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Rigidity of U -Gibbs measures near conservative Anosov diffeomorphisms on \mathbb{T}^3

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Abstract. We show that within a C^1 -neighborhood \mathcal{U} of the set of volume preserving Anosov diffeomorphisms on the three-torus \mathbb{T}^3 which are strongly partially hyperbolic with expanding center, any $f \in \mathcal{U} \cap \text{Diff}^2(\mathbb{T}^3)$ satisfies the dichotomy: either the strong-stable and strong-unstable bundles E^s, E^u of f are jointly integrable, or any fully supported u -Gibbs measure of f is SRB.

Keywords: partially hyperbolic systems, measure rigidity.

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1. Introduction

1.1. Context

Invariant foliations play a key role in partially and uniformly hyperbolic dynamics. For example, they can be used to obtain ergodicity, topological transitivity and mixing for certain systems. Trying to understand these foliations, one can investigate their topological and ergodic properties. For topological properties, one may ask about minimality, or robust minimality, of invariant foliations. In this paper we focus on understanding ergodic properties of invariant foliations for a certain type of dynamical systems.

We refer the reader to Section 2 for the definition of the dynamical objects that appear in this introduction. Let us denote by $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$ the three-dimensional torus. We let $\mathcal{A}^2(\mathbb{T}^3) \subset \text{Diff}^2(\mathbb{T}^3)$ be the set of Anosov diffeomorphisms which are strongly partially hyperbolic with uniformly expanding center, that is, a diffeomorphism f belongs to

$\mathcal{A}^2(\mathbb{T}^3)$ if f is Anosov and there is a splitting

$$T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u,$$

where E^c expands uniformly under the action of Df . A diffeomorphism in $\mathcal{A}^2(\mathbb{T}^3)$ can be viewed as an Anosov diffeomorphism and as a partially hyperbolic diffeomorphism. In this context the bundles $E^s, E^u, E^u \oplus E^c, E^s \oplus E^c$ and E^c integrate to invariant foliations respectively denoted by $\mathcal{W}^s, \mathcal{W}^u, \mathcal{W}^{cu}, \mathcal{W}^{cs}$ and \mathcal{W}^c , and called the stable, unstable, center-unstable, center-stable and center foliations (see [4, 38]). If we view f as an Anosov diffeomorphism, \mathcal{W}^{cu} is a (two-dimensional) unstable foliation. If we see it as a partially hyperbolic diffeomorphism, \mathcal{W}^u is the (one-dimensional) strong-unstable foliation.

There are two types of elements of $\mathcal{A}^2(\mathbb{T}^3)$: the *conservative elements*, which preserve some volume (which must be ergodic by Hopf's argument) and form a set denoted by $\mathcal{A}_m^2(\mathbb{T}^3)$, and the *dissipative ones*, which do not. In both cases there exists a unique invariant measure which is the "most compatible" with the volume and that is called the *SRB measure* (for Sinai–Ruelle–Bowen): *measures that are absolutely continuous with respect to the Lebesgue measure along center-unstable leaves* (see Section 2.5). In particular, they capture the "statistical" behavior of Lebesgue almost every point (see [49]). SRB measures are very important in the theory of smooth dynamics. Palis conjectured that for a typical dynamical system there are finitely many attractors, each attractor supporting a unique SRB measure, and these measures capture the behavior of Lebesgue almost every point [36]. This conjecture remains open.

1.2. Dynamics of (center-)unstable foliation

Let $f \in \mathcal{A}^2(\mathbb{T}^3)$ with a splitting $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$. The dynamics of the center-unstable foliation \mathcal{W}^{cu} is very well understood. It is minimal (i.e., every leaf is dense in \mathbb{T}^3) and there is a unique SRB measure.

On the other hand, properties of the strong-unstable foliation \mathcal{W}^u are especially interesting for dissipative dynamics: the study of topological and ergodic properties of attractors or quasi-attractors (which are \mathcal{W}^u -saturated, i.e., contain whole \mathcal{W}^u -leaves) is closely related to the problem of understanding properties of these foliations (or laminations). Apart from some finiteness results (see [10, 21, 23]), the dynamical properties of the strong-unstable foliation are not well understood, even in the uniformly hyperbolic setting.

For instance, it was only recently announced by Avila–Crovisier–Eskin–Potrie–Wilkinson–Zhang that \mathcal{W}^u is minimal for any $C^{1+\alpha}$ Anosov diffeomorphism of \mathbb{T}^3 . In higher dimensions, Avila–Crovisier–Wilkinson recently announced that C^1 -openly and C^r -densely among the transitive Anosov diffeomorphisms admitting a decomposition $E^s \oplus E^c \oplus E^u$, where E^c is one-dimensional and uniformly expanding, the strong-unstable manifold is minimal.

A type of invariant measures that is associated with \mathcal{W}^u are the so-called u -Gibbs measures. A measure is u -Gibbs if it admits conditional measures along \mathcal{W}^u -leaves that

are absolutely continuous with respect to the Lebesgue measure of these leaves. In particular, the support of such a measure is \mathcal{W}^u -saturated. Let us make a few remarks about u -Gibbs and SRB measures in our setting:

- (1) SRB measures are absolutely continuous along two-dimensional objects (\mathcal{W}^{cu} -leaves), while u -Gibbs measures are absolutely continuous along one-dimensional objects (\mathcal{W}^u -leaves).
- (2) SRB measures are also u -Gibbs measures.
- (3) In general, we do not know when a u -Gibbs measure is an SRB measure.

1.3. Main result

The goal of this paper is related to item (3) above. We are interested in knowing when the u -Gibbs property implies SRB. In other words, given a measure that is absolutely continuous along \mathcal{W}^u , when can we show that this measure is absolutely continuous along \mathcal{W}^{cu} ?

For $\mathcal{A}^2(\mathbb{T}^3)$, we say that E^s and E^u are *jointly integrable* if there exists a two-dimensional foliation \mathcal{W}^{su} tangent to $E^s \oplus E^u$. It is known that C^1 -openly and C^2 -densely in $\mathcal{A}^2(\mathbb{T}^3)$ the directions E^s and E^u are not jointly integrable (see [45]).

Our main result is the following:

Theorem A. *There exists an open neighborhood \mathcal{U} of $\mathcal{A}_m^2(\mathbb{T}^3)$ within $\text{Diff}^2(\mathbb{T}^3)$ such that for every $f \in \mathcal{U}$, either*

- (1) E^s and E^u are jointly integrable, or
- (2) any fully supported ergodic u -Gibbs measure μ is SRB.

Remark 1.1. The hypothesis that the system is near a volume preserving one is used to have C^1 -stable holonomies. The conclusion of Theorem A also holds for $f \in \mathcal{A}^2(\mathbb{T}^3)$ with C^1 -stable holonomies; see Theorem 5.10.

Remark 1.2. As we mentioned before, Avila et al. announced that for any $C^{1+\alpha}$ Anosov diffeomorphism in \mathbb{T}^3 the strong-unstable foliation is minimal. Since the support of any u -Gibbs measure is saturated by \mathcal{W}^u -leaves, their result would imply that case (2) of Theorem A could be improved to the statement that every u -Gibbs measure is SRB. We could also apply the result announced by Avila–Crovisier–Wilkinson, which we mentioned, to deduce that openly and densely in \mathcal{U} every u -Gibbs is SRB, where \mathcal{U} is the open set from Theorem A.

Let us mention one application of our result. In [20], Gogolev, Kolmogorov and Maimon consider the linear Anosov diffeomorphism on \mathbb{T}^3 induced by the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The eigenvalues of A are real and approximately 0.2, 1.55 and 3.25. In particular, the diffeomorphism induced by this matrix belong to $\mathcal{A}_m^2(\mathbb{T}^3)$. In [20], the authors did

a numerical study for two explicit families of perturbations, one conservative and one dissipative. Their numerical study indicates that for these families of perturbations of A , there is a unique u -Gibbs measure, and this measure coincides with the SRB measure. They make the following conjecture.

Conjecture ([20], Conjecture 1.3). *For all analytic diffeomorphisms f in a sufficiently small neighborhood of A there exists a unique u -Gibbs measure.*

Our result shows that for any C^2 diffeomorphism in a neighborhood of A , either E^s and E^u are jointly integrable or any u -Gibbs fully supported measure is SRB. If we assume Avila et al.'s result, then we would infer that either we have joint integrability or there is only one u -Gibbs measure. In Section 12 we introduce Gogolev–Kolmogorov–Maimon's conservative and dissipative families and prove that for neither case, E^s and E^u are jointly integrable, applying a very convenient criterion of Gan–Shi [17]. That gives a theoretical explanation for their numerical study. Let us remark that in [20] the authors also make conjectures about transitivity and minimality of \mathcal{W}^u . Rodríguez Hertz–Ures [46] confirmed their transitivity conjecture.

1.4. Related works and further results

One can think of a u -Gibbs measure as a measure that is “homogeneous” along strong-unstable manifolds and an SRB measure as being “homogeneous” along entire unstable manifolds.

In a series of celebrated works, in the homogeneous setting, Ratner classified measures that are invariant by the action of a unipotent group [40–43]. She proved that such measures are homogeneous, i.e., they are the Haar measure of some subgroup. Observe that unipotent flows parameterize unstable manifolds of the geodesic flow on surfaces with constant negative curvature (the horocycle flow). Hence, a consequence of Ratner's measure rigidity result is measure rigidity of the u -Gibbs measures of the geodesic flow on surfaces with constant negative curvature. A key idea in Ratner's approach is the so-called polynomial drift, which allowed her to obtain extra invariance of the measure from invariance along orbits of the unipotent flow.

Benoist–Quint [2] introduced the idea of exponential drift to prove a measure rigidity result for stationary measures of a Zariski dense random walk on homogeneous spaces.

Outside the homogeneous setting, Eskin–Mirzakhani [14] gave a non-trivial modification of the exponential drift strategy, which is called the factorization method, to prove measure rigidity results for the action of $\mathrm{SL}(2, \mathbb{R})$ on moduli spaces. Since then, these ideas have been carried over to various settings. Brown–Rodríguez Hertz [7] classified the hyperbolic stationary measures of random products of surface diffeomorphisms. Cantat–Dujardin [8] applied Brown–Rodríguez Hertz's result to classify random products of automorphisms of real and complex projective spaces.

In the partially hyperbolic setting, the third author [35] adapted Brown–Rodríguez Hertz's result to obtain a rigidity result for u -Gibbs measures for partially hyperbolic skew products with two-dimensional center.

Katz [30] adapted the Eskin–Mirzakhani strategy to the smooth setting. He proved that for any C^∞ Anosov diffeomorphism f having a splitting $TM = E^s \oplus E^c \oplus E^u$, where E^c is one-dimensional and expanding, any u -Gibbs measure that satisfies a technical condition called *QNI* (quantified non-integrability) is SRB. After our work, Eskin–Potrie–Zhang [16] obtained notions equivalent to QNI that are easier to work with. In our setting, their result implies that, assuming C^∞ regularity, either any fully supported u -Gibbs measure is SRB, or E^s and E^u satisfy a condition that they called joint integrability up to order l , for every $l > 0$, which is slightly weaker than the usual joint integrability. Their result will be used by Avila et al., in another ongoing project, to prove that for any Anosov diffeomorphism in $\mathcal{A}^\infty(\mathbb{T}^3)$ either E^s and E^u are jointly integrable, or every u -Gibbs measure is SRB. All of these works are stated for C^∞ regularity, but they can be obtained for C^r regularity for $r \gg 1$. Part of the goal of this paper is to obtain this type of measure rigidity result for u -Gibbs measures, but in lower regularity (in our case C^2).

1.5. *Ingredients of the proof*

The first ingredient concerns the transversality condition. We replace Katz’s QNI condition by a zero-one law for angles inspired by Brown–Rodríguez Hertz [6, Lemma 7.1] (see also [2]). We use the fact that stable holonomies $(H_{x,y}^s)_{x,y}$ are C^1 for any diffeomorphism $f \in \mathcal{A}^2(\mathbb{T}^3)$ close to a conservative one (see Lemma 2.6); thus, for x, y in the same stable manifold, we can define an (unoriented) angle (see Figure 1)

$$\alpha^s(x, y) := \angle(DH_{x,y}^s E^u(x), E^u(y)).$$

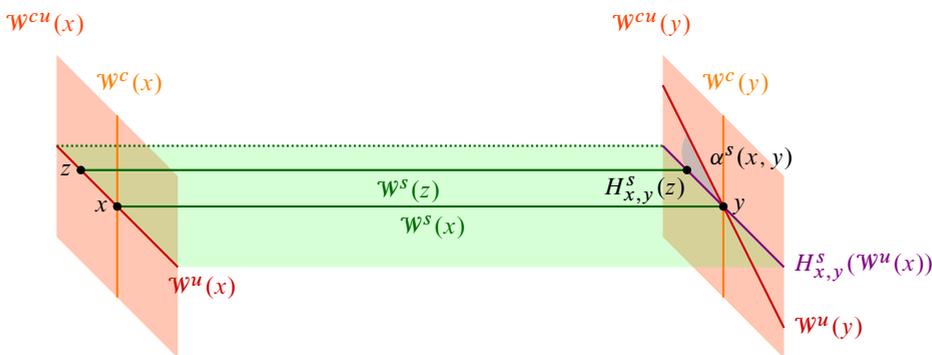


Fig. 1. Stable holonomies and the angle function.

Note that conditional measures μ_x^s on stable manifolds are not well defined. But full- and zero-measure sets for μ_x^s are well defined: see Section 2.4. We can now state our zero-one law (see Theorem 4.2 for a slightly more general statement).

Theorem B (A zero-one law for angles). *There exists an open neighborhood \mathcal{U} of $\mathcal{A}_m^2(\mathbb{T}^3)$ within $\text{Diff}^2(\mathbb{T}^3)$ such that for any $f \in \mathcal{U}$, and for any ergodic f -invariant measure μ , the following dichotomy holds: either*

- (1) *for μ -a.e. $x \in \mathbb{T}^3$ and μ_x^s -a.e. $y \in \mathcal{W}^s(x)$, $\alpha^s(x, y) = 0$; or*
- (2) *for μ -a.e. $x \in \mathbb{T}^3$ and μ_x^s -a.e. $y \in \mathcal{W}^s(x)$, $\alpha^s(x, y) > 0$.*

This theorem is stated here in terms of the angle function α^s ; yet, it seems possible to generalize it to a broader context (see Remark 4.4).

Another ingredient which plays an instrumental role in our proof is the existence of *normal forms* for the dynamics along two-dimensional unstable foliations. The dynamics along unstable manifolds is simplified when it is looked at in normal forms. The theory of non-stationary normal forms has been studied quite extensively since the pioneering work of Guysinsky–Katok [22]; see for instance [19, 26, 27]. Katz uses the result from [26] where higher regularity is needed depending on the Lyapunov spectrum.

Yet, these results do not apply directly here, due namely to the fact that f is merely assumed to be C^2 . We use instead an *ad hoc* construction, of different nature, based on one-dimensional normal forms along the center/unstable directions. Although the outcome is somewhat similar to the result in [26], we list a few differences with their work in Remark 1.3 below. We show the following.

Theorem C (Normal forms). *Let $f \in \mathcal{A}^2(\mathbb{T}^3)$. Then there exists a family $\{\Phi_x\}_{x \in \mathbb{T}^3}$ of C^1 diffeomorphisms $\Phi_x: \mathbb{R}^2 \rightarrow \mathcal{W}^{cu}(x)$ such that*

- (1) *$f \circ \Phi_x = \Phi_{f(x)} \circ N_x$, with $N_x := \begin{bmatrix} \lambda_x^u & 0 \\ 0 & \lambda_x^c \end{bmatrix}$, letting $\lambda_x^* := \|Df(x)|_{E^*}\|$;*
- (2) *$\Phi_x(0) = x$ and $D\Phi_x(0)(1, 0) = v^u(x)$, $D\Phi_x(0)(0, 1) = v^c(x)$, $v^u(x)$, resp. $v^c(x)$ being a unit vector in $E^u(x)$, resp. $E^c(x)$;*
- (3) *$\Phi_x(\cdot)$ depends continuously on x in the local C^1 topology;¹*
- (4) *Φ_x is a foliated chart for \mathcal{W}^u , i.e., for all $s \in \mathbb{R}$, $\Phi_x(\mathbb{R} \times \{s\}) = \mathcal{W}^u(\Phi_x(0, s))$, and $\Phi_x(\{0\} \times \mathbb{R}) = \mathcal{W}^c(x)$.*

Remark 1.3. Compared with [26], our construction only requires f to be C^2 , and does not need (pointwise) $1/2$ -pinching. The price to pay is that the resulting change of coordinates is merely C^1 , while it is C^∞ in [26]. Yet, this suffices for our purpose; indeed, u -Gibbs measures, when looked at in normal coordinates, still have the property of being absolutely continuous along the image of \mathcal{W}^u , namely, the horizontal foliation. In contrast to [26], the normal forms $\{\Phi_x\}_{x \in \mathbb{T}^3}$ we obtain do not in general define an affine structure on the foliation \mathcal{W}^{cu} (see Remark 6.4); yet, we will investigate in Section 6 certain invariance properties of these normal forms under changes of charts.

Note that Katz also frequently uses C^∞ regularity on other occasions in his adaptation of Eskin–Mirzakhani’s factorization method for Anosov systems. For example, at some moments, he has to approximate stable/unstable manifolds by Taylor polynomials with

¹Uniform convergence of the function and its first derivative on compact sets.

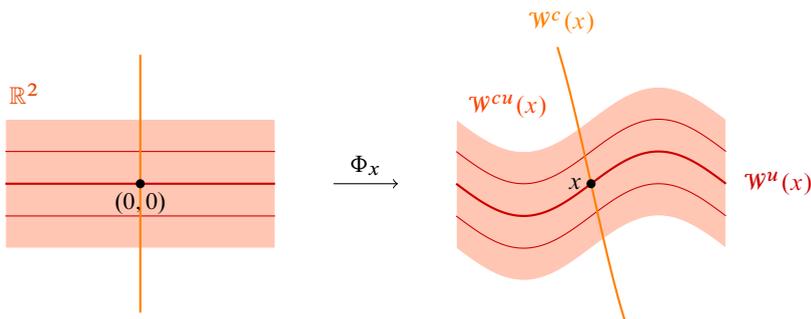


Fig. 2. Theorem C provides non-stationary C^1 linearizations of the dynamics along center-unstable leaves. They send horizontal lines onto unstable manifolds.

very high degree. In our setting, Theorems B and C are the two main reasons why we are able to adapt Eskin–Mirzakhani’s strategy in lower regularity.

We stress that, unlike previous works, we implement Y -configurations and a version of the factorization technique of [14] without using suspensions or any reparametrization. We make all estimations directly with the diffeomorphism. This is possible because we can obtain uniform estimates for the drift of leaf-wise (quotient) measures along the center as well as synchronization estimates for stopping times (see Section 9.3), using only basic distortion estimates (see Section 2.3).

1.6. Organization of the paper

This paper is organized as follows: in Section 2 we introduce the basic definitions and results we need. In Section 3 we give an outline of the proof of Theorem A. In Section 4 we establish a zero-one law for transversality between the bundles E^s and E^u . In Section 5 we reduce the proof of Theorem A to a more technical result, Theorem 5.10. In Section 6 we construct a non-stationary family of C^1 linearizations of the dynamics restricted to center-unstable manifolds and use them to construct a family $\{\hat{\nu}_x^c\}_{x \in \mathbb{T}^3}$ of measures on the real line. We reduce the proof of our main technical result to proving that these measures are Lebesgue almost surely. In Section 8, we explain an argument from [7, 25] that shows that if the measures $\hat{\nu}_x^c$ are “invariant” by certain affine maps for many points x , then the measures $\hat{\nu}_x^c$ are actually the Lebesgue measure. We then explain how this is achieved by reducing the proof to proving Proposition 8.2. In Section 9 we introduce Y -configurations and other objects crucial for our argument. In Section 10 we introduce matched Y -configurations and in Section 11 we complete the proof.

2. Partially hyperbolic diffeomorphisms with expanding center

In this section we introduce the class of dynamical systems we work with as well as the main objects, taking the opportunity to fix notations and recalling important basic facts.

2.1. Definitions

For any integer $r \geq 1$, we let $\mathcal{PH}^r(\mathbb{T}^3)$ be the set of all C^r (strongly) partially hyperbolic diffeomorphisms of \mathbb{T}^3 with one-dimensional stable/center/unstable bundles, i.e., the diffeomorphisms $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ such that there exist a continuous splitting of $T\mathbb{T}^3$ into Df -invariant line bundles,

$$T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u,$$

as well as a Riemannian metric $\|\cdot\|$ adapted to this splitting such that the functions

$$x \mapsto \lambda_x^\star := \|Df(x)|_{E^\star}\|, \quad \star \in \{s, c, u\},$$

are continuous and satisfy $\lambda_x^s < 1 < \lambda_x^u$ and $\lambda_x^s < \lambda_x^c < \lambda_x^u$, for all $x \in \mathbb{T}^3$. We will sometimes make the dependence on f explicit and write $\lambda_{f,x}^\star$ instead of λ_x^\star . We also let $E^{cs} := E^c \oplus E^s$, resp. $E^{cu} := E^c \oplus E^u$ be the center-stable, resp. center-unstable subbundle, and set $E^{su} := E^s \oplus E^u$. We refer to Katok–Hasselblatt’s book [28] for more details.

2.1.1. Anosov diffeomorphisms with uniformly expanding center. We denote by $\mathcal{A}^r(\mathbb{T}^3) \subset \mathcal{PH}^r(\mathbb{T}^3)$ the subset consisting of partially hyperbolic diffeomorphisms $f \in \mathcal{PH}^r(\mathbb{T}^3)$ with uniformly expanding center, i.e., such that $\lambda_x^c > 1$ for all $x \in \mathbb{T}^3$; in particular, any such diffeomorphism f is Anosov, for the hyperbolic splitting $E^s \oplus E^{cu}$. We also denote by $\mathcal{A}_m^r(\mathbb{T}^3) \subset \mathcal{A}^r(\mathbb{T}^3)$ the subset made up of conservative Anosov diffeomorphisms (i.e., preserving some volume).

