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The Batyrev–Manin conjecture for DM stacks

Received July 24, 2022; revised January 10, 2024

Abstract. We define a new height function on rational points of a DM (Deligne–Mumford) stack over a number field. This generalizes a generalized discriminant of Ellenberg–Venkatesh, the height function recently introduced by Ellenberg–Satriano–Zureick-Brown (as far as DM stacks over number fields are concerned), and the quasi-toric height function on weighted projective stacks by Darda. Generalizing the Manin conjecture and the more general Batyrev–Manin conjecture, we formulate a few conjectures on the asymptotic behavior of the number of rational points of a DM stack with bounded height. To formulate the Batyrev–Manin conjecture for DM stacks, we introduce the orbifold versions of the so-called a - and b -invariants. When applied to the classifying stack of a finite group, these conjectures specialize to Malle’s conjecture, except that we remove certain thin subsets from counting. More precisely, we remove breaking thin subsets, which have been studied in the case of varieties by people including Hassett, Tschinkel, Tanimoto, Lehmann and Sengupta, and can be generalized to DM stack thanks to our generalization of a - and b -invariants. The notion of breaking thin subset enables us to reinterpret Klüners’ counterexample to Malle’s conjecture.

Keywords: Batyrev–Manin conjecture, Malle’s conjecture, DM stacks, rational points, heights, twisted sectors.

Contents

1. Introduction	2352
1.1. Notation	2355
2. Twisted sectors, residue maps and ages	2355
2.1. Twisted sectors	2355
2.2. Residue maps	2359
2.3. Ages	2362
3. Line bundles and stable heights	2363
4. Raising data and unstable heights	2366
5. Fano stacks; the stacky Manin conjecture	2372
6. Compatibility with products of Fano stacks	2375
7. Compatibility with coarse moduli spaces	2378

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Mathematics Subject Classification 2020: 11G35 (primary); 11G50, 14D23, 14E16 (secondary).

8. The orbifold pseudo-effective cone	2380
9. a - and b -invariants; the stacky Batyrev–Manin conjecture	2386
9.1. a - and b -invariants and breaking thin subsets	2386
9.2. The stacky Batyrev–Manin conjecture	2388
9.3. Fano stacks revisited	2389
9.4. Malle’s conjecture revisited	2390
9.5. Klüners’ counterexample revisited	2394
9.6. Comprehensiveness	2397
References	2398

1. Introduction

The Batyrev–Manin conjecture [6, 19] and Malle’s conjecture [32, 33] are two important conjectures about asymptotic behaviors of arithmetic objects; rational points of a variety and G -extensions of a number field for a prescribed transitive subgroup G of the symmetric group S_n . There is a clear similarity between them. In a proper setting, for each real $B > 0$, the number of rational points with height at most B is finite; and similarly for the number of G -extensions of a number field with discriminant at most B . In both cases, if we denote these numbers by $N(B)$, then the conjectures claim asymptotic formulas of the form

$$N(B) \sim CB^a(\log B)^{b-1} \quad (B \rightarrow \infty)$$

for some constants $C > 0$, $a > 0$ and $b \geq 1$, where a and b are invariants admitting simple expressions. There are also works on formulas for the leading constant C , notably [7, 37] for rational points and [8] for S_n -extensions.

In [48, 49], the second-named author studied relations between these conjectures from the viewpoint of the McKay correspondence. In particular, he showed that the constants a and b in the two conjectures are closely related, and observed by a heuristic argument using zeta functions that there are nonrigorous “implications” between the conjectures. Key ingredients in this study were the *discrepancy* invariant of singularities and the *age* invariant, as is often the case in studies on the McKay correspondence and quotient singularities. Recently, the first-named author [13] and Ellenberg–Satriano–Zureick-Brown [16] started studying distributions of rational points on algebraic stacks as a step toward unifying the Batyrev–Manin conjecture and Malle’s conjecture. For this purpose, they needed to introduce new height functions on stacks, based on different ideas. The first-named author introduced the notion of *quasi-toric heights* on weighted projective stacks and derived a precise asymptotic formula for rational points of these stacks. On the other hand, Ellenberg–Satriano–Zureick-Brown introduced the *height function associated to a vector bundle* \mathcal{V} , motivated by the work of Wood–Yasuda [45]. Moreover, under certain assumptions, they defined the Fujita invariant $a(\mathcal{V})$ of a vector bundle and conjectured that for any $\varepsilon > 0$, there exist constants $C_1, C_2 > 0$ satisfying

$$C_1 B^{a(\mathcal{V})} \leq N(B) \leq C_2 B^{a(\mathcal{V})+\varepsilon},$$

where $N(B)$ is the number defined as before but with the height function associated to \mathcal{V} . They call this a weak form of the stacky Batyrev–Manin–Malle Conjecture. Note that their definition of $a(\mathcal{V})$ is not given in terms of the pseudo-effective cone as in the original Batyrev–Manin conjecture.

The purpose of the present article is to introduce yet another height function for DM (Deligne–Mumford) stacks and formulate a few conjectures on the asymptotic behavior of the number of rational points with bounded height. In particular, we suggest two compatible ways of determining the exponent of $\log B$ in relevant asymptotic formulas, while we “normalize” data necessary to define a height function in such a way that the exponent of B is expected to be 1. On the other hand, we do not try to find a formula for the leading constant C , except that we speculate upon conditions for such a formula to exist (Remark 9.17). Our key ingredient is the *stack of twisted 0-jets*, denoted by $\mathcal{J}_0\mathcal{X}$, of the given stack \mathcal{X} , which appears in motivic integration over DM stacks [46, 47] and is also known as the *cyclotomic inertia stack* in the Gromov–Witten theory for DM stacks [2]. The trivial connected component of $\mathcal{J}_0\mathcal{X}$ is called the *nontwisted sector* and the other connected components are called *twisted sectors*. The set of all sectors, denoted by $\pi_0(\mathcal{J}_0\mathcal{X})$, is equipped with the age function, $\text{age}: \pi_0(\mathcal{J}_0\mathcal{X}) \rightarrow \mathbb{Q}_{\geq 0}$. To get a meaningful height function especially when \mathcal{X} has a nontrivial generic stabilizer, we need to choose another function $c: \pi_0(\mathcal{J}_0\mathcal{X}) \rightarrow \mathbb{R}$, which we call a *raising function*, and define the function $\text{age}_c := \text{age} + c$ on $\pi_0(\mathcal{J}_0\mathcal{X})$. A line bundle \mathcal{L} on a DM stack \mathcal{X} endowed with a raising function c defines a height function $H_{\mathcal{L},c}$ on rational points of \mathcal{X} , which is unique modulo bounded functions. Specifying an adelic metric on \mathcal{L} and a raising datum c_* refining the raising function c completely determines the height function. Our height function inherits some nice properties of the classical height function on varieties, namely, multiplicativity, functoriality and the Northcott property. This new height function generalizes those of the first-named author [13] and Ellenberg–Satriano–Zureick-Brown mentioned above (as far as DM stacks over a number field are concerned), and also the “ f -discriminant” of Ellenberg–Venkatesh [17, p. 163]. Note that the paper [16] treats also Artin stacks over global fields.

Our first conjecture, Conjecture 5.6, concerns the case where our stack \mathcal{X} is *Fano*, meaning that the anti-canonical line bundle $\omega_{\mathcal{X}}^{-1}$ corresponds to an ample \mathbb{Q} -line bundle on the coarse moduli space. We also assume that the raising function is *adequate*, which means that $\text{age}_c(\mathcal{Y}) \geq 1$ for every twisted sector \mathcal{Y} and that if \mathcal{X} is of dimension zero, then there exists a sector \mathcal{Y} with $\text{age}_c(\mathcal{Y}) = c(\mathcal{Y}) = 1$. The conjecture claims that in this situation, the asymptotic formula for $H_{\omega_{\mathcal{X}}^{-1},c}$ is of the form

$$CB(\log B)^{\rho(\mathcal{X})+j_c(\mathcal{X})-1},$$

where $\rho(\mathcal{X})$ is the Picard number and $j_c(\mathcal{X})$ is the number of c -*junior sectors*, that is, twisted sectors with $\text{age}_c = 1$, provided that a suitable “accumulation thin subset” is removed. As evidence for this conjecture, we show that the conjecture is compatible with taking products of Fano stacks and with the Manin conjecture for Fano varieties with canonical singularities.

The second conjecture, Conjecture 9.10, treats a more general situation and is formulated in terms of a variant of the pseudo-effective cone, which we call the *orbifold pseudo-effective cone*, like the original Batyrev–Manin conjecture for varieties. We define a - and b -invariants for the pair (\mathcal{L}, c) of a line bundle and a raising function c and denote them by $a(\mathcal{L}, c)$ and $b(\mathcal{L}, c)$. The conjecture claims that if the pair (\mathcal{L}, c) is big (meaning that the pair gives a point lying in the interior of the orbifold pseudo-effective cone), if we assume the adequacy condition (Definition 9.4), which in particular implies $a(\mathcal{L}, c) = 1$, and if we remove a suitable thin subset, then the asymptotic formula for $H_{\mathcal{L}, c}$ is of the form

$$CB(\log B)^{b(\mathcal{L}, c)-1}.$$

We show that the second conjecture implies the first one; the proof is nontrivial unlike in the case of varieties. Of course, our conjectures are formulated in such a way that they specialize to the Manin and Batyrev–Manin conjectures for varieties. On the other hand, applying these conjectures to the classifying stack BG of a finite group G , we obtain a version of Malle’s conjecture with respect to a generalized discriminant, which was considered in [17], except that the removed “accumulation subsets” are different. We follow Peyre’s suggestion [38, p. 345] to remove a thin subset from the set of rational points. Our second conjecture also incorporates the notion of *breaking thin morphisms*, which have been studied in the case of varieties for example in [21, 27, 28, 30]. That enables us to interpret Klüners’ counterexample [23] to Malle’s conjecture in terms of breaking thin maps of stacks. A variation of this notion, weakly breaking thin morphism, is closely related to Wood’s *fairness* [44]. The slightly more general case of classifying stacks BG with G an étale finite group scheme is studied in more detail in another paper [15] of the authors. It is proved there that our conjectures hold true for BG with G commutative.

In this paper, for the sake of simplicity, we focus on the case where the base field is a number field. However, almost every argument and statement in this paper can be easily translated to the case of an arbitrary global field, provided that we restrict ourselves to *tame* DM stacks, that is, DM stacks having stabilizers of orders coprime to the characteristic of the base field.

The paper is organized as follows. In Section 2, we set the basics of twisted sectors and ages and define the residue map, which plays an important role in the definition of our height function. In Section 3, we recall the correspondence between \mathbb{Q} -line bundles of a stack and ones of the coarse moduli space. We also recall the height function associated to a line bundle with an adelic metric in a generalized context of stacks. In Section 4, we introduce the notion of raising data and define the height function associated to the pair of a line bundle and a raising datum. In Section 5, we formulate our first conjecture concerning Fano stacks. In Sections 6 and 7, we show that this conjecture is compatible with taking products of Fano stacks and with the Manin conjecture for Fano varieties with canonical singularities. In Section 8, we introduce the notions of the orbifold Néron–Severi space and the orbifold pseudo-effective cone. In Section 9, we define the a - and b -invariants of the pair of a line bundle and a raising function and formulate a few versions of the Batyrev–Manin conjecture for DM stacks. We also see how one of these conjecture

specializes to the first conjecture about Fano stacks as well as to Malle’s conjecture. We then reinterpret Klüners’ counterexample to Malle’s conjecture in our language. Lastly, we introduce the notion of c -comprehensiveness for finite groups, which gives a sufficient condition that for a finite group G , the set of connected G -torsors is disjoint from any weakly breaking thin subset.

1.1. Notation

Throughout the paper, we denote by F a number field. A place of F is usually denoted by v . We denote the set of places by M_F . The local field of F at a place v is denoted by F_v . If v is a finite place, the residue field of F_v is denoted by κ_v . We denote its cardinality by q_v . If S is a finite set of places including all infinite places, we denote the ring of S -integers by \mathcal{O}_S . We denote the absolute Galois group of F by Γ_F .

We denote by \mathcal{X} a DM stack over F satisfying a certain condition. We denote the dimension of \mathcal{X} by d . We denote by $\mathcal{X}(F)$ the groupoid of F -points, and by $\mathcal{X}\langle F \rangle$ the set of isomorphism classes of F -points.

For a scheme T , notations such as \mathcal{X}_T and G_T mean that these objects are *defined over* T . They typically arise as either the base change to T from another scheme or the extension to T from an open subscheme of T . When $T = \text{Spec } R$, we also write \mathcal{X}_R and G_R .

2. Twisted sectors, residue maps and ages

2.1. Twisted sectors

Let us fix a number field F .

Definition 2.1. A *nice stack over F* means a DM stack \mathcal{X} over F satisfying the following conditions:

- (1) \mathcal{X} is separated, geometrically irreducible, and smooth over F ,
- (2) a coarse moduli space of \mathcal{X} is a projective F -scheme,
- (3) \mathcal{X} is not isomorphic to $\text{Spec } F$.

Remark 2.2. We exclude $\text{Spec } F$ from nice stacks. One reason for this is that $\text{Spec } F$ has only one F -point and counting it is not very meaningful. Another reason is that the orbifold Néron–Severi space of $\text{Spec } F$ (see Definition 8.1) is the trivial/zero space and we cannot define meaningful a - and b -invariants. Note that we do not exclude the possibility that a nice stack may have $\text{Spec } F$ as its coarse moduli space. In fact, such stacks are of great interest in relation to Malle’s conjecture.

Throughout the rest of the paper, we denote by \mathcal{X} a nice stack, unless otherwise specified. We choose a finite set of places, $S \subset M_F$, including all infinite places such that there exists an irreducible, smooth, proper, and tame model $\mathcal{X}_{\mathcal{O}_S}$ of \mathcal{X} over \mathcal{O}_S . Here “tame”

means that for every point $x \in \mathcal{X}_{\mathcal{O}_S}(K)$ with K a field, the automorphism group $\text{Aut}(x)$ has order coprime to the characteristic of K . We see that such a finite set S exists.

Definition 2.3 (Stacks of twisted 0-jets). Let $\mu_{l,F}$ and μ_{l,\mathcal{O}_S} be the group schemes of l -th roots of unity over F and \mathcal{O}_S , respectively. Let $\mathbb{B} \mu_{l,F} := [\text{Spec } F/\mu_{l,F}]$ and $\mathbb{B} \mu_{l,\mathcal{O}_S} := [\text{Spec } \mathcal{O}_S/\mu_{l,\mathcal{O}_S}]$ be their classifying stacks, where the notation $[-/-]$ means a quotient stack. We define the *stack of twisted 0-jets* of \mathcal{X} by

$$\mathcal{J}_0 \mathcal{X} := \coprod_{l>0} \underline{\text{Hom}}_F^{\text{rep}}(\mathbb{B} \mu_{l,F}, \mathcal{X}).$$

Here $\underline{\text{Hom}}_F^{\text{rep}}(-, -)$ means the Hom stack of representable morphisms (see [36]). We also define

$$\mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S} := \coprod_{l>0} \underline{\text{Hom}}_{\mathcal{O}_S}^{\text{rep}}(\mathbb{B} \mu_{l,\mathcal{O}_S}, \mathcal{X}_{\mathcal{O}_S}).$$

Namely, for each F -scheme T , the fiber $(\mathcal{J}_0 \mathcal{X})(T)$ over T is the groupoid of representable T -morphisms $\mathbb{B} \mu_{l,T} \rightarrow \mathcal{X}_T$, with the subscript T meaning base change to T ; and similarly for $\mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S}$. From [36], $\mathcal{J}_0 \mathcal{X}$ and $\mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S}$ are DM stacks locally of finite type over F and \mathcal{O}_S , respectively. From [51, Lemma 6.5], they are, in fact, of finite type.

Remark 2.4. If $\text{I } \mathcal{X}$ denotes the inertia stack of \mathcal{X} , then we have a (noncanonical) isomorphism $(\mathcal{J}_0 \mathcal{X}) \otimes_F \bar{F} \cong (\text{I } \mathcal{X}) \otimes_F \bar{F}$ (see [47, Proposition 22]).

Remark 2.5. Stacks of twisted 0-jets were used in [46,47] to develop motivic integration over DM stacks. The same notion also appears in the orbifold Gromov–Witten theory [1,2]. In [2], it is called the *cyclotomic inertia stack*. Motivic integration (or a variant of it, p -adic integration) and Gromov–Witten theory for stacks may be regarded, respectively, as local and geometric analogues of counting rational points of stacks. This explains why looking at the stack of twisted 0-jets is natural in the context of the present paper.

Definition 2.6 (Sectors). We call the trivial connected component $\underline{\text{Hom}}_F(\mathbb{B} \mu_{1,F}, \mathcal{X})$ of $\mathcal{J}_0 \mathcal{X}$, which is canonically isomorphic to \mathcal{X} , the *nontwisted sector* and denote it again by \mathcal{X} , abusing notation. We call the other connected components *twisted sectors*. We call a connected component of $\mathcal{J}_0 \mathcal{X}$ a *sector*, whether twisted or nontwisted. We denote by $\pi_0(\mathcal{J}_0 \mathcal{X})$ (resp. $\pi_0^*(\mathcal{J}_0 \mathcal{X})$) the set of sectors (resp. twisted sectors).

Lemma 2.7. *The stack $\mathcal{J}_0 \mathcal{X}$ is smooth and proper over F . Moreover, if $\mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S}$ is flat over \mathcal{O}_S , then it is also smooth and proper over \mathcal{O}_S .*

Proof. By generic flatness, if we enlarge S , we can make $\mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S}$ flat over \mathcal{O}_S . Thus, the first assertion follows from the second. To show the second assertion, suppose that $\mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S}$ is flat over \mathcal{O}_S . It suffices to show that for each geometric point $\text{Spec } K \rightarrow \text{Spec } \mathcal{O}_S$, the fiber $\mathcal{J}_0 \mathcal{X}_K$ is smooth and proper over K . The K -stack $\mathcal{J}_0 \mathcal{X}_K$ is isomorphic to the inertia stack $\text{I } \mathcal{X}_K$ of \mathcal{X}_K , which is well-known to be smooth and proper. The properness of $\text{I } \mathcal{X}_K$ follows from the fact that the natural morphism $\text{I } \mathcal{X}_K \rightarrow \mathcal{X}_K$ is identified with the (say first) projection

$$\mathcal{X}_K \times_{\Delta, \mathcal{X}_K \times_K \mathcal{X}_K, \Delta} \mathcal{X}_K \rightarrow \mathcal{X}_K,$$

which is proper. The smoothness follows from the local description of the inertia stack [42, tag 0374]. ■

Corollary 2.8. *If $\mathcal{J}_0\mathcal{X}_{\mathcal{O}_S}$ is flat over \mathcal{O}_S , then the natural map $\pi_0(\mathcal{J}_0\mathcal{X}) \rightarrow \pi_0(\mathcal{J}_0\mathcal{X}_{\mathcal{O}_S})$ is bijective. Here $\pi_0(-)$ denotes the set of connected components.*

Proof. Let K/F be a finite Galois extension such that every connected component of $\mathcal{J}_0\mathcal{X}_K$ is geometrically connected. Let G be its Galois group. We have a natural G -action on $\pi_0(\mathcal{J}_0\mathcal{X}_K)$ and can identify $\pi_0(\mathcal{J}_0\mathcal{X})$ with the set of G -orbits in $\pi_0(\mathcal{J}_0\mathcal{X}_K)$.

Let \mathcal{O}_T be the integral closure of \mathcal{O}_S in K . Since $\mathcal{J}_0\mathcal{X}_{\mathcal{O}_T}$ is smooth over \mathcal{O}_T , its irreducible components are pairwise disjoint. This shows that connected components \mathcal{Y} of $\mathcal{J}_0\mathcal{X}_K$ have closures $\bar{\mathcal{Y}}$ in $\mathcal{J}_0\mathcal{X}_{\mathcal{O}_T}$ which are pairwise disjoint. It follows that $\pi_0(\mathcal{J}_0\mathcal{X}_K)$ and $\pi_0(\mathcal{J}_0\mathcal{X}_{\mathcal{O}_T})$ are canonically isomorphic G -sets. Accordingly, their orbit sets are identified, which implies the corollary. ■

Remark 2.9. For a more geometric (rather than algebraic) treatment of twisted sectors, we refer the reader to [4].

It is sometimes more useful to express $\mathcal{J}_0\mathcal{X}$ by using the pro-finite group scheme $\hat{\mu}_F := \varprojlim \mu_{l,F}$ instead of finite ones $\mu_{l,F}$, $l \in \mathbb{Z}_{>0}$. Note that since transition morphisms of the projective system $\mu_{l,F}$, $l \in \mathbb{Z}_{>0}$, are all affine morphisms, the projective limit $\varprojlim \mu_{l,F}$ exists as a scheme (see [20, Proposition 8.2.3] or [42, tag 01YX]) and has a natural structure of a group scheme. In other words, the limit $\varprojlim \mu_{l,F}$ as a functor from F -schemes to groups is (represented by) a group scheme; and similarly for $\hat{\mu}_{\mathcal{O}_S} := \varprojlim \mu_{l,\mathcal{O}_S}$.

To obtain such a description of $\mathcal{J}_0\mathcal{X}$, we need to slightly generalize the Hom stack as follows. Let R be a ring, let \mathbf{Sch}_R be the category of R -schemes and let \mathcal{U} and \mathcal{V} be two categories fibered in groupoids over \mathbf{Sch}_R . We define $\underline{\mathrm{Hom}}_F(\mathcal{U}, \mathcal{V})$ to be the category fibered in groupoids over \mathbf{Sch}_R as follows: for each R -scheme T , the fiber $\underline{\mathrm{Hom}}_R(\mathcal{U}, \mathcal{V})(T)$ is

$$\mathrm{Hom}_T(\mathcal{U} \times_{\mathrm{Spec} R} T, \mathcal{V} \times_{\mathrm{Spec} R} T),$$

the groupoid of T -morphisms $\mathcal{U} \times_{\mathrm{Spec} R} T \rightarrow \mathcal{V} \times_{\mathrm{Spec} R} T$.

Lemma 2.10. *Let G be a group scheme over R . Let $\mathcal{K}_{\mathcal{X},G}$ be the fibered category over \mathbf{Sch}_R such that for each F -scheme T , the fiber $\mathcal{K}_{\mathcal{X},G}(T)$ is the groupoid of pairs (x, ϕ) , where x is an object of $\mathcal{X}(T)$ and ϕ is a homomorphism $G_T^{\mathrm{op}} \rightarrow \underline{\mathrm{Aut}}_T(x)$ of group schemes over T .¹ Then*

$$\underline{\mathrm{Hom}}_F(\mathrm{B}G, \mathcal{X}) \cong \mathcal{K}_{\mathcal{X},G}.$$

Proof. Let T be an R -scheme. We will construct functors between the groupoids $\mathrm{Hom}_T(\mathrm{B}G_T, \mathcal{X}_T)$ and $\mathcal{K}_{\mathcal{X},G}(T)$ which are quasi-inverses of each other.

¹We follow the convention that a group acts on a torsor *from the left*. Then the automorphism group of a G -torsor is identified with a subgroup of the opposite group G^{op} .

