.2. The maximal boundary from the algebraic viewpoint. In this paper, we make intensive use of the maximal boundary Ω to do analysis on affine buildings, and in particular to define and study harmonic measures and Furstenberg compactifications of affine buildings.

First, recall that Ω was defined in a purely geometric way: it is the set of parallelism classes of sectors, and therefore it can be seen also as the set of chambers in the spherical building at infinity $\mathscr{X}^\infty$, i.e., $\Omega = C(\mathscr{X}^\infty)$. From this viewpoint, the topology on Ω is defined to be generated by the shadows $\Omega(x, y)$ emanating from a given special vertex x (with y varying). Nevertheless, since the resulting topology does not depend on the choice of x [36, Proposition 3.15], we can see it as generated by all shadows $\Omega(x, y)$ where the special vertices x and y both vary in V_s.

Going back to group actions, for $g \in \mathbf{G}(k)$ we have

$$y \in [x,\omega] \iff g.y \in g.[x,\omega] \iff g.y \in [g.x, g.\omega],$$

implying that $\mathbf{G}(k)$ permutes the shadows and therefore acts continuously on Ω. Since $\mathbf{G}(k)$ acts strongly transitively on $\mathscr{X}$, it does so also on $\mathscr{X}^\infty$; in particular, the $\mathbf{G}(k)$-action on Ω is transitive. As a consequence, if we pick $\omega \in \Omega$, the orbit map of ω provides a homeomorphism

$$\mathbf{G}(k)/\mathbf{P}_\omega(k) \simeq \Omega$$

where $\mathbf{P}_\omega$ is the minimal parabolic k-subgroup associated with ω (note that compactness of $\mathbf{G}(k)/\mathbf{P}_\omega(k)$ follows for instance from a suitable Iwasawa decomposition – for more details, we refer to the alternative group-theoretic definition of harmonic measures in Section 6.2).

As is well-known from Borel–Tits theory [8, Section 21], the group $\mathbf{G}(k)$ admits a Bruhat decomposition

$$\mathbf{G}(k) = \bigsqcup_{w \in W^{\mathrm{sph}}} \mathbf{P}(k) w \mathbf{P}(k)$$

where $\mathbf{P}$ is any minimal parabolic k-subgroup of $\mathbf{G}$. In fact, for each $w \in W^{\mathrm{sph}}$ the double class $\mathbf{P}(k)w\mathbf{P}(k)$ can be written in a better way, in particular avoiding redundancies. If we pick a maximal k-split torus $\mathbf{S}$ in $\mathbf{P}$, we have (positive) root subgroups with respect to $\mathbf{S}$ included in the unipotent radical $\mathbf{U}^+ = \mathrm{rad}_\mathrm{u}(\mathbf{P})$, and similarly we have (negative) root groups included in the unipotent radical $\mathbf{U}^-$ of the minimal parabolic subgroup that is opposite $\mathbf{P}$ with respect to $\mathbf{S}$. Using this, we have $\mathbf{P}(k)w\mathbf{P}(k) = \mathbf{U}^+_w(k)w\mathbf{P}(k)$ with $\mathbf{U}^+_w = \mathbf{U}^+ \cap w\mathbf{U}^- w^{-1}$. Multiplying by the longest element $\bar{w}$ in W^{sph}, we obtain the refined Birkhoff decomposition

$$\mathbf{G}(k) = \bigsqcup_{w \in W^{\mathrm{sph}}} \mathbf{U}^{-,w}(k)w\mathbf{P}(k)$$

with $\mathbf{U}^{-,w} = \mathbf{U}^- \cap w\mathbf{U}^- w^{-1}$ for each $w \in W^{\mathrm{sph}}$. Using root groups, it is also known that group multiplication provides an isomorphism of k-varieties [8, Proposition 21.9 and Theorem 21.20]

$$\prod_{b \in \Phi^{\mathrm{sph},-} \cap w\Phi^{\mathrm{sph},-}} \mathbf{U}_b \simeq \mathbf{U}^{-,w}.$$

Note that the biggest subgroup $\mathbf{U}^{-,w}$ is $\mathbf{U}^-$ and it corresponds to $w = 1$.

Going back to the maximal boundary, and denoting by $-\omega$ (or $\mathrm{opp}_\mathbf{S}(\omega)$ when necessary) the chamber which is opposite ω with respect to the apartment $\mathcal{A}(\mathbf{S})$ corresponding to $\mathbf{S}$, the isomorphism $\mathbf{G}(k)/\mathbf{P}_\omega(k) \simeq \Omega$ and the Birkhoff decomposition of $\mathbf{G}(k)$ provide a decomposition of the maximal boundary

$$\Omega = \{-\omega\} \sqcup \Big(\bigsqcup_{\substack{w \in W^{\mathrm{sph}} \\ w \neq 1, \bar{w}}} \mathbf{U}^{-,w}(k)w.\omega \Big) \sqcup \mathbf{U}^-(k).\omega,$$

which is valid for each $\omega \in \Omega$ and each k-split torus $\mathbf{S}$ such that $\mathcal{A}(\mathbf{S})^\infty$ contains ω. The last subset of the partition is nothing other than the big cell $\Omega'(-\omega)$ of chambers which are opposite $-\omega$; more generally, the above partition is indexed by the Weyl distance of chambers from $-\omega$ in the spherical building $\mathscr{X}^\infty$. The big cell $\Omega'(-\omega)$ is an open neighborhood of ω. Note that the fact that any residue can be seen as a compact subset of Ω (this fact in the group-free case will be proved in Lemma 1.1) has a more natural proof in the algebraic group context since a residue can be seen as an orbit under a well-chosen parabolic subgroup acting on Ω. In fact, this orbit is also the orbit of a Levi factor of the parabolic subgroup, and even the orbit of a maximal compact subgroup in that Levi factor: this is a useful remark when dealing with Furstenberg compactifications – see Section 6.2 below.

Some neighborhoods smaller than big cells can be produced thanks to the filtrations on coordinates of the root groups $\mathbf{U}_b(k)$. More precisely, we can start from the previous big cell

$$\Omega'(-\omega) = \mathbf{U}^-(k).\omega = \Big(\prod_{b \in \Phi^{\mathrm{sph},-}} \mathbf{U}_b(k) \Big).\omega \simeq \prod_{b \in \Phi^{\mathrm{sph},-}} \mathbf{U}_b(k)$$

to see that the root groups provide a system of coordinates (note that any root group is isomorphic, as a variety, to an affine space [8, Theorem 21.20 (i)]). Imposing some valuation conditions on the additive parameters of each root group still provides some

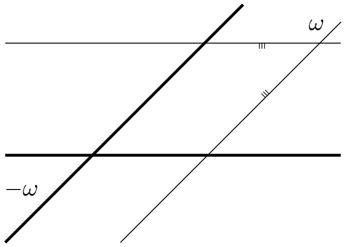

Fig. 8. Shadows as constructed with root group parametrizations.

(now compact) open subsets; in other words, we are replacing here each factor k of the coordinate system by a compact open factor $\varpi^m k^\circ$ for some $m \in \mathbb{Z}$. The choices of the parameters can be made consistent thanks to geometric considerations. For instance, we can pick a special vertex $x \in \mathcal{A}(\mathbf{S})$. This gives a sector $[x, \omega]$, and any special vertex $y \in [x, \omega]$ leads to a shadow $\Omega(x, y)$; in the above parametrization of $\Omega'(-\omega)$, this corresponds to choosing for each $b \in \Phi^{\mathrm{sph},-}$ the largest compact open subgroup of the Bruhat–Tits filtration of $\mathbf{U}_b(k)$ fixing y [13, Section 6.2].

2. Affine buildings at infinity

In this section, we adopt the geometric viewpoint on affine buildings (e.g., we recall the existence of complete, non-positively-curved metrics) and we introduce various related structures, mainly at infinity. For instance, we go back to the classical definition of the spherical building at infinity of a given affine building (Section 2.1) and, less classically, we introduce some auxiliary affine buildings of smaller rank, called *façades* (after G. Rousseau [51]). There are façades of two kinds. The outer façades are defined thanks to a wide generalization of the parallelism equivalence relation on sectors: the generalization applies to the family of subsets taken into account (we use chimneys and not only sectors, see Section 2.2), and also to the equivalence relations (roughly speaking, we take into account the value of the Hausdorff distance between these subsets). The inner façades are (inessential) subbuildings that are very useful for technical purposes. The outer façades were recently introduced in order to construct, in a purely combinatorial way and without any group considerations, the polyhedral compactification of an affine building [17], while the inner ones had already been introduced by Bruhat–Tits [13, Section 7.6] in a group-theoretic context, which we can eventually get rid of. We conclude this section by describing maximal boundaries of façades in terms of residues in the spherical building at infinity; this is useful when studying limits of harmonic measures in Section 6.

2.1. *Geometric and metric realizations*

In this paper, even though our starting point is a combinatorial definition of buildings (which is well-adapted to our analytic arguments), we are sometimes led to using geometric realizations of affine buildings, as well as the associated non-positively-curved distances. The sets of vertices on which we perform harmonic analysis are discrete subsets of their geometric realizations.

For geometric realizations, we proceed as in J. Tits' paper [56] on the classification of affine buildings. This allows us to use G. Rousseau's book [51] in which apartments are by definition Euclidean affine spaces (in order to directly treat non-discrete buildings). We will also use [17] which sticks to the case we consider, namely locally finite affine buildings. Let us fix the following notational convention: if $\mathscr{A}$ is a (discrete) apartment, then $\mathcal{A}$ denotes its geometric realization; $\mathcal{A}$ is thus a Euclidean affine space on which the affine Weyl group W^a acts by isometries. Accordingly, we denote by $\mathcal{X}$ the geometric realization of $\mathscr{X}$ and we adopt the same convention for sectors: if $\mathscr{S}$ is a sector, we denote by $\mathcal{S}$ its geometric realization. This additional structure on the geometric realization of each apartment is the suitable context to define cones in apartments (in the usual real affine sense, i.e., via stability under $\mathbb{R}_+$-action by scalars) and we can also use the notions of vectorial and conical directions in apartments. Each sector $\mathcal{S}$ is a simplicial cone, and so is each of its faces, which we call a *sector face* of $\mathcal{X}$ without referring to any specific sector containing it.

For any type of building, whenever the model for the apartments has a geometric realization admitting a distance which is invariant under the Weyl group, the geometric realization of the ambient building can be endowed with a distance which restricts to the initial one on each apartment [12, Theorem 10A.4]. If we start with a Weyl-invariant CAT(0)-distance on the apartments, we obtain a CAT(0)-distance on the building and each type-preserving automorphism extends to an isometry. This is obviously the case for affine buildings, and actually the CAT(0)-distance we will use was considered from the very beginning of affine building theory [13, Section 2.5]. The distance on $\mathcal{X}$, as for any metric space, leads to the notion of *Hausdorff distance* d_{H} between subsets. Namely, for $A, B \subset \mathcal{X}$, we set

$$d_{\mathrm{H}}(A, B) = \inf\{\varepsilon > 0 : A \subset V_\varepsilon(B) \text{ and } B \subset V_\varepsilon(A)\}$$

where $V_\varepsilon(C)$ denotes the ε-neighborhood $\{x \in \mathcal{X} : d(x, C) < \varepsilon\}$ of $C \subset \mathcal{X}$. We say that two subsets in $\mathcal{X}$ are *asymptotic* if they are at finite Hausdorff distance from one another; this is an equivalence relation.

The non-positively-curved distance d on an affine building $\mathcal{X}$ provides an intrinsic definition of geodesic segments between two points:

$$[x, y] = \{z \in \mathcal{X} : d(x, z) + d(z, y) = d(x, y)\}$$

for any $x, y \in \mathcal{X}$. Geodesic segments themselves are the starting point of convexity arguments. The convex hull of a subset Y of $\mathcal{X}$ is the smallest subset containing it and stable by taking geodesic segments between points in Y; when Y is contained in an apartment, it is also the intersection of the affine half-spaces contained in the apartment and containing Y. The discrete version of the convex hull in this context was defined in [13, Section 2.4]: the *enclosure* of a subset Y in $\mathcal{X}$ intersecting an alcove is the smallest subset of $\mathcal{X}$ containing

Y and stable by taking (closures of) minimal galleries between alcoves intersecting Y; we denote it by $\mathrm{encl}(Y)$. When Y is contained in an apartment, it is also the intersection of the (closed) affine half-spaces contained in the apartment, bounded by a wall and containing Y. The advantage of the latter definition is that it applies to arbitrary subsets of apartments [13, Proposition 2.4.5].

Let us denote by $\partial_\infty \mathcal{X}$ the set of equivalence classes of geodesic rays; we call it the *visual boundary* of $\mathcal{X}$. By [51, Section 3.2.13], $\partial_\infty \mathcal{X}$ is a geometric realization of the *spherical building at infinity* $\mathscr{X}^\infty$. This allows us to see any facet F of $\mathscr{X}^\infty$ as a set of asymptotic classes of geodesic rays. There is a unique apartment system in $\partial_\infty \mathcal{X}$ which consists of the boundaries $\partial_\infty \mathcal{A}$ of the apartments $\mathcal{A}$ in $\mathcal{X}$ (recall that we systematically deal with full apartment systems for all the affine buildings in this paper). For any cone $\mathcal{C}$ in some apartment $\mathcal{A}$ (e.g. for any sector face in $\mathcal{A}$), we denote by $\partial_\infty \mathcal{C}$ the set of asymptotic classes of rays drawn in $\mathcal{C}$. The set of asymptotic classes of sector faces is the set of facets in the visual boundary $\partial_\infty \mathcal{X}$ (see [56] where the asymptotic equivalence relation is called *parallelism*, or also [1, Section 11.8]).

2.2. *Sector faces, chimneys and equivalence relations*

Let us recall the construction of an affine building associated with each facet in $\partial_\infty \mathcal{X}$. The resulting collection of buildings gives a stratification of the boundary of almost all the compactifications we consider in this paper (the exceptions are the Gromov compactification and the Martin compactification above the bottom of the spectrum: we explain this in Section 7.5). The affine buildings at infinity, indexed by the facets in $\partial_\infty \mathcal{X}$, are called *façades* in [51, Section 3.3]; they provide an important geometric viewpoint on the study of convergence of core sequences.

Recall first that a *sector face* in the building $\mathcal{X}$ is a cone $\mathcal{F}$, in some apartment $\mathcal{A}$, of the form $\mathcal{F} = x + \vec{F}$ where $x \in \mathcal{A}$ and $\vec{F}$ is some face in the vectorial model of $\mathcal{A}$; we then say that $\vec{F}$ is the *direction* of the sector face. This notion can be generalized in the following way: for any facet σ in $\mathcal{A}$, the enclosure $\mathrm{encl}(\sigma + \vec{F})$ is called the *chimney* $\mathcal{R}(\sigma, \vec{F})$ *based* at σ and *directed* by $\vec{F}$ in $\mathcal{A}$ [17, Definition 4.2.2].

Loosely speaking, taking into account the numerical values of the Hausdorff distance between sector faces (or, more generally, between cones) in the same asymptotic class leads to a more subtle equivalence relation on cones with a given direction. Chimneys are technical tools which are used to define a simplicial structure (eventually, a building structure) on the corresponding set of equivalence classes.

More precisely, the asymptotic equivalence relation can also be formulated in more combinatorial terms [17, Definition 4.7.1]. Let $\mathcal{F} = x + \vec{F}$ be a sector face. A subsector face $\mathcal{F}'$ of $\mathcal{F}$ (i.e., a sector face $\mathcal{F}'$ contained in $\mathcal{F}$) is said to be *full* in $\mathcal{F}$ if the sector faces $\mathcal{F}$ and $\mathcal{F}'$ have the same direction. Similarly, but taking into account the fact that the base of a chimney is now a facet and not merely a point, we say that a subchimney $\mathcal{R}'$ of $\mathcal{R}$ (i.e., a chimney $\mathcal{R}'$ contained in $\mathcal{R}$) is *full* in $\mathcal{R}$ if $\mathcal{R}'$ and $\mathcal{R}$ have the same direction and generate the same affine subspace in any apartment containing $\mathcal{R}$. We define the

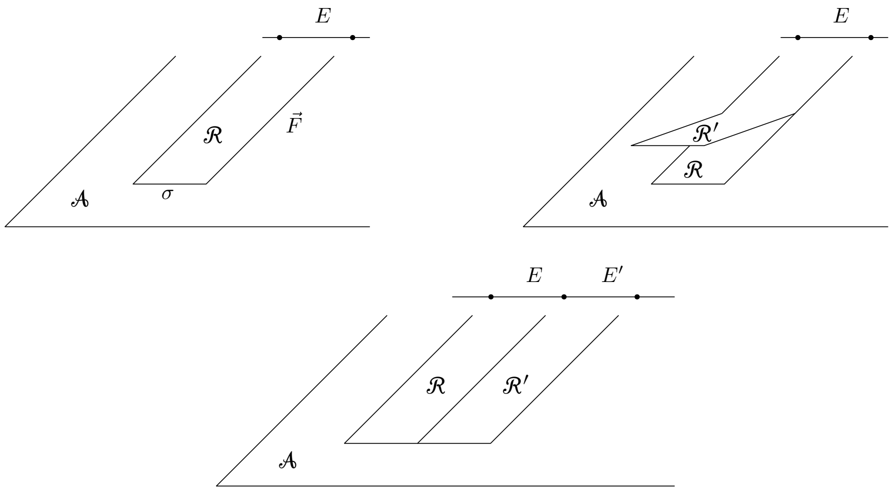

Fig. 9. Top left: a chimney. Top right: two equivalent chimneys. Bottom: adjacency of chimneys (and of their classes) in the façade.

germ (*at infinity*) $\mathrm{germ}_\infty(\mathcal{R})$ of a chimney $\mathcal{R} = \mathrm{encl}(\sigma + \vec{F})$: a chimney $\mathcal{R}'$ belongs to $\mathrm{germ}_\infty(\mathcal{R})$ if and only if $\mathcal{R} \cap \mathcal{R}'$ contains a chimney which is full in both $\mathcal{R}$ and $\mathcal{R}'$.

2.3. Affine buildings at infinity, or (outer) façades

Let us choose a facet F in $\mathscr{X}^\infty$. We denote by $\mathcal{X}(F)$ the set all germs of sector faces in the asymptotic class F, that is,

$$\mathcal{X}(F) = \{\mathrm{germ}_\infty(\mathcal{F}) : \mathcal{F} \text{ a sector face such that } \partial_\infty \mathcal{F} = F\}.$$

The set $\mathcal{X}(F)$ admits an affine building structure [51, Section 3.3.15] where the apartments are given by apartments $\mathcal{A}$ such that $\partial_\infty \mathcal{A} \supset F$; for such an $\mathcal{A}$, the corresponding apartment in $\mathcal{X}(F)$ is

$$\mathcal{A}(F) = \{\mathrm{germ}_\infty(\mathcal{F}) : \mathcal{F} \text{ a sector face contained in } \mathcal{A} \text{ such that } \partial_\infty \mathcal{F} = F\}.$$

The facets are the germs of chimneys whose direction is given by F [18, Section 2.6]. Note that the building $\mathcal{X}(F)$ is not contained in $\mathcal{X}$ since it is abstractly constructed out of germs at infinity of vector faces. Nevertheless, there is a natural map

$$\pi_F : \mathcal{X} \to \mathcal{X}(F), \quad x \mapsto \mathrm{germ}_\infty(x + F),$$

which is a homomorphism of buildings in the sense of [51, Section 2.1.13].

There is a good compatibility between this construction and buildings at infinity. Let $R(F)$ denote the residue of F in the spherical building at infinity $\mathscr{X}^\infty$. Let $\mathcal{S}$ be a sector one of whose sector faces represents F; this means that F is a facet of $\partial_\infty \mathcal{S}$, or also that the parallelism class of $\mathcal{S}$, denoted by $[\mathcal{S}]_{/\!/}$, belongs to $R(F)$. The sectors in $\mathcal{X}(F)$ are the images under the above map π_F of such sectors [18, Section 2.6]. Let $\mathcal{S}$ and $\mathcal{S}'$ be two such sectors. If we assume in addition that they are parallel, there is a subsector $\mathcal{S}''$ in $\mathcal{S} \cap \mathcal{S}'$. Then $\pi_F(\mathcal{S}'')$ is a common subsector of the sectors $\pi_F(\mathcal{S})$ and $\pi_F(\mathcal{S}')$ in $\mathcal{X}(F)$, so that $\pi_F(\mathcal{S})$ and $\pi_F(\mathcal{S}')$ are parallel. This allows us to define the surjective map

$$\varphi_F : R(F) \to \Omega_{\mathcal{X}(F)}, \quad [\mathcal{S}]_{/\!/} \mapsto [\pi_F(\mathcal{S})]_{/\!/},$$

where the notation $[\cdot]_{/\!/}$ corresponds to taking the relevant parallelism class on each side of the map. If J is the type of F, the maximal boundary $\Omega_{\mathcal{X}(F)}$ is the residue $\mathrm{res}_J(c_\infty)$ where c_∞ is any chamber at infinity whose closure contains F.

We can also put a building structure on a well-chosen subset of $\mathcal{X}$, making it a subbuilding in the sense of [51, Section 2.1.15]. This requires choosing an apartment $\mathbb{A}$ for which $F \subset \partial_\infty \mathbb{A}$. We assume that $o \in \mathbb{A}$. We also fix a facet F' opposite F in $\mathbb{A}$ (which we may write $F' = \mathrm{opp}_{\mathbb{A}}(F)$ sometimes). We consider the subset of apartments

$$\mathbb{A}(F, F') = \{\mathbb{A} \in \mathscr{A} : \partial_\infty \mathbb{A} \supset F \cup F'\}$$

and we set

$$\mathcal{X}(F, F') = \bigcup_{\mathbb{A} \in \mathbb{A}(F,F')} \mathbb{A}.$$

Then $\mathcal{X}(F, F')$ is an affine building for which $\mathbb{A}(F, F')$ is a system of apartments [51, Theorem 3.3.14.1]. In general, this building is *non-essential*, meaning that the smallest vectorial facets in any apartment are of positive dimension. Such a facet is contained in the Euclidean factor acted upon trivially when we decompose the affine Weyl group action on an apartment (this corresponds to the Euclidean factor in the de Rham decomposition of Riemannian symmetric spaces). Note that there always exists an *essentialization map* factoring out the trivial Euclidean factor of each apartment [51, Section 2.1.12]. The subbuilding $\mathcal{X}(F, F')$ is called the *inner façade* associated with F and F' [51, Section 3.3.14]. The restricted map $\pi_F|_{\mathcal{X}(F,F')}$ is the essentialization map of $\mathcal{X}(F, F')$; see also [18, Theorem 2.8], where more precise statements can be found.

Let us see what the concrete counterpart of these constructions is at the level of apartments; this will be useful when studying the convergence of core sequences for some compactifications. We work in the Euclidean space $\mathfrak{a}$ that is the vectorial model of the apartments in $\mathcal{X}$, hence in $\mathscr{X}$ (see Section 1.2). Given $J \subsetneq I_0$, let Φ_J be the set of roots $\alpha \in \Phi$ such that $\langle \alpha, \lambda_k \rangle = 0$ if $k \notin J$. Let $\tilde{\alpha}_0$ be the highest root in Φ_J. In $\mathfrak{a}$, we consider the subcollection $\mathscr{H}_J$ of walls consisting of all hyperplanes $H_{j;k}$ for $j \in J$ and $k \in \mathbb{Z}$, together with

$$H_{\tilde{\alpha}_0;k} = \{x \in \mathfrak{a} : \langle x, \tilde{\alpha}_0 \rangle = k\} \quad \text{for all } k \in \mathbb{Z}.$$

Let $\tilde{r}_0$ be the orthogonal reflection in $H_{\tilde{\alpha}_0;1}$. We denote by W_J and W_J^a the subgroups of W and W^a generated by $\{r_j : j \in J\}$ and $\{\tilde{r}_0\} \cup \{r_j : j \in J\}$, respectively. Let $C_J(\Sigma)$ be the family of all open connected components of $\mathfrak{a} \setminus \bigcup_{H \in \mathscr{H}_J} H$. Since the group W_J^a acts transitively on $C_J(\Sigma)$, we can turn it into a chamber system. Let Σ_J denote the resulting abstract simplicial complex. The space $\mathfrak{a}$ is an inessential realization of Σ_J in the sense of [10, Chapter V]. Since $\mathscr{H}_J \subset \mathscr{H}$, we can see Σ_J as a subcomplex of Σ. In particular, each half-apartment in Σ_J is a half-apartment in Σ. For each $(k_j : j \in J) \in \mathbb{Z}^J$, the set

$$\{x \in \mathfrak{a} : \langle x, \alpha_j \rangle = k_j \text{ for all } j \in J\}$$

is called a J-*vertex*. The *fundamental* J-*sector* is

$$S^J = \{x \in \mathfrak{a} : \langle x, \alpha_j \rangle \geqslant 0 \text{ for all } j \in J\}.$$

All J-sectors in Σ_J are of the form $w.S^J$ for some $w \in W_J^a$.

In order to obtain from $\mathfrak{a}$ an essential realization of Σ_J we need to introduce the orthogonal projection π_J onto $\mathfrak{a}_J = \bigoplus_{j \in J} \mathbb{R}\alpha_j$ which is a direct factor in the orthogonal decomposition $\mathfrak{a} = \mathfrak{a}_J \oplus \bigoplus_{i \in I_0 \setminus J} \mathbb{R}\lambda_i$. For $x \in \mathfrak{a}$, we have

$$\pi_J(x) = P_J(x) = \sum_{j \in J} \langle x, \alpha_j \rangle P_J(\lambda_j)$$

where

$$P_J(x) = \frac{1}{|W_J|} \sum_{w \in W_J} (x - w.x). \tag{2.1}$$

We also set $Q_J = \mathrm{id} - P_J$. To see that P_J is indeed the orthogonal projection onto $\mathfrak{a}_J$, note first that for each $j \in J$ and each $i \in I_0 \setminus J$ we have $s_j.\lambda_i = \lambda_i$, hence $w.x = x$ for each $w \in W_J$ and each $x \in \bigoplus_{i \in I_0 \setminus J} \mathbb{R}\lambda_i$; moreover, the fact that P_J acts as the identity on $\mathfrak{a}_J$ comes from the irreducibility of the standard linear representation of an irreducible Coxeter system.

To sum up, when J varies over the proper subsets of I_0, the Euclidean spaces $\mathfrak{a}$ endowed with the subcollections $\mathcal{H}_J$ of walls are models for the apartments in the inner façades $\mathcal{X}(F, F')$ associated with opposite facets at infinity; after projection onto $\mathfrak{a}_J$, we obtain the models for the apartments of the façades at infinity $\mathcal{X}(F)$ (or of the essentializations of inner façades).

Remark. Let (x_n) be an (ω, J, c)-core sequence with auxiliary sequence (u_n). This sequence obviously defines a spherical facet at infinity, say F, namely the facet attached to the J-residue of ω in the spherical building at infinity $\mathscr{X}^\infty$. Therefore we have a projection $\pi_F : \mathcal{X} \to \mathcal{X}(F)$. Let $\mathcal{A}$ be an apartment containing the sector $[o, \omega]$. Then, in view of the above description of the restriction of π_F to $\mathcal{A}$ and by definition of a core sequence, the sequence $(\pi_F(u_n) : n \in \mathbb{N})$ is constant; we denote by x_F its unique value. The facet F is called the *spherical facet* associated to the core sequence (x_n), and the point x_F of $\mathcal{X}(F)$ is called the *vertex at infinity* associated to (x_n).

2.4. Polyhedral compactification, after Rousseau

In this section we introduce a further compactification procedure which we will use as a tool and which has the advantage of being well-adapted to projections to outer façades, so that the façades naturally appear as strata at infinity. Apart from Lemma 2.1 below, what follows is a summary of [51, Section 3.4] (to which we refer for more details).