To simplify the exposition, we assume that the bundles E^\star are orientable and that f preserves their orientation (this can always be achieved by taking an orientable cover and considering powers of f). In particular, there are unitary vector fields $\mathbb{T}^3 \ni x \mapsto v^\star(x) \in E^\star(x)$ such that

$$Df(x)v^\star(x) = \lambda_x^\star v^\star(f(x)).$$

2.1.2. Notations for orbits and derivatives. To give a more friendly appearance to some long calculations we shall make, we introduce the following notation. For a point $x \in \mathbb{T}^3$ we denote

$$x_n = f^n(x) \quad \text{for } n \in \mathbb{Z}. \tag{2.1}$$

Also, for $n \in \mathbb{Z}$ and for $\star = s, c, u$ we denote the derivative of f restricted to the bundle E^\star by

$$\lambda_x^\star(n) := \|Df^n(x)|_{E^\star}\|. \tag{2.2}$$

The following cocycle property follows from the chain rule and the fact that the bundles are one-dimensional:

$$\lambda_x^\star(n + m) = \lambda_{f^n(x)}^\star(m)\lambda_x^\star(n), \quad \star \in \{s, c, u\}. \tag{2.3}$$

The following quantities associated to f will be useful for crude estimations:

$$\begin{aligned} \|Df\| &:= \max \{ \|Df(x)v\| : x \in \mathbb{T}^3, v \in T_x\mathbb{T}^3, \|v\| = 1 \}, \\ m(Df) &:= \min \{ \|Df(x)v\| : x \in \mathbb{T}^3, v \in T_x\mathbb{T}^3, \|v\| = 1 \}. \end{aligned}$$

2.1.3. *Adapted metric and hyperbolic estimates.* The following quantity will play a key role later when we introduce stopping times and Y -configurations:

$$d_x^\ell := \frac{\lambda_{x-\ell}^c(\ell)}{\lambda_{x-\ell}^u(\ell)}. \tag{2.4}$$

Notice that d_x^ℓ measures the amount of projective hyperbolicity we have for the dominated splitting $E^c \oplus E^u$.

We fix a Riemannian metric on \mathbb{T}^3 and constants $\chi_j^\star \in \mathbb{R}$, for $\star \in \{c, s, u, d\}$ and $j = 1, 2$, such that the following hold:

- (1) $\chi_1^d < \chi_2^d < 0$ and $e^{\chi_1^d \ell} < d_x^\ell < e^{\chi_2^d \ell}$ for all $x \in \mathbb{T}^3$ and $\ell \in \mathbb{Z}$.
- (2) $\chi_1^s < \chi_2^s < 0$ and $e^{\chi_1^s \ell} < \lambda_x^s(\ell) < e^{\chi_2^s \ell}$ for all $x \in \mathbb{T}^3$ and $\ell \in \mathbb{Z}$.
- (3) $\chi_1^c > \chi_2^c > 0$ and $e^{\chi_1^c \ell} > \lambda_x^c(\ell) > e^{\chi_2^c \ell}$ for all $x \in \mathbb{T}^3$ and $\ell \in \mathbb{Z}$.
- (4) $\chi_1^u > \chi_2^u > 0$ and $e^{\chi_1^u \ell} > \lambda_x^u(\ell) > e^{\chi_2^u \ell}$ for all $x \in \mathbb{T}^3$ and $\ell \in \mathbb{Z}$.

2.1.4. *Invariant manifolds.* Let $r \geq 1$ and let $f \in \mathcal{A}^r(\mathbb{T}^3)$. It is well known (see [24]) that the strong bundles E^s and E^u are uniquely integrable to f -invariant continuous foliations with C^r leaves $\mathcal{W}_f^s = \mathcal{W}^s$ and $\mathcal{W}_f^u = \mathcal{W}^u$ respectively, called the *strong-stable* and *strong-unstable foliations*. Since the splitting $E^s \oplus E^{cu}$ is Anosov, the center-unstable bundle E^{cu} also integrates uniquely to an f -invariant continuous foliation $\mathcal{W}_f^{cu} = \mathcal{W}^{cu}$, called the *center-unstable foliation*. For any $x \in \mathbb{T}^3$ and $\star = s, u, cu$, we denote by $\mathcal{W}^\star(x)$ the leaf of \mathcal{W}^\star through x ; it is an immersed C^r manifold.

We now define the concept of *joint integrability* which appears in the statement of Theorem A.

Definition 2.1 (Joint integrability). We say that $f \in \mathcal{A}^r(\mathbb{T}^3)$ is (or that the bundles E^s and E^u are) *jointly integrable* if the bundle $E^s \oplus E^u$ integrates to a continuous foliation with C^1 leaves.

Let $r \geq 1$, and fix a C^r Anosov diffeomorphism $f \in \mathcal{A}^r(\mathbb{T}^3)$. By Corollary 1.3 in [34] the non-wandering set $\Omega(f)$ of f is equal to \mathbb{T}^3 , and f is topologically conjugate to a *hyperbolic toral automorphism*. As a consequence, one has the following lemma (see [37, Theorem 1.3] for a proof).

Lemma 2.2. *The stable foliation of f is minimal, i.e., each leaf of \mathcal{W}^s is dense in \mathbb{T}^3 .*

2.1.5. *Dynamical coherence.* As remarked before, any diffeomorphism $f \in \mathcal{A}^r(\mathbb{T}^3)$ can be seen either as a strongly partially hyperbolic diffeomorphism with respect to the

splitting $E^s \oplus E^c \oplus E^u$, or as an Anosov diffeomorphism with respect to the splitting $E^s \oplus E^{cu}$. By Lemma 2.2 we know that $\Omega(f) = \mathbb{T}^3$.

By the results of Potrie [38] and Brin–Burago–Ivanov [4] on partially hyperbolic diffeomorphisms of \mathbb{T}^3 , we know that f is *dynamically coherent*. In particular, E^{cs} is also integrable to an f -invariant continuous foliation $\mathcal{W}_f^{cs} = \mathcal{W}^{cs}$, called the *center-stable foliation*. Moreover, \mathcal{W}^s subfoliates \mathcal{W}^{cs} , while \mathcal{W}^u subfoliates \mathcal{W}^{cu} , and the collection of all leaves $\mathcal{W}^c(x) := \mathcal{W}^{cs}(x) \cap \mathcal{W}^{cu}(x)$, $x \in \mathbb{T}^3$, forms a foliation $\mathcal{W}_f^c = \mathcal{W}^c$, called the *center foliation*, which integrates E^c , and subfoliates both \mathcal{W}^{cs} and \mathcal{W}^{cu} . For $\star \in \{u, c, s, cu, cs\}$, let d_\star be the leaf-wise distance, and for $x \in \mathbb{T}^3$ and $\sigma > 0$, set $\mathcal{W}_\sigma^\star(x) := \{y \in \mathcal{W}^\star(x) : d_\star(x, y) < \sigma\}$.

In our Anosov case we can show fairly easily that the foliations \mathcal{W}^c and \mathcal{W}^u are globally transverse inside each \mathcal{W}^{cu} leaf, as the two lemmas below demonstrate.

Lemma 2.3. *For any $x \in \mathbb{T}^3$, we have $\mathcal{W}^{cu}(x) = \bigcup_{y \in \mathcal{W}^c(x)} \mathcal{W}^u(y)$.*

Proof. Let $z \in \mathcal{W}^{cu}(x)$ be arbitrary. We need to show that $\mathcal{W}^c(x) \cap \mathcal{W}^u(z) \neq \emptyset$. Since the bundles E^c and E^u are integrable and the local leaves have uniform size due to hyperbolicity, and since the splitting $E^c \oplus E^u$ is dominated and backward iteration under f contracts distances uniformly along \mathcal{W}^{cu} , we must have some $n \in \mathbb{N}$ such that

$$\mathcal{W}^c(f^{-n}(x)) \cap \mathcal{W}^u(f^{-n}(z)) \neq \emptyset,$$

with transverse intersection. By forward iteration and using the fact that integral manifolds are invariant by the dynamics we obtain the conclusion of the lemma. ■

Lemma 2.4. *For every $y \in \mathcal{W}^c(x)$, we have $\mathcal{W}^c(x) \cap \mathcal{W}^u(y) = \{y\}$.*

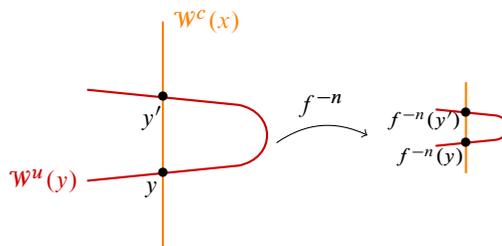


Fig. 3. Proof of Lemma 2.4: iterating the picture of the left we arrive at a small scale where the picture violates the uniformly positive angle between E^c and E^u .

Proof. Assume there exists a point $y' \neq y$ in $\mathcal{W}^c(x) \cap \mathcal{W}^u(y)$. Consider the piece γ^u of unstable manifold joining y to y' . Since the length of $f^{-n}(\gamma^u)$ decreases exponentially, there exists some n such that the curve $f^{-n}(\gamma^u)$ is entirely contained in a coordinate chart for which the line field E^c is almost vertical. Since this curve joins the points $f^{-n}(y)$ and $f^{-n}(y')$ which belong to the same local integral curve of E^c , this proves that the tangent space of $f^{-n}(\gamma^u)$ is almost vertical somewhere. This contradicts the dominated splitting $E^c \oplus E^u$ and completes the proof. ■

2.1.6. *Holonomies.* Let $x_1, x_2 \in \mathbb{T}^3$ be two *stably connected* points, i.e., $x_2 \in \mathcal{W}^s(x_1)$. Set $r := d_s(x_1, x_2)$. By transversality, for any sufficiently small $\varepsilon > 0$, there is $\sigma_1 > 0$ such that for any point $y_1 \in \mathcal{W}_{\sigma_1}^{cu}(x_1)$, there exists a unique point $y_2 \in \mathcal{W}^s(y_1) \cap \mathcal{W}_1^{cu}(x_2)$ with $d_s(y_1, y_2) \in (r - \varepsilon, r + \varepsilon)$. We denote $H_{x_1, x_2}^s(y_1) := y_2$. Thus, we get a well defined local homeomorphism

$$H_{x_1, x_2}^s: \mathcal{C}_1 \rightarrow \mathcal{C}_2,$$

from a neighborhood \mathcal{C}_1 of x_1 within $\mathcal{W}^{cu}(x_1)$ to a neighborhood \mathcal{C}_2 of x_2 within $\mathcal{W}^{cu}(x_2)$, called the (local) *stable holonomy map* between \mathcal{C}_1 and \mathcal{C}_2 . The holonomies H^u, H^{cs}, H^{cu} along $\mathcal{W}^u, \mathcal{W}^{cs}, \mathcal{W}^{cu}$ are defined in a similar way.

2.2. Regularity of extreme bundles and holonomy maps

It is crucial to our proof that certain holonomy maps are of class C^1 . This is the case when some *bunching* inequalities are satisfied between the rates of contraction/expansion of the system; they actually hold in a neighborhood of volume preserving Anosov diffeomorphisms of \mathbb{T}^3 , which is the main motivation behind this assumption in our result.

In our setting, we will say that f is *bunched* if there exists some $n \in \mathbb{N}$ such that for any $x \in \mathbb{T}^3$,

$$\lambda_{f,x}^s(n) < \frac{\lambda_{f,x}^c(n)}{\lambda_{f,x}^u(n)}, \tag{2.5}$$

where the notation $\lambda_{f,x}^*(n)$ was introduced in Section 2.1. In the inequality above, we have made explicit the dependence of the contraction and expansion rates on f . We remark that this is clearly a C^1 -open condition. This definition appears in [39] (see also [9, Section 4.7]).

2.2.1. *Regularity of the unstable bundle.* Let $f \in \mathcal{A}^2(\mathbb{T}^3)$. Then f is a strongly partially hyperbolic diffeomorphism with one-dimensional center bundle, hence f is automatically center-bunched, i.e., the lack of conformality of the center bundle is dominated by the expansion, resp. contraction rates of the strong unstable, resp. stable directions (see [39, Theorem B] for a precise definition). By [39, Theorem B], the unstable bundle E^u is C^1 when restricted to any center-unstable leaf $\mathcal{W}^{cu}(x)$. In particular, the vector field $v^u|_{\mathcal{W}^{cu}}$ is a C^1 vector field over the immersed submanifold $\mathcal{W}^{cu}(x)$ and its C^1 norm depends continuously on $x \in \mathbb{T}^3$.

As a corollary, for any two small and nearby center curves $\gamma_1, \gamma_2 \subset \mathcal{W}^{cu}(x)$ in the same center-unstable leaf, the unstable holonomy map $H^u: \gamma_1 \rightarrow \gamma_2$ is C^1 . This regularity will play a role in our argument in Section 10.4. Moreover, for some estimations in our proof it is important to quantify the Lipschitz constant of these unstable holonomy maps, as in the following result, which is simply a more precise statement of [39, Theorem B] in our case.

Lemma 2.5. *Let $f \in \mathcal{A}^2(\mathbb{T}^3)$. There exist $\rho_0, C^u > 0$ such that for any $x, y \in \mathbb{T}^3$, if $x \in \mathcal{W}_2^u(y)$ then the unstable holonomy $H_{x,y}^u$ between local center manifolds is defined*

over $W_{\rho_0}^c(x)$ and for all $z, z' \in W_{\rho_0}^c(x)$ we have

$$d(H_{x,y}^u(z), H_{x,y}^u(z')) \leq C^u d(z, z').$$

2.2.2. *Regularity of the stable bundle.* For the bundle E^s a stronger statement can be made when f is close to a volume preserving map. Indeed, fix an arbitrary $f_0 \in \mathcal{A}_m^1(\mathbb{T}^3)$.

Lemma 2.6. *There exists a neighborhood $\mathcal{U}(f_0)$ of f_0 within $\text{Diff}^2(\mathbb{T}^3)$ such that for any $f \in \mathcal{U}(f_0)$, we have $f \in \mathcal{A}^2(\mathbb{T}^3)$ and the stable bundle E^s of f is of class C^1 .*

Proof. Since f_0 is uniformly hyperbolic, there exists a continuous function $\mathbb{T}^3 \ni x \mapsto C(x) \in \text{GL}(3, \mathbb{R})$ such that for every point $x \in \mathbb{T}^3$,

$$Df_0(x) = C(f_0(x))^{-1} \begin{bmatrix} \lambda_{f_0,x}^s & 0 & 0 \\ 0 & \lambda_{f_0,x}^c & 0 \\ 0 & 0 & \lambda_{f_0,x}^u \end{bmatrix} C(x).$$

The matrix $C(x)$ is a matrix with positive determinant that sends a basis formed by unit vectors in $E^*(x)$ to the orthogonal basis $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. In particular, there exists a uniform constant $C \geq 1$ such that

$$\max \{ |\det C(x)|, |\det C(x)^{-1}| \} \leq C.$$

Consequently,

$$|\det Df_0(x)| = |\det C(f_0(x))^{-1}| \cdot |\det C(x)| \cdot \lambda_{f_0,x}^s \lambda_{f_0,x}^c \lambda_{f_0,x}^u.$$

Observe that for every $n \in \mathbb{N}$,

$$|\det Df_0^n(x)| = |\det C(f_0^n(x))^{-1}| \cdot |\det C(x)| \cdot \lambda_{f_0,x}^s(n) \lambda_{f_0,x}^c(n) \lambda_{f_0,x}^u(n).$$

The diffeomorphism f_0 is conservative so its Jacobian is a coboundary (by Livshits's Theorem [33] and [3, Theorem 4.14]). Thus, there exists a continuous function $\phi: \mathbb{T}^3 \rightarrow (0, \infty)$ bounded away from 0 and ∞ such that for any $x \in \mathbb{T}^3$, we have

$$|\det C(f_0(x))^{-1}| \cdot |\det C(x)| \cdot \lambda_{f_0,x}^s \lambda_{f_0,x}^c \lambda_{f_0,x}^u = \frac{\phi \circ f_0(x)}{\phi(x)}.$$

Hence,

$$\begin{aligned} \lambda_{f_0,x}^s(n) &= \frac{\lambda_{f_0,x}^s(n) \lambda_{f_0,x}^c(n) \lambda_{f_0,x}^u(n)}{\lambda_{f_0,x}^c(n) \lambda_{f_0,x}^u(n)} \\ &= |\det C(f_0^n(x))^{-1}|^{-1} \cdot |\det C(x)|^{-1} \cdot \frac{\phi \circ f^n(x)}{\phi(x)} \cdot \frac{1}{(\lambda_{f_0,x}^c(n))^2} \frac{\lambda_{f_0,x}^c(n)}{\lambda_{f_0,x}^u(n)} < \frac{\lambda_{f_0,x}^c(n)}{\lambda_{f_0,x}^u(n)} \end{aligned}$$

as long as n is large enough so that

$$|\det C(f_0(x))^{-1}| \cdot |\det C(x)| \frac{\phi \circ f^n(x)}{\phi(x)} < (\lambda_{f_0,x}^c(n))^2.$$

Such an integer n may be chosen independently of x because ϕ is uniformly bounded away from 0 and ∞ , $\max\{|\det C(x)^{-1}|, |\det C(x)|\} \leq C$ and E^c is uniformly expanding. In particular, if $f \in \text{Diff}^2(\mathbb{T}^3)$ is sufficiently C^1 -close to f_0 , then $f \in \mathcal{A}^2(\mathbb{T}^3)$, and for the same choice of $n \in \mathbb{N}$ and every $x \in \mathbb{T}^3$, we have $\lambda_{f,x}^s(n) < \frac{\lambda_{f,x}^c(n)}{\lambda_{f,x}^u(n)}$ and f satisfies (2.5). In other words, the hyperbolic splitting $E^s \oplus E^{cu}$ of f is bunched (see (2.5)). Then, according to the results of [39] (see also [9, Theorem 4.21]), the stable bundle E_f^s is C^1 , as also are the stable holonomy maps. ■

Remark 2.7. As noted in the above proof, the assumption that f is C^1 -close to a conservative diffeomorphism ensures that the following *bunching condition* is automatically satisfied (and hence stable holonomies H^s are C^1 , by [39]):

$$\lambda_{f,x}^s(n) < \frac{\lambda_{f,x}^c(n)}{\lambda_{f,x}^u(n)}, \quad \forall x \in \mathbb{T}^3. \tag{2.6}$$

In other words, (2.6) means that the lack of conformality of Df along E^{cu} is dominated by the contraction along E^s . In particular, the conclusion of Theorem A holds true for any Anosov diffeomorphism $f \in \mathcal{A}^2(\mathbb{T}^3)$ satisfying (2.6).

2.2.3. *Hölder regularity of cs-holonomies.* Despite the above regularity results, a substantial source of technical difficulties for our strategy comes from the absence of smoothness for *center-stable holonomies*. The quest for overcoming this is the main reason behind our *matching argument for Y-configurations* in Sections 10 and 11.3.2. In our setting, the best that can be said about center-stable holonomies comes from [39, Theorem A], which we quote below in a convenient way for our purposes.

Lemma 2.8. *Let $f \in \mathcal{A}^2(\mathbb{T}^3)$. Then there exist $\rho_0 > 0$ (which we can assume is the same as in Lemma 2.5) and $C^{cs}, \theta^{cs} > 0$ such that for any $x, y \in \mathbb{T}^3$ if $x \in \mathcal{W}_{\rho_0}^{cs}(y)$ then the center-stable holonomy $H_{x,y}^{cs}$ is defined on $\mathcal{W}_2^u(x)$ and for all $z, z' \in \mathcal{W}_2^u(x)$ we have*

$$d(H_{x,y}^{cs}(z), H_{x,y}^{cs}(z')) \leq C^{cs} d(z, z')^{\theta^{cs}}.$$

We remark that the roles of ρ_0 and 2 are “exchanged” when compared with Lemma 2.5. Here, the transversal we are looking at for the center-stable holonomy has size 2, while in Lemma 2.5 the transversal has size ρ_0 .

2.3. *Basic distortion estimates*

The goal of this subsection is to collect some classical distortion estimates and fix once and for all some constants which are going to be important all along our constructions.

Lemma 2.9 (Basic distortion lemma). *Let $\varphi: \mathbb{T}^3 \rightarrow \mathbb{R}$ be a Hölder continuous function. Then there exists a constant $C = C(\varphi) > 0$ such that if $n > 0$ and either*

- (a) $y \in \mathcal{W}_1^s(x)$, or
- (b) $f^\ell(y) \in \mathcal{W}_1^{cu}(f^\ell(x))$ for $\ell = 0, \dots, n$,

then

$$\left| \sum_{\ell=0}^{n-1} [\varphi(f^\ell(x)) - \varphi(f^\ell(y))] \right| \leq C.$$

We omit the proof as it is quite classical. Applying the lemma to the functions $\varphi = \log \|Df(\cdot)|_{E^\star}\|$, $\star = c, u$ we obtain a constant $C_0 = C_0(f) \geq 1$ such that if $n > 0$ and either $y \in \mathcal{W}_1^s(x)$ or $f^\ell(y) \in \mathcal{W}_1^{cu}(f^\ell(x))$ for every $\ell = 0, \dots, n$, then

$$C_0^{-1} \leq \frac{\|Df^n(x)|_{E^\star}\|}{\|Df^n(y)|_{E^\star}\|} \leq C_0. \tag{2.7}$$

Moreover, up to enlarging C_0 , we also find that if $y \in \mathcal{W}_1^{cu}(x)$ then

$$C_0^{-1} \leq \frac{\|Df^{-n}(x)|_{E^\star}\|}{\|Df^{-n}(y)|_{E^\star}\|} \leq C_0 \tag{2.8}$$

for every $n \in \mathbb{N}$. Another important application is obtained by considering the function $\psi(x) = \log \frac{\|Df(x)|_{E^c}\|}{\|Df(x)|_{E^u}\|}$. We can assume that the constant C_0 also satisfies the following.