Construction of $\text{Hom}_T(\text{B } G_T, \mathcal{X}_T) \rightarrow \mathcal{K}_{\mathcal{X},G}(T)$: Let $f: \text{B } G_T \rightarrow \mathcal{X}_T$ be a T -morphism. Suppose that the trivial torsor $\tau: G \times T \rightarrow T$, which is an object of $(\text{B } G_T)(T)$, maps to $\alpha \in \mathcal{X}_T(T) = \mathcal{X}(T)$ by f . The morphism f induces a morphism of group schemes over T ,

$$G_T^{\text{op}} \xrightarrow{\sim} \underline{\text{Aut}}_T(\tau) \rightarrow \underline{\text{Aut}}_T(\alpha).$$

Construction of $\mathcal{K}_{\mathcal{X},G}(T) \rightarrow \text{Hom}_T(\text{B } G_T, \mathcal{X}_T)$: Let α be an object of $\mathcal{X}(T)$ and $\phi: G_T^{\text{op}} \rightarrow \underline{\text{Aut}}_T(\alpha)$ a homomorphism. Let $[G_T^{\text{op}} \rightrightarrows T]'$ (resp. $[G_T^{\text{op}} \rightrightarrows T]$) be the prestack (resp. stack) associated to the trivial groupoid scheme $G_T^{\text{op}} \rightrightarrows T$. We define a morphism $[G_T^{\text{op}} \rightrightarrows T]' \rightarrow \mathcal{X}_T$ so that a morphism $S \rightarrow T$, regarded as an object in $[T \rightrightarrows G_T^{\text{op}}]'(S)$, maps to α_S , and a morphism $S \rightarrow G_T^{\text{op}}$, regarded as a morphism in $[T \rightrightarrows G_T^{\text{op}}]'(S)$, maps to the automorphism of α_S corresponding to the composition $S \rightarrow G_T^{\text{op}} \xrightarrow{\phi} \underline{\text{Aut}}_T(\alpha)$. This induces a morphism of stacks $\text{B } G_T = [T \rightrightarrows G_T^{\text{op}}] \rightarrow \mathcal{X}_T$.

It is easy to see that these functors between $\text{Hom}_T(\text{B } G_T, \mathcal{X}_T)$ and $\mathcal{K}_{\mathcal{X},G}(T)$ are quasi-inverses of each other, and induce the claimed isomorphism of the lemma. ■

Proposition 2.11. *We have*

$$\mathcal{J}_0 \mathcal{X} \cong \underline{\text{Hom}}_F(\text{B } \hat{\mu}_F, \mathcal{X}) \quad \text{and} \quad \mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S} \cong \underline{\text{Hom}}_{\mathcal{O}_S}(\text{B } \hat{\mu}_{\mathcal{O}_S}, \mathcal{X}_{\mathcal{O}_S}).$$

Proof. For an F -scheme T , a representable morphism $\text{B } \mu_{l,T} \rightarrow \mathcal{X}$ induces the (non-representable) morphism

$$\text{B } \hat{\mu}_T \rightarrow \text{B } \mu_{l,T} \rightarrow \mathcal{X}.$$

This induces a morphism $\mathcal{J}_0 \mathcal{X} \rightarrow \underline{\text{Hom}}_F(\text{B } \hat{\mu}_F, \mathcal{X})$. Conversely, for a morphism $\text{B } \hat{\mu}_T \rightarrow \mathcal{X}$, we have the associated morphism

$$\hat{\mu}_T \rightarrow \underline{\text{Aut}}_T(x) \quad (x \in \mathcal{X}(F))$$

from Lemma 2.10. Each point $t \in T$ has an open neighborhood $U \subset T$ such that the induced morphism $\hat{\mu}_U \rightarrow \underline{\text{Aut}}_U(x_U)$ canonically factors into the composition

$$\hat{\mu}_U \rightarrow \mu_{l,U} \twoheadrightarrow \underline{\text{Aut}}_U(x_U)$$

of the canonical surjection $\hat{\mu}_U \rightarrow \mu_{l,U}$ and an injection $\mu_{l,U} \twoheadrightarrow \underline{\text{Aut}}_U(x_U)$. Thus, we get an object of $\mathcal{J}_0 \mathcal{X}$ over U . They glue together to give an object over T . Thus, we get a morphism $\underline{\text{Hom}}_F(\text{B } \hat{\mu}_F, \mathcal{X}) \rightarrow \mathcal{J}_0 \mathcal{X}$. We see that these morphisms between $\mathcal{J}_0 \mathcal{X}$ and $\underline{\text{Hom}}_F(\text{B } \hat{\mu}_F, \mathcal{X})$ are quasi-inverses to each other, which shows the first isomorphism of the proposition. The second isomorphism is proved in the same way. ■

Remark 2.12. The stack $\text{B } \hat{\mu}_F$ is neither a DM or Artin stack.

Remark 2.13. If e is a sufficiently factorial positive integer, then we can describe $\mathcal{J}_0 \mathcal{X} = \underline{\text{Hom}}_F(\text{B } \hat{\mu}_F, \mathcal{X})$ also as $\underline{\text{Hom}}_F(\text{B } \mu_{e,F}, \mathcal{X})$.

Corollary 2.14. *For a (not necessarily representable) morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$ of nice stacks, we have a natural morphism $\mathcal{J}_0 \mathcal{Y} \rightarrow \mathcal{J}_0 \mathcal{X}$ and a natural map $\pi_0(\mathcal{J}_0 \mathcal{Y}) \rightarrow \pi_0(\mathcal{J}_0 \mathcal{X})$. Moreover, if $f_{\mathcal{O}_S}: \mathcal{Y}_{\mathcal{O}_S} \rightarrow \mathcal{X}_{\mathcal{O}_S}$ is a model of f , then we have a natural morphism $\mathcal{J}_0 \mathcal{Y}_{\mathcal{O}_S} \rightarrow \mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S}$.*

Proof. The natural morphism $\mathcal{J}_0\mathcal{Y} \rightarrow \mathcal{J}_0\mathcal{X}$ is defined by sending a morphism $h: \mathbb{B}\hat{\mu}_T \rightarrow \mathcal{Y}$ to the composition $f \circ h: \mathbb{B}\hat{\mu}_T \rightarrow \mathcal{Y} \rightarrow \mathcal{X}$. This morphism induces a map $\pi_0(\mathcal{J}_0\mathcal{Y}) \rightarrow \pi_0(\mathcal{J}_0\mathcal{X})$. The morphism $\mathcal{J}_0\mathcal{Y}_{\mathcal{O}_S} \rightarrow \mathcal{J}_0\mathcal{X}_{\mathcal{O}_S}$ is defined similarly. ■

Example 2.15. Let G be a finite étale group scheme over F and let $\mathcal{X} = \mathbb{B}G$. Let \bar{F} be an algebraic closure of F . Then we have the identifications

$$\begin{aligned} |\mathcal{J}_0\mathcal{X}_{\bar{F}}| &= \pi_0(\mathcal{J}_0\mathcal{X}_{\bar{F}}) \\ &= \text{Hom}(\hat{\mu}^{\text{op}}(\bar{F}), G^{\text{op}}(\bar{F}))/G^{\text{op}}(\bar{F}) \\ &= \text{Hom}(\hat{\mu}(\bar{F}), G(\bar{F}))/G(\bar{F}), \end{aligned}$$

where $|\ - |$ denotes the point set of a stack, the $G(\bar{F})$ -action on $\text{Hom}(\hat{\mu}(\bar{F}), G(\bar{F}))$ is induced by the conjugate action of $G(\bar{F})$ on itself, and similarly for the $G^{\text{op}}(\bar{F})$ -action on $\text{Hom}(\hat{\mu}^{\text{op}}(\bar{F}), G^{\text{op}}(\bar{F}))$. If e denotes the exponent of $G(\bar{F})$, then we also have

$$\text{Hom}(\hat{\mu}(\bar{F}), G(\bar{F}))/G(\bar{F}) = \text{Hom}(\mu_e(\bar{F}), G(\bar{F}))/G(\bar{F}).$$

The last set is denoted by G_* in [15]. These identifications are equivariant for natural Γ_F -actions. We can then identify $\pi_0(\mathcal{J}_0\mathcal{X})$ with the set of Γ_F -orbits in one of these sets. If G is a constant group, then $\pi_0(\mathcal{J}_0\mathcal{X})$ is identified with the set of F -conjugacy classes. If $G = \mu_l$, then $\pi_0(\mathcal{J}_0\mathcal{X})$ is identified with $\mathbb{Z}/l\mathbb{Z}$ (regardless of the field F). For these facts, see [15] (cf. [45, Theorem 5.4], [48, Proposition 4.5], [49, Proposition 8.5]).

2.2. Residue maps

Let v be a finite place of F with $v \notin S$ and let L be the maximal unramified extension of F_v . For a positive integer l coprime to the residue characteristic of v , let L_l denote the unique degree- l extension of L , which is given by adjoining the l -th roots of a uniformizer, and let \mathcal{O}_{L_l} denote its integer ring. There exists a natural action of μ_{l, \mathcal{O}_L} on $\text{Spec } \mathcal{O}_{L_l}$, which induces the quotient stack $[\text{Spec } \mathcal{O}_{L_l} / \mu_{l, \mathcal{O}_L}]$. Namely, this is the l -th root stack associated to $\text{Spec } \mathcal{O}_{L_l}$ and its closed point regarded as a Cartier divisor. We can regard $\text{Spec } L$ as an open substack of $[\text{Spec } \mathcal{O}_{L_l} / \mu_{l, \mathcal{O}_L}]$ by the canonical open immersion.

Lemma 2.16. *With the above notation, for an L -point $x: \text{Spec } L \rightarrow \mathcal{X}$, there exists a unique positive integer l and a unique representable morphism*

$$\tilde{x}: [\text{Spec } \mathcal{O}_{L_l} / \mu_{l, \mathcal{O}_L}] \rightarrow \mathcal{X}_{\mathcal{O}_S}$$

which extends x .

Proof. Let $X_{\mathcal{O}_S}$ be the coarse moduli space of $\mathcal{X}_{\mathcal{O}_S}$. The induced L -point, $\bar{x}: \text{Spec } L \rightarrow X$, uniquely extends to an integral point $\text{Spec } \mathcal{O}_L \rightarrow X_{\mathcal{O}_L}$, which is a closed immersion. Replacing $X_{\mathcal{O}_L}$ with an open neighborhood of this integral point, we may reduce the problem to the case where $X_{\mathcal{O}_L}$ is affine, say $X_{\mathcal{O}_L} = \text{Spec } R$. Let $\hat{X}_{\mathcal{O}_L}$ be the formal completion $\text{Spec } \hat{R}$ (not as a formal scheme but as a scheme) of $X_{\mathcal{O}_L}$

along this integral point and let $\widehat{\mathcal{X}}_{\mathcal{O}_S} := \widehat{\mathcal{X}}_{\mathcal{O}_S} \times_{\mathcal{X}_{\mathcal{O}_S}} \mathcal{X}$. We have a natural morphism $\text{Spec } L \rightarrow \widehat{\mathcal{X}}_{\mathcal{O}_S}$. Let \mathcal{D} be the relative normalization of $\widehat{\mathcal{X}}_{\mathcal{O}_S}$ in $\text{Spec } L$. Note that the relative normalization for schemes is compatible with étale base change [42, tag 0ABP], and hence generalizes to DM stacks. Then \mathcal{D} is a regular, irreducible and one-dimensional DM stack which contains $\text{Spec } L$ as an open dense substack and has $\text{Spec } \mathcal{O}_L$ as the coarse moduli space.

We claim that \mathcal{D} is of the form $[\text{Spec } \mathcal{O}_{L_l} / \mu_{l, \mathcal{O}_L}]$ for some positive integer l coprime to the residue characteristic of v . Indeed, there exists an étale atlas of \mathcal{D} of the form

$$\text{Spec } \mathcal{O}_{L_l} \rightarrow \mathcal{D}$$

for a tame finite extension L_l/L . Consider the associated groupoid scheme

$$\text{Spec } \mathcal{O}_{L_l} \times_{\mathcal{D}} \text{Spec } \mathcal{O}_{L_l} \rightrightarrows \text{Spec } \mathcal{O}_{L_l}.$$

Since L_l has an algebraically closed residue field, the scheme $\text{Spec } \mathcal{O}_{L_l} \times_{\mathcal{D}} \text{Spec } \mathcal{O}_{L_l}$ is isomorphic to a disjoint union of copies of $\text{Spec } \mathcal{O}_{L_l}$. We conclude that the above groupoid scheme is isomorphic to the groupoid scheme

$$\text{Spec } \mathcal{O}_{L_l} \times G \rightrightarrows \text{Spec } \mathcal{O}_{L_l}$$

associated to an action of a constant group G on $\text{Spec } \mathcal{O}_{L_l}$. Thus, $\mathcal{D} \cong [\text{Spec } \mathcal{O}_{L_l} / G]$. Since \mathcal{D} contains $\text{Spec } L$ as an open dense substack and has $\text{Spec } \mathcal{O}_L$ as the coarse moduli space, the G -action on $\text{Spec } L_l$ is free, and G is the Galois group of L_l/L and isomorphic to μ_{l, \mathcal{O}_L} for some l . We have proved the claim.

The induced morphism $\mathcal{D} \rightarrow \widehat{\mathcal{X}}_{\mathcal{O}_S}$ is representable, since it becomes a relative normalization of schemes after the base change to an étale atlas of the target. It follows that $\mathcal{D} \rightarrow \mathcal{X}_{\mathcal{O}_S}$ is also representable. In summary, we have proved the existence of a positive integer and a representable morphism as claimed in the lemma. To show uniqueness, consider an arbitrary morphism $\mathcal{D}' = [\text{Spec } \mathcal{O}_{L_{l'}} / \mu_{l', \mathcal{O}_L}] \rightarrow \mathcal{X}_{\mathcal{O}_S}$ as in the lemma. This induces a representable morphism $\mathcal{D}' \rightarrow \widehat{\mathcal{X}}_{\mathcal{O}_S}$, which is again a relative normalization. The uniqueness of relative normalization shows that there exists an isomorphism $\mathcal{D} \cong \mathcal{D}'$ compatible with the morphisms to $\widehat{\mathcal{X}}_{\mathcal{O}_S}$ as well as the ones to $\mathcal{X}_{\mathcal{O}_S}$, which, in turn, shows $l = l'$. This completes the proof of the lemma. ■

We keep the notation of Lemma 2.16 and let k be the residue field of L , which is an algebraic closure of κ_v . Suppose that $\mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S}$ is flat at the point of $\text{Spec } \mathcal{O}_S$ corresponding to v . The lemma shows that each L -point x of \mathcal{X} induces the representable morphism

$$\text{B } \mu_{l, k} \hookrightarrow [\text{Spec } \mathcal{O}_{L_l} / \mu_{l, \mathcal{O}_L}] \xrightarrow{\tilde{x}} \mathcal{X}_{\mathcal{O}_S},$$

which is a k -point of $\mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S}$. Thus, for each $v \in M_F \setminus S$, we get the composite functor

$$\mathcal{X}(F_v) \rightarrow \mathcal{X}(L) \rightarrow (\mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S})(k).$$

In particular, each F_v -point of \mathcal{X} determines a connected component of $\mathcal{J}_0\mathcal{X}_{\mathcal{O}_S}$. From Corollary 2.8, we have the corresponding sector of \mathcal{X} ; namely, for each $v \in M_F \setminus S$, we get a map

$$\psi_v: \mathcal{X}\langle F_v \rangle \rightarrow \pi_0(\mathcal{J}_0\mathcal{X}),$$

where $\mathcal{X}\langle F_v \rangle$ denotes the set of isomorphism classes in the groupoid $\mathcal{X}(F_v)$.

Definition 2.17. We call the map ψ_v the *residue map* of \mathcal{X} at v , and $\psi_v(x)$ the *residue* of x .

From [35, Section 2.1] (see also [11, Section 2.4]), $\mathcal{X}\langle F_v \rangle$ has a natural topology.

Proposition 2.18. *The residue map ψ_v is locally constant. Equivalently, it is continuous for the discrete topology on $\pi_0(\mathcal{J}_0\mathcal{X})$.*

Proof. We keep the notation of the proof of Lemma 2.16. Let $\mathcal{D}^l := [\mathrm{Spec} \mathcal{O}_{L_l} / \mu_{l, \mathcal{O}_L}]$. We consider the stack

$$\mathcal{Y}_l := \underline{\mathrm{Hom}}_{\mathcal{O}_L}^{\mathrm{rep}}(\mathcal{D}^l, \mathcal{X}_{\mathcal{O}_L}).$$

Let $\mathcal{Y}_l = \coprod_{i=1}^n \mathcal{Y}_{l,i}$ be the decomposition into connected components. Then the natural map

$$(\mathrm{Hom}(\mathcal{D}^l, \mathcal{X}_{\mathcal{O}_L}) / \cong) = \mathcal{Y}_l\langle \mathcal{O}_L \rangle \rightarrow \pi_0(\mathcal{J}_0\mathcal{X})$$

is constant on each open and closed subset $\mathcal{Y}_{l,i}\langle \mathcal{O}_L \rangle \subset \mathcal{Y}\langle \mathcal{O}_L \rangle$. Since

$$\mathcal{Y}_l \otimes_{\mathcal{O}_L} L = \underline{\mathrm{Hom}}_L^{\mathrm{rep}}(\mathcal{D}^l \otimes_{\mathcal{O}_L} L, \mathcal{X}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} L) = \underline{\mathrm{Hom}}_L^{\mathrm{rep}}(\mathrm{Spec} L, \mathcal{X}) = \mathcal{X}_L,$$

we have $\mathcal{Y}_l\langle L \rangle = \mathcal{X}\langle L \rangle$. From [11, Proposition 2.9 (e)], the maps

$$\mathcal{Y}_l\langle \mathcal{O}_L \rangle \rightarrow \mathcal{Y}_l\langle L \rangle = \mathcal{X}\langle L \rangle$$

are open, and hence the images of $\mathcal{Y}_{l,i}\langle \mathcal{O}_L \rangle$ in $\mathcal{X}\langle L \rangle$ are open. Since $\mathcal{X}\langle L \rangle$ is covered by the images of these maps, we see that $\mathcal{X}\langle L \rangle \rightarrow \pi_0(\mathcal{J}_0\mathcal{X})$ is locally constant and continuous. From [11, Corollary 2.7], $\mathcal{X}\langle F_v \rangle \rightarrow \mathcal{X}\langle L \rangle$ is continuous. We conclude that the composition $\psi_v: \mathcal{X}\langle F_v \rangle \rightarrow \mathcal{X}\langle L \rangle \rightarrow \pi_0(\mathcal{J}_0\mathcal{X})$ is also continuous. ■

Remark 2.19. Representable morphisms $[\mathrm{Spec} \mathcal{O}_{L,l} / \mu_{l, \mathcal{O}_L}] \rightarrow \mathcal{X}_{\mathcal{O}_S}$ as in Lemma 2.16 are a version of twisted arcs [46, 47, 50, 51]. Associating twisted 0-jets $B \mu_{l,k} \rightarrow \mathcal{X}_{\mathcal{O}_S}$ to them is a special case of the truncation map. Morphisms $[\mathrm{Spec} \mathcal{O}_{L,l} / \mu_{l, \mathcal{O}_L}] \rightarrow \mathcal{X}_{\mathcal{O}_S}$ are also an analogue of twisted stable maps [3] and of tuning stacks [16].

Proposition 2.20. *Let \mathcal{Y} and \mathcal{X} be nice stacks having models $\mathcal{Y}_{\mathcal{O}_S}$ and $\mathcal{X}_{\mathcal{O}_S}$ over \mathcal{O}_S , respectively. Let $\mathcal{Y}_{\mathcal{O}_S} \rightarrow \mathcal{X}_{\mathcal{O}_S}$ be a morphism over \mathcal{O}_S . Suppose that $\mathcal{J}_0\mathcal{Y}_{\mathcal{O}_S}$ and $\mathcal{J}_0\mathcal{X}_{\mathcal{O}_S}$ are flat over \mathcal{O}_S . Then, for every finite place $v \notin S$, the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{Y}\langle F_v \rangle & \longrightarrow & \mathcal{X}\langle F_v \rangle \\ \psi_v \downarrow & & \downarrow \psi_v \\ \pi_0(\mathcal{J}_0\mathcal{Y}) & \longrightarrow & \pi_0(\mathcal{J}_0\mathcal{X}) \end{array}$$

Here the horizontal morphisms are induced by the morphism $\mathcal{Y}_{\mathcal{O}_S} \rightarrow \mathcal{X}_{\mathcal{O}_S}$.

Proof. As before, we denote by L the maximal unramified extension of F_v and by k its residue field. An F_v -point $y: \text{Spec } F_v \rightarrow \mathcal{Y}$ induces an L -point $y_L: \text{Spec } L \rightarrow \mathcal{Y}$. It follows from Lemma 2.16 that this L -point uniquely extends to a representable morphism $\widetilde{y}_L: [\text{Spec } \mathcal{O}_{L_l} / \mu_{l, \mathcal{O}_L}] \rightarrow \mathcal{Y}_{\mathcal{O}_S}$. There exists a unique divisor m of l and a unique representable morphism $\widetilde{x}_L: [\text{Spec } \mathcal{O}_{L_m} / \mu_{m, \mathcal{O}_L}] \rightarrow \mathcal{X}_{\mathcal{O}_S}$ making the following diagram commutative:

$$\begin{array}{ccccc}
 \mathbb{B} \mu_{l,k} & \longrightarrow & [\text{Spec } \mathcal{O}_{L_l} / \mu_{l, \mathcal{O}_L}] & \xrightarrow{\widetilde{y}_L} & \mathcal{Y}_{\mathcal{O}_S} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbb{B} \mu_{m,k} & \longrightarrow & [\text{Spec } \mathcal{O}_{L_m} / \mu_{m, \mathcal{O}_L}] & \xrightarrow{\widetilde{x}_L} & \mathcal{X}_{\mathcal{O}_S}
 \end{array}$$

Here arrows without a label are natural morphisms and all horizontal arrows are representable morphisms. This is basically [47, Proposition 23] with $n = 0, \infty$, except that we work over a different base ring, but the proof there works also in our situation. We also see that \widetilde{x}_L is the same as the morphism induced from the point $x := f(y) \in \mathcal{X}(F_v)$. Let

$$y_k: \text{Spec } k \rightarrow \mathbb{B} \mu_{l,k} \rightarrow \mathcal{Y}_{\mathcal{O}_S} \quad \text{and} \quad x_k = f(y_k): \text{Spec } k \rightarrow \mathbb{B} \mu_{m,k} \rightarrow \mathcal{X}_{\mathcal{O}_S}$$

be the induced k -points. The injections $\mu_{l,k} \hookrightarrow \underline{\text{Aut}}_k(y_k)$ and $\mu_{m,k} \hookrightarrow \underline{\text{Aut}}_k(x_k)$ that correspond to $\mathbb{B} \mu_{l,k} \rightarrow \mathcal{Y}_{\mathcal{O}_S}$ and $\mathbb{B} \mu_{m,k} \rightarrow \mathcal{X}_{\mathcal{O}_S}$ respectively fit into the commutative diagram

$$\begin{array}{ccccc}
 \widehat{\mu}_k & \twoheadrightarrow & \mu_{l,k} & \hookrightarrow & \underline{\text{Aut}}_k(y_k) \\
 & \searrow & \downarrow & & \downarrow \\
 & & \mu_{m,k} & \hookrightarrow & \underline{\text{Aut}}_k(x_k)
 \end{array}$$

This shows that if $\beta \in (\mathcal{J}_0 \mathcal{Y}_{\mathcal{O}_S})(k)$ and $\alpha \in (\mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S})(k)$ denote the k -points corresponding to $\mathbb{B} \mu_{l,k} \rightarrow \mathcal{Y}_{\mathcal{O}_S}$ and $\mathbb{B} \mu_{m,k} \rightarrow \mathcal{X}_{\mathcal{O}_S}$ respectively, then β maps to α by the morphism $\mathcal{J}_0 \mathcal{Y}_{\mathcal{O}_S} \rightarrow \mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S}$.

The residue $\psi_v(y)$ of the F_v -point y is the sector containing β and the residue $\psi_v(x)$ of the F_v -point $x = f(y)$ is the sector containing α . Thus, the above fact that β maps to α proves the proposition. ■

2.3. Ages

A point \widetilde{x} of $(\mathcal{J}_0 \mathcal{X})(\overline{F})$ is represented by the pair (x, ι) of a point $x \in \mathcal{X}(\overline{F})$ and a group monomorphism $\iota: \mu_l \hookrightarrow \text{Aut}(x)$. Letting \mathcal{V} be a vector bundle of rank r on \mathcal{X} , we get a representation of the group $\mu_l \subset \overline{F}$:

$$\rho_{\widetilde{x}}: \mu_l \hookrightarrow \text{Aut}(x) \rightarrow \text{GL}(\mathcal{V}_x) \cong \text{GL}_r(\overline{F}).$$

Here \mathcal{V}_x is the fiber of \mathcal{V} over x , in particular, an \overline{F} -vector space of dimension r . If τ denotes the standard one-dimensional representation

$$\tau: \mu_l \hookrightarrow \overline{F}^* = \text{GL}_1(\overline{F}),$$

then we can write

$$\rho_{\tilde{x}} \cong \bigoplus_{i=1}^r \tau^{a_i} \quad (a_i \in \{0, 1, \dots, l - 1\}).$$

Definition 2.21. We define the *age* of \tilde{x} with respect to \mathcal{V} to be

$$\text{age}(\tilde{x}; \mathcal{V}) := \frac{1}{l} \sum_{i=1}^d a_i \in \mathbb{Q}_{\geq 0}.$$

When \mathcal{V} is the tangent bundle $T\mathcal{X}$, we simply call it the *age* of \tilde{x} and denote it by $\text{age}(\tilde{x})$. When we wish to specify the stack \mathcal{X} in question, we write $\text{age}_{\mathcal{X}}(\tilde{x})$.