First of all, recall that in any complete, proper, CAT(0) metric space, the nearest point projection onto any given convex subset is a 1-Lipschitz map [12, Chapter II.2]. This fact is the starting point for defining compactifications at this level of generality. If Z is such a space endowed with an increasing sequence $(B_k)_{k\geqslant 0}$ of compact convex subsets such that $Z=\bigcup_k B_k$, and if one denotes by pr_k the nearest point projection onto B_k, then the projective system (B_k,pr_k) provides a compactification $\overline{Z}=\varprojlim_{k\geqslant 0} B_k$. When $(B_k)_{k\geqslant 0}$ is given by a sequence of balls centered at a common point z and of radii $r_k\to+\infty$ (for the given, i.e., numerical, distance), the resulting compactification is the horospherical one [12, Chapter II.8, Definition 8.5 and Theorem 8.13].

In the case of affine buildings, Rousseau suggests using different compact exhaustions $(B_k)_{k\geqslant 0}$, replacing metric balls by balls defined in terms of vectorial distance.

2.4.1. Vectorial balls. For a point z in $\mathcal{X}$ and for a vector ξ in the model S_0 of sectors, we introduce the *vectorial ball* $B^v(z,\xi)$ of center z and vectorial radius ξ by setting

$$B^v(z,\xi)=\{y\in\mathcal{X}:\sigma(z,y)\preccurlyeq\xi\},$$

where $\sigma(z,y)\preccurlyeq\xi$ means that the vector $\xi-\sigma(z,y)$ is in the cone $\bigoplus_{i\in I_0}\mathbb{R}_+\alpha_i$ dual to S_0. Vectorial balls are compact and convex [51, Lemma 3.4.3.1] and the intersection of a vectorial ball with an apartment containing the center of the ball can be described precisely. Indeed, for each apartment $\mathcal{A}$ containing z, the (topological) boundary $\mathcal{P}$ of $\mathcal{A}\cap B^v(z,\xi)$ is an explicit polyhedron [51, Proposition 3.4.1.1]: each vectorial face F of the partition of $\mathcal{A}$ into Weyl chambers gives rise to a face of $\mathcal{P}$ which is orthogonal to F; this parametrizes bijectively the faces of the boundary of $\mathcal{A}\cap B^v(z,\xi)$. For instance, the faces corresponding to Weyl sectors are the vertices of $\mathcal{P}$, and $\mathcal{A}\cap B^v(z,\xi)$ is the convex hull of these vertices.

Moreover, the nearest point projection onto $B^v(z,\xi)$, restricted to any apartment containing the center z, can be described explicitly: the preimage of a point u in the boundary $\mathcal{P}$, more precisely in the face determined by (and orthogonal to) a vectorial face F say, is the cone $u+F$ [51, Lemma 3.4.2.2].

2.4.2. Polyhedral compactification. Keeping z and ξ as above, we can choose in addition an increasing sequence of real numbers $r_k\geqslant 1$ with $r_k\to+\infty$, and consider the dilated vectorial balls $B_k=B^v(z,r_k\xi)$. The *polyhedral compactification* of the building is then $\overline{\mathcal{X}}^{\mathrm{pol}}=\varprojlim_{k\geqslant 0} B_k$, so that a point η in $\overline{\mathcal{X}}^{\mathrm{pol}}$ is a sequence (z_k) such that $z_k\in B_k$ and $\mathrm{pr}_k(z_{k+m})=z_k$ for all $k,m\geqslant 0$; we denote $z_k=\mathrm{pr}_k(\eta)$. The projective limit topology on $\overline{\mathcal{X}}^{\mathrm{pol}}$ is Hausdorff and compact [51, Theorem 3.4.4.4] and it does not depend on the choice of the center z, the (regular) vectorial radius ξ or the increasing sequence (r_k) of real numbers going to infinity: this is seen by comparing the resulting projective limit compactifications with another one, defined more intrinsically [51, Proposition 3.4.6.3].

Finally, for each spherical facet in the spherical building at infinity $\mathscr{X}^\infty$, there is a map ι_F embedding the outer façade $\mathcal{X}(F)$ into $\overline{\mathcal{X}}^{\mathrm{pol}}$: recall that a point η in $\mathcal{X}(F)$ is the germ at infinity of a cone of direction F; then the image $\iota_F(\eta)$ is the sequence $(\mathrm{pr}_k(\eta))_{k\geqslant 0}$ where, by definition, the point $\mathrm{pr}_k(\eta)$ in the boundary ∂B_k is the image by the nearest point projection pr_k of a sufficiently small cone representing the germ η [51, Section 3.4.4.3]. It turns out that $\overline{\mathcal{X}}^{\mathrm{pol}}$ is the disjoint union of the building and of the images of the outer façades $\mathcal{X}(F)$ by ι_F when F varies over the spherical facets at infinity [51, Theorem 3.4.4.4].

2.4.3. Convergence of core sequences in the polyhedral compactification. Recall that an (ω, J, c)-core sequence $(x_n : n \in \mathbb{N})$ with auxiliary sequence $(u_n : n \in \mathbb{N})$ defines a spherical facet at infinity F of type J and a vertex x_F in the outer façade $\mathcal{X}(F)$ (Remark at the end of Section 2.3).

Lemma 2.1. *Let (x_n) be an (ω, J, c)-core sequence with auxiliary sequence (u_n). Let F be the associated spherical facet in the building at infinity $\mathscr{X}^\infty$ and let x_F be the associated point at infinity. Then, in the polyhedral compactification, we have*

$$\lim_{n\to\infty} x_n = x_F.$$

Proof. We first show that in $\overline{\mathcal{X}}^{\mathrm{pol}}$ we have $\lim_{n\to\infty} u_n = x_F$, by arguing in an apartment $\mathcal{A}$ containing the sector $[o, \omega]$. In view of the definition of a projective limit, this amounts to showing that for each $k \geqslant 0$ we have

$$\lim_{n\to\infty} \mathrm{pr}_k(u_n) = \mathrm{pr}_k(x_F).$$

It is enough to check this for large enough k. We consider an index k such that the face $\Sigma_k^{\perp F}$ of the polyhedron ∂B_k which is orthogonal to F contains a point u with $\alpha_j(u) \geqslant c_j$ for each $j \in J$. On the one hand, when n is large enough, we have $u_n \notin B_k$ and by the above description of the fibers of pr_k in $\mathcal{A}$, the point $\mathrm{pr}_k(u_n)$ is the unique point $s_k \in \Sigma_k^{\perp F}$ such that $\alpha_j(s_k) = c_j$ for each $j \in J$. On the other hand, the value of pr_k at the point at infinity x_F is the (unique) value that pr_k takes on the cone $u_n + F$ for n large enough. This implies that for n large enough, we have $\mathrm{pr}_k(u_n) = \mathrm{pr}_k(x_F)$, which gives the desired convergence.

Now we consider the (ω, J, c)-core sequence (x_n) itself. Let k be again a large enough index (in the above sense) and let n be large enough too, so that $u_n \notin B_k$. We pick an apartment $\mathcal{A}'$ containing the origin o and x_n. Since $\Omega(o, x_n) \subseteq \Omega(o, u_n)$, the point u_n lies in the convex hull of o and x_n, hence in $\mathcal{A}'$. Therefore we can argue as in the previous paragraph, using the apartment $\mathcal{A}'$ instead of $\mathcal{A}$. This implies that $\mathrm{pr}_k(x_n)$ and $\mathrm{pr}_k(u_n)$ are both equal to the unique point $s_k \in \Sigma_k^{\perp F}$ such that $\alpha_j(s_k) = c_j$ for each $j \in J$, proving the announced convergence. ∎

2.5. Some lemmas for convergence studies

We finish with two lemmas that will be useful in studying convergence of core sequences.

Lemma 2.2. *Let F be a facet in the building at infinity $\mathscr{X}^\infty$. Then the map*

$$\varphi_F : R(F) \to \Omega_{\mathcal{X}(F)}, \quad [\mathcal{S}]_{/\!/} \mapsto [\pi_F(\mathcal{S})]_{/\!/},$$

is a homeomorphism.

Proof. We already know that the map φ_F is surjective since it is obtained from factorizing by parallelism equivalence relations (at the levels of the source and of the target) the map obtained from π_F sending surjectively the set of sectors in $\mathcal{X}$ containing a sector face directed by F to the set of sectors in $\mathcal{X}(F)$. Now, let $\mathcal{S}$ and $\mathcal{S}'$ be sectors both containing a sector face directed by F. There are subsectors $\mathcal{T}$ and $\mathcal{T}'$ of $\mathcal{S}$ and $\mathcal{S}'$ respectively, contained in the same apartment $\mathcal{A}$ and both containing a sector face directed by F. We assume that $\mathcal{S}$ and $\mathcal{S}'$ are not parallel to one another. Then neither are $\mathcal{T}$ and $\mathcal{T}'$ in $\mathcal{A}$, so there exists w in the spherical Weyl group, non-trivial but stabilizing F, sending $\mathcal{T}$ to $\mathcal{T}'$. After projecting by π_F this provides two sectors in the apartment $\pi_F(\mathcal{A})$ which are images of one another under the action of a non-trivial element of the Weyl group of $\mathcal{X}(F)$, hence not parallel to one another. This proves the injectivity of φ_F.

To finish the proof, we use compactness of the source $R(F)$ of φ_F (given by Lemma 1.1). Therefore, in order to conclude that φ_F is a homeomorphism, it is enough to show that it is continuous. We pick x_∞, y_∞ in $\mathcal{X}(F) \cap \pi_F(V_s)$ and ω in $\varphi_F^{-1}(\Omega_{\mathcal{X}(F)}(x_\infty, y_\infty))$, where $\Omega_{\mathcal{X}(F)}(x_\infty, y_\infty)$ is the shadow in the maximal boundary $\Omega_{\mathcal{X}(F)}$ of $\mathcal{X}(F)$ defined by x_∞ and y_∞. By definition of a shadow, we have $y_\infty \in [x_\infty, \varphi_F(\omega)]$. The sector $[x_\infty, \varphi_F(\omega)]$ is contained in an apartment of the façade $\mathcal{X}(F)$, which we can write $\pi_F(\mathcal{A})$ for an apartment $\mathcal{A}$ in $\mathcal{X}$ such that $\partial_\infty\mathcal{A}$ contains F. Let $x \in \mathcal{A} \cap V_s$ be such that $\pi_F(x) = x_\infty$. In $\mathcal{A}$, there exists a sector tipped at x whose image by π_F is in the parallelism class $\varphi_F(\omega)$. By injectivity of φ_F this sector represents ω so we can write $[x_\infty, \varphi_F(\omega)] = \pi_F([x, \omega])$. Now, we choose $y \in [x, \omega] \cap V_s$ such that $\pi_F(y) = y_\infty$ and we consider the shadow $\Omega(x, y)$. Let $\omega' \in R(F) \cap \Omega(x, y)$. We have $y \in [x, \omega']$, so applying π_F we obtain $y_\infty \in [x_\infty, \varphi_F(\omega')]$. This shows that the open neighborhood $\Omega(x, y)$ of ω is contained in $\varphi_F^{-1}(\Omega_{\mathcal{X}(F)}(x_\infty, y_\infty))$. This proves the continuity of φ_F since ω was picked arbitrarily in the preimage $\varphi_F^{-1}(\Omega_{\mathcal{X}(F)}(x_\infty, y_\infty))$. ■

Lemma 2.3. *Let F be a facet in the building at infinity $\mathscr{X}^\infty$. Let J be its type, $J \subsetneq I_0$.*

(i) *For all $x, x' \in \mathcal{X}$ with $R(F) \cap \Omega(x, x') \neq \varnothing$, we have*

$$\sigma_{\mathcal{X}(F)}(\pi_F(x), \pi_F(x')) = P_J(\sigma(x, x'))$$

where $\sigma_{\mathcal{X}(F)}$ denotes the vectorial distance of the outer façade $\mathcal{X}(F)$.

(ii) *For all $x, x' \in \mathcal{X}$ and each $\omega \in R(F)$, we have*

$$h_{\mathcal{X}(F)}(\pi_F(x), \pi_F(x'); \varphi_F(\omega)) = P_J(h(x, x'; \omega))$$

where $h_{\mathcal{X}(F)}$ denotes the Busemann function of the outer façade $\mathcal{X}(F)$.

In the rest of the paper we are mainly interested in outer façades in order to understand boundaries of various compactifications. This is why we did not introduce the retraction $\rho : \mathcal{X} \to \mathcal{X}(F, F')$ above; it is defined carefully in [51, Section 3.3.17]. We use this reference in the proof below.

Proof of Lemma 2.3. In order to prove (i), we pick $x, x' \in \mathcal{X}$ and $\omega \in R(F) \cap \Omega(x, x')$. We have $x' \in [x, \omega]$ and the sector $[x, \omega]$ contains a face whose direction is given by F.

There exists an apartment, say $\mathcal{A}$, containing $[x, \omega]$. The apartment $\mathcal{A}$ contains a facet F' opposite F. We denote by $\mathcal{X}(F, F')$ the inner façade defined by the couple (F, F'); by construction, $\mathcal{A}$ is an apartment of $\mathcal{X}(F, F')$ so that the retraction $\rho : \mathcal{X} \to \mathcal{X}(F, F')$ induces the identity map on it. The result then follows from the fact that π_F is the composition of ρ with the essentialization map [51, Section 3.3.17], and the latter map is performed by P_J on the model of the apartments of $\mathcal{X}$.

To show (ii), we pick $x, x' \in \mathcal{X}$ and $\omega \in R(F)$. Let $z \in [x, \omega] \cap [x', \omega]$. The point z is chosen in order to write $h(x, x'; \omega) = \sigma(x, z) - \sigma(x', z)$. By the very choice of z, we have $\omega \in \Omega(x, z) \cap R(F)$, therefore by (i) we have

$$\sigma_{\mathcal{X}(F)}(\pi_F(x), \pi_F(z)) = P_J(\sigma(x, z)).$$

Similarly, by replacing x by x' we obtain

$$\sigma_{\mathcal{X}(F)}(\pi_F(x'), \pi_F(z)) = P_J(\sigma(x', z)).$$

Now $\pi_F([x, \omega])$ is a sector in $\mathcal{X}(F)$; more precisely the sector $[\pi_F(x), \varphi_F(\omega)]$. Moreover,

$$\pi_F(z) \in [\pi_F(x), \varphi_F(\omega)].$$

Similarly by replacing x by x' we obtain

$$\pi_F(z) \in [\pi_F(x'), \varphi_F(\omega)].$$

Therefore, $\pi_F(z) \in [\pi_F(x), \varphi_F(\omega)] \cap [\pi_F(x'), \varphi_F(\omega)]$, so that we can compute

$$h_{\mathcal{X}(F)}\big(\pi_F(x), \pi_F(x'); \varphi_F(\omega)\big) = \sigma_{\mathcal{X}(F)}(\pi_F(x), \pi_F(z)) - \sigma_{\mathcal{X}(F)}(\pi_F(x'), \pi_F(z)),$$

which is equal to $P_J(\sigma(x, z)) - P_J(\sigma(x', z))$ by the previous arguments. Finally, by linearity of P_J we obtain

$$h_{\mathcal{X}(F)}\big(\pi_F(x), \pi_F(x'); \varphi_F(\omega)\big) = P_J(\sigma(x, z) - \sigma(x', z)) = P_J(h(x, x'; \omega)),$$

as required. ■

We finish with a lemma relating core sequences and façades at infinity. In the following result, we use the fact that the polyhedral compactification, when defined as the projective limit of an increasing exhaustion by convex compact subsets, does not depend on the point at which the convex compact subsets are centered.

Lemma 2.4. *Let $(x_n : n \in \mathbb{N})$ be an (ω, J, c)-core sequence with auxiliary sequence $(u_n : n \in \mathbb{N})$. Let F be the associated spherical facet in the building at infinity $\mathscr{X}^\infty$. Then the sequence $(\pi_F(u_n) : n \in \mathbb{N})$ is constant in the outer façade $\mathcal{X}(F)$. Let x_F be its value. Then for each $z \in V_g$, there is a strictly increasing map $\varphi : \mathbb{N} \to \mathbb{N}$ such that*

$$\sigma_{\mathcal{X}(F)}(\pi_F(z), x_F) = P_J(\sigma(z, x_{\varphi(n)}))$$

for each $n \in \mathbb{N}$.

Proof. By Lemma 2.3 (i) we have $P_J(\sigma(o, u_n)) = \sigma_{\mathcal{X}(F)}(\pi_F(o), \pi_F(u_n))$. Moreover, by definition of a core sequence, the auxiliary sequence (u_n) satisfies $P_J(\sigma(o, u_n)) = P_J(\sigma(o, x_n))$. Therefore for all n we have

$$\sigma_{\mathcal{X}(F)}(\pi_F(o), x_F) = P_J(\sigma(o, x_n)).$$

Now, choosing the vertex $z \in V_g$ as a new origin, by Lemma 1.2 we have a strictly increasing map φ and parameters (ω', J', c') such that $(x_{\varphi(n)})$ is an (ω', J', c')-core sequence with respect to z. The argument of the previous paragraph shows that for all n we have

$$\sigma_{\mathcal{X}(F')}(\pi_{F'}(z), x_{F'}) = P_{J'}(\sigma(z, x_{\varphi(n)}))$$

where F' is the spherical facet attached to the parameters (ω', J', c') and $x_{F'}$ is the corresponding point in the façade at infinity $\mathcal{X}(F')$. By Lemma 2.1, the point x_F is the limit of (x_n) in the polyhedral compactification endowed with the projective limit topology provided by vectorial balls centered at o; similarly $x_{F'}$ is the limit of the subsequence $(x_{\varphi(n)})$ in the polyhedral compactification endowed with the projective limit topology provided by vectorial balls centered at z. But these two topologies coincide [51, Proposition 3.4.6.3] and are Hausdorff [51, Section 3.4.4.2], so $x_F = x_{F'}$. By [51, Section 3.4.4], this implies that $F = F'$ and $J = J'$, which gives the result by the second displayed formula. ∎

Lemma 2.5. *Let (x_n) be a sequence satisfying one of the following conditions:*

- *(x_n) is an angular (ω, θ)-sequence such that for all $i \in J$,*

$$\liminf_{n\to\infty} \langle \sigma(o, x_n), \alpha_i \rangle = +\infty, \text{ or}$$

- *(x_n) is an (ω, J, c)-core sequence.*

Let (u_n) be an auxiliary sequence. Let F be the associated spherical facet in the building at infinity $\mathscr{X}^\infty$. Then the sequence $(\pi_F(u_n) : n \in \mathbb{N})$ is constant in the outer façade $\mathcal{X}(F)$. Let x_F be its value. Then for each $y \in V_g$, we have

$$\lim_{n\to\infty} (\sigma(o, x_n) - \sigma(y, x_n)) = \sigma_{\mathcal{X}(F)}(\pi_F(o), x_F) - \sigma_{\mathcal{X}(F)}(\pi_F(y), x_F) + Q_J(h(o, y; \omega)) \tag{2.2}$$

where in fact the sequence $\sigma(o, x_n) - \sigma(y, x_n)$ is eventually constant.

Proof. Let (x'_n) be any (ω, J, c)-core subsequence of (x_n). We first note that $\sigma(o, x'_n) = \sigma(o, u'_n) + \sigma(u'_n, x'_n)$ because $\Omega(o, x'_n) \subseteq \Omega(o, u'_n)$. We claim that the following holds.

Claim 2.6. *For any $y \in V_s$ there is an integer n_y such that $\sigma(y, x'_n) = \sigma(y, u'_n) + \sigma(u'_n, x'_n)$ for any $n \geqslant n_y$.*

Since we are using full apartment systems, by [51, Proposition 3.1.2] there is an apartment $\mathscr{A}'$ containing y and a shortening of $u'_n + F_J$, that is, $u'_n + F_J$ for $n \geqslant n_y$ for some n_y. There is ω' in the boundary of $\mathscr{A}'$ belonging to the J-residue of ω such that the cone $[y, \omega']$ contains u'_n for $n \geqslant n_y$ possibly after increasing n_y. The intersection of the cone $[x'_n, -\omega]$ with the support of F_J contains u'_n, and since ω and ω' are J-equivalent, u'_n also belongs to the intersection of $[x'_n, -\omega']$ with the support of F_J. Therefore $[x'_n, -\omega']$ contains $[u'_n, -\omega']$ which contains y. This implies the claim.

The claim provides the equality $\sigma(o, x'_n) - \sigma(y, x'_n) = \sigma(o, u'_n) - \sigma(y, u'_n)$ for $n \geqslant n_y$, so it remains to compute the limit (2.2) with x_n replaced by u'_n. As before, there is an apartment $\mathscr{A}'$ containing y and u'_n for $n \geqslant n_y$. Let ψ be an isomorphism between this apartment and the model Σ such that $\psi([y, \omega]) = S_0$. There exists $w \in W_J$ such that $w.\psi(u'_n) \in S_0$ for $n \geqslant n_y$. Let $\tilde{u}_n$ be the ψ-preimage of $w.\psi(u'_n)$. There exists $n'_y \geqslant n_y$ such that for $n \geqslant n'_y$ we have $\tilde{u}_n \in [u'_{n_y}, \omega]$. Then $Q_J(\sigma(u'_{n_y}, u'_n)) = Q_J(\sigma(u'_{n_y}, \tilde{u}_n))$. Moreover, $\sigma(y, u'_n) = \sigma(y, \tilde{u}_n)$. Therefore,

$$Q_J(\sigma(u'_{n_y}, u'_n) - \sigma(y, u'_n)) = Q_J(\sigma(u'_{n_y}, \tilde{u}_n) - \sigma(y, \tilde{u}_n)) = Q_J(h(u'_{n_y}, y; \omega)).$$

Because $u'_{n_y} \in [o, \omega]$, by the cocycle relation we obtain

$$\begin{aligned} Q_J(\sigma(o, u'_n) - \sigma(y, u'_n)) &= Q_J(\sigma(o, u'_{n_y})) + Q_J(\sigma(u'_{n_y}, u'_n) - \sigma(y, u'_n)) \\ &= Q_J(h(o, u'_{n_y}; \omega)) + Q_J(h(u'_{n_y}, y; \omega)) = Q_J(h(o, y; \omega)). \end{aligned}$$

Finally, to complete the proof we just invoke Lemma 2.4 for the P_J-projections of $\sigma(o, u'_n)$ and $\sigma(y, u'_n)$. Since the limit is independent of the subsequence (x'_n), the lemma follows. ∎

2.6. Parabolic subgroups, Levi factors and their façades

In this section we consider the Bruhat–Tits case. In particular, we are going to explain that each outer façade $\mathcal{X}(F)$ is the Bruhat–Tits building associated with the semisimple quotient of the parabolic subgroup stabilizing F. Picking an opposite facet F' corresponds to picking a parabolic subgroup $\mathrm{Stab}(F')$ which is opposite $\mathrm{Stab}(F)$, i.e., $\mathrm{Stab}(F) \cap \mathrm{Stab}(F')$ is a Levi factor in both parabolic subgroups.

The descriptions of the Furstenberg compactification, of the combinatorial compactification and of the Martin compactification at the bottom of the spectrum make appear affine buildings at infinity. As expected, the algebraic group case, and its rich Lie-theoretic combinatorics, is also the main source of inspiration for the constructions of the auxiliary affine buildings involved.

Keeping the notation of the previous section, let again $\mathbb{A}(\mathbf{S})$ be the apartment attached to the maximal k-split torus $\mathbf{S}$. Let F be a facet of the spherical building at infinity $\mathscr{X}^\infty$, which we assume to be contained in the boundary $\mathbb{A}(\mathbf{S})^\infty$ of $\mathbb{A}(\mathbf{S})$. The stabilizer $\mathbf{P}_F(k) = \mathrm{Stab}_{\mathbf{G}(k)}(F)$ consists of the k-rational points of a parabolic k-subgroup $\mathbf{P}_F$ containing $\mathbf{S}$. The maximal k-split torus $\mathbf{S}$ provides a Levi decomposition $\mathbf{P}_F \simeq \mathbf{M}_{F,\mathbf{S}} \ltimes \mathrm{rad}_{\mathrm{u}}(\mathbf{P}_F)$ where $\mathbf{M}_{F,\mathbf{S}}$ can be defined algebraically as the centralizer of a suitable singular subtorus of $\mathbf{S}$ [8, Proposition 20.5], or geometrically as the (Zariski closure of) the stabilizer in $\mathbf{G}(k)$ of F and its opposite $-F = \mathrm{opp}_{\mathbf{S}}(F)$ with respect to the boundary $\mathbb{A}(\mathbf{S})^\infty$. Note that $\mathrm{rad}_{\mathrm{u}}(\mathbf{P}_F)(k)$ acts simply transitively on the facets of $\mathscr{X}^\infty$ which are opposite F, or equivalently on Levi factors of $\mathbf{P}_F$.

So far, we are dealing with the (unique) spherical building at infinity: this building comes from the combinatorics of the spherical Tits system constructed in Borel–Tits theory. It can be geometrically constructed as the boundary at infinity [12, Section II.8] of the Gromov compactification of $\mathscr{X}$, which is the compactification obtained by the horospherical process when using the family of usual distances to points for the Bruhat–Tits CAT(0)-metric [1, Section 11.2].

Just as for any compactification, thanks to the Cartan decomposition and its geometric interpretation (Section 1.5.1), it is enough to describe the closure of a Weyl sector in order to describe the full space. In the case of the visual boundary, the parameters of convergence are radial. The points in the boundary of a chosen Weyl sector are in bijective correspondence with the unit vectors in the sector.

Another family of parameters, which is useful to describe the three compactifications mentioned at the beginning of this section, consists of the distances to the walls bounding a chosen Weyl sector. The choice of a spherical facet at infinity F as before can be seen as a way of selecting a subfamily of distances; this is explained in Section 2.3 when introducing the façade corresponding to F and the inner façade corresponding to F and $-F$ as before. We want to give an algebraic group interpretation of these objects.

More precisely, let us go back to the Levi decomposition $\mathbf{P}_F \simeq \mathbf{M}_{F,\mathbf{S}} \ltimes \mathrm{rad_u}(\mathbf{P}_F)$ associated with the inclusion $F \subset \mathbb{A}(\mathbf{S})^\infty$. The group $\mathbf{M}_{F,\mathbf{S}}$ is a reductive k-group, therefore it admits a Bruhat–Tits building, which we denote by $\mathscr{X}(\mathbf{M}_{F,\mathbf{S}}, k)$. The connected center $\mathbf{Z}_{F,\mathbf{S}}$ of $\mathbf{M}_{F,\mathbf{S}}$ is a torus, the derived subgroup $\mathbf{G}_{F,\mathbf{S}} = [\mathbf{M}_{F,\mathbf{S}}, \mathbf{M}_{F,\mathbf{S}}]$ is a semisimple k-group and the multiplication map $\mathbf{Z}_{F,\mathbf{S}} \times \mathbf{G}_{F,\mathbf{S}} \to \mathbf{M}_{F,\mathbf{S}}$ is an isogeny: the geometric counterpart of the last fact is that the building $\mathscr{X}(\mathbf{M}_{F,\mathbf{S}}, k)$ admits a direct factor isometric to the Euclidean space $X_*(\mathbf{Z}_{F,\mathbf{S}}) \otimes_{\mathbb{Z}} \mathbb{R}$. In this case, the inner façade $\mathcal{X}(F, -F)$ associated with the inclusion $-F \cup F \subset \mathbb{A}(\mathbf{S})^\infty$ in Section 2.3 is nothing other than a natural $\mathbf{M}_{F,\mathbf{S}}(k)$-equivariant copy of (the geometric realization of) $\mathscr{X}(\mathbf{M}_{F,\mathbf{S}}, k)$ inside $\mathcal{X}$; in the abstract group-theoretic (and more general) context of valuations of root group data, it had already been constructed in [13, Section 7.6]. As a set, the inner façade $\mathcal{X}(F, -F)$ is the union of the apartments $g.\mathbb{A}(\mathbf{S})$ where g runs over $\mathbf{G}_{F,\mathbf{S}}(k)$, or equivalently over $\mathbf{M}_{F,\mathbf{S}}(k)$ since the elements of the singular torus $\mathbf{Z}_{F,\mathbf{S}}(k)$ act trivially on the Bruhat–Tits building of $\mathbf{G}_{F,\mathbf{S}}$. In fact, in the apartment $\mathbb{A}(\mathbf{S})$ the elements of $\mathbf{Z}_{F,\mathbf{S}}(k)$ act as translations along directions which are parallel to the vector subspace V_F of V given by F; in particular, they preserve the distances to any wall in the apartment $\mathbb{A}(\mathbf{S})$ whose direction contains V_F, and moreover unbounded sequences of such semisimple matrices can be used to push a given vertex in $\mathbb{A}(\mathbf{S})$ to infinity.