Corollary 2.10. *Given $\ell \in \mathbb{N}$ recall from (2.4) that $d_x^\ell = \frac{\|Df^\ell(f^{-\ell}(x))|_{E^c}\|}{\|Df^\ell(f^{-\ell}(x))|_{E^u}\|}$. If $f^{-\ell}(y) \in \mathcal{W}_1^s(f^{-\ell}(x))$, then*

$$C_0^{-1} \leq \frac{d_x^\ell}{d_y^\ell} \leq C_0.$$

2.3.1. Distortion for quadrilaterals. One of the key dynamical configurations for our strategy is the *quadrilaterals*. A *quadrilateral* is a quadruple $(x, x^u, y, y^u) \in (\mathbb{T}^3)^4$ such that $x^u \in \mathcal{W}_1^u(x)$, $y \in \mathcal{W}_1^s(x)$, and $y^u \in \mathcal{W}_1^u(y) \cap \mathcal{W}^{cs}(x^u)$. For such a quadrilateral, we define $z^u := H_{x,y}^s(x^u)$, so that $z^u \in \mathcal{W}^s(x^u) \cap \mathcal{W}^c(y^u)$.

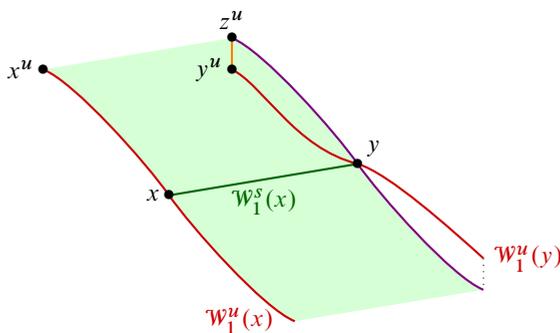


Fig. 4. A quadrilateral.

From our previous distortion results, we can take a larger constant $C_0 = C_0(f)$ in order to have the following.

Corollary 2.11. *Assume that $y \in \mathcal{W}_1^s(x)$, and that $x^u \in \mathcal{W}_1^u(x)$ and $y^u \in \mathcal{W}_1^u(y)$ are such that $y^u = H_{x,y}^{c,s}(x^u)$. Let $z^u = H_{x,y}^s(x^u)$. If $n \in \mathbb{N}$ satisfies $d(f^n(z^u), f^n(y^u)) \leq 1$ then*

$$C_0^{-1} \leq \frac{\|Df^\ell(x^u)|_{E^c}\|}{\|Df^\ell(y^u)|_{E^c}\|} \leq C_0 \quad \text{for every } \ell = 0, \dots, n.$$

Proof. The result follows from (2.7) since

$$\frac{\|Df^\ell(x^u)|_{E^c}\|}{\|Df^\ell(y^u)|_{E^c}\|} = \frac{\|Df^\ell(x^u)|_{E^c}\|}{\|Df^\ell(z^u)|_{E^c}\|} \times \frac{\|Df^\ell(z^u)|_{E^c}\|}{\|Df^\ell(y^u)|_{E^c}\|}. \quad \blacksquare$$

2.4. Subordinate partitions and disintegrations

We move now to the *ergodic theory* of Anosov diffeomorphisms of \mathbb{T}^3 with expanding center. Before giving the main definitions, we give some preliminaries on measurable partitions, disintegration and invariant measures.

2.4.1. Measurable partitions. Let μ be a probability measure on some standard Borel set X . Let ξ_1, ξ_2 be two partitions (mod 0) of X into measurable subsets. Say that ξ_1 is *finer than* (or *refines*) ξ_2 if for μ -a.e. $x \in X$ we have $\xi_1(x) \subset \xi_2(x) \pmod 0$. This will be denoted by $\xi_2 < \xi_1$.

The *join* of ξ_1 and ξ_2 is the partition defined as $\xi_1 \vee \xi_2 = \{\xi_1(x) \cap \xi_2(x) : x \in X\}$.

A partition ξ of X is *measurable* whenever there exists a sequence $(\xi_n)_{n \in \mathbb{N}}$ of finite partitions of X into Borel subsets such that

$$\xi = \bigvee_{n=0}^{\infty} \xi_n.$$

Rokhlin [47] proved the fundamental result that the measure μ can be *disintegrated* into atoms of any measurable partition ξ . This means that there exists a family $\{\mu_x^\xi\}_x$ of probability measures, called *conditional measures* of μ relative to ξ , defined for μ -a.e. $x \in X$ and satisfying, for μ -a.e. $x \in X$,

- (1) μ_x^ξ is a probability measure on X with $\mu_x^\xi(\xi(x)) = 1$;
- (2) if $y \in \xi(x)$ then $\mu_y^\xi = \mu_x^\xi$.

Moreover, for every Borel subset $A \subset X$,

- (3) $x \mapsto \mu_x^\xi(A)$ is measurable;
- (4) $\mu(A) = \int \mu_x^\xi(A) d\mu(x)$.

Furthermore, such a family is unique modulo a null set of μ .

2.4.2. Disintegration of invariant measures. Assume now that μ is invariant by some invertible measurable transformation $f: X \rightarrow X$. Let ξ be a measurable partition and, for $n \in \mathbb{Z}$, let $\xi_n := f^n \xi$. Let $\{\mu_x\}_x := \{\mu_x^\xi\}_x$ and $\{\mu_{n,x}\}_x := \{\mu_x^{\xi_n}\}_x$ be families of conditional measures of μ relative to ξ and ξ_n respectively. The following lemma will be useful.

Lemma 2.12. For every $n \in \mathbb{N}$ and μ -almost every $x \in X$, we have

$$\mu_{n,x} = f_*^n \mu_{f^{-n}(x)}.$$

Proof. Let $\varphi \in L^1(X, \mu)$. We first disintegrate μ along ξ and use f -invariance twice:

$$\begin{aligned} \int \varphi(x) d\mu(x) &= \int \varphi \circ f^n(x) d\mu(x) = \int \left(\int_{\xi(x)} \varphi \circ f^n(y) d\mu_x(y) \right) d\mu(x) \\ &= \int \left(\int_{\xi(f^{-n}(x))} \varphi \circ f^n(y) d\mu_{f^{-n}(x)}(y) \right) d\mu(x) \\ &= \int \left(\int_{\xi_n(x)} \varphi(y) d(f_*^n \mu_{f^{-n}(x)})(y) \right) d\mu(x). \end{aligned}$$

We deduce that $\{f_*^n \mu_{f^{-n}(x)}\}_x$ is a system of conditional measures of μ with respect to ξ_n . The lemma follows by uniqueness μ -a.e. of conditional measures. ■

2.4.3. Partitions subordinate to expanded and contracted foliations. Let M be a closed manifold, $f: M \rightarrow M$ be a diffeomorphism of M , and μ be an ergodic invariant measure of f . Assume that \mathcal{W}^+ is a foliation invariant by f . Assume furthermore that it is uniformly expanded, that is, $\|Df^{-1}|_{T\mathcal{W}^+}\| \leq \lambda$ for some constant $0 < \lambda < 1$.

A measurable partition ξ is *subordinate to* \mathcal{W}^+ if the following properties hold for μ -a.e. $x \in M$:

- (1) $\xi(x)$ contains an open (in the internal topology) neighborhood of x in $\mathcal{W}^+(x)$;
- (2) $\xi < f^{-1}\xi$ (we say that ξ is *increasing*);
- (3) $\bigvee_{n=0}^{\infty} f^{-n}\xi$ is the partition into points.

The existence of subordinate partitions was proven in [31] in a more general context (see also [48] and [5, Appendix D]).

Remark 2.13. The subordinate partitions constructed in [5, 48] have atoms with uniformly bounded diameter. In particular, we can take subordinate partitions with diameter bounded by 1.

Remark 2.14. Assume that $\mathcal{W}' \subset \mathcal{W}^+$ is an f -invariant subfoliation with expansion constant $0 < \lambda' \leq \lambda$ and that ξ is subordinate to \mathcal{W}^+ . Then it follows from the proofs in [5, 31, 48] that the partition $\xi' = \xi \vee \mathcal{W}'$ is subordinate to \mathcal{W}' .

Similarly, assume \mathcal{W}^- is invariant and *uniformly contracted* by f . This means that $\|Df|_{T\mathcal{W}^-}\| < \lambda$ for some $0 < \lambda < 1$. We say that ξ is *subordinate to* \mathcal{W}^- if properties (1)–(3) above hold on replacing f by f^{-1} . We then say that ξ is *decreasing* for f .

Remark 2.15. If \mathcal{W}^+ (resp. \mathcal{W}^-) has dimension 1 then the atoms of the subordinate partition constructed in [5, 31, 48] are intervals.

2.4.4. Superposition property of subordinate partitions. Let M, f, \mathcal{W}^+ be as in the above Section 2.4.3. Let ξ be a partition subordinate to \mathcal{W}^+ . Then, for every $n \geq 0$, $\xi_{-n} := f^{-n}\xi$

is also a partition subordinate to \mathcal{W}^+ . Since $\xi_{-n}(x)$ contains an open neighborhood of x for μ -a.e. $x \in M$, an atom of ξ contains at most countably many atoms of ξ_{-n} . In the terminology of [12, Definition 5.15], we say that ξ and ξ_{-n} are *countably equivalent*.

From [12, Proposition 5.17] we obtain the following *superposition property*.

Lemma 2.16 (Superposition property). *For μ a.e. $x \in M$ we have*

$$\mu_x(\xi_{-n}(x)) > 0$$

for every $n \in \mathbb{N}$. Moreover, if $\mu_{-n,x}$ is the disintegration of μ with respect to the partition ξ_{-n} , then for any measurable set A ,

$$\mu_{-n,x}(A) = \frac{\mu_x(A \cap \xi_{-n}(x))}{\mu_x(\xi_{-n}(x))}.$$

By the definition of a subordinate partition, for μ -a.e. $x \in M$, $\{\xi_{-n}(x)\}_{n \geq 0}$ contains a basis of open neighborhoods of x .

Corollary 2.17. *For μ -a.e. $x \in M$, μ_x charges every open neighborhood of x .*

Remark 2.18. If \mathcal{W}^- is contracting, and ξ is subordinate to \mathcal{W}^- , then statements analogous to Lemma 2.16 and Corollary 2.17 hold for \mathcal{W}^- , on replacing ξ_{-n} with ξ_n for $n \geq 0$.

2.4.5. Uniform growth property. Another useful property of the sequence of partitions ξ_n subordinate to a uniformly expanded foliation \mathcal{W}^+ is the following *uniform growth property*.

Lemma 2.19. *Given a real number $R > 0$, for μ -almost every $x \in M$ there exists $n_0 = n_0(x, R) > 0$ such that if $n > n_0$ then*

$$\mathcal{W}_R^+(x) \subset \xi_n(x).$$

Proof. Define $\Lambda(\varepsilon) = \{x \in \mathbb{T}^3 : \mathcal{W}_\varepsilon^+(x) \subset \xi(x)\}$, and observe that $\mu(\Lambda(\varepsilon)) \rightarrow 1$ as $\varepsilon \rightarrow 0$, due to property (1) in the definition of subordinate partitions. The conclusion holds whenever $f^{-n_k}(x) \in \Lambda(1/k)$ where $n_k > \frac{\log(kR)}{-\log \lambda}$ (recall that $\|Df^{-1}|_{T\mathcal{W}^+}\| < \lambda < 1$). So the conclusion of the lemma holds in the set $\bigcup_k f^{n_k}(\Lambda(1/k))$ which has measure 1 by f -invariance of μ . ■

Remark 2.20. The above lemma has an analogous version for measurable partitions subordinate to uniformly contracted foliations: the integer n_0 is then negative and the conclusion holds for all $n < n_0$.

2.5. SRB and u -Gibbs measures

We return now to our partially hyperbolic setting where $f \in \mathcal{A}^2(\mathbb{T}^3)$. Let ξ^{cu} be a measurable partition of \mathbb{T}^3 subordinate to the center-unstable foliation \mathcal{W}^{cu} . Let μ be an ergodic

invariant measure for f . We say that μ is an *SRB measure* if its disintegration $\{\mu_x^{cu}\}_{x \in \mathbb{T}^3}$ with respect to the partition ξ^{cu} has the property that μ_x^{cu} is absolutely continuous with respect to the inner Lebesgue measure Leb_x^{cu} of $\mathcal{W}^{cu}(x)$ for μ -a.e. $x \in \mathbb{T}^3$.

Consider now a measurable partition ξ^u subordinate to the unstable foliation \mathcal{W}^u and let $\{\mu_x^u\}_{x \in \mathbb{T}^3}$ denote the disintegration. Moreover, suppose that ξ^u has diameter bounded from above by 1 (see Remark 2.13). We say that μ is a *u -Gibbs measure* provided that μ_x^u is absolutely continuous with respect to the inner Lebesgue (length) measure Leb_x^u of $\mathcal{W}^u(x)$ for μ -a.e. $x \in \mathbb{T}^3$.

Remark 2.21. By Ledrappier–Young [32] none of the above definitions depends on the choice of subordinate partition.

2.5.1. *Conditional measures along unstable leaves of an u -Gibbs measure.* We can (and will) consider the particular case in which the atoms of ξ^u are $\xi^u(x) = \mathcal{W}^u(x) \cap \xi^{cu}(x)$. By Remark 2.14 we know that this defines a partition subordinate to the unstable foliation \mathcal{W}^u . This is especially important in our setting, in which we want to prove that a given u -Gibbs measure is SRB. In a similar fashion, we consider the partition ξ^c whose atoms are $\xi^c(x) = \mathcal{W}^c(x) \cap \xi^{cu}(x)$.

Set $\xi_n^\star := f^n(\xi^\star)$, for $\star = u, cu, c$ and $n \geq 0$, with $\xi_0^\star = \xi^\star$. Consider a u -Gibbs measure μ and let $\mu_{n,x}^{cu}$ denote the conditional measures along the partition ξ_n^{cu} . Notice that $\{\xi_n^u(y)\}_{y \in \xi_n^{cu}(x)}$ is a measurable partition of the probability space $(\xi_n^{cu}(x), \mu_{n,x}^{cu}, \mathcal{B}|_{\xi_n^{cu}(x)})$, where $\mathcal{B}|_{\xi_n^{cu}(x)}$ denotes the Borel sigma-algebra of \mathbb{T}^3 restricted to the atom $\xi_n^{cu}(x)$. Rokhlin’s disintegration theorem in this case give us a probability measure $\mu_{n,x}^c$ defined on $\xi_n^c(x)$ and a family $\{\mu_{n,y}^u\}_{y \in \xi_n^c(x)}$ of probability measures such that for every Borel measurable set $A \subset \xi_n^{cu}(x)$ we have

$$\mu_{n,x}^{cu}(A) = \int_{A \cap \mathcal{W}^c(x)} \mu_{n,y}^u(A \cap \mathcal{W}^u(y)) d\mu_{n,x}^c(y).$$

Remark 2.22. For simplicity of exposition in many situations, where no confusion may arise, given a set $A \subset \mathcal{W}^\star(x)$, we shall write

$$|A| := \text{Leb}_x^\star(A) \quad \text{for } \star \in \{s, c, u\}.$$

An easy consequence of the u -Gibbs property is the following.

Lemma 2.23. *There exists $\beta > 1$ depending only on the diffeomorphism f such that for μ -almost every $x \in \mathbb{T}^3$,*

$$\frac{1}{\beta \text{Leb}_x^u[\xi^u(x)]} < \frac{d\mu_x^u}{d\text{Leb}_x^u} < \frac{\beta}{\text{Leb}_x^u[\xi^u(x)]}$$

inside $\xi^u(x)$.

Proof. This follows from the fact that μ_x^u is a probability measure with a uniformly log-Hölder continuous density with respect to Leb_x^u and that the diameter of $\xi^u(x)$ is less than 1. ■

3. Heuristics of the proof

This section can be used to get an overview of the proof of Theorem A and also as a guide to read our paper.

From now on let $f \in \mathcal{A}^2(\mathbb{T}^3)$ be a diffeomorphism that is close to a conservative one. In this case, the stable holonomies $H_{x,y}^s$ are C^1 , and we can define angles $\alpha^s(x, y)$ for two points x, y in the same stable manifold (see Definition 4.1). The vanishing of the angle α^s can be seen as a kind of infinitesimal joint integrability.

Let μ be an ergodic u -Gibbs measure for f .

Starting point: A zero-one law

We first use Theorem B, which yields the following dichotomy.

Case 1: $\alpha^s(x, y) = 0$ for μ -a.e. pair (x, y) in the same stable manifold;

Case 2: $\alpha^s(x, y) > 0$ for μ -a.e. pair (x, y) in the same stable manifold.

The proof of this theorem will be given in Section 4 and relies on a martingale argument. In words, either we have positive angles almost everywhere along the stable manifold of almost every point, or we have zero angles almost everywhere along the stable manifold of almost every point.

The rest of the proof is divided into two parts:

- [Case 1 + $\text{supp}(\mu) = \mathbb{T}^3$] \Rightarrow joint integrability;
- Case 2 $\Rightarrow \mu$ is SRB.

Part 1: Joint integrability

Given a point $x \in \mathbb{T}^3$ we define the *bad set* as

$$\mathbf{B} := \{x \in \mathbb{T}^3 : \mu_x^s\{\alpha_x^s > 0\} = 0\},$$

where $\alpha_x^s := [y \in \mathcal{W}^s(x) \mapsto \alpha^s(x, y)]$, and μ_x^s is the conditional measure of x along a stable manifold (see Section 5.1 for more details). The bad set is the set of points such that there is an “infinitesimal” joint integrability with almost every other point in its stable manifold.

The first part is an argument by contraposition that goes as follows (see Proposition 5.5):

- if there is no joint integrability, then for any $x \in \mathbb{T}^3$, the set $\{\alpha_x^s > 0\}$ is open and dense within the stable manifold $\mathcal{W}^s(x)$ (see Lemma 5.8);
- if, furthermore, the bad set \mathbf{B} has measure 1, the continuity of the angles implies that whenever $\alpha^s(x_0, y_0) > 0$, there exists a small foliated chart \mathcal{V} around y_0 for \mathcal{W}^s of measure 0 (see Lemma 5.9);
- hence, combining the previous two points, if there is no joint integrability and \mathbf{B} has full measure, then the support of μ must have empty interior.

The last point is the only place in the paper where the support condition on μ is used.

Part 2: Transversality implies SRB

Most of this paper is dedicated to proving that if μ is a u -Gibbs measure such that $\mu(\mathbf{B}) = 0$ then μ is SRB (Theorem 5.10).

To check that a u -Gibbs measure μ is SRB, one can consider the disintegration of this measure along center-unstable manifolds $\{\mu_x^{cu}\}$ and then quotient it by the strong-unstable manifolds $\{\hat{\mu}_x^c\}$. These are called the transverse measures. Then μ is SRB if and only if the transverse measures are equivalent to Lebesgue. In this approach, it is really convenient to consider certain parametrizations of center-unstable manifolds that simplify the dynamics, the so-called *normal forms* (see Section 6). These coordinates allow us to identify the quotient measures $\{\hat{\mu}_x^c\}$ with measures $\{\hat{\nu}_x^c\}$ in \mathbb{R} . To conclude that the measures $\{\hat{\mu}_x^c\}$ are equivalent to Lebesgue, we will show that the measures $\{\hat{\nu}_x^c\}$ are proportional to the Lebesgue measure on \mathbb{R} , where we say that two locally finite Borel measures ν, η are *proportional*, and we write $\nu \propto \eta$, if $\nu = c\eta$ for some $c > 0$.

To do so, it is enough to prove that for μ -a.e. $x \in \mathbb{T}^3$, $\hat{\nu}_x^c$ is invariant by translation. Actually, thanks to an argument due originally to Katok–Spatzier [29] and Kalinin–Katok [25] (see also [7, Proposition 7.1]), which is a beautiful mixture of ergodic and Lie-theoretic arguments, it is enough to prove something weaker: *for a set G of points $x \in \mathbb{T}^3$ with positive measure, there exist affine maps $\psi = as + b$, with $|a|$ uniformly bounded from above and $0 < |b|$ arbitrarily small, such that $\hat{\nu}_x^c \propto \psi_* \hat{\nu}_x^c$* . This is Lemma 8.1 (largely inspired by [6, Lemma 6.1]).

The construction of these affine maps is where our key arguments are located. Our strategy is to use the Y -configurations introduced by Eskin–Mirzakhani [14] (see also Eskin–Lindenstrauss [15]). Below we outline this proof, splitting the explanation in two parts, for the sake of clarity. First, we describe Y -configurations. Then we explain how to use them in order to get invariance by affine maps in the way we described in the previous paragraph.

Y-configurations. See Figure 5. Fixing $\ell \in \mathbb{N}$ we find points x and y that are typical for μ such that

- (1) $y \in \mathcal{W}_{\text{loc}}^s(x)$ and $d(x, y) \approx \|Df^\ell|_{E^s}\|$;
- (2) if $x_{-\ell} = f^{-\ell}(x)$ and $y_{-\ell} = f^{-\ell}(y)$, then $d(x_{-\ell}, y_{-\ell}) \approx 1$ and the angle $\alpha^s(x_{-\ell}, y_{-\ell})$ is more than some constant $1/C$.

As a consequence of (1) and (2), $\alpha^s(x, y) \geq \frac{1}{C} \frac{\|Df^\ell(x_{-\ell})|_{E^c}\|}{\|Df^\ell(x_{-\ell})|_{E^u}\|}$. Normal forms help a lot here. This computation is performed in the proof of Lemma 9.2.

We can then find points $x^u \in \mathcal{W}_{\text{loc}}^u(x)$ and $y^u \in \mathcal{W}_{\text{loc}}^u(y)$ such that

- there exists a point z^u with $\{z^u\} = \mathcal{W}_{\text{loc}}^s(x^u) \cap \mathcal{W}_{\text{loc}}^c(y^u)$;
- $d(z^u, y^u) \approx \frac{\|Df^\ell(x_{-\ell})|_{E^c}\|}{\|Df^\ell(x_{-\ell})|_{E^u}\|}$.