Lemma 2.22. *If we fix a vector bundle \mathcal{V} , then the age $\text{age}(\tilde{x}; \mathcal{V})$ depends only on the sector to which \tilde{x} belongs.*

Proof. We omit the proof, as this is well-known for $V = T\mathcal{X}$ (for example, see [47, p. 743]) and there is no essential difference in the general case. ■

Definition 2.23. For a sector \mathcal{Y} , we define $\text{age}(\mathcal{Y}; \mathcal{V})$ to be $\text{age}(\tilde{x}; \mathcal{V})$ for any $\tilde{x} \in \mathcal{Y}(\bar{F})$. When $\mathcal{V} = T\mathcal{X}$, we write simply $\text{age}(\mathcal{Y})$.

3. Line bundles and stable heights

Let \mathcal{X} be a nice stack over F .

Definition 3.1. The *Picard group* of \mathcal{X} , denoted by $\text{Pic}(\mathcal{X})$, is the group of isomorphism classes of line bundles on \mathcal{X} ; the group structure is given by tensor product. We define $\text{Pic}(\mathcal{X})_{\mathbb{Q}} := \text{Pic}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and call its elements *\mathbb{Q} -line bundles*.

A \mathbb{Q} -line bundle is thus represented by a formal product $\bigotimes_{i=1}^n \mathcal{L}_i^{s_i}$ of line bundles with $s_i \in \mathbb{Q}$ or even by a formal power \mathcal{L}^s , $s \in \mathbb{Q}$, of a single line bundle \mathcal{L} . Let $\pi: \mathcal{X} \rightarrow \bar{\mathcal{X}} = X$ be the coarse moduli space morphism. We have the pullback map $\pi^*: \text{Pic}(X)_{\mathbb{Q}} \rightarrow \text{Pic}(\mathcal{X})_{\mathbb{Q}}$.

Proposition 3.2. *The map $\pi^*: \text{Pic}(X)_{\mathbb{Q}} \rightarrow \text{Pic}(\mathcal{X})_{\mathbb{Q}}$ is an isomorphism.*

Proof. Let r be a positive integer factorial enough so that the automorphism group of every point of \mathcal{X} has order dividing r . For any line bundle \mathcal{L} on \mathcal{X} , $\pi_*(\mathcal{L}^r)$ is a line bundle on X . The \mathbb{Q} -linear map $\text{Pic}(\mathcal{X})_{\mathbb{Q}} \rightarrow \text{Pic}(X)_{\mathbb{Q}}$ sending \mathcal{L} to the \mathbb{Q} -line bundle $(\pi_*(\mathcal{L}^r))^{1/r}$ is the inverse of π^* . ■

Since \mathcal{X} is smooth, it has the *canonical line bundle* $\omega_{\mathcal{X}} := \det(\Omega_{\mathcal{X}/F})$. On the other hand, the coarse moduli space X is not generally smooth, but has quotient singularities. In particular, X is \mathbb{Q} -factorial. The canonical sheaf ω_X is defined to be the unique reflexive sheaf such that $\omega_X|_{X_{\text{sm}}} = \omega_{X_{\text{sm}}}$ with X_{sm} denoting the smooth locus. This is not a line bundle in general. However, for some positive integer r , the r -th reflexive power $\omega_X^{[r]}$,

which can be constructed as the double dual $(\omega_X^r)^{\vee\vee}$ of ω_X^r , is an invertible sheaf. See [22, 24] for more details. Thus, we get the canonical \mathbb{Q} -line bundle of X as $(\omega_X^{[r]})^{1/r}$. The corresponding element of $\text{Pic}(X)_{\mathbb{Q}}$ is independent of the choice of r . The canonical line bundle ω_X of X and the canonical \mathbb{Q} -line bundle $(\omega_X^{[r]})^{1/r}$ of X do not generally correspond to each other under the isomorphism $\text{Pic}(X)_{\mathbb{Q}} \cong \text{Pic}(X)_{\mathbb{Q}}$. However, if $X \rightarrow X$ is étale in codimension 1, then they do. This is the case, for example, when $X \rightarrow X$ is a gerbe or when X has the trivial generic stabilizer and has no reflection (Definition 7.1).

For a place $v \in M_F$, we denote by $|\cdot|_v$ the v -adic norm on F_v normalized as follows. For finite places, we normalize as $|\pi_v|_v = q_v^{-1}$ with π_v a uniformizer of F_v . For a real place, we let $|\cdot|_v$ be the standard absolute value. For a complex place, we let $|\cdot|_v$ be the square of the standard absolute value. Let \mathcal{L} be a line bundle on X and let $v \in M_F$. We have a continuous map

$$\eta_v: \mathcal{L}\langle F_v \rangle \rightarrow X\langle F_v \rangle.$$

This is not quite a line bundle, but has a structure close to it. For an isomorphism class $[x] \in X\langle F_v \rangle$, the fiber $\eta_v^{-1}([x])$ is the quotient of the line $x^*\mathcal{L} \cong F_v$ by the action of $\text{Aut}(x)$. Since this action preserves the zero vector, we get the zero section $X\langle F_v \rangle \rightarrow \mathcal{L}\langle F_v \rangle$ and it makes sense to say that an element of $\mathcal{L}\langle F_v \rangle$ is zero or not. The following definition restates [13, Definition 4.3.1.1] in a slightly different way:

Definition 3.3. A v -adic metric on \mathcal{L} means a continuous map $\|\cdot\|_v: \mathcal{L}\langle F_v \rangle \rightarrow \mathbb{R}_{\geq 0}$ such that

- (1) for $a \in F_v$ and $[s] \in \mathcal{L}\langle F_v \rangle$, $\|[as]\|_v = |a|_v \cdot \|[s]\|_v$,
- (2) for each F_v -morphism $f: U \rightarrow X_{F_v}$ from an F_v -scheme U of finite type and for every section $s: U(F_v) \rightarrow \mathcal{L}_U(F_v)$, the composition

$$U(F_v) \xrightarrow{s} \mathcal{L}_U(F_v) \rightarrow \mathcal{L}\langle F_v \rangle \xrightarrow{\|\cdot\|_v} \mathbb{R}_{\geq 0}$$

is continuous.

As in the case of varieties, for almost every place v , there exists a canonical choice of v -adic metric on the given \mathcal{L} derived from a model. Let $X_{\mathcal{O}_S}$ be a model of X over \mathcal{O}_S and let $\mathcal{L}_{\mathcal{O}_S}$ be a line bundle on $X_{\mathcal{O}_S}$ which is a model of \mathcal{L} . For $v \in M_F \setminus S$ and for $x \in X(F_v)$, the \overline{F}_v -point $x_{\overline{F}_v} \in X(\overline{F}_v)$ induced from x uniquely extends to an $\overline{\mathcal{O}_{F_v}}$ -point $x_{\overline{\mathcal{O}_{F_v}}} \in X(\overline{\mathcal{O}_{F_v}})$. Therefore, the one-dimensional \overline{F}_v -vector space $(x_{\overline{F}_v})^*\mathcal{L}$ has the canonical lattice $(x_{\overline{\mathcal{O}_{F_v}}})^*\mathcal{L}_{\mathcal{O}_S} \cong \overline{\mathcal{O}_{F_v}}$. We denote again by $|\cdot|_v$ the unique extension of $|\cdot|_v$ to \overline{F}_v . We choose an isomorphism $(x_{\overline{F}_v})^*\mathcal{L} \cong \overline{F}_v$ under which $(x_{\overline{\mathcal{O}_{F_v}}})^*\mathcal{L}_{\mathcal{O}_S}$ and $\overline{\mathcal{O}_{F_v}}$ correspond to each other, which induces a norm $\|\cdot\|_v$ on $(x_{\overline{F}_v})^*\mathcal{L}$ corresponding to $|\cdot|_v$ on \overline{F}_v .

$$\begin{array}{ccc}
 \mathcal{O}_{\overline{F}_v} & \hookrightarrow & \overline{F}_v & \xrightarrow{|\cdot|_v} & \mathbb{R} \\
 \parallel \wr & & \parallel \wr & \nearrow & \\
 (x_{\overline{\mathcal{O}_{F_v}}})^*\mathcal{L}_{\mathcal{O}_S} & \hookrightarrow & (x_{\overline{F}_v})^*\mathcal{L} & &
 \end{array}$$

The norm $\|\cdot\|_v$ given in this way is independent of the choice of the isomorphism $(x_{\overline{F}_v})^* \mathcal{L} \cong \overline{F}_v$ as above and invariant under the actions of $\text{Aut}(x_{\overline{F}_v})$ and $\text{Aut}(x)$. Thus, we get a continuous map $\|\cdot\|_v: \mathcal{L}(F_v) \rightarrow \mathbb{R}_{\geq 0}$.

Definition 3.4 (Adelic metric). For $v \in M_F \setminus S$, the v -adic metric on \mathcal{L} induced by the model $\mathcal{L}_{\mathcal{O}_S}$ is defined to be the composite map

$$\mathcal{L}(F_v) \rightarrow \mathcal{L}(\overline{F}_v) \xrightarrow{\|\cdot\|_v} \mathbb{R}_{\geq 0};$$

we denote it again by $\|\cdot\|_v$. An *adelic metric* on \mathcal{L} is a collection $(\|\cdot\|_v)_{v \in M_F}$ of v -adic metrics such that for almost every v , $\|\cdot\|_v$ is induced from a model.

Remark 3.5. When \mathcal{X} is a variety, the v -adic metric $\|\cdot\|_v$ induced by a model takes values in $\{0\} \cup q_v^{\mathbb{Z}} \subset \mathbb{R}_{\geq 0}$. This is no longer true for stacks. For a point $x \in \mathcal{X}(F_v)$, the map

$$F_v \cong x^* \mathcal{L} \rightarrow \mathcal{L}(F_v) \xrightarrow{\|\cdot\|_v} \mathbb{R}_{\geq 0}$$

takes values in $\{0\} \cup q_v^{r+\mathbb{Z}}$ for some $r \in \mathbb{Q}$.

Definition 3.6 (Stable height). Let \mathcal{L} be a line bundle on \mathcal{X} given with an adelic metric $(\|\cdot\|_v)_{v \in M_F}$ and let $x \in \mathcal{X}(F)$. We choose a nonzero element $0 \neq s \in x^* \mathcal{L} \cong F$. For each place $v \in M_F$, we get the induced F_v -point $x_v \in \mathcal{X}(F_v)$ and $0 \neq s_v \in (x_v)^* \mathcal{L} \cong F_v$. We define the *height function* $H_{\mathcal{L}}: \mathcal{X}(F) \rightarrow \mathbb{R}$ by

$$H_{\mathcal{L}}(x) := \prod_{v \in M_F} \|s_v\|_v^{-1}.$$

This function is well-defined; it is independent of the choice of an F -point x from an isomorphism class as well as of the choice of s . If \mathcal{L} and \mathcal{L}' are line bundles given with adelic metrics, then the tensor product $\mathcal{L} \otimes \mathcal{L}'$ has the naturally induced adelic metric. The height function has the following multiplicativity property:

$$H_{\mathcal{L} \otimes \mathcal{L}'}(x) = H_{\mathcal{L}}(x) \cdot H_{\mathcal{L}'}(x).$$

The height function also has the following functoriality. For a morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$ of nice stacks and a line bundle \mathcal{L} on \mathcal{X} given with an adelic metric, the pullback $f^* \mathcal{L}$ has a naturally induced adelic metric. For every $y \in \mathcal{Y}(F)$,

$$H_{f^* \mathcal{L}}(y) = H_{\mathcal{L}}(f(y)).$$

The above constructions have natural generalizations to \mathbb{Q} -line bundles:

Definition 3.7. An *adelic metric* on a \mathbb{Q} -line bundle $\bigotimes_{i=1}^n \mathcal{L}_i^{s_i}$ is a collection consisting of an adelic metric for each \mathcal{L}_i . For a \mathbb{Q} -line bundle $\mathcal{L} = \bigotimes_{i=1}^n \mathcal{L}_i^{s_i}$ with an adelic metric, the *height function* of \mathcal{L} is defined by

$$H_{\mathcal{L}} = \prod_i (H_{\mathcal{L}_i})^{s_i}.$$

Remark 3.8 (Stability properties). The height function $H_{\mathcal{X}}$ has the following stability properties, which were observed already in [13, 16].

- (1) Let K/F be a finite extension. For a line bundle \mathcal{L} on \mathcal{X} , let \mathcal{L}_K be the induced line bundle on \mathcal{X}_K . Suppose that \mathcal{L}_K is given an adelic metric, which induces one on \mathcal{L} in the obvious way. In this situation, the associated height functions $H_{\mathcal{X}}$ and $H_{\mathcal{X}_K}$ are related as follows: for an F -point $x \in \mathcal{X}\langle F \rangle$ and for the induced K -point $x_K \in \mathcal{X}_K\langle K \rangle$,

$$H_{\mathcal{X}_K}(x_K) = H_{\mathcal{X}}(x)^{[K:F]}.$$

- (2) Let \mathcal{L} and L be \mathbb{Q} -line bundles on \mathcal{X} and X corresponding to each other. An adelic metric on \mathcal{L} induces one on L , and vice versa. When the bundles are given adelic metrics corresponding to each other in this way, the height functions $H_{\mathcal{X}}$ and H_L are related by

$$H_{\mathcal{X}} = H_L \circ \pi,$$

where π is the map $\mathcal{X}\langle F \rangle \rightarrow X(F)$.

Remark 3.9. Even if a line bundle \mathcal{L} on \mathcal{X} corresponds to an *ample* \mathbb{Q} -line bundle on X , the height function $H_{\mathcal{X}}: \mathcal{X}\langle F \rangle \rightarrow \mathbb{R}$ does not generally satisfy the Northcott property.² This follows from either of the two stability properties in Remark 3.8. Indeed, for a map $\mathcal{X}\langle K \rangle \rightarrow \mathcal{X}\langle F \rangle$ or $\mathcal{X}\langle F \rangle \rightarrow X(F)$, it may happen that infinitely many distinct points map to a single point. This is the main reason why we need to introduce “unstable” heights in the next section.

4. Raising data and unstable heights

Definition 4.1. A *raising function* of \mathcal{X} is a function

$$c: \pi_0(\mathcal{J}_0\mathcal{X}) \rightarrow \mathbb{R}$$

satisfying $c(\mathcal{X}) = 0$. A *raising datum* of \mathcal{X} is a collection $c_* = (c, (c_v)_{v \in M_F})$ of a raising function c and continuous functions

$$c_v: \mathcal{X}\langle F_v \rangle \rightarrow \mathbb{R}$$

such that for almost every v , c_v factors as $c_v = c \circ \psi_v$ with ψ_v the residue map (see Definition 2.17):

$$c_v: \mathcal{X}\langle F_v \rangle \xrightarrow{\psi_v} \pi_0(\mathcal{J}_0\mathcal{X}) \xrightarrow{c} \mathbb{R}.$$

We call c the *generic raising function* of c_* . We say that a raising function c is *positive* if $c(\mathcal{Y}) > 0$ for every twisted sector \mathcal{Y} . We say that a raising datum is *positive* if its generic raising function is positive.

²We say that a function $H: U \rightarrow \mathbb{R}$ has the *Northcott property* if for every $B \in \mathbb{R}$, the set $\{x \in U \mid H(x) \leq B\}$ is finite.

Note that from [11, Proposition 2.9], $\mathcal{X}\langle F_v \rangle$ are compact and the continuous maps c_v are bounded. Note also that if ψ_v are surjective for infinitely many v , then the generic raising function c of $c_* = (c, (c_v)_{v \in M_F})$ is determined by the subcollection $(c_v)_{v \in M_F}$. If $\mathcal{X} = \mathbf{B}G$ for an étale group scheme G over F , then ψ_v is surjective for infinitely many v . In this case, the raising function is essentially the same as what is called the rational class function in [17] and the counting function in [15, 44].

Definition 4.2. A *raised line bundle* (resp. *strictly raised line bundle*) on \mathcal{X} means the pair (\mathcal{L}, c) (resp. (\mathcal{L}, c_*)) of a line bundle \mathcal{L} on \mathcal{X} and a raising function c (resp. a raising datum c_*).

Definition 4.3 (Unstable height). For $x \in \mathcal{X}(F)$ and $v \in F_v$, let x_v denote the induced F_v -point $x_v \in \mathcal{X}(F_v)$. Let (\mathcal{L}, c_*) be a strictly raised line bundle. We suppose that \mathcal{L} is given an adelic metric so that the height function $H_{\mathcal{L}}$ is defined as in Definition 3.6. Then we define the *associated height function* of (\mathcal{L}, c_*) to be

$$H_{\mathcal{L}, c_*}(x) := H_{\mathcal{L}}(x) \times \prod_{v \in M_F} q_v^{c_v(x_v)}.$$

Here q_v is the cardinality of the residue field κ_v for a finite place v and we put $q_v := e$ for an infinite place v .

Note that if we fix $x \in \mathcal{X}(F)$, then for almost every v , the residue $\psi_v(x_v)$ is the non-twisted sector, and hence $c_v(x_v) = 0$. Thus, the above product $\prod_{v \in M_F} q_v^{c_v(x_v)}$ is a finite product. For strictly raised line bundles (\mathcal{L}, c_*) and (\mathcal{L}', c'_*) , the pair $(\mathcal{L} \otimes \mathcal{L}', c_* + c'_*)$ is also a strictly raised line bundle. The height function satisfies the multiplicativity

$$H_{\mathcal{L} \otimes \mathcal{L}', c_* + c'_*}(x) = H_{\mathcal{L}, c_*}(x) \cdot H_{\mathcal{L}', c'_*}(x).$$

The height function has the following functoriality.

Proposition 4.4. Let $f: \mathcal{Y} \rightarrow \mathcal{X}$ be a morphism of nice stacks, and let (\mathcal{L}, c_*) be a strictly raised line bundle on \mathcal{X} , which induces a strictly raised line bundle on \mathcal{Y} . For every F -point y of \mathcal{Y} , we have

$$H_{f^* \mathcal{L}, f^* c_*}(y) = H_{\mathcal{L}, c_*}(f(y)).$$

Proof. Note that the raising datum $c_* = (c, (c_v)_{v \in M_F})$ for \mathcal{X} induces the collection of maps, $f^* c_* = (f^* c, (f^* c_v)_v)$, by natural maps $\pi_0(\mathcal{J}_0 \mathcal{Y}) \rightarrow \pi_0(\mathcal{J}_0 \mathcal{X})$ and $\mathcal{Y}(F_v) \rightarrow \mathcal{X}(F_v)$. Using Proposition 2.20, we can show that this collection is again a raising datum. Suppose that we have a morphism $f: \mathcal{Y}_{\mathcal{O}_S} \rightarrow \mathcal{X}_{\mathcal{O}_S}$ of models over \mathcal{O}_S from nice stacks and that $\mathcal{J}_0 \mathcal{Y}_{\mathcal{O}_S}$ and $\mathcal{J}_0 \mathcal{X}_{\mathcal{O}_S}$ are flat over \mathcal{O}_S . The desired equality follows from the functoriality of stable height functions mentioned in Section 3 and the definition of unstable height functions. ■

If two raising data c_* and c'_* have the same generic raising function c , then the function $H_{\mathcal{L}, c_*} / H_{\mathcal{L}, c'_*}$ is bounded. Thus, a height function $H_{\mathcal{L}, c}$ of a (nonstrictly) raised line bundle (\mathcal{L}, c) is determined modulo bounded functions. In what follows, we mainly consider (nonstrictly) raised line bundles and their height functions.

Remark 4.5 (Unstability, cf. Remark 3.8). Let K/F be a finite extension. Suppose that a line bundle \mathcal{L} on \mathcal{X} and the induced line bundle \mathcal{L}_K on \mathcal{X}_K have raising data c_* and d_* which are compatible with each other in the obvious way. Then, for an F -point $x \in \mathcal{X}(F)$ and for the induced K -point $x_K \in \mathcal{X}(K) = \mathcal{X}_K(K)$, the formula

$$H_{\mathcal{L}_K, d_*}(x_K) = H_{\mathcal{L}, c_*}(x)^{[K:F]}$$

does *not* hold unlike the case of stable height as in Remark 3.8. For example, if $\mathcal{X} = \text{BG}$ for a finite group G and if $x \in \mathcal{X}(F)$ is a G -torsor over F , then $H_{\mathcal{L}, c_*}(x)$ is a generalized discriminant as will be explained in Example 4.12 and can become arbitrarily large as x varies. On the other hand, x is trivialized by some finite extension K/F and the corresponding K -point $x_K \in \mathcal{X}(K)$ has height bounded by a constant independent of x .

Lemma 4.6. *Let T be a connected Dedekind scheme and let \mathcal{X} be a separated DM stack over T . Let $a, b, c \in \mathcal{X}(T)$. Let $\underline{\text{Iso}}_T(a, b)^\circ$ denote the open and closed subscheme of the Iso scheme $\underline{\text{Iso}}_T(a, b)$ consisting of one-dimensional connected components; and similarly for $\underline{\text{Iso}}_T(a, c)^\circ$, $\underline{\text{Aut}}_T(a)^\circ$, etc.*

- (1) *The scheme $\underline{\text{Iso}}_T(a, b)^\circ$ is finite and étale over T . In particular, $\underline{\text{Aut}}_T(a)^\circ$ is a finite étale group scheme over T .*
- (2) *If the scheme $\underline{\text{Iso}}_T(a, b)^\circ$ is not empty, then it has a natural structure of $\underline{\text{Aut}}_T(b)^\circ$ -torsor as well as one of $\underline{\text{Aut}}_T(a)^\circ$ -torsor. In particular, $\underline{\text{Iso}}_T(a, b)^\circ$ is étale and finite over T .*
- (3) *Suppose that there exists an isomorphism $\underline{\text{Iso}}_T(a, b)^\circ \rightarrow \underline{\text{Iso}}_T(a, c)^\circ$ which is equivariant for $\underline{\text{Aut}}_T(a)^\circ$ -actions. Then b and c are isomorphic in $\mathcal{X}(T)$.*

Proof. Note that $\underline{\text{Iso}}_T(a, b)$ is finite and unramified over T . From the local structure of unramified morphisms [42, tag 04HJ], $\underline{\text{Iso}}_T(a, b)^\circ$ is étale over T . It is easy to see that $\underline{\text{Aut}}_T(a)^\circ$ is a subgroup scheme of $\underline{\text{Aut}}_T(a)$. The first assertion follows.

As for the second assertion, we easily see that there is a natural $\underline{\text{Aut}}_T(b)^\circ$ -action on $\underline{\text{Iso}}_T(a, b)^\circ$ and that this action makes $\underline{\text{Iso}}_T(a, b)^\circ$ an $\underline{\text{Aut}}_T(b)^\circ$ -torsor over an open dense subscheme of T , provided that $\underline{\text{Iso}}_T(a, b)^\circ \neq \emptyset$. For a geometric point $t \in T(K)$, the action of $\underline{\text{Aut}}_T(b)^\circ(t)$ on $\underline{\text{Iso}}_T(a, b)^\circ(t)$ is free. Since $\underline{\text{Aut}}_T(b)^\circ(t)$ and $\underline{\text{Iso}}_T(a, b)^\circ(t)$ have the same cardinality, the action is also transitive. It follows that $\underline{\text{Iso}}_T(a, b)^\circ$ is an $\underline{\text{Aut}}_T(b)^\circ$ -torsor over the whole T ; and similarly for the $\underline{\text{Aut}}_T(a)^\circ$ -action.