This is a suitable place to mention a group-theoretic interpretation of distances to walls. Let x be a special vertex in $\mathbb{A}(\mathbf{S})$ and let ω be a chamber at infinity lying in $\mathbb{A}(\mathbf{S})^\infty$ and belonging to the residue of F; the chamber ω defines a basis $(a_i : i \in I_0)$ of the root system of $\mathbf{G}$. The Weyl cone $[x, \omega]$ is simplicial and there is a bijective correspondence between the simple roots $(a_i : i \in I_0)$ and the sector panels (codimension 1 faces) of $[x, \omega]$: the affine subspace spanned by a sector panel is directed by the kernel of a simple root and each simple root appears in this way. If we denote by J the type of F, the subroot system Φ_J with a basis $(a_i : i \in J)$ corresponds to the directions of the walls containing V_F. Concretely, the sector $[x, \omega]$ is bounded by two kinds of sector panels: those whose direction is the kernel of a simple root in Φ_J and the remaining ones. By construction each element in $\mathbf{S}(k)$ stabilizes $\mathcal{A}(\mathbf{S})$ and acts as a translation (Section 1.5). Among those translations, the semisimple matrices in $\mathbf{Z}_{F,\mathbf{S}}(k)$ act with a direction parallel to the sector face F. More precisely, let a_i be a simple root and let Π_i be the corresponding sector panel. Recall from Section 1.5.1 that we have a decreasing filtration of $\mathbf{U}_{a_i}(k)$ by a countable family of compact open subgroups indexed by affine linear forms of given vectorial part a_i. Then for a special vertex $y \in [x, \omega]$, the (non-negative) difference between the affine root whose zero-set contains Π_i (hence x) and the affine root whose zero-set contains y is a discrete version of the Bruhat–Tits distance for special vertices in $[x, \omega]$. Recalling that for any $n \in \mathbf{N}(k)$ we have $nX_\alpha n^{-1} = X_{\alpha\circ\xi(n)}$, we see that the elements $s \in \mathbf{Z}_{F,\mathbf{S}}(k)$ act as translations along directions which are parallel to the vector subspace V_F, since any such s centralizes each root group $\mathbf{U}_{a_j}(k)$ with $j \in J$. Figure 10 illustrates the dynamics of semisimple matrices along a sector face.

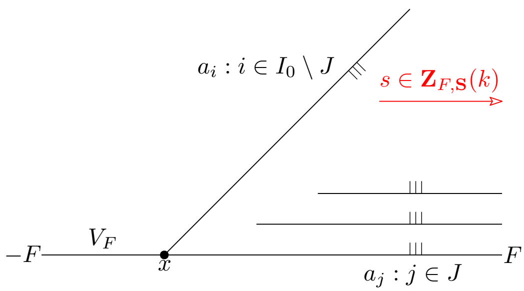

Fig. 10. Dynamics of semisimple matrices along a sector face.

Going back to façades, we now consider the affine building $\mathcal{X}(F)$ associated with the facet F only. Strictly speaking, this (essential) building is not contained in $\mathcal{X}$ but it is a stratum of the boundary of any of the compactifications mentioned above. In the natural $\mathbf{G}(k)$-action on these compactifications, the subgroup $\mathbf{P}_F(k)$ stabilizes the stratum $\mathcal{X}(F)$, which can thus be identified with the Bruhat–Tits building of the semisimple quotient of $\mathbf{P}_F$. This is made precise in the theorem below, which also summarizes many available compactifications and contains the first half of Theorem B stated in the Introduction.

Theorem 2.7. *Let* $\mathbf{G}$ *be a connected semisimple linear algebraic group defined over a non-Archimedean local field* k *and let* $\mathcal{X}(\mathbf{G}, k)$ *be its Bruhat–Tits building.*

(i) *The maximal Berkovich compactification* $\overline{\mathcal{X}}(\mathbf{G}, k)$ *is* $\mathbf{G}(k)$*-equivariantly homeomorphic to the wonderful compactification, the maximal Satake compactification, as well as the polyhedral compactification of* $\mathcal{X}(\mathbf{G}, k)$*. The closure of the set of vertices in* $\overline{\mathcal{X}}(\mathbf{G}, k)$ *is* $\mathbf{G}(k)$*-equivariantly homeomorphic to the group-theoretic compactification of* $\mathcal{X}(\mathbf{G}, k)$*.*

(ii) *For any proper parabolic k-subgroup* $\mathbf{P}$*, there exists a natural, injective, continuous map*

$$\mathcal{X}(\mathbf{P}/\mathrm{rad}(\mathbf{P}), k) \to \overline{\mathcal{X}}(\mathbf{G}, k)$$

whose image lies in the boundary. These maps altogether provide the stratification

$$\overline{\mathcal{X}}(\mathbf{G}, k) = \bigsqcup_{\mathbf{P} \text{ a parabolic } k\text{-subgroup}} \mathcal{X}(\mathbf{P}/\mathrm{rad}(\mathbf{P}), k)$$

where the union is over all parabolic k-subgroups in $\mathbf{G}$*.*

(iii) *Any two points x, y in $\overline{\mathcal{X}}(\mathbf{G}, k)$ lie in a common compactified apartment $\overline{\mathcal{A}}(\mathbf{S})$ and we have*

$$\mathbf{G}(k) = \mathrm{Stab}_{\mathbf{G}(k)}(x)\mathbf{N}(k)\,\mathrm{Stab}_{\mathbf{G}(k)}(y)$$

where $\mathbf{N}$ *is the normalizer of the maximal split torus* $\mathbf{S}$ *defining the apartment* $\mathcal{A}(\mathbf{S})$*.*

Proof. Berkovich compactifications of Bruhat–Tits buildings were defined in [7] in the split case, and in [46] in full generality. The idea is to embed $\mathcal{X}(\mathbf{G}, k)$ into the analytic space $\mathbf{G}^{\mathrm{an}}$ attached to $\mathbf{G}$ and then into the (compact) analytic space $(\mathbf{G}/\mathbf{P})^{\mathrm{an}}$ attached to a (projective) flag variety of $\mathbf{G}$; when the parabolic k-subgroup $\mathbf{P}$ is minimal, the closure of the image of the composed map is maximal among Berkovich compactifications. The wonderful compactification is obtained by replacing the flag variety $\mathbf{G}/\mathbf{P}$ by the de Concini–Procesi wonderful compactification $\overline{\mathbf{G}}$ and then by using the resulting equivariant embedding from $\mathcal{X}(\mathbf{G}, k)$ to $\overline{\mathbf{G}}^{\mathrm{an}}$. It was defined in [48] in the split case, and in [16] in full generality. The polyhedral compactification in the Bruhat–Tits case was first constructed in [31]; it coincides with the compactification we presented in Section 2.4 [51, Section 3.4.5.1]. The Satake compactifications were defined in [58] and then revisited in the context of non-Archimedean analytic geometry in [47]. The identifications between Berkovich and Satake compactifications are given by [47, Theorem 2.1], while the identification between the wonderful (resp. polyhedral) and the maximal Berkovich compactification is in [16, Corollary 15] (resp. [58] or [47, Proposition 5.4]). Therefore the description of the closure of vertices (in any of the previous compactifications) by means of the group-theoretic compactification is the content of [25, Theorem 20]. This provides (i), and once this is obtained, (ii) and (iii) follow from [46, Theorem 2 of Introduction]. ∎

3. Gromov compactification

In this section we describe the Gromov compactification that can be constructed for any CAT(0)-space X. For this purpose we embed X into the space of continuous functions on X. The Gromov boundary corresponds to the Busemann functions. The method was originally introduced in [6, Section 3] for complete Riemannian manifolds of non-positive curvature, and it has been extended to CAT(0)-spaces in [12, Chapter II.8]. It is often called the *horofunction procedure*. In particular, it depends on the choice of the metric. In our construction we use an intrinsic and natural metric which is well-defined for special vertices and which avoids the use of any geometric realization of the buildings.

First, let us observe that

$$d(x, y) = |\sigma(x, y)| \quad \text{for } x, y \in V_s$$

defines a metric on special vertices of $\mathscr{X}$. Since $\sigma(x, y) = -w_0.\sigma(y, x)$, it is enough to justify the triangle inequality. Given $x, y, z \in V_s$, let $\omega \in \Omega(x, y)$. Then by (1.6) and (1.5), we get

$$\begin{aligned} |\sigma(x, y)| = |h(x, y; \omega)| &\leqslant |h(x, z; \omega)| + |h(z, y; \omega)| \\ &\leqslant |\sigma(x, y)| + |\sigma(z, y)|. \end{aligned} \tag{3.1}$$

Analogously, one can show that

$$\langle \sigma(x, y), \tilde{\rho} \rangle \leqslant \langle \sigma(x, y), \tilde{\rho} \rangle + \langle \sigma(z, y), \tilde{\rho} \rangle. \tag{3.2}$$

Lemma 3.1. *For all $x, y, z \in V_s$,*

$$|\sigma(x, y) - \sigma(x, z)| \leqslant |\sigma(y, z)|.$$

Proof. Let $\omega \in \Omega(x, y)$, and let $\mathscr{A}$ be any apartment containing $[x, \omega)$. Let $\psi : \mathscr{A} \to \Sigma$ be the isomorphism such that $\psi([x, \omega)) = S_0$. Suppose that $z \in V_s(\mathscr{A})$. Let $w \in W$ be such that $z_0 = w.\psi(z) \in S_0$. Then

$$\sigma(x, y) - \sigma(x, z) = \sigma(x, y) - \sigma(x, z_0) = \psi(y) - \psi(z_0).$$

Let $w = r_{i_1} \cdots r_{i_k}$ be a minimal representation of w. For $j = 1, \ldots, k$, we set

$$z_j = \psi^{-1}(r_{i_j} \psi(z_{j-1})).$$

In particular, $\psi(y)$ and $\psi(z_{j-1})$ are on the same side of the wall $H_{\alpha_{i_j};0}$, which does not contain $\psi(z_j)$. Let v_j be the unique point on the line segment between $\psi(y)$ and $\psi(z_j)$ which belongs to $H_{\alpha_{i_j};0}$. Since

$$|v_j - \psi(z_{j-1})| = |v_j - \psi(z_j)|,$$

we obtain

$$\begin{aligned} |\psi(y) - \psi(z_{j-1})| &\leqslant |\psi(y) - v_j| + |v_j - \psi(z_{j-1})| \leqslant |\psi(y) - v_j| + |v_j - \psi(z_j)| \\ &= |\psi(y) - \psi(z_j)|. \end{aligned}$$

Consequently, $|\psi(y) - \psi(z_0)| \leqslant |\psi(y) - \psi(z)|$, which completes the proof when $x, y, z \in V_s(\mathscr{A})$.

Next, assume that $z \notin V_s(\mathscr{A})$. Let $c \in \operatorname{St}(x)$ be a chamber in $\mathscr{A}$ having all vertices in $V \cap [x, \omega)$. Let $\rho_{c,\mathscr{A}}$ be the retraction of $\mathscr{X}$ onto $\mathscr{A}$ with center c; see [51, Section 2.1.7] for the definition. We set $z_0 = \rho_{c,\mathscr{A}}(z)$. Since $\sigma(x, z) = \sigma(x, z_0)$, by the first part of the proof we have

$$|\sigma(x, y) - \sigma(x, z)| \leqslant |\sigma(y, z_0)|.$$

Now, we invoke [51, Theorem 2.1.8] to get $|\sigma(y, z_0)| \leqslant |\sigma(y, z)|$, and the conclusion follows. ■

Now, let us denote by $\mathcal{C}_*(V_s)$ the quotient space of real-valued 1-Lipschitz functions on V_s equipped with the topology of pointwise convergence by the one-dimensional subspace of constant functions. We introduce the action of $\mathrm{Aut}(\mathscr{X})$ on $\mathcal{C}^*(V_s)$ by setting, for $g \in \mathrm{Aut}(\mathscr{X})$,

$$g.[f] = [g.f]$$

where $[f]$ is the equivalence class represented by f, and

$$(g.f)(y) = f(g^{-1}.y), \quad y \in V_s.$$

Now, we define the embedding

$$\iota : V_s \to \mathcal{C}_*(V_s), \quad x \mapsto [f_x],$$

where

$$f_x(y) = -d(y, x), \quad y \in V_s.$$

It is easy to check that the map ι is equivariant and injective, and $\iota(V_s)$ is discrete. Let us denote by $\overline{\mathscr{X}}_V$ the closure of $\iota(V_s)$ in $\mathcal{C}_*(V_s)$. The space $\mathcal{C}_*(V_s)$ is metrizable, and thus by Lemma 1.3 when studying the Gromov boundary we can restrict attention to angular sequences.

The following theorem provides a connection between angular sequences and spherical buildings at infinity, as introduced and described in Section 2.1 (see also [51, Section 3.2]). To put the result below in perspective recall that in suitable (metric) cases, applying the horofunction procedure and taking asymptotic classes of geodesic rays leads to the same compactification [12, Chapter II.8, Theorem 8.13 and Corollary 8.20]. In the specific case of affine buildings, we see that the convergence parameters for unbounded sequences are angular and compatible with the spherical building structure, formulated in terms of residues.

Theorem 3.2. *Let (x_n) be an angular (ω, θ)-sequence. Then*

$$\lim_{n\to\infty} [f_{x_n}(y)] = [\langle \theta, h(o, y; \omega)\rangle] \quad \textit{for any } y \in V_s.$$

If (x'_n) is an (ω', θ')-sequence such that $(f_{x'_n})$ converges to the same limit then $\theta = \theta'$ and $\omega' \sim_J \omega$ where $J = \{i \in I_0 : \langle \theta, \alpha_i \rangle = 0\}$.

Proof. We write

$$\begin{aligned} f_{x_n}(y) - f_{x_n}(o) &= \frac{|\sigma(o, x_n)|^2 - |\sigma(y, x_n)|^2}{|\sigma(y, x_n)| + |\sigma(o, x_n)|} \\ &= \frac{2\langle \sigma(o, x_n), \sigma(o, x_n) - \sigma(y, x_n)\rangle - |\sigma(o, x_n) - \sigma(y, x_n)|^2}{|\sigma(y, x_n)| + |\sigma(o, x_n)|}. \end{aligned} \tag{3.3}$$

Since $|\sigma(o, y)| = |\sigma(y, o)|$, by Lemma 3.1 we have

$$|\sigma(y, x_n) - \sigma(o, x_n)| \leqslant |\sigma(o, y)|. \tag{3.4}$$

Therefore, by (3.3),

$$\lim_{n\to\infty} f_{x_n}(y) - f_{x_n}(o) = \lim_{n\to\infty} \frac{\langle \sigma(o, x_n), \sigma(o, x_n) - \sigma(y, x_n)\rangle}{|\sigma(o, x_n)|}.$$

In view of (3.4),

$$\lim_{n\to\infty} \frac{\langle \sigma(o, x_n), \sigma(o, x_n) - \sigma(y, x_n)\rangle}{|\sigma(o, x_n)|} = \lim_{n\to\infty} \langle \theta, \sigma(o, x_n) - \sigma(y, x_n)\rangle.$$

Since (x_n) is an angular (ω, θ)-sequence, the inequality $\langle \theta, \lambda_i\rangle > 0$ implies that

$$\lim_{n\to\infty} \langle \sigma(o, x_n), \alpha_i\rangle = +\infty,$$

and thus by Lemma 2.5,

$$\lim_{n\to\infty} \langle \theta, \sigma(o, x_n) - \sigma(y, x_n)\rangle = \langle \theta, h(o, y; \omega)\rangle$$

as claimed.

Let us now turn to the proof of the second part of the theorem. Suppose that

$$\langle \theta, h(o, y; \omega)\rangle = \langle \theta', h(o, y; \omega')\rangle \quad \text{for all } y \in V_s. \tag{3.5}$$

Let $\mathscr{A}$ be an apartment containing ω and ω', and let o' be any good vertex in $\mathscr{A}$. By the cocycle relation and (3.5), we get

$$\langle \theta, h(o', y; \omega)\rangle = \langle \theta', h(o', y; \omega')\rangle \quad \text{for all } y \in V_s.$$

Hence, there is $w \in W$ such that

$$\langle \theta, \lambda\rangle = \langle \theta', w^{-1}.\lambda\rangle \quad \text{for all } \lambda \in P. \tag{3.6}$$

Let $k = \ell(w)$. We write $w = w_k = w_{k-1} r_{\beta_k}$ with $\ell(w_k) > \ell(w_{k-1})$ and $\beta_k \in \Phi^+$. Then

$$\begin{aligned}\langle \theta', w_k^{-1}.\lambda\rangle &= \langle r_{\beta_k}.\theta', w_{k-1}^{-1}.\lambda\rangle \\ &= \langle \theta', w_{k-1}^{-1}.\lambda\rangle - \langle \theta', \beta_k^\vee\rangle\langle \beta_k, w_{k-1}^{-1}.\lambda\rangle.\end{aligned}$$

Hence, arguing by induction we arrive at

$$\langle \theta', w^{-1}.\lambda\rangle = \langle \theta', \lambda\rangle - \sum_{j=1}^{k} \langle \theta', \beta_j^\vee\rangle\langle \lambda, w_{j-1}.\beta_j\rangle \tag{3.7}$$

where for each $j \in \{1, \ldots, k\}$, we have $w_j = w_{j-1} r_{\beta_j}$ with $\ell(w_j) > \ell(w_{j-1})$ and $\beta_j \in \Phi^+$. By [28, Proposition in Section 5.7], we have

$$w_{j-1}.\beta_j \in \Phi^+. \tag{3.8}$$

Hence, by (3.6) and (3.7) we get $\langle \theta, \lambda \rangle \leqslant \langle \theta', \lambda \rangle$. Since we can swap ω and θ with ω' and θ', respectively, we conclude that $\theta = \theta'$. Thus, by (3.6) and (3.7),

$$\sum_{j=1}^{k} \langle \theta, \beta_j^\vee \rangle \langle \lambda, w_{j-1}.\beta_j \rangle = 0.$$

As $\theta \in \mathbb{S}_+^{r-1}$, by (3.8), $\langle \theta, \beta_j^\vee \rangle = 0$ for all $j \in \{1, \ldots, k\}$, and the theorem follows. ■

Since both $\mathbb{S}^{r-1}$ and Ω are compact, in view of Theorem 3.2 the set $\bar{\mathscr{X}}_V$ is a compact subset of $\mathcal{C}_*(V_s)$. It is called the *Gromov compactification* of $\mathscr{X}$. The action of $\operatorname{Aut}(\mathscr{X})$ is continuous on $\bar{\mathscr{X}}_V$. At this stage, the maximal boundary Ω, so far a mere compact topological space, can also be seen as the set of chambers of the spherical building at infinity of $\mathscr{X}$, which is also the boundary of the Gromov compactification of $\mathscr{X}$.

4. Combinatorial compactifications

We use yet another construction described in [15] for a wide class of buildings not necessarily of affine type. Since we are interested in compactifying special vertices only, we can make the approach more explicit. In particular, we show convergence of core sequences and identify when the limits are the same. Thanks to this we immediately conclude in Section 6 that the combinatorial and Furstenberg compactifications are $\operatorname{Aut}(\mathscr{X})$-equivariantly isomorphic.

For this purpose, let

$$\Gamma(\mathscr{X}) = \Big\{\gamma : V(\mathscr{X}) \to \bigsqcup_{x \in V(\mathscr{X})} \operatorname{St}(x) : \gamma(x) \in \operatorname{St}(x) \text{ for all } x \in V(\mathscr{X})\Big\}. \tag{4.1}$$

Since $V(\mathscr{X})$ is countable and $\operatorname{St}(x)$ is finite for each $x \in V(\mathscr{X})$, by Tikhonov's theorem $\Gamma(\mathscr{X})$ is a compact metrizable space. For $g \in \operatorname{Aut}(\mathscr{X})$ and $\gamma \in \Gamma(\mathscr{X})$, we set

$$(g.\gamma)(x) = g.\gamma(g^{-1}.x) \quad \text{for } x \in V(\mathscr{X}).$$

Let us consider the map

$$\iota : V_s \to \Gamma(\mathscr{X}), \quad x \mapsto \gamma_x,$$

where

$$\gamma_x(y) = \bigcap_{c \in \mathcal{C}(x)} \operatorname{proj}_y(c), \quad y \in V(\mathscr{X}); \tag{4.2}$$

here $\mathcal{C}(x)$ is the set of chambers in $\operatorname{St}(x)$, and $\operatorname{proj}_y(c)$ is the unique chamber in $\operatorname{St}(y)$ that is the closest to c. The map ι is equivariant, injective and has discrete image. The last two properties are consequences of

$$\gamma_x(y) = \text{maximal simplex in } \operatorname{St}(y) \text{ with vertices in } [x, y], \tag{4.3}$$

which leads to $\gamma_x(y) = x$ if and only if $y = x$. In particular, the codimension of $\gamma_x(y)$ is the number of independent walls containing x and y. For the proof of (4.3) see [15, Lemma 1.1]. The closure of $\iota(V_s)$ in $\Gamma(\mathscr{X})$ is called the *combinatorial compactification* and denoted by $\overline{\mathscr{X}}_C$. The map γ_x can be interpreted as a discrete vector field. This analogy was already mentioned in [15, Introduction] and we make it more concrete below. In the following theorem we study the corresponding boundary in terms of core sequences.

Theorem 4.1. *Suppose that (x_n) is an (ω, J, c)-core sequence. Then for every $y \in V(\mathscr{X})$, the sequence $(\gamma_{x_n}(y) : n \in \mathbb{N}_0)$ is eventually constant, and thus*

$$\gamma(y) = \lim_{n\to\infty} \gamma_{x_n}(y) = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \gamma_{x_n}(y).$$

If (x'_n) is an (ω', J', c')-core sequence such that $(\gamma_{x'_n})$ converges to the same limit then $J' = J$, $\omega' \sim_J \omega$ and $c' = c$.

Proof. Fix a core sequence (x_n) with auxiliary sequence (u_n). Let

$$Q = \bigcup_{n=1}^{\infty} [o, u_n].$$

The set Q is convex. Let $\mathscr{A}$ be any apartment containing Q. Let $\omega_1 \in \Omega$ be such that $Q \subset [o, \omega_1] \subset \mathscr{A}$. First, we show that for each $y \in V(\mathscr{A})$, the sequence $(\gamma_{u_n}(y) : n \in \mathbb{N}_0)$ is eventually constant. Indeed, there is n_0 such that every half-apartment in $\mathscr{A}$ containing y with boundary parallel to an α-wall for $\alpha \in \Phi^+ \setminus \Phi_J$ contains u_n for all $n \geqslant n_0$. On the other hand, if $\alpha \in \Phi_J$, then the half-apartment containing y with the boundary parallel to the α-wall which passes through u_n is independent of n. Hence, by (4.3), we easily conclude that $(\gamma_{u_n}(y))$ is eventually constant. In fact, if $x \in V(\mathscr{A})$ is such that

$$\begin{aligned} \langle h(o, x; \omega_1), \alpha_j \rangle &= \langle h(o, u_{n_0}; \omega_1), \alpha_j \rangle \quad \text{for all } j \in J, \\ \langle h(o, x; \omega_1), \alpha_i \rangle &\geqslant \langle h(o, u_{n_0}; \omega_1), \alpha_i \rangle \quad \text{for all } i \in I_0 \setminus J, \end{aligned}$$

then $\gamma_{u_n}(y) = \gamma_x(y)$ for all $n \geqslant n_0$.

If ω'_1 is opposite ω_1 in $\mathscr{A}$, then $\rho_{\mathscr{A}}^{o,\omega'_1}$ restricted to the set

$$\{x_n\} \cup \bigcap_{\alpha \in \Phi^- \setminus \Phi_J} L^\alpha_{u_n}$$

is an isomorphism onto its image, where for $\alpha \in \Phi^-$ and a good vertex $x \in V(\mathscr{A})$ we denote by L^α_x the half-apartment in $\mathscr{A}$ with boundary parallel to the α-wall passing through x. Thus the sequence $(\gamma_{x_n}(y))$ is eventually constant.

Next, let us consider $y \in V(\mathscr{X})$ such that there is no apartment containing y and Q. Let $(c_1, \ldots, c_k)$ be a shortest gallery between Q and y. Let c_j be the last chamber in this gallery that can be put in one apartment with Q. Take $\mathscr{A}$ to be an apartment containing both c_j and Q, and let L^α be the half-apartment in $\mathscr{A}$ containing c_j whose boundary contains the panel shared between c_j and c_{j+1}. Since c_{j+1} cannot be put into the common apartment with Q, the boundary ∂L^α intersects Q. Let L be the half-apartment in $\mathscr{A}$ with boundary ∂L^α which contains ω_1. Since $\rho_{\mathscr{A}}^{o,\omega_1}(y) \in V(\mathscr{A})$, the sequence

$$\left(\gamma_{x_n}(\rho_{\mathscr{A}}^{o,\omega_1}(y)) : n \in \mathbb{N}_0\right)$$

is eventually constant. Moreover, $\rho_{\mathscr{A}}^{o,\omega_1}$ restricted to conv $\{L, y\}$ is an isomorphism onto its image, and thus the sequence $(\gamma_{x_n}(y))$ is eventually constant too. This completes the proof of the first part of the theorem.

Next, let us define Q_x for $x \in V(\mathscr{X})$ by the formula

$$Q_x = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} [x, u_n]. \tag{4.4}$$

The following fact can be viewed as the construction of a discrete geodesic flow.

Claim 4.2. *For each $x \in V(\mathscr{X})$, Q_x is the minimal set $Q' \subset V(\mathscr{X})$ such that*

(i) *$x \in Q'$,*

(ii) *if a vertex y is in Q' then $\gamma(y) \subset Q'$.*

Moreover, Q_x consists of all vertices $y \in V(\mathscr{X})$ such that there is a sequence $(v_0, v_1, \ldots, v_n)$ of vertices such that $v_0 = x$, $v_n = y$ and $v_{j+1} \in \gamma(v_j)$ for all $j \in \{0, 1, \ldots, n-1\}$.

For the proof, let us denote by $\tilde{Q}_x$ the intersection of all subsets of $V(\mathscr{X})$ satisfying (i) and (ii). Observe that $\tilde{Q}_x \subseteq Q_x$. To see this it is enough to show that Q_x satisfies (ii). If $y \in Q_x$ then there is $m \geqslant 1$ such that $y \in [x, u_n]$ for all $n \geqslant m$, and thus by (4.3) we have $\gamma(y) \subset [x, u_n]$ for all $n \geqslant m$. Next, to show that $Q_x \subseteq \tilde{Q}_x$, we prove that for each $y \in Q_x$ there is a sequence $(v_0, \ldots, v_m)$ of vertices such that $v_0 = x$, $v_m = y$ and $v_{j+1} \in \gamma(v_j)$ for all $j \in \{0, 1, \ldots, m-1\}$. We proceed by induction on the length of a minimal gallery between x and y. The assertion is trivial if the length equals 1. Suppose that it holds true for all vertices x and y in a minimal gallery between them of length $k \geqslant 2$. Let x' be any vertex in $[x, y]$ other than x and y. Since $y \in Q_x$, there is n_0 such that $y \in [x, u_n]$ for all $n \geqslant n_0$. Hence, $x' \in [x, u_n]$ for all $n \geqslant n_0$, and so $x' \in Q_x$. Since the length of a minimal gallery between x and x' is smaller than k, by the inductive hypothesis there is a sequence $(v'_0, \ldots, v'_{m'})$ of vertices such that $v'_0 = x$, $v'_{m'} = x'$ and $v'_{j+1} \in \gamma(v'_j)$ for all $j \in \{0, \ldots, m'-1\}$. Moreover, $y \in [x', u_n]$ for all $n \geqslant n_0$, and thus $y \in Q_{x'}$. Since the length of a minimal gallery between x' and y is smaller than k, by the inductive hypothesis there is a sequence $(v''_0, \ldots, v''_{m''})$ of vertices such that $v''_0 = x'$, $v''_{m''} = y$ and $v''_{j+1} \in \gamma(v''_j)$ for all $j \in \{0, \ldots, m''-1\}$. Therefore, the sequence $(v'_0, \ldots, v'_{m'}, v''_1, \ldots, v''_{m''})$ is as desired in the claim.