So far we have obtained points x^u and y^u with some estimate of the displacement in the center direction that we get when projecting x^u to $\mathcal{W}_{\text{loc}}^u(y)$ by center-stable holonomy.

Next, fixing $\varepsilon > 0$, we define the stopping time

$$\tau(\ell) = \tau(x, x^u, \varepsilon, \ell) := \inf \left\{ n \in \mathbb{N} : \frac{\|Df^\ell(x_{-\ell})\|_{E^c}}{\|Df^\ell(x_{-\ell})\|_{E^u}} \|Df^n(x^u)\|_{E^c} \geq \varepsilon \right\}$$

and consider the points $f^{\tau(\ell)}(x^u)$ and $f^{\tau(\ell)}(y^u)$. The stopping time $\tau(\ell)$ is so chosen that

$$d(f^{\tau(\ell)}(x^u), f^{\tau(\ell)}(y^u)) \approx \varepsilon; \tag{3.1}$$

see Section 9.3.2. Essentially, these points are related to the translational part of size ε of the affine maps that we desire to construct. To control the derivative of the affine maps and to actually obtain “invariance” of the measures $\hat{\nu}_x^c$, as we mentioned before, we also define another stopping time

$$t(\ell) = t(x, x^u, \varepsilon, \ell) := \inf \left\{ n \in \mathbb{N} : \frac{\|Df^n(x)\|_{E^c}}{\|Df^{\tau(\ell)}(x^u)\|_{E^c}} \geq 1 \right\}.$$

In particular, this condition implies that

$$\|Df^{t(\ell)}(x)\|_{E^c} \approx \|Df^{\tau(\ell)}(x^u)\|_{E^c}; \tag{3.2}$$

see Section 9.1.

How to use Y-configurations to get invariance by affine maps. By construction of measures $\{\hat{\nu}_p^c\}_{p \in \mathbb{T}^3}$ we understand how they change under three basic moves.

- (1) *Applying the dynamics.* For μ -a.e. $p \in \mathbb{T}^3$, $\hat{\nu}_{f^n(p)}^c \propto \Lambda_* \hat{\nu}_p^c$ for every $n \in \mathbb{N}$, where $\Lambda(s) = \|Df^n(p)\|_{E^c} \|s\|$ is a linear map of \mathbb{R} (see Lemma 7.13).
- (2) *Moving along unstable manifolds.* If $q \in \mathcal{W}^u(p)$ then $\hat{\nu}_q^c \propto L_* \hat{\nu}_p^c$ where $L(s) = \beta s$ is a linear map, for some $C_u^{-1} \leq \beta \leq C_u$, and $C_u = C_u(d_u(p, q)) > 1$ is a number which is bounded from above and it depends on the unstable distance $d_u(p, q)$ (see Lemma 7.14).
- (3) *Moving along center manifolds.* If $q \in \mathcal{W}^c(p)$ then $\hat{\nu}_q^c \propto \Psi_* \hat{\nu}_p^c$ where $\Psi(s) = as + b$, with $C_c^{-1} \leq a \leq C_c$ for some constant $C_c = C_c(d_c(p, q)) > 1$ bounded from above by $d_c(p, q)$, and with $b \approx d_c(p, q)$ (see Lemma 7.12).

Now we look at the points on the top part of Figure 5 and deduce two facts.

- (a) $\hat{\nu}_{f^{\tau(\ell)}(x^u)}^c \propto (\Lambda_1)_* \hat{\nu}_{f^{t(\ell)}(x)}^c$ for some linear map $\Lambda_1: \mathbb{R} \rightarrow \mathbb{R}$ with norm bounded independently of ℓ .
- (b) $\hat{\nu}_{f^{\tau(\ell)}(y^u)}^c \propto (\Lambda_2)_* \hat{\nu}_{f^{t(\ell)}(y)}^c$ for some linear map $\Lambda_2: \mathbb{R} \rightarrow \mathbb{R}$ with norm bounded independently of ℓ .

There is a subtlety here: the stopping times $\tau(\ell)$ and $t(\ell)$ depend on x, x^u and not on y, y^u . This is dealt with by our *synchronization estimates* (Lemma 9.4). Recall that $z^u \in \mathcal{W}^s(x^u)$ and $y \in \mathcal{W}^s(x)$. Now note that as $\ell \rightarrow \infty$, $\tau(\ell), t(\ell) \rightarrow \infty$ so

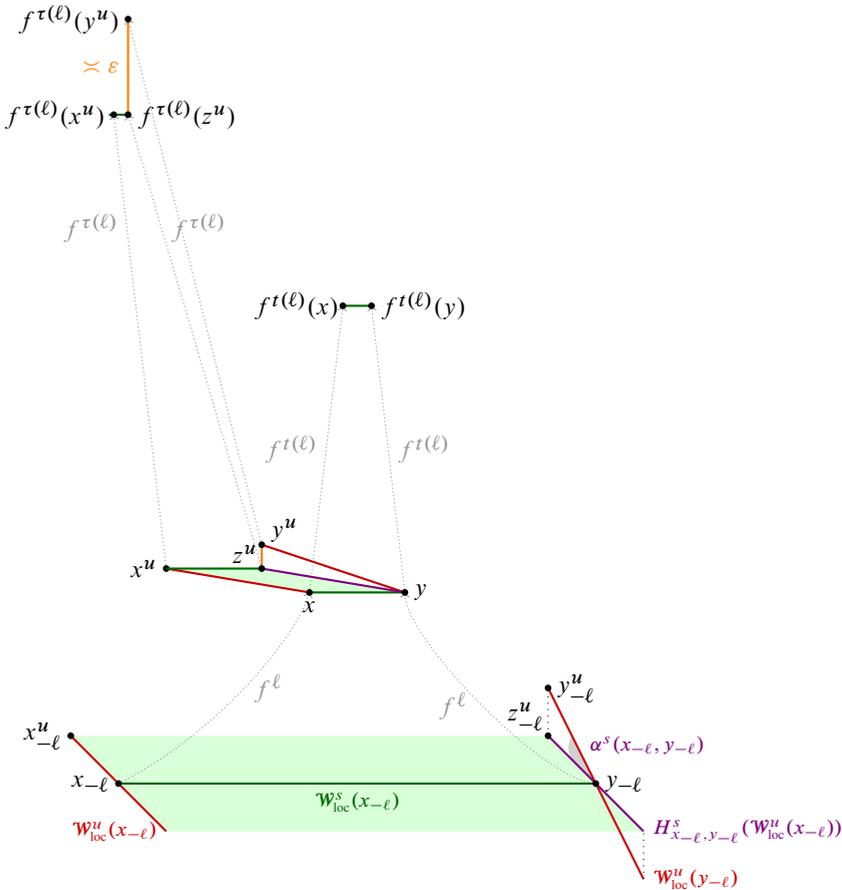


Fig. 5. The points $x, x^u, x_{-l}, f^{\tau(l)}(x^u)$ and $f^{t(l)}(x)$ are called a Y -configuration (and similarly for the points involving y 's).

$d(f^{\tau(l)}(x^u), f^{\tau(l)}(z^u)) \rightarrow 0$ and $d(f^{t(l)}(x), f^{t(l)}(y)) \rightarrow 0$. If we knew that the family of measures $\hat{\nu}_z^c$ were continuous in z then we could hope to compare $\hat{\nu}_{f^{\tau(l)}(x^u)}^c$ with $\hat{\nu}_{f^{\tau(l)}(y^u)}^c$ and take accumulation points to construct the desired set G .

But the objects we are working with are only measurable, so we must first fix a large Luzin set for which the map $z \mapsto \hat{\nu}_z^c$ is continuous (among other dynamical objects that appear in the proof). We want to do the constructions of all the points mentioned above in such a way that all of them belong to this Luzin set. For this, it is essential to obtain *quasi-isometric estimates* for the functions $\tau(\cdot)$ and $t(\cdot)$ (see Lemma 9.1; see also Lemma 11.5 where the quasi-isometric estimates for stopping times are used in a crucial way).

By continuity, we have $\hat{\nu}_{f^{t(l)}(x)}^c \approx \hat{\nu}_{f^{t(l)}(y)}^c$ and $\hat{\nu}_{f^{\tau(l)}(x^u)}^c \approx \hat{\nu}_{f^{\tau(l)}(z^u)}^c$. We conclude that $\hat{\nu}_{f^{\tau(l)}(x^u)}^c$ is *almost proportional* to $\hat{\nu}_{f^{\tau(l)}(y^u)}^c$, up to some linear map with controlled derivative.

After considering a subsequence ℓ_k we obtain points p and q (obtained as the limit of $f^{\tau(\ell_k)}(x^u)$ and $f^{\tau(\ell_k)}(y^u)$) such that

- $q \in \mathcal{W}_{\text{loc}}^c(p)$;
- $d(p, q) \approx \varepsilon$;
- p and q belong to the Luzin set;
- $\hat{v}_q^c \propto \hat{\Lambda}_* \hat{v}_p^c$, $\hat{\Lambda}$ being linear with uniformly bounded norm (see Section 10.4).

Finally, we explained that $\hat{v}_p^c \propto \Psi_* \hat{v}_q^c$ for an affine map $\Psi = as + b$ with $|a|$ uniformly bounded and $|b| \approx \varepsilon$: we find $\hat{v}_p^c \propto (\Psi \hat{\Lambda})_* \hat{v}_q^c$: $\Psi \hat{\Lambda}$ is the desired affine map (with bounded slope and translational part $\approx \varepsilon$).

A technical difficulty. There is one delicate point in the above argument. Since the center-stable foliation is not absolutely continuous, in general we cannot choose the points x^u and y^u at the same time in the Luzin set and such that $y^u \in \mathcal{W}_{\text{loc}}^{cs}(x^u)$. To overcome this, we introduce the notion of matched Y -configurations in Section 10 (the picture changes just a little from the one described above) and use it to prove our theorem (see Section 11).

Let us also emphasize that the control we obtain on the distance between the points z^u and y^u is possible because H^s is C^1 , so essentially we can control this distance by looking at the angle between $DH_{x,y}^s E^u(x)$ and $E^u(y)$. This allows us to explicitly define the stopping time τ in our construction. This is not possible in Katz’s proof since the holonomies are only Hölder. To get around this problem, he defines a much more complicated stopping time built from an operator using ideas of Eskin–Mirzakhani.

4. A zero-one law for angles

The goal of this section is to prove Theorem B. In a slightly more general context, i.e., for some $f \in \mathcal{P}\mathcal{H}^2(\mathbb{T}^3)$ with C^1 stable holonomies (for instance, if f is close to some $f_0 \in \mathcal{A}_m^2(\mathbb{T}^3)$), as we saw in Lemma 2.6), we introduce the angle function α^s , which measures the “twist” of unstable manifolds along stable manifolds.

Definition 4.1 (The angle function). Let $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^s(x)$. The *angle function* is the assignment $(x, y) \mapsto \alpha^s(x, y) := \angle(DH_{x,y}^s(x)E^u(x), E^u(y))$.

In this context, we have the following zero-one law, whose proof is inspired by the work of Brown–Rodríguez Hertz [6, Lemma 7.1].

Theorem 4.2 (A zero-one law for angles). *Let $f \in \mathcal{P}\mathcal{H}^2(\mathbb{T}^3)$ be a partially hyperbolic diffeomorphism with a splitting $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$, whose stable holonomies H^s along the strong-stable foliation \mathcal{W}^s are C^1 . Fix an ergodic f -invariant measure μ . Let ξ^s be a measurable partition subordinate to \mathcal{W}^s and $\{\mu_x^s\}_x$ be a system of conditional measures relative to ξ^s . Then the following dichotomy holds: either*

- (1) for μ -almost every $x \in \mathbb{T}^3$,

$$\mu_x^s\{y \in \xi^s(x) : \alpha^s(x, y) = 0\} = 1;$$

or

(2) for μ -almost every $x \in \mathbb{T}^3$,

$$\mu_x^s \{y \in \xi^s(x) : \alpha^s(x, y) = 0\} = 0.$$

The second alternative will allow us to build “twisted” quadrilaterals (such as the one depicted in Figure 4) with points x, y in some good Luzin set while for x^u and y^u we will find points of the Luzin set arbitrarily close to them, so that the twisted quadrilateral will be part of a *matched Y -configuration* associated to x and y .

4.1. Conditional expectation, σ -algebras and martingales

The proof of Theorem 4.2 requires some preliminaries about martingales. If ξ is a measurable partition of a measurable space X then we let \mathcal{F}_ξ denote the σ -algebra generated by unions of atoms of ξ . Let $\{\mu_x\}_x$ be a system of conditional measures of μ with respect to ξ . We define the *conditional expectation* of $\varphi \in L^1(X, \mu)$ to be the L^1 function

$$\mathbb{E}_\mu[\varphi \mid \mathcal{F}_\xi]: x \mapsto \int_{\xi(x)} \varphi \, d\mu_x^\xi.$$

Note that if $(\xi_n)_n$ is an increasing sequence of partitions (in the sense that $\xi_n < \xi_{n+1}$) then $\mathcal{F}_{\xi_n} \subset \mathcal{F}_{\xi_{n+1}}$.

The following result is a consequence of the increasing martingale theorem (for which we refer to [13, Theorem 5.5, p. 126]).

Theorem 4.3 (Increasing martingale theorem). *Let $(\xi_n)_{n \in \mathbb{N}}$ be an increasing sequence of measurable partitions of X such that $\bigvee_{n=0}^\infty \xi_n$ is the partition into points. Then, for every $\varphi \in L^1(X, \mu)$ and μ -almost every $x \in X$, we have*

$$\lim_{n \rightarrow \infty} \mathbb{E}_\mu[\varphi \mid \mathcal{F}_{\xi_n}](x) = \varphi(x).$$

4.2. Proof of the zero-one law

We are now ready to prove Theorem 4.2. We fix a measurable partition ξ^s subordinate to \mathcal{W}^s . For $n \in \mathbb{N}$, set $\xi_n^s := f^n \xi^s$. Systems of conditional measures relative to ξ^s and ξ_n^s are denoted respectively by $\{\mu_x^s\}_x$ and $\{\mu_{n,x}^s\}_x$. For μ -almost every $x \in \mathbb{T}^3$ and every $n \in \mathbb{N}$, set

$$\begin{aligned} \mathcal{P}^{\xi^s}(x) &:= \{y \in \xi^s(x) : \alpha^s(x, y) = 0\}, \\ \mathcal{P}_n^{\xi^s}(x) &:= \{y \in \xi_n^s(x) : \alpha^s(x, y) = 0\}. \end{aligned}$$

By definition of ξ_n^s and invariance of the unstable bundle and of the stable foliation, we have the following commutation relation for μ -almost every $x \in \mathbb{T}^3$:

$$f^n \mathcal{P}^{\xi^s}(f^{-n}(x)) = \mathcal{P}_n^{\xi^s}(x). \tag{4.1}$$

Let $A := \{x \in \mathbb{T}^3 : \mu_x^s[\mathcal{P}^{\xi^s}(x)] > 0\}$. This is a Borel set. We must prove that if $\mu(A) > 0$, then

$$\mu(A) = 1, \quad \text{and for } \mu\text{-almost every } x \in \mathbb{T}^3, \quad \mu_x^s[\mathcal{P}^{\xi^s}(x)] = 1. \tag{4.2}$$

So let us suppose $\mu(A) > 0$ and prove (4.2). With that goal in mind, we claim that A is an f -invariant set (mod 0). Indeed, take a μ -generic point $x \in A$. Notice that $\mathcal{P}_1^{\xi^s}(x) \subset \mathcal{P}^{\xi^s}(x)$, since $\xi_1^s(x) \subset \xi^s(x)$. Also, recall from Lemma 2.12 that

$$\mu_{1,f(x)}^s = f_*\mu_x^s.$$

Combining this with (4.1) one obtains

$$\mu_{1,f(x)}^s(\mathcal{P}_1^{\xi^s}(f(x))) = \mu_x^s(f^{-1}(\mathcal{P}_1^{\xi^s}(f(x)))) = \mu_x^s(\mathcal{P}^{\xi^s}(x)) > 0.$$

Thus, from the superposition property (Lemma 2.16 and Remark 2.18) one gets

$$\mu_{f(x)}^s(\mathcal{P}^{\xi^s}(f(x))) \geq \mu_{f(x)}^s(\mathcal{P}_1^{\xi^s}(f(x))) = \mu_{1,f(x)}^s(\mathcal{P}_1^{\xi^s}(f(x)))\mu_{f(x)}^s(\xi_1^s(f(x))) > 0,$$

which implies that $f(x) \in A$, proving our claim.

From the ergodicity of μ , we have $\mu(A) = 1$. We define functions in $L^1(\mathbb{T}^3, \mu)$ by

- $\phi: x \mapsto \mu_x^s[\mathcal{P}^{\xi^s}(x)]$;
- $\phi_n: x \mapsto \mu_{n,x}^s[\mathcal{P}_n^{\xi^s}(x)] = \mu_{n,x}^s[\mathcal{P}^{\xi^s}(x)]$ for each $n \in \mathbb{N}$.

For μ -almost every $x \in \mathbb{T}^3$ and every $n \in \mathbb{N}$ we consider

- $\mathcal{F}_x^s = \mathcal{F}_{\xi^s(x)}^s = \{\emptyset, \xi^s(x)\}$, the trivial σ -algebra over $\xi^s(x)$;
- $\mathcal{F}_{n,x}^s = \mathcal{F}_{n,\xi^s(x)}^s$, the σ -algebra generated by unions of atoms $\xi_n^s(y)$, $y \in \xi^s(x)$ (note that $\mathcal{F}_{n,x}^s \subset \mathcal{F}_{n+1,x}^s$);
- $\mathcal{F}_{\infty,x}^s = \mathcal{F}_{\infty,\xi^s(x)}^s$, the smallest σ -algebra containing $\bigcup_{n=0}^{\infty} \mathcal{F}_{n,x}^s$. This is the Borel σ -algebra of $\xi^s(x)$.

For μ -almost every $x \in \mathbb{T}^3$ and $n \in \mathbb{N}$, we define the function $\psi_{n,x} \in L^1(\xi^s(x), \mu_x^s)$ by the following formula, for μ_x^s -almost every $y \in \xi^s(x)$:

$$\psi_{n,x}(y) = \mathbb{E}_{\mu_x^s}[\mathbf{1}_{\mathcal{P}^{\xi^s}(x)} | \mathcal{F}_{n,x}^s](y) = \mu_{n,y}^s[\mathcal{P}^{\xi^s}(x)].$$

Note that $\psi_{n,x}(y) = \phi_n(y)$ for all n and μ_x^s -almost every $y \in \mathcal{P}^{\xi^s}(x)$ (note that in that case $\mathcal{P}^{\xi^s}(x) = \mathcal{P}^{\xi^s}(y)$).

On the one hand, for μ -almost every $x \in \mathbb{T}^3$, by the increasing martingale theorem (we apply Theorem 4.3 to the probability space $(\xi^s(x), \mu_x^s)$), $\psi_{n,x}$ converges to $\mathbf{1}_{\mathcal{P}^{\xi^s}(x)}$ μ_x^s -almost surely as $n \rightarrow \infty$. In particular,

$$\text{for } \mu_x^s\text{-a.e. } y \in \mathcal{P}^{\xi^s}(x), \quad \psi_{n,x}(y) = \phi_n(y) \rightarrow 1 \text{ as } n \rightarrow \infty. \tag{4.3}$$

Define

$$\mathcal{S} := \{x \in \mathbb{T}^3 : \phi_n(x) \rightarrow 1 \text{ as } n \rightarrow \infty\}.$$

For each $n \in \mathbb{N}$, ϕ_n is measurable, hence \mathcal{S} is a Borel set. Assume that $\mu[\mathcal{S}] = 0$. Since $\mu[\mathcal{S}] = \int \mu_x^s[\mathcal{S}] d\mu(x)$, we would then have $\mu_x^s[\mathcal{S}] = 0$ for μ -a.e. $x \in \mathbb{T}^3$. But (4.3) implies that for μ -a.e. $x \in A$, $\mu_x^s[\mathcal{S}] \geq \mu_x^s[\mathcal{S} \cap \mathcal{P}^{\xi^s}(x)] = \mu_x^s[\mathcal{P}^{\xi^s}(x)] > 0$, and by our assumption that $\mu[A] > 0$, we reach a contradiction. Therefore,

$$\mu[\mathcal{S}] = \mu\{x \in \mathbb{T}^3 : \phi_n(x) \rightarrow 1 \text{ as } n \rightarrow \infty\} > 0.$$

On the other hand, we deduce from (4.1) and from Lemma 2.12 that

$$\phi_n(x) = \mu_{n,x}^s[\mathcal{P}_n^{\xi^s}(x)] = f_*^n \mu_{f^{-n}(x)}^s[f^n \mathcal{P}(f^{-n}(x))] = \phi(f^{-n}(x)).$$

The latter proves that $\phi \circ f^{-k}$ converges to 1 μ -almost surely on \mathcal{S} as $k \rightarrow \infty$. Therefore, by considering the Cesàro averages $\frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^{-k}(x)$ for a μ -generic point $x \in \mathcal{S}$, and by Birkhoff’s theorem, we conclude that $\int_{\mathbb{T}^3} \phi d\mu = 1$. As ϕ takes values in $[0, 1]$, we must have $\phi(x) = 1$ for μ -almost every $x \in \mathbb{T}^3$. ■

Remark 4.4. It is clear that $\alpha^s(x, y) = 0$ is an equivalence relation on stable leaves, so Theorem 4.2 can be generalized as follows. Let f be as in Theorem 4.2. Let \mathcal{R} be a measurable equivalence relation on stable leaves (i.e., $x \mathcal{R} y \Rightarrow y \in \mathcal{W}^s(x)$) such that $x \mathcal{R} y \Rightarrow f^n(x) \mathcal{R} f^n(y)$ for any $n \in \mathbb{N}$. Fix a measurable partition ξ^s subordinate to \mathcal{W}^s , a system $\{\mu_x^s\}_x$ of conditional measures relative to ξ^s , and for $x \in \mathbb{T}^3$, let $\mathcal{P}^{\xi^s}(x)$ be its \mathcal{R} -equivalence class. Then the following dichotomy holds: either

- (1) for μ -almost every $x \in M$, $\mu_x^s[\mathcal{P}^{\xi^s}(x)] = 1$; or
- (2) for μ -almost every $x \in M$, $\mu_x^s[\mathcal{P}^{\xi^s}(x)] = 0$.