We show the third assertion. Note that we have the obvious T -isomorphism $\underline{\text{Iso}}_T(a, b)^\circ \cong \underline{\text{Iso}}_T(b, a)^\circ$. Consider the morphism

$$\underline{\text{Iso}}_T(b, a)^\circ \times \underline{\text{Iso}}_T(a, c)^\circ \rightarrow \underline{\text{Iso}}_T(b, c)^\circ, \quad (f, g) \mapsto g \circ f.$$

This morphism is invariant under the $\underline{\text{Aut}}_T(a)^\circ$ -action given by $\alpha(f, g) := (\alpha \circ f, g \circ \alpha^{-1})$ and induces a morphism

$$(\underline{\text{Iso}}_T(b, a)^\circ \times \underline{\text{Iso}}_T(a, c)^\circ) / \underline{\text{Aut}}_T(a)^\circ \rightarrow \underline{\text{Iso}}_T(b, c)^\circ.$$

The last morphism is a morphism of $\underline{\text{Aut}}_T(c)^\circ$ -torsors, and hence an isomorphism. From the assumption, we have the induced morphism

$$\underline{\text{Iso}}_T(b, a)^\circ \rightarrow \underline{\text{Iso}}_T(b, a)^\circ \times \underline{\text{Iso}}_T(a, c)^\circ,$$

which is equivariant for $\underline{\text{Aut}}_T(a)^\circ$ -actions. We get the composite morphism

$$T = \underline{\text{Iso}}_T(a, b)^\circ / \underline{\text{Aut}}_T(a)^\circ \rightarrow (\underline{\text{Iso}}_T(b, a)^\circ \times \underline{\text{Iso}}_T(a, c)^\circ) / \underline{\text{Aut}}_T(a)^\circ \rightarrow \underline{\text{Iso}}_T(b, c)^\circ.$$

This shows that $\underline{\text{Iso}}_T(b, c)^\circ(T)$ is nonempty, and that b and c are isomorphic in $\mathcal{X}(T)$. ■

Lemma 4.7. *Let T be a connected Dedekind scheme and let G be a finite étale group scheme of order r over T . Let P be a finite étale T -scheme of degree r . Then the number of G -torsor structures that can be given to P is at most $(r!)^r$.*

Proof. A G -torsor structure on P is determined by a homomorphism $G \rightarrow \underline{\text{Aut}}_T(P)$ of group schemes over T . The Aut scheme $\underline{\text{Aut}}_T(P)$ is a finite étale T -scheme of degree $r!$. We can see this, for example, by taking a base change to reduce to the case where P is the trivial T -scheme $T \sqcup \cdots \sqcup T \rightarrow T$ of degree r . If $t: \text{Spec } K \rightarrow T$ is the geometric generic point, then there are $(r!)^r$ K -morphisms $G_K \rightarrow \underline{\text{Aut}}_T(P)_K$. It follows that the number of T -morphisms $G \rightarrow \underline{\text{Aut}}_T(P)$ as well as the number of T -homomorphisms $G \rightarrow \underline{\text{Aut}}_T(P)$ is at most $(r!)^r$. ■

Proposition 4.8 (Northcott property). *Let (\mathcal{L}, c) be a raised line bundle on \mathcal{X} and let L be the \mathbb{Q} -line bundle corresponding to \mathcal{L} on the coarse moduli space $\bar{\mathcal{X}}$. Suppose that L is ample and c is positive. Let $H_{\mathcal{L},c}$ be the height function of (\mathcal{L}, c) . Then, for each real number B , there are only finitely many points $x \in \mathcal{X}(F)$ with $H_{\mathcal{L},c}(x) \leq B$. Moreover, there exist positive constants C and m such that for every $B > 0$,*

$$\#\{x \in \mathcal{X}(F) \mid H_{\mathcal{L},c}(x) \leq B\} \leq CB^m.$$

Proof. We choose a raising datum c_* refining c and prove the proposition for $H_{\mathcal{L},c} = H_{\mathcal{L},c_*}$. Note that the second factor $\prod_{v \in M_F} q_v^{c_v(x_v)}$ in the definition of $H_{\mathcal{L},c_*}$ is at least 1, and hence $H_{\mathcal{L}}(x) \leq H_{\mathcal{L},c_*}(x)$ for every x . On the other hand, the first factor $H_{\mathcal{L}}(x)$ is equal to $H_L(\pi(x))$ with π denoting the natural map $\mathcal{X}(F) \rightarrow X(F)$. Using [39, Corollary, p. 447] and basic properties of the Weil height machine, we can see that for some positive constants C_1 and m_1 ,

$$\#\pi(\{x \in \mathcal{X}(F) \mid H_{\mathcal{L},c}(x) \leq B\}) \leq \#\{y \in \bar{\mathcal{X}}(F) \mid H_L(y) \leq B\} \leq C_1 B^{m_1}.$$

If $H_{\mathcal{L},c_*}(x) \leq B$, then $\prod_v q_v^{c_v(x_v)} \leq a_1 B$ for some constant a_1 . It follows that there are some positive constants a_2 and a_3 such that if $H_{\mathcal{L},c}(x) \leq B$, then the F -point x extends to an \mathcal{O}_T -point of \mathcal{X} for a finite set T of places satisfying

$$\prod_{v \in T \text{ finite places}} q_v \leq a_2 B^{a_3}.$$

Let $x_i \in \mathcal{X}(F)$, $i \in \{0, 1\}$, be F -points of \mathcal{X} with $H_{\mathcal{X},c_*}(x_i) \leq B$ which have the same image $y \in \tilde{\mathcal{X}}(F)$. Suppose that x_i extends to an \mathcal{O}_{T_i} -point with $\prod_{v \in T_i} q_v \leq a_2 B^{a_3}$. Then the $\text{Aut}_F(x_0)$ -torsor $\text{Iso}_F(x_0, x_1)$ extends to $\text{Aut}_{\mathcal{O}_{T_0 \cup T_1}}(x'_0)^\circ$ -torsor $\text{Iso}_{\mathcal{O}_{T_0 \cup T_1}}(x'_0, x'_1)^\circ$, where x'_i are the $\mathcal{O}_{T_0 \cup T_1}$ -points induced from x_i respectively. Let N be an integer such that the automorphism group of every point of \mathcal{X} has order $\leq N$ and let M be an integer such that for every finite place v , every finite étale cover $\text{Spec } L \rightarrow \text{Spec } F_v$ of degree $\leq N$ has discriminant exponent at most M . Then the torsor $\text{Iso}_F(x_0, x_1)^\circ$ has discriminant at most $(a_2 B^{a_3})^{2M}$. From [41] and Lemma 4.7, there exist positive constants C_2 and m_2 such that the number of $\text{Aut}_F(x_0)$ -torsors with discriminant at most $(a_2 B^{a_3})^{2M}$ is at most $C_2 B^{m_2}$. Note that the result in [41] is about counting number fields, but it is easy to generalize it to counting torsors. From Lemma 4.6, the number of lifts of y in $\mathcal{X}(F)$ is at most $C_2 B^{m_2}$. In summary, we have

$$\#\{x \in \mathcal{X}(F) \mid H_{\mathcal{X},c_*}(x) \leq B\} \leq (C_1 B^{m_1})(C_2 B^{m_2}) = (C_1 C_2) B^{m_1+m_2}. \quad \blacksquare$$

Remark 4.9. A slight change of the above proof yields a stronger version of the above proposition: for every positive integer n , there exist positive constants C and m such that for every $B > 0$,

$$\#\left\{x \in \bigsqcup_{L/F \text{ finite field ext.}} \mathcal{X}\langle L \rangle \mid [L : F] \leq n \text{ and } H_{\mathcal{X},c}(x) \leq B\right\} \leq CB^m.$$

For this purpose, we need to use a result in [40], a polynomial bound for algebraic points with bounded degree and bounded height, in place of one in [39]. Note that for a finite extension L'/L , the map $\mathcal{X}\langle L \rangle \rightarrow \mathcal{X}\langle L' \rangle$ is not generally injective, and the height $H_{\mathcal{X},c_*}$ is not preserved by this map (Remark 4.5). This is why we take the disjoint union $\bigsqcup_{L/F} \mathcal{X}\langle L \rangle$.

Example 4.10 (The height function associated to a vector bundle). Ellenberg–Satriano–Zureick-Brown [16] introduced a new height function on stacks which is associated to a vector bundle. They also consider Artin stacks, but if we restrict ourselves to DM stacks over number fields, then our height function $H_{\mathcal{X},c_*}$ generalizes theirs. Let \mathcal{V} be a vector bundle on a nice stack \mathcal{X} . Translating their definition into the setting of multiplicative heights, we can write the height function $H_{\mathcal{V}}$ associated to \mathcal{V} as

$$H_{\mathcal{V}}(x) = H_{\det(\mathcal{V})}(x) \times \prod_v q_v^{\varepsilon_v(x_v)}.$$

The exponent $\varepsilon_v(x_v)$ is generalization of the w - and v -functions in [45, 50] to semilinear representations. We claim that $(\varepsilon_v)_v$ together with some raising function is a raising datum. Indeed, at tame places v , $\varepsilon_v(x_v)$ is nothing but the age, $\text{age}(-; \mathcal{V})$ (cf. [50, Example 6.7], [45, Lemma 4.3]). To see this, let $v \in M_F$ be a general finite place and let $x \in \mathcal{X}(L)$ with $L = F_v^{\text{nr}}$, following the notation of Lemma 2.16. We take the induced representable morphism $\tilde{x}: [\text{Spec } \mathcal{O}_{L_i}/\mu_{l,\mathcal{O}_L}] \rightarrow \mathcal{X}_{\mathcal{O}_S}$. The vector bundle $\tilde{x}^* \mathcal{V}$ on $[\text{Spec } \mathcal{O}_{L_i}/\mu_{l,\mathcal{O}_L}]$ corresponds to a free \mathcal{O}_{L_i} -module of finite rank with a semilinear

action of $\mu_l = \mathbb{Z}/l\mathbb{Z}$. This module is equivariantly isomorphic to

$$M = \bigoplus_{i=1}^r \mathfrak{m}_{L_l}^{a_i} \quad (0 \leq a_i < l),$$

where \mathfrak{m}_{L_l} denotes the maximal ideal of \mathcal{O}_{L_l} . If we put

$$b_i := \begin{cases} 0 & (a_i = 0), \\ l & (a_i \neq 0), \end{cases}$$

then

$$\begin{aligned} \varepsilon_v(x_v) &= \frac{1}{l} \text{length} \frac{M}{M^{\mu_l} \cdot \mathcal{O}_{L_l}} = \frac{1}{l} \text{length} \bigoplus_{i=1}^r \mathfrak{m}_{L_l}^{a_i} / \mathfrak{m}_{L_l}^{b_i} \\ &= \frac{1}{l} \sum_{i=1}^r (b_i - a_i). \end{aligned}$$

Note that the special fiber of $\tilde{\mathcal{X}}^* \mathcal{V}$ is identified with the representation $(\bigoplus_i \mathfrak{m}_{L_l}^{a_i} / \mathfrak{m}_{L_l}^{a_i+1})^\vee$. The age computed from the last representation is equal to $\varepsilon_v(x_v)$. From Lemma 2.22, the age depends only on the sector associated to x_v . Thus, for almost every place v , ε_v is the composite map

$$\mathcal{X}(F_v) \xrightarrow{\psi_v} \pi_0(\mathcal{J}_0 \mathcal{X}) \xrightarrow{\text{age}(-; \mathcal{V})} \mathbb{R}_{\geq 0}.$$

Namely, the collection $(\text{age}(-; \mathcal{V}), (\varepsilon_v)_v)$ is a raising datum. The height function $H_{\mathcal{V}}$ is equal to $H_{\det(\mathcal{V}), \varepsilon_*}$.

Example 4.11 (Raising functions not coming from any vector bundle). The raising function $\text{age}(-; \mathcal{V})$ in the last example associated to a vector bundle \mathcal{V} is \mathbb{Q} -valued. Thus, any raising function with nonrational values is not associated to any vector bundle. If a raising function takes a rational value at some twisted sector and a nonrational value at another twisted sector, then this raising function is not obtained by multiplying $\text{age}(-; \mathcal{V})$ with a positive real constant. There is also a more interesting example of raising function that takes only rational values, but not coming from a vector bundle. Consider $\mathcal{X} = \mathbb{B} \mu_{p,F}$ for a prime number p . Then $\pi_0(\mathcal{J}_0 \mathcal{X}) = \mathbb{Z}/p\mathbb{Z} = \{0, 1, \dots, p-1\}$. Consider a vector bundle of \mathcal{X} associated to the representation

$$V = \bigoplus_{i=1}^d L^{\otimes a_i} \quad (0 \leq i < p).$$

Here L is the standard one-dimensional representation of μ_p . Since the summands with $a_i = 0$ do not contribute to the raising function, we may assume that $a_i > 0$ for every i , without loss of generality. Then the raising function associated to \mathcal{V} , $c_{\mathcal{V}}: \pi_0(\mathcal{J}_0 \mathcal{X}) = \{0, 1, \dots, p-1\} \rightarrow \mathbb{Q}$, is given by $c_{\mathcal{V}}(j) = \sum_{i=1}^d \{ja_i/p\}$, where $\{-\}$ denotes the fractional part. In particular,

$$d/p \leq c_{\mathcal{V}}(j) \leq (p-1)d/p \quad (j \in \{1, \dots, p-1\}).$$

This implies that for $j, j' \in \{1, \dots, p - 1\}$,

$$c_{\mathcal{V}}(j) \leq (p - 1)c_{\mathcal{V}}(j').$$

If $c: \pi_0(\mathcal{J}_0\mathcal{X}) = \{0, 1, \dots, p - 1\} \rightarrow \mathbb{Q}$ is any function violating the last inequality, it never comes from a vector bundle, nor can it be a scalar multiple of a raising function associated to a vector bundle. An example of such raising functions is the following one for $p \geq 3$:

$$c(j) = \begin{cases} 0 & (j = 0), \\ 1 & (j = 1, \dots, p - 2), \\ p & (j = p - 1). \end{cases}$$

Example 4.12 (Generalized discriminants). Let $G \subset S_n$ be a transitive subgroup, let $G_1 \subset G$ be the stabilizer subgroup of $1 \in \{1, \dots, n\}$, and let $\mathcal{X} = \mathbf{B}G_F$. An object of $\mathcal{X}(F)$ is a G -torsor K/F . Then K^{G_1}/F is an étale F -algebra of degree n . Similarly for an object of $\mathcal{X}(F_v)$. For a finite place v and for a G -torsor K_v/F_v , let $d_v(K_v) \in \mathbb{Z}$ denote the discriminant exponent of the degree n algebra $K_v^{G_1}/F_v$. For an infinite place, we put $d_v \equiv 0$. Then $d_* = (d_v)$ is a raising datum and its generic raising function is given by

$$c: \pi_0(\mathcal{J}_0\mathcal{X}) = F\text{-Conj}(G) \rightarrow \mathbb{Z}_{\geq 0}, \quad [g] \mapsto \text{ind}(g).$$

Here $\text{ind}(g)$ denotes the index appearing in the context of Malle’s conjecture [32,33]. The height function associated to a general raising function c (with $c_v \equiv 0$ at wild places v) is the modified discriminant considered by Ellenberg–Venkatesh [17, p. 163]. We refer the reader to [15] for more details.

Remark 4.13. Landesman [25] shows that the Faltings height on $\overline{\mathcal{M}}_{1,1}$ in characteristic 3 is not induced from a vector bundle in the sense of [16]. The Faltings height in characteristic 3 would not fit into our definition of height either. Indeed, the stack $\overline{\mathcal{M}}_{1,1}$ in characteristic 3 is wild. In that case, the stack $\mathcal{J}_0\mathcal{X} = \underline{\text{Hom}}(\mathbf{B}\hat{\mu}, \mathcal{X})$ would not be the right object to look at. In [51], the stack of twisted 0-jets is considered also in the wild case, but it is of infinite dimension and has a structure considerably more complex than $\underline{\text{Hom}}(\mathbf{B}\hat{\mu}, \mathcal{X})$. One might be able to generalize our height by using this version of the stack of twisted 0-jets or something similar to incorporate “wild aspects,” in particular, the Faltings height in characteristic 3.

5. Fano stacks; the stacky Manin conjecture

Definition 5.1. We mean by a *Fano stack* a nice stack \mathcal{X} such that the anti-canonical line bundle $\omega_{\mathcal{X}}^{-1}$ corresponds to an ample \mathbb{Q} -line bundle on the coarse moduli space $\bar{\mathcal{X}}$ by the isomorphism $\text{Pic}(\mathcal{X})_{\mathbb{Q}} \cong \text{Pic}(\bar{\mathcal{X}})_{\mathbb{Q}}$ in Proposition 3.2.

Typical examples of Fano stacks are Fano varieties (except $\text{Spec } F$) and nice stacks of dimension zero.

Definition 5.2. Let c be a raising function of a Fano stack \mathcal{X} . We define a function age_c on $\pi_0(\mathcal{J}_0\mathcal{X})$ by $\text{age}_c := \text{age} + c$. We say that c is *adequate* if

- (1) $\text{age}_c(\mathcal{Y}) \geq 1$ for every twisted sector \mathcal{Y} ,
- (2) if $\dim \mathcal{X} = 0$, then $\min \{c(\mathcal{Y}) \mid \mathcal{Y} \text{ twisted sector}\} = 1$.

Note that if $\dim \mathcal{X} = 0$, then the height function $H_{\mathcal{X},c}$ depends only on $c = \text{age}_c$ (up to bounded functions) and the choice of \mathcal{L} plays no role. Therefore, we usually choose the structure sheaf $\mathcal{O} = \mathcal{O}_{\mathcal{X}}$ (say with the standard adelic metric). For a constant $r \in \mathbb{R}_{\geq 0}$, we have

$$H_{\mathcal{O},rc} = (H_{\mathcal{O},c})^r.$$

Thus, the second condition in the above definition is just a normalization condition and does not lead to a loss of generality. As for the first condition, we speculate that without this condition, it would be more difficult to control the asymptotic behavior of the number of rational points with bounded height (see Remark 7.8).

Definition 5.3. Let \mathcal{X} be a Fano stack given with an adequate raising function c . We say a twisted sector \mathcal{Y} of \mathcal{X} is *c-junior* if $\text{age}_c(\mathcal{Y}) = 1$. We denote by $j_c(\mathcal{X})$ the number of c -junior twisted sectors of \mathcal{X} :

$$j_c(\mathcal{X}) := \#\{\mathcal{Y} \in \pi_0^*(\mathcal{J}_0\mathcal{X}) \mid \text{age}_c(\mathcal{Y}) = 1\}.$$

Definition 5.4. Let $f: \mathcal{Y} \rightarrow \mathcal{X}$ be a morphism of nice stacks. We say that f is *birational* if there exist open dense substacks $\mathcal{V} \subset \mathcal{Y}$ and $\mathcal{U} \subset \mathcal{X}$ such that f restricts to an isomorphism $\mathcal{V} \xrightarrow{\sim} \mathcal{U}$. We say that f is *thin* if it is nonbirational, representable, and generically finite onto the image. A *thin subset* of $\mathcal{X}\langle F \rangle$ is a subset contained in $\bigcup_{i=1}^n f_i(\mathcal{Y}_i\langle F \rangle)$ for finitely many thin morphisms $f_i: \mathcal{Y}_i \rightarrow \mathcal{X}$.

Definition 5.5. For a nice stack \mathcal{X} , we denote its *Néron–Severi space* by $N^1(\mathcal{X})_{\mathbb{R}}$, that is, the \mathbb{R} -vector space generated by the numerical classes of line bundles. We denote its dimension by $\rho(\mathcal{X})$ and call it the *Picard number* of \mathcal{X} .

From Proposition 3.2, the Néron–Severi space of \mathcal{X} is identified with the corresponding space $N^1(\bar{\mathcal{X}})_{\mathbb{R}}$ of the coarse moduli space $\bar{\mathcal{X}}$. In particular, \mathcal{X} and $\bar{\mathcal{X}}$ have the same Picard number. Below is our first conjecture, which concerns the number of rational points with bounded height on a Fano stack.

Conjecture 5.6 (The Manin conjecture for Fano stacks). *Let \mathcal{X} be a Fano stack and let c be an adequate raising function. Suppose that $\mathcal{X}\langle F \rangle$ is Zariski dense. Then there exists a thin subset $T \subset \mathcal{X}\langle F \rangle$ and a positive constant C such that*

$$\#\{x \in \mathcal{X}\langle F \rangle \setminus T \mid H_{\omega_{\mathcal{X}}^{-1},c}(x) \leq B\} \sim CB(\log B)^{\rho(\mathcal{X})+j_c(\mathcal{X})-1} \quad (B \rightarrow \infty).$$

Remark 5.7. The above conjecture, in particular, says that the left hand side is a finite number, even if c is not necessarily positive. On the other hand, we have proved the Northcott property in Proposition 4.8 only when the raising function is positive and the line bundle corresponds to an ample \mathbb{Q} -line bundle on the coarse moduli space. Note

that a version of the Northcott property (the Northcott property with a certain Zariski closed subset removed) holds for a big (but not necessarily ample) divisor on a variety. We speculate that the same would be true for a big raised line bundle defined later in Definition 8.15, where the raising function is allowed to have negative values. In [14], we proved this for the anti-canonical raised line bundle for a toric stack.

Example 5.8 (Malle’s conjecture). Let G be a finite group and let G_F be the corresponding constant group scheme over F . Consider the DM stack $\mathcal{X} := \mathbb{B} G_F$ over F . From Example 2.15, we may identify $\pi_0(\mathcal{J}_0 \mathcal{X})$ with the set $F\text{-Conj}(G)$ of F -conjugacy classes of G . From Example 4.12, the height function $H_{\mathcal{O},c}$ of the raised line bundle (\mathcal{O}, c) is a generalization of discriminant. For a proper subgroup $H \subsetneq G$, the induced morphism $\mathbb{B} H_F \rightarrow \mathbb{B} G_F$ is a thin morphism. Thus,

$$\bigcup_{H \subsetneq G} \text{Im}((\mathbb{B} H_F)(F) \rightarrow \mathcal{X}(F))$$

is a thin subset of $\mathcal{X}(F)$, whose complement is the set of connected G -torsors over F , that is, G -Galois extensions of F . If we can take this thin subset as T , then Conjecture 5.6 is the same as Malle’s conjecture for a generalized discriminant associated to c . See also Section 9.4 for a discussion of the choice of a thin subset and of the value of the leading constant.

Example 5.9 (Weighted projective stacks). For $\mathbf{a} = (a_0, \dots, a_n) \in (\mathbb{Z}_{>0})^n$, we consider the weighted projective stack

$$\mathcal{X} = \mathcal{P}(\mathbf{a}) = \mathcal{P}(a_0, \dots, a_n) = [(\mathbb{A}_F^{n+1} \setminus \{0\})/\mathbb{G}_{m,F}].$$

Here we follow the convention that $\mathbb{G}_{m,F}$ acts on \mathbb{A}_F^{n+1} by $t \cdot (x_i)_i = (t^{-a_i} x_i)_i$. Note that using the alternative action $t \cdot (x_i)_i = (t^{a_i} x_i)_i$ leads to an isomorphic stack via an automorphism of $\mathbb{G}_{m,F}$ given by $t \mapsto t^{-1}$. Sectors of $\mathcal{P}(\mathbf{a})$ are in one-to-one correspondence with elements of

$$I := \left(\bigcup_i \frac{1}{a_i} \mathbb{Z} \right) \cap [0, 1).$$

When the base field is \mathbb{C} , this is proved in [34, Section 3.b], [12, p. 148]. But this is true over any field, because the correspondence is valid over \bar{F} and the natural action of the absolute Galois group Γ_F on $\pi_0(\mathcal{J}_0 \mathcal{X}_{\bar{F}})$ is trivial. From [12, p. 149] or [34, Proposition 3.9], for the sector \mathcal{Y}_r of \mathcal{X} corresponding to $r \in I$ we have

$$\text{age}(\mathcal{Y}_r) = \sum_{j=0}^n \{-a_j r\}.$$

Here $\{-\}$ denotes the fractional part. Let $|\mathbf{a}| := \sum_{j=0}^r a_j$, which is an important number since $\omega_{\mathcal{X}}^{-1} = \mathcal{O}_{\mathcal{X}}(|\mathbf{a}|)$. Consider the raising function of \mathcal{X} given by

$$c(\mathcal{Y}_r) = r \cdot |\mathbf{a}|.$$

Then

$$\text{age}_c(\mathcal{Y}_r) = r \sum_{j=0}^n a_j + \sum_{j=0}^n \{-a_j r\} = \sum_j \lceil a_j r \rceil.$$

Here $\lceil - \rceil$ denotes the ceiling function. In particular, if $\dim \mathcal{X} = n > 0$, then $\text{age}_c(\mathcal{Y}_r) \geq 2$ for $r \neq 0$. If $\dim \mathcal{X} = 0$, then

$$\text{age}_c(\mathcal{Y}_r) \begin{cases} = 0 & (r = 0), \\ = 1 & (r = 1/a_0), \\ > 1 & (r > 1/a_0). \end{cases}$$

We see that $(\omega_{\mathcal{X}}^{-1}, c)$ is adequate and

$$\rho(\mathcal{X}) + j_c(\mathcal{X}) = 1.$$

The height function associated to a quasi-toric degree $|\mathbf{a}|$ family of smooth functions considered in [13] is, in our language, the height function $H_{\omega_{\mathcal{X}}^{-1}, c}$ associated to the raised line bundle $(\omega_{\mathcal{X}}^{-1}, c)$ with the raising function c described above. The equality $\rho(\mathcal{X}) + j_c(\mathcal{X}) = 1$ explains the result [13, Theorem 8.2.2.12] that the asymptotic formula for this height is linear (that is, there is no log factor).

