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Positive surface group representations in $\mathsf{PO}(p,q)$

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Abstract. We show that Θ-positive Anosov representations $\rho : \Gamma \to \mathsf{PO}(p,q)$ of a surface group Γ satisfy root versus weight collar lemmas for all the Anosov roots, and are positively ratioed with respect to all such roots. We deduce from this, using a result of Beyrer–Pozzetti (2014), that Θ-positive Anosov representations $\rho : \Gamma \to \mathsf{PO}(p,q)$ form connected components of character varieties.

Keywords: positive representations, collar lemma, cross ratio, Anosov representations.

1. Introduction

Higher rank Teichmüller theory stems from the seminal work of Labourie [30], Fock–Goncharov [17], and Burger–Iozzi–Wienhard [11]; they discovered that, for some classes of Lie groups, there exist *higher rank Teichmüller spaces:* connected components of character varieties of fundamental groups Γ of closed surfaces S of genus at least 2 in higher rank semisimple Lie groups that only consist of injective representations with discrete image. More specifically, Labourie and Fock–Goncharov showed that for split real Lie groups the *Hitchin components*, discovered by Hitchin [24], form higher rank Teichmüller spaces, while Burger–Iozzi–Wienhard discovered the *maximal components* for Hermitian Lie groups and proved that they also form higher rank Teichmüller spaces. These components consist only of representations with further remarkable geometric properties bearing strong similarities with holonomies of hyperbolizations [12, 16, 25, 31, 32, 40]; furthermore, since they form connected components of character varieties, they can be studied with an array of different tools including Higgs bundles [7, 24] and real algebraic geometry [9].

A recent breakthrough in the field was given by the insight of Guichard–Wienhard [22]. Partially in collaboration with Labourie, they developed a beautiful and

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clear conjectural picture of all higher rank Teichmüller spaces [19, 23]. They give a complete list of pairs of Lie groups and parabolic subgroups that admit a Θ*-positive structure*;[1] when this is the case they define Θ*-positive representations*. They conjecture that higher rank Teichmüller spaces are precisely the connected components of character varieties that contain a Θ-positive representation [22, Conjecture 5.4], and prove that the limit set of Θ-positive representations has important positivity features, generalizing [17]. The latter are behind the good geometric properties of these representations. Simultaneously Bradlow–Collier–García-Prada–Gothen–Oliveira, partially in collaboration with Aparicio-Arroyo, discovered and parametrized special connected components of the moduli space of Higgs bundles on a compact Riemann surface X that conjecturally detected all examples of higher rank Teichmüller spaces, up to some exceptional Lie groups [1, 6].

In this work we study aspects of the higher rank Teichmüller theory of $\mathsf{PO}(p,q)$; the classification of [23] shows that $\mathsf{PO}(p,q)$ is the only family of classical groups that carries a positive structure besides split real Lie groups and Hermitian Lie groups of tube type; in the last two cases Θ-positive representations are, respectively, Hitchin and maximal representations. We make three major contributions: we show that every Θ-positive Anosov representation satisfies collar lemmas comparing roots and weights, we show that every Θ-positive Anosov representation is positively ratioed, and, also relying on results of the companion paper [4], we confirm Guichard–Wienhard's conjecture that Θ-positive Anosov representations form connected components of character varieties.[2]

The first two results generalize familiar properties of holonomies of hyperbolizations to Θ-positive representations, and can be regarded as additional geometric justification for the name *higher rank Teichmüller theory*. The third result gives the first proof that a connected component of a character variety in a higher rank Lie group consists only of discrete and injective representations, not relying on Higgs bundles[3] or bounded cohomology. The new strategy we develop could also be applied to Hitchin representations in $\mathsf{PSL}(n,\mathbb{R})$, after extracting a suitable formulation of a root versus weight collar lemma from [32, Proposition 2.12]. All our results also hold and are new for Hitchin representations into $\mathsf{PO}(p,p)$; see Appendix A for details.

Θ-*Positive representations are positively ratioed*

To describe our results in more detail, we need to recall some basic facts from Guichard–Wienhard's theory of Θ-positivity in the special case of the group $\mathsf{PO}(p,q)$. For technical

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[1] In Guichard–Wienhard's theory the letter Θ refers to the subset of simple roots associated to the parabolic subgroup playing a role for the notion of positivity, but for the purpose of the paper it is harmless to understand "Θ-positive" as "positive in the sense of Guichard–Wienhard".

[2] In an independent paper Guichard–Labourie–Wienhard proved that Θ-positive representations for any admissible target G form connected components of the set of non-parabolic representations [19]. Relying on the aforementioned results by Aparicio-Arroyo–Bradlow–Collier–García-Prada–Gothen–Oliveira [1, 6] this gives examples of higher rank Teichmüller spaces.

[3] Labourie's proof that Hitchin representations are injective relies on Higgs bundles techniques to guarantee that Hitchin representations are irreducible and thus non-parabolic [30, Lemma 10.1].

reasons, we assume from now on that $1 < p < q$. However, the case $p = q$ works similarly; we discuss it in Appendix A. We denote by $\mathcal{F}_{p-1}(\mathbb{R}^{p,q})$ the set of partial flags of isotropic subspaces of $\mathbb{R}^{p,q}$, which consist of isotropic subspaces of all dimensions but the maximal one (see Section 3.1). The Θ-positive structure associated to $\mathsf{PO}(p,q)$ is defined with respect to the stabilizer of a point in $\mathcal{F}_{p-1}(\mathbb{R}^{p,q})$. Since $\mathsf{PO}(p,q)$ has a Θ-positive structure, one can define *Θ-positive n-tuples* in $\mathcal{F}_{p-1}(\mathbb{R}^{p,q})$. This generalizes the notion of cyclically oriented n-tuple on $\mathbb{RP}^1 \simeq \partial_\infty\mathbb{H}^2$, which is crucial in classical Teichmüller theory. A map $\xi : \mathbb{S}^1 \to \mathcal{F}_{p-1}(\mathbb{R}^{p,q})$ is *Θ-positive* if it maps positive n-tuples in $\mathbb{S}^1$ to positive n-tuples in $\mathcal{F}_{p-1}(\mathbb{R}^{p,q})$. Let Γ be the fundamental group of a closed hyperbolic surface, so that in particular $\partial_\infty\Gamma \simeq \mathbb{S}^1$. Following [22, Definition 5.3] we say that a representation $\rho : \Gamma \to \mathsf{PO}(p,q)$ is *Θ-positive* if it admits an equivariant Θ-positive map. In this paper we will only consider Θ-positive representations that are furthermore Anosov;[4] while any Θ-positive map is automatically transverse, in order to be Anosov it needs to additionally be continuous and dynamics preserving (see Section 2.1 for the precise definition).

The first result of the paper establishes an additional positivity property of Θ-positive representations: they are *positively ratioed* in the sense of Martone–Zhang ([34, Definition 2.25], see also Definition 2.7). This amounts to saying that the restriction of the natural cross ratio on the set $\mathrm{Iso}_k(\mathbb{R}^{p,q})$ of k-dimensional isotropic planes to the image of the boundary map induces a positive cross ratio; a Θ-positive boundary map induces boundary maps to the isotropic k-planes for $k < p$ through the natural projection $\mathcal{F}_{p-1}(\mathbb{R}^{p,q}) \to \mathrm{Iso}_k(\mathbb{R}^{p,q})$.

Theorem A (Theorem 4.9). *Let $\rho : \Gamma \to \mathsf{PO}(p,q)$ be a Θ-positive Anosov representation. Then ρ is k-positively ratioed for all $k < p$.*

Being positively ratioed implies that suitable Finsler length functions associated to the representations can be computed as intersections with a geodesic current [34, Theorem 1.1]. In turn this has a number of geometric consequences: length shortening under surgery [34, Corollary 1.3], relations between systole and entropy [34, Corollary 1.2], as well as domains of discontinuity for the mapping class group action on suitable compactifications [8, Corollary 1.10].

It follows from our results of [3] that Θ-positive representations into $\mathsf{PO}(p,q)$ are k-positively ratioed for $k \leq p-2$; thus Theorem A is only new for $k = p-1$. However, in this case a new substantial difficulty has to be overcome, as the boundary maps are only Lipschitz regular and not C^1. As a result, rather than relying on continuity of the derivative, we need to perform a much more careful analysis. We also include a new proof for $k \leq p-2$, since it is better adapted to the Θ-positive structure and builds on two results of independent interest: First, for all $k < p$, we show that any Θ-positive triple

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[4] In the aforementioned independent paper Guichard–Labourie–Wienhard proved that a Θ-positive representation is necessarily Anosov, so this assumption is not really needed [19]. However, since our two papers are independent we will keep the assumption.

(x, y, z) in $\mathcal{F}_{p-1}(\mathbb{R}^{p,q})$ naturally defines a tangent cone $c_k^+(x)$ in $\mathrm{T}_x\mathrm{Iso}_k(\mathbb{R}^{p,q})$, and the induced boundary map of a Θ-positive representation containing (x, y, z) is almost everywhere tangent to $c_k^+(x)$ (Proposition 4.4, see also Remark 4.6). This relies on the fact that the boundary map is Lipschitz regular as shown in [38], and was inspired by discussions of the second author with Wienhard while working on [38]. Second, we prove that the cross ratio (infinitesimally) increases along those cones (Proposition 4.8).

Collar lemmas

An important property of holonomies of hyperbolizations is the collar lemma [28]: any simple curve g admits an embedded collar neighbourhood of width logarithmic in the inverse of the hyperbolic length of g. In particular, this has the algebraic consequence that the length of any curve h crossing g must be at least the logarithm of the inverse of the length of g. As a result, only simple curves can be very short in a hyperbolic structure. We generalize here the algebraic formulation of the collar lemma to Θ-positive representations and prove an asymmetric strengthening: we show that the k-th Finsler length, which is the logarithm of the product of the first k eigenvalues, of an element g already controls the k-th eigenvalue gap of all linked elements h. We say that two elements $g, h \in \Gamma$ are *linked* if the attracting and repelling fixed points $h_+, h_- \in \partial_\infty\Gamma$ of h are in different connected components of $\partial_\infty\Gamma \setminus \{g_-, g_+\}$. Given $A \in \mathsf{PO}(p, q)$ we denote by $\lambda_1(A), \ldots, \lambda_{p+q}(A)$ the generalized eigenvalues of a lift of A to $\mathrm{O}(p, q)$ ordered so that their moduli are non-decreasing, i.e. $|\lambda_i(A)| \geq |\lambda_{i+1}(A)|$.

Theorem B (Theorem 5.11). *Let $\rho : \Gamma \to \mathsf{PO}(p, q)$ be a Θ-positive Anosov representation and $g, h \in \Gamma$ a linked pair. Then, for any $k \leq p - 1$,*

$$\left(1 - \left|\frac{\lambda_{k+1}}{\lambda_k}(\rho(h))\right|\right)^{-1} < \lambda_1^2 \cdots \lambda_k^2(\rho(g)).$$

We can rephrase the statement in Lie-theoretic terms: Let $\alpha_k(\rho(h))$ be the k-th *restricted root* of $\mathsf{PO}(p, q)$ (for the standard numeration) applied to the Jordan projection of $\rho(h)$, so that $\alpha_k(\rho(h)) = \log|\frac{\lambda_k}{\lambda_{k+1}}(\rho(h))|$, and let $\omega_k(\rho(g))$ be the k-th *fundamental weight* applied to the Jordan projection of $\rho(g)$, namely $\omega_k(\rho(g)) = \log(\lambda_1^2 \cdots \lambda_k^2(\rho(g)))$. Theorem B, after elementary algebraic operations, states that for any linked pair g, h and $k \leq p - 1$,

$$(e^{\alpha_k(\rho(h))} - 1)(e^{\omega_k(\rho(g))} - 1) > 1. \tag{1.1}$$

In higher rank many measures of the magnitude of an element play an important role in understanding geometric features of actions, and different measures generalize different properties of hyperbolizations. On the one hand, the fundamental weights ω_k describe the translation length on the symmetric space with respect to suitable Finsler distances [26], and, as mentioned above, behave like the hyperbolic length function under surgery for representations in higher rank Teichmüller spaces. On the other hand, the roots α_k, albeit not being induced by a distance, are, for Anosov representations, coarsely equivalent to the stable length with respect to any generating system [27], and their entropy is constant and equal to one on higher rank Teichmüller spaces [36, 38]. Adding to this, Theorem B encodes a powerful generalization of another feature of holonomies of hyperbolizations – the collar lemma – to Θ-positive representations into $\mathrm{PO}(p,q)$; analogous results were previously established for Hitchin representations [32], maximal representations [12] and representations that satisfy some partial hyperconvexity properties, a class containing representations that are not Θ-positive, and does not always consist of connected components of the character variety [3].

The asymmetry comparing two different length functions in Theorem B is key and indicates that both α_k and ω_k play an important role in the study of Θ-positive representations. Equation (1.1) implies that also a weight versus weight collar lemma holds: for any linked pair g, h and $k \leq p-1$ we have

$$(e^{\omega_k(\rho(h))}-1)(e^{\omega_k(\rho(g))}-1) > 1.$$

This generalizes the collar lemma from [14] in the case of $\mathrm{PO}(2,q)$. Since it is not expected that a root versus root collar lemma holds (see [3, Section 7]), the asymmetric version is the strongest version to hope for. We will indeed need this strong version, as in the proof of Theorem C it is crucial that the collar lemma allows us to bound the root length from below.

Theorem B follows from our results in [3] for $k < p-2$, and is new for $k = p-2, p-1$. To deal with the two additional cases we need a different approach compared to [3] since Θ-positive representations are in general not p-Anosov and do not satisfy additional transversality properties. There are three key new tools, building on the Θ-positive structure, that allow us to circumvent this problem. First we need that the cross ratio is increasing along cones defined by Θ-positive triples, as mentioned above. The second tool is the construction of a *hybrid* flag associated to a pair of transverse flags in $\mathcal{F}_{p-1}$ (Definition 5.4). In Proposition 5.9 we establish in which sense positivity is preserved under the hybrid construction. This result is of independent interest, and we believe it will be useful in further study of geometric properties of Θ-positive representations and more general classes of Anosov representations. With Proposition 5.9 at hand we can reduce the proof of Theorem B to an inequality for the Θ-positive structure for $\mathrm{PO}(2,q)$, which we establish in Lemma 5.3, and which is the third key building block of our proof. It constitutes the main step in the proof of a root versus weight collar lemma for maximal representations in $\mathrm{PO}(2,q)$ (Theorem 5.1), which is substantially different from the available proof for $\mathrm{Sp}(2n,\mathbb{R})$ [12].

As it does not cost any extra effort, we include proofs of the collar lemmas for the cases $k < p-2$ as well, which are simpler and more direct than the ones in [3], making this work independent of [3].

Θ-*Positive Anosov representations form connected components*

A key advantage of a collar lemma controlling eigenvalue gaps is that it guarantees that a limit of representations satisfying such collar lemma remains proximal. This is particularly important since we showed in the companion paper [4] that a proximal limit of positively ratioed representations admits continuous, dynamics preserving equivariant boundary maps [4, Theorem B]. Since thanks to Theorems A and B we can apply such result, we obtain the following.

Theorem C. *The set of* Θ*-positive Anosov representations is closed in* $\mathrm{Hom}(\Gamma, \mathsf{PO}(p,q))$.

We show that a limit of Θ-positive Anosov representations is Θ-positive Anosov in two steps. First we use Theorems A and B to show that [4, Theorem B] is applicable. This guarantees that the limit representation admits continuous, dynamics preserving equivariant boundary maps, which we prove being additionally Θ-positive using properties of Θ-positivity. In the second step of the proof we show that the representation is Anosov, by using a criterion due to Guéritaud–Guichard–Kassel–Wienhard [18] based on eigenvalue gap growth. Here we use once again the positivity of hybrid flags, and we introduce a new idea, allowing us to read the eigenvalue gap of an element as a cross ratio involving a hybrid flag; we use this to control the growth of the eigenvalue gaps (see Proposition 6.5 for details).

Since by the work of Guichard–Wienhard positivity of n-tuples is an open condition [22, Theorem 4.7] (see also Corollary 3.8 below), and Anosov representations are open [21, Theorem 1.2], the set of Θ-positive Anosov representations is open in the character variety. A consequence of Theorem C is thus the following.

Corollary 1.1 (Corollary 6.6). *Being* Θ*-positive Anosov is an open and closed condition in the character variety*

$$\Xi(\Gamma, \mathsf{PO}(p,q)) := \mathrm{Hom}^{\mathrm{red}}(\Gamma, \mathsf{PO}(p,q))/\mathsf{PO}(p,q)$$

and thus the set of Θ*-positive Anosov representations is a union of connected components of the character variety.*

Recall that the character variety $\Xi(\Gamma, \mathsf{PO}(p,q))$ can be defined equivalently as the semialgebraic GIT quotient of the representation space $\mathrm{Hom}(\Gamma, \mathsf{PO}(p,q))$ by the $\mathsf{PO}(p,q)$ action by conjugation [10, 39], or by the quotient of the subset $\mathrm{Hom}^{\mathrm{red}}(\Gamma, \mathsf{PO}(p,q)) \subset \mathrm{Hom}(\Gamma, \mathsf{PO}(p,q))$ consisting of reductive representations, on which the $\mathsf{PO}(p,q)$-action is separated [35, Section 4.1].

Remark 1.2. A representation into $\mathsf{O}(p,q)$,[5] a group that also naturally acts on $\mathcal{F}_{p-1}(\mathbb{R}^{p,q})$, is Θ-positive if and only if its projectivization is positive. In particular, all the results in this work also hold for Θ-positive representations in $\mathsf{O}(p,q)$. The case where the image is in $\mathsf{SO}(p,q)$ is particularly interesting, since for those groups connected components of character varieties have been studied in detail with Higgs bundles techniques [1] (see Remark 6.7).

Combining Corollary 1.1 with [1, Theorem 7.6, Proposition 7.13] we find that a conjugacy class of reductive representations $\rho : \Gamma \to \mathsf{SO}(p,q)$ consists of Θ-positive Anosov

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[5] Observe that, when $p+q$ is odd, $\mathsf{SO}(p,q) = \mathsf{PO}(p,q)$ and $\mathsf{O}(p,q) = \mathsf{SO}(p,q) \times \mathbb{Z}/2\mathbb{Z}$, but that things are more subtle when $p+q$ is even, and delicate liftability questions arise.

representations if and only if it belongs to one of the special connected components parametrized in [1, Theorem 4.1]. This gives the following result.

Theorem (see [1, Section 7.2]). *Let X be a Riemann surface structure on S. The subset of $\Xi(\Gamma, \mathsf{SO}(p,q))$ consisting of conjugacy classes of reductive Θ-positive Anosov representations is parametrized by*

$$\mathcal{M}_{K^p}(\mathsf{SO}(1,q-p+1)) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j}),$$

where K is the canonical bundle of X, $\mathcal{M}_{K^p}(\mathsf{SO}(1,q-p+1))$ denotes the moduli space of K^p-twisted $\mathsf{SO}(1,q-p+1)$-Higgs bundles on X and $H^0(K^{2j})$ is the vector space of holomorphic sections of K^{2j}.

Θ-positive representations of open surfaces

It is possible to define good notions of Θ-positive representations of more general surfaces that are not necessarily compact, by requiring the existence of a positive boundary map defined on some associated cyclically ordered set such as the circle with the action induced by a finite volume hyperbolization, or the subset of cusps (see [17] for representations in split real Lie groups and [11] for maximal representations in Hermitian Lie groups). We expect that Theorem B works verbatim in this setting, and that Theorem A and Theorem C admit suitable generalizations, respectively that the natural pullback of the cross ratio is positive, and that the space of Θ-positive representations forms connected components of relative character varieties.