In particular, by Claim 4.2, the set Q_x depends only on the vertex x and the map γ. To finish the proof of the theorem, let us assume that there are two core sequences (x_n) and (x'_n) such that

$$\lim_{n\to\infty} \gamma_{x_n} = \lim_{n\to\infty} \gamma_{x'_n} = \gamma.$$

Thanks to Claim 4.2,

$$\bigcup_{n=1}^{\infty} [o, u_n] = Q_o = \bigcup_{n=1}^{\infty} [o, u'_n],$$

which easily leads to $c' = c$, $J' = J$ and $\omega' \sim_J \omega$. ∎

5. Harmonic measures

In this section we introduce harmonic measures which we use for Furstenberg compactifications: there is one such probability measure on the maximal boundary for each special point in the building. Then we study big cells in the maximal boundary from the topological and measure-theoretic viewpoints.

5.1. *Construction of harmonic measures*

The following proposition is well-known (see e.g. [36, 45]). It introduces the family of harmonic measures that are used to define the Furstenberg compactification. The subtlety is that we cannot rely on group actions and integration on homogeneous spaces to define these measures.

Proposition 5.1. *For every special vertex $x \in V_s$, there is a unique Borel probability measure ν_x on Ω such that for all $y, y' \in V_s$, if $\sigma(x, y) = \sigma(x, y')$ then*

$$\nu_x(\Omega(x, y)) = \nu_x(\Omega(x, y')). \tag{5.1}$$

Moreover, for any $x, y \in V_s$ the measures ν_x and ν_y are mutually absolutely continuous. If $x, y \in V_g$ then the Radon–Nikodym derivative equals

$$\frac{\mathrm{d}\nu_y}{\mathrm{d}\nu_x}(\omega) = \chi(h(x, y; \omega)), \quad \omega \in \Omega. \tag{5.2}$$

The proof below gives in fact all Radon–Nikodym derivatives, but the formulation uses notions from Section 1.3 such as the permutation ε.

Proof of Proposition 5.1. Let us consider a map Λ_x defined on the set $\{\mathbb{1}_{\Omega(x,y)} : y \in V_s\}$ of characteristic functions by the formula

$$\Lambda_x(\mathbb{1}_{\Omega(x,y)}) = \frac{1}{\#\{y' \in V_s : \sigma(x, y') = \sigma(x, y)\}}. \tag{5.3}$$

Recall that for every $z \in V_s$ such that $\sigma(x, z) = \sigma(x, y) + \sigma(y, z)$, we have

$$\#\{z \in V_s : \sigma(x, z) = \sigma(x, z')\} = \#Z' \cdot \#\{y' \in V_s : \sigma(x, y') = \sigma(x, y)\}$$

where $Z' = \{z' \in V_s : \sigma(x, z') = \sigma(x, z),\ \sigma(y, z') = \sigma(y, z)\}$. Since

$$\Omega(x, y) = \bigsqcup_{z' \in Z'} \Omega(x, z'),$$

we obtain

$$\Lambda_x(\mathbb{1}_{\Omega(x,y)}) = \sum_{z' \in Z} \Lambda_x(\mathbb{1}_{\Omega(x,z')}).$$

Consequently, Λ_x extends to a linear operator acting on locally constant functions on Ω. Since locally constant functions on Ω are dense in the space of continuous functions on Ω, the operator has a unique extension to a positive linear operator $\tilde{\Lambda}_x$ defined on all continuous complex-valued functions on Ω. Hence, by the Riesz–Markov–Kakutani theorem, there is a unique regular Borel measure ν_x on Ω such that

$$\tilde{\Lambda}_x(f) = \int_\Omega f(\omega)\, \nu_x(\mathrm{d}\omega)$$

for any continuous function $f : \Omega \to \mathbb{C}$.

To prove (5.2), we fix $x, y \in V_g$. Let $\omega \in \Omega$. By [45, Lemma 3.13], there is a good vertex $z \in [x, \omega] \cap [y, \omega]$ such that $\Omega(x, z) = \Omega(y, z)$. We may assume that $h(x, z; \omega)$ and $h(y, z; \omega)$ are strongly dominant coweights. Then, by (1.2) and (1.4),

$$\nu_x(\Omega(x,z)) = \frac{1}{W(q^{-1})} \chi(-h(x,z;\omega)) = \frac{1}{W(q^{-1})} \chi(-\sigma(x,z))$$

and

$$\nu_y(\Omega(x,z)) = \nu_y(\Omega(y,z)) = \frac{1}{W(q^{-1})} \chi(-h(y,z;\omega)) = \frac{1}{W(q^{-1})} \chi(-\sigma(y,z)).$$

Hence,

$$\frac{\nu_y(\Omega(x,z))}{\nu_x(\Omega(x,z))} = \chi(-h(y,z;\omega))\chi(h(x,z;\omega)) = \chi(h(x,y;\omega))$$

where we have used the cocycle relation (1.6).

If $x \in V_g$ and $z \in V_g^\varepsilon$, we set

$$Z' = \left\{ z' \in V_g : \exists \omega \in \Omega(x,z) \text{ such that } h(z,z';\omega) = \tfrac{1}{2}\lambda_r \right\}.$$

Let c be the unique chamber in $\mathrm{St}(z)$ which is closest to x. Let w_0 and w_{0r} be the longest elements in W and W_{λ_r}, respectively. Each vertex $z' \in Z'$ belongs to a certain chamber from $\mathrm{St}(z)$ which is opposite c. There are $q_{\varepsilon;w_0}$ chambers with this property. However, for a fixed $z' \in Z'$, there are $q_{\varepsilon;w_{0r}}$ distinct chambers sharing the vertices z' and z that are opposite c. Hence,

$$\#Z' = \frac{q_{\varepsilon;w_0}}{q_{\varepsilon;w_{0r}}}.$$

Since

$$\Omega(x,z) = \bigsqcup_{z' \in Z'} \Omega(x,z'),$$

we get

$$\nu_x(\Omega(x,z)) = \sum_{z' \in Z'} \nu_x(\Omega(x,z')) = \frac{q_{\varepsilon;w_0}}{q_{\varepsilon;w_{0r}}} \frac{1}{W(q^{-1})} \chi\left(-\tfrac{1}{2}\lambda_r\right) \chi(-h(x,z;\omega)).$$

Therefore, for $y \in V_g^\varepsilon$ and $x \in V_g$ we obtain

$$\frac{\nu_y(\Omega(x,z))}{\nu_x(\Omega(x,z))} = \frac{W(q^{-1})}{W(q_\varepsilon^{-1})} \frac{q_{\varepsilon;w_{0r}}}{q_{\varepsilon;w_0}} \chi\big(\tfrac{1}{2}\lambda_r\big) \chi(h(x,y;\omega)).$$

This completes the proof. ■

Remark. We can weaken the hypothesis in Proposition 5.1, by imposing (5.1) for all $y \in V_g$ having fixed type, say $\tau(y) = j \in I$. To see this, let $z \in V_g$ with $\tau(z) \neq j$. There is $y_0 \in V_g$ with $\tau(y_0) = j$ such that

$$\Omega(x, y_0) \subset \Omega(x, z).$$

Fix $\omega_0 \in \Omega(x, y_0)$. Then

$$Y = \{y \in V_g : \exists \omega \in \Omega(x,z) \text{ such that } h(x,y;\omega) = h(x,y_0;\omega_0)\}.$$

Observe that each vertex $y \in Y$ has type j, and

$$\Omega(x,z) = \bigsqcup_{y \in Y} \Omega(x,y).$$

Now, the linear operator Λ defined in (5.3) for $y \in V_g$ with $\tau(y) = j$ can be uniquely extended to $\mathbb{1}_{\Omega(x,z)}$ for any $z \in V_g$. The rest of the poof is unchanged.

The measures $(\nu_x : x \in V_g)$ naturally appear while studying harmonic functions with respect to vertex averaging operators; see Section 7.1. To be more precise, for $\lambda \in P^+$ and a function $F : V_g \to \mathbb{C}$, we set

$$A_\lambda F(x) = \frac{1}{N_\lambda} \sum_{y \in V_\lambda(x)} F(y), \quad x \in V_g. \tag{5.4}$$

Then for any $f \in L^\infty(\Omega)$, the function $F : V_g \to \mathbb{C}$ defined by

$$F(x) = \int_\Omega f(\omega)\, \nu_x(\mathrm{d}\omega), \quad x \in V_g,$$

satisfies

$$A_\lambda F(x) = F(x)$$

for all $x \in V_g$ and $\lambda \in P^+$. We have a similar characterization for vertices in V_g^ε.

5.2. *Disintegration of harmonic measures*

Let us fix $J \subset I_0$. We denote by Ω^J the set of all spherical residues of type J and thus we have a quotient map $\pi^J : \Omega \twoheadrightarrow \Omega^J$ defined by $\omega \mapsto \operatorname{res}_J(\omega)$. To each J-residue R is associated a facet F of type J in the spherical building at infinity; the intersection of all closed chambers at infinity in R is equal to F. Moreover, according to Section 2.3, to each J-residue R we can associate an affine building $\mathscr{X}_R = \mathscr{X}(F)$ and we have a homomorphism of buildings

$$\pi_R : \mathscr{X} \to \mathscr{X}_R,$$

together with a homeomorphism

$$\varphi_R : R \to \Omega_R$$

identifying the residue R with the maximal boundary $\Omega_R = \Omega_{\mathscr{X}(F)}$ of the outer façade $\mathscr{X}_R$ (Lemma 2.2).

Now pick in addition $x, y \in V_g$ and let $\omega \in \Omega(x, y)$. If y' is another vertex, we say that y' is *J-related to y* (with respect to R) if there is an apartment containing $[x, \omega]$ and y' such that y' belongs to the orbit of the subgroup of the spherical Weyl group fixing the sector face of type J contained in $[x, \omega]$ (this subgroup is also the subgroup fixing the facet F in the corresponding apartment at infinity). We then have $\sigma(x, y') = \sigma(x, y)$. We denote by $[y]_J$ the set of vertices which are J-related to y with respect to R. The projection $\pi_R|_{[y]_J}$ provides a bijection between $[y]_J$ and the vectorial sphere in $\mathscr{X}_R$ centered at $\pi_R(x)$ and of radius $P_J\sigma(x, y)$; we denote by $N^J_{P_J\sigma(x,y)}$ the common cardinality of these two sets.

We introduce on Ω^J a probability measure η^J_x by setting

$$\eta^J_x(\Omega(x, y)^J) = \frac{N^J_{P_J\sigma(x,y)}}{N_{\sigma(x,y)}} \tag{5.5}$$

where

$$\Omega(x, y)^J = \{R \in \Omega^J : \Omega(x, y) \cap R \neq \varnothing\}.$$

Analogously to the proof of Proposition 5.1 one can show that the measures $\eta^J_x \in \mathscr{P}(\Omega^J)$ are well-defined for each $J \subset I_0$ and each $x \in V_g$. Having in mind the partition

$$\Omega = \bigsqcup_{R\in\Omega^J} R = \bigsqcup_{R\in\Omega^J} \varphi_R^{-1}(\Omega_R),$$

we want to disintegrate each harmonic measure $\nu_x \in \mathscr{P}(\Omega)$ by integrating first on each J-residue – seen as the maximal boundary of the corresponding outer façade – against a suitable harmonic measure of the façade and by using the measure η^J_x for the second integration step.

Proposition 5.2. *With the above notation, we have the equality of measures $\eta^J_x = (\pi^J)_*\nu_x$ in $\mathscr{P}(\Omega^J)$. Moreover, for each Borel set $A \subset \Omega$, we have the disintegration formula*

$$\nu_x(A) = \int_{A^J} \nu_{\pi_R(x)}(\varphi_R(R \cap A))\, \eta^J_x(\mathrm{d}R) \tag{5.6}$$

where $\nu_{\pi_R(x)}$ is the harmonic measure on the maximal boundary $\Omega_R = \varphi_R(R)$ of the outer façade $\mathscr{X}_R$ attached to the vertex $\pi_R(x)$.

Proof. In order to show the equality $\eta_x^J = (\pi^J)_* \nu_x$ in $\mathscr{P}(\Omega^J)$, it is enough to prove that for any $y \in V_g$ we have

$$((\pi^J)_* \nu_x)(\Omega(x, y)^J) = \frac{N^J_{P_J \sigma(x,y)}}{N_{\sigma(x,y)}}, \tag{5.7}$$

that is,

$$\nu_x((\pi^J)^{-1}(\Omega(x, y)^J)) = \frac{N^J_{P_J \sigma(x,y)}}{N_{\sigma(x,y)}}.$$

We first note that

$$\Omega(x, y)^J = \{R \in \Omega^J : \Omega(x, y) \cap R \neq \varnothing\} = \{\mathrm{res}_J(\omega) : \omega \in \Omega(x, y)\},$$

which implies that

$$(\pi^J)^{-1}(\Omega(x, y)^J) = \bigcup_{\omega \in \Omega(x,y)} \mathrm{res}_J(\omega).$$

We claim that

$$(\pi^J)^{-1}(\Omega(x, y)^J) = \bigsqcup_{y' \in [y]_J} \Omega(x, y'). \tag{5.8}$$

Since $\#[y]_J = N^J_{P_J \sigma(x,y)}$ and each shadow $\Omega(x, y')$ with $y' \in [y]_J$ has the same ν_x-mass, the equality (5.8) immediately leads to (5.7). To prove (5.8), we notice that if $\omega' \in \mathrm{res}_J(\omega)$ with $\omega \in \Omega(x, y)$, then the sectors $[x, \omega]$ and $[x, \omega']$ share their vectorial face of type J. Hence, the stabilizer in the Weyl group of that vectorial face, W_J say, sends y to a vertex y' contained in $[x, \omega']$ and thus $y' \in [y]_J$. Conversely, if ω' belongs to a shadow $\Omega(x, y')$ with $y' \in [y]_J$, then we can pick an apartment containing $[x, \omega']$ and y. In this apartment W_J sends the sector $[x, \omega']$ to a J-adjacent one which contains y, so that denoting this sector by $[x, \omega]$ we see that $\omega' \in \mathrm{res}_J(\omega)$ with $\omega \in \Omega(x, y)$, as desired.

We turn now to the proof of the disintegration formula. Let us consider the Borel measure μ on Ω defined by

$$\mu(A) = \int_{A^J} \nu_{\pi_R(x)}(\varphi_R(R \cap A))\, \eta_x^J(\mathrm{d}R)$$

where $A^J = \{R \in \Omega^J : R \cap A \neq \varnothing\}$. Since for all $x, y \in V_g$ and $R \in \Omega(x, y)^J$, by Lemma 2.3 we have

$$\sigma_R(\pi_R(x), \pi_R(y)) = P_J \sigma(x, y),$$

it follows that

$$\begin{aligned}
\mu(\Omega(x, y)) &= \int_{\Omega(x,y)^J} \nu_{\pi_R(x)}\big(\Omega_R(\pi_R(x), \pi_R(y))\big)\, \eta_x^J(\mathrm{d}R) \\
&= \int_{\Omega(x,y)^J} (N^J_{\sigma_R(\pi_R(x), \pi_R(y))})^{-1}\, \eta_x^J(\mathrm{d}R) \\
&= (N^J_{P_J \sigma(x,y)})^{-1} \int_{\Omega(x,y)^J} \eta_x^J(\mathrm{d}R) \\
&= (N_{\sigma(x,y)})^{-1} = \nu_x(\Omega(x, y)).
\end{aligned}$$

Finally, by Proposition 5.1 characterizing harmonic measures, we have $\mu = \nu_x$. ■

5.3. Big cells

In the study of convergence of unbounded sequences of harmonic measures, we will make use of analogues of contraction arguments in the case of symmetric spaces and Bruhat–Tits buildings. The subspaces of Ω on which the arguments are valid are analogues of the so-called big cells in flag varieties. The following fact saying that big cells are conull in maximal boundaries has already been proved in [27, Proposition 2.13]; however, our proof is different in that it is conducted inside the building, without using points at infinity.

Theorem 5.3. *For each $\omega_0 \in \Omega$ and $x \in V_s$, we have $\nu_x(\Omega'(\omega_0)) = 1$.*

Proof. Without loss of generality we assume that $x \in V_g$. Given $y \in V_g$, we denote by $\Omega_y(\omega_0)$ the set of all $\omega \in \Omega$ such that there is an apartment $\mathscr{A}$ containing both sectors $[y, \omega_0]$ and $[y, \omega]$. Then

$$\Omega = \bigcup_{y \in V_g} \Omega_y(\omega_0),$$

and so

$$\Omega \setminus \Omega'(\omega_0) = \bigcup_{y \in V_g} \left(\Omega_y(\omega_0) \setminus \Omega'_y(\omega_0)\right) \tag{5.9}$$

where we have set $\Omega'_y(\omega_0) = \Omega'(\omega_0) \cap \Omega_y(\omega_0)$. Since the set of vertices is countable, it is therefore enough to show that for all $y \in V_g$,

$$\nu_x(\Omega_y(\omega_0) \setminus \Omega'_y(\omega_0)) = 0. \tag{5.10}$$

Since the measures ν_x are mutually absolutely continuous, it is sufficient to consider $x = y = o$.

Next, let us observe that

$$\Omega_o(\omega_0) \setminus \Omega'_o(\omega_0) = \bigcap_{n=0}^{\infty} \bigcup_{\substack{w \in W \\ w \neq w_0}} \bigcup_{y \in B_w^{n\rho}} \Omega(o, y). \tag{5.11}$$

where for $w \in W$ and $\lambda \in P^+$, we denote by B_w^λ the set of all $x \in V_g$ such that $h(o, x; \omega_0) = w.\lambda$ and there is an apartment containing $[o, \omega_0]$ and x. It is easy to verify the inclusion $\subseteq$. To show the reverse one, let us consider ω belonging to the right-hand side of (5.11). Then there are sequences $(y_n : n \in \mathbb{N})$ and $(w_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, we have $w_n \in W \setminus \{w_0\}$, $y_n \in B_{w_n}^{n\rho}$ and $\omega \in \Omega(o, y_n)$. Therefore,

$$\omega \in \bigcap_{n=1}^{\infty} \Omega(o, y_n).$$

In particular, $[o, y_n] \subset [o, y_{n+1}] \subset [o, \omega]$. Now, let Λ_n be the set of all sectors $[o, \omega']$ with $\omega' \in \Omega'_o(\omega_0)$ such that the apartment $[\omega_0, \omega']$ contains y_n. Since $y_n \in B_{w_n}^{n\rho}$, each set Λ_n is non-empty. Moreover, $\Lambda_{n+1} \subset \Lambda_n$ because the convex hull of o and y_{n+1} contains y_n.

Hence, by compactness of Ω, there is ω_0' such that the apartment $[\omega_0, \omega_0']$ contains the convex hull of o and y_n for all $n \in \mathbb{N}$, and thus it contains $[o, \omega]$. This proves (5.11).

Notice that for a fixed $n \in \mathbb{N}$, the sets $\Omega(o, y)$ are disjoint provided that $\sigma(o, y) = n\rho$, and thus by (5.11),

$$\begin{aligned}\nu_o(\Omega_o(\omega_0) \setminus \Omega_o'(\omega_0)) &= \lim_{n\to\infty} \nu_o\Big(\bigcup_{\substack{w\in W\\ w\neq w_0}} \bigcup_{y\in B_w^{n\rho}} \Omega(o, y)\Big)\\ &= \lim_{n\to\infty} \sum_{\substack{w\in W\\ w\neq w_0}} \sum_{y\in B_w^{n\rho}} \nu_o(\Omega(o, y)).\end{aligned}$$

Therefore to finish the proof of the theorem we show the following claim.

Claim 5.4. *For each $w \in W$, $w \neq w_0$, we have*

$$\lim_{n\to\infty} \frac{\# B_w^{n\rho}}{N_{n\rho}} = 0.$$

For the proof, let us notice that for a given $y \in B_w^{(n+1)\rho}$ there is exactly one $x \in B_w^{n\rho}$ such there is an apartment containing $[o, \omega_0]$, x and y. Therefore, the problem is reduced to estimating the number of y's that correspond to a given x. To do so, we observe that there is a wall passing through x such that o and ω_0 are on its opposite sides. Let $\beta \in \Phi^+$ be the corresponding root. Since the convex hull of $[o, \omega_0]$ and x is contained in the intersection of all half-apartments having x on the boundary and containing the sector $[o, \omega_0]$, the intersection of the link[2] of x with the convex hull contains at least two chambers that are β-adjacent. Let c_0 and c_1 be the chambers with vertex x that are closest to o and ω_0, respectively. Each minimal gallery between x and y starts with a certain chamber c' having vertex x and which is opposite c_0. All minimal galleries have the same type f. For a given y the chamber c' is unique. Moreover, for each c' there is a minimal gallery between c_0 and c' which contains c_1. Therefore, there are at most $q_\beta^{-1} q_{w_0}$ possible choices for c', and hence there are at most $q_\beta^{-1} q_{w_0} q_{w_f}$ vertices $y \in B_w^{(n+1)\rho}$ such that the apartment containing y and $[o, \omega_0]$ also contains x. Consequently,

$$\# B_w^{(n+1)\rho} \leqslant \# B_w^{n\rho} \cdot q_\beta^{-1} \chi(\rho).$$

In view of (1.2), we get

$$\frac{\# B_w^{(n+1)\rho}}{N_{(n+1)\rho}} \leqslant q_\beta^{-1} \frac{\# B_w^{n\rho}}{N_{n\rho}},$$

which completes the proof of the claim.

Now, using Claim 5.4 and Proposition 5.1, we get

$$\nu_o(\Omega_o(\omega_0) \setminus \Omega_o'(\omega_0)) = \sum_{\substack{w\in W\\ w\neq w_0}} \lim_{n\to\infty} \frac{\# B_w^{n\rho}}{N_{n\rho}} = 0,$$

which proves (5.10), and the theorem follows. ∎

[2] The *link* of x is the collection of chambers sharing the vertex x.

Corollary 5.5. *For each $\omega_0 \in \Omega$, the big cell $\Omega'(\omega_0)$ is dense in Ω.*

Proof. Indeed, otherwise there are $\omega' \in \Omega \setminus \Omega'(\omega_0)$ and its neighborhood $\Omega(x, y)$ for some distinct $x, y \in V_g$ such that $\Omega(x, y) \subset \Omega \setminus \Omega'(\omega_0)$. Hence, by Theorem 5.3,

$$0 < \nu_x(\Omega(x, y)) \leqslant \nu_x(\Omega \setminus \Omega'(\omega_0)) = 0,$$

which is a contradiction. ∎

6. Furstenberg compactification

In this section we describe the Furstenberg compactification. It originates in the study of harmonic functions on semisimple Lie groups [21]. For Bruhat–Tits buildings associated with semisimple groups over local fields, it has been constructed in [25]. Our approach is purely geometric and provides the maximal Furstenberg compactification for a large class of affine buildings including exotic ones.

Let $\mathscr{P}(\Omega)$ be the space of Borel probability measures on Ω endowed with the weak-$*$ topology. An automorphism $g \in \operatorname{Aut}(\mathscr{X})$ acts on a measure $\nu \in \mathscr{P}(\Omega)$ by pushing forward, that is,

$$(g.\nu)(A) = (g_*\nu)(A) = \nu(g^{-1}.A)$$

for any Borel set A.

We define

$$\iota : V_s \to \mathscr{P}(\Omega), \quad x \mapsto \nu_x,$$

where ν_x is the harmonic measure defined in Proposition 5.1. The map ι is an equivariant embedding. Moreover, $\iota(V_s)$ is discrete in $\mathscr{P}(\Omega)$ since there is $\varepsilon > 0$ such that the set

$$\bigcap_{j=1}^{r} \bigcap_{y_j \in V_{\lambda_j}(x)} \{\nu \in \mathscr{P}(\Omega) : \nu(\Omega(x, y_j)) > (1-\varepsilon)\nu_x(\Omega(x, y_j))\}$$

is open and its intersection with $\iota(V_s)$ is the singleton ν_x. Let $\overline{\mathscr{X}}_F$ be the closure of $\iota(V_s)$ in $\mathscr{P}(\Omega)$. Then $\overline{\mathscr{X}}_F$ equipped with the induced topology is a compact Hausdorff space called the *Furstenberg compactification* of $\mathscr{X}$. Let us describe the structure of $\overline{\mathscr{X}}_F$. Since $\mathscr{P}(\Omega)$ is metrizable, it is sufficient to consider sequences (x_n) approaching infinity such that $(\iota(x_n))$ converges in $\mathscr{P}(\Omega)$. Furthermore, in view of Lemma 1.2, we restrict our attention to core sequences.

6.1. *Degenerations of harmonic measures*

In order to state our main theorem on convergence of sequences of harmonic measures (which is Theorem C in the Introduction), it is convenient to use the notions and terminology introduced in Section 2, including façades indexed by spherical facets, i.e., residues at infinity. We freely use the notation of Section 2.3. Given a J-residue R in the building at infinity we define the map

$$\phi_R : \Omega \to \Omega_R, \quad \omega \mapsto \varphi_R(\mathrm{proj}_R(\omega)),$$

where $\mathrm{proj}_R(\omega)$ is the unique chamber closest to ω in R.

Theorem 6.1. *Suppose that (x_n) is an (ω, J, c)-core sequence. Let R be the J-residue in $\mathscr{X}^\infty$ containing ω and let F be the corresponding spherical facet. Denote by $\mathscr{X}(F)$ the façade associated with F, and by x_F the vertex in $\mathscr{X}(F)$ defined by (x_n). Then the sequence (ν_{x_n}) of harmonic measures weakly converges in $\mathscr{P}(\Omega)$ to the measure μ characterized by the following two conditions:*

(i) $\mathrm{supp}(\mu) = \mathrm{res}_J(\omega)$;

(ii) $(\phi_R)_*\mu$ *is the harmonic measure on the maximal boundary of $\mathscr{X}(F)$ attached to the vertex x_F.*

Moreover, if (x'_n) is an (ω', J', c')-core sequence such that $(\nu_{x'_n})$ weakly converges to the same limit μ as above, then $J' = J$, $\omega' \sim_J \omega$ and $c' = c$.

Proof. The proof uses the weak-$*$ compactness of the set $\mathscr{P}(\Omega)$ of probability measures on the compact metrizable maximal boundary Ω. Convergence is obtained by proving uniqueness of the cluster value in two steps. First one shows that any cluster value μ of a sequence of harmonic measures as above has support in the residue R, which then allows us to push μ forward by the homeomorphism φ_R in order to prove, in the second step, that $(\phi_R)_*\mu$ must be the announced harmonic measure on the façade $\mathscr{X}(F)$.

Without loss of generality we assume that each x_n is in V_g (see Section 1.3). Let μ be any cluster point of (ν_{x_n}). By selecting a further subsequence we can guarantee that for each $j \in I_0 \setminus J$, the sequence $(\eta_{x_n}^{I_0 \setminus \{j\}})$ of probability measures weakly converges (see (5.5) for the definition).

Before we embark on the proof, let us observe the following fact.