5. Joint integrability and the bad set

In this section we start the formal proof of Theorem A. So we let $f \in \mathcal{A}^2(\mathbb{T}^3)$ be an Anosov diffeomorphism, strongly partially hyperbolic with expanding center and C^1 stable bundle. Recall from Lemma 2.6 that this is always satisfied when f is close to a conservative map f_0 . Let μ denote an ergodic u -Gibbs measure, with full support.

We are going to reduce the proof of Theorem A to a more technical version of the result by studying the set of points x for which one sees almost no twist of the bundle E^u by the application of the stable holonomy. Our main technical result says that if the measure of this *bad set* is zero then μ is SRB.

5.1. The bad set

For any $x \in \mathbb{T}^3$, we denote

$$\mathcal{P}(x) := \{y \in \mathcal{W}^s(x) : \alpha^s(x, y) = 0\}, \quad \mathcal{N}(x) := \{y \in \mathcal{W}^s(x) : \alpha^s(x, y) > 0\}.$$

Observe that given any measurable partition ξ^s subordinate to \mathcal{W}^s we have, for μ -a.e. $x \in \mathbb{T}^3$, $\mathcal{P}^{\xi^s}(x) = \mathcal{P}(x) \cap \xi^s(x)$, where $\mathcal{P}^{\xi^s}(x)$ was defined in the proof of Theorem 4.2, our zero-one law: see Section 4.2.

Remark 5.1. Since the unstable bundle E^u is invariant under Df , and holonomy maps are equivariant with the dynamics, i.e.,

$$f \circ H_{x,y}^s = H_{f(x),f(y)}^s \circ f, \tag{5.1}$$

for all x, y we have

$$\alpha^s(x, y) = 0 \iff \forall n \in \mathbb{Z}, \alpha^s(f^n(x), f^n(y)) = 0. \tag{5.2}$$

For any $x \in \mathbb{T}^3$, the function $\alpha^s(x, \cdot)$ on $\mathcal{W}^s(x)$ is continuous. Therefore, the sets $\mathcal{P}(x)$ and $\mathcal{N}(x)$ are respectively closed and open. Moreover, by Remark 5.1,

$$f^n(\mathcal{P}(x)) = \mathcal{P}(f^n(x)), \quad f^n(\mathcal{N}(x)) = \mathcal{N}(f^n(x)), \quad \forall n \in \mathbb{Z}. \tag{5.3}$$

Definition 5.2 (Bad set). Let ξ^s be a measurable partition subordinate to the stable foliation \mathcal{W}^s , and let $\{\mu_x^s\}_{x \in \mathbb{T}^3}$ be a system of conditional measures relative to ξ^s . The bad set $\mathbf{B} = \mathbf{B}(\xi^s, \mu)$ is defined as

$$\mathbf{B} := \{x \in \mathbb{T}^3 : \mu_x^s(\mathcal{N}(x)) = 0\}.$$

Recall that in Section 4.2, we introduced the set $A = \{x \in \mathbb{T}^3 : \mu_x^s[\mathcal{P}^{\xi^s}(x)] > 0\}$. Using our zero-one law, and the results established in Section 4.2, we have the following properties:

- (1) \mathbf{B} is μ -almost everywhere equal to A ;
- (2) \mathbf{B} is measurable and f -invariant, and for all $\ell \in \mathbb{Z}$,

$$\mathbf{B}(\xi^s, \mu) = \mathbf{B}(\xi_\ell^s, \mu), \tag{5.4}$$

where we recall $\xi_\ell^s = f^\ell(\xi^s)$ is still a measurable partition subordinate to \mathcal{W}^s ;

- (3) \mathbf{B} has measure 0 or 1.

A priori the bad set $\mathbf{B} = \mathbf{B}(\xi^s, \mu)$ depends on the particular choice of the measurable partition subordinate to \mathcal{W}^s . Nevertheless, the next lemma shows that the bad set associated to another subordinate partition is equal to \mathbf{B} modulo sets of measure 0.

Lemma 5.3. For every measurable partition η^s subordinate to \mathcal{W}^s we have

$$\mu(\mathbf{B}(\xi^s, \mu)) = \mu(\mathbf{B}(\eta^s, \mu)) \in \{0, 1\}.$$

Proof. For $\ell \in \mathbb{Z}$, set $\xi_\ell^s := f^\ell(\xi^s)$. Then all the partitions ξ_ℓ^s are measurable partitions subordinate to \mathcal{W}^s , and by (5.4) we have $\mathbf{B}(\xi^s, \mu) = \mathbf{B}(\xi_\ell^s, \mu)$ for all $\ell \in \mathbb{Z}$. These sets have measure 0 or 1; assume they have measure 1. Consider a μ -generic point x such that

- $\eta^s(x)$ contains an open neighborhood of x in $\mathcal{W}^s(x)$ so $\mu_x^{\xi_\ell^s}(\eta^s(x)) > 0$ for all $\ell \in \mathbb{Z}$ (see Corollary 2.17);
- $\eta^s(x)$ has diameter less than 1;
- $\mu_x^{\xi_\ell^s}(\mathcal{N}(x)) = 0$ for all $\ell \in \mathbb{Z}$.

By Lemma 2.19 (specifically Remark 2.20) there exists $m < 0$ such that $\eta^s(x) \subset \xi_m^s(x)$ so we can apply the superposition property and

$$\mu_x^{\eta^s}(\mathcal{N}(x)) = \frac{\mu_x^{\xi_m^s}(\mathcal{N}(x) \cap \eta^s(x))}{\mu_x^{\xi_m^s}(\eta^s(x))} \leq \frac{\mu_x^{\xi_m^s}(\mathcal{N}(x))}{\mu_x^{\xi_m^s}(\eta^s(x))} = 0.$$

As a result, $x \in \mathbf{B}(\eta^s, \mu)$. This proves that $\mu(\mathbf{B}(\eta^s, \mu)) = 1$, confirming the second assertion. ■

Although we always need to fix a partition to speak about the bad set, Lemma 5.3 allows us to speak about *the measure of the bad set* without fixing a particular choice of a partition.

Remark 5.4. The conclusion of Lemma 5.3 remains valid even if the measurable partition η^s is only required to have the following properties for μ -a.e. $x \in \mathbb{T}^3$:

- (1) $\eta^s(x) \subset \mathcal{W}^s(x)$;
- (2) $\eta^s(x)$ contains an open neighborhood of $\mathcal{W}^s(x)$ in the internal topology;
- (3) $\eta^s(x)$ has uniformly bounded diameter.

Indeed, the proof of Lemma 5.3 does not use the assumption that η^s is decreasing.

There are many (non-decreasing) measurable partitions satisfying these three properties. For example, consider any finite foliated atlas \mathcal{A} for \mathcal{W}^s composed of open charts U_i such that $\mu(\partial U_i) = 0$. Define a finite partition mod 0 of \mathbb{T}^3 as

$$\mathcal{Q} = \bigvee_i \{U_i, \mathbb{T}^3 \setminus \overline{U_i}\},$$

Each atom $\mathcal{Q}(x)$ is an open set included in a foliated chart for \mathcal{W}^s . As a result it is trivially foliated by stable plaques; this defines a measurable partition of $\mathcal{Q}(x)$. We obtain a refinement of \mathcal{Q} , denoted by η^s , that satisfies the desired properties, and is referred to as *the measurable partition associated to \mathcal{A}* .

5.2. Joint integrability

When the bad set has positive measure, ergodicity implies that for almost every x , we see almost no twist of E^u along $\mathcal{W}^s(x)$. This can be read as an infinitesimal form of joint integrability between E^s and E^u . In this subsection we shall improve this to actual joint integrability.

Proposition 5.5. *Let $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a C^2 Anosov diffeomorphism, strongly partially hyperbolic with expanding center and C^1 stable holonomies. Let μ be a fully supported ergodic f -invariant measure. If the bad set has full measure then E^s and E^u are jointly integrable.*

It is worth pointing out that the proof of this proposition is the only place in our argument towards Theorem A where the full support assumption of the u -Gibbs measure is used. Moreover, in Proposition 5.5 the ergodic invariant measure μ need not be u -Gibbs.

5.2.1. *Local joint integrability.* In the proof of Proposition 5.5 we shall apply a criterion for joint integrability that comes from [11]. See also [44, Section 2.3].

Definition 5.6. Given $x \in \mathbb{T}^3$ we say that the bundles E^s and E^u are *jointly integrable at x* if there exist $\delta, \varepsilon > 0$ such that for all $z \in \mathcal{W}_\delta^s(x)$ and $y \in \mathcal{W}_\delta^u(x)$,

$$\mathcal{W}_\varepsilon^u(z) \cap \mathcal{W}^s(y) \neq \emptyset.$$

The result below follows directly from [11, Lemma 5] (see also [44, Lemma 2.3.7]).

Lemma 5.7. *Let $f \in \mathcal{A}^2(\mathbb{T}^3)$. Assume that E^s and E^u are jointly integrable at each $x \in \mathbb{T}^3$, with uniform constants δ, ε . Then f is jointly integrable (in the sense of Definition 2.1).*

5.2.2. *Proof of Proposition 5.5.* This proposition follows directly from Lemmas 5.8 and 5.9 below.

On the topological level, we have the following result.

Lemma 5.8. *Let $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a C^2 Anosov diffeomorphism, strongly partially hyperbolic with expanding center and C^1 stable holonomies. If for some $x \in \mathbb{T}^3$ the set $\mathcal{N}(x)$ is not open and dense in $\mathcal{W}^s(x)$, then E^s and E^u are jointly integrable.*

Proof. Assume that there exists $x_0 \in \mathbb{T}^3$ such that $\mathcal{P}(x_0)$ contains a non-trivial open interval, i.e., for some $x \in \mathcal{W}^s(x_0)$ and $\varepsilon > 0$, we have $\mathcal{W}_\varepsilon^s(x) \subset \mathcal{P}(x_0)$. Since $\mathcal{P}(x) = \mathcal{P}(x_0)$, we also have $\mathcal{P}(x) \supset \mathcal{W}_\varepsilon^s(x)$, hence by (5.3), for any integer $n \geq 1$,

$$\mathcal{P}(f^{-n}(x)) \supset f^{-n}(\mathcal{W}_\varepsilon^s(x)). \tag{5.5}$$

By compactness, we can take a subsequence $(f^{-n_k}(x))_{k \geq 0}$ such that $f^{-n_k}(x) \rightarrow y$ as $k \rightarrow \infty$, for some point $y \in \mathbb{T}^3$. Since the restriction of f^{-1} to stable leaves is uniformly expanding, we deduce from (5.5) that

$$\mathcal{P}(y) = \mathcal{W}^s(y). \tag{5.6}$$

Let us now show that the same holds for any point, i.e., $\mathcal{P}(z) = \mathcal{W}^s(z)$ for any $z \in \mathbb{T}^3$. Fix $z \in \mathbb{T}^3$ and $z' \in \mathcal{W}^s(z)$. By minimality of the stable foliation, the leaf $\mathcal{W}^s(y)$ is dense in \mathbb{T}^3 , hence there exists a sequence $(y_n)_{n \geq 0} \in (\mathcal{W}^s(y))^\mathbb{N}$ such that $\lim_{n \rightarrow \infty} y_n = z$. For each integer $n \geq 0$, we let $y'_n := H_{z,z'}^s(y_n) \in \mathcal{W}^{cu}(z') \cap \mathcal{W}^s(y_n)$. In particular, we also have $\lim_{n \rightarrow \infty} y'_n = H_{z,z'}^s(\lim_n y_n) = z'$. By continuity of the angle function α^s , we deduce that

$$\alpha^s(z, z') = \lim_{n \rightarrow \infty} \alpha^s(y_n, y'_n) = 0,$$

i.e., $z' \in \mathcal{P}(z)$. Since $z' \in \mathcal{W}^s(z)$ was chosen arbitrarily within $\mathcal{W}^s(z)$, we deduce that $\mathcal{W}^s(z) = \mathcal{P}(z)$ for all $z \in \mathbb{T}^3$.

Now, fix an arbitrary $x \in \mathbb{T}^3$. Let $0 < \delta < \varepsilon$ be chosen so that the stable holonomy map $H_{x,z}^s: \mathcal{W}_\delta^{cu}(x) \rightarrow \mathcal{W}_\varepsilon^{cu}(z)$ is well defined for every $z \in \mathcal{W}_\delta^s(x)$. Notice that for any $y \in \mathcal{W}_\delta^u(x)$, for any $z \in \mathcal{W}_\delta^s(x)$, if we set $y' := H_{x,z}^s(y) \in \mathcal{W}_\varepsilon^s(y)$, then we have

$$\angle(DH_{x,z}^s E^u(y), E^u(y')) = \alpha^s(y, y') = 0.$$

We claim that $\nu(\mathcal{V}_{y_0}) = 0$. To see this, we consider a finite foliated atlas \mathcal{A} for \mathcal{W}^s such that

- (1) $\overline{\mathcal{U}}$ is included in a foliated chart \mathcal{U}' of \mathcal{A} with $\mu(\partial\mathcal{U}') = 0$;
- (2) any other foliated chart of \mathcal{A} is disjoint from \mathcal{U} , and its boundary has measure 0.

To obtain such an atlas, first observe that from our construction, \mathcal{U} can be taken with closure in an open set \mathcal{U}' trivially foliated by \mathcal{W}^s . Next, we cover the compact set $\mathbb{T}^3 \setminus \mathcal{U}'$ by finitely many open balls B_i that trivialize \mathcal{W}^s , have $\mu(\partial B_i) = 0$ and are disjoint from \mathcal{U} . Let η be the measurable partition associated to \mathcal{A} (see Remark 5.4). By construction, the stable plaques $\mathcal{U}(x)$ are contained in atoms of η , and for almost every $x \in T$ we have

$$\eta(x) \cap \mathcal{V}_{y_0} = \mathcal{U}(x) \cap \mathcal{V}_{y_0} \subset \mathcal{N}(x),$$

hence $\nu_x^\eta(\eta(x) \cap \mathcal{V}_{y_0}) \leq \nu_x^\eta(\mathcal{N}(x)) = 0$ for μ -a.e. x . By the definition of conditional measures, this implies that $\nu(\mathcal{V}_{y_0}) = 0$. In other words, any point $y \in \mathcal{N}(x_0)$ has an open neighborhood \mathcal{V}_y such that $\mathbb{T}^3 \setminus \mathcal{V}_y$ has full measure and so

$$\text{supp}(\nu) \subset \mathbb{T}^3 \setminus \bigcup_{y \in \mathcal{N}(x_0)} \mathcal{V}_y.$$

As $\mathcal{N}(x_0)$ is open and dense in $\mathcal{W}^s(x_0)$, and $\mathcal{W}^s(x_0)$ is dense in \mathbb{T}^3 , the set $\bigcup_{y \in \mathcal{N}(x_0)} \mathcal{V}_y$ is open and dense in \mathbb{T}^3 . Thus, $\text{supp}(\nu)$ has empty interior. ■

5.3. Main technical theorem

We shall now show that our main result, Theorem A, can be reduced to a more technical statement involving the bad set.

Theorem 5.10. *Let $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a C^2 Anosov diffeomorphism, strongly partially hyperbolic with expanding center and C^1 stable holonomies. Let μ be an ergodic u -Gibbs measure. If the bad set of μ has zero measure then μ is an SRB measure.*

5.3.1. Proof of Theorem A assuming Theorem 5.10. Recall that by Lemma 2.6 for every $f_0 \in \mathcal{A}_m^2(\mathbb{T}^3)$ there exists a small neighborhood $\mathcal{U}(f_0)$ in $\text{Diff}^2(\mathbb{T}^3)$ such that every $f \in \mathcal{U}(f_0)$ is an Anosov diffeomorphism, strongly partially hyperbolic with expanding center and C^1 stable holonomies. Let $\mathcal{U} := \bigcup_{f_0 \in \mathcal{A}_m^2(\mathbb{T}^3)} \mathcal{U}(f_0)$. Take $f \in \mathcal{U}$. Then f fulfils the assumptions of Theorem 5.10. Assume that E^s and E^u are not jointly integrable and let μ be a fully supported ergodic u -Gibbs measure. Then Lemma 5.9 implies that the bad set has zero measure. By Theorem 5.10 it follows that μ is SRB. ■

It is important to remark that in this main technical result we *do not assume* that our u -Gibbs measure is fully supported. The result says that the existence of positive angles $\alpha^s(x, y)$ for almost every $y \in \mathcal{W}^s(x)$ and almost every $x \in \mathbb{T}^3$ suffices to convert μ into an SRB measure.

6. Normal forms

In this section, our goal is to prove Theorem C. This section is independent of the rest of the paper and we emphasize that we do not require any regularity of holonomies. We begin by revisiting the normal forms along one-dimensional expanding foliations, as developed by Kalinin–Katok [25]. Then, essentially using the fact that E^u is C^1 along the center-unstable manifold, we extend this construction to two dimensions. In the final part of the section, we study the change of coordinates for points on the same leaf of \mathcal{W}^{cu} .

Let $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a C^2 Anosov diffeomorphism with a decomposition $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$, where E^c is uniformly expanded.

Let us recall some notations that will be frequently used in this section.

- Given a point $x \in \mathbb{T}^3$ we will write $\lambda_x^\star := \|Df(x)|_{E^\star}\|$ for $\star = c, u$. Observe that $\|Df^{-1}(x)|_{E^\star}\| = (\lambda_{x^{-1}}^\star)^{-1}$.
- Given $\ell \in \mathbb{Z}$ we write $x_\ell = f^\ell(x)$.
- Given $n \in \mathbb{Z}$ and $\star = c, u$, we write

$$\lambda_x^\star(n) := \|Df^n(x)|_{E^\star}\| = \begin{cases} \prod_{\ell=0}^{n-1} \lambda_{x_\ell}^\star & \text{when } n \geq 0, \\ \prod_{\ell=1}^{-n} \frac{1}{\lambda_{x_{-\ell}}^\star} = \frac{1}{\lambda_{x_n}^\star(-n)} & \text{when } n < 0. \end{cases} \tag{6.1}$$

6.1. One-dimensional normal forms

Given $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^{cu}(x)$, we consider the functions

$$\rho_x^\star(y) := \prod_{\ell=1}^\infty \frac{\lambda_{x_{-\ell}}^\star}{\lambda_{y_{-\ell}}^\star} = \lim_{n \rightarrow \infty} \frac{\lambda_y^\star(-n)}{\lambda_x^\star(-n)} \quad \text{for } \star = c, u. \tag{6.2}$$

It follows readily from the definition that $\rho_y^\star(x) = (\rho_x^\star(y))^{-1}$. Furthermore, for any $z \in \mathcal{W}^{cu}(x) = \mathcal{W}^{cu}(y)$, we have

$$\rho_x^\star(z) = \rho_y^\star(z) \rho_x^\star(y). \tag{6.3}$$

The basic distortion result (Lemma 2.9) implies that $\rho_x^\star: \mathcal{W}^{cu}(x) \rightarrow (0, \infty)$ is a continuous map that depends continuously on the base point (for further details we refer to [25]).

We then define

$$\mathcal{H}_x^\star(y) := \int_x^y \rho_x^\star(\hat{y}) d\hat{y}, \tag{6.4}$$

which gives a C^1 diffeomorphism from $\mathcal{W}^\star(x)$ to the real line. Now, consider the inverse map $\Phi_x^\star = (\mathcal{H}_x^\star)^{-1}$.

Proposition 6.1 (Kalinin–Katok [25, Section 3.1]). *For $\star = c, u$, the family $\{\Phi_x^\star\}_{x \in \mathbb{T}^3}$ of C^1 diffeomorphisms $\Phi_x^\star: \mathbb{R} \rightarrow \mathcal{W}^\star(x)$ satisfies*

- (1) $f \circ \Phi_x^\star(s) = \Phi_{x_1}^\star(\lambda_{x_1}^\star s)$ for all $s \in \mathbb{R}$;
- (2) $\Phi_x^\star(0) = x$;

(3) $D\Phi_x^*(0)e_1 = v^*(x)$, where e_1 denotes the unit tangent vector field on \mathbb{R} .

For any $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^*(x)$, we define the *change of charts* as

$$\mathcal{H}_{x,y}^* := \mathcal{H}_y^* \circ (\mathcal{H}_x^*)^{-1} = (\Phi_y^*)^{-1} \circ \Phi_x^* : \mathbb{R} \rightarrow \mathbb{R}. \tag{6.5}$$

We remark that $\mathcal{H}_{x,y}^*(0) = \mathcal{H}_x^*(x)$.