Some of our techniques require however that the surface is closed: we use differential arguments based on the regularity of the image of the boundary map, which cannot be directly generalized to the open surfaces where, in most cases, the boundary map is not even continuous. While new ideas are needed to treat the general case, we expect that our general strategy, as well as some of the tools we develop, such as the study of hybrid flags, and the discovery of cross ratios that read eigenvalue gaps, will also be precious for dealing with open surfaces.

Relation to Guichard–Wienhard's work

We build on Guichard–Wienhard's theory of positivity for general Lie groups G, which generalizes aspects of Lusztig positivity [33] to this setting. We freely use results from [23] where general properties of positive n-tuples of flags, as well as of the positive semigroup, are established. In a previous version of this work we re-established many properties of Θ-positive triples only building on results announced in [22], but now that [23] is available we decided to avoid duplicates. As already remarked, we also build upon [38, Theorem D], by the second author in collaboration with Sambarino and Wienhard, and the ideas behind positivity that are crucial in that proof.

Our work is independent from the work of Guichard–Labourie–Wienhard, who proved that there exist higher rank Teichmüller components consisting of Θ-positive representations in any Lie group G admitting a Θ-positive structure [19]. In their paper they prove that general Θ-positive representations are necessarily Θ-Anosov, and form connected components of the subset of representations that are not contained in a parabolic subgroup. Since we show that Θ-positive Anosov representations are closed in the whole representation variety, our papers are complementary and together prove [22, Conjecture 5.4] for the group $\mathsf{PO}(p,q)$. Combining our results with [1] we obtain a statement stronger than [22, Conjecture 5.4]: for $1 < p < q$ the only higher rank Teichmüller components of the $\mathsf{PO}(p,q)$-character variety are the ones which contain a Θ-positive representation (see also Remark 6.7).

2. Preliminaries

We list here some notation that we keep throughout this work.

The group Γ

- Γ denotes a *surface group*, i.e. the fundamental group of a closed connected orientable surface of genus at least 2. Its *Gromov boundary* $\partial_\infty\Gamma$ is homeomorphic to the circle $\mathbb{S}^1$.
- $\partial_\infty\Gamma^{(j)}$ denotes the set of j-tuples of $\partial_\infty\Gamma$ consisting of pairwise distinct points.
- $\partial_\infty\Gamma^{[j]} \subset \partial_\infty\Gamma^{(j)}$ denotes the set of cyclically ordered j-tuples, namely the j-tuples that are either positively or negatively ordered for the standard cyclic orientation of the circle.
- $(x, y)_z \subset \partial_\infty\Gamma$ is the interval of $\partial_\infty\Gamma \setminus \{x, y\}$ that does *not* contain z, for $(x, y, z) \in \partial_\infty\Gamma^{(3)}$ [34].

Isotropic subspaces and eigenvalues

- $\mathbb{R}^{p,q}$ is the vector space $\mathbb{R}^{p+q}$ equipped with a non-degenerate symmetric bilinear form Q of signature (p,q) – apart from the appendix, we will assume $1 < p < q$. We denote by $\cdot^\perp$ the orthogonal complement with respect to Q. The isometry group of $\mathbb{R}^{p,q}$ is denoted by $\mathsf{O}(p,q)$, its subroup consisting of matrices of determinant 1 by $\mathsf{SO}(p,q)$, and the quotient of $\mathsf{O}(p,q)$ by its center is denoted by $\mathsf{PO}(p,q)$.
- $\lambda_1(g), \dots, \lambda_{p+q}(g)$ denote the eigenvalues of an element $g \in \mathsf{O}(p,q)$ counted with multiplicity and ordered so that their absolute values are non-increasing, i.e. $|\lambda_i(g)| \geq |\lambda_{i+1}(g)|$. Since $g \in \mathsf{O}(p,q)$, it follows that $\lambda_i(g) = \lambda_{p+q+1-i}^{-1}(g)$. If $g \in \mathsf{PO}(p,q)$ denote by $\tilde{g} \in \mathsf{O}(p,q)$ a lift, then

$$\frac{\lambda_i}{\lambda_{i+1}}(g) := \frac{\lambda_i(\tilde{g})}{\lambda_{i+1}(\tilde{g})}$$

  does not depend on the choice of the lift.

- $\mathrm{Iso}_k(\mathbb{R}^{p,q})$ denotes the set of *isotropic k-planes* in $\mathbb{R}^{p,q}$, i.e. k-planes on which the form Q is identically zero. Given $V \in \mathrm{Iso}_k(\mathbb{R}^{p,q})$ and $W \in \mathrm{Iso}_k(\mathbb{R}^{p,q})$ we say that V and W are *transverse* if the sum $V + W^\perp$ is direct (equivalently $V^\perp + W$ is direct); we denote this by $V \pitchfork W$. If two subspaces are not transverse we write $V \not\pitchfork W$.

2.1. Anosov representations

The *stable length* of $\gamma \in \Gamma$ is $|\gamma|_\infty := \lim_{n\to\infty} |\gamma^n|_\Gamma / n$, where $|\cdot|_\Gamma$ is a fixed word metric on the Cayley graph of Γ. Anosov representations in $\mathsf{PO}(p,q)$ admit the following characterization [18, Theorem 1.7] which we will use as a definition.

Definition 2.1. A homomorphism $\rho : \Gamma \to \mathsf{PO}(p,q)$ is *k-Anosov* for $k \in \{1,\dots,p\}$ if there exists a ρ-equivariant continuous boundary map $\xi^k : \partial_\infty\Gamma \to \mathrm{Iso}_k(\mathbb{R}^{p,q})$ such that

(1) $\xi^k(x) \pitchfork \xi^k(y)$ for all $x \neq y \in \partial_\infty\Gamma$;

(2) ξ^k is *dynamics preserving*, i.e. for every infinite order element $\gamma \in \Gamma$ with attracting fixed point $\gamma_+ \in \partial_\infty\Gamma$ the point $\xi^k(\gamma_+)$ is an attracting fixed point for the action of $\rho(\gamma)$ on $\mathrm{Iso}_k(\mathbb{R}^{p,q})$;

(3) $|\frac{\lambda_k}{\lambda_{k+1}}(\rho(\gamma_i))| \to \infty$ if $|\gamma_i|_\infty \to \infty$.

Anosov representations have many interesting geometric and dynamical properties, for instance they are discrete and faithful and form an open subset of $\mathrm{Hom}(\Gamma, \mathsf{PO}(p,q))$.

Remark 2.2. Anosov representations are defined for general reductive Lie groups; we do not introduce the general theory here as we only work with $\mathsf{PO}(p,q)$. Via the natural inclusion $\mathsf{PO}(p,q) \to \mathsf{PGL}(\mathbb{R}^{p,q})$, a representation $\rho : \Gamma \to \mathsf{PO}(p,q)$ is k-Anosov if and only if it is k-Anosov in $\mathsf{PGL}(\mathbb{R}^{p,q})$. In particular, any result for k-Anosov representations in $\mathsf{PGL}(\mathbb{R}^{p,q})$ applies to our context.

We say that a representation is *Θ-Anosov* if it is k-Anosov for all $k = 1,\dots,p-1$. Since the boundary maps of Anosov representations are dynamics preserving, for all non-trivial $\gamma \in \Gamma$ we have $\xi^k(\gamma_+) \subset \xi^{k+1}(\gamma_+)$. Thus for any Θ-Anosov representation the continuity of the boundary map and the density of the fixed points in $\partial_\infty\Gamma$ imply that the map

$$\xi = (\xi^1, \dots, \xi^{p-1}) : \partial_\infty\Gamma \to \mathcal{F}_{p-1}$$

is well defined, equivariant, continuous and transverse. Here, and in the rest of the paper, $\mathcal{F}_{p-1}$ denotes the partial flag manifold consisting of flags of isotropic subspaces of dimensions $1,\dots,p-1$.

Notation. We will write x^k_ρ and x^k for $\xi^k(x)$, where ρ is a k-Anosov representation and ξ^k the associated boundary map. Similarly we may write γ_ρ instead of $\rho(\gamma)$ for $\gamma \in \Gamma$. Moreover, we will write x^{p+q-k} for $\xi^k(x)^\perp$.

2.2. *Cross ratios*

In this paper we use the cross ratios defined on

$$\mathcal{A}_k := \{(V_1, W_1, W_2, V_2) \in \mathrm{Iso}_k^4(\mathbb{R}^{p,q}) \mid V_i \pitchfork W_j,\ i, j = 1, 2\}.$$

Definition 2.3. The cross ratio $\mathrm{cr}_k : \mathcal{A}_k \to \mathbb{R} \setminus \{0\}$ is defined by

$$\mathrm{cr}_k(V_1, W_1, W_2, V_2) := \frac{V_1 \wedge W_2^\perp}{V_1 \wedge W_1^\perp} \frac{V_2 \wedge W_1^\perp}{V_2 \wedge W_2^\perp}.$$

Here $V_i \wedge W_j^\perp$ denotes the element $v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_{p+q-k} \in \bigwedge^{p+q} \mathbb{R}^{p+q} \simeq \mathbb{R}$ for bases $\{v_1, \ldots, v_k\}, \{w_1, \ldots, w_{p+q-k}\}$ of V_i and $W_j^\perp$, and a fixed identification $\bigwedge^{p+q} \mathbb{R}^{p+q} \simeq \mathbb{R}$. The value of cr_k is independent of all choices made.

If $(V_1, W_1, W_2, V_2) \in \mathcal{A}_1 \subset \mathrm{Iso}_1^4(\mathbb{R}^{p,q})$, the cross ratio cr_1 can be expressed as

$$\mathrm{cr}_1(V_1, W_1, W_2, V_2) := \frac{Q(\tilde{V}_1, \tilde{W}_2)}{Q(\tilde{V}_1, \tilde{W}_1)} \frac{Q(\tilde{V}_2, \tilde{W}_1)}{Q(\tilde{V}_2, \tilde{W}_2)},$$

where $\tilde{V}_i, \tilde{W}_j \in \mathbb{R}^{p,q} \setminus \{0\}$ are such that $\tilde{V}_i \in V_i$ and $\tilde{W}_j \in W_j$.

The following properties are classical and easy to verify.

Proposition 2.4. *Let* $V_1, V_2, V_3, W_1, W_2, W_3 \in \mathrm{Iso}_k(\mathbb{R}^{p,q})$. *Then whenever all quantities are defined we have:*

(1) $\mathrm{cr}_k(V_1, W_1, W_2, V_2)^{-1} = \mathrm{cr}_k(V_2, W_1, W_2, V_1) = \mathrm{cr}_k(V_1, W_2, W_1, V_2)$, *in particular* $\mathrm{cr}_k(V_1, W_1, W_1, V_2) = 1$.

(2) $\mathrm{cr}_k(V_1, W_1, W_2, V_2) \cdot \mathrm{cr}_k(V_2, W_1, W_2, V_3) = \mathrm{cr}_k(V_1, W_1, W_2, V_3)$.

(3) $\mathrm{cr}_k(V_1, W_1, W_2, V_2) \cdot \mathrm{cr}_k(V_1, W_2, W_3, V_2) = \mathrm{cr}_k(V_1, W_1, W_3, V_2)$.

(4) $\mathrm{cr}_k(V_1, W_1, W_2, V_2) = \mathrm{cr}_k(gV_1, gW_1, gW_2, gV_2)$ *for all* $g \in \mathsf{PO}(p, q)$.

(5) *The cross ratio is algebraic.*

The identities (3) *and* (4) *will be called* cocycle identities.

Proof. (1)–(3) are straightforward computations from the expression in Definition 2.3, while (4) follows since $\mathsf{PO}(p, q)$ preserves orthogonal complement and induces multiplication by a scalar on $\bigwedge^{p+q} \mathbb{R}^{p+q}$.

(5) The expression for the cross ratio is clearly algebraic when defined on the frame manifold, and descends to an algebraic function on $\mathcal{A}_k$ since it does not depend on the choice of a lift. ■

In the context of Anosov representations, the cross ratio computes the fundamental weights. It is easy to check the following.

Lemma 2.5 ([34, p. 19]). *If* ρ *is* k*-Anosov, then*

$$\mathrm{cr}_k(\gamma_-^k, x^k, \gamma x^k, \gamma_+^k) = \lambda_1^2(\gamma) \cdots \lambda_k^2(\gamma)$$

for all non-trivial $\gamma \in \Gamma$ *and* $x \in \partial_\infty \Gamma \setminus \{\gamma_\pm\}$.

A computation as in [3, proof of Proposition 3.11] yields the following.

Proposition 2.6. *Let* $(V_1, W_1, W_2, V_2) \in \mathcal{A}_k$ *with* $\dim V_1 \cap V_2 = k-1$. *Set*

$$V := V_1 \cap V_2, \quad V_+ := V_1 + V_2, \quad \widehat{V}_+ := V_+/V, \quad \widehat{V}_\perp := V^\perp/V.$$

Denote by $[V_1], [V_2], [W_1 \cap V^\perp], [W_2 \cap V^\perp] \in \mathrm{Iso}_1(\widehat{V}_\perp)$ *and* $[V_1], [V_2], [W_1^\perp \cap V_+], [W_2^\perp \cap V_+] \in \mathbb{P}(\widehat{V}_+) \simeq \mathbb{RP}^1$ *the associated subspaces. Then*

$$\begin{aligned} \mathrm{cr}_k(V_1, W_1, W_2, V_2) &= \mathrm{cr}_1^{\widehat{V}_\perp}([V_1], [W_1 \cap V^\perp], [W_2 \cap V^\perp], [V_2]) \\ &= \mathrm{cr}_1^{\widehat{V}_+}([V_1], [W_1^\perp \cap V_+], [W_2^\perp \cap V_+], [V_2]), \end{aligned}$$

where we define $\mathrm{cr}_1^{\widehat{V}_+}$ *on* $\mathbb{P}(\widehat{V}_+)$ *to be the usual projective cross ratio on* $\mathbb{RP}^1$.

Martone–Zhang [34] introduced a class of Anosov representations that satisfy a positivity condition with respect to the cross ratio. Recall that a quadruple (x, y, z, w) in $\partial_\infty\Gamma$ is *cyclically ordered* if it is positively or negatively oriented for the standard cyclic order on $\mathbb{S}^1 = \partial_\infty\Gamma$.

Definition 2.7 ([34, Definition 2.25]). A representation $\rho : \Gamma \to \mathsf{PO}(p,q)$ is called *k-positively ratioed* if it is k-Anosov and for all cyclically ordered quadruples $(x, y, z, w) \in \partial_\infty\Gamma^{[4]}$,

$$\mathrm{cr}_k(x^k, y^k, z^k, w^k) \geq 1. \tag{2.1}$$

It is shown in [34] that the inequalities in (2.1) are necessarily strict.

2.3. Property H_k

The following transversality property of boundary maps was introduced by Labourie in his work on Hitchin representations, and was further studied in [37,41] where relations to differentiability of boundary maps were established.

Definition 2.8 ([30, Section 7.1.4]). A representation $\rho : \Gamma \to \mathsf{PO}(p,q)$ satisfies *property H_k* for $2 \leq k \leq p-1$ if it is $\{k-1, k, k+1\}$-Anosov and for all $(x, y, z) \in \partial_\infty\Gamma^{(3)}$ the following sum is direct and thus equal to $\mathbb{R}^{p,q}$:

$$x^k + (y^k \cap z^{p+q-k+1}) + z^{p+q-k-1}. \tag{2.2}$$

A representation satisfies *property H_1* if it is $\{1,2\}$-Anosov and for all $(x, y, z) \in \partial_\infty\Gamma^{(3)}$ the following sum is direct and thus equal to $\mathbb{R}^{p,q}$:

$$x^1 + y^1 + z^{p+q-2}.$$

The transversality of the Anosov boundary maps implies that the sum in (2.2) is direct if and only if the sum $(x^k \cap z^{p+q-k+1}) + (y^k \cap z^{p+q-k+1}) + z^{p+q-k-1}$ is direct. Moreover, taking the orthogonal complement, one easily checks that the sum is direct if and only if the following sum is direct:

$$(x^{p+q-k} \cap z^{k+1}) + (y^{p+q-k} \cap z^{k+1}) + z^{k-1}. \tag{2.3}$$

It was shown in [37] that representations satisfying property H_k have boundary maps with C^1 image, with tangent spaces prescribed by the Anosov boundary maps. The tangent space to $\mathrm{Gr}_k(\mathbb{R}^{p+q})$ at a subspace V can be naturally identified with $\mathrm{Hom}(V, W)$ where W is a vector subspace of dimension $p+q-k$ transverse to V. Since $\mathrm{Iso}_k(\mathbb{R}^{p,q})$ is a submanifold of $\mathrm{Gr}_k(\mathbb{R}^{p+q})$, we identify its tangent space $T_{x^k}\mathrm{Iso}_k(\mathbb{R}^{p,q})$ with a subspace of $\mathrm{Hom}(x^k, y^{p+q-k})$.

Proposition 2.9 ([37, Proposition 8.11]). *If $\rho : \Gamma \to \mathsf{PO}(p,q)$ satisfies property H_k, then the boundary curve ξ^k has C^1 image and the tangent space is given by*

$$T_{x^k}\xi^k(\partial_\infty\Gamma) = \{\phi \in \mathrm{Hom}(x^k, y^{p+q-k}) \mid x^{k-1} \subseteq \ker\phi,\ \mathrm{Im}\,\phi \subseteq x^{k+1} \cap y^{p+q-k}\}$$

for any $y \neq x \in \partial_\infty\Gamma$.

Proof. To deduce this statement from [37, Proposition 8.11], we set in their notation $p = k$ and $s = k+1$, and observe that condition (i) in [37, Proposition 8.11] is guaranteed since ρ is $(p-1)$-Anosov. [37, Proposition 8.11] then shows that $\bigwedge^k \rho$ is $(1,1,2)$-hyperconvex and thus has C^1 1-Anosov boundary map by [37, Proposition 7.4] with tangent space given by the 2-Anosov boundary map of $\bigwedge^k \rho$. Since the 1-Anosov boundary map of $\bigwedge^k \rho$ is given by the composition of the k-Anosov boundary map of ρ with $\bigwedge^k : \mathrm{Iso}_k(\mathbb{R}^{p,q}) \to \mathbb{P}(\bigwedge^k \mathbb{R}^{p,q})$, and the latter is algebraic, we deduce that the k-Anosov boundary map is itself C^1. The statement about the tangent space follows from the proof of [37, Proposition 8.9] where it is observed that the image of the vector subspace

$$\{\phi \in \mathrm{Hom}(x^k, y^{p+q-k}) \mid x^{k-1} \subseteq \ker\phi,\ \mathrm{Im}\,\phi \subseteq x^{k+1} \cap y^{p+q-k}\} < T_{x^k}\mathrm{Iso}_k(\mathbb{R}^{p,q})$$

under $d\bigwedge^k$ is the tangent space at $\xi^2_{\bigwedge^k\rho}(x)$ of the submanifold $\xi^2_{\bigwedge^k\rho}$. ■

3. Θ-Positive structure for $\mathsf{PO}(p,q)$

In this section we recall the definition and general facts about the Θ-positive structure, Θ-positive triples and Θ-positive representations associated to $\mathsf{PO}(p,q)$, restricting, as always, to the case $1 < p < q$. We additionally prove several basic facts that will be relevant for us in the the rest of the paper.