6. Compatibility with products of Fano stacks

For $i = 1, 2$, let \mathcal{X}_i be nice stacks, and let (\mathcal{L}_i, c_i) be raised line bundles on \mathcal{X}_i , respectively. These line bundles induce the line bundle $\mathcal{L}_1 \boxtimes \mathcal{L}_2 := \text{pr}_1^* \mathcal{L}_1 \otimes \text{pr}_2^* \mathcal{L}_2$ on the product $\mathcal{X}_1 \times \mathcal{X}_2$. The following lemma enables us to derive a raising function of $\mathcal{X}_1 \times \mathcal{X}_2$ from c_1 and c_2 :

Lemma 6.1. *There exists a natural isomorphism $\mathcal{J}_0(\mathcal{X}_1 \times \mathcal{X}_2) \cong (\mathcal{J}_0 \mathcal{X}_1) \times (\mathcal{J}_0 \mathcal{X}_2)$.*

Proof. From Proposition 2.11,

$$\begin{aligned} \mathcal{J}_0(\mathcal{X}_1 \times \mathcal{X}_2) &\cong \underline{\text{Hom}}(\mathbf{B} \hat{\mu}, \mathcal{X}_1 \times \mathcal{X}_2) \\ &\cong \underline{\text{Hom}}(\mathbf{B} \hat{\mu}, \mathcal{X}_1) \times \underline{\text{Hom}}(\mathbf{B} \hat{\mu}, \mathcal{X}_2) \\ &\cong (\mathcal{J}_0 \mathcal{X}_1) \times (\mathcal{J}_0 \mathcal{X}_2). \end{aligned} \quad \blacksquare$$

From the above lemma, there exist projections

$$\text{pr}_i : \mathcal{J}_0(\mathcal{X}_1 \times \mathcal{X}_2) \rightarrow \mathcal{J}_0 \mathcal{X}_i.$$

The raising functions c_i of \mathcal{X}_i induce raising functions $\text{pr}_i^*(c_i)$ of $\mathcal{X}_1 \times \mathcal{X}_2$.

Definition 6.2. We define a raising function $c_1 \boxplus c_2$ of $\mathcal{X}_1 \times \mathcal{X}_2$ to be $\text{pr}_1^*(c_1) + \text{pr}_2^*(c_2)$.

It is easy to see that we have the following equality of functions on $(\mathcal{X}_1 \times \mathcal{X}_2)\langle F \rangle$:

$$H_{\mathcal{X}_1 \boxtimes \mathcal{X}_2, c_1 \boxplus c_2} = \prod_{i=1}^2 H_{\mathcal{X}_i, c_i} \circ \text{pr}_i.$$

Lemma 6.3. *We have the following equality of functions on $\pi_0(\mathcal{J}_0(\mathcal{X}_1 \times \mathcal{X}_2))$:*

$$\text{age}_{\mathcal{X}_1 \times \mathcal{X}_2} = \text{age}_{\mathcal{X}_1} \boxplus \text{age}_{\mathcal{X}_2}.$$

Proof. For an algebraically closed field K , points $p_i \in (\mathcal{J}_0 \mathcal{X}_i)(K)$ determine $\hat{\mu}(K)$ -representations $T_{x_i} \mathcal{X}_i$, where $x_i \in \mathcal{X}_i(K)$ are points induced by p_i . The point $(p_1, p_2) \in (\mathcal{J}_0(\mathcal{X}_1 \times \mathcal{X}_2))(K)$ then determines a $\hat{\mu}(K)$ -representation $T_{(x_1, x_2)}(\mathcal{X}_1 \times \mathcal{X}_2)$, which is isomorphic to $T_{x_1} \mathcal{X}_1 \oplus T_{x_2} \mathcal{X}_2$. This shows the lemma. ■

As a consequence of this section, we obtain the following result:

Proposition 6.4. *For $i \in \{1, 2\}$, let \mathcal{X}_i be a Fano stack whose F -points are Zariski dense and let c_i be an adequate raising function of \mathcal{X}_i . Suppose that Conjecture 5.6 holds for pairs (\mathcal{X}_1, c_1) and (\mathcal{X}_2, c_2) . Then $\mathcal{X}_1 \times \mathcal{X}_2$ is again a Fano stack whose F -points are Zariski dense, $c_1 \boxplus c_2$ is an adequate raising function of $\mathcal{X}_1 \times \mathcal{X}_2$, and Conjecture 5.6 holds for $(\mathcal{X}_1 \times \mathcal{X}_2, c_1 \boxplus c_2)$.*

Proof. It is easy to see that $\mathcal{X}_1 \times \mathcal{X}_2$ is again a Fano stack whose F -points are Zariski dense. From Lemma 6.3, we know that $\text{age}_{c_1 \boxplus c_2} = \text{age}_{c_1} \boxplus \text{age}_{c_2}$. If \mathcal{Y}_1 and \mathcal{Y}_2 are sectors of \mathcal{X}_1 and \mathcal{X}_2 respectively, then the sector $\mathcal{Y}_1 \times \mathcal{Y}_2$ of $\mathcal{X}_1 \times \mathcal{X}_2$ is twisted if and only if either \mathcal{Y}_1 or \mathcal{Y}_2 is twisted. If this is the case, then

$$\text{age}_{c_1 \boxplus c_2}(\mathcal{Y}_1 \times \mathcal{Y}_2) = \text{age}_{c_1}(\mathcal{Y}_1) + \text{age}_{c_2}(\mathcal{Y}_2) \geq 1.$$

If $\mathcal{X}_1 \times \mathcal{X}_2$ is of dimension zero, then \mathcal{X}_1 is also of dimension zero and has a twisted sector \mathcal{Y}_1 with $c_1(\mathcal{Y}_1) = 1$. Therefore, the twisted sector $\mathcal{Y}_1 \times \mathcal{X}_2$ of $\mathcal{X}_1 \times \mathcal{X}_2$ satisfies

$$(c_1 \boxplus c_2)(\mathcal{Y}_1 \times \mathcal{X}_2) = c_1(\mathcal{Y}_1) + c_2(\mathcal{X}_2) = 1.$$

Thus, $c_1 \boxplus c_2$ is adequate.

To show the last assertion of the proposition, we first claim that

$$j_{c_1 \boxplus c_2}(\mathcal{X}_1 \times \mathcal{X}_2) = j_{c_1}(\mathcal{X}_1) + j_{c_2}(\mathcal{X}_2).$$

Indeed, a $c_1 \boxplus c_2$ -junior sector of $\mathcal{X}_1 \times \mathcal{X}_2$ is either $\mathcal{Y}_1 \times \mathcal{X}_2$ with \mathcal{Y}_1 a c_1 -junior sector or $\mathcal{X}_1 \times \mathcal{Y}_2$ with \mathcal{Y}_2 a c_2 -junior sector. Moreover, sectors of these forms are only $c_1 \boxplus c_2$ -junior sectors, which shows the claim. As is well-known, we have $\rho(\mathcal{X}_1 \times \mathcal{X}_2) = \rho(\mathcal{X}_1) + \rho(\mathcal{X}_2)$.

Let $T_i \subset \mathcal{X}_i\langle F \rangle$, $i \in \{1, 2\}$, be thin subsets as in Conjecture 5.6. Then $T_1 \times \mathcal{X}_2\langle F \rangle$ and $\mathcal{X}_1\langle F \rangle \times T_2$ are both thin subsets of $(\mathcal{X}_1 \times \mathcal{X}_2)\langle F \rangle$ and we have

$$(\mathcal{X}_1 \times \mathcal{X}_2)\langle F \rangle \setminus ((T_1 \times \mathcal{X}_2\langle F \rangle) \cup (\mathcal{X}_1\langle F \rangle \times T_2)) = (\mathcal{X}_1\langle F \rangle \setminus T_1) \times (\mathcal{X}_2\langle F \rangle \setminus T_2).$$

From the assumption that Conjecture 5.6 holds for pairs (\mathcal{X}_1, c_1) and (\mathcal{X}_2, c_2) and from [19, Section 1, 2. Proposition], we find that for some constant $C > 0$,

$$\begin{aligned} & \#\{x \in (\mathcal{X}_1 \times \mathcal{X}_2)(F) \setminus ((T_1 \times \mathcal{X}_2(F)) \cup (\mathcal{X}_1(F) \times T_2)) \mid H_{\omega_{\mathcal{X}_1 \times \mathcal{X}_2, c_1 \boxplus c_2}}^{-1}(x) \leq B\} \\ &= \#\{x \in (\mathcal{X}_1(F) \setminus T_1) \times (\mathcal{X}_2(F) \setminus T_2) \mid H_{\omega_{\mathcal{X}_1, c_1}}^{-1}(x) \cdot H_{\omega_{\mathcal{X}_2, c_2}}^{-1}(x) \leq B\} \\ &\sim CB(\log B)^{\rho(\mathcal{X}_1) + \rho(\mathcal{X}_2) + j_{c_1}(\mathcal{X}_1) + j_{c_2}(\mathcal{X}_2) - 1} = CB(\log B)^{\rho(\mathcal{X}_1 \times \mathcal{X}_2) + j_{c_1 \boxplus c_2}(\mathcal{X}_1) - 1}. \end{aligned}$$

This completes the proof of the proposition. ■

Example 6.5. Taking products enables us to construct new examples from known cases where Conjecture 5.6 holds. For example, let $\mathcal{X}_1 = \mathcal{P}(a_0, \dots, a_n)$ be a weighted projective stack endowed with the raising function described in Example 5.9 and let $\mathcal{X}_2 = \mathbb{B}\mu_{m,F}$ be the classifying stack of $\mu_{m,F}$ endowed with an arbitrary adequate raising function c_2 . Then the product $\mathcal{X}_1 \times \mathcal{X}_2$ is the weighted projective stack $\mathcal{P}(ma_0, \dots, ma_n)$. Conjecture 5.6 holds for the Fano stack $\mathcal{X}_1 \times \mathcal{X}_2$ endowed with the raising function $c_1 \boxplus c_2$. The stack $\mathcal{X}_1 \times \mathcal{X}_2$ has at least one $c_1 \boxplus c_2$ -junior sector, provided that $m \geq 2$; the $c_1 \boxplus c_2$ -junior sectors are the sectors of the form $\mathcal{X}_1 \times \mathcal{Y}$ with \mathcal{Y} a c_2 -junior sector of \mathcal{X}_2 . Thus, the function $c_1 \boxplus c_2$ is different from the raising function of $\mathcal{P}(ma_0, \dots, ma_n)$ described in Example 5.9. For example, applying this construction to the moduli stack of elliptic curves, $\overline{\mathcal{M}}_{1,1} = \mathcal{P}(4, 6) = \mathcal{P}(2, 3) \times \mathbb{B}\mu_{2,F}$, defines a height function for which the asymptotic formula has the form $\#\{\dots\} \sim CB(\log B)$. Note that the notation $\overline{\mathcal{M}}_{1,1}$ does not mean the coarse moduli space. Under the identifications

$$\begin{aligned} \pi_0(\mathcal{J}_0(\mathcal{P}(2, 3))) &= \{0, 1/3, 1/2, 2/3\}, \pi_0(\mathcal{J}_0(\mathbb{B}\mu_{2,F})) = \{0, 1/2\}, \\ \pi_0(\mathcal{J}_0(\mathcal{P}(2, 3) \times \mathbb{B}\mu_{2,F})) &= \{0, 1/3, 1/2, 2/3\} \times \{0, 1/2\} \end{aligned}$$

(see Example 5.9), invariants of sectors of $\mathcal{P}(2, 3)$, $\mathbb{B}\mu_{2,F}$ and $\mathcal{P}(2, 3) \times \mathbb{B}\mu_{2,F}$ are summarized in Table 1.

Sectors of $\mathcal{X}_1 = \mathcal{P}(2, 3)$	0	1/3	1/2	2/3	Sectors of $\mathcal{X}_2 = \mathbb{B}\mu_{2,F}$	0	1/2	
c_1	0	5/3	5/2	10/3	c_2	0	1	
age	0	1/3	1/2	2/3	age	0	0	
age_{c_1}	0	2	3	4	age_{c_2}	0	1	
Sectors of $\mathcal{X}_1 \times \mathcal{X}_2$	(0, 0)	(0, 1/2)	(1/3, 0)	(1/3, 1/2)	(1/2, 0)	(1/2, 1/2)	(2/3, 0)	(2/3, 1/2)
$c_1 \boxplus c_2$	0	1	5/3	8/3	5/2	7/2	10/3	13/3
age	0	0	1/3	1/3	1/2	1/2	2/3	2/3
$\text{age}_{c_1 \boxplus c_2}$	0	1	2	3	3	4	4	5

Tab. 1. Sectors and their invariants of $\mathcal{P}(2, 3)$, $\mathbb{B}\mu_{2,F}$ and $\mathcal{P}(2, 3) \times \mathbb{B}\mu_{2,F}$.

7. Compatibility with coarse moduli spaces

We consider a Fano stack \mathcal{X} with the trivial raising function $c \equiv 0$. Let X be the coarse moduli space of \mathcal{X} . The function $H_{\mathcal{X},c}$ is identical to the stable height function $H_{\mathcal{X}}$ considered in Section 3. In particular, it factors as

$$\mathcal{X}\langle F \rangle \rightarrow X(F) \xrightarrow{H_L} \mathbb{R},$$

where L is the \mathbb{Q} -line bundle on X corresponding to \mathcal{L} . We suppose that c is adequate. This means that $\text{age}(\mathcal{Y}) \geq 1$ for every twisted sector \mathcal{Y} .

Definition 7.1. Let \mathcal{W} be a DM stack. Let $w \in \mathcal{W}(\bar{F})$ and let $\alpha \in \text{Aut}(w) \setminus \{1\}$. We say that α is a *reflection* if the automorphism of $T_w \mathcal{W}$ induced by α is a reflection, that is, the fixed point locus has codimension 1. We say that \mathcal{W} *has no reflection* if there is no such pair (w, α) .

Lemma 7.2. *The stack \mathcal{X} has no reflection.*

Proof. If $x \in \mathcal{X}(\bar{F})$ and if $\alpha \in \text{Aut}(x) \setminus \{1\}$ is a reflection, then its action on $T_x \mathcal{X}$ is represented by a diagonal matrix $\text{diag}(\zeta, 1, \dots, 1)$ with ζ a root of unity for a suitable basis. This shows that the age of α is less than 1, which contradicts our assumption of adequacy. ■

This lemma shows that the morphism $\mathcal{X} \rightarrow X$ is an isomorphism in codimension 1. In particular, the canonical line bundle $\omega_{\mathcal{X}}$ of \mathcal{X} corresponds to the canonical \mathbb{Q} -line bundle of X by the isomorphism $\text{Pic}(\mathcal{X})_{\mathbb{Q}} \cong \text{Pic}(X)_{\mathbb{Q}}$ (see Section 3); we denote the canonical \mathbb{Q} -line bundle of X by ω'_X . Let $\mathcal{U} \subset \mathcal{X}$ and $U \subset X$ be the largest open substacks such that $\mathcal{X} \rightarrow X$ restricts to an isomorphism $\mathcal{U} \rightarrow U$. The height function $H_{\omega_{\mathcal{X}}^{-1}}|_{\mathcal{U}\langle F \rangle}$ and the height function $H_{(\omega'_X)^{-1}}|_{U\langle F \rangle}$ are then identical through the identification $\mathcal{U}\langle F \rangle = U\langle F \rangle$. We check that the asymptotic formula for $H_{\omega_{\mathcal{X}}^{-1}}|_{\mathcal{U}\langle F \rangle}$ expected by Conjecture 5.6 coincides with the one for $H_{(\omega'_X)^{-1}}|_{U\langle F \rangle}$ expected by a version of the Manin conjecture for singular Fano varieties.

We now briefly review the terminology on singularities appearing in the minimal model program, which we will use. For details, we refer the reader to [24]. Let Y be a variety over F with the \mathbb{Q} -Cartier canonical divisor K_Y . A *divisor over Y* means a prime divisor on some resolution of Y . We identify two divisors over Y when they determine the same valuation on the function field $K(Y)$. For a divisor over Y , say lying on a resolution Z of Y , the *discrepancy* $a(E; Y)$ is defined to be the multiplicity of E in the relative canonical divisor $K_{Z/Y}$, which is a rational number. We say that Y *has only canonical singularities* if $a(E; Y) \geq 0$ for every divisor E over Y . A divisor E over Y is said to be *crepant* if it has zero discrepancy. When Y has canonical singularities, there are at most finitely many crepant divisors over Y up to identification explained above. We denote by $\gamma(Y)$ the number of crepant divisors over Y .

Remark 7.3. A divisor over Y may not be geometrically irreducible. By the base change to \bar{F} , a crepant divisor over Y may split into several distinct crepant divisors over $Y_{\bar{F}}$. Thus, we may have the strict inequality $\gamma(Y_{\bar{F}}) > \gamma(Y)$.

Lemma 7.4. *The coarse moduli space X has only canonical singularities.*

Proof. This follows from the Reid–Shepherd–Barron–Tai criterion [24, Theorem 3.21] and the adequacy condition that $\text{age}(\mathcal{Y}) \geq 1$. ■

Proposition 7.5 (cf. [48, Proposition 4.5], [49, Proposition 8.5]). *We have $\gamma(X) = j_c(\mathcal{X})$.*

Proof. We only sketch the proof. Let $W \subset X$ be the singular locus and let $\mathcal{W} \subset \mathcal{X}$ be its preimage, which is nothing but the locus of stacky points. From [47, Theorem 3], we have $\Sigma(X)_W = \Sigma(\mathcal{X})_{\mathcal{W}}$ (these invariants are also denoted by $M_{\text{st}}(X)_W$ and $M_{\text{st}}(\mathcal{X})_{\mathcal{W}}$ in the literature). The left hand side has an expression in terms of a log resolution and exceptional divisors, while the right hand side has an expression in terms of sectors. Taking the E-polynomial or Poincaré polynomial realization of this equality and comparing the coefficients of the leading terms, we see that the number of geometric crepant divisors (that is, crepant divisors over $X_{\bar{F}}$) is equal to the number of geometric junior sectors (that is, junior sectors of $\mathcal{X}_{\bar{F}}$). In fact, not only the numbers are equal, but also there is a natural one-to-one correspondence

$$\{\text{geometric crepant divisors}\} \leftrightarrow \{\text{geometric junior sectors}\}. \tag{7.1}$$

This follows from the correspondence between arcs on a resolution of X and twisted arcs of \mathcal{X} (see [47]). Correspondence 7.1 is Γ_F -equivariant. In particular, both sides have the same number of Γ_F -orbits. This implies the proposition. ■

Let us recall the following version of the Manin conjecture for Fano varieties with canonical singularities:

Conjecture 7.6 ([48, Conj. 2.3], [49, Conj. 5.6]). *Let W be a Fano variety over F with only canonical singularities, that is, a normal projective variety with only canonical singularities such that the anti-canonical \mathbb{Q} -line bundle $(\omega'_W)^{-1}$ is ample. Suppose that W has Zariski dense F -points. Then there exists a thin subset $T \subset W(F)$ such that*

$$\#\{x \in W(F) \setminus T \mid H_{(\omega'_W)^{-1}}(x) \leq B\} \sim CB(\log B)^{\rho(W)+\gamma(W)-1}.$$

We have the following equivalence of conjectures:

Proposition 7.7. *Let $\mathcal{Z} \subset \mathcal{X}$ be the locus of stacky points given with the structure of a reduced closed substack and let $Z \subset X$ be the singular locus. Then the following conditions are equivalent:*

- (1) *Conjecture 5.6 holds for the pair (\mathcal{X}, c) with a thin subset T containing $\mathcal{Z}(F)$.*
- (2) *Conjecture 7.6 holds for X with a thin subset T containing $Z(F)$.*

Proof. We first note that \mathcal{Z} and Z are of codimension ≥ 2 in \mathcal{X} and X respectively, and Z is the image of \mathcal{Z} . If we identify $(\mathcal{X} \setminus \mathcal{Z})\langle F \rangle$ and $(X \setminus Z)(F)$, then the height functions $H_{\omega_{\mathcal{X}}^{-1}, c}|_{(\mathcal{X} \setminus \mathcal{Z})\langle F \rangle}$ and $H_{(\omega_X')^{-1}}|_{(X \setminus Z)(F)}$ are the same. If $T = \mathcal{Z}\langle F \rangle \sqcup T'$ is a thin subset of $\mathcal{X}\langle F \rangle$ containing $\mathcal{Z}\langle F \rangle$, then T' is also a thin subset of $X(F)$ through the identification $(\mathcal{X} \setminus \mathcal{Z})\langle F \rangle = (X \setminus Z)(F)$. We also have $\rho(\mathcal{X}) = \rho(X)$ and $j_c(\mathcal{X}) = \gamma(X)$. It follows that if Conjecture 5.6 holds for the pair (\mathcal{X}, c) with a thin $T = \mathcal{Z}\langle F \rangle \sqcup T'$, then Conjecture 7.6 holds for X with the thin set $Z(F) \sqcup T'$; and similarly for the opposite implication. ■

Remark 7.8. If a nice stack \mathcal{X} has trivial generic stabilizer and has no reflection, and if some twisted sector of it has age < 1 , then singularities of the coarse moduli space X are not canonical but only log terminal. An observation in [49, Section 13.3] suggests that it is more difficult to control the asymptotic of heights of rational points on such a variety and to formulate a Batyrev–Manin-type conjecture for stacks which is compatible with one for varieties. That is why we impose the condition of adequacy.

Example 7.9. Let $\mathcal{X} := \mathcal{P}(1, 1, 2)$. This stack has only one twisted sector, which has age 1. Its coarse moduli space $X = \mathbb{P}(1, 1, 2)$, the weighted projective space, is a toric surface with a unique singular point. Applying a result in [5] (see also [7, Example 5.1.1]) to the minimal resolution of X , we get an asymptotic formula of the form $\#\{\dots\} \sim CB(\log B)$ for the height function of the anti-canonical sheaf ω_X^{-1} , which is a genuine line bundle (rather than a \mathbb{Q} -line bundle), since X is Gorenstein. It follows that Conjecture 5.6 holds for the stack \mathcal{X} with the trivial raising function $c \equiv 0$; the exponent of the log factor in this case is

$$\rho(\mathcal{X}) + j_c(\mathcal{X}) - 1 = 1 + 1 - 1 = 1.$$

Note that the c -junior sector of \mathcal{X} is of dimension zero, while ones in Example 6.5 have the same dimension as the product stack $\mathcal{X}_1 \times \mathcal{X}_2$ in question. Combining examples of this kind with the product construction in Example 6.5 provides more cases where Conjecture 5.6 holds.