Claim 6.2. *For each $\omega_1 \in \Omega$, $i \in I_0$ and $n \in \mathbb{N}_0$, if*

$$\langle h(o, u_{n+2}; \omega_1), \alpha_i \rangle \geqslant \langle h(o, u_{n+1}; \omega_1), \alpha_i \rangle$$

then

$$\langle h(o, u_{n+1}; \omega_1), \alpha_i \rangle \geqslant \langle h(o, u_n; \omega_1), \alpha_i \rangle.$$

For the proof, let us denote by $\mathscr{A}$ an apartment containing $[u_n, \omega_1]$. Since $u_{n+1} \in [u_n, u_{n+2}]$, the inequalities

$$\langle h(u_n, u_{n+1}; \omega_1), \alpha_i \rangle < 0 \quad \text{and} \quad \langle h(u_{n+1}, u_{n+2}; \omega_1), \alpha_i \rangle \geqslant 0$$

cannot both hold true because the α_i-wall passing through $\rho_{\mathscr{A}}^{u_n,\omega_1}(u_{n+1})$ cannot have $\rho_{\mathscr{A}}^{u_n,\omega_1}(u_{n+2})$ on the same side as $[u_n, \omega_1]$.

Observe now that if $i \in I_0 \setminus J$, then the sequence $(\langle h(o, u_n; \omega_1), \alpha_i \rangle : n \in \mathbb{N}_0)$ cannot be eventually constant. Furthermore, as a consequence of Claim 6.2, either it is unbounded, or there is $n_0 \geqslant 0$ such that for all $n > n_0$,

$$\langle h(o, x_n; \omega_1), \alpha_i \rangle < \langle h(o, u_{n_0}; \omega_1), \alpha_i \rangle,$$

and

$$\langle h(o, u_{n_0}; \omega_1), \alpha_i \rangle = \sup_{n \in \mathbb{N}} \langle h(o, u_n; \omega_1), \alpha_i \rangle.$$

Step 1. In this step we show that the support of μ is contained in R. Let $\omega_0 \in \Omega \setminus R$. Then there is $i \in I_0 \setminus J$ such that

$$\sup_{n \in \mathbb{N}_0} \langle h(o, u_n; \omega_0), \alpha_i \rangle < \infty. \tag{6.1}$$

We are going to construct $U \subset \Omega$, an open neighborhood of ω_0, such that $\mu(U) = 0$.

In view of (6.1) and Claim 6.2, there are $N \geqslant 1$ and $n_0 \in \mathbb{N}$ such that

$$\langle h(o, u_{n_0+1}; \omega_0), \alpha_i \rangle < \langle h(o, u_{n_0}; \omega_0), \alpha_i \rangle = \sup_{n \geqslant 1} \langle h(o, u_n; \omega_0), \alpha_i \rangle \leqslant N.$$

Let

$$U = \left\{ \omega_1 \in \Omega : \begin{array}{c} \langle h(o, u_{n_0+1}; \omega_1), \alpha_i \rangle < \langle h(o, u_{n_0}; \omega_1), \alpha_i \rangle, \\ \langle h(o, u_{n_0-1}; \omega_1), \alpha_i \rangle \leqslant \langle h(o, u_{n_0}; \omega_1), \alpha_i \rangle, \\ \langle h(o, u_{n_0}; \omega_1), \alpha_i \rangle \leqslant N \end{array} \right\}.$$

By Claim 6.2, for all $\omega_1 \in U$, we have

$$\sup_{n \geqslant n_0} \langle h(o, u_n; \omega_1), \alpha_i \rangle = \langle h(o, u_{n_0}; \omega_1), \alpha_i \rangle \leqslant N.$$

Since the Busemann function takes values in $\frac{1}{2}P$, the set U is an open neighborhood of ω_0. Furthermore, as a sublevel set of a continuous function it is also closed. Hence,

$$\mu(U) = \lim_{n \to \infty} \nu_{x_n}(U).$$

Next, we are going to compute the limit. Setting $U' = U \cap \Omega'(\omega)$, by conullity of big cells (see Theorem 5.3) we get

$$\nu_{x_n}(U) = \nu_{x_n}(U') = \int_{(U')^{\{i\}}} \nu_{\pi_{R_1}(x_n)}(\varphi_{R_1}(R_1 \cap U'))\, \eta_{x_n}^{\{i\}}(\mathrm{d}R_1)$$

where in the second equality we have used (5.6). Recall that we know explicitly the Radon–Nikodym derivatives between harmonic measures (Proposition 5.1), that is, for each $R_1 \in (U')^{\{i\}}$, we have

$$\begin{aligned} \nu_{\pi_{R_1}(x_n)}(\varphi_{R_1}(R_1 \cap U')) &= \int_{\varphi_{R_1}(R_1 \cap U')} \chi^{\{i\}}\big(h_{R_1}(\pi_{R_1}(o), \pi_{R_1}(x_n); \omega_1)\big)\, \nu_{\pi_{R_1}(o)}(\mathrm{d}\omega_1) \\ &= \int_{\varphi_{R_1}(R_1 \cap U')} \chi\big(P_{\{i\}} h(o, x_n; \varphi_{R_1}^{-1}(\omega_1))\big)\, \nu_{\pi_{R_1}(o)}(\mathrm{d}\omega_1) \end{aligned}$$

where the last equality is a consequence of Lemma 2.3.

By Claim 6.2, for $\omega_1 \in U$ and $n > n_0$,

$$\langle h(o, x_n; \omega_1), \alpha_i \rangle < \langle h(o, u_{n_0}; \omega_1), \alpha_i \rangle \leqslant N,$$

thus

$$\sup_{\omega_1 \in \varphi_{R_1}(R_1 \cap U)} \sup_{n > n_0} \chi\big(P_{\{i\}} h(o, x_n; \varphi_{R_1}^{-1}(\omega_1))\big) \leqslant N.$$

Our next aim is to show that for every sequence $(\omega^n : n \in \mathbb{N}) \subset U'$ tending to $\omega_1 \in U'$, we have

$$\lim_{n \to \infty} \chi\big(P_{\{i\}} h(o, x_n; \varphi_{R_1}^{-1}(\omega^n))\big) = 0. \tag{6.2}$$

Since

$$\lim_{n \to \infty} \langle h(o, u_n; \omega_1), \alpha_i \rangle = -\infty,$$

by Claim 6.2, for each $A > 0$ there is $n' \geqslant 1$ such that for all $n > n'$,

$$\langle h(o, u_n; \omega_1), \alpha_i \rangle < \langle h(o, u_{n'}; \omega_1), \alpha_i \rangle \leqslant -A.$$

By continuity, there is an open set $B \subset \Omega$ with $\omega_1 \in B$ such that for all $\omega_2 \in B$,

$$\langle h(o, u_{n'}; \omega_2), \alpha_i \rangle \leqslant -A.$$

Let

$$V = \{\omega_2 \in B \cap U' : \langle h(o, u_{n'+1}; \omega_2), \alpha_i \rangle < \langle h(o, u_{n'}; \omega_2), \alpha_i \rangle\}.$$

The set V is an open neighborhood of ω_1 such that for all $\omega_2 \in V$ and $n > n'$,

$$\langle h(o, x_n; \omega_2), \alpha_i \rangle < \langle h(o, u_{n'}; \omega_2), \alpha_i \rangle \leqslant -A,$$

which leads to (6.2). Now, by Lebesgue's dominated convergence theorem for weakly convergent probability measures [53, Theorem 3.5], we obtain

$$\mu(U) = \lim_{n \to \infty} \int_{(U')^{\{i\}}} \int_{\varphi_{R_1}(R_1 \cap U')} \chi\big(P_{\{i\}} h(o, x_n; \varphi_{R_1}(\omega_1))\big)\, \nu_{\pi_{R_1}(o)}(\mathrm{d}\omega_1)\, \eta_{x_n}^{\{i\}}(\mathrm{d}R_1) = 0,$$

proving that $U \cap \operatorname{supp} \mu = \varnothing$. Consequently, $\operatorname{supp} \mu \subseteq R$.

Step 2. In this step, we show that any cluster value μ as in Step 1, which we write $\mu = \lim \nu_{x_n}$ for simplicity, is such that $(\phi_R)_* \mu$ is the harmonic measure of $\mathscr{X}(F)$ attached to the vertex x_F.

First, let us consider the map

$$\pi^J : \Omega \to \Omega^J, \quad \omega \mapsto \operatorname{res}_J(\omega).$$

Since $(\pi^J)_* \nu_x = \eta_x^J$ by Proposition 5.2, we get

$$\lim_{n \to \infty} \eta_{x_n}^J = \lim_{n \to \infty} (\pi^J)_* \nu_{x_n} = (\pi^J)_* \mu = \delta_R. \tag{6.3}$$

Let $x_0 = x_F$. Let y_0 be any special vertex of $\mathscr{X}(F)$. Let $\omega'_R \in \Omega_R(x_0, y_0)$. There is an apartment containing the cone $u_1 + F$ and the germ y_0 [51, Section 3.1]. Then there exists ω' such that the sector $[u_1, \omega']$ projects onto $[x_0, \omega_R]$. Let us pick $y' \in [u_1, \omega']$ such that $\pi_R(y') = y_0$. By construction, $\omega' \in \Omega(u_1, y')$; moreover, by convexity any $\omega''_R \in \Omega_R(x_0, y_0)$ will be the image of some $\omega'' \in \Omega(u_1, y')$.

Now our aim is to compute $((\phi_R)_*\mu)(\Omega_R(x_0, y_0))$. First for $j \in J$, we set

$$V_j = \{\omega_1 \in \Omega : \langle h(u_1, u_2; \omega_1), \alpha_j \rangle \leqslant \langle h(u_1, u_1; \omega_1), \alpha_j \rangle = 0\}.$$

Since each V_j is clopen and contains R, the set

$$U = \Omega(u_1, y') \cap \bigcap_{j \in J} V_j$$

is clopen and $\phi_R(U) = \Omega_R(x_0, y_0)$. Moreover, we have

$$\begin{aligned} ((\phi_R)_*\mu)(\Omega_R(x_0, y_0)) &= \mu(\phi_R^{-1}(\Omega_R(x_0, y_0))) \\ &= \mu\big(\phi_R^{-1}(\Omega_R(x_0, y_0)) \cap R\big) = \mu(U \cap R) = \mu(U). \end{aligned}$$

Hence, we can write

$$((\phi_R)_*\mu)(\Omega_R(x_0, y_0)) = \lim_{n \to \infty} \nu_{x_n}(U).$$

Now, by (5.6),

$$\nu_{x_n}(U) = \int_{U^J} \nu_{\pi_{R_1}(x_n)}(\varphi_{R_1}(U \cap R_1)) \nu^J_{x_n}(\mathrm{d}R_1).$$

In view of Proposition 5.1,

$$\nu_{\pi_{R_1}(x_n)}(\varphi_{R_1}(U \cap R_1)) = \int_{\varphi_{R_1}(U \cap R_1)} \chi^J\big(h_{R_1}(\pi_{R_1}(u_1), \pi_R(x_n); \omega_1)\big)\, \nu_{\pi_R(u_1)}(\mathrm{d}\omega_1).$$

To complete the proof, we need the following fact.

Claim 6.3. *For all* $\lambda \in P^+$,

$$\chi(P_J \lambda) = \prod_{\alpha \in \Phi_J^+} \tau_\alpha^{\langle \lambda, \alpha \rangle}.$$

To see this it is enough to notice that for all $w \in W_J$ and $\alpha \in \Phi^+ \setminus \Phi_J$,

$$w.\alpha \in \Phi^+ \setminus \Phi_J \quad \text{and} \quad \tau_{w.\alpha} = \tau_\alpha.$$

Next, thanks to Claim 6.3, for every $\omega_1 \in U$,

$$\begin{aligned} \chi^J\big(h_{R_1}(\pi_{R_1}(u_1), \pi_{R_1}(x_n); \omega_1)\big) &= \chi(P_J h(u_1, x_n; \omega_1)) \\ &= \prod_{\alpha \in \Phi_J^+} \tau_\alpha^{\langle h(u_1, x_n; \omega_1), \alpha \rangle} \leqslant 1, \end{aligned}$$

therefore

$$\nu_{\pi_{R_1}(x_n)}(\varphi_{R_1}(U \cap R_1)) \leqslant \nu_{\pi_{R_1}(u_1)}(\varphi_{R_1}(U \cap R_1)),$$

and so

$$\nu_{x_n}(U) \leqslant \int_{U^J} \nu_{\pi_{R_1}(u_1)}(\varphi_{R_1}(U \cap R_1))\, \eta^J_{x_n}(\mathrm{d}R_1).$$

In view of (6.3), we get

$$(\phi_R)_*\mu(\Omega_R(x_0, y_0)) \leqslant \nu_{x_0}(\Omega_R(x_0, y_0)).$$

Since y_0 was arbitrary and both $(\phi_R)_*\mu$ and ν_{x_0} are probability measures, we must have $(\phi_R)_*\mu = \nu_{x_0}$.

To show the uniqueness statement, let us suppose that there are two core sequences (x_n) and (x'_n) of type (ω, J, c) and (ω', J', c'), respectively, converging to the same limit, say μ. By (i), we have $\mathrm{res}_J(\omega) = \mathrm{supp}(\mu) = \mathrm{res}_{J'}(\omega')$, and since types of residues are well-defined we deduce that $J' = J$ and $\omega' \sim_J \omega$. Moreover, in view of the description of $(\varphi_J)_*\mu$ as a harmonic measure, and since Proposition 5.1 implies that the vertex defining such a measure is well-defined, we have $x_{F_J} = x_{F_{J'}}$ and therefore the parameters c and c' must be the same. This completes the proof. ■

Theorem 6.4. *Let $\mathscr{X}$ be a thick regular locally finite affine building. The closure of the collection of harmonic measures on $\mathscr{X}$ in the space $\mathscr{P}(\Omega)$ of probability measures on the maximal boundary Ω endowed with the weak-$*$ topology is* $\mathrm{Aut}(\mathscr{X})$*-equivariantly isomorphic to the polyhedral and to the combinatorial compactification of $\mathscr{X}$. More precisely, the maximal boundary of each affine building at infinity, or stratum, can be seen as a residue in Ω and any cluster value of any unbounded sequence of harmonic measures in $\mathscr{X}$ is a harmonic measure on a well-defined stratum.*

Proof. In the three compactifications considered, core sequences converge: for the polyhedral compactification this follows from Lemma 2.1, for the Caprace–Lécureux compactification it follows from Theorem 4.1 and for the Furstenberg compactification it follows from Theorem 6.1. As a consequence, combining a standard topological argument (see e.g. the domination criterion given by [24, Lemma 3.28] in both directions between two compactifications) and the uniqueness assertions in these results provide the identifications between the three compactifications, hence the first part of Theorem 6.4. In view of Lemma 2.2, we can identify the maximal boundary of $\mathscr{X}(F)$ with $\mathrm{res}_J(\omega)$, and thus the rest of Theorem 6.4 is a consequence of (i) and (ii) in Theorem 6.1. ■

6.2. *Furstenberg compactification for Bruhat–Tits buildings*

We provide here the Bruhat–Tits aspect of the previous section. The measure-theoretic compactification procedures are among the oldest ones for Riemannian symmetric spaces [21]. The idea, due to H. Furstenberg, is beautiful: it consists in using probability measures on the (maximal) boundary Ω by seeing it as a homogeneous space for as many compact subgroups as possible in the ambient Lie group. In order to be more precise, we need to combine Iwasawa decompositions of $\mathbf{G}(k)$ and Levi decompositions of parabolic k-subgroups. For any chamber at infinity $\omega \in \Omega$, we saw that $\Omega \simeq \mathbf{G}(k)/\mathbf{P}_\omega(k)$. We can furthermore choose a maximal k-split torus $\mathbf{S}$ such that the spherical apartment $\mathcal{A}(\mathbf{S})^\infty$ contains ω and we can pick a special vertex x in the affine apartment $\mathcal{A}(\mathbf{S})$. While we have an Iwasawa decomposition (Section 1.5.1)

$$\mathbf{G}(k) = K_x \mathbf{S}(k) \mathbf{U}^\omega(k),$$

we also have a Levi decomposition (Section 2.6)

$$\mathbf{P}_\omega(k) = \big([\mathbf{Z_G}(\mathbf{S}), \mathbf{Z_G}(\mathbf{S})](k) \cdot \mathbf{S}(k)\big) \ltimes \mathbf{U}^\omega(k).$$

We have used here the decomposition of the centralizer $\mathbf{Z_G}(\mathbf{S})$ into its part (derived subgroup) $[\mathbf{Z_G}(\mathbf{S}), \mathbf{Z_G}(\mathbf{S})]$ whose k-rational points fix the apartment $\mathcal{A}(\mathbf{S})$ pointwise and the part $\mathbf{S}$ whose k-rational points provide the translations of the affine Weyl group.

Putting together these decompositions and then varying $\mathbf{S}$ and $x \in \mathcal{A}(\mathbf{S})$, we see that the stabilizer K_x of any special vertex x acts upon Ω continuously and transitively. It then follows from general integration theory on homogeneous spaces [11, VII, §2, 6, Théorème 3] that there is a unique K_x-invariant probability measure on Ω, which we denote by μ_x; we call μ_x the *homogeneous measure* associated with the special vertex x. As a result, if we also use the construction from Proposition 5.1 we can associate to each special vertex x the harmonic measure ν_x and the homogeneous measure μ_x. Here is the Bruhat–Tits version of Theorem C in the Introduction.

Theorem 6.5. *Let* $\mathbf{G}$ *be a simply connected semisimple algebraic group defined over a non-Archimedean local field* k *and let* $\mathcal{X}(\mathbf{G}, k)$ *be the associated Bruhat–Tits building. Denote by* Ω *the maximal boundary of* $\mathcal{X}(\mathbf{G}, k)$ *and by* $\mathcal{P}(\Omega)$ *the set of probability measures on it, endowed with the weak-$*$ topology. For any special vertex* $x \in \mathcal{X}(\mathbf{G}, k)$*, the harmonic measure* ν_x *and the homogeneous measure* μ_x *coincide, therefore the Furstenberg compactification is also the closure of the set of homogeneous measures in* $\mathcal{P}(\Omega)$.

Remark. The proof below is valid in the more general case when $\mathbf{G}(k)$ is replaced by a type-preserving and strongly transitive automorphism group G acting on a locally finite affine building $\mathscr{X}$.

Proof of Theorem 6.5. By the uniqueness of homogeneous measures checked above, it is enough to show that for any $x \in V_s$ the probability measure ν_x is K_x-invariant. Let us pick $x \in V_s$ and $k \in K_x$; we need to show that $k_*\nu_x = \nu_x$, which we do by checking harmonicity of $k_*\nu_x$ (using the uniqueness given by Proposition 5.1). Let $y, z \in V_s$ be such that $\sigma(x, y) = \sigma(x, z)$. By the geometric interpretation of the Cartan decomposition (Section 1.5.1), the group K_x acts transitively on the sectors tipped at x, and since it preserves types (because $\mathbf{G}$ is assumed to be simply connected) we have $\sigma(x, k.y) = \sigma(x, k.z)$. By harmonicity of ν_x, this implies that $\nu_x(\Omega(x, k.y)) = \nu_x(\Omega(x, k.z))$. But $\Omega(x, k.y) = \Omega(k.x, k.y) = k.\Omega(x, y)$, and similarly $\Omega(x, k.z) = \Omega(k.x, k.z) = k.\Omega(x, z)$, so the previous equality says that for any $y, z \in V_s$ such that $\sigma(x, y) = \sigma(x, z)$, we have $({k^{-1}}_*\nu_x)(\Omega(x, y)) = ({k^{-1}}_*\nu_x)(\Omega(x, z))$; this is the required harmonicity, proving the first statement. ∎

One complementary issue is to try to attach a natural measure on Ω to an arbitrary point of the building. The problem is the lack of transitivity of the action on Ω for an arbitrary facet stabilizer. More precisely, when we described geometrically the Cartan decomposition in Section 1.5.1, we saw that the first step (the one using foldings given by root group actions, see Figure 4) showed that if c is an alcove in a given apartment $\mathbb{A}$, then the Iwahori subgroup $\mathrm{Stab}_{\mathbf{G}(k)}(c)$ acts on $\mathscr{X}(\mathbf{G}, k)$ with $\mathbb{A}$ as a fundamental domain; this is the geometric counterpart of the Bruhat–Tits decomposition $\mathbf{G}(k) = \bigsqcup_{w\in W^a} \mathrm{Stab}_{\mathbf{G}(k)}(c) w\, \mathrm{Stab}_{\mathbf{G}(k)}(c)$ [13, Proposition 4.2.1]. By approximating geodesic rays by geodesic segments and passing to the limit, the outcome is that $\mathrm{Stab}_{\mathbf{G}(k)}(c)$ acts on Ω with a fundamental set of representatives given by the chambers at infinity lying in $\mathbb{A}^\infty$. Keeping the group $\mathrm{Stab}_{\mathbf{G}(k)}(c)$, which does not contain any lift of elements of the vectorial part of W^a, we cannot do better because the second step is not available. This shows that the set of $\mathrm{Stab}_{\mathbf{G}(k)}(c)$-orbits on Ω is indexed by the spherical Weyl group of $\mathbf{G}$ over k (the combinatorial counterpart here is [13, Théorème 5.1.3 (vi)]). In particular, we cannot see Ω as a homogeneous space for the Iwahori subgroup $\mathrm{Stab}_{\mathbf{G}(k)}(c)$ and deduce that it carries a unique invariant probability measure: there is a simplex of possible choices according to the mass given to each orbit. Note that the problem persists for non-special vertices, whose stabilizers in W^a do not act transitively on $\mathbb{A}^\infty$. This explains, at least in the Bruhat–Tits case, why we only considered the set of special vertices when defining Furstenberg compactifications. We intend to go back to this problem in future work.

7. Martin compactification

Our aim is to construct Martin compactifications for the set of special vertices of affine buildings. This relies on the asymptotics of the Green function recently obtained by the second author [57]. It covers a large class of random walks on good vertices. Consequently, we can achieve our program for all affine buildings with reduced root systems. For the non-reduced case, by changing the origin o, we can compactify only half of the special vertices at the same time. To complete the program in the non-reduced case, we introduce a certain *distinguished* random walk described in Appendix A.

Let us start by recalling the basics of random walks on discrete structures, and then we immediately specialize to affine buildings by describing the asymptotics of ground spherical functions. This enables us to provide the uniqueness statements for limits of Martin kernels, which lead to the identifications announced in Theorem A of the Introduction. The last two parts contain the convergence theorems which define and describe the Martin compactifications of affine buildings, at and above the bottom of the spectrum.

7.1. Martin embeddings

We say that a random walk is *isotropic* if the transition probabilities $p(x, y)$ only depend on $\sigma(x, y)$, i.e., they are constant on

$$\{(x', y') \in V_g \times V_g : \sigma(x', y') = \sigma(x, y)\}.$$

We set

$$p(0; x, y) = \delta_x(y),$$
$$p(n; x, y) = \sum_{z \in V_p} p(n-1; x, z) p(z, y), \quad n \geqslant 1.$$

A random walk is *irreducible* if for any $x, y \in V_g$ there is $n \in \mathbb{N}$ such that $p(n; x, y) > 0$. It is *aperiodic* if for every $x \in V_g$,

$$\gcd\{n \in \mathbb{N} : p(n; x, x) > 0\} = 1.$$

Lastly, a walk has *finite range* if for every $x \in V_g$, we have

$$\#\{y \in V_g : p(x, y) > 0\} < \infty.$$

For each isotropic finite range random walk on good vertices of $\mathscr{X}$ with transition density p, we define the corresponding operator acting on $f : V_g \to \mathbb{C}$ by

$$Af(x) = \sum_{y \in V_g} p(x, y) f(y). \tag{7.1}$$

Let $\mathscr{A}_0 = \mathbb{C}\text{-span}\{A_\lambda : \lambda \in P^+\}$ be the commutative $*$-subalgebra of the algebra of bounded linear operators on $\ell^2(V_g)$ where the A_λ are defined in (5.4). The multiplicative functionals on $\mathscr{A}_0$ are given in terms of Macdonald spherical functions. The latter are defined for $\lambda \in P^+$ by

$$P_\lambda(z) = \frac{\chi^{-1/2}(\lambda)}{W(q^{-1})} \sum_{w \in W} e^{\langle w.z, \lambda \rangle} \mathbf{c}(w.z), \quad z \in \mathbb{C}^r, \tag{7.2}$$

where

$$\begin{aligned} \mathbf{c}(z) &= \prod_{\alpha \in \Phi^+} \frac{1 - \tau_\alpha^{-1} \tau_{\alpha/2}^{-1/2} e^{-\langle z, \alpha^\vee \rangle}}{1 - \tau_{\alpha/2}^{-1/2} e^{-\langle z, \alpha^\vee \rangle}} \\ &= \prod_{\alpha \in \Phi^{++}} \frac{(1 - \tau_{2\alpha}^{-1} \tau_\alpha^{-1/2} e^{-\frac{1}{2}\langle z, \alpha^\vee \rangle})(1 + \tau_\alpha^{-1/2} e^{-\frac{1}{2}\langle z, \alpha^\vee \rangle})}{1 - e^{-\langle z, \alpha^\vee \rangle}}. \end{aligned} \tag{7.3}$$

The values of P_λ when the denominator of the $\mathbf{c}$-function equals zero can be obtained by taking appropriate limits. The mapping

$$h_z(A_\lambda) = P_\lambda(z), \quad z \in \mathbb{C}^r, \lambda \in P^+,$$

extends to a multiplicative functional, still denoted by h_z, on $\mathscr{A}_0$. Moreover, all multiplicative functions on $\mathscr{A}_0$ are of this form. For more details about spherical harmonic analysis on $\mathscr{X}$ we refer the interested reader to [33, 45].

In fact, formula (7.2) defines Macdonald spherical functions for a given root system Φ and parameters $(\tau_\alpha : \alpha \in \Phi)$ invariant under the action of the Weyl group W. In particular, the definition is valid without any underlying building: see e.g. [38].

We henceforth fix an isotropic finite range random walk on good vertices of $\mathscr{X}$, with transition function p. Since the walk has finite range, there are a finite set $\mathcal{V} \subset P$ and positive real numbers $\{c_v : v \in \mathcal{V}\}$ such that

$$\kappa(z) := \varrho^{-1} h_z(A) = \sum_{v \in \mathcal{V}} c_v e^{\langle z, v \rangle}, \quad z \in \mathbb{C}^r, \tag{7.4}$$

where

$$\varrho = h_0(A) \tag{7.5}$$

with A given by (7.1). Let us observe that ϱ is the spectral radius of A. Indeed, by the Gelfand theorem, $\varrho = \sup_{z \in M_2} |h_z(A)|$ where M_2 denotes the spectrum of the commutative C^*-algebra $\mathscr{A}_2$, and since A is a finite convex combination of A_λ, we get

$$h_0(A) \leqslant \sup_{z \in M_2} |h_z(A)| \leqslant \sum_\lambda a_\lambda \sup_{z \in M_2} |h_z(A_\lambda)| = \sum_\lambda a_\lambda h_0(A_\lambda) = h_0(A) \tag{7.6}$$

where the penultimate equality follows by [45, Theorem 6.5].