We fix a basis $(e_1, \dots, e_{p+q})$ such that the form Q is represented by the matrix

$$Q = \begin{pmatrix} 0 & 0 & K \\ 0 & J & 0 \\ K^t & 0 & 0 \end{pmatrix}, \tag{3.1}$$

where

$$K = \begin{pmatrix} 0 & 0 & 0 & (-1)^{p-1} \\ 0 & 0 & \iddots & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\mathrm{Id}_{q-p} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

An important role for the Θ-positive structure will be played by the vector subspace $V_J := \mathrm{span}\{e_p, \dots, e_{q+1}\}$ together with the bilinear form[6] b_J of signature $(1, q-p+1)$ induced by J, and the induced quadratic form q_J. Namely,

$$b_J(\mathsf{w}, \mathsf{z}) := \tfrac{1}{2}\mathsf{w}^t J \mathsf{z}, \quad q_J(\mathsf{w}) := b_J(\mathsf{w}, \mathsf{w}), \quad \mathsf{w}, \mathsf{z} \in V_J.$$

We will denote by

$$c_J(V_J) = \{\mathsf{w} = (\mathsf{w}_p, \dots, \mathsf{w}_{q+1}) \in V_J \mid \mathsf{w}_p > 0,\ q_J(\mathsf{w}) > 0\} \tag{3.2}$$

the cone of Q-positive vectors with positive first entry and by $\overline{c}_J(V_J)$ its partial closure, where the second inequality is not required to be strict. For every w in $c_J(V_J)$ we have $\mathsf{w}_{q+1} \geq 0$: indeed, denoting by $\overline{\mathsf{w}}$ the vector $(\mathsf{w}_{p+1}, \dots, \mathsf{w}_q)$ we have

$$q_J(\mathsf{w}) = \mathsf{w}_p \mathsf{w}_{q+1} - \|\overline{\mathsf{w}}\|^2 > 0,$$

from which the claim follows.

The following basic observation will be important later on.

Lemma 3.1. *Let* $\mathsf{v} \in c_J(V_J)$, *and* $\mathsf{w} \in \overline{c}_J(V_J)$. *Then* $b_J(\mathsf{v}, \mathsf{w}) > 0$.

Proof. If $\mathsf{v} \in c_J(V_J)$ and $\mathsf{w} \in \overline{c}_J(V_J)$, then the quadratic form $b_J|_{\langle \mathsf{v}, \mathsf{w} \rangle}$ has signature $(1,1)$. Indeed, since b_J has signature $(1, q-p+1)$ and $q_J(\mathsf{v}) > 0$, the restriction of b_J to $\mathsf{v}^\perp$ is negative definite.

In particular, we can choose a basis of $\langle \mathsf{v}, \mathsf{w} \rangle$ with respect to which b_J is represented by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Since v, w are (semi-)positive vectors for b_J, they belong to the closure of either the positive or the negative quadrant; the (semi-)positivity of the last coordinates of a vector in $c_J(V_J)$ guarantees that they furthermore belong to the same quadrant, and thus their inner product is positive. ■

3.1. Θ-*Positive elements*

Guichard–Wienhard defined the notion of Θ-positive structure. We introduce here the Θ-positive structure of $\mathsf{PO}(p,q)$ following [22, Section 4.5].

We denote by Θ the first $p-1$ simple roots of $\mathsf{PO}(p,q)$ in the standard way of drawing the Dynkin diagram,[7] and by $\mathcal{F}_{p-1} = \mathcal{F}_{p-1}(\mathbb{R}^{p,q})$ the associated flag manifold,

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[6] By construction this represents the restriction of Q to V_J.

[7] Namely, Θ consists of all roots but the simple root that is only connected to one other root by a double arrow; equivalently of all the long roots.

which we realize as the set of flags of subspaces of $\mathbb{R}^{p,q}$ of dimension $\{1,\dots,p-1, q+1,\dots,q+p-1\}$ such that the first $p-1$ subspaces are isotropic for Q and the other subspaces are their orthogonals with respect to Q. Clearly an element in $\mathcal{F}_{p-1}$ is uniquely determined by the first $p-1$ subspaces. Throughout we will denote by Z and X the partial flags in $\mathcal{F}_{p-1}$ defined by

$$\mathsf{Z}^l = \langle e_1,\dots,e_l\rangle \quad \text{and} \quad \mathsf{X}^l = \langle e_{p+q},\dots,e_{p+q-l+1}\rangle. \tag{3.3}$$

Here, as above, l ranges in the set $\{1,\dots,p-1,q+1,\dots,q+p\}$. Observe that we have

$$\mathsf{X}^l \cap \mathsf{Z}^{p+q-l+1} = \langle e_{p+q-l+1}\rangle.$$

Given a positive real number s and an integer $1 \le k \le p-2$ the *elementary matrix* $E_k(s)$ is the matrix that differs from the identity only in positions $(k,k+1)$ and $(p+q-k, p+q-k+1)$ where it is equal to s. Instead, for $k=p-1$, we choose a vector $\mathsf{s} \in c_J(V_J)$, denote by s^t the transposed vector and set

$$E_{p-1}(\mathsf{s}) = \begin{pmatrix} \mathrm{Id}_{p-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & \mathsf{s}^t & q_J(\mathsf{s}) & 0 \\ 0 & 0 & \mathrm{Id}_{q-p+2} & J\mathsf{s} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id}_{p-2} \end{pmatrix}. \tag{3.4}$$

Our next goal is to recall the definition of the Θ*-positive semigroup* $U_\Theta^{>0}$ of the unipotent subgroup U_Θ of the stabilizer in $\mathsf{PO}(p,q)$ of the partial flag Z (see [22, Theorem 4.5]). To this end, we consider the cone

$$V_\Theta := \{v = (s_1,\dots,s_{p-2}, s_{p-1}^t)^t \in \mathbb{R}^q \mid s_1,\dots,s_{p-2} \in \mathbb{R}_{>0},\ s_{p-1} \in c_J(V_J)\}.$$

Given $v \in V_\Theta$ we set

$$a(v) = \prod_{j \le p-1,\, j \text{ odd}} E_j(s_j),$$
$$b(v) = \prod_{j \le p-1,\, j \text{ even}} E_j(s_j).$$

Since the matrices in the product defining a, respectively b, commute, their order plays no role. It will be convenient to also use the notation

$$ab(v) := a(v)b(v).$$

Definition 3.2. Let $\vec{v} = (v_1,\dots,v_{p-1}) \in V_\Theta^{p-1}$. The Θ*-positive element* $P(\vec{v})$ is the product

$$P(\vec{v}) = ab(v_1)\cdots ab(v_{p-1}).$$

The Θ-*positive semigroup* $U_\Theta^{>0}$ is the set of Θ-positive elements defined above, which forms a semigroup [23, Corollary 8.16 (2)].[8]

Remark 3.3. There are 2^{p-1} possible different choices for the cone V_Θ, depending on the choice of the sign of each of the first $p-2$ entries and of the sign of the first coordinate of the vector s_{p-1}. Each such choice gives a different choice of a Θ-positive semigroup and any two choices are conjugate in $\mathsf{PO}(p,q)$.

3.2. Θ-*Positivity of triples of partial flags*

We can now define positivity for triples and n-tuples of flags associated to $\mathsf{PO}(p,q)$ (see [22, Definition 4.6]).

Definition 3.4. A triple $(x,y,z)\in\mathcal{F}_{p-1}^3$ is Θ-*positive* if there exists $g\in\mathsf{PO}(p,q)$ and a Θ-positive element $P\in U_\Theta^{>0}$ such that

$$(gx,gy,gz)=(\mathsf{X},P\mathsf{X},\mathsf{Z}).$$

An n-tuple $(x_1,\dots,x_n)\in\mathcal{F}_{p-1}^n$ is Θ-*positive* if there exist $P_2,\dots,P_{n-1}\in U_\Theta^{>0}$ and $g\in\mathsf{PO}(p,q)$ such that

$$(gx_1,gx_2,\dots,gx_{n-1},gx_n)=(\mathsf{X},P_2\mathsf{X},\dots,P_2\cdots P_{n-1}\mathsf{X},\mathsf{Z}). \tag{3.5}$$

A *standard* Θ-*positive* n-*tuple* is an n-tuple of the form

$$(\mathsf{X},P_2\mathsf{X},\dots,P_2\cdots P_{n-1}\mathsf{X},\mathsf{Z})\quad \text{for } P_2,\dots,P_{n-1}\in U_\Theta^{>0}.$$

Since $U_\Theta^{>0}$ is a semigroup, $P_1P_2\in U_\Theta^{>0}$ for $P_1,P_2\in U_\Theta^{>0}$.

In order to obtain a notion that is invariant under the $\mathsf{PO}(p,q)$-action by conjugation it is necessary to consider as Θ-positive also negatively oriented triples: already in the case of $\mathsf{PO}(1,2)=\mathrm{Isom}(\mathbb{H}^2)$ both positively and negatively oriented triples in the circle belong to the same $\mathsf{PO}(1,2)$-orbit. In general, the following holds.

Proposition 3.5 ([23, Proposition 10.15 (3)]). *If $(x_1,x_2,x_3)\in\mathcal{F}_{p-1}^3$ is Θ-positive, then for any permutation σ the triple $(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})$ is positive.*

In order to choose a coherent orientation for various triples one uses positivity of n-tuples for $n\geq 4$.

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[8] In order to match this with the discussion in [22, Definition 6.5] it is useful to note that the Coxeter number of the root system of type B_{p-1}, $p\geq 3$, is equal to $2(p-1)$ [5, pp. 150–151]. Furthermore, if $W_{B_{p-1}}$ denotes the Weyl group associated to the root system B_{p-1}, we choose the standard generating system S of $W_{B_{p-1}}$ given by $S=\{s_1,\dots,s_{p-1}\}$, where s_{p-1} corresponds to the reflection along the only short root in a set of simple roots. Then the longest element w_0 of $W_{B_{p-1}}$ can be expressed as $w_0=(ab)^{p-1}$, where a is the product of all the elements of S with odd index and b is the product of all elements of S with even index, and the latter is a reduced expression [5, pp.150–151] (see also [15, Lemma 4.3]); such reduced expression is different from the one chosen in [23, Appendix B], but it is proven in [23, Theorem 8.1 (8)] that the positive semigroup does not depend on the choice of the reduced expression.

Proposition 3.6 ([23, Lemma 10.24]). *For every n, any Θ-positive n-tuple $(x_1,\dots,x_n) \in \mathcal{F}_{p-1}^n$, and any permutation σ in the n-th dihedral group, the triple $(x_{\sigma(1)},\dots,x_{\sigma(n)})$ is positive.*

We conclude this subsection with another useful characterization of positive triples proven by Guichard–Wienhard. Given a flag $A \in \mathcal{F}_{p-1}$ we denote by Ω_A the set of flags transverse to A, that is, the set of flags $F \in \mathcal{F}_{p-1}$ such that for every $1 \le k \le p-1$, the sum $A^k + (F^k)^\perp$ is direct.

Theorem 3.7 ([23, Theorem 9.2]). *The set*

$$\{F \in \mathcal{F}_{p-1} \mid F = P\mathsf{X},\ P \in U_\Theta^{>0}\}$$

is a connected component of $\Omega_\mathsf{X} \cap \Omega_\mathsf{Z}$. Thus, if A, B are transverse flags, the set

$$\{F \in \mathcal{F}_{p-1} \mid (A,F,B) \text{ is } \Theta\text{-positive}\}$$

is a union of connected components of $\Omega_A \cap \Omega_B$.

We record the following useful corollary.

Corollary 3.8. *Let $c : \mathbb{R} \supset [0,a] \to \mathcal{F}_{p-1}^n(\mathbb{R}^{p,q})$, $c(t) = (x_1(t),\dots,x_n(t))$, be a continuous path such that $x_i(t) \pitchfork x_j(t)$ for $i \neq j$ and all $t \in [0,a]$. If $c(0)$ is a Θ-positive n-tuple, then $c(t)$ is a Θ-positive n-tuple for all $t \in [0,a]$.*

Proof. Let $g \in \mathsf{PO}(p,q)$ with $gc(0) = (\mathsf{X}, P_2\mathsf{X},\dots,P_2\cdots P_{n-1}\mathsf{X},\mathsf{Z})$. Since the identity component $\mathsf{PO}_\circ(p,q)$ acts transitively on transverse pairs, we find a continuous curve $t \mapsto g_t \in \mathsf{PO}(p,q)$ such that $g_t x_1(t) = \mathsf{X}$, $g_t x_n(t) = \mathsf{Z}$ and $g_0 = g$. Thus for all $t \in [0,a]$, the point $g_t x_2(t)$ is in the same connected component of $\Omega_\mathsf{X} \cap \Omega_\mathsf{Z}$ as $P_2\mathsf{X}$. Hence we find a continuous map $t \mapsto P_2(t) \in U_\Theta^{>0}$ such that $g_t x_2(t) = P_2(t)\mathsf{X}$. By the same reasoning we find a continuous map $t \mapsto P_3(t) \in U_\Theta^{>0}$ such that

$$(P_2(t)^{-1} g_t x_2(t), P_2(t)^{-1} g_t x_3(t), g_t x_n(t)) = (\mathsf{X}, P_3(t)\mathsf{X}, \mathsf{Z}),$$

i.e.

$$(g_t x_1(t), g_t x_2(t), g_t x_3(t), g_t x_n(t)) = (\mathsf{X}, P_2(t)\mathsf{X}, P_2(t)P_3(t)\mathsf{X}, \mathsf{Z}).$$

The claim follows by induction. ■

3.3. Θ-Positive triples in $\mathsf{PO}(2,q)$

In the case $p = 2$ the flag manifold $\mathcal{F}_1 = \mathrm{Iso}_1(\mathbb{R}^{2,q})$ is also known as the Einstein universe (see for instance [2] for an introduction to this space). In this case a transverse pair $(x,z) \in \mathrm{Iso}_1(\mathbb{R}^{2,q})$ determines a positive cone $C_{x,z}^+ \subset \mathrm{Iso}_1(\mathbb{R}^{2,q})$; this is the unique connected component of the set of points transverse to both x and z whose intersection with a small neighbourhood of x is in the future of x and similarly it is locally in the past of z. The pair (x,z) also determines a negative cone $C_{x,z}^- = C_{z,x}^+ \subset \mathrm{Iso}_1(\mathbb{R}^{2,q})$. Note that the choice of sign depends on the time orientation of the Einstein universe, and is thus only invariant by an index 2 subgroup of $\mathsf{PO}(2,q)$, while there are elements in $\mathsf{PO}(2,q)$ exchanging $C^+_{x,z}$ and $C^-_{x,z}$. A triple (x,y,z) is Θ-positive if y belongs either to the positive or to the negative cone. If a 4-tuple (x,y,z,w) is positive then necessarily not only y and z belong to the same of the two cones determined by x,w, say $C_{x,w}(z)$, but also $C_{y,w}(z)\subset C_{x,w}(z)$.

We will not need the interpretation in terms of the Einstein universe, but we need several geometric properties of the positive semigroup $U_\Theta^{>0}$ and of positive n-tuples in $\mathrm{Iso}_1(\mathbb{R}^{2,q})$ which we collect in this section.

Example 3.9 (Case $p=2$). Any element $P\in U_{\Theta_2}^{>0}$ can be written as

$$P(\mathsf{s})=E_1(\mathsf{s})=\begin{pmatrix}1 & \mathsf{s}^t & q_J(\mathsf{s})\\ 0 & \mathrm{Id}_q & J\mathsf{s}\\ 0 & 0 & 1\end{pmatrix}. \tag{3.6}$$

for some $\mathsf{s}\in V_{\Theta_2}=c_J(V_J)$.

For the sake of readability we denote by e_i also the line generated by the i-th basis vector. Any element $x\in\mathcal{F}_1(\mathbb{R}^{2,q})=\mathrm{Iso}_1(\mathbb{R}^{2,q})$ transverse to e_1 has a unique representative of the form $(q_J(\mathsf{s}_x),\mathsf{s}_x,1)$. The following elementary lemma will be useful in the proof of Proposition 5.3.

Lemma 3.10. *The* 4*-tuple* $(e_{q+2},x,y,e_1)\in\mathcal{F}_1^4(\mathbb{R}^{2,q})$ *is equal to a standard* Θ*-positive* 4*-tuple* $(\mathsf{X},P(\mathsf{s}_1)\mathsf{X},P(\mathsf{s}_1)P(\mathsf{s}_2)\mathsf{X},\mathsf{Z})$ *for some* $\mathsf{s}_1,\mathsf{s}_2\in U_{\Theta_2}^{>0}$ *if and only if both* s_x *and* $\mathsf{s}_y-\mathsf{s}_x$ *are in* $c_J(V_J)$*, i.e. are positive for* q_J *and have positive first entry.*

Proof. Observe that $e_{q+2}=\mathsf{X}$ and $e_1=\mathsf{Z}$. Moreover, $x=E_1(J\mathsf{s}_x)\mathsf{X}$ and $y=E_1(J\mathsf{s}_y)\mathsf{X}$ $=E_1(J\mathsf{s}_x)E_1(J\mathsf{s}_y-J\mathsf{s}_x)\mathsf{X}$. Setting $\mathsf{s}_1=J\mathsf{s}_x$ and $\mathsf{s}_2=J(\mathsf{s}_y-\mathsf{s}_x)$, the claim follows from the fact that J preserves $c_J(V_J)$. ■

As a result we obtain the following compatibility of cross ratio and positivity, which will be useful later.

Proposition 3.11. *If a* 4*-tuple* $(a,b,c,d)\in\mathrm{Iso}_1^4(\mathbb{R}^{2,q})$ *is positive, then*

$$\mathrm{cr}_1(a,b,c,d)>1.$$

Proof. Since both notions are invariant by the $\mathsf{PO}(2,q)$-action, we can assume that $(a,b,c,d)=(e_{q+2},x,y,e_1)$ is a standard Θ-positive 4-tuple. A direct computation gives

$$\mathrm{cr}_1(e_{q+2},x,y,e_1)=\frac{q_J(\mathsf{s}_y)}{q_J(\mathsf{s}_x)}=\frac{q_J(\mathsf{s}_x)+q_J(\mathsf{s}_y-\mathsf{s}_x)+2b_J(\mathsf{s}_x,\mathsf{s}_y-\mathsf{s}_x)}{q_J(\mathsf{s}_x)}.$$

The result follows from the definition of $c_J(V_J)$, which ensures that $q_J(\mathsf{s}_y-\mathsf{s}_x)$ is positive, and from Lemma 3.1 which ensures that $2b_J(\mathsf{s}_x,\mathsf{s}_y-\mathsf{s}_x)$ is positive. ■

Given $g\in\mathsf{PO}(2,q)$ we denote by g^t its transpose, that is, the element represented by the transposed matrix with respect to the standard basis. By our choice of quadratic form, $g^t\in\mathsf{PO}(2,q)$ as well. Observe that for every $P\in U_\Theta=\mathrm{Stab}(\mathsf{Z})$, $P^t\in\mathrm{Stab}(\mathsf{X})$.

Lemma 3.12. *For any $P \in U_\Theta^{>0}$ there is $Q \in U_\Theta^{>0}$ such that $P\mathsf{X} = Q^t\mathsf{Z}$ and $P^{-1}\mathsf{X} = (Q^t)^{-1}\mathsf{Z}$. Conversely, for any $Q \in U_\Theta^{>0}$ there is $P \in U_\Theta^{>0}$ such that $Q^t\mathsf{Z} = P\mathsf{X}$ and $(Q^t)^{-1}\mathsf{Z} = P^{-1}\mathsf{X}$.*

Proof. This is a direct computation. For example, if $P = P(v)$ for some $v \in c_J(V_J)$ then $Q = P(q_J(v)^{-1}Jv)$. Since $P^{-1} = P(-v)$, we also get $P^{-1}\mathsf{X} = P(-q_J(v)^{-1}Jv)^t\mathsf{Z}$, as desired. ∎

Lemma 3.13. *For every $Q \in U_\Theta^{>0}$, Q^t acts trivially on $T_\mathsf{X}\mathrm{Iso}_1(\mathbb{R}^{2,q})$.*

Proof. We identify $T_\mathsf{X}\mathrm{Iso}_1(\mathbb{R}^{2,q})$ with V_J via the inverse of the map $v \mapsto \frac{d}{ds}\big|_{s=0}P(sv)\mathsf{X}$. We compute

$$P(w)^tP(sv)\mathsf{X} = \begin{pmatrix} \frac{s^2q_J(v)}{s^2q_J(v)q_J(w)+s(v^tw)+1} \\ \frac{s^2q_J(v)w+sJv}{s^2q_J(v)q_J(w)+s(v^tw)+1} \\ 1 \end{pmatrix}.$$

Taking $\frac{d}{ds}\big|_{s=0}$ gives the desired result. ∎

3.4. Θ-Positive representations and Θ-positive curves

Definition 3.14 ([22, Definition 5.3]). A map $\xi : \mathbb{S}^1 \to \mathcal{F}_{p-1}$ is Θ-*positive* if, for every positively oriented n-tuple $(x_1, \dots, x_n) \in (\mathbb{S}^1)^n$, the n-tuple $(\xi(x_1), \dots, \xi(x_n)) \in \mathcal{F}_{p-1}^n$ is Θ-positive.