8. The orbifold pseudo-effective cone

Our next goal is to formulate the Batyrev–Manin conjecture for DM stacks which treats non-Fano stacks and line bundles different from the anti-canonical line bundle. Recall that the original Batyrev–Manin conjecture was formulated in terms of the pseudo-effective cone. In this section, we introduce an “orbifold version” of the pseudo-effective cone. Let \mathcal{X} be a nice stack over a number field F as before.

Definition 8.1. We define the orbifold Néron–Severi space to be

$$N_{\text{orb}}^1(\mathcal{X})_{\mathbb{R}} := N^1(\mathcal{X})_{\mathbb{R}} \oplus \bigoplus_{\mathcal{Y} \in \pi_0^*(\mathcal{I}_0 \mathcal{X})} \mathbb{R}[\mathcal{Y}].$$

This is a finite-dimensional \mathbb{R} -vector space. Since $\mathcal{X} \neq \text{Spec } F$ from the definition of nice stack, this space has positive dimension; if $\dim \mathcal{X} > 0$, then $N^1(\mathcal{X})_{\mathbb{R}} \neq 0$, and if $\dim \mathcal{X} = 0$, then $\pi_0^*(\mathcal{J}_0 \mathcal{X}) \neq \emptyset$. Each element $\theta \in N^1_{\text{orb}}(\mathcal{X})_{\mathbb{R}}$ is uniquely written as

$$\theta = \theta_0 + \sum_y \theta_y [y] \quad (\theta_0 \in N^1(\mathcal{X})_{\mathbb{R}}, \theta_y \in \mathbb{R}).$$

Remark 8.2. It is worth noting that the orbifold Néron–Severi space $N^1_{\text{orb}}(\mathcal{X})_{\mathbb{R}}$ can be regarded as a subspace of the Chen–Ruan orbifold cohomology $\bigoplus_{i \in \mathbb{Q}} H^i_{\text{CR}}(\mathcal{X}(\mathbb{C}), \mathbb{R})$ (for this notion, see [4]), like the usual Néron–Severi space $N^1_{\text{orb}}(X)_{\mathbb{R}}$ of a smooth variety X can be regarded as a subspace of $H^2(X(\mathbb{C}), \mathbb{R})$.

Definition 8.3. For a raised line bundle (\mathcal{L}, c) , we define its *numerical class* to be

$$[\mathcal{L}, c] := [\mathcal{L}] + \sum_y c(y)[y] \in N^1_{\text{orb}}(\mathcal{X})_{\mathbb{R}}.$$

For a \mathbb{Q} -factorial projective variety X , the pseudo-effective cone $\overline{\text{Eff}}(X) \subset N^1(X)_{\mathbb{R}}$ is by definition the closure of the cone generated by the classes of effective divisors. By [9], this cone is dual to the cone of moving curves.³ The pseudo-effective cone plays an essential role in formulation of the Batyrev–Manin conjecture. This would be partly because moving curves are a geometric analogue of “general” rational points. We consider the following generalization of moving curves to stacks:

Definition 8.4. A *stacky curve* on $\mathcal{X}_{\bar{F}}$ is a representable \bar{F} -morphism $\mathcal{C} \rightarrow \mathcal{X}_{\bar{F}}$, where \mathcal{C} is a one-dimensional, irreducible, proper, and smooth DM stack over \bar{F} which has trivial generic stabilizer. A *covering family of stacky curves* of $\mathcal{X}_{\bar{F}}$ is a pair

$$(\pi: \tilde{\mathcal{C}} \rightarrow T, \tilde{f}: \tilde{\mathcal{C}} \rightarrow \mathcal{X}_{\bar{F}})$$

of \bar{F} -morphisms of DM stacks such that

- (1) π is smooth and surjective,
- (2) T is an integral scheme of finite type over \bar{F} ,
- (3) for each point $t \in T(\bar{F})$, the morphism

$$\tilde{f}|_{\pi^{-1}(t)}: \pi^{-1}(t) \rightarrow \mathcal{X}_{\bar{F}}$$

is a stacky curve on $\mathcal{X}_{\bar{F}}$,

- (4) \tilde{f} is dominant.

We often omit π and write a covering family of stacky curves simply as $\tilde{f}: \tilde{\mathcal{C}} \rightarrow \mathcal{X}_{\bar{F}}$.

Let $f: \mathcal{C} \rightarrow \mathcal{X}_{\bar{F}}$ be a stacky curve and let $p \in \mathcal{C}(\bar{F})$ be a stacky point. The formal neighborhood of p is written as $[\text{Spec } \bar{F}[[t]]/\mu_{l, \bar{F}}]$ for some positive integer l . The morphism f induces a twisted arc of $\mathcal{X}_{\bar{F}}$, that is, a representable morphism

$$[\text{Spec } \bar{F}[[t]]/\mu_{l, \bar{F}}] \rightarrow \mathcal{X}_{\bar{F}}.$$

³The paper [9] proves this for smooth varieties, and the singular case is easily reduced to the smooth case by desingularization.

In turn, this induces a twisted 0-jet

$$\mathbf{B} \mu_{l, \bar{F}} \hookrightarrow [\mathrm{Spec} \bar{F}[[t]]/\mu_{l, \bar{F}}] \rightarrow \mathcal{X}_{\bar{F}},$$

which is an \bar{F} -point of $\mathcal{G}_0 \mathcal{X}$. We denote the sector of \mathcal{X} (not of $\mathcal{X}_{\bar{F}}$) containing this point by \mathcal{Y}_p . Note that this construction is an analogue of construction of the residue map ψ_ν (see Definition 2.17). As an analogue of our unstable height function (Definition 4.3), we introduce the following notion.

Definition 8.5. Let $\theta = \theta_0 + \sum_{\mathcal{Y}} \theta_{\mathcal{Y}}[\mathcal{Y}]$ be an element of $N_{\mathrm{orb}}^1(\mathcal{X})_{\mathbb{R}}$ and let $f: \mathcal{C} \rightarrow \mathcal{X}_{\bar{F}}$ be a stacky curve. The *intersection number* $(f, \theta) = (\mathcal{C}, \theta)$ is defined by

$$(\mathcal{C}, \theta) := \deg f^* \theta_0 + \sum_{p \in \mathcal{C}(\bar{F}) \text{ stacky point}} \theta_{\mathcal{Y}_p}.$$

Lemma 8.6. Let $(\pi: \tilde{\mathcal{C}} \rightarrow T, \tilde{f}: \tilde{\mathcal{C}} \rightarrow \mathcal{X}_{\bar{F}})$ be a covering family of stacky curves and let $\theta \in N_{\mathrm{orb}}^1(\mathcal{X})$. Then there exists an open dense subscheme $T_0 \subset T$ such that for all $t \in T_0(\bar{F})$, the intersection numbers $(\tilde{f}|_{\pi^{-1}(t)}, \theta)$ are the same.

Proof. We let f vary among the morphisms $\tilde{f}|_{\pi^{-1}(t)}: \pi^{-1}(t) \rightarrow \mathcal{X}_{\bar{F}}, t \in T(\bar{F})$. The first term $\deg f^* \theta_0$ of the intersection number is constant on the entire $T(\bar{F})$. We show that the second term $\sum_p \theta_{\mathcal{Y}_p}$ is also constant on some open dense subset $T_0 \subset T$. There exists a dominant morphism $T' \rightarrow T$ of \bar{F} -varieties satisfying

- (1) the base change $\pi_{T'}: \tilde{\mathcal{C}}_{T'} \rightarrow T'$ of $\pi: \tilde{\mathcal{C}} \rightarrow T$ has sections $s_1, \dots, s_l: T' \rightarrow \tilde{\mathcal{C}}_{T'}$ such that the union of their images, $\bigcup_{i=1}^l \mathrm{Im}(s_i)$, is precisely the stacky locus of $\tilde{\mathcal{C}}_{T'}$,
- (2) each reduced closed substack $\mathrm{Im}(s_i) \subset \tilde{\mathcal{C}}_{T'}$ is isomorphic to $\mathbf{B} \mu_{l_i, T'}$ over T' for some $l_i > 0$.

The morphism

$$\mathbf{B} \mu_{l_i, T'} \hookrightarrow \tilde{\mathcal{C}}_{T'} \rightarrow \mathcal{X}_{\bar{F}}$$

defines a T' -point of $\mathcal{G}_0 \mathcal{X}_{\bar{F}}$ and one of $\mathcal{G}_0 \mathcal{X}$, which in turn determines a sector of \mathcal{X} . This shows that when an \bar{F} -point $p \in \tilde{\mathcal{C}}_{T'}(\bar{F})$ varies along $\mathrm{Im}(s_i)$, then the associated sector \mathcal{Y}_p is unchanged. This shows that the second term $\sum_p \theta_{\mathcal{Y}_p}$ of the intersection number is constant, when t varies in the image of $T'(\bar{F}) \rightarrow T(\bar{F})$. ■

Definition 8.7. For a covering family $(\pi: \tilde{\mathcal{C}} \rightarrow T, \tilde{f}: \tilde{\mathcal{C}} \rightarrow \mathcal{X}_{\bar{F}})$ of stacky curves and $\theta \in N_{\mathrm{orb}}^1(\mathcal{X})$, we define the *intersection number* (\tilde{f}, θ) to be $(f|_{\pi^{-1}(t)}, \theta)$ for a general point $t \in T(\bar{F})$.

Definition 8.8 (Orbifold pseudo-effective cone). We define the *orbifold pseudo-effective cone* $\overline{\mathrm{Eff}}_{\mathrm{orb}}(\mathcal{X})$ to be the cone in $N_{\mathrm{orb}}^1(\mathcal{X})$ consisting of elements θ such that for every covering family $\tilde{f}: \tilde{\mathcal{C}} \rightarrow \mathcal{X}_{\bar{F}}$ of stacky curves, $(\tilde{f}, \theta) \geq 0$. Elements in $\overline{\mathrm{Eff}}_{\mathrm{orb}}(\mathcal{X})$ are called *pseudo-effective classes*. We also define the (nonorbifold) *pseudo-effective cone* $\overline{\mathrm{Eff}}(\mathcal{X})$ to be $\overline{\mathrm{Eff}}(\mathcal{X}) := \overline{\mathrm{Eff}}_{\mathrm{orb}}(\mathcal{X}) \cap N^1(\mathcal{X})_{\mathbb{R}}$.

We see that by the identification $N^1(\mathcal{X})_{\mathbb{R}} = N^1(\bar{\mathcal{X}})_{\mathbb{R}}$, we have $\overline{\mathrm{Eff}}(\mathcal{X}) = \overline{\mathrm{Eff}}(\bar{\mathcal{X}})$.

Remark 8.9 (Intersection pairing). If $N_{1,\text{orb}}(\mathcal{X})_{\mathbb{R}}$ denotes the dual space of $N_{\text{orb}}^1(\mathcal{X})_{\mathbb{R}}$, we may define the intersection number in terms of the pairing

$$N_{\text{orb}}^1(\mathcal{X})_{\mathbb{R}} \times N_{1,\text{orb}}(\mathcal{X})_{\mathbb{R}} \rightarrow \mathbb{R}.$$

The space $N_{1,\text{orb}}(\mathcal{X})_{\mathbb{R}}$ is expressed as

$$N_{1,\text{orb}}(\mathcal{X})_{\mathbb{R}} = N_1(\mathcal{X})_{\mathbb{R}} \oplus \bigoplus_{\mathcal{Y} \in \pi_0^*(\mathcal{I}_0 \mathcal{X})} \mathbb{R}[\mathcal{Y}]^*,$$

where $N_1(\mathcal{X})_{\mathbb{R}}$ is the space of real 1-cycles modulo numerical equivalence and $\{[\mathcal{Y}]^*\}_{\mathcal{Y}}$ is the dual basis of $\{[\mathcal{Y}]\}_{\mathcal{Y}}$, which is supposed to be orthogonal to $N^1(\mathcal{X})_{\mathbb{R}}$. With a covering family $\tilde{f}: \tilde{\mathcal{C}} \rightarrow \mathcal{X}_{\bar{F}}$ of stacky curves, we can associate an element

$$[\tilde{f}]_{\text{orb}} := [\tilde{f}|_{\pi^{-1}(t)}] + \sum_{p \in (\pi^{-1}(t))(\bar{F}) \text{ stacky point}} [\mathcal{Y}_p]^* \quad ([\tilde{f}|_{\pi^{-1}(t)}] \in N_1(\mathcal{X})_{\mathbb{R}})$$

so that for every $\theta \in N_{\text{orb}}^1(\mathcal{X})_{\mathbb{R}}$, the intersection number (\tilde{f}, θ) is expressed as the pairing $([\tilde{f}]_{\text{orb}}, \theta)$. The dual cone of $\bar{\text{Eff}}_{\text{orb}}(\mathcal{X})$ is then the closure of the cone generated by the classes $[\tilde{f}]_{\text{orb}}$ of covering families $\tilde{f}: \tilde{\mathcal{C}} \rightarrow \mathcal{X}_{\bar{F}}$ of stacky curves.

Lemma 8.10. *Let $\theta = \theta_0 + \sum_{\mathcal{Y}} \theta_{\mathcal{Y}}[\mathcal{Y}] \in N_{\text{orb}}^1(\mathcal{X})_{\mathbb{R}}$. If θ_0 is not pseudo-effective as an element of $N^1(\bar{\mathcal{X}})$, then θ is not pseudo-effective either.*

Proof. There exists a finite morphism $W \rightarrow \mathcal{X}$ from a scheme W (see [42, tag 04V1]). We may suppose that $W \rightarrow \bar{\mathcal{X}}$ is a Galois cover, say with Galois group G . Then W is \mathbb{Q} -factorial and we have an isomorphism

$$(N^1(W)_{\mathbb{R}})^G \cong N^1(\bar{\mathcal{X}})_{\mathbb{R}}.$$

Moreover, this isomorphism preserves pseudo-effectivity. Thus, if $\theta_0 \in N^1(\mathcal{X})_{\mathbb{R}} = N^1(\bar{\mathcal{X}})_{\mathbb{R}}$ is not pseudo-effective, then the corresponding $\eta \in (N^1(W)_{\mathbb{R}})^G$ is not pseudo-effective either. Therefore, there exists a moving curve C of W with $(C, \eta) < 0$. Then $C \rightarrow \mathcal{X}$ induces a covering family of stacky curves whose source is a scheme. Since C has no stacky point, $\theta_{\mathcal{Y}}$'s do not contribute to the intersection number (C, θ) and we have

$$(C, \theta) = (C, \eta) < 0.$$

We see that θ is not pseudo-effective. ■

Corollary 8.11. *We have*

$$\bar{\text{Eff}}(\mathcal{X}) + \sum_{\mathcal{Y}} \mathbb{R}_{\geq 0}[\mathcal{Y}] \subset \bar{\text{Eff}}_{\text{orb}}(\mathcal{X}) \subset \bar{\text{Eff}}(\mathcal{X}) + \sum_{\mathcal{Y}} \mathbb{R}[\mathcal{Y}].$$

Proof. The left inclusion is obvious. The right inclusion is the last lemma. ■

Lemma 8.12. *Let K be an infinite field and let n be a positive integer coprime to the characteristic of K . Let C be a geometrically irreducible, smooth, and proper curve over K . Let $p_0, p_1, \dots, p_m \in C(K)$ be K -points of C . Then there exist a geometrically irreducible, smooth, and proper curve D over K and a finite morphism $f: D \rightarrow C$ of degree n such that for $i \in \{1, \dots, m\}$, $f^{-1}(p_i)_{\text{red}} \cong \text{Spec } K$ and f is étale over p_0 .*

Proof. We take an affine open subscheme $C' = \text{Spec } R \subset C$ containing all of p_0, \dots, p_m and suppose that it is embedded in the affine space \mathbb{A}_K^l . Since K is an infinite field, for each $i > 0$ there exists a hyperplane $H_i = \{f_i = 0\} \subset \mathbb{A}_K^l$ which intersects C' transversally at p_i , but does not meet p_j for any $j \neq i$. Let $f \in R$ be the restriction of $\prod_{i>0} f_i$ to C' . This function on C' has zeroes of order 1 at each of p_1, \dots, p_m and does not vanish at p_0 . It follows that the finite cover

$$D' := \text{Spec } R[x]/(x^n - f) \rightarrow C'$$

has the desired properties except properness. We only need to take projective compactification. ■

Proposition 8.13. *Let \mathcal{X} be a nice stack and let \mathcal{Y}_0 be a twisted sector of \mathcal{X} . Then, for any positive integer n , there exists a covering family $\tilde{f}: \tilde{\mathcal{C}} \rightarrow \mathcal{X}_{\bar{F}}$ of stacky curves such that if we write*

$$[\tilde{f}]_{\text{orb}} = [\tilde{f}] + \sum_{\mathcal{Y} \in \pi_0^*(\mathcal{J}_0 \mathcal{X})} \theta_{\tilde{f}, \mathcal{Y}} [\mathcal{Y}]^* \quad (\theta_{\mathcal{Y}} \in \mathbb{Z}_{\geq 0})$$

(see Remark 8.9), then

$$\theta_{\tilde{f}, \mathcal{Y}_0} > n \sum_{\mathcal{Y} \neq \mathcal{Y}_0} \theta_{\tilde{f}, \mathcal{Y}}.$$

Proof. In this proof, we identify geometric points of a DM stack and ones of the coarse moduli space. Using a local description of $\mathcal{X}_{\bar{F}}$ as a quotient stack, we can construct a covering family $(\pi: \tilde{\mathcal{C}} \rightarrow T, \tilde{f}: \tilde{\mathcal{C}} \rightarrow \mathcal{X}_{\bar{F}})$ such that $\theta_{\tilde{f}, \mathcal{Y}_0} > 0$. Let $\tilde{\mathcal{C}}$ denote the coarse moduli space of $\tilde{\mathcal{C}}$. Let $L = \overline{K(T)}$ be an algebraic closure of the function field $K(T)$ of T . Let

$$p_0, \dots, p_m \in \tilde{\mathcal{C}}_L \langle L \rangle = \tilde{\mathcal{C}}_L(L)$$

be stacky points of $\tilde{\mathcal{C}}_L$. For an integer $n' > nm$, we take a finite cover $\tilde{D}_L \rightarrow \tilde{\mathcal{C}}_L$ of degree n' as in Lemma 8.12. The induced rational map $\tilde{D}_L \rightarrow \tilde{\mathcal{C}}_L \dashrightarrow \mathcal{X}_{\bar{F}}$ uniquely extends to a representable morphism

$$\tilde{g}_L: \tilde{D}_L \rightarrow \mathcal{X}_{\bar{F}},$$

where \tilde{D}_L is a smooth proper DM stack over L with the coarse moduli space morphism $\tilde{D}_L \rightarrow \tilde{D}_L$. The only stacky L -points of \tilde{D}_L are the n' points $r_{0,1}, \dots, r_{0,n'}$ lying over p_0 and possibly some of the points r_1, \dots, r_m lying over p_1, \dots, p_m respectively. If we

define sectors of \mathcal{X} associated to stacky L -points of $\tilde{\mathcal{D}}_L$ in the same way as we did for stacky \bar{F} -points before, then for every $j \in \{1, \dots, n'\}$, we have $\mathcal{Y}_{r_{0,j}} = \mathcal{Y}_0$. If we define $[\tilde{g}_L]_{\text{orb}} \in N_{1,\text{orb}}(\mathcal{X})$ as before, then we get

$$\begin{aligned} [\tilde{g}_L]_{\text{orb}} &= [\tilde{g}_L] + \sum_{j=1}^{n'} [\mathcal{Y}_{r_{0,j}}]^* + \sum_{i=1}^m [\mathcal{Y}_{r_i}]^* \\ &= [\tilde{g}] + n'[\mathcal{Y}_0]^* + \sum_{i=1}^m [\mathcal{Y}_{r_i}]^*. \end{aligned}$$

It follows that

$$\theta_{\tilde{g}_L, \mathcal{Y}_0} \geq n' > nm \geq n \sum_{\mathcal{Y} \neq \mathcal{Y}_0} \theta_{\tilde{g}_L, \mathcal{Y}}.$$

This is basically the desired inequality, except that $\tilde{\mathcal{D}}_L$ is defined not over an \bar{F} -variety, but over $L = \overline{K(T)}$. By a standard argument, we can find a finitely generated \bar{F} -subalgebra $R \subset L$ such that $\tilde{\mathcal{D}}_L$ is the base change of an R -stack $\tilde{\mathcal{D}}_R$ and the morphism \tilde{g}_L is induced from a morphism $\tilde{\mathcal{D}}_R \rightarrow \mathcal{X}_{\bar{F}}$. The pair $(\tilde{\mathcal{D}}_R \rightarrow \text{Spec } R, \tilde{\mathcal{D}}_R \rightarrow \mathcal{X}_{\bar{F}})$ is a covering family of stacky curves that has the desired property. ■

Proposition 8.14. *Consider an element $\eta \in N_{\text{orb}}^1(\mathcal{X})$ of the form $\sum_{\mathcal{Y}} \eta_{\mathcal{Y}}[\mathcal{Y}]$ with $\eta_{\mathcal{Y}} \in \mathbb{R}$ and suppose that $\eta_{\mathcal{Y}_0} < 0$ for some twisted sector \mathcal{Y}_0 . Then $\eta \notin \overline{\text{Eff}}_{\text{orb}}(\mathcal{Y})$. Hence,*

$$\overline{\text{Eff}}_{\text{orb}}(\mathcal{X}) \cap \bigoplus_{\mathcal{Y} \in \pi_0^*(\mathcal{J}_0 \mathcal{X})} \mathbb{R}[\mathcal{Y}] = \sum_{\mathcal{Y} \in \pi_0^*(\mathcal{J}_0 \mathcal{X})} \mathbb{R}_{\geq 0}[\mathcal{Y}].$$

Proof. For each $n > 0$, we take a covering family $\tilde{f}_n: \tilde{\mathcal{C}}_n \rightarrow \mathcal{X}_{\bar{F}}$ of stacky curves as in Proposition 8.13. Then

$$\begin{aligned} (\tilde{f}_n, \eta) &= \sum_{\mathcal{Y}} \theta_{\tilde{f}_n, \mathcal{Y}} \cdot \eta_{\mathcal{Y}} = \theta_{\tilde{f}_n, \mathcal{Y}_0} \cdot \eta_{\mathcal{Y}_0} + \sum_{\mathcal{Y} \neq \mathcal{Y}_0} \theta_{\tilde{f}_n, \mathcal{Y}} \cdot \eta_{\mathcal{Y}} \\ &\leq \theta_{\tilde{f}_n, \mathcal{Y}_0} \cdot \eta_{\mathcal{Y}_0} + \left(\sum_{\mathcal{Y} \neq \mathcal{Y}_0} \theta_{\tilde{f}_n, \mathcal{Y}} \right) \max \{ \eta_{\mathcal{Y}} \mid \mathcal{Y} \neq \mathcal{Y}_0 \}. \end{aligned}$$

For sufficiently large n , the last expression is negative, which implies that $\eta \notin \overline{\text{Eff}}_{\text{orb}}(\mathcal{Y})$. The equality of the proposition follows from Corollary 8.11. ■

Definition 8.15. We say that a raised line bundle (\mathcal{L}, c) on \mathcal{X} is *big* if its numerical class $[\mathcal{L}, c]$ lies in the interior of $\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})$.

Lemma 8.16. *Let (\mathcal{L}, c) be a raised line bundle on \mathcal{X} . Suppose that \mathcal{X} has positive dimension. Then the following conditions are equivalent:*

- (1) \mathcal{L} is big, that is, $[\mathcal{L}]$ is in the interior of the cone $\overline{\text{Eff}}(\mathcal{X})$.
- (2) $[\mathcal{L}, c]$ is big.