For each $\zeta \geqslant \varrho$, the *Green function* G_ζ is defined by

$$G_\zeta(x, y) = \sum_{n \geqslant 0} \zeta^{-n} p(n; x, y), \quad x, y \in V_g.$$

Without loss of generality we assume that the random walk is aperiodic. Indeed, otherwise we consider

$$\tilde{p}(x, y) = \tfrac{1}{2} \delta_x(y) + \tfrac{1}{2} p(x, y).$$

Then

$$G_\zeta(x, y) = \frac{\zeta}{\zeta + 1} \tilde{G}_{\frac{\zeta+1}{2}}(x, y).$$

Next, let us observe that for each $y \in V_g$, the function $V_g \ni x \mapsto G_\zeta(x, y)$ is ζ-*harmonic*, that is,

$$\sum_{v \in V_g} p(x, v) G_\zeta(v, y) = \zeta G_\zeta(x, y).$$

Recall that a function $f : V_g \to \mathbb{R}$ is called ζ-*superharmonic* if $Af \leqslant \zeta f$. Let us denote by $\mathscr{B}_\zeta(V_g)$ the set of positive ζ-superharmonic functions on good vertices of $\mathscr{X}$, normalized to take value 1 at the origin o. The set $\mathscr{B}_\zeta(V_g)$ endowed with the topology of pointwise convergence is a compact second countable Hausdorff space, and thus it is metrizable. For $g \in \operatorname{Aut}(\mathscr{X})$ and $f \in \mathscr{B}_\zeta(V_g)$ we set

$$(g.f)(x) = \frac{f(g^{-1}.x)}{f(g^{-1}.o)} \tag{7.7}$$

provided that $f(g^{-1}.o) \neq 0$. Let us define the map

$$\iota : V_g \to \mathscr{B}_\zeta(V_g), \quad y \mapsto K_\zeta(\cdot, y),$$

where for $x, y \in V_g$ we set

$$K_\zeta(x, y) = \frac{G_\zeta(x, y)}{G_\zeta(o, y)}.$$

Since the random walk is transient, the map ι is injective. Moreover, for $g \in \operatorname{Aut}(\mathscr{X})$, we have

$$g.\iota(y) = g.K_\zeta(\cdot, y) = \frac{K_\zeta(g^{-1}\cdot, y)}{K_\zeta(g^{-1}.o, y)} = \frac{K_\zeta(\cdot, g.y)}{K_\zeta(o, g.y)} = K_\zeta(\cdot, g.y) = \iota(gy)$$

and thus ι is equivariant. Notice that $\iota(V_g)$ is also discrete, because for $y, y' \in V_g$ with $y \neq y'$, we have

$$\begin{aligned} K_\zeta(y, y) - K_\zeta(y, y') &= \frac{G_\zeta(y, y)G_\zeta(o, y') - G_\zeta(o, y)G_\zeta(y, y')}{G_\zeta(o, y)G_\zeta(o, y')} \\ &= \frac{(G_\zeta(o, o) - 1)G_\zeta(o, y') + G_\zeta(o, y') - G_\zeta(o, y)G_\zeta(y, y')}{G_\zeta(o, y)G_\zeta(o, y')} \\ &\geqslant \frac{G_\zeta(o, o) - 1}{G_\zeta(o, y)}. \end{aligned}$$

Let $\overline{\mathscr{X}}_{M,\zeta}$ be the closure of $\iota(V_g)$ in $\mathscr{B}_\zeta(V_g)$. The space $\mathscr{B}_\zeta(V_g)$ is metrizable, thus by Lemma 1.2, while studying $\overline{\mathscr{X}}_{M,\zeta}$ we restrict attention to core sequences. By [24, Proposition 6.4], the group $\operatorname{Aut}(\mathscr{X})$ acts continuously on $\overline{\mathscr{X}}_{M,\zeta}$.

7.2. *Asymptotic behavior of ground state spherical functions*

Before embarking on computing the Martin kernels, we need the following auxiliary result. Given a subset $J \subseteq I_0$, let us define

$$\Xi_J(\mu) = \lim_{\theta \to 0} \frac{1}{|W_J|} \sum_{w \in W_J} e^{\langle w.\theta, \mu \rangle} \mathbf{c}_J(w.\theta), \qquad \mu \in P,$$

where $\mathbf{c}_J$ denotes the $\mathbf{c}$-function for the root system Φ_J, that is,

$$\mathbf{c}_J(z) = \prod_{\alpha \in \Phi_J^+} \frac{1 - \tau_\alpha^{-1} \tau_{\alpha/2}^{-1/2} e^{-\langle z, \alpha^\vee \rangle}}{1 - \tau_{\alpha/2}^{-1/2} e^{-\langle z, \alpha^\vee \rangle}}.$$

Let $\mathbf{b}_J(z) = e^{\langle z, \rho_J \rangle} \mathbf{c}_J(z) \Delta_J(z)$ where $\rho_J = \frac{1}{2} \sum_{\alpha \in \Phi_J^{++}} \alpha^\vee$ and

$$\Delta_J(z) = \prod_{\alpha \in \Phi_J^{++}} (e^{\frac{1}{2}\langle z, \alpha^\vee \rangle} - e^{-\frac{1}{2}\langle z, \alpha^\vee \rangle}).$$

If $J = I_0$ we drop the index J from the notation. By (7.2) we have

$$\Xi(\mu) = \frac{W(q^{-1})}{|W|} \chi^{1/2}(\mu) P_\mu(0). \tag{7.8}$$

We start with the following lemma.

Lemma 7.1. *For each* $\lambda \in P$,

$$\Xi(\lambda) = \frac{1}{a}\Big\{\prod_{\alpha \in \Phi^{++}} \langle \nabla, \alpha^\vee \rangle\Big\}\{e^{\langle \theta, \lambda+\rho\rangle}\mathbf{b}(\theta)\}_{\theta=0}$$

where

$$a = \Big\{\prod_{\alpha \in \Phi^{++}} \langle \nabla, \alpha^\vee \rangle\Big\}\{\Delta(\theta)\}_{\theta=0}.$$

Proof. First, let us write

$$\sum_{w \in W} e^{\langle w.\theta, \lambda \rangle}\mathbf{c}(w.\theta) = \sum_{w \in W} e^{\langle w.\theta, \lambda+\rho\rangle}\mathbf{b}(w.\theta)\frac{1}{\Delta(w.\theta)}.$$

Since $\Delta(\theta)$ is a W-anti-invariant exponential polynomial, we have

$$\sum_{w \in W} e^{\langle w.\theta, \lambda \rangle}\mathbf{c}(w.\theta) = \frac{1}{\Delta(\theta)} \sum_{w \in W} (-1)^{\ell(w)} e^{\langle w.\theta, \lambda+\rho\rangle}\mathbf{b}(w.\theta).$$

Now, by multiple applications of L'Hôpital's rule we get (see e.g. [2])

$$\begin{aligned}
&\lim_{\theta \to 0} \frac{a}{|W|} \sum_{w \in W} e^{\langle w.\theta, \lambda \rangle}\mathbf{c}(w.\theta) \\
&\qquad = \Big\{\prod_{\alpha \in \Phi^{++}} \langle \nabla, \alpha^\vee \rangle\Big\}\Big\{\frac{1}{|W|} \sum_{w \in W} (-1)^{\ell(w)} e^{\langle w.\theta, \lambda+\rho\rangle \mathbf{b}(w.\theta)}\Big\}_{\theta=0} \\
&\qquad = \Big\{\prod_{\alpha \in \Phi^{++}} \langle \nabla, \alpha^\vee \rangle\Big\}\{e^{\langle \theta, \lambda+\rho\rangle}\mathbf{b}(\theta)\}_{\theta=0}
\end{aligned}$$

where the last equality follows by W-anti-invariance of the differential operator. This completes the proof of the lemma. ■

We are now ready to prove the asymptotic formula for the auxiliary functions Ξ_J, which will be used in the next section to describe the asymptotics of ground spherical functions. The following proposition is motivated by [3, Remark 2.2.13].

Proposition 7.2. *Let* $J \subsetneq I_0$. *Suppose that* $(\gamma_n : n \in \mathbb{N})$ *is a sequence of dominant coweights such that*

$$\sup_{n \in \mathbb{N}} \langle \gamma_n, \alpha \rangle < \infty \qquad \text{for all } \alpha \in \Phi_J^+, \tag{7.9}$$

$$\lim_{n \to \infty} \langle \gamma_n, \alpha \rangle = +\infty \quad \text{for all } \alpha \in \Phi^+ \setminus \Phi_J^+. \tag{7.10}$$

Then there is a positive constant a_J *such that*

$$\Xi(\gamma_n) = \Big(\prod_{\alpha \in \Phi^{++} \setminus \Phi_J} \langle \gamma_n, \alpha^\vee \rangle\Big)\Xi_J(\gamma_n)(a_J + o(1)). \tag{7.11}$$

Proof. The proof is in two steps.

Step 1. Let us first prove (7.11) assuming that $\lim_{n\to\infty}|\gamma_n| = +\infty$ and

$$\liminf_{n\to\infty} \left\langle \frac{\gamma_n}{|\gamma_n|}, \alpha_j \right\rangle = 0 \quad \text{if and only if} \quad j \in J, \tag{7.12}$$

instead of (7.9) and (7.10). Let P_J and Q_J be the projections defined by (2.1). We set

$$\tilde{a}_J = \Big\{ \prod_{\alpha\in\Phi_J^{++}} \langle \nabla, \alpha^\vee \rangle \Big\} \{\Delta_J(\theta)\}_{\theta=0}. \tag{7.13}$$

Since for each $\lambda \in P$,

$$\langle \theta, \lambda + \rho \rangle = \langle \theta, P_J(\lambda + \rho) \rangle + \langle \theta, Q_J(\lambda + \rho) \rangle,$$

Lemma 7.1 shows that $\Xi(\lambda)$ is equal to the sum over $A \subseteq \Phi^{++}$ of terms

$$\frac{1}{\tilde{a}_{I_0}} \prod_{\alpha\in A} \langle Q_J(\lambda+\rho), \alpha^\vee \rangle \Big\{ \prod_{\alpha\in\Phi^{++}\setminus A} \langle \nabla, \alpha^\vee \rangle \Big\} \{ e^{\langle \theta, P_J(\lambda+\rho)\rangle} \mathbf{b}(\theta) \}_{\theta=0}. \tag{7.14}$$

Observe that (7.14) equals zero if $A \cap \Phi_J^{++} \neq \varnothing$. Moreover, if $A \subseteq \Phi^{++} \setminus \Phi_J^{++}$, then the term (7.14) is $\mathcal{O}(|\lambda|^{|A|})$. Hence, taking $\lambda = \gamma_n$, we get

$$\begin{aligned} &\lim_{\theta\to 0} \Xi(\gamma_n) \\ &\quad = \frac{1}{\tilde{a}_{I_0}} \Big(\prod_{\alpha\in\Phi^{++}\setminus\Phi_J} \langle \gamma_n, \alpha^\vee \rangle \Big) \Big\{ \prod_{\alpha\in\Phi_J^{++}} \langle \nabla, \alpha^\vee \rangle \Big\} \{ e^{\langle \theta, P_J(\gamma_n+\rho)\rangle} \mathbf{b}(\theta) \}_{\theta=0} (1 + o(1)). \end{aligned}$$

Since $\mathbf{b}/\mathbf{b}_J$ is W_J-invariant, for each $\lambda \in P$ we have

$$\begin{aligned} \frac{\mathbf{b}(0)}{\mathbf{b}_J(0)} \Xi_J(\lambda) &= \lim_{\theta\to 0} \frac{1}{|W_J|} \sum_{w\in W_J} e^{\langle w.\theta, P_J(\lambda)\rangle} \mathbf{c}_J(w.\theta) \frac{\mathbf{b}(w.\theta)}{\mathbf{b}_J(w.\theta)} \\ &= \lim_{\theta\to 0} \frac{1}{|W_J|} \sum_{w\in W_J} e^{\langle w.\theta, P_J(\lambda+\rho)\rangle} \mathbf{b}(w.\theta) \frac{1}{\Delta_J(w.\theta)} \\ &= \lim_{\theta\to 0} \frac{1}{\Delta_J(\theta)} \frac{1}{|W_J|} \sum_{w\in W_J} (-1)^{\ell(w)} e^{\langle w.\theta, P_J(\lambda+\rho)\rangle} \mathbf{b}(w.\theta) \\ &= \frac{1}{\tilde{a}_J} \Big\{ \prod_{\alpha\in\Phi_J^{++}} \langle \nabla, \alpha^\vee \rangle \Big\} \{ e^{\langle \theta, P_J(\lambda+\rho)\rangle} \mathbf{b}(\theta) \}_{\theta=0} \end{aligned}$$

where the last equality follows by multiple applications of L'Hôpital's rule. Hence,

$$\lim_{\theta\to 0} \Xi(\gamma_n) = \frac{\tilde{a}_J}{\tilde{a}_{I_0}} \cdot \frac{\mathbf{b}(0)}{\mathbf{b}_J(0)} \Big(\prod_{\alpha\in\Phi^{++}\setminus\Phi_J} \langle \gamma_n, \alpha^\vee \rangle \Big) \Xi_J(\gamma_n)(1 + o(1)),$$

which completes the proof of the first step.

Step 2. Let $(\gamma_n : n \in \mathbb{N})$ satisfy (7.9) and (7.10). The proof is by induction on the rank. If $r = 2$, the conclusion easily follows by Step 1. Suppose that Proposition 7.2 holds true for all root systems of rank smaller than r. We define

$$J_1 = \left\{ j \in I_0 : \liminf_{n\to\infty} \left\langle \frac{\gamma_n}{|\gamma_n|}, \alpha_j \right\rangle = 0 \right\}.$$

Notice that $J \subseteq J_1$. In view of Step 1, we have

$$\Xi(\gamma_n) = \frac{\tilde{a}_{J_1}}{\tilde{a}_{I_0}} \cdot \frac{\mathbf{b}(0)}{\mathbf{b}_{J_1}(0)} \Big(\prod_{\alpha\in\Phi^{++}\setminus\Phi_{J_1}} \langle \gamma_n, \alpha^\vee \rangle \Big) \Xi_{J_1}(\gamma_n)(1 + o(1)). \tag{7.15}$$

This completes the proof in the case $J = J_1$. If $J \subsetneq J_1$, we consider J_1 in place of I_0. By the inductive hypothesis we have

$$\Xi_{J_1}(\gamma_n) = \frac{\tilde{a}_{J_1}}{\tilde{a}_{J_1}} \cdot \frac{\mathbf{b}(0)}{\mathbf{b}_J(0)} \Big(\prod_{\alpha\in\Phi_{J_1}^{++}\setminus\Phi_J} \langle \gamma_n, \alpha^\vee \rangle \Big) \Xi_J(\gamma_n) \frac{\tilde{a}_J \mathbf{b}_{J_1}(0)}{\tilde{a}_{J_1} \mathbf{b}_J(0)} (1 + o(1)),$$

which together with (7.15) finishes the proof. ■

7.3. *Uniqueness of limit functions*

We here introduce partial ground state spherical functions attached to subsets of simple roots. These functions will appear naturally when constructing the Martin boundary. This fits in with the fact that façades at infinity (as defined in Section 2.3) can be used to describe Martin compactifications. The partial ground spherical functions are combined with partial horospherical functions; the presence of the latter factors is explained by the fact that the corresponding affine buildings of smaller rank lie at infinity. The difference between the analytic behaviors of the factors enables us to provide a precise parametrization of the functions in the Martin boundaries.

Let us define the J-*ground state* spherical function by

$$\mathbf{\Phi}_J(\lambda) = \frac{|W_J|}{W_J(q^{-1})} \chi_J^{-1/2}(\lambda) \Xi_J(\lambda), \quad \lambda \in P^+.$$

Theorem 7.3. *Let $J, J' \subsetneq I_0$, $\omega, \omega' \in \Omega$ and $y \in [o, \omega]$, $y' \in [o, \omega']$. If for all $x \in V_g$, we have*

$$\frac{\mathbf{\Phi}_J(\sigma(x, y))}{\mathbf{\Phi}_J(\sigma(o, y))} \chi^{1/2}(Q_J h(o, x; \omega)) = \frac{\mathbf{\Phi}_{J'}(\sigma(x, y'))}{\mathbf{\Phi}_{J'}(\sigma(o, y'))} \chi^{1/2}(Q_{J'} h(o, x; \omega')) \tag{7.16}$$

then $J' = J$, $\omega' \sim_J \omega$ and $\pi_F(y') = \pi_F(y)$ where $\pi_F : \mathscr{X} \to \mathscr{X}(F)$ is the projection to the façade at infinity $\mathscr{X}(F)$, and F is the spherical facet at infinity corresponding to the residue $\mathrm{res}_J(\omega)$.

Note that the ratio of values of partial spherical ground functions above corresponds to the fact that, in the group case, there is a projective action on the function space used to define Martin compactifications (see [24, p. 101, paragraph before Proposition 6.4]). The corresponding uniqueness statement in the case of symmetric spaces is [24, Theorem 7.22]; this result is more analytic and group-theoretic in nature since it establishes the uniqueness of eigenfunctions of Laplace operators satisfying some invariance under the action of a well-chosen unipotent subgroup.

Proof of Theorem 7.3. In view of the cocycle relation, for all $x, x' \in V_s$, we have

$$\begin{aligned}&\frac{\mathbf{\Phi}_J(\sigma(x',y))}{\mathbf{\Phi}_J(\sigma(o,y))}\chi^{1/2}(Q_J h(o,x';\omega))\\ &\qquad = \frac{\mathbf{\Phi}_J(\sigma(x',y))}{\mathbf{\Phi}_J(\sigma(x,y))}\chi^{1/2}(Q_J h(x,x';\omega))\frac{\mathbf{\Phi}_J(\sigma(x,y))}{\mathbf{\Phi}_J(\sigma(o,y))}\chi^{1/2}(Q_J h(o,x;\omega)).\end{aligned}$$

Consequently, (7.16) implies that for all $x, x' \in V_s$,

$$\frac{\mathbf{\Phi}_J(\sigma(x',y))}{\mathbf{\Phi}_J(\sigma(x,y))}\chi^{1/2}(Q_J h(x,x';\omega)) = \frac{\mathbf{\Phi}_{J'}(\sigma(x',y'))}{\mathbf{\Phi}_{J'}(\sigma(x,y'))}\chi^{1/2}(Q_{J'} h(x,x';\omega')). \tag{7.17}$$

Let $\mathscr{A}$ be an apartment containing ω and ω' in its boundary. Let $\tilde{\omega}$ be opposite ω and such that $[\omega, \tilde{\omega}] = \mathscr{A}$. We select $x \in V_s(\mathscr{A}) \cap [y', \tilde{\omega}] \cap [y, \tilde{\omega}]$. For $\lambda \in P^+$ and $n \in \mathbb{N}$, we denote by $x_{\lambda,n}$ the unique point in $[x, \tilde{\omega}]$ such that $\sigma(x_{\lambda;n}, x) = n\lambda$. Then

$$\begin{aligned}\sigma(x_{\lambda;n}, y) &= \sigma(x_{\lambda;n}, x) + \sigma(x, y) = n\lambda + \sigma(x, y),\\ \sigma(x_{\lambda;n}, y') &= n\lambda + \sigma(x, y').\end{aligned}$$

Next, there is $w \in W$ such that

$$h(x, x_{\lambda;n}; \omega) = -n\lambda \quad \text{and} \quad h(x, x_{\lambda;n}; \omega') = -nw.\lambda.$$

Therefore taking $x = x_{\lambda;n}$ in (7.17), we obtain

$$\frac{\mathbf{\Phi}_J(\sigma(x,y)+n\lambda)}{\mathbf{\Phi}_J(\sigma(x,y))}\chi^{1/2}(nQ_J\lambda) = \frac{\mathbf{\Phi}_{J'}(\sigma(x,y')+n\lambda)}{\mathbf{\Phi}_{J'}(\sigma(x,y'))}\chi^{1/2}(nQ_{J'}w.\lambda). \tag{7.18}$$

Suppose that, contrary to our claim, there is $j \in J \setminus J'$. Select $\lambda = \lambda_j$. Then the left-hand side of (7.18) takes the form

$$\frac{\Xi_J(\sigma(x,y)+n\lambda_j)}{\Xi_J(\sigma(x,y))}\chi^{1/2}(-n\lambda_j),$$

which has a factor that is a non-trivial polynomial in n, while the right-hand side equals

$$\chi^{1/2}(-nQ_{J'}w.\lambda_j).$$

Therefore, (7.18) cannot be satisfied for n sufficiently large. Hence $J = J'$ and (7.18) takes the form

$$\frac{\Xi_J(\sigma(x,y)+n\lambda)}{\Xi_J(\sigma(x,y))} \cdot \frac{\Xi_J(\sigma(x,y'))}{\Xi_J(\sigma(x,y')+n\lambda)} = \chi^{1/2}(nQ_J(\lambda - w^{-1}.\lambda)). \tag{7.19}$$

However, (7.19) cannot be satisfied unless both sides are constant sequences for n sufficiently large. Checking the right-hand side of (7.19) we conclude that it has to be constantly 1 for $n \geqslant 1$. Since for each $\lambda \in P^+$,

$$\chi(Q_J\lambda) = e^{\langle \eta, Q_J\lambda\rangle},$$

we conclude that

$$\langle \eta, Q_J\lambda\rangle = \langle \eta, Q_J w^{-1}.\lambda\rangle. \tag{7.20}$$

First, let us show that (7.20) entails that $w \in W_J$. Let $k = \ell(w)$ and suppose that $w = w_k = w_{k-1}r_{\beta_k}$ is such that $\ell(w_k) > \ell(w_{k-1})$. We have

$$\begin{aligned}\langle \eta, Q_J w_k^{-1}.\lambda\rangle &= \langle Q_J\eta, w_k^{-1}.\lambda\rangle = \langle Q_J\eta, r_{\beta_k} w_{k-1}^{-1}.\lambda\rangle = \langle r_{\beta_k} Q_J\eta, w_{k-1}^{-1}.\lambda\rangle \\ &= \langle Q_J\eta, w_{k-1}^{-1}.\lambda\rangle - \langle Q_J\eta, \beta_k^\vee\rangle\langle \beta_k, w_{k-1}^{-1}.\lambda\rangle.\end{aligned}$$

By induction on k, we get

$$\langle \eta, Q_J w_k^{-1}.\lambda\rangle = \langle \eta, Q_J\lambda\rangle - \sum_{i=1}^{k} \langle Q_J\eta, \beta_i^\vee\rangle\langle \beta_i, w_{i-1}^{-1}.\lambda\rangle. \tag{7.21}$$

Therefore,

$$\sum_{i=1}^{k} \langle Q_J\eta, \beta_i^\vee\rangle\langle \beta_i, w_{i-1}^{-1}.\lambda\rangle = 0. \tag{7.22}$$

For each $i \in \{1,\ldots,k\}$, $w_i = w_{i-1}r_{\beta_i}$ and $\ell(w_i) > \ell(w_{i-1})$, hence by [28, Section 5.7, Proposition] we have

$$w_{i-1}.\beta_i \in \Phi^+. \tag{7.23}$$

Moreover, in view of (1.1), $Q_J\eta \in S_0$, and $\langle Q_J\eta, \alpha_j\rangle = 0$ for all $j \in J$. Hence, (7.22) implies that $\beta_i \in \Phi_J^+$ for all $i \in \{1,\ldots,k\}$. Consequently, $w \in W_J$ and so $\omega' \sim_J \omega$.

To complete the proof, we need to show that $\pi_F(y) = \pi_F(y')$. Since ω and ω' are not opposite each other, there is an apartment $\mathscr{A}$ containing both ω and ω' on its boundary such that $y \in V_s(\mathscr{A})$. Select $\omega'' \in \Omega$ such that $[y, \omega''] \subset \mathscr{A}$ and $y \in [y', \omega'']$. For $\lambda \in P^+$ and $n \in \mathbb{N}$, we take $x_{\lambda;n} \in [y, \omega'']$ such that

$$\sigma(x_{\lambda;n}, y) = n\lambda.$$

Then

$$\sigma(x_{\lambda;n}, y') = n\lambda + \sigma(y, y').$$

By taking $x = y$ and $x' = x_{\lambda;n}$ in (7.19), we obtain

$$\Xi_J(n\lambda) = \frac{\Xi_J(\sigma(y,y')+n\lambda)}{\Xi_J(\sigma(y,y'))} \tag{7.24}$$

for all $n \in \mathbb{N}$ and $\lambda \in P^+$. Now, by Lemma 7.1, we get

$$\begin{aligned}\tilde{a}_J \Xi_J(\sigma(y,y')+n\lambda) = \mathbf{b}_J(0) \prod_{\alpha\in\Phi_J^{++}} \langle \sigma(y,y')+\rho_J+n\lambda, \alpha^\vee\rangle \\ + \sum_{\beta\in\Phi_J^{++}} \langle \nabla, \beta^\vee\rangle \mathbf{b}_J(0) \cdot \prod_{\substack{\alpha\in\Phi_J^{++}\\ \alpha\neq\beta}} \langle \sigma(y,y')+\rho_J+n\lambda, \alpha^\vee\rangle \\ + \text{lower powers of } n\end{aligned}$$

where $\tilde{a}_J$ is given by (7.13). Hence,

$$\begin{aligned}\tilde{a}_J \Xi_J(\sigma(y,y')+n\lambda) = n^{|\Phi_J^{++}|}\mathbf{b}_J(0) \prod_{\alpha\in\Phi_J^{++}} \langle \lambda, \alpha^\vee\rangle \\ + n^{|\Phi_J^{++}|-1}\mathbf{b}_J(0) \sum_{\beta\in\Phi_J^{++}} \langle \sigma(y,y')+\rho_J, \beta^\vee\rangle \prod_{\substack{\alpha\in\Phi_J^{++}\\ \alpha\neq\beta}} \langle \lambda, \alpha^\vee\rangle \\ + n^{|\Phi_J^{++}|-1} \sum_{\beta\in\Phi_J^{++}} \langle \nabla, \beta^\vee\rangle \mathbf{b}_J(0) \cdot \prod_{\substack{\alpha\in\Phi_J^{++}\\ \alpha\neq\beta}} \langle \lambda, \alpha^\vee\rangle \\ + \text{ lower powers of } n.\end{aligned}$$

For $\lambda = \rho$, by comparing the leading terms in (7.24), we immediately get

$$\Xi_J(\sigma(y,y')) = 1.$$

Now, the equality of the following terms implies that for all $\lambda \in P^+$, we have

$$\sum_{\beta\in\Phi_J^{++}} \langle \sigma(y,y'), \beta^\vee\rangle \prod_{\substack{\alpha\in\Phi_J^{++}\\ \alpha\neq\beta}} \langle \lambda, \alpha^\vee\rangle = 0. \tag{7.25}$$

We notice that for each $j \in J$, if $\alpha \in \Phi_J^{++}$ satisfies $\langle \alpha, \rho-\lambda_j\rangle = 0$, then $\alpha = \alpha_j$. Thus, taking $\lambda = \rho - \lambda_j$ in (7.25), we obtain

$$\langle \sigma(y,y'), \alpha_j^\vee\rangle \prod_{\substack{\alpha\in\Phi_J^{++}\\ \alpha\neq\alpha_j}} \langle \rho-\lambda_j, \alpha^\vee\rangle = 0.$$

Hence, for each $\alpha \in \Phi_J^+$,

$$\langle \sigma_J(\pi_F(y), \pi_F(y')), \alpha\rangle \le \langle \sigma(y,y'), \alpha\rangle = 0,$$

which implies that $\pi_F(y) = \pi_F(y')$ and the theorem follows. ∎

Here is a variant of the previous uniqueness result which takes into account an additional radial parameter; it will be used to describe the Martin boundary above the bottom of the spectrum. Recall that the sector face $[o, F_J]$ is the subset of $\mathfrak{a}$ consisting of the vectors such that $\langle x, \alpha\rangle \geqslant 0$ for all $\alpha \in \Phi^+$ and $\langle x, \alpha\rangle = 0$ for all $\alpha \in \Phi_J$.

Theorem 7.4. *Let $J, J' \subsetneq I_0$, $\omega, \omega' \in \Omega$, $s, s' \in S_0$, and $y \in [o, \omega]$, $y' \in [o, \omega']$. If for all $x \in V_g$, we have*

$$\frac{\mathbf{\Phi}_J(\sigma(x, y))}{\mathbf{\Phi}_J(\sigma(o, y))} \chi^{1/2}(Q_J h(o, x; \omega)) e^{\langle s, Q_J h(o,x;\omega)\rangle} = \frac{\mathbf{\Phi}_{J'}(\sigma(x, y'))}{\mathbf{\Phi}_{J'}(\sigma(o, y'))} \chi^{1/2}(Q_J h(o, x; \omega')) e^{\langle s', Q_J h(o,x;\omega')\rangle}$$

then $J' = J$, $\omega' \sim_J \omega$, $Q_J s' = Q_J s$ and $\pi_F(y') = \pi_F(y)$ where $\pi_F : \mathscr{X} \to \mathscr{X}(F)$ is the projection to the façade at infinity $\mathscr{X}(F)$, and F is the spherical facet at infinity corresponding to the residue $\operatorname{res}_J(\omega)$.