The following is an immediate consequence of Corollary 3.8 and the fact that the set of positively oriented n-tuples in $\mathbb{S}^1$ is connected.

Corollary 3.15. *Let $\xi : \mathbb{S}^1 \to \mathcal{F}_{p-1}$ be a continuous transverse curve. If the image of one positively oriented n-tuple in $\mathbb{S}^1$ under ξ is a Θ-positive n-tuple in $\mathcal{F}_{p-1}$, then every positively oriented n-tuple in $\mathbb{S}^1$ is mapped to a Θ-positive n-tuple in $\mathcal{F}_{p-1}$.*

In particular, such a map ξ is Θ-positive if and only if for every $n \geq 3$ there is one positively oriented n-tuple in $\mathbb{S}^1$ which is mapped to a Θ-positive n-tuple in $\mathcal{F}_{p-1}$.

Definition 3.16 (see [22, Definition 5.3]). A representation $\rho : \Gamma \to \mathsf{PO}(p,q)$ is Θ-*positive Anosov* if it is Θ-Anosov and the Anosov boundary map $\xi : \partial_\infty\Gamma \to \mathcal{F}_{p-1}$ is a Θ-positive map. We will sometimes write just *positive Anosov representation.*[9]

This is a conjugation invariant notion: the conjugate of a Θ-Anosov representation is Θ-Anosov, and the conjugate of a Θ-positive representation is Θ-positive.

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[9] In the original definition of Θ-positive representations there is no requirement on the representation being Anosov. In the work of Guichard–Labourie–Wienhard [19, Proposition 5.8], which appeared on the arXiv at the same time as this paper, the authors show that any Θ-positive representation is Θ-positive Anosov. It is not immediate that a representation that is Θ-positive and Θ-Anosov is Θ-positive Anosov according to our definition, as, a priori, the Θ-positive boundary map and the Θ-Anosov boundary map might be different: the Θ-positive boundary map might not, a priori, be dynamics preserving.

Important examples of Θ-positive Anosov representations are representations in Fuchsian loci, namely the subloci of the set of Θ-positive Anosov representations arising through the following construction:

Example 3.17 (see [1, Section 7]). Let $\mathbb{R}^{p,p-1} \oplus \mathbb{R}^{q-p+1} = \mathbb{R}^{p,q}$ be an orthogonal splitting of $\mathbb{R}^{p,q}$, denote by $\mathsf{O}(p,p-1) \times \mathsf{O}(q-p+1) \subset \mathsf{O}(p,q)$ the subgroup preserving this splitting. For every irreducible representation $\tau : \mathsf{SL}(2,\mathbb{R}) \to \mathsf{SO}(p,p-1)$, every discrete and faithful representation $\iota : \Gamma \to \mathsf{SL}(2,\mathbb{R})$ and any representation $\alpha : \Gamma \to \mathsf{O}(q-p+1)$, the projectivization

$$\rho : \Gamma \to \mathsf{PO}(p,q), \quad \rho := (\tau \circ \iota) \oplus \alpha,$$

is Θ-positive Anosov. More generally, $\tau \circ \iota$ can be replaced by any Hitchin representation $\eta : \Gamma \to \mathsf{SO}(p,p-1)$.

Combining Theorem C and the classification of connected components of the $\mathsf{SO}(p,q)$-character variety given in [1] we will be able to deduce that if $q \neq p+1$, then every Θ-positive Anosov representation into $\mathsf{SO}(p,q)$ can be deformed to such a representation (Remark 6.7) – see Remark 1.2 for Θ-positive representations in $\mathsf{O}(p,q)$ and $\mathsf{SO}(p,q)$.

3.5. The boundary map of a Θ-positive representation

The Anosov boundary map of a Θ-positive Anosov representation has remarkable additional properties, which were proven by Sambarino, Wienhard and the second author and will be useful in our work as well.

We first consider the k-boundary maps for $k < p-1$. The following was shown in [38, proof of Theorem 10.3]. We include the simple argument here for the reader's convenience; this step of the proof does not require the assumption that the Θ-positive curve is equivariant with respect to some representation.

Let $z \in \mathcal{F}_{p-1}$ and $k < p-1$. We set

$$\mathrm{Iso}_k^{\pitchfork z} := \left\{ y^k \in \mathrm{Iso}_k(\mathbb{R}^{p,q}) \;\middle|\; \dim((y^k)^\perp \cap z^{k+1}) = 1,\ (y^k)^\perp \cap z^{k-1} = \{0\} \right\}.$$

We then have a well defined projection

$$\pi_z^k : \mathrm{Iso}_k^{\pitchfork z} \to \mathbb{P}(z^{k+1}/z^{k-1}) \simeq \mathbb{RP}^1, \quad y^k \mapsto [((y^k)^\perp \cap z^{k+1}) + z^{k-1}].$$

This yields a well defined projection $\pi_z^k(x)$ for any $x \in \mathcal{F}_{p-1}$ transverse to $z \in \mathcal{F}_{p-1}$ by restricting to x^k.

Proposition 3.18 (see [38, Theorem 10.3]). *For every Θ-positive $(n+1)$-tuple $(x_1, \ldots, x_n, z) \in \mathcal{F}_{p-1}^{n+1}$, and for every $k < p-1$, the $(n+1)$-tuple*

$$(\pi_z^k(x_1), \ldots, \pi_z^k(x_n), [z^k])$$

is cyclically ordered in $\mathbb{P}(z^{k+1}/z^{k-1}) \simeq \mathbb{RP}^1$.

Proof. This is a consequence of the explicit expression for a standard Θ-positive triple recalled in Section 3.1: we can assume without loss of generality that $(x_1, \dots, x_n, z) = (\mathsf{X}, P(\vec{v}_2)\mathsf{X}, \dots, P(\vec{v}_2)\cdots P(\vec{v}_n)\mathsf{X}, \mathsf{Z})$ for Θ-positive elements $P(\vec{v}_i)$, $\vec{v}_i \in V_\Theta^{p-1}$, for $i = 2, \dots, n$. Then, since $P(\vec{v})$ is an element of the stabilizer of Z, we get

$$((P(\vec{v})\mathsf{X})^{q+p-k} \cap \mathsf{Z}^{k+1}) + \mathsf{Z}^{k-1} = P(\vec{v})(\mathsf{X}^{q+p-k} \cap \mathsf{Z}^{k+1}) + \mathsf{Z}^{k-1}.$$

In particular, it is enough to verify that the k-th coordinate of the vector $P(\vec{v}_2)e_{q+p-k}$ is positive and that the k-th coordinate of $P(\vec{v}_2)\cdots P(\vec{v}_i)\mathsf{X}$ is greater than the one of $P(\vec{v}_2)\cdots P(\vec{v}_{i+1})\mathsf{X}$. Recalling the definitions one directly computes that the k-th coordinate of the vector $P(\vec{v})e_{q+p-k}$ is $\sum_{i=1}^{p-1} s_k^i > 0$, where we set $\vec{v} = (v_1, \dots, v_{p-1})$ and $v_i = (s_1^i, \dots, s_{p-1}^i)$ and the k-th coordinate of $P(\vec{v})P(\vec{w})e_{q+p-k}$ is the sum of the k-th coordinate of $P(\vec{v})e_{q+p-k}$ and the one of $P(\vec{w})e_{q+p-k}$. Inductively the claim follows. ■

As a consequence, we recover the following result.

Corollary 3.19 ([38, Theorem 10.3]). *Let $\xi : \mathbb{S}^1 \to \mathcal{F}_{p-1}$ be a Θ-positive map. Then ξ satisfies property H_k for $k < p-1$, i.e. the transversalities of* (2.2) *are satisfied.*

Proof. This is a direct consequence of Lemma 3.18 and (2.3). ■

For every $z \in \mathcal{F}_{p-1}$, the quotient z^{q+2}/z^{p-2} is naturally endowed with a bilinear form of signature $(2, q-p+2)$. Our next goal is to define a natural projection from a big subset of $\mathrm{Iso}_{p-1}(\mathbb{R}^{p,q})$ to $\mathrm{Iso}_1(z^{q+2}/z^{p-2})$. Observe first that z^{p-1} induces a point $[z^{p-1}] \in \mathrm{Iso}_1(z^{q+2}/z^{p-2})$. We denote by $\mathrm{Iso}_{p-1}^{\pitchfork z}$ the subset of $\mathrm{Iso}_{p-1}(\mathbb{R}^{p,q})$ consisting of subspaces in general position with respect to z^{p-2}:

$$\mathrm{Iso}_{p-1}^{\pitchfork z} := \left\{ y^{p-1} \in \mathrm{Iso}_{p-1}(\mathbb{R}^{p,q}) \;\middle|\; \dim(y^{p-1} \cap z^{q+2}) = 1,\ y^{p-1} \cap z^{p-2} = \{0\} \right\}.$$

We then have a well defined projection

$$\pi_z^{p-1} : \mathrm{Iso}_{p-1}^{\pitchfork z} \to \mathrm{Iso}_1(z^{q+2}/z^{p-2}), \quad y^{p-1} \mapsto [(y^{p-1} \cap z^{q+2}) + z^{p-2}].$$

For the projection π_z we have:

Proposition 3.20 (see [38, Proposition 10.5]). *If the $(n+1)$-tuple $(x_1, \dots, x_n, z) \in \mathcal{F}_{p-1}^{n+1}$ is Θ-positive, then*

$$(\pi_z^{p-1}(x_1), \dots, \pi_z^{p-1}(x_n), [z^{p-1}]) \in \mathrm{Iso}_1(z^{q+2}/z^{p-2})^{n+1}$$

is Θ-positive.

Proof. It follows from (3.5) that we can assume, up to conjugating with a suitable $g \in \mathsf{PO}(p,q)$, that there exist $\vec{v}_2, \dots, \vec{v}_n \in V_\Theta^{p-1}$ such that

$$(x_1, \dots, x_n, z) = (\mathsf{X}, P(\vec{v}_2)\mathsf{X}, \dots, P(\vec{v}_2)\cdots P(\vec{v}_n)\mathsf{X}, \mathsf{Z}).$$

For $1 \leq k \leq p-2$ the elementary matrices $E_k(s)$ act as the identity on $\mathsf{Z}^{q+2}/\mathsf{Z}^{p-2}$, while the elementary matrix $E_{p-1}(\mathsf{s})$ acts by the corresponding Θ-positive element (as in Example 3.9), which we still denote, with a slight abuse of notation, by $E_{p-1}(\mathsf{s})$. Given $\vec{v} \in V_\Theta^{p-1}$ we denote $\vec{v}^{p-1} = \sum_{i=1}^{p-1} \mathsf{s}_{p-1}^i \in c_J(V_J)$. It then follows from the definition that the $(n+1)$-tuple $(\pi_z^{p-1}(x_1), \dots, \pi_z^{p-1}(x_n), [z^{p-1}])$ is given by

$$([e_{q+2}], P(\vec{v}_2^{p-1})[e_{q+2}], \dots, P(\vec{v}_2^{p-1}) \cdots P(\vec{v}_n^{p-1})[e_{q+2}], [e_{p-1}]),$$

which concludes the proof. ∎

Let now $\rho : \Gamma \to \mathsf{PO}(p,q)$ be $\{p-2, p-1\}$-Anosov. For every $x \in \partial_\infty \Gamma$ we can use the projection π_{x_ρ} to define a curve $\xi_x : \partial_\infty \Gamma \to \mathrm{Iso}_1(x_\rho^{q+2}/x_\rho^{p-2})$ by

$$\begin{cases} \xi_x(y) := \pi_{x_\rho}^{p-1}(y_\rho^{p-1}), \\ \xi_x(x) := [x_\rho^{p-1}]. \end{cases} \tag{3.7}$$

The following is an immediate consequence of Proposition 3.20.

Corollary 3.21 ([38, Proposition 10.5]). *Let $\rho : \Gamma \to \mathsf{PO}(p,q)$ be Θ-positive Anosov with boundary map $\xi : \partial_\infty \Gamma \to \mathcal{F}_{p-1}$. Then $\xi_z : \partial_\infty \Gamma \to \mathrm{Iso}_1(z_\rho^{q+2}/z_\rho^{p-2})$ is a Θ-positive curve for every $z \in \partial_\infty \Gamma$.*

The following regularity property of the map ξ was obtained in [38] as a consequence of Corollaries 3.19 and 3.21. We record it here for future reference.

Theorem 3.22 ([38, Corollary 10.4 and Proposition 10.5]). *Let $\rho : \Gamma \to \mathsf{PO}(p,q)$ be Θ-*positive Anosov. *Then the curve $\xi^{p-1} : \partial_\infty \Gamma \to \mathrm{Iso}_{p-1}(\mathbb{R}^{p,q})$ has Lipschitz image, and the curves $\xi^k : \partial_\infty \Gamma \to \mathrm{Iso}_k(\mathbb{R}^{p,q})$ for $k \leq p-2$ have C^1 image.*

In particular, the image of ξ is a Lipschitz submanifold of $\mathcal{F}_{p-1}$, since it is the graph of monotone functions between Lipschitz circles.

4. Θ-Positive representations are positively ratioed

In this section we describe tangent cones defined by Θ-positive triples in $\mathcal{F}_{p-1}$ that naturally come from the Θ-positive structure. The main technical result of this section is that the cross ratio increases along these tangent cones (Proposition 4.8). This implies that Θ-positive Anosov representations into $\mathsf{PO}(p,q)$ are k-positively ratioed for $k = 1, \dots, p-1$ (Theorem 4.9). Proposition 4.8 will also be crucially used in the proof of the collar lemma.

The proof of Proposition 4.8 will build upon the Lipschitz regularity of the image of the boundary maps (Theorem 3.22). As a result, for every k, we can (and will) choose a Lipschitz structure on $\partial_\infty \Gamma \simeq \mathbb{S}^1$ defining a Lebesgue measure class with the property that for almost every point $x \in \partial_\infty \Gamma$, the curve ξ^k has a non-zero derivative $\dot{\xi}^k(x) \in T_{\xi^k(x)} \mathrm{Iso}_k$. The Lipschitz structures on $\partial_\infty \Gamma$ induced by the boundary maps ξ^k are, except for very special cases, mutually singular.

In our analysis we will need a more precise understanding of where these derivatives lie; this is provided in the next subsection. In Section 4.2 we will use this information to show that the derivative of the cross ratio is positive along explicit paths that we construct with the given derivative. Thanks to Rademacher's theorem, this is enough to deduce the main result of this section (Theorem 4.9).

4.1. *Tangent cones*

The goal of this subsection is to define, for every Θ-positive triple $(x, y, z) \in \mathcal{F}^3_{p-1}$ and every $1 \leq k \leq p-1$, a cone $c^+_k(x, y, z)$ in a linear subspace of $T_{x^k}\mathrm{Iso}_k(\mathbb{R}^{p,q})$ with the property that, for each equivariant Θ-positive curve ξ containing (x, y, z) in its image and differentiable at x, one has $\dot{\xi}^k(x) \in c^+_k(x, y, z)$. We will also show that if x, y, z belong to the image of a positive curve $\xi : \partial_\infty\Gamma \to \mathcal{F}_{p-1}$, the cone $c^+_k(x, y, z)$ only depends on the point x, and on the orientation of $\partial_\infty\Gamma$.

We focus first on *the case* $p = 2$, so that $\mathcal{F}_1(\mathbb{R}^{2,q}) = \mathrm{Iso}_1(\mathbb{R}^{2,q})$. Given a Θ-positive triple (x, y, z), we choose $g \in \mathsf{PO}(2, q)$ such that $(gx, gy, gz) = (\mathsf{X}, P\mathsf{X}, \mathsf{Z})$ for some $P \in U^{>0}_\Theta$. The unipotent radical U_Θ of the stabilizer of Z acts freely and transitively on the open subset of $\mathcal{F}_{p-1}$ consisting of elements transverse to Z; through this action we obtain an open chart around the point X identified with U_Θ. Differentiating this chart and composing with the differential action of g^{-1} we obtain a linear isomorphism between the Lie algebra $\mathfrak{u}_\Theta$ of U_Θ and $T_{x^1}\mathrm{Iso}_1(\mathbb{R}^{2,q})$. Recall from Section 3 that $\mathfrak{u}_\Theta$ consists of matrices of the form

$$\left\{ \begin{pmatrix} 0 & \mathsf{w}^t & 0 \\ 0 & 0 & J\mathsf{w} \\ 0 & 0 & 0 \end{pmatrix} \middle| \, \mathsf{w} \in \mathbb{R}^q \right\},$$

and thus we obtain an isomorphism $f_g : T_{x^1}\mathrm{Iso}_1(\mathbb{R}^{2,q}) \to V_J$. Recalling (3.2), we define

$$c^+_1(x, y, z) := f_g^{-1}(\overline{c}_J(V_J)). \tag{4.1}$$

In the interpretation of $\mathrm{Iso}_1(\mathbb{R}^{2,q})$ as the Einstein universe from Section 3.3, the cone $c^+_1(x, y, z)$ is the tangent at x to the cone $C_{x,z}(y)$. The following lemma should be clear for the experts in Einstein geometry; we include a proof in our framework.

Lemma 4.1. (1) *The cone* $c^+_1(x, y, z)$ *does not depend on the element* g.

(2) *For every* $g \in \mathsf{PO}(2, q)$, $c^+_1(x, y, z) = g^{-1}_* c^+_1(gx, gy, gz)$.

(3) *If* $(x, y, z, w) \in \mathrm{Iso}_1(\mathbb{R}^{2,q})^{(4)}$ *is positive, then* $c^+_1(x, y, w) = c^+_1(x, z, w) = c^+_1(x, y, z)$. *Furthermore,* $c^+_1(x, y, z) = -c^+_1(x, z, y)$.

Proof. (1) Let g_1, g_2 be such that $(g_i x, g_i y, g_i z) = (\mathsf{X}, P_i\mathsf{X}, \mathsf{Z})$ some $P_i \in U^{>0}_\Theta$. Then $h = g_2 g_1^{-1}$ belongs to the stabilizer of (X, Z), and since $hP_1 = P_2 \in U^{>0}_\Theta$, h preserves the cone $\overline{c}_J(V_J)$. This implies that

$$f^{-1}_{g_1}(\overline{c}_J(V_J)) = f^{-1}_{g_2}(h^{-1}_*\overline{c}_J(V_J)) = f^{-1}_{g_2}(\overline{c}_J(V_J)).$$

(2) follows directly from (1).