Proof. From Corollary 8.11, the second condition implies the first one. We now assume the first condition and prove the second condition. For $\alpha \in \overline{\text{Eff}}_{\text{orb}}(\mathcal{X})^\vee$, we have

$$(\alpha, [\mathcal{L}, c]) \geq (\alpha, [\mathcal{L}]) > 0.$$

Corollary 8.11 implies

$$\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})^\vee \subset \overline{\text{Eff}}(\mathcal{X})^\vee + \sum_{\mathcal{Y}} \mathbb{R}_{\geq 0}[\mathcal{Y}]^*.$$

It follows that there exists a hyperplane $H \subset N_{1,\text{orb}}(\mathcal{X})_{\mathbb{R}}$ such that H does not contain the origin and the intersection $\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})^\vee \cap H$ is a nonempty compact set. From the extreme value theorem, the function given by the intersection pairing with $[\mathcal{L}, c]$,

$$\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})^\vee \cap H \rightarrow \mathbb{R}_{>0}, \quad \alpha \mapsto (\alpha, [\mathcal{L}, c]),$$

has a positive minimum value. Perturbing $[\mathcal{L}, c]$ a little in $N_{1,\text{orb}}^1(\mathcal{X})_{\mathbb{R}}$ in an arbitrary direction does not affect the positivity of the minimum value of the associated function $\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})^\vee \cap H \rightarrow \mathbb{R}$. This means that $[\mathcal{L}, c]$ is not pushed out from $\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})$ by a small perturbation in an arbitrary direction. Hence, $[\mathcal{L}, c]$ is in the interior of $\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})$. ■

9. *a*- and *b*-invariants; the stacky Batyrev–Manin conjecture

9.1. *a*- and *b*-invariants and breaking thin subsets

We keep denoting by \mathcal{X} a nice stack over F .

Definition 9.1. We define the *orbifold canonical class* of \mathcal{X} to be

$$[K_{\mathcal{X},\text{orb}}] := [\omega_{\mathcal{X}}] + \sum_{\mathcal{Y}} (\text{age}(\mathcal{Y}) - 1)[\mathcal{Y}] \in N_{1,\text{orb}}^1(\mathcal{X})_{\mathbb{R}}.$$

Here $\omega_{\mathcal{X}}$ denotes the canonical line bundle of \mathcal{X} .

Remark 9.2. Generalizing Proposition 7.5, we can show that if \mathcal{X} has trivial generic stabilizer, some twisted sectors \mathcal{Y} of \mathcal{X} correspond to divisors E over the coarse moduli space $\overline{\mathcal{X}}$. Then, for \mathcal{Y} and E corresponding to each other, $\text{age}(\mathcal{Y}) - 1$ is equal to the discrepancy of E . This explains why the coefficient $\text{age}(\mathcal{Y}) - 1$ in the above definition is natural.

Now we are ready to define the *a*-invariant in the context of stacks:

Definition 9.3. We define the *a*-invariant of a big raised line bundle (\mathcal{L}, c) , denoted by $a(\mathcal{L}, c)$, to be the unique real number a such that $a[\mathcal{L}, c] + [K_{\mathcal{X},\text{orb}}]$ lies on the boundary of $\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})$.

As for a (nonraised) big line bundle \mathcal{L} on \mathcal{X} , we define the *a*-invariant $a(\mathcal{L})$ in a similar way by using the (nonorbifold) pseudo-effective cone $\overline{\text{Eff}}(\mathcal{X}) = \overline{\text{Eff}}(\overline{\mathcal{X}})$ in

$N^1(\mathcal{X})_{\mathbb{R}} = N^1(\overline{\mathcal{X}})_{\mathbb{R}}$ and the canonical class $[\omega_{\mathcal{X}}]$. Note that $a(\mathcal{L})$ may be different from $a(L)$ of the corresponding \mathbb{Q} -line bundle L on $\overline{\mathcal{X}}$. This is because $[\omega_{\mathcal{X}}]$ may be different from $[\omega'_{\overline{\mathcal{X}}}]$, unless the morphism $\mathcal{X} \rightarrow \overline{\mathcal{X}}$ is étale in codimension 1.

We generalize the notion of adequacy for a raising function of a Fano stack as follows:

Definition 9.4. Let (\mathcal{L}, c) be a big raised line bundle. We say that (\mathcal{L}, c) is *adequate* if

- (1) $\text{age}_c(\mathcal{Y}) \geq 1$ for every twisted sector \mathcal{Y} ,
- (2) if $\dim \mathcal{X} > 0$, then $a(\mathcal{L}) = 1$,
- (3) if $\dim \mathcal{X} = 0$, then $\min \{c(\mathcal{Y}) \mid \mathcal{Y} \text{ twisted sector}\} = 1$.

When \mathcal{X} is a Fano stack, the raised line bundle $(\omega_{\mathcal{X}}^{-1}, c)$ is adequate if and only if c is adequate.

Remark 9.5. The condition $a(\mathcal{L}) = 1$ may be viewed as a normalization condition. Indeed, for $r \in \mathbb{Q}$, $a(\mathcal{L}^r) = a(\mathcal{L})/r$.

Proposition 9.6. *If (\mathcal{L}, c) is adequate, then $a(\mathcal{L}, c) = 1$.*

Proof. The zero-dimensional case follows from Proposition 9.25 proved later. We prove the case $\dim \mathcal{X} > 0$ here. For $a' < 1$,

$$a'[\mathcal{L}, c] + [K_{\mathcal{X}, \text{orb}}] = (a'[\mathcal{L}] + [\omega_{\mathcal{X}}]) + \sum_{\mathcal{Y}} (a' \cdot c(\mathcal{Y}) + \text{age}(\mathcal{Y}) - 1)[\mathcal{Y}].$$

Since $a(\mathcal{L}) = 1 > a'$, we find that $a'[\mathcal{L}] + [\omega_{\mathcal{X}}]$ is not pseudo-effective. From Lemma 8.10, $a'[\mathcal{L}, c] + [K_{\mathcal{X}, \text{orb}}] \notin \overline{\text{Eff}}_{\text{orb}}(\mathcal{X})$. This shows that $a(\mathcal{L}, c) \geq 1$. On the other hand,

$$[\mathcal{L}, c] + [K_{\mathcal{X}, \text{orb}}] = ([\mathcal{L}] + [\omega_{\mathcal{X}}]) + \sum_{\mathcal{Y}} (\text{age}_c(\mathcal{Y}) - 1)[\mathcal{Y}].$$

Since $[\mathcal{L}] + [\omega_{\mathcal{X}}] = a(\mathcal{L})[\mathcal{L}] + [\omega_{\mathcal{X}}]$ is pseudo-effective and $\text{age}_c(\mathcal{Y}) - 1 \geq 0$, from Corollary 8.11, $[\mathcal{L}, c] + [K_{\mathcal{X}, \text{orb}}]$ is pseudo-effective. This shows that $a(\mathcal{L}, c) \leq 1$. ■

Definition 9.7. We define the *b-invariant* $b(\mathcal{L}, c)$ of a big raised line bundle (\mathcal{L}, c) to be the codimension of the minimal face of $\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})$ containing $a(\mathcal{L}, c)[\mathcal{L}, c] + [K_{\mathcal{X}, \text{orb}}]$, that is, the dimension of the following face of the dual cone $\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})^{\vee} \subset N_{1, \text{orb}}(\mathcal{X})_{\mathbb{R}}$:

$$\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})^{\vee} \cap (a(\mathcal{L}, c)[\mathcal{L}, c] + [K_{\mathcal{X}, \text{orb}}])^{\perp}.$$

Recall that for a morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$ of nice stacks, we have a natural morphism $\mathcal{J}_0 \mathcal{Y} \rightarrow \mathcal{J}_0 \mathcal{X}$, and hence a raising function c of \mathcal{X} induces a raising function f^*c of \mathcal{Y} .

Definition 9.8. Let us fix a big raised line bundle (\mathcal{L}, c) of \mathcal{X} . A thin morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$ of nice stacks is called a *breaking thin morphism* (resp. *weakly breaking thin morphism*) if the raised line bundle $(f^*\mathcal{L}, f^*c)$ is big and

$$(a(f^*\mathcal{L}, f^*c), b(f^*\mathcal{L}, f^*c)) > (a(\mathcal{L}, c), b(\mathcal{L}, c))$$

$$(\text{resp. } (a(f^*\mathcal{L}, f^*c), b(f^*\mathcal{L}, f^*c)) \geq (a(\mathcal{L}, c), b(\mathcal{L}, c)))$$

in the lexicographic order. A *breaking thin* (resp. *weakly breaking thin*) subset of $\mathcal{X}\langle F \rangle$ means a nonempty subset of $\mathcal{X}\langle F \rangle$ which is the image of the map $\mathcal{Y}\langle F \rangle \rightarrow \mathcal{X}\langle F \rangle$ associated to a breaking thin (resp. weakly breaking thin) morphism $\mathcal{Y} \rightarrow \mathcal{X}$.

Remark 9.9. A breaking thin morphism is basically the same as a breaking thin map in [27] except that we work with stacks and assume the representability condition. If we did not assume the representability condition, then a morphism of the form $\mathcal{X} \times \mathrm{B}G \rightarrow \mathcal{X}$ for a finite group scheme G would become a breaking thin morphism. We do not want to include such a map, since it induces a surjection of F -point sets.

9.2. The stacky Batyrev–Manin conjecture

We now formulate the first version of the Batyrev–Manin conjecture for DM stacks as follows:

Conjecture 9.10 (The stacky Batyrev–Manin conjecture I). *Let \mathcal{X} be a nice stack over F whose coarse moduli space is geometrically rationally connected. Let (\mathcal{L}, c) be a raised line bundle which is big and adequate. Suppose that $\mathcal{X}\langle F \rangle$ is Zariski dense. Then the union T of breaking thin subsets of $\mathcal{X}\langle F \rangle$ is a thin subset. Moreover, there exists a constant $C > 0$ such that*

$$\#\{x \in \mathcal{X}\langle F \rangle \setminus T \mid H_{\mathcal{L},c}(x) \leq B\} \sim CB(\log B)^{b(\mathcal{L},c)-1} \quad (B \rightarrow \infty).$$

Remark 9.11. In the conjecture, we impose the geometric rational connectedness on the coarse moduli space rather than the more restrictive (and more common) Fano condition, following the version of the Batyrev–Manin conjecture in [29]. Since we would like to allow singular coarse moduli spaces, it would be more suitable to use a condition which is invariant under birational equivalences like geometric rational connectedness.

Remark 9.12. If \mathcal{X} is a smooth variety, then this conjecture is a version of the Batyrev–Manin conjecture for a smooth variety and a big line bundle L with $a(L) = 1$ such that the removed accumulation subset is a thin subset. When the a -invariant is not equal to 1 but positive, then we can reduce it to the case with $a = 1$ by considering the \mathbb{R} -line bundle $L^{1/a(L)}$ and generalizing the above conjecture to \mathbb{R} -line bundles in the obvious way.

Remark 9.13. One may be tempted to remove also *weakly* breaking thin subsets. An evidence for this idea in the case of varieties was provided by the work of Le Rudulier [31]. She showed that for some algebraic surface, the leading constant becomes the same as the one conjectured by Peyre [37] only after removing a weakly breaking thin subset. However, if we remove all weakly breaking thin subsets, then it may happen that all F -points are removed, and no point is left to count. See Remark 9.33.

Remark 9.14. Checking the first assertion of Conjecture 9.10, that the union of breaking thin subsets is a thin subset, is an interesting problem on its own. When the target \mathcal{X} is a geometrically uniruled variety, this problem was affirmatively solved by Lehmann–Sengupta–Tanimoto [27]. The problem has an affirmative answer also when \mathcal{X} is $\mathrm{B}G$

for a commutative group scheme G (see Corollary 9.29), and for some constant group scheme G (see Proposition 9.39).

We also formulate a variant of Conjecture 9.10, incorporating the following notions.

Definition 9.15. A subset $U \subset \mathcal{X}(F)$ is said to be *cothin* if its complement $\mathcal{X}(F) \setminus U$ is thin. When a raised line bundle (\mathcal{L}, c) on \mathcal{X} is fixed, we say that an element of $\mathcal{X}(F)$ is *secure* (resp. *strongly secure*) if it is not contained in any breaking thin subset (resp. any weakly breaking thin subset) of $\mathcal{X}(F)$. We say that a subset of $\mathcal{X}(F)$ is *secure* (resp. *strongly secure*) if it contains only secure (resp. strongly secure) elements.

The following is the second version of the Batyrev–Manin conjecture for DM stacks, which allows some freedom in the choice of the set of counted F -points.

Conjecture 9.16 (The stacky Batyrev–Manin conjecture II). *Let \mathcal{X} be a nice stack over F whose coarse moduli space is geometrically rationally connected. Let (\mathcal{L}, c) be a raised line bundle which is big and adequate. Suppose that $\mathcal{X}(F)$ is Zariski dense. Let $U \subset \mathcal{X}(F)$ be a secure cothin subset. Then there exists a constant $C > 0$ such that*

$$\#\{x \in U \mid H_{\mathcal{L},c}(x) \leq B\} \sim CB(\log B)^{b(\mathcal{L},c)-1} \quad (B \rightarrow \infty).$$

Remark 9.17 (Speculation on a formula for the leading constant C). In the situation of Conjecture 9.16, we speculate that if U is also *strongly secure* and if we use the height function $H_{\mathcal{L},c_*}$ for a strictly raised line bundle (\mathcal{L}, c_*) equipped with an adelic metric on \mathcal{L} , then the constant C may admit an explicit expression similar to ones of Peyre [37] and Bhargava [8]. Note that there are nice stacks \mathcal{X} without any strongly secure F -point. See Remark 9.32 and Section 9.39 for a related discussion in the case of zero-dimensional stacks.

9.3. Fano stacks revisited

Proposition 9.18. *Let \mathcal{X} be a Fano stack and let c be a raising function of \mathcal{X} such that $(\omega_{\mathcal{X}}^{-1}, c)$ is adequate. Then*

$$b(\omega_{\mathcal{X}}^{-1}, c) = \rho(\mathcal{X}) + j_c(\mathcal{X}).$$

In particular, Conjecture 9.10 implies Conjecture 5.6.

Proof. We have

$$\begin{aligned} \eta &:= a(\omega_{\mathcal{X}}^{-1}, c)[\omega_{\mathcal{X}}^{-1}, c] + [K_{\mathcal{X},\text{orb}}] = [\omega_{\mathcal{X}}^{-1}, c] + [K_{\mathcal{X},\text{orb}}] \\ &= \sum_{\mathcal{Y}} (\text{age}_c(\mathcal{Y}) - 1)[\mathcal{Y}]. \end{aligned}$$

For a vector

$$w \in \bigoplus_{\mathcal{Y} \text{ not } c\text{-junior}} \mathbb{R}[\mathcal{Y}] =: W$$

and for $0 < \varepsilon \ll 1$, $\eta \pm \varepsilon w \in \overline{\text{Eff}}_{\text{orb}}(\mathcal{X})$. On the other hand, if w is chosen outside W , then from Proposition 8.14 and Lemma 8.10, either of $\eta \pm \varepsilon w$ is not contained in $\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})$. This shows that the minimal face containing η has dimension equal to $\dim W$. We have proved the first assertion of the proposition.

To show the second assertion, we only need to check that the coarse moduli space X of \mathcal{X} is geometrically rationally connected. Let D be the branch divisor of the coarse moduli space morphism $\pi: \mathcal{X} \rightarrow X$ with standard \mathbb{Q} -coefficients. Then the \mathbb{Q} -divisor $K_X + D$ corresponds to the canonical line bundle $\omega_{\mathcal{X}}$ of \mathcal{X} and the pair (X, D) has only Kawamata log terminal singularities. From [52, Theorem 1], X is geometrically rationally connected. ■

Remark 9.19. To give a conjectural expression of the leading constant C in the asymptotic formula for rational points of a Fano variety X , Peyre [37] uses the three ingredients; the volume of $\overline{X}(F) \subset X(\mathbb{A})$, the volume of the intersection of $\overline{\text{Eff}}(X)$ and the hyperplane $(-K_X, *) = 1$, and the Γ_F -module $N^1(X_{\overline{F}})$. We may speculate upon how to generalize it to stacks. For a nice stack \mathcal{X} , as soon as we find the correct topology and measure on $\mathcal{X}(\mathbb{A})$, we can define the volume of $\overline{\mathcal{X}}(F) \subset X(\mathbb{A})$. The pseudo-effective cone should be replaced with the orbifold pseudo-effective cone $\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})$. The volume of the hyperplane section $\overline{\text{Eff}}_{\text{orb}}(\mathcal{X}) \cap \{([K_{\mathcal{X}, \text{orb}}], *) = 1\}$ for an appropriate measure would be the counterpart of the second ingredient. As for the Γ_F -module, we may consider

$$N^1_{\text{orb}, c\text{-jun}}(\mathcal{X}_{\overline{F}})_{\mathbb{R}} := N^1(\mathcal{X}_{\overline{F}})_{\mathbb{R}} \oplus \bigoplus_{\mathcal{Y} \text{ } c\text{-junior sector}} \mathbb{R}[\mathcal{Y}].$$

From Proposition 7.5, if \mathcal{X} has trivial generic stabilizer and if the raising function $c \equiv 0$ is adequate, then this module is isomorphic to the Γ_F -module $N^1(X_{\overline{F}})_{\mathbb{R}} \oplus \bigoplus_E \mathbb{R}[E]$, where E runs over crepant divisors over the coarse moduli space $X_{\overline{F}}$. Thanks to Proposition 7.5, this Galois module is an analogue of \mathcal{L} -Picard group considered by Batyrev and Tschinkel [7]. We do not know whether the literal translation of Peyre’s constant by using the above orbifold versions of ingredients gives the correct value of the leading constant, but we expect that the above speculation is heading in the right direction.

9.4. Malle’s conjecture revisited

In this subsection, we suppose that \mathcal{X} over F has dimension zero and has at least one F -point. From [42, tag 06QK], this is equivalent to saying that \mathcal{X} is a neutral gerbe over F . From [26, (3.21)], for an F -point $x \in \mathcal{X}(F)$, there exists a canonical isomorphism $\mathcal{X} \cong \mathbb{B} \underline{\text{Aut}}_F(x)$. In particular, \mathcal{X} is isomorphic to the classifying stack $\mathbb{B}G$ of a finite group scheme G over F . Note that the isomorphism class of such a group scheme G is not generally uniquely determined by \mathcal{X} , as the next result shows.

Lemma 9.20 ([10, Proposition 2.2.3.6], [18, p. 127]). *Let G be a finite group scheme over F , let $x \in (\mathbb{B}G)(F)$ and let $x' \in (\mathbb{B} \underline{\text{Aut}}(G^{\text{op}}))(F)$ be the $\underline{\text{Aut}}(G^{\text{op}})$ -torsor derived from x and the conjugation morphism $G \rightarrow \underline{\text{Aut}}(G) = \underline{\text{Aut}}(G^{\text{op}})$. Then $\underline{\text{Aut}}_F(x)$ is isomorphic to the twisted form of G^{op} associated to x' .*

Corollary 9.21. *If G is commutative, then for every $x \in (\mathbf{B}G)(F)$, we have $\underline{\text{Aut}}_F(x) \cong G$.*

Proof. This follows from the last lemma and the fact that in the commutative case, the conjugation map $G \rightarrow \underline{\text{Aut}}(G)$ is the trivial map onto the identity point. ■

Our principal interest is in the stack $\mathcal{X} = \mathbf{B}G_F$, where G is a finite group and G_F is the corresponding constant group scheme over F . In this case, from Example 2.15, we may identify $\pi_0(\mathcal{I}_0\mathcal{X})$ with the set $F\text{-Conj}(G)$ of F -conjugacy classes of G . Thus, we may write

$$N_{\text{orb}}^1(\mathcal{X}) = \bigoplus_{[1] \neq [g] \in F\text{-Conj}(G)} \mathbb{R}[g].$$

Proposition 9.22. *Suppose that $\mathcal{X} = \mathbf{B}G_F$ for a finite group G . Suppose that F contains all the $\#G$ -th roots of unity. Then*

$$\overline{\text{Eff}}_{\text{orb}}(\mathcal{X}) = \sum_{[1] \neq [g] \in \text{Conj}(G)} \mathbb{R}_{\geq 0}[g].$$

Proof. We first note that from the assumption, we have

$$N_{\text{orb}}^1(\mathcal{X}) = \bigoplus_{[1] \neq [g] \in \text{Conj}(G)} \mathbb{R}[g].$$

We claim that for any twisted sector \mathcal{Y} of $\mathcal{X}_{\overline{F}}$, there exists a stacky curve $f: \mathcal{C} \rightarrow \mathcal{X}_{\overline{F}}$ such that for every stacky point $p \in \mathcal{C}(\overline{F})$, the associated sector \mathcal{Y}_p is \mathcal{Y} . To show this, suppose that \mathcal{Y} corresponds to an injection $\iota: \mu_l \hookrightarrow G$. Let $f := \prod_{i=0}^{l-1} (x - i)$ and let $L := \overline{F}(x)(f^{1/l})$ be the l -cyclic extension of $\overline{F}(x)$ associated to f . Let $C \rightarrow \mathbb{P}_{\overline{F}}^1$ be the associated μ_l -cover. This is ramified exactly at $x = 0, 1, \dots, l - 1$. Note that the cover is unramified over $\infty \in \mathbb{P}_{\overline{F}}^1$, since f has degree l . The morphism $C \rightarrow \text{Spec } \overline{F}$ is ι -equivariant and induces

$$f: \mathcal{C} := [C/\mu_l] \rightarrow \mathcal{X}_{\overline{F}} = [\text{Spec } \overline{F}/G].$$

By construction, stacky points of \mathcal{C} are the images of $0, 1, \dots, l - 1 \in \mathbb{P}_{\overline{F}}^1$. Moreover, the twisted sector corresponding to each stacky point is \mathcal{Y} . This proves the claim.

For a twisted sector \mathcal{Y}_0 , we choose $f_0: \mathcal{C} \rightarrow \mathcal{X}_{\overline{F}}$ as above. For $\theta = \sum \theta_{\mathcal{Y}}[\mathcal{Y}] \in N_{\text{orb}}^1(\mathcal{X})$,

$$(f_0, \theta) = b\theta_{\mathcal{Y}_0} \quad (b > 0).$$

Note that f_0 is a covering family of stacky curves of $\mathcal{X}_{\overline{F}}$. Thus, if $\theta \in \overline{\text{Eff}}_{\text{orb}}(\mathcal{X})$, then $\theta_{\mathcal{Y}} \geq 0$ for every twisted sector \mathcal{Y} . The converse is easy to show. ■

Corollary 9.23. *We have*

$$\overline{\text{Eff}}_{\text{orb}}(\mathcal{X}) = \sum_{\mathcal{Y} \in \pi_0^*(\mathcal{I}_0\mathcal{X})} \mathbb{R}_{\geq 0}[\mathcal{Y}].$$

In particular, if $\mathcal{X} = \mathbf{B}G_F$ for a finite group G , then

$$\overline{\text{Eff}}_{\text{orb}}(\mathcal{X}) = \sum_{[1] \neq [g] \in F\text{-Conj}(G)} \mathbb{R}_{\geq 0}[g].$$

Proof. There exists a finite Galois extension K/F such that $\mathcal{X}_K \cong \text{B}G_K$ for a finite group G , and K contains all the $\#G$ -th roots of unity. From the last proposition,

$$\overline{\text{Eff}}_{\text{orb}}(\mathcal{X}_K) = \sum_{\mathcal{Y}' \in \pi_0^*(\mathcal{J}_0 \mathcal{X}_K)} \mathbb{R}_{\geq 0}[\mathcal{Y}'],$$

We have an isomorphism $N_{\text{orb}}^1(\mathcal{X}) \cong (N_{\text{orb}}^1(\mathcal{X}_K))^{\text{Gal}(K/F)}$ such that for a sector \mathcal{Y} of \mathcal{X} , the class $[\mathcal{Y}]$ corresponds to the sum $\sum_{i=1}^m [\mathcal{Y}'_i]$, where $\{\mathcal{Y}'_1, \dots, \mathcal{Y}'_m\} \subset \pi_0(\mathcal{J}_0 \mathcal{X}_K)$ is the Galois orbit corresponding to \mathcal{Y} . By the isomorphism, $\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})$ corresponds to

$$\overline{\text{Eff}}_{\text{orb}}(\mathcal{X}_K) \cap (N_{\text{orb}}^1(\mathcal{X}_K))^{\text{Gal}(K/F)}.$$

The last cone is generated by such sums $\sum_{i=1}^m [\mathcal{Y}'_i]$ as above. This shows the corollary. ■

As in Section 5, we only consider the structure sheaf \mathcal{O} as a line bundle on \mathcal{X} . Corollary 9.23 implies the following result.