Lemma 6.2. *For any $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^*(x)$, and for any $s \in \mathbb{R}$,*

$$\mathcal{H}_{x,y}^*(s) - \mathcal{H}_{x,y}^*(0) = \rho_y^*(x) \cdot s. \tag{6.6}$$

In other words, the change of one-dimensional normal form coordinates is affine with derivative $\rho_y^(x)$.*

Proof. Fix $s \in \mathbb{R}$, and let $z := (\mathcal{H}_x^*)^{-1}(s) = (\mathcal{H}_y^*)^{-1}(\mathcal{H}_{x,y}^*(s)) \in \mathcal{W}^*(x) = \mathcal{W}^*(y)$. By the definition of $\mathcal{H}_x^*(x)$ (see (6.4)), after differentiating $\mathcal{H}_{x,y}^* = \mathcal{H}_y^* \circ (\mathcal{H}_x^*)^{-1}$ at s , we get

$$\frac{d\mathcal{H}_{x,y}^*}{ds}(s) = \frac{(\mathcal{H}_y^*)'(z)}{(\mathcal{H}_x^*)'(z)} = \frac{\rho_y^*(z)}{\rho_x^*(z)} = \rho_y^*(x),$$

thus $\mathcal{H}_{x,y}^*(s) - \mathcal{H}_{x,y}^*(0) = \int_0^s \rho_y^*(x) dt = \rho_y^*(x) \cdot s$. ■

6.2. Two-dimensional normal forms

Recall that there are unitary vector fields $\mathbb{T}^3 \ni x \mapsto v^*(x) \in E^*(x)$ such that

$$Df(x)v^*(x) = \lambda_x^* v^*(x_1).$$

Notice that since f is C^2 , the bundle E^u is C^1 inside \mathcal{W}^{cu} and since $\|v^u(x)\| > 0$ the map $x \mapsto \lambda_x^u \in \mathbb{R}$ is C^1 when restricted to center-unstable manifolds. Moreover, for a fixed $R > 0$ the C^1 norm

$$\sup \{ \|D(\lambda^u|_{\mathcal{W}^{cu}(x)})(y)\| : y \in \mathcal{W}_R^{cu}(x), x \in M \}$$

is bounded by some uniform constant $C = C(R, f) > 0$. These results follow from [39].

We now restate Theorem C, which gives a center-unstable version of Proposition 6.1.

Theorem 6.3. *There exists a family of C^1 diffeomorphisms $\Phi_x : \mathbb{R}^2 \rightarrow \mathcal{W}^{cu}(x)$, depending continuously on x , such that*

- (1) $f \circ \Phi_x(t, s) = \Phi_{f(x)}(\lambda_x^u t, \lambda_x^c s)$ for all $(s, t) \in \mathbb{R}^2$;
- (2) $\Phi_x(0, 0) = x$;
- (3) $D\Phi_x(0, 0)e_1 = v^u(x)$ and $D\Phi_x(0, 0)e_2 = v^c(x)$;
- (4) $\Phi_x(\cdot, \cdot)$ depends continuously on x in the C^1 topology;
- (5) for all $s \in \mathbb{R}$, $\Phi_x(\mathbb{R} \times \{s\}) = \mathcal{W}^u(\Phi_x(0, s))$, and $\Phi_x(\{0\} \times \mathbb{R}) = \mathcal{W}^c(x)$ (see Fig. 2).

The drawback of our construction is that the natural generalization of Lemma 6.2 is no longer true, as the remark below demonstrates.

Remark 6.4. Note that in general, we cannot expect the change of normal charts $\mathcal{H}_{x,y} := \Phi_y^{-1} \circ \Phi_x: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ to be affine for any $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^{cu}(x)$.

Suppose it were the case. We claim that it implies that for any $x \in \mathbb{T}^3$, the center foliation \mathcal{W}^c is C^1 within the center-unstable leaf $\mathcal{W}^{cu}(x)$. Indeed, for any $y \in \mathcal{W}^{cu}(x)$, by Theorem 6.3 (5), we have

$$\Phi_x^{-1}(\mathcal{W}^c(y)) = \mathcal{H}_{y,x} \circ \Phi_y^{-1}(\mathcal{W}^c(y)) = \mathcal{H}_{y,x}(\{0\} \times \mathbb{R}),$$

which is a straight line since the map $\mathcal{H}_{y,x}$ is affine. Since two distinct center leaves cannot cross, and $\Phi_x^{-1}(\mathcal{W}^c(x)) = \{0\} \times \mathbb{R}$ is vertical, it follows that $\Phi_x^{-1}(\mathcal{W}^c(y))$ is vertical for any $y \in \mathcal{W}^{cu}(x)$. In particular, since the normal chart Φ_x is C^1 , it does imply that \mathcal{W}^c is C^1 within $\mathcal{W}^{cu}(x)$.

However, regularity of the center foliation \mathcal{W}^c is a rare phenomenon. For instance, in [18, Lemma 3], it is shown that for $r \geq 2$, there exists a C^1 -open and C^r -dense subset $\mathcal{V} \subset \mathcal{A}_m^r(\mathbb{T}^3)$ of the set of C^r conservative Anosov diffeomorphisms on \mathbb{T}^3 considered in the present work such that for $f \in \mathcal{A}_m^r(\mathbb{T}^3)$, the center foliation \mathcal{W}^c is Lipschitz inside the center-unstable foliation \mathcal{W}^{cu} if and only if $f \in \mathcal{A}_m^r(\mathbb{T}^3) \setminus \mathcal{V}$.

We now proceed to the proof of Theorem 6.3.

6.2.1. A foliated chart. A natural way of obtaining a parametrization (t, s) of $\mathcal{W}^{cu}(x)$ is to consider the arc-length parameter t along the curve $\mathcal{W}^u(\Phi_x^c(s))$. As we shall see, it is not hard to prove that this indeed gives a C^1 identification with nice properties (for instance, horizontal lines are mapped onto unstable manifolds). The difficulty in proving Theorem 6.3 is that in these coordinates the map f does not act linearly. For this reason in the forthcoming paragraphs we will perform suitable reparametrizations.

More formally, let $\phi: \mathbb{R} \times \mathcal{W}^{cu}(x) \rightarrow \mathcal{W}^{cu}(x)$ denote the flow of the vector field $x \mapsto v^u(x)$. Notice that $v^u|_{\mathcal{W}^{cu}(x)}$ is C^1 , with its C^1 norm depending continuously on x on compact subsets of $\mathcal{W}^{cu}(x)$. With this flow at hand we define a map $\Gamma_x: \mathbb{R}^2 \rightarrow \mathcal{W}^{cu}(x)$ by

$$\Gamma_x(t, s) := \phi^t(\Phi_x^c(s)), \quad \forall (t, s) \in \mathbb{R}^2,$$

where $\{\Phi_x^c\}_{x \in \mathbb{T}^3}$ are Kalinin–Katok’s normal forms in \mathcal{W}^c (see Theorem 6.3).

Lemma 6.5. *For each $x \in \mathbb{T}^3$, the map Γ_x is a C^1 diffeomorphism whose C^1 norm depends continuously on x .*

Proof. Let us show that this defines a C^1 diffeomorphism. Indeed, Γ_x is injective and surjective due to Lemmas 2.3 and 2.4 respectively. Therefore it suffices to prove that Γ_x is C^1 with invertible derivative at each point.

Choosing an arbitrary coordinate system locally in $\mathcal{W}^{cu}(x)$ we can think of Γ_x as a map from \mathbb{R}^2 to \mathbb{R}^2 . Now, observe that the Jacobian matrix of the map $(t, s) \mapsto \phi^t(\Phi_x^c(s))$ is the 2×2 matrix whose columns are the vectors $v^u(\phi^t(\Phi_x^c(s)))$ and $D\phi^t(\Phi_x^c(s)) \frac{d}{ds} \Phi_x^c(s)$. Since ϕ^t is a C^1 flow and Φ_x^c is a C^1 curve, this proves that Γ_x has

continuous derivative. Moreover, since $v^u(\Phi_x^c(s))$ and $\frac{d}{ds}\Phi_x^c(s)$ are transverse and since $D\phi^t(\Phi_x^c(s))$ is a linear isomorphism, the vectors

$$v^u(\phi^t(\Phi_x^c(s))) = D\phi^t(\Phi_x^c(s))v^u(\Phi_x^c(s)) \quad \text{and} \quad D\phi^t(\Phi_x^c(s))\frac{d}{ds}\Phi_x^c(s)$$

are also transverse. Thus the Jacobian matrix of Γ_x is invertible, proving the assertion. The continuous dependence of the C^1 norm follows from that of v^u and Φ^c . ■

6.2.2. Preliminary construction. We shall start the construction of the map Φ with a slight modification of the construction of the foliated chart Γ_x . Instead of using the arc-length parameter along unstable manifolds, we shall use normal forms. This choice will allow us to perform the required reparametrization for linearizing the action of $f|_{\mathcal{W}^{cu}(x)}$.

Lemma 6.6. *The map $\Psi_x: \mathbb{R}^2 \rightarrow \mathcal{W}^{cu}(x)$ given by $\Psi_x(t, s) := \Phi_{\Phi_x^c(s)}^u(t)$ is a C^1 diffeomorphism, with C^1 norm depending continuously on x .*

Proof. Notice that Ψ_x is bijective because we can define the inverse map directly. Indeed, given $y \in \mathcal{W}^{cu}(x)$ we consider the point $y^c \in \mathcal{W}^c(x)$, whose existence is ensured by Lemmas 2.3 and 2.4, such that

$$\{y^c\} = \mathcal{W}^u(y) \cap \mathcal{W}^c(x).$$

Define then a map $\Xi_x: \mathcal{W}^{cu}(x) \rightarrow \mathbb{R}^2$ by

$$\Xi_x(y) := (\mathcal{H}_{y^c}^u(y), \mathcal{H}_x^c(y^c)).$$

From the definitions we have

$$\Psi_x \circ \Xi_x(y) = \Phi_{\Phi_x^c(\mathcal{H}_x^c(y^c))}^u(\mathcal{H}_{y^c}^u(y)) = \Phi_{y^c}^u(\mathcal{H}_{y^c}^u(y)) = y.$$

In a similar way one sees that $\Xi_x \circ \Psi_x(t, s) = (t, s)$. It suffices then to check that Ξ_x is C^1 with invertible derivative at each point.

To prove that Ξ_x is a C^1 diffeomorphism it suffices to establish that $\Xi_x \circ \Gamma_x: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a C^1 diffeomorphism. To compute the derivative of this map fix a point $y = \Gamma_x(t, s) \in \mathcal{W}^{cu}(x)$. Then, since Γ_x is a foliated chart, we have $y^c = \Gamma_x(0, s)$. Thus,

$$\Xi_x \circ \Gamma_x(t, s) = (\mathcal{H}_{\Gamma_x(0,s)}^u(\Gamma_x(t, s)), \mathcal{H}_x^c(\Gamma_x(0, s))).$$

Note that $(t, s) \mapsto \mathcal{H}_x^c(\Gamma_x(0, s)) = s$ is a C^1 function. Moreover, as

$$\mathcal{H}_{\Gamma_x(0,s)}^u(\Gamma_x(t, s)) = \int_{\Gamma_x(0,s)}^{\Gamma_x(t,s)} \rho_{\Gamma_x(0,s)}(\Gamma_x(r, s)) \, dr,$$

the Leibniz rule implies that $\Xi_x \circ \Gamma_x$ is a C^1 map provided we prove that $(t, s) \mapsto \rho_{\Gamma_x(0,s)}(\Gamma_x(t, s)) \in (0, \infty)$ is C^1 . To verify this assertion, recall that

$$\rho_{\Gamma_x(0,s)}^u(\Gamma_x(t, s)) = \prod_{\ell=0}^{\infty} \frac{\|Df^{-1}(f^{-\ell}(\Gamma_x(t, s)))\|_{E^u}}{\|Df^{-1}(f^{-\ell}(\Gamma_x(0, s)))\|_{E^u}}.$$

Consider the auxiliary function

$$g(t, s) = \log \rho_{\Gamma_x(0,s)}^u(\Gamma_x(t, s)).$$

Then we can write

$$g(t, s) = \sum_{\ell=0}^{\infty} (\log \lambda_{f^{-\ell-1}(\Gamma_x(0,s))}^u - \log \lambda_{f^{-\ell-1}(\Gamma_x(t,s))}^u).$$

First notice that as $\lambda_{(\cdot)}^u: \mathcal{W}^{cu} \rightarrow \mathbb{R}$ is C^1 and since $d(f^{-\ell}(\Gamma_x(0, s)), f^{-\ell}(\Gamma_x(t, s))) \rightarrow 0$ exponentially fast (with uniform rate, depending only on f), we deduce that g is the uniform limit on compact sets of the sequence of partial sums. Consider the function $g_\ell(t, s) = \log \lambda_{f^{-\ell-1}(\Gamma_x(t,s))}^u$. By the chain rule,

$$Dg_\ell(t, s) = \frac{D\lambda_{f^{-\ell-1}(\Gamma_x(t,s))}^u Df^{-\ell-1}(\Gamma_x(t, s)) D\Gamma_x(t, s)}{\lambda_{f^{-\ell-1}(\Gamma_x(t,s))}^u}.$$

Therefore, for $(t, s) \in B_R(0)$ there is a uniform constant $C = C(R, f) > 0$ such that

$$\|Dg_\ell(t, s)\| \leq C \|Df^{-\ell-1}(\Gamma_x(t, s))|_{E^{cu}}\|.$$

The right-hand side above has a uniform bound decreasing exponentially as ℓ increases, because $\|Df^{-\ell}(x)|_{E^{cu}}\| \leq e^{-\chi_2^c \ell}$ for every $x \in \mathbb{T}^3$ with $\chi_2^c > 0$ (recall our notations from Section 2.1.3). Thus,

$$\|Dg_\ell(t, s)\| \leq C e^{-\chi_2^c \ell}.$$

An analogous estimate holds for the function $\tilde{g}_\ell(t, s) = \log \lambda_{f^{-\ell-1}(\Gamma_x(0,s))}^u$. This proves that the derivative of the truncated series defining g also converges uniformly on compact sets. By elementary calculus (see for instance [1, Proposition 1.41]) this proves that g is C^1 , and therefore we conclude that $(t, s) \mapsto \rho_{\Gamma_x(0,s)}^u(\Gamma_x(t, s)) \in (0, \infty)$ is C^1 , as desired. To complete the proof, observe that the derivative of $\Xi_x \circ \Gamma_x$ is an upper triangular matrix with non-zero entries on the diagonal. By the inverse function theorem, as $\Xi_x \circ \Gamma_x$ is bijective, this proves that this map is in fact a C^1 diffeomorphism. ■

6.2.3. A reparametrization function. Here again, the diffeomorphism Ψ_x constructed in Lemma 6.6 does not satisfy all the requirements we need. Indeed, to obtain condition (1) of Theorem 6.3 it is necessary to perform a suitable reparametrization. The design of such a map is the content of the next lemma.

Lemma 6.7. *For each $x \in \mathbb{T}^3$, there exists a C^1 function $\beta_x: \mathbb{R} \rightarrow \mathbb{R}^+$ such that*

$$Df(\Phi_x^c(s)) \cdot \beta_x(s) v^u(\Phi_x^c(s)) = \lambda_x^u \beta_{x_1}(\lambda_{x_1}^c s) v^u(\Phi_{x_1}^c(\lambda_{x_1}^c s)). \tag{6.7}$$

Proof. Fix $x \in \mathbb{T}^3$. Consider the sequence of functions $\mathbb{R} \ni s \mapsto \psi_{-\ell}^x(s) \in \mathcal{W}^c(x_{-\ell})$ where

$$\psi_{-\ell}^x(s) := \Phi_{x_{-\ell}}^c((\lambda_{x_{-\ell}}^c \cdots \lambda_{x_{-1}}^c)^{-1} s).$$

For coherence of notation we also make the convention that $\psi_0^x(s) = \Phi_x^c(s)$.

Remark 6.8. Note that by the properties of Kalinin–Katok’s normal forms, if $y = \psi_0^x(s) = \Phi_x^c(s)$ for some $s \in \mathbb{R}$ and $\ell \in \mathbb{N}$, then

$$f^{-\ell}(y) = \psi_{-\ell}^x(s).$$

Using this sequence we define $h_{n,x}: \mathbb{R} \rightarrow \mathbb{R}$ by

$$h_{n,x}(s) := \sum_{\ell=1}^n (\log \lambda_{\psi_{-\ell}^x(s)}^u - \log \lambda_{x_{-\ell}}^u).$$

Notice that $y \mapsto \lambda_y^u$ is uniformly bounded from above and from below, and the lower bound is larger than 1. Thus

$$|\log \lambda_{\psi_{-\ell}^x(s)}^u - \log \lambda_{x_{-\ell}}^u| \leq |\lambda_{\psi_{-\ell}^x(s)}^u - \lambda_{x_{-\ell}}^u|.$$

Now, as $y \mapsto \lambda_y^c$ is also uniformly bounded with a lower bound larger than 1, since the family $\{\Phi_x^c\}$ is continuous in the C^1 topology and $\Phi_{x_{-\ell}}^c(0) = x_{-\ell}$, it follows that $d(\psi_{-\ell}^x(s), x_{-\ell}) \rightarrow 0$ exponentially fast with uniform constants (keep in mind Remark 6.8). Moreover, as $y \mapsto \lambda_y^u$ is uniformly C^1 in \mathcal{W}^{cu} , there exists some constant $C > 0$ such that

$$|\lambda_{\psi_{-\ell}^x(s)}^u - \lambda_{x_{-\ell}}^u| \leq Cd(\psi_{-\ell}^x(s), x_{-\ell}).$$

This proves that $h_{n,x}$ converges uniformly to a continuous function h_x . We define $\beta_x(s) := e^{h_x(s)}$. Observe that with notation (6.2), we can write

$$\beta_x(s) = \prod_{\ell=1}^{\infty} \frac{\lambda_{\psi_{-\ell}^x(s)}^u}{\lambda_{x_{-\ell}}^u} = \rho_{\Phi_x^c(s)}^u(x) = \frac{1}{\rho_x^u(\Phi_x^c(s))}. \tag{6.8}$$

We claim that $\lambda_{\Phi_x^c(s)}^u \beta_x(s) = \lambda_x^u \beta_{x_1}(\lambda_x^c s)$. This formula immediately gives (6.7). Indeed, by the properties of Kalinin–Katok’s normal forms,

$$Df(\Phi_x^c(s)) \cdot v^u(\Phi_x^c(s)) = \lambda_{\Phi_x^c(s)}^u v^u(f(\Phi_x^c(s))) = \lambda_{\Phi_x^c(s)}^u v^u(\Phi_{x_1}^c(\lambda_x^c s)),$$

and combining this equality with the claim provides (6.7).

To prove the claim we first remark that

$$\psi_{-\ell}^{x_1}(\lambda_x^c s) = \Phi_{x_{-\ell+1}}^c((\lambda_{x_{-\ell+1}}^c \cdots \lambda_{x_{-1}}^c \lambda_x^c)^{-1} \lambda_x^c s) = \psi_{-\ell+1}^x(s).$$

We deduce that

$$\beta_{x_1}(\lambda_x^c s) = \prod_{\ell=1}^{\infty} \frac{\lambda_{\psi_{-\ell}^{x_1}(\lambda_x^c s)}^u}{\lambda_{x_{-\ell+1}}^u} = \prod_{\ell=1}^{\infty} \frac{\lambda_{\psi_{-\ell+1}^x(s)}^u}{\lambda_{x_{-\ell+1}}^u} = \frac{\lambda_{\Phi_x^c(s)}^u}{\lambda_x^u} \beta_x(s),$$

proving our claim.

To finish the lemma it remains to show that β_x is a C^1 function, depending continuously (with respect to the local C^1 topology) on x . With that goal in mind we compute

$$\frac{d}{ds} \psi_{-\ell}^x(s) = (\lambda_{x_{-\ell}}^c \cdots \lambda_{x_{-1}}^c)^{-1} D\Phi_{x_{-\ell}}^c((\lambda_{x_{-\ell}}^c \cdots \lambda_{x_{-1}}^c)^{-1} s) e_1.$$

Applying the chain rule we obtain

$$\begin{aligned} \frac{d}{ds} \lambda_{\psi_{-\ell}^x}^u &= D(\lambda_{\psi_{-\ell}^x}^u) \frac{d}{ds} \psi_{-\ell}^x(s) \\ &= (\lambda_{x_{-\ell}}^c \cdots \lambda_{x_{-1}}^c)^{-1} D(\lambda_{\psi_{-\ell}^x}^u) D\Phi_{x_{-\ell}}^c ((\lambda_{x_{-\ell}}^c \cdots \lambda_{x_{-1}}^c)^{-1} s) e_1, \end{aligned}$$

where $D(\lambda_y^u)$ denotes the derivative of the function $y \mapsto \lambda_y^u$ at y . As $(\lambda_{x_{-\ell}}^c \cdots \lambda_{x_{-1}}^c)^{-1} \rightarrow 0$ exponentially fast with uniform constants, the left-hand side above also vanishes exponentially fast because λ_y^u is C^1 and $x \mapsto \max_{|s| \leq 1} \|D\Phi_x^c(s)\|$ is uniformly bounded. As a consequence, the sequence

$$h'_{n,x}(s) = \sum_{\ell=1}^n \frac{\frac{d}{ds} \lambda_{\psi_{-\ell}^x}^u}{\lambda_{\psi_{-\ell}^x}^u}$$

is uniformly bounded by a convergent geometric series, because λ_y^u is lower bounded by a constant larger than 1. Since $h_{n,x} \rightarrow h_x$ uniformly on compact sets, by elementary calculus (see for instance [1, Proposition 1.41]) this proves that h_x is C^1 with $h'_x = \lim_{n \rightarrow \infty} h'_{n,x}$. Moreover, the uniform estimates obtained show that the C^1 norm of f varies continuously with x . ■

6.2.4. *Defining the normal form.* With Lemmas 6.6 and 6.7 at hand we are in a position to define our two-dimensional parametrization of center-unstable manifolds $\mathcal{W}^{cu}(x)$. For each $x \in \mathbb{T}^3$ we consider

$$\Phi_x: \mathbb{R}^2 \rightarrow \mathcal{W}^{cu}(x), \quad (t, s) \mapsto \Psi_x(\beta_x(s)t, s) = \Phi_{\Phi_x^c(s)}^u(\beta_x(s)t).$$

Since by Lemma 6.7, $s \mapsto \beta_x(s)$ is a C^1 positive function, it follows that $(t, s) \mapsto (\beta_x(s)t, s)$ is a C^1 diffeomorphism of \mathbb{R}^2 . Therefore, Lemma 6.6 shows that Φ_x is a C^1 diffeomorphism. We denote $\mathcal{H}_x := \Phi_x^{-1}$.