(3) Pick $g \in \mathsf{PO}(2,q)$ with $(gx, gy, gw, gz) = (\mathsf{X}, P_2\mathsf{X}, P_2P_3\mathsf{X}, \mathsf{Z})$. Since $P_2P_3 \in U_\Theta^{>0}$, it follows by definition that

$$c_1^+(x,y,w) = f_g^{-1}(\overline{c}_J(V_J)) = c_1^+(x,z,w).$$

Next we show that $c_1^+(x,y,z) = -c_1^+(x,z,y)$. Since by (2) we have $c_1^+(x,y,z) = g_*^{-1}c_1^+(gx,gy,gz)$ for any $g \in \mathsf{PO}(2,q)$ with differential g_*, it is enough to show that $c_1^+(\mathsf{X}, P\mathsf{X}, \mathsf{Z}) = -c_1^+(\mathsf{X}, \mathsf{Z}, P\mathsf{X})$. Let

$$H = \begin{pmatrix} -1 & & \\ & \mathrm{Id}_q & \\ & & -1 \end{pmatrix} \in \mathsf{PO}(2,q).$$

Then H stabilizes X and Z and acts as $-\mathrm{Id}$ on $T_\mathsf{X}\mathrm{Iso}_1(\mathbb{R}^{2,q})$. Let $P = P(v)$ for some $v \in c_J(V_J)$. Then

$$HP(v)\mathsf{X} = HP(v)H\mathsf{X} = P(-v)\mathsf{X} = P^{-1}\mathsf{X}.$$

Thus $c_1^+(\mathsf{X}, P^{-1}\mathsf{X}, \mathsf{Z}) = -c_1^+(\mathsf{X}, P\mathsf{X}, \mathsf{Z})$. Let $Q \in U_\Theta^{>0}$ be such that $P\mathsf{X} = Q^t\mathsf{Z}$; it exists by Lemma 3.12. According to Lemma 3.12 we also have $(Q^t)^{-1}\mathsf{Z} = P^{-1}\mathsf{X}$. Thus

$$(\mathsf{X}, \mathsf{Z}, P\mathsf{X}) = (\mathsf{X}, \mathsf{Z}, Q^t\mathsf{Z}) = Q^t \cdot (\mathsf{X}, (Q^t)^{-1}\mathsf{Z}, \mathsf{Z}) = Q^t \cdot (\mathsf{X}, P^{-1}\mathsf{X}, \mathsf{Z}).$$

Therefore $c_1^+(\mathsf{X}, \mathsf{Z}, P\mathsf{X}) = Q_*^t c_1^+(\mathsf{X}, P^{-1}\mathsf{X}, \mathsf{Z})$. Since the differential Q_*^t acts trivially on $T_\mathsf{X}\mathrm{Iso}_1(\mathbb{R}^{2,q})$ by Lemma 3.13, Q_*^t preserves the cone $c_1^+(\mathsf{X}, P^{-1}\mathsf{X}, \mathsf{Z}) = -c_1^+(\mathsf{X}, P\mathsf{X}, \mathsf{Z})$. This proves the claim.

Finally, we show $c_1^+(x,y,z) = c_1^+(x,y,w)$. For this recall that by Proposition 3.6, (x,w,z,y) is also a positive quadruple. Thus by what we have shown so far,

$$c_1^+(x,y,z) = -c_1^+(x,z,y) = -c_1^+(x,w,y) = c_1^+(x,y,w). \qquad \blacksquare$$

In particular, if $x = \xi(a)$ is in the image of a Θ-positive curve $\xi : \mathbb{S}^1 \to \mathrm{Iso}_1(\mathbb{R}^{2,q})$, then the cone $c_1^+(x, \xi(b), \xi(c))$ does not depend on b, c, but only on the orientation induced on the circle by the triple (a,b,c). For this reason, in this situation, we will simply write

$$c_1^+(x) := c_1^+(x, \xi(b), \xi(c)) \quad \text{for any } b, c \text{ inducing the given orientation.}$$

Lemma 4.2. *Let $\xi : \mathbb{S}^1 \to \mathrm{Iso}_1(\mathbb{R}^{2,q})$ be Θ-positive. If ξ is differentiable at $x \in \mathbb{S}^1$ with non-zero derivative, then*

$$\dot{\xi}(x) \in c_1^+(x).$$

Thus if ξ is the boundary map of a Θ-positive representation, then $\dot{\xi}(x) \in c_1^+(x)$ for almost all $x \in \partial_\infty\Gamma \simeq \mathbb{S}^1$.

Proof. Let x be a differentiability point for ξ. We complete x to a positively oriented triple $(x,y,z) \in (\mathbb{S}^1)^3$, and choose $g \in \mathsf{PO}(2,q)$ so that $(g\xi(x), g\xi(y), g\xi(z)) = (\mathsf{X}, P\mathsf{X}, \mathsf{Z})$.

In this way we obtain a smooth chart

$$V_J \to U_x \subset \mathrm{Iso}_1(\mathbb{R}^{2,q}), \quad \mathsf{w} \mapsto g^{-1}\exp(\mathsf{w}) \cdot \mathsf{X}.$$

Working in this smooth chart, the derivative of ξ, where defined, is the limit of rescaled partial increments. Since each of those belongs to $c_J(V_J)$ (compare Example 3.9), the limit belongs to the closure $\overline{c}_J(V_J)$. It follows from Theorem 3.22 that almost every point $x \in \partial_\infty \Gamma$ is a differentiability point for ξ. ■

We now turn to *the general situation*. Recall that for any $x \in \mathcal{F}_{p-1}$, the quotient x^{q+2}/x^{p-2} is naturally endowed with a form of signature $(2, q-p+2)$, and there is a natural inclusion

$$\phi_x : \mathrm{Iso}_1(x^{q+2}/x^{p-2}) \to \mathrm{Iso}_{p-1}(\mathbb{R}^{p,q})$$

whose image is the analytic submanifold consisting of $(p-1)$-dimensional isotropic subspaces contained in x^{q+2} and containing x^{p-2} as a subspace. In order to improve readability we will denote by $\mathrm{Iso}_{p-1}(x^{q+2}/x^{p-2})$ the image of ϕ_x.

Realizing $\mathrm{Iso}_{p-1}(\mathbb{R}^{p,q})$ as a subset of the $(p-1)$-Grassmannian of $\mathbb{R}^{p+q}$, we obtain, for any z^{q+1} transverse to x^{p-1}, an inclusion

$$T_{x^{p-1}}\mathrm{Iso}_{p-1}(\mathbb{R}^{p,q}) < \mathrm{Hom}(x^{p-1}, z^{q+1}).$$

The tangent space to $\phi_x(\mathrm{Iso}_1(x^{q+2}/x^{p-2}))$ is then precisely given by those linear maps in $T_{x^{p-1}}\mathrm{Iso}_{p-1}(\mathbb{R}^{p,q})$ that vanish on x^{p-2} and have image in x^{q+2}:

$$T_{x^{p-1}}\mathrm{Iso}_{p-1}(x^{q+2}/x^{p-2}) := \left\{ \Phi \in T_{x^{p-1}}\mathrm{Iso}_{p-1}(\mathbb{R}^{p,q}) \;\middle|\; \begin{array}{l} \Phi(x^{p-2}) = 0, \\ \Phi(x^{p-1}) < x^{q+2} \cap z^{q+1} \end{array} \right\}.$$

Let now $(x, y, z) \in \mathcal{F}_{p-1}^3$ be Θ-positive. Recall from Proposition 3.20 that we denote by $\pi_x^{p-1} : \mathrm{Iso}_{p-1}^{\pitchfork z} \to \mathrm{Iso}_1(x^{q+2}/x^{p-2})$ the natural projection, and that if $(x, y, z) \in \mathcal{F}_{p-1}^3$ is Θ-positive, then y, z belong to $\mathrm{Iso}_{p-1}^{\pitchfork z}$ and the triple $([x^{p-1}], \pi_x^{p-1}(y), \pi_x^{p-1}(z)) \in \mathrm{Iso}_1(x^{q+2}/x^{p-2})$ is Θ-positive. In the last statement we are also using the fact that any permutation of a Θ-positive triple is Θ-positive (Corollary 3.5).

Definition 4.3. For a Θ-positive triple $(x, y, z) \in \mathcal{F}_{p-1}^3$ we set

$$c_{p-1}^+(x, y, z) := d\phi_x(c_1^+([x^{p-1}], \pi_x^{p-1}(y), \pi_x^{p-1}(z))).$$

It is clear from the construction that for every Θ-positive triple $(x, y, z) \in \mathcal{F}_{p-1}^3$ and for any $g \in \mathsf{PO}(p, q)$,

$$c_{p-1}^+(gx, gy, gz) = g_* c_{p-1}^+(x, y, z),$$

where g_* denotes the induced action of g on $T_*\mathrm{Iso}_{p-1}(\mathbb{R}^{p,q})$.

Combining Lemma 4.1 and Proposition 3.20, which ensures that the image under π_x^{p-1} of a positive n-tuple is positive, we deduce that if $x = \xi(a)$ is in the image of a Θ-positive curve $\xi : \mathbb{S}^1 \to \mathcal{F}_{p-1}(\mathbb{R}^{p,q})$, then the cone $c_{p-1}^+(x, \xi(b), \xi(c))$ does not depend on b, c, but only on the orientation induced on the circle by the triple (a, b, c). As in the case $p = 2$, in this situation we will write

$$c^+_{p-1}(x) := c^+_{p-1}(x, \xi(b), \xi(c)) \quad \text{for any } b, c \text{ inducing the given orientation.}$$

While in the case $p = 2$ the cone $c^+_1(x)$ has full dimension, for $p \geq 3$ it is supported in a proper vector subspace.

In the following proposition we use the Θ-positive curves $\xi_x : \partial_\infty\Gamma \to \mathrm{Iso}_1(x^{q+2}/x^{p-2})$ associated to a Θ-positive curve $\xi : \partial_\infty\Gamma \to \mathcal{F}_{p-1}(\mathbb{R}^{p,q})$ through (3.7).

Proposition 4.4. *Let $\rho : \Gamma \to \mathsf{PO}(p,q)$ be Θ-positive Anosov. If the derivative $\dot\xi^{p-1}(x)$ of ξ at x exists, we have*

$$\dot\xi^{p-1}(x) \in c^+_{p-1}(x).$$

Proof. We divide the proof into two steps. First we show that if $\dot\xi^{p-1}(x)$ exists, then it is in $T_{x^{p-1}}\mathrm{Iso}_{p-1}(x^{q+2}/x^{p-2})$.

We denote by $\Psi \in \mathrm{Hom}(x^{p-1}, z^{q+1})$ the vector corresponding to $\dot\xi^{p-1}(x)$. Since ρ has property H_{p-2}, the curve ξ^{p-2} is C^1 and the tangent space $T_{x^{p-2}}\xi^{p-2}$ is generated by a vector $\Phi \in \mathrm{Hom}(x^{p-2}, z^{q+2})$ with $\mathrm{Im}\,\Phi < x^{p-1}$ (see Proposition 2.9). In turn this means that infinitesimal variations of vectors in x^{p-2} are contained in x^{p-1}, and thus $\ker\Psi = x^{p-2}$. Since x^{p-1} is isotropic, it follows that $\mathrm{Im}\,\Psi < (x^{p-2})^\perp = x^{q+2}$. As a result, $\dot\xi^{p-1}(x) \in T_{x^{p-1}}\mathrm{Iso}_{p-1}(x^{q+2}/x^{p-2})$.

We conclude by showing that then $\dot\xi^{p-1}(x)$ necessarily belongs to the cone. By definition, we have $c^+_{p-1}(x) \subset T_{x^{p-1}}\mathrm{Iso}_{p-1}(x^{q+2}/x^{p-2})$. To show that $\dot\xi^{p-1}(x) \in c^+_{p-1}(x)$ we can assume that, up to the action of $\mathsf{PO}(p,q)$, $(\xi(x), \xi(y), \xi(z)) = (\mathsf{X}, P\mathsf{X}, \mathsf{Z})$ is a standard Θ-positive triple. Then consider the projection

$$\pi_\mathsf{Z} : \mathrm{Iso}^{\pitchfork\mathsf{Z}}_{p-1} \to \mathrm{Iso}_1(\mathsf{Z}^{q+2}/\mathsf{Z}^{p-2}), \quad Y^{p-1} \mapsto [(Y^{p-1} \cap \mathsf{Z}^{q+2}) + \mathsf{Z}^{p-2}],$$

where, as always,

$$\mathrm{Iso}^{\pitchfork\mathsf{Z}}_{p-1} := \left\{Y^{p-1} \in \mathrm{Iso}_{p-1} \,\middle|\, \dim(Y^{p-1} \cap \mathsf{Z}^{q+2}) = 1,\ Y^{p-1} \cap \mathsf{Z}^{p-2} = \{0\}\right\}.$$

The set $\mathrm{Iso}_{p-1}(\mathsf{X}^{q+2}/\mathsf{X}^{p-2})$ is contained in the domain of definition of π_Z, and π_Z induces a diffeomorphism between $\mathrm{Iso}_{p-1}(\mathsf{X}^{q+2}/\mathsf{X}^{p-2})$ and $\mathrm{Iso}_1(\mathsf{Z}^{q+2}/\mathsf{Z}^{p-2})$, so that it is enough to check that

$$d\pi_\mathsf{Z}(\dot\xi^{p-1}(x)) \in d\pi_\mathsf{Z}(c^+_{p-1}(\mathsf{X})).$$

Observe that $\pi_\mathsf{Z} \circ \xi = \xi_z$ on $\partial_\infty\Gamma \setminus \{z\}$. Moreover, Proposition 3.20 guarantees that $(\xi_z(x), \xi_z(y), \xi_z(z))$ is a standard Θ-positive triple in $\mathrm{Iso}_1(\mathsf{Z}^{q+2}/\mathsf{Z}^{p-2})$. Since ξ_z is Θ-positive and differentiable at x (because ξ is differentiable at x by assumption), we deduce from Lemma 4.2 that

$$d\pi_\mathsf{Z}(\dot\xi^{p-1}(x)) \in c^+_1([\mathsf{X}]) = c^+_1(\xi_z(x)),$$

where $c^+_1([\mathsf{X}])$ is the cone defined by the standard Θ-positive triple in $\mathrm{Iso}_1(\mathsf{Z}^{q+2}/\mathsf{Z}^{p-2})$.

We claim that $d\pi_{\mathsf{Z}}(c^+_{p-1}(\mathsf{X})) = c^+_1(\xi_z(x))$. Indeed,

$$\exp(c^+_{p-1}(\mathsf{X})) = \{E_{p-1}(\mathsf{s}) \cdot \mathsf{X} \mid \mathsf{s} \in c_J(V_J)\},$$

where $E_{p-1}(\mathsf{s})$ is as in (3.4). Thus

$$\pi_{\mathsf{Z}}(\exp(c^+_{p-1}(\mathsf{X}))) = \{E^{\mathsf{Z}}_1(\mathsf{s}) \cdot \xi_z(x) \mid \mathsf{s} \in c_J(V_J)\},$$

where $E^{\mathsf{Z}}_1(\mathsf{s})$ is the (only) elementary matrix for the Θ-positive structure on $\mathsf{Z}^{q+2}/\mathsf{Z}^{p-2}$ (as in Example 3.9). Since

$$\{E^{\mathsf{Z}}_1(\mathsf{s}) \cdot \xi_z(x) \mid \mathsf{s} \in c_J(V_J)\} = \exp(c^+_1([\mathsf{X}])) = \exp(c^+_1(\xi_z(x)))$$

and exp is a diffeomorphism, the claim follows.

Thus, as desired,

$$d\pi_{\mathsf{Z}}(\dot{\xi}^{p-1}(x)) \in d\pi_{\mathsf{Z}}(c^+_{p-1}(\mathsf{X})). \qquad \blacksquare$$

The same analysis can be done for the other boundary maps $\xi^k : \partial_\infty\Gamma \to \mathrm{Iso}_k(\mathbb{R}^{p,q})$. In this case, since the Θ-positive curve has property H_k (Corollary 3.19), the curve ξ^k is already C^1, with tangent space at x given by the line (Proposition 2.9)

$$\dot{x}^k \in T_{x^k}[x^{k+1}/x^{k-1}] \subset T_{x^k}\mathrm{Iso}_k(\mathbb{R}^{p,q}).$$

In this case the cone $c^+_k(x)$ is the ray corresponding to the positive orientation of the circle (compare [3, p. 31]) induced by the choice of the triple. In particular, the following holds.

Proposition 4.5. *Let $\rho : \Gamma \to \mathsf{PO}(p,q)$ be Θ-positive Anosov. Then for all $k = 1, \ldots, p-2$ and all $x \in \partial_\infty\Gamma$,*

$$\dot{\xi}^k(x) \in c^+_k(x).$$

Remark 4.6 (Interpretations of the tangent cones in terms of positivity). A more Lie-theoretic way of defining the cones $c^+_k(x)$ is as follows. Given a Lie group G, a parabolic subgroup P_θ which is conjugate to its opposite and a transverse pair $x, z \in \mathsf{G}/\mathsf{P}_\theta$, the tangent space $T_x\mathsf{G}/\mathsf{P}_\theta$ is canonically identified with the Lie algebra $\mathfrak{u}_\theta$ of the unipotent radical U_θ of the stabilizer in G of the point z.

If now the group G admits a Θ-positive structure, then, by definition, there are L^0_Θ-invariant convex cones $c_\beta \subset \mathfrak{u}_\beta$, where $\mathfrak{u}_\beta \subset \mathfrak{u}_\Theta$ is the sum of all the root spaces $\mathfrak{g}_\alpha$ that are equal to β modulo the span of $\Delta \setminus \Theta$. In particular, for every $\beta \in \Theta$, $\mathfrak{u}_\beta \subset \mathfrak{u}_{\{\beta\}} = T_x\mathsf{G}/\mathsf{P}_{\{\beta\}}$, and we have $c^+_k = c_{\alpha_k}$.

4.2. The derivative of the cross ratio

We can now compute the variation of the cross ratio along paths tangent to the cones c^+_i introduced in the previous section, thus proving Proposition 4.8. We begin with an explicit computation in the case $p = 2$.

Lemma 4.7. *Let* $(x,y,z)\in \mathrm{Iso}_1^3(\mathbb{R}^{2,q})$ *be a* Θ*-positive triple and* $\Phi\in c_1^+(x)\subset T_x\mathrm{Iso}_1(\mathbb{R}^{2,q})$. *Then*

$$d_x\,\mathrm{cr}_1(z,x,\cdot,y)(\Phi)>0.$$

Proof. Up to the action of an element in $\mathsf{PO}(2,q)$ we can assume that $z=e_1$, $x=e_d$ and there exist $\mathsf{y}\in c_J(V_J)$ and $\mathsf{w}\in \overline{c}_J(V_J)$ such that

$$y=\left\langle\begin{pmatrix} q_J(\mathsf{y})\\ \mathsf{y}\\ 1\end{pmatrix}\right\rangle,\quad \Phi=\left.\frac{d}{dt}\right|_{t=0}x_t\quad \text{for } x_t:=\left\langle\begin{pmatrix} t^2q_J(\mathsf{w})\\ t\mathsf{w},\\ 1\end{pmatrix}\right\rangle.$$

As in the proof of Proposition 3.11, we have

$$\mathrm{cr}_1(z,x,x_t,y)=\frac{-q_J(\mathsf{y})}{-t^2q_J(\mathsf{w})+2tb_J(\mathsf{w},\mathsf{y})-q_J(\mathsf{y})},$$

and therefore

$$d_x\,\mathrm{cr}_1(z,x,\cdot,y)(\Phi)=\left.\frac{d}{dt}\right|_{t=0}\mathrm{cr}_1(z,x,x_t,y)=\frac{2b_J(\mathsf{w},\mathsf{y})}{q_J(\mathsf{y})}>0$$

by Lemma 3.1. ∎

For $k=p-1$ we reduce the general case to Lemma 4.7 using Proposition 3.20.