Corollary 9.24. *A raised line bundle (\mathcal{O}, c) on \mathcal{X} is big if and only if c is positive.*

Proposition 9.25. *For a positive raising function c of \mathcal{X} , we have*

$$\begin{aligned} a(\mathcal{O}, c) &= \max \{c(\mathcal{Y})^{-1} \mid \mathcal{Y} \in \pi_0^*(\mathcal{J}_0 \mathcal{X})\} \\ &= (\min \{c(\mathcal{Y}) \mid \mathcal{Y} \in \pi_0^*(\mathcal{J}_0 \mathcal{X})\})^{-1} \end{aligned}$$

and

$$b(\mathcal{O}, c) = \#\{\mathcal{Y} \in \pi_0^*(\mathcal{J}_0 \mathcal{X}) \mid c(\mathcal{Y}) = a(\mathcal{O}, c)^{-1}\}.$$

Proof. For $a \in \mathbb{R}$, we have

$$v_a := a[\mathcal{O}, c] + [K_{\mathcal{X}, \text{orb}}] = \sum_{\mathcal{Y} \in \pi_0^*(\mathcal{J}_0 \mathcal{X})} (a \cdot c(\mathcal{Y}) - 1)[\mathcal{Y}].$$

Therefore, $v_a \in \overline{\text{Eff}}_{\text{orb}}(\mathcal{X})$ if and only if $a \cdot c(\mathcal{Y}) - 1 \geq 0$ for every twisted sector \mathcal{Y} . The minimum real number a satisfying the last condition is exactly the value of $a(\mathcal{O}, c)$ stated in the proposition.

The minimal face of $\overline{\text{Eff}}_{\text{orb}}(\mathcal{X})$ containing $a(\mathcal{O}, c)[\mathcal{O}, c] + [K_{\mathcal{X}, \text{orb}}]$ is the cone generated by the classes $[\mathcal{Y}]$ of those twisted sectors that *do not* have the minimal c -value. Thus, the codimension of this face is equal to the number of those twisted sectors that *do* have the minimal c -value. The desired formula for the b -invariant follows. ■

Remark 9.26 (Malle’s conjecture revisited). If $\mathcal{X} = \text{B}G_F$ for a finite group G , then $\pi_0(\mathcal{J}_0 \mathcal{X}) = F\text{-Conj}(G)$ as already mentioned. In this situation, Conjecture 9.10 with a - and b -invariants computed in Proposition 9.25 is the same as the version of Malle’s conjecture for the generalized discriminant (see Example 4.12) associated to the given raising function c , except that we count G -torsors over F giving points of $\mathcal{X}(F) \setminus T$, while the

original conjecture of Malle concerns G -fields over F (corresponding to connected G -torsors over F). Counting only G -fields amounts to removing the thin subset T' given by

$$T' = \bigcup_{H \subsetneq G} ((\mathbf{B} H_F)\langle F \rangle \rightarrow (\mathbf{B} G_F)\langle F \rangle).$$

Points coming from breaking thin morphisms $\mathcal{Y} \rightarrow \mathbf{B} G_F$ (see Section 9.5 for an example) are expected to give an asymptotic larger than the expectation of Malle’s conjecture and should be removed. The same remark shows that T may include some G -fields.

Remark 9.27. Let $f: \mathcal{Y} \rightarrow \mathcal{X}$ be a thin morphism with $\mathcal{Y}\langle F \rangle \neq \emptyset$ and let $y \in \mathcal{Y}\langle F \rangle$. Then we have the injection $\underline{\text{Aut}}_F(y) \rightarrow \underline{\text{Aut}}_F(f(y))$ of group schemes over F . Through the canonical isomorphisms $\mathcal{Y} \cong \mathbf{B} \underline{\text{Aut}}_F(y)$ and $\mathcal{X} \cong \mathbf{B} \underline{\text{Aut}}_F(f(y))$, the morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$ is identified with the natural morphism $\mathbf{B} \underline{\text{Aut}}_F(y) \rightarrow \mathbf{B} \underline{\text{Aut}}_F(f(y))$.

The following proposition enables us to check the secureness of each point $x \in \mathcal{X}\langle F \rangle$ by looking only at (necessarily finitely many) subgroup schemes of $\underline{\text{Aut}}_F(x)$.

Proposition 9.28. *A point $x \in \mathcal{X}\langle F \rangle$ is secure (resp. strongly secure) if and only if for any proper subgroup scheme H of $\underline{\text{Aut}}_F(x)$, the induced morphism $\mathbf{B} H \rightarrow \mathbf{B} \underline{\text{Aut}}_F(x) \cong \mathcal{X}$ is not breaking thin (resp. weakly breaking thin).*

Proof. This result also appears in [15], but stated and proved in a slightly different language. For completeness, we give a proof using our language. If there exists a proper subgroup scheme $H \subsetneq \underline{\text{Aut}}_F(x)$ such that $\mathbf{B} H \rightarrow \mathbf{B} \underline{\text{Aut}}_F(x) \cong \mathcal{X}$ is a breaking thin morphism, then by definition, x is not secure. Conversely, if x is not secure, then there exist a breaking thin morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$ and an F -point $y \in \mathcal{Y}\langle F \rangle$ with $f(y) = x$. From Remark 9.27, f is identified with $\mathbf{B} H \rightarrow \mathbf{B} \underline{\text{Aut}}_F(x)$ for some proper subgroup scheme $H \subsetneq \underline{\text{Aut}}_F(x)$. We have proved the “secure” version of the assertion. The “strongly secure” version is proved similarly. ■

Corollary 9.29. *If G is commutative, then $(\mathbf{B} G_F)\langle F \rangle$ has no breaking thin subset. Hence, $(\mathbf{B} G_F)\langle F \rangle$ is a secure cothin subset of itself.*

Proof. From Corollary 9.21 and Remark 9.27, every thin morphism $\mathcal{Y} \rightarrow \mathcal{X}$ with $\mathcal{Y}\langle F \rangle \neq \emptyset$ is of the form $\mathbf{B} H \rightarrow \mathbf{B} G$ for a proper subgroup scheme H of G . From the commutativity and from Example 2.15, the map $\pi_0(\mathcal{J}_0(\mathbf{B} H)) \rightarrow \pi_0(\mathcal{J}_0(\mathbf{B} G))$ is identified with

$$\text{Hom}(\hat{\mu}(\bar{F}), H(\bar{F}))/\Gamma_F \rightarrow \text{Hom}(\hat{\mu}(\bar{F}), G(\bar{F}))/\Gamma_F.$$

This is injective, which implies that $\mathcal{Y} \rightarrow \mathcal{X}$ is not a breaking thin morphism. ■

Remark 9.30. Suppose that $\mathcal{X} = \mathbf{B} G$ with G a commutative finite group scheme over F . From Corollary 9.29, the subset T in Conjecture 9.10 is empty. In this case, Conjecture 9.10 holds (see [15] and references therein).

Example 9.31. Let p be a prime number and let $\mathcal{X} = \mathbf{B}\mu_{p^2, F}$. We can identify $\pi_0(\mathcal{J}_0\mathcal{X})$ with $\mathbb{Z}/p^2\mathbb{Z}$. For $0 < c_1 < c_2$, we define a raising function c by

$$c(i) = \begin{cases} 0 & (i = 0), \\ c_1 & (i \in p\mathbb{Z}/p^2\mathbb{Z}), \\ c_2 & (\text{otherwise}). \end{cases}$$

Then the natural morphism $\mathbf{B}\mu_{p, F} \rightarrow \mathcal{X}$ is weakly breaking thin with respect to c .

Remark 9.32. For an abelian group G , there may exist a *weakly* breaking thin morphism $\mathcal{Y} \rightarrow \mathbf{B}G_F$. Wood [44] introduces the notion of fairness to have a nice leading constant for counting abelian extensions of a number field. If we adapt this notion into our language and if a raising function c of $\mathbf{B}G$ for an abelian finite group G is fair, then there is no weakly breaking thin morphism $\mathcal{Y} \rightarrow \mathbf{B}G$ with $\mathcal{Y}\langle F \rangle \neq \emptyset$.

Remark 9.33. Let $\mathcal{X} = \mathbf{B}G$ with G a commutative finite group scheme over F . Then, for any two points $x, x' \in \mathcal{X}\langle F \rangle$, there exists an automorphism $\mathcal{X} \xrightarrow{\sim} \mathcal{X}$ mapping x to x' . It follows that if x is in the image of the map $\mathcal{Y}\langle F \rangle \rightarrow \mathcal{X}\langle F \rangle$ associated to a morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$ of stacks, then x' is in the image of the map $\mathcal{Y}\langle F \rangle \rightarrow \mathcal{X}\langle F \rangle$ associated to the composite $\mathcal{Y} \xrightarrow{f} \mathcal{X} \xrightarrow{g} \mathcal{X}$ of f and some automorphism g . Thus, if c is a raising function of \mathcal{X} preserved by automorphisms of \mathcal{X} and if \mathcal{X} contains a weakly breaking thin subset (see Example 9.31), then $\mathcal{X}\langle F \rangle$ is covered by weakly breaking thin subsets. In particular, $\mathcal{X}\langle F \rangle$ has no strongly secure element. Note that this pathology does not occur for breaking thin morphisms, since there is no breaking thin morphism to \mathcal{X} at all.

9.5. Klüners' counterexample revisited

In this subsection, we explain Klüners' counterexample to Malle's conjecture [23] in our language. Consider the wreath product

$$G := C_3 \wr C_2 = (C_3 \times C_3) \rtimes C_2.$$

Here C_n denotes the cyclic group of order n . This is a group of order 18 and realized as the transitive subgroup of S_6 generated by permutations $(1, 2, 3)$, $(4, 5, 6)$, and $(1, 4)(2, 5)(3, 6)$. Recall that the index function $\text{ind}: G \rightarrow \mathbb{Z}_{\geq 0}$ is defined by

$$\text{ind}(g) := 6 - \#\{g\text{-orbits in } \{1, \dots, 6\}\}.$$

This function restricted to $G \setminus \{1\}$ takes the minimal value 2 exactly on the following four elements:

$$(1, 2, 3), (1, 3, 2), (4, 5, 6), (4, 6, 5), \tag{9.1}$$

all of which have order 3. Note that there are also elements of order 3 having index 4, for example, $(1, 2, 3)(4, 5, 6)$.

Let us now consider the stack $\mathcal{X} := \mathbf{B}G_{\mathbb{Q}}$. Its set of sectors, $\pi_0(\mathcal{J}_0\mathcal{X})$, is identified with the set of \mathbb{Q} -conjugacy classes, $\mathbb{Q}\text{-Conj}(G)$. The four elements of index 2 are divided

into two conjugacy classes $\{(1, 2, 3), (4, 5, 6)\}$ and $\{(1, 3, 2), (4, 6, 5)\}$. In turn, these two conjugacy classes form one \mathbb{Q} -conjugacy class. In summary, there is a unique \mathbb{Q} -conjugacy class of index 2.

We now discuss possible forms of breaking thin morphisms $\mathcal{Y} \rightarrow \mathcal{X}$ with $\mathcal{Y}(F) \neq \emptyset$. The stack \mathcal{Y} has to be isomorphic to $B H'$ where H' is a twisted form of a subgroup $H \not\subseteq G$ which contains an element of index 2. We can easily see that no subgroup of order 2 or 6 contains an element of index 2. Thus, the order of H is either 3 or 9. It follows that H is contained in the unique 3-Sylow subgroup

$$N := \langle (1, 2, 3), (4, 5, 6) \rangle \triangleleft G.$$

Let $x: \text{Spec } L \rightarrow \text{Spec } \mathbb{Q}$ be a G -torsor and let $\text{Spec } K := \text{Spec } L^N \rightarrow \text{Spec } \mathbb{Q}$ be the induced C_2 -torsor. We claim that the normal subgroup scheme $N' \triangleleft \underline{\text{Aut}}_{\mathbb{Q}}(x)$ corresponding to N is isomorphic to the twisted form $N^{(K)}$ of $N_{\mathbb{Q}}$ induced by

$$\Gamma_{\mathbb{Q}} \xrightarrow{\tau_K} C_2 \xrightarrow{\phi} \text{Aut}(N_{\mathbb{Q}}), \tag{9.2}$$

where τ_K is the map corresponding to the C_2 -torsor $\text{Spec } K \rightarrow \text{Spec } \mathbb{Q}$ and ϕ is the one used in the definition of the wreath product $G = C_3 \wr C_2$. Since it induces the N -torsor $\text{Spec } L \rightarrow \text{Spec } K$, the group scheme N' is trivialized by the scalar extension K/\mathbb{Q} ;

$$N'_K \cong N_K \cong \overbrace{\text{Spec } K \amalg \cdots \amalg \text{Spec } K}^{9 \text{ copies}}.$$

The Galois action on this trivial group is again induced by

$$\Gamma_{\mathbb{Q}} \xrightarrow{\tau_K} C_2 \xrightarrow{\phi} \text{Aut}(N_{\mathbb{Q}}) = \text{Aut}(N_K).$$

This shows the claim. Let

$$f_x: B N^{(K)} \rightarrow B \underline{\text{Aut}}_F(x) \cong \mathcal{X}$$

be the induced morphism. Then the image of the induced map $f_x(\mathbb{Q}): (B N^{(K)})(\mathbb{Q}) \rightarrow \mathcal{X}(\mathbb{Q})$ is exactly the set of G -torsors $\text{Spec } M \rightarrow \text{Spec } \mathbb{Q}$ such that the associated C_2 -torsor $\text{Spec } M^N \rightarrow \text{Spec } \mathbb{Q}$ is isomorphic to $\text{Spec } K \rightarrow \text{Spec } \mathbb{Q}$; namely, the fiber of $\mathcal{X}(\mathbb{Q}) \rightarrow (B C_2)(\mathbb{Q})$ over the isomorphism class of $\text{Spec } K \rightarrow \text{Spec } \mathbb{Q}$. In particular, the image of $f_x(\mathbb{Q})$ depends only on the isomorphism class of K .

We next show that there is no breaking thin morphism coming from a group scheme of order 3. If $K = \mathbb{Q}^2$, that is, if $\text{Spec } K \rightarrow \text{Spec } \mathbb{Q}$ is the trivial C_2 -torsor, then N' is the constant group $N_{\mathbb{Q}}$. It contains (necessarily constant) subgroup schemes of order 3 which contains elements of index 2. But since $B C_{3, \mathbb{Q}}$ has only one twisted sector, no morphism $B C_{3, \mathbb{Q}} \rightarrow \mathcal{X}$ is breaking thin. If K is a field, then N has six connected components and is of the form

$$\text{Spec } \mathbb{Q} \amalg \text{Spec } \mathbb{Q} \amalg \text{Spec } \mathbb{Q} \amalg \text{Spec } K \amalg \text{Spec } K \amalg \text{Spec } K.$$

The first three components correspond to the diagonal elements of $N = C_3 \times C_3$, those elements fixed by the C_2 -action. The other three components correspond to the following C_2 -orbits, respectively:

$$\{(1, 2, 3), (4, 5, 6)\}, \quad \{(1, 3, 2), (4, 6, 5)\}, \quad \{(1, 2, 3)(4, 6, 5), (1, 3, 2)(4, 5, 6)\}.$$

The only subgroup schemes of N of order 3 are the diagonal one $\text{Spec } \mathbb{Q} \amalg \text{Spec } \mathbb{Q} \amalg \text{Spec } \mathbb{Q}$ and the union $\text{Spec } \mathbb{Q} \amalg \text{Spec } K$ of the identity component $\text{Spec } \mathbb{Q}$ and the component $\text{Spec } K$ corresponding to

$$\{(1, 2, 3)(4, 6, 5), (1, 3, 2)(4, 5, 6)\}.$$

The diagonal elements, $(1, 2, 3)(4, 6, 5)$ and $(1, 3, 2)(4, 5, 6)$, do not have index 2. Thus, the induced morphisms $B C_{3, \mathbb{Q}} \rightarrow \mathcal{X}$ and $B(\text{Spec } \mathbb{Q} \amalg \text{Spec } K) \rightarrow \mathcal{X}$ are not breaking thin. In conclusion, there is no breaking thin morphism of the form $B H' \rightarrow \mathcal{X}$, where H' is of order 3.

It remains possible that H' is of order 9 and isomorphic to $N^{(K)}$, the twisted form of N associated to a C_2 -torsor $\text{Spec } K \rightarrow \text{Spec } \mathbb{Q}$. Since N contains elements of index 2, the morphism $B N^{(K)} \rightarrow \mathcal{X}$ is always weakly breaking thin. To see when it is also breaking thin, we now compute $\pi_0(\mathcal{J}_0(B N^{(K)}))$. Recall that this set is identified with

$$\text{Hom}(\mu_9(\overline{\mathbb{Q}}), N^{(K)}(\overline{\mathbb{Q}})) / \Gamma_{\mathbb{Q}}.$$

Fixing a generator of $\mu_9(\overline{\mathbb{Q}})$, we may identify

$$\text{Hom}(\mu_9(\overline{\mathbb{Q}}), N^{(K)}(\overline{\mathbb{Q}})) = N^{(K)}(\overline{\mathbb{Q}}) = N = C_3 \times C_3.$$

The $\Gamma_{\mathbb{Q}}$ -action on this set is obtained by combining two actions; the one induced from the $\Gamma_{\mathbb{Q}}$ -action on $\mu_9(\overline{\mathbb{Q}})$ and the one induced from the action on $N^{(K)}(\overline{\mathbb{Q}})$. Through the above identification, the first action transitively interchanges a nonidentity element $g \in N$ with its square g^2 . The second action is given by (9.2). We see that the four elements of index 2, listed in (9.1), are transitively permuted by the combined $\Gamma_{\mathbb{Q}}$ -action except the two cases: $K = \mathbb{Q}^2$ and $K = \mathbb{Q}(\zeta_3)$ with ζ_3 a primitive cubic root of 1. In the former case, the second action is trivial, and hence the four elements of index 2 are divided into the two $\Gamma_{\mathbb{Q}}$ -orbits

$$\{(1, 2, 3), (1, 3, 2)\}, \quad \{(4, 5, 6), (4, 6, 5)\}.$$

In the latter case, since $\mu_3 = \text{Spec } \mathbb{Q} \amalg \text{Spec } \mathbb{Q}(\zeta_3)$, the two $\Gamma_{\mathbb{Q}}$ -actions are *synchronized*: whenever $\gamma \in \Gamma_F$ maps $(1, 2, 3) \leftrightarrow (1, 3, 2)$ and $(4, 5, 6) \leftrightarrow (4, 6, 5)$ by the first action, then the same element γ maps $(1, 2, 3) \leftrightarrow (4, 5, 6)$ and $(1, 3, 2) \leftrightarrow (4, 5, 6)$ by the second action. Similarly, whenever $\gamma \in \Gamma_F$ fixes the four elements by the first action, then the same element fixes them also by the second action. Consequently, the combined action divides the four elements into two orbits

$$\{(1, 2, 3), (4, 6, 5)\}, \quad \{(1, 3, 2), (4, 5, 6)\}.$$

Thus, the morphisms $BN(\mathbb{Q}^2) \rightarrow \mathcal{X}$ and $BN(\mathbb{Q}(\zeta_3)) \rightarrow \mathcal{X}$ are the only breaking thin morphisms. In particular, the union T of breaking thin subsets of $\mathcal{X}(\mathbb{Q})$, which appears in Conjecture 9.10, is a thin subset. Note that the image of $(BN(\mathbb{Q}^2))(\mathbb{Q}) \rightarrow \mathcal{X}(\mathbb{Q})$ contains only nonconnected torsors, which are removed also in the usual version of Malle’s conjecture. We can summarize these results as follows:

Proposition 9.34. *For a C_2 -torsor $\text{Spec } K \rightarrow \text{Spec } \mathbb{Q}$, let T_K be the image of the map $(BN^{(K)}) (\mathbb{Q}) \rightarrow \mathcal{X}(\mathbb{Q})$. Then $\mathcal{X}(\mathbb{Q}) = \bigsqcup_K T_K$. Moreover, the following assertions hold:*

- (1) *Every T_K is a weakly breaking thin subset.*
- (2) *The subset T_K is a breaking thin subset exactly when K is isomorphic to \mathbb{Q}^2 or $\mathbb{Q}(\zeta_3)$.*
- (3) *The subset $\bigsqcup_{K \neq \mathbb{Q}^2, \mathbb{Q}(\zeta_3)} T_K \subset \mathcal{X}(\mathbb{Q})$ and each of its cothin subsets is a secure cothin subset.*
- (4) *There is no strongly secure subset of $\mathcal{X}(\mathbb{Q})$.*

Remark 9.35. Türkelli [43] takes a different approach to modify Malle’s conjecture incorporating Klüners’ counterexample; he proposes changing the exponent of the log factor instead of keeping the exponent unchanged and removing a thin subset.

9.6. Comprehensiveness

Definition 9.36. Let G be a finite group and let $c: \text{Conj}(G) \rightarrow \mathbb{R}_{\geq 0}$ be a function such that for $C \in \text{Conj}(G)$, $c(C) = 0$ if and only if $C = [1]$. We say that G is c -comprehensive if for any nonidentity conjugacy class $C \in \text{Conj}(G)$ with the minimal c -value, the elements in C generate G .

Example 9.37. The subgroup generated by the elements of a conjugacy class is a normal subgroup. Therefore, a simple subgroup is c -comprehensive for any c .

Example 9.38. For the symmetric group S_n , consider the index function, $\text{ind}: \text{Conj}(S_n) \rightarrow \mathbb{Z}_{\geq 0}$. This takes the minimal value only at the conjugacy class of all transpositions. Since S_n is generated by transpositions, S_n is ind-comprehensive.

Proposition 9.39. *Let G be a finite group, let $\mathcal{X} := BG_F$ and let $x: \text{Spec } L \rightarrow \text{Spec } F$ be a connected G -torsor, that is, an F -point of \mathcal{X} . Let $c: \pi_0(\mathcal{G}_0 \mathcal{X}) = F\text{-Conj}(G) \rightarrow \mathbb{R}_{\geq 0}$ be a raising function. Denoting the composition $\text{Conj}(G) \rightarrow F\text{-Conj}(G) \xrightarrow{c} \mathbb{R}_{\geq 0}$ again by c , suppose that G is c -comprehensive. Then x is a secure element of $\mathcal{X}(F)$. Hence, the subset of $\mathcal{X}(F)$ consisting of all connected G -torsors is a strongly secure cothin subset. In particular, the union of weakly breaking thin subsets of $\mathcal{X}(F)$, and the union of breaking thin subsets, is a thin subset.*

Proof. As for the first assertion, it suffices to show that the twisted form $\underline{\text{Aut}}_F(x)$ of G_F^{op} does not contain a proper subgroup scheme containing a c -minimal element. Since x is connected, the map $\phi_x: \Gamma_F \rightarrow G$ corresponding to x is surjective. From Lemma 9.20,

$$\pi_0(\underline{\text{Aut}}_F(x)) = \underline{\text{Aut}}_F(x)(\bar{F})/\Gamma_F = \text{Conj}(G).$$

From c -comprehensiveness, if a subgroup scheme H of $\underline{\text{Aut}}_F(x)$ contains a c -minimal element, then it is in fact the entire group scheme $\underline{\text{Aut}}_F(x)$. Thus, the first assertion of the proposition holds.

The second and third assertions follow from the fact that the set of nonconnected G -torsors over F is the union of the images of the maps $(B H_F)\langle F \rangle \rightarrow (B G_F)\langle F \rangle$ associated to proper subgroups $H \subsetneq G$. ■

Remark 9.40. For a c -comprehensive group G , the stack $\mathcal{X} = B G_F$ may have a breaking thin morphism. For example, let $n \geq 4$ and let $H := \langle (1, 2), (3, 4) \rangle \subset S_n$. The induced morphism $B H_F \rightarrow B(S_n)_F$ is breaking thin. However, every F -point in its image is a nonconnected S_n -torsor.

Acknowledgments. The authors would like to thank Aaron Landesman, Daniel Loughran, Julian Lyczak, Emmanuel Peyre and an anonymous referee for useful comments.

Funding. This work was supported by JSPS KAKENHI Grant Number JP18H01112. The first named author was supported by JSPS Postdoctoral Fellowship for Research in Japan.

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