Proof of Theorem 6.3. Properties (2)-(4) are automatic from the construction of Φ_x . To prove (1) we apply Proposition 6.1 to obtain

$$f(\Phi_x(t, s)) = f(\Phi_{\Phi_x^c(s)}^u(\beta_x(s)t)) = \Phi_{f \circ \Phi_x^c(s)}^u(\lambda_{\Phi_x^c(s)}^u \beta_x(s)t).$$

Now, equality (6.7) implies that

$$\beta_x(s) \lambda_{\Phi_x^c(s)}^u v^u(f \circ \Phi_x^c(x)) = \lambda_x^u \beta_{x_1}(\lambda_x^c s) v^u(\Phi_{x_1}^c(\lambda_x^c s)).$$

Proposition 6.1 applied with Φ_x^c shows that $f \circ \Phi_x^c(s) = \Phi_{x_1}^c(\lambda_x^c s)$, where $x_1 = f(x)$. Combining we get $\beta_x(s) \lambda_{\Phi_x^c(s)}^u = \lambda_x^u \beta_{x_1}(\lambda_x^c s)$. One then deduces

$$f(\Phi_x(t, s)) = \Phi_{\Phi_{x_1}^c(\lambda_x^c s)}^u(\lambda_x^u \beta_{x_1}(\lambda_x^c s)t).$$

The very definition of Φ_x says that

$$\Phi_{\Phi_{x_1}^c(\lambda_x^c s)}^u(\lambda_x^u \beta_{x_1}(\lambda_x^c s)t) = \Phi_{x_1}(\lambda_x^u t, \lambda_x^c s). \quad \blacksquare$$

Remark 6.9. It follows readily from our construction that the inverse map $\mathcal{H}_x: \mathcal{W}^{cu}(x) \rightarrow \mathbb{R}^2$ sends each unstable manifold $\mathcal{W}^u(y)$ for $y \in \mathcal{W}^c(x)$ onto the horizontal line $\{s = \mathcal{H}_x^c(y)\}$.

6.3. *Change of normal form coordinates*

As in (6.5) and Remark 6.4, for any $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^{cu}(x)$, we define the change of charts

$$\mathcal{H}_{x,y} := \mathcal{H}_y \circ \mathcal{H}_x^{-1} = \Phi_y^{-1} \circ \Phi_x: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (t, s) \mapsto (h_{x,y}^1(t, s), h_{x,y}^2(t, s)). \quad (6.9)$$

The function $h_{x,y}^2$ only depends on the second coordinate, as demonstrated by the following lemma.

Lemma 6.10. *For any $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^{cu}(x)$, the diffeomorphism $\mathcal{H}_{x,y}$ preserves the foliation of \mathbb{R}^2 into horizontal lines $\{s = \text{const}\}$.*

Proof. This is a straightforward consequence of the fact that normal forms send horizontal lines to unstable manifolds (see Remark 6.9). More precisely, given $s \in \mathbb{R}$, using Theorem 6.3 (5) we have

$$\mathcal{W}^u(\Phi_x^c(s)) = \Phi_x(\mathbb{R} \times \{s\}).$$

Now define $\hat{y} = \mathcal{W}^c(y) \cap \mathcal{W}^u(\Phi_x^c(s))$ (recall Lemma 2.4). Applying Theorem 6.3 (5) once more we obtain

$$\mathcal{H}_y(\mathcal{W}^u(\hat{y})) = \mathbb{R} \times \{\mathcal{H}_y^c(\hat{y})\}.$$

Since $\mathcal{W}^u(\hat{y}) = \mathcal{W}^u(\Phi_x^c(s))$, this shows that

$$\mathcal{H}_{x,y}(\mathbb{R} \times \{s\}) = \mathbb{R} \times \{\mathcal{H}_y^c(\hat{y})\},$$

which concludes the proof. ■

Lemma 6.11. *The diffeomorphism $\mathcal{H}_{x,y}(\cdot, \cdot)$ depends continuously in the C^1 topology on the pair (x, y) , $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^{cu}(x)$. Moreover, for any $\bar{x} \in \mathbb{T}^3$ and $(\bar{t}, \bar{s}) \in \mathbb{R}^2$,*

- (1) $\mathcal{H}_{x,y}(t, s) \rightarrow (\bar{t}, \bar{s})$ as $(x, y) \rightarrow (\bar{x}, \bar{x})$ and $(t, s) \rightarrow (\bar{t}, \bar{s})$;
- (2) $D\mathcal{H}_{x,y}(t, s) \rightarrow \text{Id}_{\mathbb{R}^2}$ as $(x, y) \rightarrow (\bar{x}, \bar{x})$ and $(t, s) \rightarrow (\bar{t}, \bar{s})$.

Proof. This follows immediately from Theorem 6.3 (4), since $\mathcal{H}_{x,y} = \mathcal{H}_y \circ \mathcal{H}_x^{-1}$ and $D\mathcal{H}_{x,y}(t, s) = D\mathcal{H}_y(\mathcal{H}_x^{-1}(t, s)) \circ D\mathcal{H}_x^{-1}(t, s)$ with

$$D\mathcal{H}_x(t, s) \rightarrow D\mathcal{H}_{\bar{x}}(\bar{t}, \bar{s}) \quad \text{as } x \rightarrow \bar{x} \text{ and } (t, s) \rightarrow (\bar{t}, \bar{s}),$$

and

$$D\mathcal{H}_y(\mathcal{H}_x^{-1}(t, s)) \rightarrow D\mathcal{H}_{\bar{y}}(\mathcal{H}_{\bar{x}}^{-1}(\bar{t}, \bar{s})) = (D\mathcal{H}_{\bar{x}}^{-1}(\bar{t}, \bar{s}))^{-1} \quad \text{as } (x, y) \rightarrow (\bar{x}, \bar{x}) \text{ and } (t, s) \rightarrow (\bar{t}, \bar{s}). \quad \blacksquare$$

6.3.1. *Change of normal charts for two points in the same center leaf.* We first study the change of normal charts between two points of the same center leaf.

Lemma 6.12. *Let $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^c(x)$. For any $(t, s) \in \mathbb{R}^2$,*

- (1) $h_{x,y}^1(t, s) = \rho_y^u(x)t$;
- (2) $h_{x,y}^2(s) = \mathcal{H}_{x,y}^c(s) = \rho_y^c(x)s + \mathcal{H}_y^c(x)$.

Proof. We start by proving (2). Consider the function $h_{x,y}^1$. As x and y belong to the same center leaf, for any $s \in \mathbb{R}$ we have $h_{x,y}^1(0, s) = 0$ and

$$\Phi_x^c(s) = \Phi_x(0, s) = \Phi_y(h_{x,y}^1(0, s), h_{x,y}^2(s)) = \Phi_y(0, h_{x,y}^2(s)) = \Phi_y^c(h_{x,y}^2(s)),$$

hence by (6.6),

$$h_{x,y}^2(s) = (\Phi_y^c)^{-1} \circ \Phi_x^c(s) = \mathcal{H}_{x,y}^c(s).$$

The second claimed equality follows from Lemma 6.2.

Now, let us prove (1). Fix $(t, s) \in \mathbb{R}^2$, and denote $t' := h_{x,y}^1(t, s)$ and $s' := h_{x,y}^2(s)$ for simplicity. By definition, and as $\Phi_y^c(h_{x,y}^2(s)) = \Phi_x^c(s)$, we obtain

$$\Phi_{\Phi_x^c(s)}^u(\beta_x(s)t) = \Phi_x(t, s) = \Phi_y(t', s') = \Phi_{\Phi_y^c(s')}^u(\beta_y(s')t'),$$

hence

$$\beta_x(s)t = \beta_y(s')t'. \tag{6.10}$$

On the other hand, by (6.8) we have

$$\begin{aligned} \beta_y(s') &= \rho_{\Phi_y^c(s')}^u(y) = \rho_{\Phi_x^c(s)}^u(y) \\ &= \rho_{\Phi_x^c(s)}^u(x)\rho_x^u(y) = \beta_x(s)\rho_x^u(y), \end{aligned} \tag{6.11}$$

where in the second line we have applied (6.3). Combining this with (6.10) we deduce $t' = \rho_y^u(x)t$. ■

6.3.2. *Change of normal forms for two points in the same unstable leaf.* Let us now study the change of normal charts $\mathcal{H}_{x,x'}: (t, s) \mapsto (h_{x,x'}^1(t, s), h_{x,x'}^2(s))$ between two points x, x' of the same unstable leaf. Unlike the previous case, here we do not get an affine map.

Lemma 6.13. *Let $x \in \mathbb{T}^3$ and $x' \in \mathcal{W}^u(x)$. Then there exists a C^1 function $a_{x,x'}: \mathbb{R} \rightarrow \mathbb{R}$ such that for any $(t, s) \in \mathbb{R}^2$,*

$$h_{x,x'}^1(t, s) = \rho_{x'}^u(x)t + a_{x,x'}(s).$$

Proof. Consider $(t, s) \in \mathbb{R}^2$, and let us abbreviate $t' := h_{x,x'}^1(t, s)$, $s' := h_{x,x'}^2(s)$, $y = y(s) := \Phi_x^c(s)$, $y' = y'(s) := \Phi_{x'}^c(s')$, and $z = z(t, s) := \Phi_x(t, s) = \Phi_{x'}(t', s')$. Recall that $z = \Phi_x(t, s) = \Phi_y^u(\beta_x(s)t)$. By (6.4), we thus obtain

$$\beta_x(s)t = \int_y^z \rho_y^u(\hat{y}) d\hat{y}. \tag{6.12}$$

Similarly, $z = \Phi_{x'}(t', s') = \Phi_{y'}^u(\beta_{x'}(s')t')$, and

$$\beta_{x'}(s')t' = \int_{y'}^z \rho_{y'}^u(\hat{y}) d\hat{y}. \quad (6.13)$$

Observe that $y' \in \mathcal{W}^u(y)$. By (6.3), for any point $\hat{y} \in \mathcal{W}^u(y) = \mathcal{W}^u(y')$, we have $\rho_{y'}^u(\hat{y}) = \rho_y^u(\hat{y})\rho_{y'}^u(y)$. Hence,

$$\begin{aligned} \beta_{x'}(s')t' &= \int_{y'}^z \rho_{y'}^u(\hat{y}) d\hat{y} = \int_{y'}^z \rho_{y'}^u(y)\rho_y^u(\hat{y}) d\hat{y} \\ &= \rho_{y'}^u(y) \left(\int_{y'}^z \rho_y^u(\hat{y}) d\hat{y} \right) = \rho_{y'}^u(y) \left(\int_{y'}^y \rho_y^u(\hat{y}) d\hat{y} + \int_y^z \rho_y^u(\hat{y}) d\hat{y} \right) \\ &= \rho_{y'}^u(y)\beta_x(s)t + \rho_{y'}^u(y) \int_{y'}^y \rho_y^u(\hat{y}) d\hat{y}, \end{aligned}$$

where in the last equality we have used (6.12). We conclude that

$$t' = \rho_{y'}(y) \frac{\beta_x(s)}{\beta_{x'}(s')} \cdot t + \frac{\rho_{y'}^u(y)}{\beta_{x'}(s')} \int_{y'}^y \rho_y^u(\hat{y}) d\hat{y}.$$

Set $a_{x,x'}(s) := \frac{\rho_{y'}^u(y)}{\beta_{x'}(s')} \int_{y'}^y \rho_y^u(\hat{y}) d\hat{y}$. This is the translation part of the change of coordinates. Recall that

$$\beta_x(s) = \rho_{\Phi_x^u(s)}^u(x) = \rho_y^u(x).$$

Similarly,

$$\beta_{x'}(s') = \rho_{\Phi_{x'}^u(s')}^u(x') = \rho_{y'}^u(x') = (\rho_{x'}^u(y'))^{-1}.$$

Therefore, by (6.3),

$$\rho_{y'}^u(y) \frac{\beta_x(s)}{\beta_{x'}(s')} = \rho_y^u(x)\rho_{y'}^u(y)\rho_{x'}^u(y') = \rho_y^u(x)\rho_{x'}^u(y) = \rho_{x'}^u(x).$$

We conclude that $t' = \rho_{x'}^u(x) \cdot t + a_{x,x'}(s)$. Moreover,

$$\begin{aligned} a_{x,x'}(s) &:= \frac{\rho_{y'}^u(y)}{\beta_{x'}(s')} \int_{y'}^y \rho_y^u(\hat{y}) d\hat{y} = \frac{\rho_{x'}^u(x)}{\beta_x(s)} \int_{y'}^y \rho_y^u(\hat{y}) d\hat{y} \\ &= \frac{\rho_{x'}^u(x)}{\rho_y^u(x)} \int_{y'}^y \rho_y^u(\hat{y}) d\hat{y} = \rho_{x'}^u(y) \int_{y'}^y \rho_y^u(\hat{y}) d\hat{y} = \int_{y'}^y \rho_y^u(\hat{y})\rho_{x'}^u(y) d\hat{y} \\ &= \int_{\mathcal{W}^u(y',y)} \rho_{x'}^u(\hat{y}) d\hat{y}, \end{aligned}$$

where in the last integral, \hat{y} ranges over the segment $\mathcal{W}^u(y', y) \subset \mathcal{W}^u(y)$ of the unstable manifold connecting the points y' and y . ■

Let us now study the map $h_{x,x'}^2: \mathbb{R} \rightarrow \mathbb{R}$. The first step is to characterize it in terms of unstable holonomies.

Lemma 6.14. Fix $x \in \mathbb{T}^3$ and $x' \in \mathcal{W}^u(x)$ and let $H_{x,x'}^u : \mathcal{W}^c(x) \rightarrow \mathcal{W}^c(x')$ denote the unstable holonomy map. Define the map $L_{x,x'}^u : \mathbb{R} \rightarrow \mathbb{R}$ by

$$L_{x,x'}^u(s) := \mathcal{H}_{x'}^c \circ H_{x,x'}^u \circ \Phi_x^c(s).$$

Then $h_{x,x'}^2(s) = L_{x,x'}^u(s)$.

Proof. Consider an $s \in \mathbb{R}$ and define $y = \Phi_x^c(s)$ and $y' = H_{x,x'}^u(y)$. Set $s' := \mathcal{H}_{x'}^c(y')$. Notice that, by definition, $s' = L_{x,x'}^u(s)$. By Remark 6.9 we know that $\Phi_x(\mathbb{R} \times \{s\}) = \mathcal{W}^u(y)$, and similarly $\Phi_{x'}$ sends $\mathbb{R} \times \{s'\}$ onto $\mathcal{W}^u(y')$. Since $\mathcal{W}^u(y) = \mathcal{W}^u(y')$, this shows that

$$\mathcal{H}_{x'} \circ \Phi_x(\mathbb{R} \times \{s\}) = \mathbb{R} \times \{s'\},$$

and therefore $h_{x,x'}^2(s) = s'$. ■

We now prove that unstable holonomies, when conjugated by normal forms (as in the previous lemma), become linear maps of the real line.

Lemma 6.15. Fix $x \in \mathbb{T}^3$ and $x' \in \mathcal{W}^u(x)$. Then $L_{x,x'}^u(s) = \rho_{x'}^c(x)s$.

Proof. For ease of notation, set $L(s) = L_{x,x'}^u(s)$. By definition,

$$\Phi_{x'}^c \circ L(s) = H_{x,x'}^u \circ \Phi_x^c(s).$$

Now, fix $k \in \mathbb{Z}$ and define $g_k : \mathbb{R} \rightarrow \mathbb{R}$ by $s \mapsto \lambda_x^c(k) \cdot s$ (recall (6.1)). Similarly define $g'_k : \mathbb{R} \rightarrow \mathbb{R}$ by $s \mapsto \lambda_{x'}^c(k) \cdot s$. By applying f^k to both sides of the above equation and using the equivariance properties of normal forms and holonomy maps, we obtain

$$\Phi_{x'_k}^c(g'_k \circ L(s)) = H_{x_k,x'_k}^u \circ \Phi_{x_k}^c(g_k(s)).$$

Taking derivatives with respect to $s \in \mathbb{R}$ on both sides we get

$$D\Phi_{x'_k}^c(g'_k(s))\lambda_{x'}^c(k) \frac{dL}{ds}(s) = DH_{x_k,x'_k}^u(\Phi_{x_k}^c(g_k(s)))D\Phi_{x_k}^c(g_k(s))\lambda_x^c(k). \quad (6.14)$$

Recall that $\det D\Phi_{x_k}^c(g_k(s)) = \frac{1}{\rho_{x_k}^c(\Phi_{x_k}^c(g_k(s)))}$, and $\det D\Phi_{x'_k}^c(g'_k(s)) = \frac{1}{\rho_{x'_k}^c(\Phi_{x'_k}^c(g'_k(s)))}$.

Passing to the determinant in (6.14), we deduce that

$$\frac{dL}{ds}(s) = \frac{\rho_{x'_k}^c(\Phi_{x'_k}^c(g'_k(s)))}{\rho_{x_k}^c(\Phi_{x_k}^c(g_k(s)))} \cdot \frac{\lambda_x^c(k)}{\lambda_{x'}^c(k)} \cdot \det DH_{x_k,x'_k}^u(\Phi_{x_k}^c(g_k(s))). \quad (6.15)$$

We now let $k \rightarrow -\infty$. By compactness, up to considering a subsequence, we can assume that $x_k, x'_k \rightarrow q \in \mathbb{T}^3$ as $k \rightarrow -\infty$. Moreover, $g_k(s), g'_k(s) \rightarrow 0$ as $k \rightarrow -\infty$, and by the properties of $\Phi_{(\cdot)}^c(\cdot)$ stated in Proposition 6.1, we have $\Phi_{x_k}^c(g_k(s)), \Phi_{x'_k}^c(g'_k(s)) \rightarrow \Phi_q^c(0) = q$ as $k \rightarrow -\infty$. By continuity of $\rho_{(\cdot)}^c(\cdot)$, we deduce that

$$\lim_{k \rightarrow -\infty} \frac{\rho_{x'_k}^c(\Phi_{x'_k}^c(g'_k(s)))}{\rho_{x_k}^c(\Phi_{x_k}^c(g_k(s)))} = \frac{\rho_q^c(q)}{\rho_q^c(q)} = 1.$$

Since the holonomy maps H_{x_k, x'_k}^u converge uniformly to $H_{q, q}^u = \text{Id}|_{\mathcal{W}^c(q)}$ in the C^1 topology as $k \rightarrow -\infty$, we also have $\lim_{k \rightarrow -\infty} \det DH_{x_k, x'_k}^u(\Phi_{x_k}^c(g_k(s))) = 1$.

By (6.15), (6.1) and (6.2), we conclude that

$$\frac{dL}{ds}(s) = \lim_{k \rightarrow -\infty} \frac{\lambda_x^c(k)}{\lambda_{x'}^c(k)} = \prod_{\ell=1}^{\infty} \frac{\lambda_{x'-\ell}^c}{\lambda_{x-\ell}^c} = \rho_{x'}^c(x).$$

This completes the proof. ■

7. Leaf-wise quotient measures

Let us denote by $\mathcal{M}(\mathbb{R})$ the set of locally finite Borel measures on the real line. Using the families $\{\Phi_x\}, \{\Phi_x^c\}_{x \in \mathbb{T}^3}$ of parametrizations we will construct a family of elements of $\mathcal{M}(\mathbb{R})$ which will be the main object of study in this paper. We summarize the result of the construction with the following statement.

Theorem 7.1. *Let $f \in \mathcal{A}^2(\mathbb{T}^3)$ be an Anosov diffeomorphism with expanding center. Let μ be an ergodic u -Gibbs measure of f . Then there exists a family $\{\hat{\nu}_x^c\}_{x \in \mathbb{T}^3} \subset \mathcal{M}(\mathbb{R})$ of locally finite Borel measures on the real line with the following property: if, for μ -almost every x , the measure $\hat{\nu}_x^c$ is proportional to the Lebesgue measure of \mathbb{R} then μ is an SRB measure.*

The family $\{\hat{\nu}_x^c\}_{x \in \mathbb{T}^3}$ is referred to as *leaf-wise quotient measures*. Once Theorem 7.1 is established, our main result is reduced to proving that, for μ -almost every x , each $\hat{\nu}_x^c$ is a translation-invariant measure on the real line. The proposition below summarizes the properties of the family $\{\hat{\nu}_x^c\}_{x \in \mathbb{T}^3} \subset \mathcal{M}(\mathbb{R})$ that enable us to achieve this result.

Proposition 7.2 (Basic moves). *The family $\{\hat{\nu}_x^c\}_{x \in \mathbb{T}^3} \subset \mathcal{M}(\mathbb{R})$ satisfies the following properties on a full measure subset Λ of \mathbb{T}^3 , which is \mathcal{W}^u -saturated. For every $x \in \Lambda$,*

- (1) *if $y \in \mathcal{W}^c(x) \cap \Lambda$ then $\hat{\nu}_x^c \propto (\mathcal{H}_{x,y}^c)_* \hat{\nu}_y^c$, where $\mathcal{H}_{x,y}^c: \mathbb{R} \rightarrow \mathbb{R}$ is the change of normal form coordinates on the manifold $\mathcal{W}^c(x)$, which is an affine map;*
- (2) *$\hat{\nu}_{f(x)}^c \propto (\Lambda_x^c)_* \hat{\nu}_x^c$, where $\Lambda_x^c: \mathbb{R} \rightarrow \mathbb{R}$ is the linear map $\Lambda_x^c(s) = \lambda_x^c s$;*
- (3) *if $x' \in \mathcal{W}^u(x)$ then $\hat{\nu}_{x'}^c \propto (L_{x,x'})_* \hat{\nu}_x^c$, where $L_{x,x'}: \mathbb{R} \rightarrow \mathbb{R}$ is the linear map $L_{x,x'}(s) = \rho_{x'}^c(x)s$.*

The remainder of this section is dedicated to proving Theorem 7.1 and Proposition 7.2. Thus, we fix $f \in \mathcal{A}^2(\mathbb{T}^3)$ and an ergodic u -Gibbs measure μ .