Proposition 4.8. *Let* $(x,y,z)\in\mathcal{F}_{p-1}^3$ *be a* Θ*-positive triple and let* $\Phi_k\in c_k^+(x)\subset T_{x^k}\mathrm{Iso}_k(\mathbb{R}^{p,q})$ *for* $k=1,\dots,p-1$. *Then*

$$d_{x^k}\,\mathrm{cr}_k(z^k,x^k,\cdot,y^k)(\Phi_k)>0.$$

Proof. As the cross ratio is invariant under the action of $\mathsf{PO}(p,q)$, we assume without loss of generality that $x=\mathsf{X}$, $z=\mathsf{Z}$, and $y=P\mathsf{X}$ for some $P\in U_\Theta^{>0}$. For $k=1,\dots,p-2$ and $t\in\mathbb{R}$ we consider the element

$$x_t^k:=x^{k-1}\oplus\langle e_{p+q-k+1}+te_{p+q-k}\rangle$$

of $\mathrm{Iso}_k(\mathbb{R}^{p,q})$, while for $k=p-1$ we consider the element

$$x_t^{p-1}:=x^{p-2}\oplus\langle e_{q+2}+t\mathsf{w}\rangle$$

of $\mathrm{Iso}_{p-1}(\mathbb{R}^{p,q})$ for some $\mathsf{w}\in\overline{c}_J(V_J)$ (recall that $V_J<\mathbb{R}^{p,q}$).

By construction, for $1\le k\le p-1$, the map $t\mapsto x_t^k$ gives a curve in $\mathrm{Iso}_k(\mathbb{R}^{p,q})$ whose derivative lies in the cone c_k^+, and for every $\Phi_k\in c_k^+$ we can find such a curve tangent to Φ_k.

Thus it is enough to check

$$\left.\frac{d}{dt}\right|_{t=0}\mathrm{cr}_k(z^k,x^k,x_t^k,y^k)>0.$$

Let $X_k := x^{k+1}/x^{k-1}$ for $k \leq p-2$ and $X_{p-1} := x^{q+2}/x^{p-2}$. We denote by $[\cdot]$ the obvious projection to $\mathbb{P}(X_k)$ (resp. $\mathrm{Iso}_1(X_{p-1})$), so that

$$[y] := \begin{cases} [y^{p+q-k} \cap x^{k+1}], & k \leq p-2, \\ [y^{p-1} \cap x^{q+2}], & k = p-1, \end{cases}$$

and the same for $[z]$. Then

$$\begin{aligned} \mathrm{cr}_k(z^k, x^k, x^k_t, y^k) &= \mathrm{cr}_k(x^k, z^k, y^k, x^k_t) = \mathrm{cr}_1^{X_k}([x], [z], [y], [x_t]) \\ &= \mathrm{cr}_1^{X_k}([z], [x], [x_t], [y]), \end{aligned}$$

where the first and last equalities follow from Proposition 2.4 (2), and the second is a consequence of the second statement in Proposition 2.6.

For $k \leq p-2$, property H_k (Corollary 3.19) guarantees that $[z], [x], [y]$ are distinct (see (2.3)); by definition $[\Phi_k] = \frac{d}{dt}\big|_{t=0}[x_t] \neq 0 \in T_{[x]}X_k$ is directed towards $[y]$, i.e. $[z], [x], [x_t], [y]$ are cyclically ordered on $\mathbb{P}(X_k) \simeq \mathbb{RP}^1$. As $\mathrm{cr}_1^{X_k}$ is the usual projective cross ratio on $\mathbb{P}(X_k)$, this is enough to guarantee the claim (compare [3, Lemma 3.2]).

In the case $k = p-1$, Lemma 4.7 guarantees that

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{cr}_1^{X_{p-1}}([z], [x], [x_t], [y]) > 0,$$

which proves the claim. ∎

Theorem 4.9. *Let* $\rho : \Gamma \to \mathsf{PO}(p, q)$ *be* Θ*-positive Anosov. Then* ρ *is* k*-positively ratioed for* $k = 1, \ldots, p-1$.

Proof. Let (x, y, z, w) be cyclically ordered on $\partial_\infty\Gamma$. Theorem 3.22 implies that, for all $1 \leq k \leq p-1$, the curve $u \mapsto \xi^k(u)$ has rectifiable image. Thus, for each k, we can parametrize the interval $(y, z) \subset \partial_\infty\Gamma$ not containing x, w by a map $t \mapsto y_t$ such that $t \mapsto \xi^k(y_t)$ is Lipschitz and $y_0 = y$, $y_1 = z$. Then also $f : [0, 1] \to \mathbb{R}$ defined by $f(t) := \mathrm{cr}_k(x^k, y^k, y^k_t, w^k)$ is Lipschitz and thus by Rademacher's theorem and the fundamental theorem of calculus we get

$$\begin{aligned} \mathrm{cr}_k(x^k, y^k, z^k, w^k) &= f(1) = f(0) + \int_0^1 f'(s)\, ds \\ &= \mathrm{cr}_k(x^k, y^k, y^k, w^k) + \int_0^1 \frac{d}{dt}\bigg|_{t=s} \mathrm{cr}_k(x^k, y^k, y^k_t, w^k)\, ds. \end{aligned}$$

We have $\mathrm{cr}_k(x^k, y^k, y^k, w^k) = 1$. Using the cocycle identity, we deduce that (almost everywhere)

$$\frac{d}{dt}\bigg|_{t=t_0} \mathrm{cr}_k(x^k, y^k, y^k_t, w^k) = \mathrm{cr}_k(x^k, y^k, y^k_{t_0}, w^k) \frac{d}{dt}\bigg|_{t=t_0} \mathrm{cr}_k(x^k, y^k_{t_0}, y^k_t, w^k). \tag{4.2}$$

From Proposition 4.8, whose assumptions are satisfied thanks to Proposition 4.4 (resp. Proposition 4.5), it follows that (almost everywhere)

$$\left.\frac{d}{dt}\right|_{t=t_0} \mathrm{cr}_k(x^k, y^k_{t_0}, y^k_t, w^k) > 0.$$

For every t_0 the coefficient $\mathrm{cr}_k(x^k, y^k, y^k_{t_0}, w^k)$ in (4.2) is positive since the 4-tuple $(x^k, y^k, y^k_{t_0}, w^k)$ is in the same connected component of the domain of definition of cr_k as the 4-tuple (x^k, y^k, y^k, w^k) for which $\mathrm{cr}_k(x^k, y^k, y^k, w^k) = 1$ and the cross ratio is continuous and does not vanish (recall Definition 2.3). Hence $f'(t) > 0$ almost everywhere and therefore $\mathrm{cr}_k(x^k, y^k, z^k, w^k) > 1$, as desired. ■

5. Collar lemmas for Θ-positive representations

The goal of this section is to prove the collar lemma, stated as Theorem B in the introduction. As a warm up, we give in Section 5.1 a complete proof in the case $p = 2$, which builds on the technical Lemma 5.3, a key ingredient also in the proof of the general case. In Section 5.2 we introduce the notion of hybrid flags, and discuss the general strategy of proof, which is carried out in the remaining three subsections.

5.1. *Collar lemma for* $\mathsf{PO}(2,q)$*-positive representations*

In this subsection we focus on the case $p = 2$; in this case the group $\mathsf{PO}(2,q)$ is of Hermitian type, and Θ-positive Anosov representations $\rho : \Gamma \to \mathsf{PO}(2,q)$ are nothing other than maximal representations as in [11,14]. Our aim is to give a self-contained proof of the collar lemma in this case, which is considerably simpler, but already illustrates the topological input needed for the general proof, and sheds some light on the general strategy. Recall from the introduction that we say that two elements $g, h \in \Gamma$ are *linked* if the attracting and repelling fixed points $h_+, h_- \in \partial_\infty\Gamma$ of h are in different connected components of $\partial_\infty\Gamma \setminus \{g_-, g_+\}$. Theorem B in this context can be restated as follows.

Theorem 5.1. *Let* $\rho : \Gamma \to \mathsf{PO}(2,q)$ *be* Θ*-positive Anosov and* $g, h \in \Gamma$ *be a linked pair. Then*

$$\left(1 - \left|\frac{\lambda_2}{\lambda_1}(\rho(h))\right|\right)^{-1} < \lambda_1^2(\rho(g)).$$

Remark 5.2. A (slightly weaker) collar lemma for maximal representations $\rho : \Gamma \to \mathrm{Sp}(2n, \mathbb{R})$ was proven in [12], but [12] does not cover maximal representations in Hermitian Lie groups different from $\mathrm{Sp}(2n, \mathbb{R})$, and our techniques of proof are different. Our techniques are also different from the technique of the proof of the collar lemma for Hitchin representations in [32], albeit all the three proofs build on the topological input from hyperbolic geometry giving the orientation of a well chosen 6-tuple of fixed points of hyperbolic elements recalled in Figure 3 below.

An important advantage of the projective cross ratio $\mathrm{cr}_{\mathbb{RP}^1}$ over the cross ratios cr_k, which is at the basis of an easy proof of the collar lemma for holonomies of hyperbolizations, is its additional symmetry:

$$\mathrm{cr}_{\mathbb{RP}^1}(d,a,b,c) = (1-\mathrm{cr}_{\mathbb{RP}^1}(a,b,c,d)^{-1})^{-1}.$$

While this does not always hold for cr_1, we establish the following generalization, which has the advantage of involving the exponential of the root (recall from Lemma 2.5 that $\mathrm{cr}_1(h_+,hx,x,h_-) = \lambda_1(h)^2$).

Lemma 5.3. *Let $h \in \mathsf{PO}(2,q)$ be such that $|\lambda_1(h)| > |\lambda_2(h)|$. Denote by $h_\pm \in \mathrm{Iso}_1(\mathbb{R}^{2,q})$ the eigenlines corresponding to the eigenvalues of highest and lowest absolute value. Then for every $x \in \mathrm{Iso}_1(\mathbb{R}^{2,q})$ such that (h_+,hx,x,h_-) is positive,*

$$\left(1-\left|\frac{\lambda_2}{\lambda_1}(h)\right|\right)^{-1} < \mathrm{cr}_1(h_-,h_+,hx,x).$$

Proof. Since the 4-tuple (h_+,hx,x,h_-) is positive, in particular the triple (h_+,x,h_-) is positive. Thus we can assume without loss of generality that, with respect to the standard basis, $h_+ = e_{q+2}$, $h_- = e_1$ and we can write, as in Example 3.9,

$$x = [q_J(\mathsf{s}_x) : \mathsf{s}_x : 1]$$

for some vector $\mathsf{s}_x \in c_J(V_J)$.

We choose the lift of h to $\mathsf{O}(2,q)$, which we also denote by h, such that $\lambda_1 := \lambda_1(h) > 0$. Since $h_+ = e_{q+2}$ and $h_- = e_1$, the element h acts on the subspace $V_J = \langle e_1, e_{q+2}\rangle^\perp$. We denote by $h_0 : V_J \to V_J$ the induced linear map, which preserves the form b_J. Then

$$hx = [\lambda_1^{-1} q_J(\mathsf{s}_x) : h_0\mathsf{s}_x : \lambda_1] = [\lambda_1^{-2} q_J(\mathsf{s}_x) : \lambda_1^{-1} h_0\mathsf{s}_x : 1].$$

Since the quadruple (e_{q+2},hx,x,e_1) is positive, in particular the triple (e_{q+2},hx,e_1) is positive, and thus $\lambda_1^{-1}h_0\mathsf{s}_x \in c_J(V_J)$.

Recall that $\tilde{x} \in \mathbb{R}^{2,q}$ denotes a non-trivial lift of $x \in \mathrm{Iso}_1(\mathbb{R}^{2,q})$. We can now explicitly compute[10]

$$\begin{aligned}\mathrm{cr}_1(h_-,h_+,hx,x) &= \frac{Q(\tilde{h}_-,\tilde{hx})Q(\tilde{h}_+,\tilde{x})}{Q(\tilde{h}_-,\tilde{h}_+)Q(\tilde{x},\tilde{hx})}\\ &= \frac{\lambda_1^2 q_J(\mathsf{s}_x)}{(\lambda_1^2+1)q_J(\mathsf{s}_x) - 2b_J(\mathsf{s}_x,\lambda_1 h_0\mathsf{s}_x)}.\end{aligned} \tag{5.1}$$

In order to conclude the proof we need to find a good lower bound on the value $2b_J(\mathsf{s}_x,\lambda_1h_0\mathsf{s}_x)$, which is positive by Lemma 3.1. Setting $v_x := \frac{1}{\sqrt{q_J(\mathsf{s}_x)}}\mathsf{s}_x$, we note that v_x and h_0v_x belong to $q_J^{-1}(1)$. Therefore v_x and h_0v_x are in the -1 level set of the

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[10] Recall from (3.1) that $Q((a,\mathsf{s},b)^t,(c,\mathsf{t},d)^t) = -ad - bc + 2b_J(\mathsf{s},\mathsf{t})$.

quadratic form $-q_J$ of signature $(q,1)$. Since, by assumption, v_x and h_0v_x both have positive first entry with respect to the basis $(e_2,\dots,e_{q+1})$, they are also in the same connected component, denoted by S^+, of $-q_J^{-1}(-1)$, which is preserved by h_0. This also implies that the eigenvalue of h_0 of greatest modulus, which we denote by λ_2 and which coincides with $\lambda_2(h)$, is positive.

Observe that S^+ is the hyperboloid model of real hyperbolic $(q-1)$-space $\mathbb{H}^{q-1}$. Therefore

$$\operatorname{arccosh}(b_J(v_x,h_0v_x)) = d_{\mathbb{H}^q}(v_x,h_0v_x) \geq \ell_{\mathbb{H}^q}(h_0)$$

where $\ell_{\mathbb{H}^q}$ denotes the translation length of the hyperbolic isometry h_0. Basic hyperbolic geometry yields

$$\cosh(\ell_{\mathbb{H}^q}(h_0)) = \tfrac{1}{2}(\lambda_2+\lambda_2^{-1})$$

and therefore

$$2b_J(\mathsf{s}_x,\lambda_1h_0\mathsf{s}_x) = 2q_J(\mathsf{s}_x)\lambda_1(v_x,h_0v_x) \geq q_J(\mathsf{s}_x)\lambda_1(\lambda_2+\lambda_2^{-1}).$$

As a result, using the fact that $\lambda_2>0$ and thus also $\lambda_2+\lambda_2^{-1}>0$, we obtain

$$\begin{aligned}
\mathrm{cr}_1(h_-,h_+,hx,x) &= \frac{\lambda_1^2q_J(\mathsf{s}_x)}{(\lambda_1^2+1)q_J(\mathsf{s}_x)-2b_J(\mathsf{s}_x,\lambda_1h_0\mathsf{s}_x)}\\
&\geq \frac{\lambda_1^2q_J(\mathsf{s}_x)}{(\lambda^2+1)q_J(\mathsf{s}_x)-\lambda_1(\lambda_2+\lambda_2^{-1})q_J(\mathsf{s}_x)}\\
&= \frac{\lambda_1}{(\lambda_1-\lambda_2)(1-\lambda_1^{-1}\lambda_2^{-1})}\\
&> \frac{\lambda_1}{\lambda_1-\lambda_2} = \frac{1}{1-\lambda_2/\lambda_1} = \frac{1}{1-|\lambda_2/\lambda_1|}.
\end{aligned}$$ ∎

Theorem 5.1 follows by combining Lemma 5.3 and the positivity of the cross ratio (Theorem 4.9). Given $x\in\partial_\infty\Gamma$ we denote as usual by $x^1\in\mathrm{Iso}_1(\mathbb{R}^{2,q})$ the image of the boundary map, and for $g\in\Gamma$ we denote by $g_\rho\in\mathrm{PO}(2,q)$ the element $\rho(g)$.

Proof of Theorem 5.1. We know from Lemma 5.3 that

$$\left(1-\left|\frac{\lambda_2}{\lambda_1}(h)\right|\right)^{-1} < \mathrm{cr}_1(h_-^1,h_+^1,h_\rho g_+^1,g_+^1).$$

Since the points $(h_-,g_-,h_+,hg_+,gh_+,g_+)\in\partial_\infty\Gamma^6$ are cyclically ordered (compare Figure 1, and [32, Lemma 2.2]), and cr_1 is positive along the image of the boundary map, we deduce using the cocycle identity (Proposition 2.4 (3, 4)) that

$$\begin{aligned}
\mathrm{cr}_1(h_-^1,h_+^1,h_\rho g_+^1,g_+^1) &< \mathrm{cr}_1(g_-^1,h_+^1,h_\rho g_+^1,h_-^1)\,\mathrm{cr}_1(h_-^1,h_+^1,h_\rho g_+^1,g_+^1)\\
&= \mathrm{cr}_1(g_-^1,h_+^1,h_\rho g_+^1,g_+^1)\\
&< \mathrm{cr}_1(g_-^1,h_+^1,h_\rho g_+^1,g_+^1)\,\mathrm{cr}_1(g_-^1,h_\rho g_+^1,g_\rho h_+^1,g_+^1)\\
&= \mathrm{cr}_1(g_-^1,h_+^1,g_\rho h_+^1,g_+^1).
\end{aligned}$$

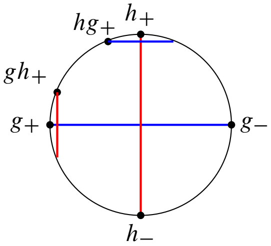

Fig. 1. The proof of Theorem 5.1.

The theorem follows from Lemma 2.5, stating

$$\mathrm{cr}_1(g^1_-, h^1_+, g_\rho h^1_+, g^1_+) = \lambda_1^2(g_\rho). \qquad \blacksquare$$

5.2. *Hybrid flags and strategy of proof in the general case*

For the proof of the collar lemma we will need the following construction.

Definition 5.4. Given two transverse flags $x, y \in \mathcal{F}_{p-1}$, the (x,k)-*hybrid flag* is

$$x \lhd_k y := (x^1, \dots, x^{k-1}, x^{k-1} \oplus (x^{k+1} \cap y^{p+q-k}), x^{k+1}, \dots, x^{p-1})$$

if $k = 1, \dots, p-2$ and

$$x \lhd_{p-1} y := (x^1, \dots, x^{p-2}, x^{p-2} \oplus (x^{q+2} \cap y^{p-1}))$$

otherwise.

In the case of the standard flags X, Z where $\mathsf{Z}^k = \langle e_1, \dots, e_k\rangle$ the hybrid flags are

$$\mathsf{Z} \lhd_k \mathsf{X} = \begin{cases} (\mathsf{Z}^1, \dots, \mathsf{Z}^{k-1}, \langle e_1, \dots, e_{k-1}, e_{k+1}\rangle, \mathsf{Z}^{k+1}, \dots, \mathsf{Z}^{p-1}) & \text{if } k < p-1, \\ (\mathsf{Z}^1, \dots, \mathsf{Z}^{p-2}, \langle e_1, \dots, e_{p-2}, e_{q+2}\rangle) & \text{if } k = p-1. \end{cases}$$

The goal of the section is to prove the collar lemmas; more specifically, we want to show that for any Θ-positive representation ρ, any linked pair $g, h \in \Gamma \setminus \{e\}$, and any $k < p-1$,

$$\left(1 - \left|\frac{\lambda_{k+1}}{\lambda_k}(h_\rho)\right|\right)^{-1} < \lambda_1^2 \cdots \lambda_k^2(g_\rho).$$

We will prove the collar lemma in two steps: First we show that

$$\left(1 - \left|\frac{\lambda_{k+1}}{\lambda_k}(h_\rho)\right|\right)^{-1} < \mathrm{cr}_k(h^k_-, h^k_+, gh^k_+, (h_- \lhd_k g_+)^k); \tag{5.2}$$

in the case $k = p-1$ we show in Section 5.3 how to reduce this claim to Lemma 5.3, building on the study of some level sets of the cross ratio cr_k carried out in Proposition 2.6. If $k < p-1$ then (5.2) is simpler and follows directly from property H_k [3].

In the second step we prove that $\mathrm{cr}_k(h_-^k, h_+^k, gh_+^k, g_+^k)$ is greater than the right hand side. The main ingredients used in this step are a technical statement (Proposition 5.9) about k-hybrid flags, which we prove in Section 5.4, and the connection between positivity of triples and positivity of the cross ratio (Proposition 4.8). We conclude the proof using the fact that the representations are k-positively ratioed and Lemma 2.5.