Proof. By the same line of reasoning as in the proof of Theorem 7.3, we can show that $J = J'$. Moreover, there is $w \in W$ such that for all $\lambda \in P^+$,

$$\chi^{1/2}(Q_J \lambda) e^{\langle s, Q_J \lambda\rangle} = \chi^{1/2}(Q_J w^{-1}.\lambda) e^{\langle s', Q_J w^{-1}.\lambda\rangle}. \tag{7.26}$$

Therefore, there is $w \in W$ such that

$$Q_J(s + \eta) = w.Q_J(s' + \eta).$$

We are going to conclude that $Q_J s = Q_J s'$ and $w \in W_J$. Using the notation from the proof of Theorem 7.3, we write

$$\begin{aligned} w_i.Q_J(s' + \eta) &= w_{i-1} r_{\beta_i} Q_J(s' + \eta) \\ &= w_{i-1} Q_J(s' + \eta) - \langle Q_J(s' + \eta), \beta_i^\vee\rangle w_{i-1}.\beta_i. \end{aligned}$$

Hence,

$$w.Q_J(s' + \eta) = Q_J(s' + \eta) - \sum_{i=1}^k \langle Q_J(s' + \eta), \beta_i^\vee\rangle w_{i-1}.\beta_i.$$

Therefore, by (7.26) we obtain

$$Q_J(s - s') = -\sum_{i=1}^k \langle Q_J(s' + \eta), \beta_i^\vee\rangle w_{i-1}\beta_i.$$

Since $s' + \eta \in S_0$, in view of (7.23) we conclude that $Q_J(s - s')$ is a non-negative combination of positive roots. Since we can swap J, ω, s and y with J', ω', s' and y', respectively, we deduce that $Q_J(s' - s)$ is also a non-negative combination of positive roots. Consequently, $Q_J s = Q_J s'$, and

$$\sum_{i=1}^k \langle Q_J(s' + \eta), \beta_i^\vee\rangle w_{i-1}\beta_i = 0.$$

Since $Q_J\eta \in S_0$ and $\langle Q_J\eta, \alpha_j\rangle = 0$ for all $j \in J$, we conclude that $\beta_i \in \Phi_J^+$ for all $i \in \{1, \dots, k\}$. Consequently, $w \in W_J$ and $\omega' \sim_J \omega$.

Now, by the same reasoning as in the proof Theorem 7.3 we show that $\pi_F(y) = \pi_F(y')$, which completes the proof. ■

7.4. Martin compactification for $\zeta = \varrho$

In this section we describe the Martin compactification at the bottom of the spectrum corresponding to the isotropic finite range random walk on good vertices of the building $\mathscr{X}$ chosen in Section 7.1. We set $\zeta = \varrho$ where ϱ is defined in (7.5).

As for Furstenberg compactifications, it is convenient to use the notions and terminology introduced in Section 2, including façades indexed by spherical facets at infinity. We again use the notation of Section 2.3.

Theorem 7.5. *Suppose that (y_n) is an (ω, J, c)-core sequence. Denote by F the spherical facet at infinity corresponding to the residue $\operatorname{res}_J(\omega)$ and by $\pi_F : \mathscr{X} \to \mathscr{X}(F)$ the projection to the façade at infinity $\mathscr{X}(F)$. Then for all $x \in V_g$, we have*

$$\lim_{n\to\infty} K_\varrho(x, y_n) = \frac{\mathbf{\Phi}_J(\sigma_{\mathscr{X}(F)}(x_F, y_F))}{\mathbf{\Phi}_J(\sigma_{\mathscr{X}(F)}(o_F, y_F))}\chi^{1/2}(Q_J h(o, x; \omega)) \tag{7.27}$$

where $x_F = \pi_F(x)$, $y_F = \lim_{n\to\infty} \pi_F(y_n)$ (limit of a constant sequence). If (y'_n) is an (ω', J', c')-core sequence such that $(K_\varrho(\cdot, y'_n))$ converges to the same limit, then $J' = J$, $\omega' \sim_J \omega$, and $c' = c$.

Proof. Let $\gamma_n = \sigma(o, y_n)$ and $\eta_n = \sigma(x, y_n)$. By [57, Theorem 6], we have

$$G_\varrho(o, y_n) = P_{\gamma_n}(0)|\gamma_n|^{-r-2|\Phi^{++}|+2}(D_0 + o(1)),$$
$$G_\varrho(x, y_n) = P_{\eta_n}(0)|\eta_n|^{-r-2|\Phi^{++}|+2}(D_0 + o(1)),$$

where D_0 is a certain positive constant. Hence, by (7.8),

$$K_\varrho(x, y_n) = \frac{\mathcal{P}(\eta_n)}{\mathcal{P}(\gamma_n)}\left(\frac{|\gamma_n|}{|\eta_n|}\right)^{r+2|\Phi^{++}|-2}\chi^{-1/2}(\eta_n - \gamma_n)(1 + o(1)).$$

By Lemma 3.1, for all $\alpha \in \Phi^+$, we have

$$|\langle\gamma_n, \alpha\rangle - \langle\eta_n, \alpha\rangle| \leqslant |\gamma_n - \eta_n|\,|\alpha| \leqslant |\sigma(o, x)|\,|\alpha|.$$

Consequently, for all $\alpha \in \Phi^+ \setminus \Phi_J$,

$$\frac{\langle\gamma_n, \alpha\rangle}{\langle\eta_n, \alpha\rangle} = 1 + o(1). \tag{7.28}$$

Moreover, by the triangle inequality and Lemma 3.1,

$$\big||\gamma_n| - |\eta_n|\big| \leqslant |\gamma_n - \eta_n| \leqslant |\sigma(o, x)|,$$

and thus

$$|\gamma_n| = |\eta_n|(1 + o(1)). \tag{7.29}$$

In view of Lemma 2.4, by taking a subsequence, we can assume that for each $\alpha \in \Phi_J$,

$$\langle \sigma_{\mathscr{X}(F)}(x_F, y_F), \alpha \rangle = \lim_{n\to\infty} \langle \eta_n, \alpha \rangle \quad \text{and} \quad \langle \sigma_{\mathscr{X}(F)}(o_F, y_F), \alpha \rangle = \lim_{n\to\infty} \langle \gamma_n, \alpha \rangle.$$

Therefore, by Proposition 7.2,

$$\begin{aligned}\frac{\mathcal{P}(\eta_n)}{\mathcal{P}(\gamma_n)} &= \frac{\mathcal{P}_J(\eta_n)}{\mathcal{P}_J(\gamma_n)} \Bigg(\prod_{\alpha \in \Phi^{++} \setminus \Phi_J} \frac{\langle \eta_n, \alpha^\vee \rangle}{\langle \gamma_n, \alpha^\vee \rangle} \Bigg) (1 + o(1)) \\ &= \frac{\mathcal{P}_J(\sigma_{\mathscr{X}(F)}(x_F, y_F))}{\mathcal{P}_J(\sigma_{\mathscr{X}(F)}(o_F, y_F))} (1 + o(1)).\end{aligned}$$

Finally, we invoke Lemmas 2.5 and 2.3 (ii) to get

$$\begin{aligned}\chi^{-1/2}(\eta_n - \gamma_n) &= \frac{\chi^{-1/2}(P_J \eta_n)}{\chi^{-1/2}(P_J \gamma_n)} \chi^{-1/2}(Q_J(\eta_n - \gamma_n)) \\ &= \frac{\chi^{-1/2}(\sigma_{\mathscr{X}(F)}(x_F, y_F))}{\chi^{-1/2}(\sigma_{\mathscr{X}(F)}(o_F, y_F))} \chi^{1/2}(Q_J(h(o, x; \omega))), (1 + o(1))\end{aligned}$$

which completes the proof of (7.27). The uniqueness part of the theorem follows from Theorem 7.3. ∎

At this stage, we can prove Theorem A (i) in the case $\zeta = \varrho$. Recall that when the finite root system associated with the building $\mathscr{X}$ is reduced, all special vertices are good.

Theorem 7.6. *Let $\mathscr{X}$ be a thick regular locally finite affine building. Then for any isotropic irreducible finite range random walk on $\mathscr{X}$, the Martin compactification $\overline{\mathscr{X}}_{M,\varrho}$ is* Aut($\mathscr{X}$)*-equivariantly isomorphic to the Furstenberg* (*measure-theoretic*) *and the Caprace–Lécureux* (*combinatorial*) *compactification of the set V_g of good vertices. Moreover, if there exists a locally compact closed subgroup of* Aut($\mathscr{X}$)*, say G, whose action on $\mathscr{X}$ is strongly transitive and type-preserving, then the Guivarc'h* (*group-theoretic*) *compactification $\overline{\mathscr{X}}_G$ of V_g is G-isomorphic to the Martin compactification $\overline{\mathscr{X}}_{M,\varrho}$.*

Proof. The proof mainly consists in putting together previous results. In all the compactifications considered, core sequences converge: for the Martin compactification this follows from Theorem 7.5, for the Furstenberg compactification it follows from Theorem 6.1, and for the Caprace–Lécureux compactification from Theorem 4.1. As a consequence, combining a standard topological argument (see e.g. the domination criterion given by [24, Lemma 3.28] in both directions between two compactifications) and the uniqueness assertions in the previous theorems provides the identifications between the first three compactifications. Moreover, since the full automorphism group Aut($\mathscr{X}$) acts continuously on each compactification and permutes the set of core sequences, the identifications are equivariant, whatever the size of Aut($\mathscr{X}$). Finally, for the identification with the Guivarc'h compactification when the action of Aut($\mathscr{X}$) on $\mathscr{X}$ is strongly transitive and type-preserving, it remains to apply [15, Theorem II]. ∎

Let us remark that the automorphism group of an irreducible affine building always acts strongly transitively on $\mathscr{X}$ if the rank of the affine building is at least 4 [54, p. 274].

7.5. Martin compactification for $\zeta > \varrho$

In this section we describe the Martin compactification above the bottom of the spectrum corresponding to the isotropic finite range random walk on good vertices of the building $\mathscr{X}$ chosen in Section 7.1. Recall the definitions of ϱ in (7.5) and of κ in (7.4). It is here that we have to use angular core sequences of good vertices defined at the end of Section 1.4. In order to describe the Martin kernel, let us define

$$\mathscr{C} = \{x \in \mathfrak{a} : \kappa(x) = \zeta\varrho^{-1}\}.$$

Notice that $\mathscr{C}$ is the boundary of a convex body such that for each $x \in \mathscr{C}$, the gradient $\nabla\kappa(x)$ is well-defined. Hence, for each $\theta \in \mathbb{S}^{r-1}$, unit sphere in $\mathfrak{a}$ centered at the origin, there is a unique point $s_\theta \in \mathscr{C}$ such that

$$\nabla\kappa(s_\theta) = |\nabla\kappa(s_\theta)|\theta. \tag{7.30}$$

Moreover, if $\theta \in \mathbb{S}^{r-1}_+$ then $s_\theta \in S_0$. In the next theorem we describe the Martin kernel for $\zeta > \varrho$.

Theorem 7.7. *Let (y_n) be an angular core (ω, J, c, θ)-sequence. Denote by F the spherical facet at infinity corresponding to the residue $\operatorname{res}_J(\omega)$ and by $\pi_F : \mathscr{X} \to \mathscr{X}(F)$ the projection to the façade at infinity $\mathscr{X}(F)$. Then for all $x \in V_g$, we have*

$$\lim_{n\to\infty} K_\zeta(x, y_n) = \frac{\mathbf{\Phi}_J(\sigma_{\mathscr{X}(F)}(x_F, y_F))}{\mathbf{\Phi}_J(\sigma_{\mathscr{X}(F)}(o_F, y_F))}\chi^{1/2}(Q_J h(o, x; \omega))e^{\langle s_\theta, Q_J h(o,x;\omega)\rangle} \tag{7.31}$$

where $x_F = \pi_F(x)$, $y_F = \lim_{n\to\infty} \pi_F(y_n)$ (limit of a constant sequence). If $(y'_n : n \in \mathbb{N})$ is an angular core $(\omega', J', c', \theta')$-sequence such that $(K_\zeta(\,\cdot\,, y'_n))$ converges to the same limit then $J' = J$, $\omega' \sim_J \omega$, $c' = c$ and $Q_J\theta' = Q_J\theta$.

Proof. First, let us introduce some notation. For $s \in \mathfrak{a}$, we define a quadratic form on $\mathfrak{a}$ by

$$B_s(y, y) = \frac{1}{2}\sum_{v,v'\in\mathcal{V}} \frac{c_v e^{\langle s,v\rangle}}{\kappa(s)} \cdot \frac{c_{v'} e^{\langle s,v'\rangle}}{\kappa(s)} \langle y, v - v'\rangle^2, \quad y \in \mathfrak{a},$$

where κ is given by (7.4). Let

$$J_1 = \{j \in I_0 : \langle u, \alpha_j\rangle = 0\}.$$

In particular, $J \subseteq J_1 \subsetneq I_0$. For $\theta \in \mathbb{S}^{r-1}$, we set

$$\mathcal{R}(\theta) = \sqrt{2\pi}|\nabla\log\kappa(s_\theta)|^{\frac{r-3}{2}+|\Phi^{++}_{J_1}|}(B_{s_\theta}(\theta,\theta))^{-1/2}\mathcal{Q}(s_\theta)$$

where for $s \in \mathfrak{a}$,

$$\mathcal{Q}(s) = \left(\frac{1}{2\pi}\right)^r \int_{\mathfrak{a}} e^{-\frac{1}{2}B_s(z,z)} |\pi_{J_1}(z)|^2 \, \mathrm{d}z$$

$$\cdot \frac{1}{|\mathbf{b}_{J_1}(0)|^2} \cdot \prod_{\alpha \in \Phi^+ \setminus \Phi_{J_1}} \frac{1 - \tau_{\alpha/2}^{-1/2} e^{-\langle s, \alpha^\vee \rangle}}{1 - \tau_\alpha^{-1} \tau_{\alpha/2}^{-1/2} e^{-\langle s, \alpha^\vee \rangle}},$$

$$\pi_{J_1}(s) = \prod_{\alpha \in \Phi_{J_1}^{++}} \langle s, \alpha^\vee \rangle.$$

Observe that $\mathcal{R}(\theta) \neq 0$. Indeed, since

$$\prod_{\alpha \in \Phi^+ \setminus \Phi_{J_1}} \frac{1 - \tau_{\alpha/2}^{-1/2} e^{-\langle s, \alpha^\vee \rangle}}{1 - \tau_\alpha^{-1} \tau_{\alpha/2}^{-1/2} e^{-\langle s, \alpha^\vee \rangle}}$$

$$= \prod_{\alpha \in \Phi^{++} \setminus \Phi_{J_1}} \frac{1 - e^{-\langle s, \alpha^\vee \rangle}}{(1 - \tau_{2\alpha}^{-1} \tau_\alpha^{-1/2} e^{-\langle s, \alpha^\vee \rangle/2})(1 + \tau_\alpha^{-1/2} e^{-\langle s, \alpha^\vee \rangle/2})},$$

the equality $\mathcal{R}(\theta) = 0$ implies that there is $\alpha \in \Phi^{++} \setminus \Phi_{J_1}$ such that $\langle s_\theta, \alpha^\vee \rangle = 0$. Hence, by (7.30), we would have $\langle \theta, \alpha^\vee \rangle = 0$, which is impossible.

Let $\gamma_n = \sigma(o, y_n)$ and $\eta_n = \sigma(x, y_n)$. By Lemma 2.4, by taking a subsequence, we can assume that for each $\alpha \in \Phi_J^+$,

$$\lim_{n\to\infty} \langle \gamma_n, \alpha \rangle = \langle \sigma_{\mathcal{X}(F)}(o_F, y_F), \alpha \rangle \text{ and } \lim_{n\to\infty} \langle \eta_n, \alpha \rangle = \langle \sigma_{\mathcal{X}(F)}(x_F, y_F), \alpha \rangle. \tag{7.32}$$

We set

$$a_n = \frac{\gamma_n}{|\gamma_n|} \quad \text{and} \quad b_n = \frac{\eta_n}{|\eta_n|}.$$

By (7.28) and (7.29), we have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = \theta.$$

Since $\mathcal{R}$ is a continuous function on $\mathbb{S}^{r-1}$ and $\mathcal{R}(\theta) \neq 0$, we obtain

$$\mathcal{R}(a_n) = \mathcal{R}(\theta)(1 + o(1)) \quad \text{and} \quad \mathcal{R}(b_n) = \mathcal{R}(\theta)(1 + o(1)).$$

Now, by [57, Theorem 5], we get

$$G_\zeta(o, y_n) = |\gamma_n|^{-\frac{r-1}{2} - |\Phi_{J_1}^{++}|} \chi^{-1/2}(\gamma_n) \mathcal{P}_{J_1}(\gamma_n) \mathcal{R}(a_n) e^{-\langle s_{a_n}, \gamma_n \rangle} (1 + o(1)),$$

$$G_\zeta(x, y_n) = |\eta_n|^{-\frac{r-1}{2} - |\Phi_{J_1}^{++}|} \chi^{-1/2}(\eta_n) \mathcal{P}_{J_1}(\eta_n) \mathcal{R}(b_n) e^{-\langle s_{b_n}, \eta_n \rangle} (1 + o(1)).$$

Hence,

$$K_\zeta(x, y_n) = \left(\frac{|\gamma_n|}{|\eta_n|}\right)^{\frac{r-1}{2} + |\Phi_{J_1}^{++}|} \frac{\mathcal{P}_{J_1}(\eta_n)}{\mathcal{P}_{J_1}(\gamma_n)} \chi^{-1/2}(\eta_n - \gamma_n) e^{-\langle s_{b_n}, \eta_n \rangle + \langle s_{a_n}, \gamma_n \rangle} (1 + o(1)).$$

By Proposition 7.2, we have

$$\frac{\mathcal{P}_{J_1}(\eta_n)}{\mathcal{P}_{J_1}(\gamma_n)} = \frac{\mathcal{P}_J(\eta_n)}{\mathcal{P}_J(\gamma_n)}\Bigg(\prod_{\alpha\in\Phi_{J_1}^{++}\setminus\Phi_J} \frac{\langle\eta_n,\alpha\rangle}{\langle\gamma_n,\alpha\rangle}\Bigg)(1+o(1)) = \frac{\mathcal{P}_J(\eta_n)}{\mathcal{P}_J(\gamma_n)}(1+o(1)).$$

Moreover, by Lemma 2.5 and (7.32),

$$\chi^{-1/2}(\eta_n-\gamma_n) = \frac{\chi^{-1/2}(\sigma_{\mathscr{X}(F)}(x_F,y_F))}{\chi^{-1/2}(\sigma_{\mathscr{X}(F)}(o_F,y_F))}\chi^{1/2}(Q_J h(o,x;\omega))(1+o(1))$$

where $Q_J = \mathrm{id} - P_J$, and P_J is given by (2.1). Therefore,

$$K_\xi(x,y_n) = \frac{\mathbf{\Phi}_J(\sigma_{\mathscr{X}(F)}(x_F,y_F))}{\mathbf{\Phi}_J(\sigma_{\mathscr{X}(F)}(o_F,y_F))}\chi^{1/2}(Q_J h(o,x;\omega))e^{-\langle s_{b_n},\eta_n\rangle+\langle s_{a_n},\gamma_n\rangle}(1+o(1)).$$

Next, we show the following claim.

Claim 7.8.

$$\langle s_{a_n},\gamma_n\rangle - \langle s_{b_n},\eta_n\rangle = \langle s_u, Q_J(\gamma_n-\eta_n)\rangle + o(1). \tag{7.33}$$

To see this, we write

$$\begin{aligned}\langle s_{a_n},\gamma_n\rangle - \langle s_{b_n},\eta_n\rangle &= |\gamma_n|\langle s_{a_n},a_n\rangle - |\eta_n|\langle s_{b_n},b_n\rangle \\ &= (|\gamma_n|-|\eta_n|)\langle s_{a_n},a_n\rangle + |\eta_n|(\langle s_{a_n},a_n\rangle - \langle s_{b_n},b_n\rangle).\end{aligned} \tag{7.34}$$

We start by considering the second term in (7.34). Let us denote by $\mathcal{M}$ the interior of the convex hull of $\mathcal{V}$. For $\xi\in\mathbb{S}^{r-1}$, we set $t_0 = \min\{t>0 : t^{-1}\xi\in\mathcal{M}\}$ and define a function on (t_0,∞) by

$$\psi_\xi(t) = t\big(\log(\zeta^{-1}\rho) - \phi(t^{-1}\xi)\big) \tag{7.35}$$

where

$$\phi(\delta) = \min\{\langle x,\delta\rangle - \log\kappa(x) : x\in\mathfrak{a}\}, \quad \delta\in\mathcal{M}.$$

For the properties of ϕ, see [57, Section 2.1]. The function ψ_ξ attains its unique maximum at $t_\xi > t_0$. In particular, $\psi_\xi'(t_\xi) = 0$ and $\psi_\xi(t_\xi) = -\langle s_\xi,\xi\rangle$. Thus the gradient of the function $\mathbb{S}^{r-1}\ni\xi\mapsto\psi_\xi(t_\xi)$ equals $-\nabla\phi(t_\xi^{-1}\xi) = -s_\xi$. Hence, by the Taylor formula we obtain

$$\begin{aligned}\langle s_{a_n},a_n\rangle - \langle s_{b_n},b_n\rangle &= \psi_{b_n}(b_{v_n}) - \psi_{a_n}(t_{a_n}) \\ &= -\langle s_{a_n},b_n-a_n\rangle + \mathcal{O}(|b_n-a_n|^2).\end{aligned} \tag{7.36}$$

Now, we compute

$$|\eta_n|(a_n-b_n) = \gamma_n - \eta_n + a_n(|\eta_n|-|\gamma_n|).$$

Since by Lemma 3.1, $|\eta_n-\gamma_n|$ is bounded and (a_n) approaches u, we obtain

$$\begin{aligned}|\eta_n|-|\gamma_n| &= \frac{|\eta_n|^2-|\gamma_n|^2}{|\eta_n|+|\gamma_n|} \\ &= \frac{2\langle\eta_n-\gamma_n,\gamma_n\rangle + |\eta_n-\gamma_n|^2}{|\eta_n|+|\gamma_n|} = \langle\eta_n-\gamma_n,u\rangle + o(1).\end{aligned}$$

In particular, $|\eta_n|\,|a_n - b_n|$ is bounded. Hence,

$$\begin{aligned} |\eta_n|\langle s_{a_n}, a_n - b_n\rangle &= \langle s_{a_n}, \gamma_n - \eta_n\rangle + \langle s_{a_n}, a_n\rangle(|\eta_n| - |\gamma_n|) \\ &= \langle s_u, \gamma_n - \eta_n\rangle + \langle s_{a_n}, a_n\rangle(|\eta_n| - |\gamma_n|) + o(1), \end{aligned}$$

which together with (7.36) implies that

$$|\eta_n|(\langle s_{a_n}, a_n\rangle - \langle s_{b_n}, b_n\rangle) = \langle s_u, \gamma_n - \eta_n\rangle + \langle s_{a_n}, a_n\rangle(|\eta_n| - |\gamma_n|) + o(1).$$

Therefore, by (7.34), we obtain

$$\langle s_{a_n}, \gamma_n\rangle - \langle s_{b_n}, \eta_n\rangle = \langle s_\theta, \gamma_n - \eta_n\rangle + o(1),$$

proving (7.33).

Now, Claim 7.8 together with Lemma 2.5 implies that

$$\langle s_{a_n}, \gamma_n\rangle - \langle s_{b_n}, \eta_n\rangle = \langle s_\theta, \gamma_n - \eta_n\rangle + o(1) = \langle s_\theta, h(o, x; \omega)\rangle + o(1)$$

where we have also used $Q_J(s_\theta) = s_\theta$. This establishes the limit (7.31). The uniqueness part of the theorem follows from Theorem 7.4. ∎

At this stage, we can prove Theorem A (i) in the case $\zeta > \varrho$. Recall that when the finite root system associated with the building $\mathscr{X}$ is reduced, all special vertices are good. Recall also that thanks to Theorem 7.6, the Martin compactification $\overline{\mathscr{X}}_{M,\varrho}$ below is equivariantly isomorphic to the combinatorial and to the measure-theoretic compactification.

Theorem 7.9. *Let $\mathscr{X}$ be a thick regular locally finite affine building. Then for any isotropic irreducible finite range random walk on $\mathscr{X}$ and for any $\zeta > \varrho$, the Martin compactification $\overline{\mathscr{X}}_{M,\zeta}$ is* Aut$(\mathscr{X})$*-isomorphic to the compactification $\overline{\mathscr{X}}_{M,\varrho} \vee \overline{\mathscr{X}}_V$, where $\overline{\mathscr{X}}_{M,\varrho}$ is the Martin compactification at the bottom of the spectrum and $\overline{\mathscr{X}}_V$ is the Gromov compactification.*

Proof. Recall that the join $\overline{\mathscr{X}}_{M,\varrho} \vee \overline{\mathscr{X}}_V$ is the compactification obtained by taking the closure of the image of the diagonal embedding of V_g in the product $\overline{\mathscr{X}}_{M,\varrho} \times \overline{\mathscr{X}}_V$ [24, Section 3.45]. As a consequence, the image of an unbounded sequence converges in the join if and only if it converges in each of the two factors of the topological product. In view of Theorems 3.2, 7.5 and 7.7, we conclude first that angular core (ω, J, c, u)-sequences converge both in the Martin compactification $\overline{\mathscr{X}}_{M,\zeta}$ for $\zeta > \varrho$ and in the join $\overline{\mathscr{X}}_{M,\varrho} \vee \overline{\mathscr{X}}_V$, and then that the uniqueness statements in the theorems provide the identification (see e.g. the domination criterion given by [24, Lemma 3.28] in both directions between two compactifications). This identification is equivariant because Aut$(\mathscr{X})$ acts continuously on the compactifications and permutes angular core sequences. ∎

7.6. *Martin compactifications for Bruhat–Tits buildings*

We finally consider the Bruhat–Tits context. Namely, we present the non-Archimedean counterpart to the study done on Riemannian symmetric spaces by Y. Guivarc'h and collaborators [23, 24]. In those references, the Archimedean case of potential-theoretic compactifications is fully treated in the following sense:

(i) Martin compactifications of symmetric spaces are defined, both by means of differential operators (i.e., eigenfunctions of Laplace operators) and via random walks [24, Chapters VI–VIII and XIV];

(ii) for a given symmetric space, the Martin compactification at the bottom of the spectrum is shown to be equivariantly homeomorphic to the maximal Satake compactification (representation-theoretic), the maximal Furstenberg compactification (measure-theoretic) or the Guivarc'h compactification (group-theoretic) [23, Theorems 2.13 and 3.20] (see also [40]);

(iii) Martin compactifications above the bottom of the spectrum are shown to be equivariantly homeomorphic to the join of the Gromov compactification with any compactification discussed in (ii) [24, Theorems 8.2 and 8.21];

(iv) Martin compactifications at the bottom of the spectrum are used to parametrize geometrically two classes of remarkable subgroups, namely maximal distal and maximal amenable subgroups [23, Theorem 2.14] (see also [41]);

(v) an integral formula for eigenfunctions of the Laplace operator is given by means of suitable Poisson kernels [24, Theorems 13.1 and 13.28] and an analogous result is given from the viewpoint of random walks [24, Theorem 13.33].

We now consider the Bruhat–Tits analogues of these results. These problems were mentioned, together with some hints, in [24, Chapter XV] and [23, Section 4]. We wish to explain here where the intuitions there could be implemented and where we took another path.

Of course, the use of techniques from partial differential equations is not directly efficient when dealing with buildings instead of Riemannian symmetric spaces. The viewpoint of random walks together with non-Archimedean harmonic analysis as developed in [33] becomes the main tool. In the probabilistic part of their work, Guivarc'h–Ji–Taylor use the notion of a well-behaved measure on a symmetric space $X = G/K$, or more precisely on the connected semisimple Lie group G: a positive measure on G is called *well-behaved* if it has a continuous density (with respect to the Haar measure) and if its support S, assumed to be compact, satisfies $G = \bigcup_{n\geqslant 0} S^n$. If we are given a bi-$K$-invariant well-behaved probability measure p, then the convolution operator associated with p provides a generalization of the Laplace operator [22, Proposition 1]. The associated random walk has finite range whenever p has compact support, and is irreducible whenever the probability measure p is well-behaved; moreover, the trick in Section 7.1 modifies, if necessary, the random walk attached to p in such a way that it becomes aperiodic but still provides the same Martin boundary. To sum up, a compactly supported bi-K-invariant well-behaved probability measure on $\mathbf{G}(k)$ defines a random walk for which Theorems 7.5 and 7.7 provide explicit descriptions of Martin boundaries by means of core sequences (at the bottom and above the bottom of the spectrum, respectively). This settles (i) above and allows us to identify the Martin compactification at the bottom of the spectrum according to Theorem 6.5 above, which settles (ii). While (iv) was established in [25], we intend to go back to (v), namely integral representation of harmonic functions, in future work. Finally, for (iii), we have the following statement which contains the second half of Theorem B of the Introduction.