Remark 5.5. Note that since the Jordan projections of g and g^{-1} agree, it is enough to prove the collar lemma only for one of the pairs (g,h) and (g^{-1},h). As a result we can and will assume, possibly replacing g with g^{-1}, that the quadruple (h_-, h_+, gh_+, g_+) is positive.

5.3. *Step 1*

The first step for $k < p-1$ follows directly from [3].

Proposition 5.6 ([3, Corollary 6.3]). *Let $\rho : \Gamma \to \mathsf{PO}(p,q)$ satisfy property H_k. Then for $k < p-1$ and any linked pair $g, h \in \Gamma \setminus \{e\}$ we have*

$$\left(1 - \left|\frac{\lambda_{k+1}}{\lambda_k}(h_\rho)\right|\right)^{-1} < \mathrm{cr}_k(h_-^k, h_+^k, gh_+^k, (h_- \lhd_k g_+)^k).$$

Thus we are left to consider the case $k = p-1$.

Proposition 5.7. *Assume that $\rho : \Gamma \to \mathsf{PO}(p,q)$ is $\{p-2, p-1\}$-Anosov and the projections ξ_x to x^{q+2}/x^{p-2} are Θ-positive curves for all $x \in \partial_\infty\Gamma$. Then for any linked pair $g, h \in \Gamma \setminus \{e\}$,*

$$\left(1 - \left|\frac{\lambda_p}{\lambda_{p-1}}(h_\rho)\right|\right)^{-1} < \mathrm{cr}_{p-1}(h_-^{p-1}, h_+^{p-1}, gh_+^{p-1}, (h_- \lhd_{p-1} g_+)^{p-1}).$$

Proof. We consider the vector space $H = h_-^{q+2}/h_-^{p-2}$ with the induced non-degenerate form of signature $(2, q-p+2)$. Of course h_ρ induces an element $\overline{h} \in \mathsf{PO}(2, q-p+2) = \mathsf{PO}(H)$ satisfying $\frac{\lambda_{p-1}}{\lambda_p}(h_\rho) = \frac{\lambda_1}{\lambda_2}(\overline{h})$.

We consider the Θ-positive curve $\xi_{h_-} : \partial_\infty\Gamma \to \mathcal{F}_1(h_-^{q+2}/h_-^{p-2})$ defined as in Corollary 3.21: $\xi_{h_-}(x) = [(x^{p-1} \cap h_-^{q+2}) \oplus h_-^{p-2}]$, $\xi_{h_-}(h_-) = [h_-^{p-1}]$. In order to make the notation lighter we will denote the image $\xi_{h_-}(x)$ by $[x^{p-1}] \in \mathrm{Iso}_1(H)$.

Since ξ_{h_-} is h_ρ-equivariant and positive (Corollary 3.21), we deduce from Lemma 5.3 that

$$\left(1 - \left|\frac{\lambda_p}{\lambda_{p-1}}(h_\rho)\right|\right)^{-1} < \mathrm{cr}_1^H([h_-^{p-1}], [h_+^{p-1}], [hg_+^{p-1}], [g_+^{p-1}]). \tag{5.3}$$

Since ξ_{h_-} is Θ-positive and the 5-tuple $(h_-, h_+, hg_+, gh_+, g_+) \in \partial_\infty\Gamma^{(5)}$ is positively oriented (see [32, Lemma 2.2], and Figure 3 below), we can apply Proposition 3.11 to derive

$$\mathrm{cr}_1^H([h_-^{p-1}], [hg_+^{p-1}], [gh_+^{p-1}], [g_+^{p-1}]) > 1.$$

Since the cocycle identity implies that $\mathrm{cr}_1^H([h_-^{p-1}], [h_+^{p-1}], [gh_+^{p-1}], [g_+^{p-1}])$ equals

$$\mathrm{cr}_1^H([h_-^{p-1}],[h_+^{p-1}],[hg_+^{p-1}],[g_+^{p-1}])\,\mathrm{cr}_1^H([h_-^{p-1}],[hg_+^{p-1}],[gh_+^{p-1}],[g_+^{p-1}]),$$

we deduce

$$\mathrm{cr}_1^H([h_-^{p-1}],[h_+^{p-1}],[hg_+^{p-1}],[g_+^{p-1}]) < \mathrm{cr}_1^H([h_-^{p-1}],[h_+^{p-1}],[gh_+^{p-1}],[g_+^{p-1}]). \quad (5.4)$$

We can then use Proposition 2.6 to compute the right hand side:

$$\begin{aligned}&\mathrm{cr}_1^H([h_-^{p-1}],[h_+^{p-1}],[gh_+^{p-1}],[g_+^{p-1}])\\ &\qquad\qquad = \mathrm{cr}_{p-1}(h_-^{p-1},h_+^{p-1},gh_+^{p-1},h_-^{p-2}\oplus(h_-^{q+2}\cap g_+^{p-1})).\end{aligned}$$

This concludes the proof once we observe that, by definition,

$$(h_- \lhd_{p-1} g_+)^{p-1} = h_-^{p-2}\oplus(g_+^{p-1}\cap h_-^{q+2}). \qquad \blacksquare$$

5.4. *Positivity of hybrid flags*

Recall from Definition 5.4 that to a pair of flags $x, y \in \mathcal{F}_{p-1}$ and an integer $1 \le k \le p-1$ we associate the (x,k)-hybrid flag $x \lhd_k y$. The goal of this section is to prove that positivity is preserved under hybridization. We will need the following basic lemma.

Lemma 5.8. *Let $(w, x, y, z) \in \mathcal{F}_1^4$ be Θ-positive. Then there is a continuous path $\zeta : [0,1] \to \mathcal{F}_1$ from x to w such that $\zeta(t)$ is transverse to y, z for all $t \in [0,1]$.*

Proof. Recall the notation from Example 3.9. Up to the action by an element in $\mathsf{PO}(2,q)$, we can assume that $x = \mathsf{X}$, $z = \mathsf{Z}$, $y = P(\mathsf{s}_y)\mathsf{X}$ and $w = P(-\mathsf{s}_w)\mathsf{X}$ for $\mathsf{s}_y, \mathsf{s}_w \in c_J(V_J)$. Then $\zeta(t) := P(-t\mathsf{s}_w)\mathsf{X}$ is a continuous path from x to w. Moreover, $\zeta(t)$ is transverse to y, z for all $t \in [0,1]$, since $(P(-t\mathsf{s}_w)\mathsf{X}, P(\mathsf{s}_y)\mathsf{X}, \mathsf{Z})$ is a positive triple; the image of this triple under $P(t\mathsf{s}_w)$ is $(\mathsf{X}, P(t\mathsf{s}_w + \mathsf{s}_y)\mathsf{X}, \mathsf{Z})$. $\blacksquare$

Proposition 5.9. *Let (w, x, y, z) be a Θ-positive quadruple in $\mathcal{F}_{p-1}^4$. For all $k = 1, \dots, p-1$ the triple $(x \lhd_k w, y, z) \in \mathcal{F}_{p-1}^3$ is positive.*

Proof. Since $(x, y, z) \in \mathcal{F}_{p-1}^3$ is positive, by Corollary 3.8 it is enough to construct a continuous path $\eta : [0,1] \to \mathcal{F}_{p-1}$ with $\eta(0) = x$ and $\eta(1) = x \lhd_k w$ such that $\eta(t)$ is transverse to y, z for all $t \in [0,1]$.

We consider the projection $\pi_x^k : \mathrm{Iso}_k^{\pitchfork x} \to \mathbb{P}(x^{k+1}/x^{k-1})$ for $k < p-1$ (respectively $\pi_x^{p-1} : \mathrm{Iso}_{p-1}^{\pitchfork x} \to \mathcal{F}_1(x^{q+2}/x^{p-2})$) defined in Section 3.5. Choose a continuous path $\zeta_{p-1} : [0,1] \to \mathcal{F}_1(x^{q+2}/x^{p-2})$ from $[x^{p-1}]$ to $\pi_x^{p-1}(w)$ as in Lemma 5.8 for the quadruple $(\pi_x^{p-1}(w), [x^{p-1}], \pi_x^{p-1}(y), \pi_x^{p-1}(z))$. The latter quadruple is positive by Proposition 3.20. For $k < p-1$ choose a continuous path $\zeta_k : [0,1] \to \mathbb{P}(x^{k+1}/x^{k-1})$ from $[x^k]$ to $\pi_x^k(w)$ not passing through $\pi_x^k(y)$ and $\pi_x^k(z)$. This is possible according to Proposition 3.18.

We set $\eta(0) := x$. For $t > 0$, if $k < p-1$, we set $\eta(t)^i := x^i$ for $i \neq k$ and $\eta^k(t)$ to be the unique point of Iso_k that contains x^{k-1}, is contained in x^{k+1} and satisfies

$\pi_x^k(\eta(t)^k) = \zeta_k(t)$ in $\mathbb{P}(x^{k+1}/x^{k-1})$. Analogously, for $k = p-1$, we set $\eta(t)^i := x^i$ for $i \neq p-1$ and $\eta^{p-1}(t)$ to be the unique point of Iso_{p-1} that contains x^{p-2}, is contained in x^{q+2} and satisfies $\pi_x^k(\eta(t)^k) = \zeta_k(t)$ in $\mathrm{Iso}_1(x^{q+2}/x^{p-2})$.

By construction, η is a continuous path from x to $x \lhd_k w$. Since $\eta(t)^i = x^i$ for $i \neq k$ and x is transverse to y and z, it is enough to observe that $\eta(t)^k$ is transverse to y^k and z^k, which is equivalent to $\pi_x^k(\eta(t))$ being transverse to $\pi_x^k(y^k)$ and $\pi_x^k(z^k)$. The latter holds by construction. ∎

5.5. *Step 2*

To finish the proof of the collar lemma we need the following fact, which is essentially a combination of Propositions 5.9 and 4.8.

Proposition 5.10. *Let ρ be Θ-positive and (w,x,y,z) be a Θ-positive quadruple in $(\partial_\infty\Gamma)^4$. Then for any $k \leq p-1$,*

$$\mathrm{cr}_k(x^k, y^k, z^k, (x \lhd_k w)^k) \leq \mathrm{cr}_k(x^k, y^k, z^k, w^k).$$

Proof. Proposition 5.9 applied to the Θ-positive quadruple $(\xi(w), \xi(x), \xi(y), \xi(z))$ guarantees that $(x \lhd_k w)^k$ is transverse to y^k and z^k. Since also x^k is transverse to y^k and z^k, the cross ratio on the left hand side of the statement is defined. Using the cocycle identity for the cross ratio, the claim reduces to showing that

$$\mathrm{cr}_k((x \lhd_k w)^k, y^k, z^k, w^k) \geq 1.$$

We consider $\mathrm{cr}_k((x \lhd_k w)^k, y^k, y_t^k, w^k)$ as a function of the point y_t varying between y and z. Since ξ^k has Lipschitz image, by Rademacher's theorem and the fundamental theorem of calculus we know that $\mathrm{cr}_k((x \lhd_k w)^k, y^k, z^k, w^k)$ is the integral of the derivative of this function – see the proof of Theorem 4.9. As a result, using again the cocycle identity, it is enough to show that, for every $y_0 \in (y,z)_x$ such that $\dot{\xi}^k(y_0)$ is defined, and for some C^1-approximation $t \mapsto y_t^k$ of ξ^k at y_0^k, we have

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{cr}_k((x \lhd_k w)^k, y_0^k, y_t^k, w^k) \geq 0;$$

here once again Proposition 5.9 applied to the 4-tuple $(\xi(w), \xi(x), \xi(y_0), \xi(z))$ ensures that the cross ratio is defined.

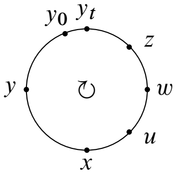

Fig. 2. The order of x, y, y_0, y_t, z, w, u for $t > 0$.

Proposition 5.9 together with Proposition 3.5 implies that for all $u \in (w, x)_y$ the triple $(\xi(y_0), \xi(w), \xi(x) \lhd_k \xi(u))$ is Θ-positive. Thus we can apply Proposition 4.8 to derive

$$\frac{d}{dt}\Big|_{t=0} \operatorname{cr}_k((x \lhd_k u)^k, y_0^k, y_t^k, w^k) > 0.$$

Now the regularity of the cross ratio yields

$$\frac{d}{dt}\Big|_{t=0} \operatorname{cr}_k((x \lhd_k u)^k, y_0^k, y_t^k, w^k) \to \frac{d}{dt}\Big|_{t=0} \operatorname{cr}_k((x \lhd_k w)^k, y_0^k, y_t^k, w^k)$$

as $u \to w$. This proves the claim. ■

The collar lemmas can now be proved following the same lines as the proof of Theorem 5.1:

Theorem 5.11. *Let $\rho : \Gamma \to \mathsf{PO}(p, q)$ be a Θ-positive Anosov representation and $g, h \in \Gamma$ a linked pair. Then for any $k \leq p - 1$,*

$$\left(1 - \left|\frac{\lambda_{k+1}}{\lambda_k}(h_\rho)\right|\right)^{-1} < \lambda_1^2 \cdots \lambda_k^2(g_\rho).$$

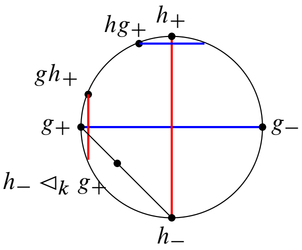

Fig. 3. The proof of Theorem 5.11.

Proof. From Propositions 5.6 and 5.7 we know that for any $k \leq p - 1$,

$$\left(1 - \left|\frac{\lambda_{k+1}}{\lambda_k}(h_\rho)\right|\right)^{-1} < \operatorname{cr}_k(h_-^k, h_+^k, gh_+^k, (h_- \lhd_k g_+)^k).$$

Proposition 5.10 implies

$$\operatorname{cr}_k(h_-^k, h_+^k, gh_+^k, (h_- \lhd_k g_+)^k) \leq \operatorname{cr}_k(h_-^k, h_+^k, gh_+^k, g_+^k).$$

Since ρ is k-positively ratioed and h_-, g_-, h_+, gh_+, g_+ are in that order on $\partial_\infty\Gamma$ (which implies $\operatorname{cr}_k(g_-^k, h_+^k, gh_+^k, h_-^k) > 1$), we deduce from the cocycle identity that

$$\operatorname{cr}_k(h_-^k, h_+^k, gh_+^k, g_+^k) < \operatorname{cr}_k(g_-^k, h_+^k, gh_+^k, g_+^k).$$

The theorem follows from Lemma 2.5 which shows

$$\operatorname{cr}_k(g_-^k, h_+^k, gh_+^k, g_+^k) = \lambda_1^2 \cdots \lambda_k^2(g_\rho).$$ ■

6. Higher rank Teichmüller theory for $\mathsf{PO}(p, q)$

In this section we show that Θ-positive Anosov representations into $\mathsf{PO}(p, q)$ form higher rank Teichmüller spaces. In other words, we show that being Θ-positive Anosov is a closed condition in the character variety

$$\Xi(\mathsf{PO}(p, q)) = \mathrm{Hom}^{\mathrm{red}}(\Gamma, \mathsf{PO}(p, q))/\mathsf{PO}(p, q).$$

A direct consequence of Corollary 3.8 is that the set of Θ-positive representations is open in the set of Anosov representations, and thus the set of Θ-positive Anosov representations is open in the character variety. We then deduce that Θ-positive Anosov representations form connected components of $\Xi(\mathsf{PO}(p, q))$.

We begin with an easy consequence of Theorem 3.7 due to Guichard–Wienhard [23].

Proposition 6.1. *The set of Θ-positive Anosov representations is a union of connected components of the set* $\mathrm{Hom}_\Theta \subset \mathrm{Hom}(\Gamma, \mathsf{PO}(p, q))$ *of Θ-Anosov representations.*

Proof. Since open connected subsets of a real algebraic variety are path connected, it is enough to verify that if $\rho_* : [0, T] \to \mathrm{Hom}_\Theta$ is a continuous path, and $\rho_0 : \Gamma \to \mathsf{PO}(p, q)$ is Θ-positive, then ρ_T is Θ-positive. We denote by $\xi_t : \partial_\infty \Gamma \to \mathcal{F}_{p-1}$ the boundary map associated to the representation ρ_t, which is transverse because ρ_t is Anosov. It is well known that ξ_t depends continuously on ρ_t [21, Theorem 5.13]; as a result, given a Θ-positive n-tuple $(x_1, \ldots, x_n) \in \partial_\infty \Gamma^{(n)}$, the path $(\xi_t(x_1), \ldots, \xi_t(x_n))$ is a continuous transverse path. Corollary 3.8 applies, showing that also $(\xi_T(x_1), \ldots, \xi_T(x_n))$ is a Θ-positive n-tuple. ∎

A *semisimplification* $\rho^{\mathrm{ss}} : \Gamma \to \mathsf{PO}(p, q)$ of a representation $\rho : \Gamma \to \mathsf{PO}(p, q)$ can be characterized as a representation in the unique closed orbit inside the closure $\overline{\mathsf{PO}(p, q) \cdot \rho} \subset \mathrm{Hom}(\Gamma, \mathsf{PO}(p, q))$, where the action of $\mathsf{PO}(p, q)$ is by conjugation (see [18, Proposition 2.39]). Thanks to the semisimplification, it is enough to consider reductive limits of representations, instead of general limits: if $\{\rho_n\}_{n\in\mathbb{N}}$ converges to ρ_0 in $\mathrm{Hom}(\Gamma, \mathsf{PO}(p, q))$, then we can find conjugates $\{g_n \rho_n g_n^{-1}\}_{n\in\mathbb{N}}$ converging to ρ_0^{ss}. Moreover, any semisimplification is reductive.

Corollary 6.2. *A representation $\rho : \Gamma \to \mathsf{PO}(p, q)$ is Θ-positive Anosov if and only if all its semisimplifications ρ^{ss} are.*

Proof. It is known that ρ is Θ-Anosov if and only if ρ^{ss} is [18, Proposition 2.39]. Since ρ and ρ^{ss} belong to the same connected component of Hom_Θ, the result follows from Proposition 6.1. ∎

Since the set of Θ-Anosov representations is open in $\mathrm{Hom}(\Gamma, \mathsf{PO}(p, q))$, it follows again from Proposition 6.1 that also the set of Θ-positive Anosov representations, a union of connected components of the former set, is open in $\mathrm{Hom}(\Gamma, \mathsf{PO}(p, q))$. It remains to show that the set of Θ-positive Anosov representations is also closed in $\mathrm{Hom}(\Gamma, \mathsf{PO}(p, q))$. For this we crucially need the following result from [4]: any proximal and reductive limit of positively ratioed representations admits a well behaved boundary map.

Theorem 6.3 ([4, Theorem B]). *Let* $\{\rho_n : \Gamma \to \mathsf{PO}(p,q)\}_{n\in\mathbb{N}}$ *be a sequence of* k*-positively ratioed representations converging to a* k*-proximal reductive representation* $\rho_0 : \Gamma \to \mathsf{PO}(p,q)$*. Then* ρ_0 *admits an equivariant transverse continuous boundary map* $\xi^k_{\rho_0} : \partial_\infty\Gamma \to \mathrm{Iso}_k(\mathbb{R}^{p,q})$*, which is dynamics preserving*[11] *at* $\gamma_+ \in \partial_\infty\Gamma$ *for any* $\gamma \in \Gamma$ *with* $\rho_0(\gamma)$ k*-proximal.*

An element $g \in \mathsf{PO}(p,q)$ is *k-proximal* if it has a unique attracting point in $\mathrm{Iso}_k(\mathbb{R}^{p,q})$, equivalently if $|\frac{\lambda_k}{\lambda_{k+1}}(g)| > 1$; a representation is *$k$-proximal* if its image contains a k-proximal element. We prove closedness in two steps. First we deduce from Theorems A, B and 6.3 that any limiting representation admits well behaved boundary maps.

Proposition 6.4. *Let* $\{\rho_n : \Gamma \to \mathsf{PO}(p,q)\}_{n\in\mathbb{N}}$ *be* Θ*-positive Anosov representations converging to a reductive representation* ρ_0 *in* $\mathrm{Hom}(\Gamma, \mathsf{PO}(p,q))$*. Then* ρ_0 *admits equivariant, dynamics preserving,* Θ*-positive maps.*

Proof. From the collar lemma (Theorem B) it follows that $\rho_0(\gamma)$ is k-proximal for every $k = 1,\dots,p-1$ and every $\gamma \in \Gamma\setminus\{e\}$. Indeed, we can find a linked element $g \in \Gamma\setminus\{e\}$ and thus Theorem 5.11 gives

$$\left(1-\left|\frac{\lambda_{k+1}}{\lambda_k}(\rho_n(\gamma))\right|\right)^{-1} < \lambda_1^2\cdots\lambda_k^2(\rho_n(g)).$$

As both quantities converge to the corresponding quantities for the representation ρ_0, we deduce that $|\frac{\lambda_k}{\lambda_{k+1}}(\rho_0(\gamma))| > 1$.

Since every ρ_n is k-positively ratioed for $k = 1,\dots,p-1$ (Theorem A), we can apply Theorem 6.3 and deduce that ρ_0 admits an equivariant, transverse, continuous, dynamics preserving boundary map

$$\xi_{\rho_0} : \partial_\infty\Gamma \to \mathcal{F}_{p-1}.$$

It follows from Corollary 3.8 that this map is furthermore Θ-positive. Indeed, since ξ_{ρ_0} is dynamics preserving, we have $\xi_{\rho_n}(\gamma_+) \to \xi_{\rho_0}(\gamma_+)$ for every attracting fixed point $\gamma_+ \in \partial_\infty\Gamma$ of $\gamma \in \Gamma\setminus\{e\}$. The same argument as in the proof of Proposition 6.1 shows that the restriction of ξ_{ρ_0} to the set of fixed points of elements in $\Gamma\setminus\{e\}$ is a Θ-positive map, and thus Corollary 3.15 implies that ξ_{ρ_0} itself is Θ-positive. ∎

Second, we show that this is enough to guarantee that the limiting representation is Anosov; for this we introduce a new way to use the Θ-positive boundary map to read eigenvalue gaps for the representation ρ_0.

Proposition 6.5. *A representation* ρ *admitting equivariant, continuous,* Θ*-positive, dynamics preserving boundary maps* ξ_ρ *is* Θ*-positive Anosov.*

---

[11] Recall Definition 2.1 (ii).

Proof. Since Θ-positive boundary maps are by definition transverse, it remains to show the third condition in our definition of Anosov representations, namely that for a sequence $\gamma_n \in \Gamma$ with $|\gamma_n|_\infty \to \infty$ we have

$$\left|\frac{\lambda_k}{\lambda_{k+1}}(\rho_0(\gamma_n))\right| \to \infty.$$

Up to conjugating and extracting a subsequence we can assume that $\gamma_n^+ \to l_+$, $\gamma_n^- \to l_-$, and $l_- \neq l_+$. Pick $x \in \partial_\infty\Gamma$ different from l_- and l_+.

It follows from the proofs of Propositions 5.6 and 5.7 (see Lemma 5.3) that

$$1 - \left|\frac{\lambda_{k+1}}{\lambda_k}(\rho(\gamma_n))\right| \geq \mathrm{cr}_k\big((\gamma_n^-)^k, (\gamma_n x)^k, (\gamma_n^+)^k, (\gamma_n^- \lhd_k x)^k\big).$$

It is enough to show that the right hand side converges to 1. If $k < p-1$ set $L := l_-^k / l_-^{k-1}$ and if $k = p-1$ set $L := l_-^{q+2}/l_-^{p-2}$. The boundary dynamics of the surface group guarantees that $\gamma_n x$ converges to l_+; using continuity of the cross ratio we deduce

$$\mathrm{cr}_k\big((\gamma_n^-)^k, (\gamma_n x)^k, (\gamma_n^+)^k, (\gamma_n^- \lhd_k x)^k\big) \to \mathrm{cr}_k\big((l_-)^k, (l_+)^k, (l_+)^k, (l_- \lhd_k x)^k\big).$$

We apply Proposition 2.6 to the latter cross ratio. If $k = p-1$ the cross ratio in the limit equals

$$\mathrm{cr}_1^L([l_-^{p-1}], [l_+^{p-1} \cap l_-^{q+2}], [l_+^{p-1} \cap l_-^{q+2}], [x^{p-1} \cap l_-^{q+2}]).$$

Since ξ_ρ is a Θ-positive map, by Proposition 3.20 the last cross ratio is defined (i.e. $[l_+^{p-1} \cap l_-^{q+2}] \pitchfork [l_-^{p-1}], [x^{p-1} \cap l_-^{q+2}]$), thus it is equal to 1, as desired.

If $k < p-1$ the cross ratio in the limit equals

$$\mathrm{cr}_1^L([l_-^k], [l_+^{p+q-k} \cap l_-^{k+1}], [l_+^{p+q-k} \cap l_-^{k+1}], [x^{p+q-k} \cap l_-^{k+1}]).$$

As ξ_ρ satisfies property H_k (Corollary 3.19), it follows that

$$[l_+^{p+q-k} \cap l_-^{k+1}] \pitchfork [x^{p+q-k} \cap l_-^{k+1}], [l_-^k].$$

Therefore the cross ratio is again defined and equal to 1. In particular, ρ_0 is k-Anosov for $k \leq p-1$, as desired. ∎

This concludes the proof of Theorem C, that the set of Θ-positive Anosov representations is closed in the representation variety: it follows from Propositions 6.4 and 6.5 that any reductive limit of Θ-positive Anosov representations is Θ-positive Anosov. Corollary 6.2 and the discussion just before the corollary imply that *any* limit of Θ-positive Anosov representations is Θ-positive Anosov.

Theorem C has applications to character varieties, which, as in the introduction, we realize as the quotient of the set of reductive representations by the $\mathsf{PO}(p,q)$-action by conjugation. Combining Corollary 6.6 and the work of [1] it follows that Θ-positive Anosov representations in $\mathsf{SO}(p,q)$ are necessarily reductive, since it is proven in [1] that all representations in the components of the caracter variety corresponding to Θ-positive Anosov representations are non-parabolic, while a non-reductive representation, as well as its semisimplification, is parabolic.

Corollary 6.6. *The subset consisting of* (*the conjugacy classes of*) Θ*-positive Anosov representations is a union of connected components of*

$$\mathrm{Hom}(\Gamma, \mathsf{PO}(p,q)) \quad \textit{and} \quad \Xi(\mathsf{PO}(p,q)).$$

Corollary 6.6 also shows that Θ-positive Anosov representations form connected components in $\Xi(\mathsf{SO}(p,q))$ (see Remark 1.2).

Remark 6.7. Aparicio-Arroyo, Bradlow, Collier, García-Prada, Gothen and Oliveira [1] used Higgs bundle techniques to count the connected components of the character variety $\Xi(\mathsf{SO}(p,q))$ and checked for the existence of Θ-positive representations. The component count in [1] is as follows, denoting by g the genus of the Riemann surface:

- There are 2^{2g+2} *mundane* components. Each of these compontents contains a representation with image contained in a compact subgroup of $\mathsf{SO}(p,q)$ – such a representation is clearly non-discrete and thus not Anosov. It follows from Corollary 6.6 that those connected components contain no Θ-positive Anosov representation.
- If $q > p+1$, then there are 2^{2g+1} additional *exceptional* components. Each such component contains a Θ-positive Anosov representation, namely one as in Example 3.17.[12] Thus every representation in those connected components is Θ-positive Anosov. The exceptional components are then, as conjectured, the higher rank Teichmüller spaces associated to the group $\mathsf{SO}(p,q)$ for $p+1<q$.
- If $q = p+1$ there are $2^{2g+1}-1+2p(g-1)$ further exceptional components, which are smooth and conjectured to consist entirely of *Zariski dense* representations [13, Conjecture 1.7]. According to [1, Remark 7.14], Corollary 6.6 implies that these also consist only of Θ-positive Anosov representations: For the natural embedding $\mathsf{SO}(p,p+1) \to \mathsf{SO}(p,p+2)$ these components inject into Θ-positive components of $\mathsf{SO}(p,p+2)$, in particular all representations are positive, when composed with the embedding $\mathsf{SO}(p,p+1) \to \mathsf{SO}(p,p+2)$. As a result those representations are also Θ-positive in $\mathsf{SO}(p,p+1)$. It would be interesting to construct explicit *hybrid* representations in these components, in analogy to [20]. See [29] for results in this direction from the Higgs bundle perspective.

Appendix A. Positive representations in the split group $\mathsf{PO}(p,p)$

We restricted throughout the paper to the case $p<q$ because the root system of the split group $\mathsf{PO}(p,p)$ has a different expression, and in this case the ratio $\log|\frac{\lambda_{p-1}}{\lambda_p}\rho(g)|$ is not a simple root of $\mathsf{PO}(p,p)$, but rather the minimum of the last two simple roots. Furthermore, the manifold $\mathcal{F}_{p-1}$ we deal with in this paper identifies, in this case, with the full flag manifold (the stabilizer of a point is a Borel subgroup); this is due to the

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[12] Actually, using the notation of Example 3.17, a representation of the form $\rho : \Gamma \to \mathsf{SO}(p,q)$, $\rho := \det(\alpha)(\tau \circ \iota) \oplus \alpha$.

fact that in a space of signature (p, p) every $(p-1)$-dimensional isotropic subspace is contained in precisely two p-dimensional isotropic subspaces.

However, it is easy to check that we never use the root system formalism in this paper, but only the linear algebraic interpretation of the group $\mathsf{PO}(p,q)$ and of the associated flags of isotropic subspaces. Moreover, if $p = q$, then the Θ-positive structure we describe is the positive structure of the split group $\mathsf{PO}(p, p)$ (see e.g. [13, proof of Proposition 7.10]). As a result, our proofs of Theorems A, B and C work in the case $p = q$ as well.[13] In particular, this shows that the Hitchin component in $\mathsf{PO}(p, p)$, which so far had been elusive, only consists of Anosov representations (Theorem C) which are furthermore positively ratioed with respect to all symmetrized weights (Theorem A) and satisfy root versus (symmetrized) weight collar lemmas for all simple roots in the root system: Theorem B for $k = p-1$ compares the minimum of the last two roots to the symmetrized fundamental weight $\omega_{p-1} + \omega_p$ (i.e. an opposition involution invariant weight). Taking the minimum of the roots can be dropped to get collar lemmas comparing the last two simple roots to the symmetrized weight. We believe that all these results are new for $\mathsf{PO}(p, p)$-Hitchin representations.

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References

[1] Aparicio-Arroyo, M., Bradlow, S., Collier, B., García-Prada, O., Gothen, P. B., Oliveira, A.: $\mathsf{SO}(p,q)$-Higgs bundles and higher Teichmüller components. Invent. Math. **218**, 197–299 (2019) Zbl 1473.14017 MR 3994589

[2] Barbot, T., Charette, V., Drumm, T., Goldman, W. M., Melnick, K.: A primer on the $(2+1)$ Einstein universe. In: Recent developments in pseudo-Riemannian geometry, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 179–229 (2008) Zbl 1154.53047 MR 2436232

[3] Beyrer, J., Pozzetti, B.: A collar lemma for partially hyperconvex surface group representations. Trans. Amer. Math. Soc. **374**, 6927–6961 (2021) Zbl 1487.22012 MR 4315593

[4] Beyrer, J., Pozzetti, B.: Degenerations of k-positive surface group representations. J. Topol. **17**, article no. e12352, 46 pp. (2024) Zbl 1552.22033 MR 4822920

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[13] Alternatively one can deduce these results from the results of this paper composing a representation in $\mathsf{PO}(p,p)$ with the standard embedding $\mathsf{PO}(p,p) \to \mathsf{PO}(p,p+1)$; the image under this homomorphism of the composition of a hyperbolization and the principal $\mathsf{SL}(2,\mathbb{R})$-embedding is then Θ-positive, and it follows from Corollary 1.1 that the whole $\mathsf{PO}(p,p)$-Hitchin component only consists of Θ-positive Anosov representations.

[5] Bourbaki, N.: Lie groups and Lie algebras. Chapters 4–6. Elements of Mathematics (Berlin), Springer, Berlin (2002) Zbl 0983.17001 MR 1890629

[6] Bradlow, S., Collier, B., García-Prada, O., Gothen, P., Oliveira, A.: A general Cayley correspondence and higher Teichmüller spaces. arXiv:2101.09377 (2021)

[7] Bradlow, S. B., García-Prada, O., Gothen, P. B.: Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces. Geom. Dedicata **122**, 185–213 (2006) Zbl 1132.14029 MR 2295550

[8] Burger, M., Iozzi, A., Parreau, A., Pozzetti, M. B.: Currents, systoles, and compactifications of character varieties. Proc. London Math. Soc. (3) **123**, 565–596 (2021) Zbl 1487.32068 MR 4368682

[9] Burger, M., Iozzi, A., Parreau, A., Pozzetti, M. B.: The real spectrum compactification of character varieties: characterizations and applications. C. R. Math. Acad. Sci. Paris **359**, 439–463 (2021) Zbl 1541.32002 MR 4278899

[10] Burger, M., Iozzi, A., Parreau, A., Pozzetti, M. B.: The real spectrum compactification of character varieties. arXiv:2311.01892 (2023)

[11] Burger, M., Iozzi, A., Wienhard, A.: Surface group representations with maximal Toledo invariant. Ann. of Math. (2) **172**, 517–566 (2010) Zbl 1208.32014 MR 2680425

[12] Burger, M., Pozzetti, M. B.: Maximal representations, non-Archimedean Siegel spaces, and buildings. Geom. Topol. **21**, 3539–3599 (2017) Zbl 1402.22010 MR 3692972

[13] Collier, B.: $\mathsf{SO}(n, n+1)$-surface group representations and Higgs bundles. Ann. Sci. École Norm. Sup. (4) **53**, 1561–1616 (2020) Zbl 1466.53031 MR 4203035

[14] Collier, B., Tholozan, N., Toulisse, J.: The geometry of maximal representations of surface groups into $\mathrm{SO}_0(2, n)$. Duke Math. J. **168**, 2873–2949 (2019) Zbl 1516.32004 MR 4017517

[15] Dumas, D., Sanders, A.: Geometry of compact complex manifolds associated to generalized quasi-Fuchsian representations. Geom. Topol. **24**, 1615–1693 (2020) Zbl 1457.32061 MR 4173918

[16] Fanoni, F., Pozzetti, M. B.: Basmajian-type inequalities for maximal representations. J. Differential Geom. **116**, 405–458 (2020) Zbl 1461.53041 MR 4182893

[17] Fock, V., Goncharov, A.: Moduli spaces of local systems and higher Teichmüller theory. Publ. Math. Inst. Hautes Études Sci. **103**, 1–211 (2006) Zbl 1099.14025 MR 2233852

[18] Guéritaud, F., Guichard, O., Kassel, F., Wienhard, A.: Anosov representations and proper actions. Geom. Topol. **21**, 485–584 (2017) Zbl 1373.37095 MR 3608719

[19] Guichard, O., Labourie, F., Wienhard, A.: Positivity and representations of surface groups. arXiv:2106.14584 (2021)

[20] Guichard, O., Wienhard, A.: Topological invariants of Anosov representations. J. Topol. **3**, 578–642 (2010) Zbl 1225.57012 MR 2684514

[21] Guichard, O., Wienhard, A.: Anosov representations: domains of discontinuity and applications. Invent. Math. **190**, 357–438 (2012) Zbl 1270.20049 MR 2981818

[22] Guichard, O., Wienhard, A.: Positivity and higher Teichmüller theory. In: European Congress of Mathematics, Eur. Math. Soc., Zürich, 289–310 (2018) Zbl 1404.22034 MR 3887772

[23] Guichard, O., Wienhard, A.: Generalizing Lusztig's total positivity. arXiv:2208.10114v2 (2024)

[24] Hitchin, N. J.: Lie groups and Teichmüller space. Topology **31**, 449–473 (1992) Zbl 0769.32008 MR 1174252

[25] Huang, Y., Sun, Z.: McShane identities for higher Teichmüller theory and the Goncharov–Shen potential. Mem. Amer. Math. Soc. **286**, no. 1422, v+116 pp. (2023) Zbl 1517.30008 MR 4595283

[26] Kapovich, M., Leeb, B., Porti, J.: Anosov subgroups: dynamical and geometric characterizations. Eur. J. Math. **3**, 808–898 (2017) Zbl 1403.22013 MR 3736790

[27] Kassel, F., Potrie, R.: Eigenvalue gaps for hyperbolic groups and semigroups. J. Modern Dynam. **18**, 161–208 (2022) Zbl 07564671 MR 4446005

[28] Keen, L.: Collars on Riemann surfaces. In: Discontinuous groups and Riemann surfaces (College Park, MD, 1973), Ann. of Math. Stud. 79, Princeton University Press, Princeton, NJ, 263–268 (1974) Zbl 0304.30014 MR 0379833

[29] Kydonakis, G.: From hyperbolic Dehn filling to surgeries in representation varieties. In: In the tradition of Thurston II. Geometry and groups, Springer, Cham, 201–260 (2022) Zbl 1506.57011 MR 4472051

[30] Labourie, F.: Anosov flows, surface groups and curves in projective space. Invent. Math. **165**, 51–114 (2006) Zbl 1103.32007 MR 2221137

[31] Labourie, F., McShane, G.: Cross ratios and identities for higher Teichmüller–Thurston theory. Duke Math. J. **149**, 279–345 (2009) Zbl 1182.30075 MR 2541705

[32] Lee, G.-S., Zhang, T.: Collar lemma for Hitchin representations. Geom. Topol. **21**, 2243–2280 (2017) Zbl 1367.57010 MR 3654108

[33] Lusztig, G.: Total positivity in reductive groups. In: Lie theory and geometry, Progr. Math. 123, Birkhäuser Boston, Boston, MA, 531–568 (1994) Zbl 0845.20034 MR 1327548

[34] Martone, G., Zhang, T.: Positively ratioed representations. Comment. Math. Helv. **94**, 273–345 (2019) Zbl 1422.37018 MR 3942786

[35] Parreau, A.: Compactification d'espaces de représentations de groupes de type fini. Math. Z. **272**, 51–86 (2012) Zbl 1322.22022 MR 2968214

[36] Potrie, R., Sambarino, A.: Eigenvalues and entropy of a Hitchin representation. Invent. Math. **209**, 885–925 (2017) Zbl 1380.30032 MR 3681396

[37] Pozzetti, M. B., Sambarino, A., Wienhard, A.: Conformality for a robust class of non-conformal attractors. J. Reine Angew. Math. **774**, 1–51 (2021) Zbl 1483.37037 MR 4250471

[38] Pozzetti, M. B., Sambarino, A., Wienhard, A.: Anosov representations with Lipschitz limit set. Geom. Topol. **27**, 3303–3360 (2023) Zbl 07777462 MR 4668098

[39] Richardson, R. W., Slodowy, P. J.: Minimum vectors for real reductive algebraic groups. J. London Math. Soc. (2) **42**, 409–429 (1990) Zbl 0675.14020 MR 1087217

[40] Vlamis, N. G., Yarmola, A.: Basmajian's identity in higher Teichmüller–Thurston theory. J. Topol. **10**, 744–764 (2017) Zbl 1376.32016 MR 3797594

[41] Zhang, T., Zimmer, A.: Regularity of limit sets of Anosov representations. J. Topol. **17**, article no. e12355, 72 pp. (2024) Zbl 1552.22043 MR 4822923