Theorem 7.10. *Let* $\mathbf{G}$ *be a semisimple simply connected algebraic group over* k*, a locally compact non-Archimedean valued field, and let* $\mathscr{X}$ *be its Bruhat–Tits building. Choose a good vertex in* $\mathscr{X}$ *and denote by* K *its stabilizer. Pick a compactly supported bi-*K*-invariant well-behaved probability measure on* $\mathbf{G}(k)$*. Then, for every* ζ *above the spectral radius* ρ *of the measure, the corresponding Martin compactification* $\overline{\mathscr{X}}_{M,\zeta}$ *is the join of* $\overline{\mathscr{X}}_{M,\varrho}$ *with the Gromov compactification.*

Proof. In view of Theorem 7.9, it is enough to see that the operator associated to a bi-K-invariant well-behaved probability measure on $\mathbf{G}(k)$ is an averaging operator as in Section 7.1. Let $\varphi(g)\mathrm{d}g$ be a compactly supported bi-K-invariant well-behaved probability measure on $\mathbf{G}(k)$. Then φ is a compactly supported bi-K-invariant continuous function on G. Let o be the good vertex in $\mathscr{X}$ whose stabilizer is the subgroup K and let V_o be the set of (good) vertices of the same type as o. Each $v \in V_o$ can be written as $g.o$ for some $g \in \mathbf{G}(k)$ which is well-defined up to right multiplication by elements in K. We wish to introduce an operator A acting on suitable functions $f : V_o \to \mathbb{C}$ by the formula (see [24, Remark 2, p. 171])

$$Af(v) = Af(g.o) = \int_G f(gh.o)\varphi(h)\,\mathrm{d}h.$$

By left invariance of the Haar measure, we can also write

$$Af(g.o) = \int_G f(h.o)\varphi(g^{-1}h)\,\mathrm{d}h,$$

which, since φ is left K-invariant, shows that the definition does not depend on the element $g \in \mathbf{G}(k)$ such that $v = g.o$.

We need to show that the operator A is as in Section 7.1. The function φ, being compactly supported and bi-K-invariant, is a (finite) linear combination of characteristic functions of double classes modulo K. We are thus reduced to the situation where $\varphi = \frac{1}{\mathrm{vol}(K\xi^{-1}(t)K)}\mathbb{1}_{K\xi^{-1}(t)K}$ for some $t \in Y^+$ in the notation of Section 1.5.1, and where vol denotes volume with respect to the Haar measure. We thus have

$$Af(g.o) = \frac{1}{\mathrm{vol}(K\xi^{-1}(t)K)} \int_{K\xi^{-1}(t)K} f(gh.o)\,\mathrm{d}h.$$

The group K acts transitively on the vertices at vectorial distance t from o, so when h runs over $K\xi^{-1}(t)K$, the vertices $(gh).o$ describe the combinatorial sphere $V_t(v)$ centered at $v = g.o$ and of vectorial radius t. Moreover, we have an identification of (finite) K-homogeneous spaces $K\xi^{-1}(t)K/K \simeq K/\mathrm{Stab}_K(t.o)$ on which the invariant measure is the counting measure. Altogether, this provides the integration formula

$$\int_{K\xi^{-1}(t)K} f(gh.o)\,\mathrm{d}h = \int_{\mathrm{Stab}_K(t.o)} \left(\int_{K/\mathrm{Stab}_K(t.o)} f(g\bar{k}.(t.o))\,\mathrm{d}\bar{k} \right) \mathrm{d}k$$
$$= \int_{\mathrm{Stab}_K(t.o)} \left(\sum_{v' \in V_t(v)} f(v') \right) \mathrm{d}k.$$

Taking f to be constantly 1 gives $\mathrm{vol}(K\xi^{-1}(t)K) = \mathrm{vol}(\mathrm{Stab}_K(t.o)) \times |V_t(v)|$, and thus going back to the previous expression for $Af(g.o)$ we obtain

$$Af(g.o) = \frac{1}{|V_t(v)|} \sum_{v' \in V_t(v)} f(v'),$$

which shows that A is indeed an averaging operator, as desired. In general, φ is a finite linear combination of characteristic functions of double classes modulo K, and therefore A is a finite linear combination of averaging operators as in Section 7.1. ■

In the proofs here, we have used more general results from Sections 7.4 and 7.5. Recall that we do not require group action in order to define and understand Martin compactifications of affine buildings.

We conclude by discussing in slightly more detail the differences from the hints and intuitions provided in [23, 24]. More precisely, not only the use of group actions is crucial in those references, but even in the Bruhat–Tits framework chosen here, some differences from the case of symmetric spaces should be mentioned.

The main difference is that for affine buildings, thanks to [57] which provides exact asymptotics of the Green functions on affine buildings, we could perform exact computations of limits of Martin kernels. One consequence is that, uniformly with respect to any chosen procedure of compactification, we can use the same parametrizing system for limits, whatever the target space of the embedding map, so that finally we can identify or describe compactifications by arguments using these parameters only (which are basically radial directions and distances to sector panels of a given Weyl sector). This is what we do at the end of each section dealing with a given type of compactification: Theorem 7.6 at the end of Section 7.4 (Martin compactification at the bottom of the spectrum), Theorem 7.9 at the end of Section 7.5 (Martin compactification above the bottom of the spectrum) and Theorem A.2 at the end of Appendix A (the case of non-reduced root systems). In the case of symmetric spaces, the available asymptotics due to Anker–Ji [3] (Green kernels) and Anker [2] (ground state spherical functions) are good enough to describe the Martin compactifications in [23, 24], but the computation of limits is not direct. This is related to the well-known fact that Harish-Chandra's integral formula for spherical functions remains an integral formula in the Archimedean case, while it can be made algebraic in the non-Archimedean case [33, Chapter IV]: this is, so to speak, the analytic approach to the theory of Macdonald spherical functions [34]. Note also that the idea to develop an abstract harmonic analysis dealing with (Iwahori–)Hecke algebras, and avoiding automorphism groups as much as possible, goes back to H. Matsumoto [38].

In order to make the comparison with the Archimedean case in more detail, let us separate two steps in the study of Martin compactifications: the computation of limits of Martin kernels, and the description of the boundaries. Already at the bottom of the spectrum, the computation of limits is not purely analytic [24, Proposition 7.26] since it is based on an argument of uniqueness of cluster value (by compactness), which itself uses a characterization of limit functions by means of conditions mixing harmonicity properties and knowledge of stabilizers [24, Theorem 7.22] (the convergence of measures for Furstenberg compactifications uses a similar uniqueness argument based on the knowledge of the support and of part of the stabilizer of the limit measure). Still at the bottom of the spectrum, the description of the Martin compactification [24, Theorem 7.33] uses the group action in a crucial way since the identification with other compactifications is based on an explicit comparison of stabilizers and of complete sets of representatives. Above the bottom of the spectrum, the computation of limits of Martin kernels [24, Theorem 8.2] is not direct since it uses the Anker–Ji and Anker asymptotics, which are given up to multiplicative constants. One then knows that a cluster value of Martin kernels attached to a core sequence is a multiple of the expected limit, but one still has to use representing measures and again knowledge of stabilizers in order to conclude; see [24, p. 124]. The description of Martin compactifications above the bottom of the spectrum is also based on a precise understanding of stabilizers and complete sets of representatives (note that for the last point an argument due to Karpelevich is systematically used: see [24, Proposition 7.20] and [29]).

In the non-Archimedean case, thanks to stronger asymptotics obtained in [57] we make in Section 7 the computation of limits of Martin kernels in a purely analytic way. The various factors in the resulting formulas (see Theorem 7.5 for the bottom spectrum and Theorem 7.7 otherwise) can be understood geometrically in terms of façades at infinity.

These factors do not have the same asymptotic behaviors: in both cases, the factor with polynomial growth mimics the initial situation in the sense that it corresponds to the ground state spherical function on the stratum at infinity involved (an affine building of smaller rank). The remaining factors have exponential growth and precisely this analytic difference is exploited to obtain the needed uniqueness results (see Section 7.3, in particular the way equation (7.19) is exploited). This allows us to directly use the geometric parameters of core sequences to describe Martin compactifications and compare them with the previous ones.

Appendix A. Distinguished random walk for $\widetilde{\mathrm{BC}}_r$

In Section 7 we study Martin compactifications of affine buildings for isotropic finite range random walks of good vertices. Unfortunately, in the non-reduced case we obtain two different boundaries corresponding to V_g and V_g^{ε}. In particular, there are no harmonic-analytic tools which allow studying functions on the whole set V_s at the same time. This limitation is a consequence of the lack of Green function's asymptotics for more general random walks. In this appendix we define a certain random walk on *all* special vertices in the non-reduced case, for which we compute the limits of Martin kernels. This allows us to obtain Martin compactifications of all special vertices of an affine building (see Theorem A.2).

Let us recall that for each $r \geqslant 1$ there is only one non-reduced finite root system, BC_r, that is,

$$\Phi = \{\pm e_i, \pm 2e_i, \pm e_j \pm e_k : 1 \leqslant i \leqslant r,\ 1 \leqslant j < k \leqslant r\}$$

where $\{e_1, \dots, e_r\}$ is the standard basis of $\mathfrak{a}$. The standard base of Φ consists of the roots

$$\alpha_j = \begin{cases} e_j - e_{j+1} & \text{if } 1 \leqslant j \leqslant r-1, \\ e_r & \text{if } j = r. \end{cases}$$

Thus the fundamental coweights are

$$\lambda_j = e_1 + \cdots + e_j.$$

Special vertices have type 0 or r, but only type 0 is good.

Let $\mathscr{X}$ be an affine building of non-reduced type. Given a chamber $c \in \mathcal{C}(\mathscr{X})$, let us denote by $v_j(c)$ the vertex of c having type j. For a vertex $v \in V(\mathscr{X})$, let $\mathcal{C}(v)$ be the set of all chambers sharing the vertex v. For $x \in V_s$, we set

$$\mathcal{V}_r(x) = \{v_r(c) : c \in \mathcal{C}(x)\} \quad \text{if } \tau(x) = 0,$$
$$\mathcal{V}_0(x) = \{v_0(c) : c \in \mathcal{C}(x)\} \quad \text{if } \tau(x) = r.$$

Observe that

$$N_r = \#\mathcal{V}_r(x) = \frac{W(q)}{W_{\lambda_r}(q)} \quad \text{and} \quad N_0 = \#\mathcal{V}_0(x) = \frac{W^a_{\lambda_r}(q)}{W_{\lambda_r}(q)}$$

where $W^a_\lambda = \{w \in W^a : w.\lambda = \lambda\}$. Now, for each $c \in \mathcal{C}(\mathscr{X})$ we set

$$p(v_0(c), v_r(c)) = \frac{1}{N_r} \quad \text{and} \quad p(v_r(c), v_0(c)) = \frac{1}{N_0}.$$

Then $P = (p(x, y) : x, y \in V_s)$ generates a reversible Markov chain on special vertices of $\mathscr{X}$. Since the random walk has period 2, it is natural to consider

$$\tilde{p}(x, y) = \sum_{z \in V_g^\varepsilon} p(x, z)p(z, y), \quad x, y \in V_g.$$

For each y belonging to

$$\bigcup_{c \in \mathcal{C}(x)} \{v_0(d) : d \in \mathcal{C}(v_r(c))\}$$

there is $z \in V_g^\varepsilon$ such that

$$\sigma(x, z) = \tfrac{1}{2}\lambda_r \quad \text{and} \quad \sigma(z, y) = \tfrac{1}{2}\lambda_r.$$

Moreover, there is $\omega \in \Omega$ such that $h(x, y; \omega) \in P^+$. Since

$$\sigma(x, y) = h(x, z; \omega) + h(z, y; \omega) \in P^+,$$

we conclude that $\sigma(x, y) = 0$ or λ_k for a certain $k \in I_0$. Hence,

$$\bigcup_{c \in \mathcal{C}(x)} \{v_0(d) : d \in \mathcal{C}(v_r(c))\} = \{x\} \sqcup \bigsqcup_{j \in I'} V_{\lambda_j}(x)$$

for a certain subset I' of I_0.

Now, let us fix $y \in V_{\lambda_j}(x)$. There are $W_{\lambda_j}(q)$ distinct chambers $c \in \mathcal{C}(x)$ such that there is $d \in \mathcal{C}(v_r(c))$ with $v_0(d) = y$, but $W_{\lambda_j \lambda_r}(q)$ of them share the vertex $v_r(c)$ where $W_{\lambda_j \lambda_r} = W_{\lambda_j} \cap W_{\lambda_r}$. Therefore

$$\sum_{y \in V_g} \tilde{p}(x, y) f(y) = \frac{1}{N_0 N_r} \left(N_r f(x) + \sum_{j \in I'} \frac{W_{\lambda_j}(q)}{W_{\lambda_j \lambda_r}(q)} \sum_{y \in V_{\lambda_j}(x)} f(y) \right).$$

Hence, $\tilde{P} = (\tilde{p}(x, y) : x, y \in V_g)$ generates a reversible Markov chain on good vertices of $\mathscr{X}$. The corresponding averaging operator belongs to the algebra $\mathscr{A}_0$. Moreover, if $x, y \in V_g$, then

$$G_\zeta(x, y) = \sum_{n=0}^{\infty} \zeta^n p(n; x, y) = \sum_{n=0}^{\infty} \zeta^{2n} \tilde{p}(n; x, y) = \tilde{G}_\zeta(x, y).$$

Hence, if $\tau(y) = 0$, then

$$K_\zeta(x, y) = \begin{cases} \tilde{K}_\zeta(x, y) & \text{if } \tau(x) = 0, \\ \frac{1}{N_0} \sum_{x' \in \mathcal{V}_0(x)} \tilde{K}_\zeta(x', y) & \text{if } \tau(x) = r, \end{cases}$$

where we have set

$$\tilde{K}_\zeta(x, y) = \frac{\tilde{G}_\zeta(x, y)}{\tilde{G}_\zeta(o, y)}, \quad x, y \in V_g.$$

Analogously, we can introduce a random walk whose transition function is given by

$$\tilde{p}^\varepsilon(x, y) = \sum_{z \in V_g} p(x, z) p(z, y), \quad x, y \in V_g^\varepsilon.$$

For a chosen $o^\varepsilon \in \mathcal{V}_r(o)$, we set

$$\tilde{K}_\zeta^\varepsilon(x, y) = \frac{\tilde{G}_\zeta^\varepsilon(x, y)}{\tilde{G}_\zeta^\varepsilon(o^\varepsilon, y)}, \quad x, y \in V_g^\varepsilon,$$

$$K_\zeta^\varepsilon(x, y) = \frac{G_\zeta(x, y)}{G_\zeta(o^\varepsilon, y)}, \quad x, y \in V_s.$$

Hence, for $x, y \in V_s$,

$$K_\zeta(x, y) = \frac{G_\zeta(x, y)}{G_\zeta(o, y)} = \frac{K_\zeta^\varepsilon(x, y)}{K_\zeta^\varepsilon(o, y)}. \tag{A.1}$$

Since $G_\zeta(x,y) = \tilde{G}^\varepsilon_\zeta(x,y)$ for $x, y \in V^\varepsilon_g$, if $\tau(y) = r$ we have

$$K^\varepsilon_\zeta(x,y) = \begin{cases} \tilde{K}^\varepsilon_\zeta(x,y) & \text{if } \tau(x) = r, \\ \frac{1}{N_r}\sum_{x'\in\mathcal{V}_r(x)} \tilde{K}^\varepsilon_\zeta(x',y) & \text{if } \tau(x) = 0. \end{cases} \tag{A.2}$$

Let us denote by $\mathcal{B}_\zeta(V_s)$ the set of positive ζ-superharmonic functions on special vertices of $\mathscr{X}$, normalized to take value 1 at o. The set $\mathcal{B}_\zeta(V_s)$ endowed with the topology of pointwise convergence is a compact second countable Hausdorff space. Let us define the map

$$\iota : V_s \to \mathcal{B}_\zeta(V_s), \quad y \mapsto K_\zeta(\cdot, y).$$

Since the random walk generated by P is transient, the map ι is an equivariant embedding with discrete image. Let $\overline{\mathscr{X}}_{M,\zeta}$ be the closure of $\iota(V_s)$ in $\mathcal{B}_\zeta(V_s)$. The space $\mathcal{B}_\zeta(V_s)$ is metrizable, and thus by Lemma 1.2, while studying $\overline{\mathscr{X}}_{M,\zeta}$ we can restrict attention to core and angular core sequences.

Now, to $\tilde{K}_\zeta$ and $\tilde{K}^\varepsilon_\zeta$ we can apply Theorems 7.5 and 7.7 proving the existence of the limits of $(K_\zeta(\cdot, y_n))$ for core and angular core sequences. It remains to describe sequences having the same limit (this is the analogue of the uniqueness statements in Theorems 7.5 and 7.7).

Suppose that there are two core sequences (y_n) and (y'_n) with parameters (ω, J, c) and (ω', J', c'), respectively, that have the same limit. We claim that $J' = J$, $\omega' \sim_J \omega$ and $c' = c$. If $\tau(y_n) = \tau(y'_n) = 0$ for all n, this is a direct consequence of Theorem 7.3. Suppose that $\tau(y_n) = \tau(y'_n) = r$ for all n. Then by (A.1), for all $x \in V^\varepsilon_g$,

$$\Big(\lim_{n\to\infty} K^\varepsilon_\zeta(o, y'_n)\Big) \lim_{n\to\infty} K^\varepsilon_\zeta(x, y_n) = \Big(\lim_{n\to\infty} K^\varepsilon_\zeta(o, y_n)\Big) \lim_{n\to\infty} K^\varepsilon_\zeta(x, y'_n),$$

and we can repeat the same reasoning as in Theorem 7.3 to obtain the desired conclusion.

Finally, consider $\tau(y_n) = 0$ and $\tau(y'_n) = r$. Then for all $x \in V_g$, by (A.2) we get

$$\Big(\lim_{n\to\infty} K^\varepsilon_\zeta(o, y'_n)\Big) \lim_{n\to\infty} \tilde{K}_\zeta(x, y_n) = \frac{1}{N_r} \sum_{x'\in\mathcal{V}_r(x)} \lim_{n\to\infty} \tilde{K}^\varepsilon_\zeta(x', y'_n).$$

Let $y = y_m$ and $y' = y'_m$ for m sufficiently large. Let $\mathscr{A}$ be an apartment containing $[o, \omega]$, and let $\tilde{\omega}$ opposite ω be such that $[\tilde{\omega}, \omega] = \mathscr{A}$. Let us consider $x \in V_g \cap [y', \tilde{\omega}] \cap [o, \tilde{\omega}]$. By repeating the line of reasoning used in the proof of Theorem 7.3, we show that $J' = J$, and

$$\frac{\mathcal{P}_J(\sigma(x,y) + n\lambda)}{\mathcal{P}_J(\sigma(x,y))} = \frac{\sum_{x'\in\mathcal{V}_r(x)} \mathcal{P}^\varepsilon_J(\sigma(x',y') + n\lambda)\chi^{1/2}(h(x,x';\omega'))}{\sum_{x'\in\mathcal{V}_r(x)} \mathcal{P}^\varepsilon_J(\sigma(x',y'))\chi^{1/2}(h(x,x';\omega'))}$$

for all $\lambda \in P^+$ and $n \in \mathbb{N}_0$. By comparing the coefficients of $n^{|\Phi^{++}_J|}$ we get

$$\frac{\mathbf{b}_J(0)}{\mathcal{P}_J(\sigma(x,y))} = \frac{\mathbf{b}^\varepsilon_J(0)\sum_{x'\in\mathcal{V}_r(x)} \chi^{1/2}(h(x,x';\omega'))}{\sum_{x'\in\mathcal{V}_r(x)} \mathcal{P}^\varepsilon_J(\sigma(x',y'))\chi^{1/2}(h(x,x';\omega'))}.$$

Since $r \in J$, by comparing the coefficients of $n^{|\Phi_J^{++}|-1}$ we arrive at

$$\langle \sigma(x, y), \alpha_r \rangle + \langle \nabla, \alpha_r \rangle \log \mathbf{b}_J(0) = \langle \sigma(x, y'), \alpha_r \rangle + \langle \nabla, \alpha_r \rangle \log \mathbf{b}_J^{\varepsilon}(0) - \frac{\sum_{x' \in \mathcal{V}_r(x)} \langle h(x, x'; \omega'), \alpha_r \rangle \chi^{1/2}(h(x, x'; \omega'))}{\sum_{x' \in \mathcal{V}_r(x)} \chi^{1/2}(h(x, x'; \omega'))}. \tag{A.3}$$

We claim that the following holds true.

Claim A.1. *For all $x \in V_g$ and $\omega \in \Omega$,*

$$\frac{\sum_{x' \in \mathcal{V}_r(x)} \langle h(x, x'; \omega), \alpha_r \rangle \chi^{1/2}(h(x, x'; \omega))}{\sum_{x' \in \mathcal{V}_r(x)} \chi^{1/2}(h(x, x'; \omega))} = \frac{1}{2} \cdot \frac{\sqrt{q_{\alpha_0}} - \sqrt{q_{\alpha_r}}}{\sqrt{q_{\alpha_0}} + \sqrt{q_{\alpha_r}}}. \tag{A.4}$$

For the proof, we recall that $\alpha_r = e_r$, and thus

$$\sum_{x' \in \mathcal{V}_r(x)} \langle h(x, x'; \omega), \alpha_r \rangle \chi^{1/2}(h(x, x'; \omega)) = \frac{1}{2} \sum_{\eta \in \{-1,1\}^r} \eta_r \chi^{1/4}\Big(\sum_{j=1}^r \eta_j e_j\Big) N_\eta$$

where for $\eta \in \{-1, 1\}^r$, we have set

$$N_\eta = \#\Big\{x' \in \mathcal{V}_r(x) : h(x, x'; \omega) = \frac{1}{2} \sum_{j=1}^r \eta_j e_j\Big\}.$$

Let us observe that for $\eta' \in \{-1, 1\}^{r-1}$,

$$N_{(\eta',-1)} = q_{\alpha_r} N_{(\eta',1)},$$

and by [36, Proposition 5.3],

$$\chi(e_r) = q_{\alpha_0} q_{\alpha_r}.$$

Hence,

$$\chi^{1/4}\Big(\sum_{j=1}^{r-1} \eta_j e_j + e_r\Big) N_{(\eta',1)} - \chi^{1/4}\Big(\sum_{j=1}^{r-1} \eta_j e_j - e_r\Big) N_{(\eta',-1)} = \bigg(1 - \sqrt{\frac{q_{\alpha_r}}{q_{\alpha_0}}}\bigg) \chi^{1/4}\Big(\sum_{j=1}^{r-1} \eta_j e_j + e_r\Big) N_{(\eta',1)},$$

which leads to

$$\sum_{x' \in \mathcal{V}_r(x)} \langle h(x, x'; \omega), \alpha_r \rangle \chi^{1/2}(h(x, x'; \omega)) = \frac{1}{2}\bigg(1 - \sqrt{\frac{q_{\alpha_r}}{q_{\alpha_0}}}\bigg) \sum_{\eta' \in \{-1,1\}^{r-1}} \chi^{1/4}\Big(\sum_{j=1}^{r-1} \eta'_j e_j + e_r\Big) N_{(\eta',1)}$$

and

$$\sum_{x'\in\mathcal{V}_r(x)} \chi^{1/2}(h(x,x';\omega)) = \Big(1+\sqrt{\frac{q_{\alpha_r}}{q_{\alpha_0}}}\Big) \sum_{\eta'\in\{-1,1\}^{r-1}} \chi^{1/4}\Big(\sum_{j=1}^{r-1}\eta'_j e_j + e_r\Big) N_{(\eta',1)},$$

and the claim follows.

Next, by [45, Section 5.2], for $\theta = \sum_{j=1}^r \theta_j e_j$ we have

$$\log \mathbf{b}_J^{\varepsilon}(\theta) - \log \mathbf{b}_J(\theta) = \sum_{j=1}^r \log\Big(1+\sqrt{\frac{q_{\alpha_r}}{q_{\alpha_0}}}e^{-\theta_j}\Big) - \log\Big(1+\sqrt{\frac{q_{\alpha_0}}{q_{\alpha_r}}}e^{-\theta_j}\Big).$$

Hence,

$$\langle \nabla, \alpha_r\rangle \log \mathbf{b}_J^{\varepsilon}(0) - \langle \nabla, \alpha_r\rangle \log \mathbf{b}_J(0) = \frac{\sqrt{q_{\alpha_0}}-\sqrt{q_{\alpha_r}}}{\sqrt{q_{\alpha_0}}+\sqrt{q_{\alpha_r}}}.$$

Therefore, by (A.4), formula (A.3) takes the form

$$\langle \sigma(x,y),\alpha_r\rangle - \langle \sigma(x,y'),\alpha_r\rangle = \frac{1}{2}\cdot\frac{\sqrt{q_{\alpha_0}}-\sqrt{q_{\alpha_r}}}{\sqrt{q_{\alpha_0}}+\sqrt{q_{\alpha_r}}}.$$

Since the left-hand side belongs to $\frac{1}{2}\mathbb{Z}$, we must have $q_{\alpha_0} = q_{\alpha_r}$, which leads to a contradiction. Analogously, we treat angular core sequences.

Theorem A.2. *Assume that the finite root system associated with the building $\mathscr{X}$ has type* BC_r. *Then for the random walk generated by P as constructed above, the following dichotomy holds:*

- [At the bottom of the spectrum] *If $\zeta = \varrho$, then $\overline{\mathscr{X}}_{M,\varrho}$ is* $\mathrm{Aut}(\mathscr{X})$*-equivariantly isomorphic to any of the Furstenberg* (*measure-theoretic*) *or the Caprace–Lécureux* (*combinatorial*) *compactifications of the set V_s of special vertices.*
- [Above the bottom of the spectrum] *If $\zeta > \varrho$, then $\overline{\mathscr{X}}_{M,\zeta}$ is* $\mathrm{Aut}(\mathscr{X})$*-equivariantly isomorphic to the join of any of the above compactifications with the Gromov* (*horofunction*) *compactification of the set V_s of special vertices.*

Moreover, if there exists a locally compact closed subgroup of $\mathrm{Aut}(\mathscr{X})$*, say G, whose action on the building $\mathscr{X}$ is strongly transitive and type-preserving, then the Guivarc'h compactification $\overline{\mathscr{X}}_G$ is G-isomorphic to the Martin compactification $\overline{\mathscr{X}}_{M,\varrho}$.*

Proof. In view of Theorem 6.1 the Martin compactification $\overline{\mathscr{X}}_{M,\varrho}$ for the random walk generated by P is $\mathrm{Aut}(\mathscr{X})$-isomorphic to the Furstenberg compactification $\overline{\mathscr{X}}_F$. Moreover, if there is a locally compact group acting strongly transitively on $\mathscr{X}$ then $\overline{\mathscr{X}}_{M,\varrho}$ is G-isomorphic to the Guivarc'h compactification. Lastly, by Theorems 3.2 and 6.1 we conclude that $\overline{\mathscr{X}}_{M,\zeta}$ with $\zeta > \varrho$ is $\mathrm{Aut}(\mathscr{X})$-isomorphic to $\overline{\mathscr{X}}_F \vee \overline{\mathscr{X}}_V$ and $\overline{\mathscr{X}}_{M,\varrho} \vee \overline{\mathscr{X}}_V$. ∎

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