7.1. Conditional measures of μ in normal forms

We will show that when we push the conditional measures of μ along center-unstable manifolds using the normal forms and subsequently disintegrate this measure along the horizontal lines (corresponding to unstable manifolds), we obtain measures *proportional* to the Lebesgue measure. This fact plays a fundamental role in the proofs of Theorem 7.1 and Proposition 7.2.

7.1.1. *Some preliminary facts about the conditional measures $\mu_{n,x}^{cu}$.* Let us start by recalling the notation from Section 2.5.1 and stating some basic lemmas that will be useful throughout this section. Start with ξ_0^{cu} , an increasing measurable partition subordinate to the uniformly expanded foliation \mathcal{W}^{cu} . When we apply the dynamics, we obtain a sequence $\xi_n^{cu} := f^n(\xi_0^{cu})$ of measurable partitions that remain subordinate to \mathcal{W}^{cu} . Moreover, for every $n \in \mathbb{N}$, we have $\xi_{n+1}^{cu} < \xi_n^{cu}$. Now let $\{\mu_{n,x}^{cu}\}_{x \in \mathbb{T}^3}$ be the disintegration of μ with respect to the partition ξ_n^{cu} . Similar to Lemma 2.16, we find that the following *superposition property* holds for μ -a.e. $x \in \mathbb{T}^3$:

$$\text{if } m > n, \text{ then } \mu_{n,x}^{cu}(A) = \frac{\mu_{m,x}^{cu}(A)}{\mu_{m,x}^{cu}(\xi_n^{cu}(x))} \tag{7.1}$$

for every measurable set $A \subset \xi_n^{cu}(x)$. Furthermore, as μ is f -invariant, the disintegrations $\{\mu_{n,x}^{cu}\}_{n \in \mathbb{N}, x \in \mathbb{T}^3}$ satisfy the following.

Lemma 7.3. *For every $n \in \mathbb{N}$ and for μ -almost every $x \in \mathbb{T}^3$, we have*

$$\mu_{n+1,f(x)}^{cu} = f_*\mu_{n,x}^{cu}.$$

Proof. The proof follows a similar approach to that of Lemma 2.12. ■

By intersecting with the center foliation \mathcal{W}^c , or unstable foliation \mathcal{W}^u , we can refine each partition ξ_n^{cu} , $n \in \mathbb{N}$. More precisely, for $\star = c, u$, we consider a partition $\xi_n^{\star} < \xi_n^{cu} = \{\xi_n^{\star}(y)\}_{y \in \mathbb{T}^3}$ with

$$\xi_n^{\star}(y) := \mathcal{W}^{\star}(y) \cap \xi_n^{cu}(y), \quad \forall y \in \mathbb{T}^3.$$

By Remark 2.14, we obtain the following result.

Lemma 7.4. *For $\star = c, u$, the family ξ_n^{\star} of sets is a measurable partition subordinate to \mathcal{W}^{\star} .*

For every $n \in \mathbb{N}$ and for μ -almost every $x \in \mathbb{T}^3$, we define $\zeta_n^{\star}(x) := \mathcal{H}_x(\xi_n^{\star}(x)) \subset \mathbb{R}^2$ and $\nu_{n,x}^{cu} := (\mathcal{H}_x)_*\mu_{n,x}^{cu}$. Note that due to Theorem 6.3 (1), for each $x \in \mathbb{T}^3$,

$$N_x \circ \mathcal{H}_x|_{\mathcal{W}^{cu}(x)} = \mathcal{H}_{f(x)} \circ f|_{\mathcal{W}^{cu}(x)}, \quad \text{where } N_x: (t, s) \mapsto (\lambda_x^u t, \lambda_x^c s). \tag{7.2}$$

As a consequence of (7.2) and Lemma 7.3, we obtain the following result.

Lemma 7.5. *For every $n \in \mathbb{N}$ and for μ -a.e. $x \in \mathbb{T}^3$,*

$$\nu_{n+1,f(x)}^{cu} = (N_x)_*\nu_{n,x}^{cu}.$$

7.1.2. *Disintegration of the conditional measures in normal forms.* As in Section 2.5.1, when we disintegrate $\mu_{n,x}^{cu}$ along the atoms $\{\xi_n^u(y)\}_{y \in \xi_n^{cu}(x)}$, we get a probability measure $\mu_{n,x}^c$ over $\xi_n^c(x)$ and a family $\{\mu_{n,y}^u\}_{y \in \xi_n^{cu}(x)}$, where $\mu_{n,y}^u$ is a probability measure on $\xi_n^u(y)$.

Now, let $\mathcal{H}_x: \mathcal{W}^{cu}(x) \rightarrow \mathbb{R}^2$ be the inverse map of the normal form coordinate chart. Define the measures

$$v_{n,x}^\star := (\mathcal{H}_x)_* \mu_{n,x}^\star \quad \text{for } \star = c, cu,$$

and

$$v_{n,s}^u := (\mathcal{H}_x)_* \mu_{n,y}^u \quad \text{for each } s = \mathcal{H}_x^c(y) \text{ with } y \in \xi_n^c(x).$$

A remarkable property of u -Gibbs measures is that their conditional measures along unstable manifolds in normal forms are *proportional* to Lebesgue measure.

Lemma 7.6. *For μ -a.e. $x \in \mathbb{T}^3$ and $v_{n,x}^{cu}$ -a.e. $(t, s) \in \zeta_n^{cu}(x)$, there exists a constant $\gamma_{n,x}(s) > 0$ (independent of t) such that for every Borel set $B \subset \zeta_n^u(\Phi_x^c(s))$,*

$$v_{n,s}^u(B) = \gamma_{n,x}(s) \text{Leb}(B).$$

Proof. Given that μ is a u -Gibbs measure, for μ -a.e. $x \in \mathbb{T}^3$ and $\mu_{n,x}^{cu}$ -a.e. $y \in \xi_n^{cu}(x)$, the measure $\mu_{n,y}^u$ is absolutely continuous. Its density, up to a constant, is given by (referring to (6.2))

$$\rho_y^u \cdot \xi_n^u(y) \ni z \mapsto \rho_y^u(z) = \prod_{j=0}^{\infty} \frac{\|Df^{-1}(f^{-j}(z))|_{E^u}\|}{\|Df^{-1}(f^{-j}(y))|_{E^u}\|}.$$

Let $(t, s) := \mathcal{H}_x(y)$, so that $v_{n,s}^u = (\mathcal{H}_x)_* \mu_{n,y}^u$. Restricted to $\mathbb{R} \times \{s\}$, $\Phi_x = \mathcal{H}_x^{-1}$ is given by the map $\Phi_{\Phi_x^c(s)}^u(\beta_x(s)t)$ for some $\beta_x(s) \in \mathbb{R}$. Therefore, up to a constant that depends only on s , the density of $v_{n,s}^u = (\mathcal{H}_x)_* \mu_{n,y}^u$ is given by

$$\tilde{\rho}_s^u: t \mapsto \rho_y^u(\Phi_{\Phi_x^c(s)}^u(t)) D\Phi_{\Phi_x^c(s)}^u(t).$$

Since $D\Phi_{\Phi_x^c(s)}^u(t) = (D\mathcal{H}_y^u)^{-1}(\Phi_{\Phi_x^c(s)}^u(t)) = \rho_y^u(\Phi_{\Phi_x^c(s)}^u(t))^{-1}$, $\tilde{\rho}_s^u$ is constant. ■

7.2. Construction of leaf-wise quotient measures

The measures we define here are similar to the quotient measures $v_{n,x}^c$, but we will use unstable segments of a fixed length together with a normalization that allows for stabilization when n is sufficiently large (see Lemma 7.8 below). Let $I \subset \mathbb{R}$ be any bounded interval centered at 0. It determines the length of unstable segments used in defining the leaf-wise quotient measures. Given a bounded Borel set $B \subset \mathbb{R}$ and a μ -typical point $x \in \mathbb{T}^3$ there exists $n_0 = n_0(x, I, B)$ such that for every $n \geq n_0$, $I \times B \subset \zeta_n^{cu}(x)$ and $I \times [-1, 1] \subset \zeta_n^{cu}(x)$: see Lemma 2.19. We first show that the choice of length of unstable segments, and thus of the interval $I \subset \mathbb{R}$, will not modify the definition.

Lemma 7.7. *For any other interval $J \subset \mathbb{R}$ centered at 0, there exists $n_0(J)$ such that*

$$\frac{v_{n,x}^{cu}(I \times B)}{v_{n,x}^{cu}(I \times [-1, 1])} = \frac{v_{n,x}^{cu}(J \times B)}{v_{n,x}^{cu}(J \times [-1, 1])} \quad \text{for every } n \geq n_0(J).$$

Proof. Consider real numbers α and β such that the affine map $\psi(t) = \alpha t + \beta$ satisfies $\psi(I) = J$. Applying the disintegration in conditional measures and using Lemma 7.6 we obtain

$$\begin{aligned} \frac{\nu_{n,x}^{cu}(J \times B)}{\nu_{n,x}^{cu}(J \times [-1, 1])} &= \frac{\int_B \nu_{n,s}^u(J) d\nu_{n,x}^c(s)}{\int_{[-1,1]} \nu_{n,s}^u(J) d\nu_{n,x}^c(s)} \\ &= \frac{\int_B \gamma_{n,x}(s) \text{Leb}(\alpha I + \beta) d\nu_{n,x}^c(s)}{\int_{[-1,1]} \gamma_{n,x}(s) \text{Leb}(\alpha I + \beta) d\nu_{n,x}^c(s)} \\ &= \frac{\int_B \gamma_{n,x}(s) \text{Leb}(I) d\nu_{n,x}^c(s)}{\int_{[-1,1]} \gamma_{n,x}(s) \text{Leb}(I) d\nu_{n,x}^c(s)} \\ &= \frac{\int_B \nu_{n,s}^u(I) d\nu_{n,x}^c(s)}{\int_{[-1,1]} \nu_{n,s}^u(I) d\nu_{n,x}^c(s)} \\ &= \frac{\nu_{n,x}^{cu}(I \times B)}{\nu_{n,x}^{cu}(I \times [-1, 1])}. \end{aligned}$$

This equality holds for sufficiently large n , ensuring that $J \times B$ and $J \times [-1, 1]$ are both subsets of $\zeta_n^{cu}(x)$. Notice that this can always be achieved due to Lemma 2.19. This completes the proof. ■

Given a bounded Borel measurable subset $B \subset \mathbb{R}$ and $n \in \mathbb{N}$, we define

$$\hat{\nu}_{n,x}^c(B) := \frac{\nu_{n,x}^{cu}(I \times B)}{\nu_{n,x}^{cu}(I \times [-1, 1])}. \tag{7.3}$$

We now use the superposition property to show that this sequence stabilizes for large n .

Lemma 7.8. *For μ -almost every $x \in \mathbb{T}^3$, every bounded Borel set $B \subset \mathbb{R}$, and every interval I , there exists $n_0 = n_0(x, I, B) \in \mathbb{N}$ such that for all integers $m > n \geq n_0$ we have $\hat{\nu}_{m,x}^c(B) = \hat{\nu}_{n,x}^c(B)$.*

Proof. First, observe that for each pair $m > n$, there exists a μ -full measure set of points $x \in \mathbb{T}^3$ for which $\mu_{m,x}^{cu}(\xi_{n,x}^{cu}(x)) > 0$. This follows from the superposition property (7.1). By taking the countable intersection of all these sets, and applying (7.1) again, we deduce that for every bounded Borel set $B \subset \mathbb{R}$ and $m > n \geq n_0(x, I, B)$ such that $I \times B, I \times [-1, 1] \subset \zeta_n^{cu}(x) \subset \zeta_m^{cu}(x)$, we have

$$\begin{aligned} \hat{\nu}_{n,x}^c(B) &= \frac{\mu_{n,x}^{cu}(\Phi_x(I \times B))}{\mu_{n,x}^{cu}(\Phi_x(I \times [-1, 1]))} \\ &= \frac{\mu_{m,x}^{cu}(\Phi_x(I \times B))}{\mu_{m,x}^{cu}(\xi_{n,x}^{cu}(x))} \times \frac{\mu_{m,x}^{cu}(\xi_{n,x}^{cu}(x))}{\mu_{m,x}^{cu}(\Phi_x(I \times [-1, 1]))} = \hat{\nu}_{m,x}^c(B). \end{aligned} \quad \blacksquare$$

Hence, the construction above defines a family of measures on \mathbb{R} .

Definition 7.9 (Leaf-wise quotient measure). For μ -a.e. $x \in \mathbb{T}^3$, let $\hat{\nu}_x^c$ be the unique locally finite Borel measure defined in \mathbb{R} such that, for each bounded Borel $B \subset \mathbb{R}$,

$$\hat{\nu}_x^c(B) := \hat{\nu}_{n_0(x,B),x}^c(B).$$

By Lemma 7.8 the right-hand side above is well defined.

Remark 7.10 (Choice of normalization). By construction, we have $\hat{\nu}_x^c([-1, 1]) = 1$.

Remark 7.11. By construction, for μ -a.e. $x \in \mathbb{T}^3$, $\hat{\nu}_x^c$ gives positive measure to any open neighborhood of 0. This follows from the definition and Corollary 2.17.

7.3. Obtaining the SRB property: Proof of Theorem 7.1

To complete the proof of Theorem 7.1 we assume that for μ -a.e. $x \in \mathbb{T}^3$ the measure $\hat{\nu}_x^c$ is proportional to the Lebesgue measure on \mathbb{R} . To conclude that μ is an SRB measure it suffices to show that the conditional measures $\mu_{0,x}^{cu}$ are absolutely continuous with respect to $\text{Leb}_{\mathcal{W}^{cu}(x)}$ for μ -a.e. x . To achieve this goal, consider $a < 0 < b$ with $b - a$ large enough so that, denoting $I = [a, b]$, we have

$$\xi_0^{cu}(x) \subset \Phi_x(I \times I) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{T}^3.$$

This can be done because the atoms of ξ_0^{cu} have uniformly bounded diameter and the C^1 norm of Φ_x depends continuously on x . Let $R = \max_{x \in \mathbb{T}^3} \text{diam}(\Phi_x(I \times I))$. Then applying Lemma 2.19 with this number we find that for μ -a.e. x , there exists $n_0(x, I)$ such that if $n \geq n_0(x, I)$ then

- (1) $I \subset \zeta_n^c(x)$;
- (2) $I \times \{s\} \subset \zeta_n^u(\Phi_x^c(s))$ for every $s \in I$.

Now, given such x and $n \geq n_0(x, I)$ take $s \in I$. We observe that

$$1 = \nu_{n,s}^u(\zeta_n^u(\Phi_x^c(s))) \geq \nu_{n,s}^u(I) = \gamma_{n,x}(s) \text{Leb}(I) = \gamma_{n,x}(s)(b - a). \tag{7.4}$$

Therefore,

$$0 < \gamma_{n,x}(s) < \frac{1}{b - a}.$$

We claim that $\nu_{n,x}^c|_I$ is absolutely continuous with respect to the leaf-wise quotient measure $\hat{\nu}_x^c|_I$. Indeed, set for simplicity $\alpha = (\nu_{n,x}^{cu}(I \times [-1, 1]))^{-1}$. If $B \subset I$ we have

$$\begin{aligned} \hat{\nu}_x^c(B) &= \alpha \nu_{n,x}^{cu}(I \times B) = \alpha \int_B \nu_{n,s}^u(I) d\nu_{n,x}^c(s) \\ &= \alpha \text{Leb}(I) \int_B \gamma_{n,x}(s) d\nu_{n,x}^c(s). \end{aligned} \tag{7.5}$$

Since $\gamma_{n,x} > 0$, if $\hat{\nu}_x^c(B) = 0$ then $\nu_{n,x}^c(B) = 0$, which proves our claim. Since, by assumption, $\hat{\nu}_x^c \propto \text{Leb}$, we conclude that indeed $\nu_{n,x}^c|_I$ is absolutely continuous with respect to $\text{Leb}|_I$. As $d\nu_{n,x}^{cu}(t, s) = d\nu_{n,s}^u(t) d\nu_{n,x}^c(s)$, it follows immediately from Lemma 7.6 that $\nu_{n,x}^{cu}|_{I \times I}$ is absolutely continuous with respect to the two-dimensional Lebesgue measure $\text{Leb}|_{I \times I}$. Since Φ_x is C^1 , we see that $\mu_{n,x}^{cu}|_{\Phi_x(I \times I)}$ is absolutely continuous with respect to $\text{Leb}_{\mathcal{W}^{cu}(x)}$. By the superposition property we deduce that $\mu_{0,x}^{cu}$ is absolutely continuous, as desired. The theorem is proved. ■

7.4. Basic moves for leaf-wise measures

We now proceed with the proof of Proposition 7.2. Our goal is to investigate how the measures $\hat{\nu}_x^c$ change as we move the base point x . In this analysis, the understanding of how the normal form coordinates change with the base point on the same center-unstable leaf plays a crucial role.

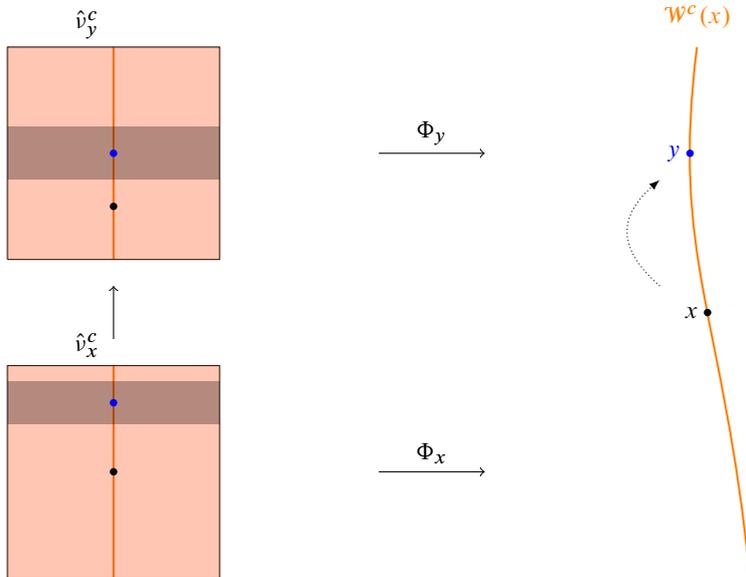


Fig. 7. Moving the base point within a central leaf changes the leaf-wise measure by an affine map up to renormalization.

7.4.1. *First basic move: Moving along center manifolds.* In the following lemma we show that for points on the same center leaf there exists an affine map that makes the corresponding leaf-wise quotient measures proportional to each other: see Figure 7. It is important to note that the derivative of this affine map is determined by normal form coordinates along the center manifold.

Lemma 7.12. *For μ -a.e. $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^c(x)$, we have*

$$\hat{\nu}_y^c \propto (h_{x,y}^2)_* \hat{\nu}_x^c,$$

where $h_{x,y}^2: \mathbb{R} \rightarrow \mathbb{R}$ is the affine map defined by $s \mapsto \rho_y^c(x) \cdot s + \mathcal{H}_y^c(x)$, corresponding to the change of normal forms on the second coordinate given by Lemma 6.12.

Proof. Let I be a bounded interval centered at 0 and B be a bounded Borel subset of \mathbb{R} . Let $n \in \mathbb{N}$ be large enough that $y \in \xi_n^{cu}(x)$ and

$$\Phi_y(I \times B) \cup \Phi_y(I \times [-1, 1]) \subset \xi_n^{cu}(y) = \xi_n^{cu}(x).$$

This is possible by Lemma 2.19. Recall the normal form coordinate change $\mathcal{H}_{x,y}(t, s) = \mathcal{H}_y \circ \Phi_x = (h_{x,y}^1(t, s), h_{x,y}^2(s))$, which is given by (see Lemma 6.12)

$$h_{x,y}^1(t, s) = \rho_y^u(x)t \quad \text{and} \quad h_{x,y}^2(s) = \rho_y^c(x)s + \mathcal{H}_y^c(x).$$

Let $\tilde{B} = h_{x,y}^2(B)$, $I_1 = [-1, 1]$ and $\tilde{I}_1 = h_{x,y}^2(I_1)$. Then we can apply Lemma 7.6 as before to argue that

$$\begin{aligned} \hat{\nu}_x^c(B) &= \frac{\mu_{n,x}^{cu}(\Phi_x(I \times B))}{\mu_{n,x}^{cu}(\Phi_x(I \times I_1))} = \frac{\mu_{n,y}^{cu}(\Phi_y \circ \mathcal{H}_{x,y}(I \times B))}{\mu_{n,y}^{cu}(\Phi_y \circ \mathcal{H}_{x,y}(I \times I_1))} \\ &= \frac{\nu_{n,y}^{cu}(\rho_y^u(x)I \times \tilde{B})}{\nu_{n,y}^{cu}(\rho_y^u(x)I \times \tilde{I}_1)} = \frac{\int_{\tilde{B}} \nu_{n,s}^u(\rho_y^u(x)I) d\nu_{n,x}^c(s)}{\int_{\tilde{I}_1} \nu_{n,s}^u(\rho_y^u(x)I) d\nu_{n,x}^c(s)} \\ &= \frac{\int_{\tilde{B}} \nu_{n,s}^u(I) d\nu_{n,x}^c(s)}{\int_{\tilde{I}_1} \nu_{n,s}^u(I) d\nu_{n,x}^c(s)} = \frac{\nu_{n,x}^{cu}(I \times \tilde{B})}{\nu_{n,x}^{cu}(I \times \tilde{I}_1)} \\ &= \alpha(x, y) \hat{\nu}_y^c(\tilde{B}), \end{aligned}$$

where $\alpha(x, y) = \frac{\nu_{n,x}^{cu}(I \times I_1)}{\nu_{n,x}^{cu}(I \times \tilde{I}_1)}$. This is exactly the desired conclusion. ■

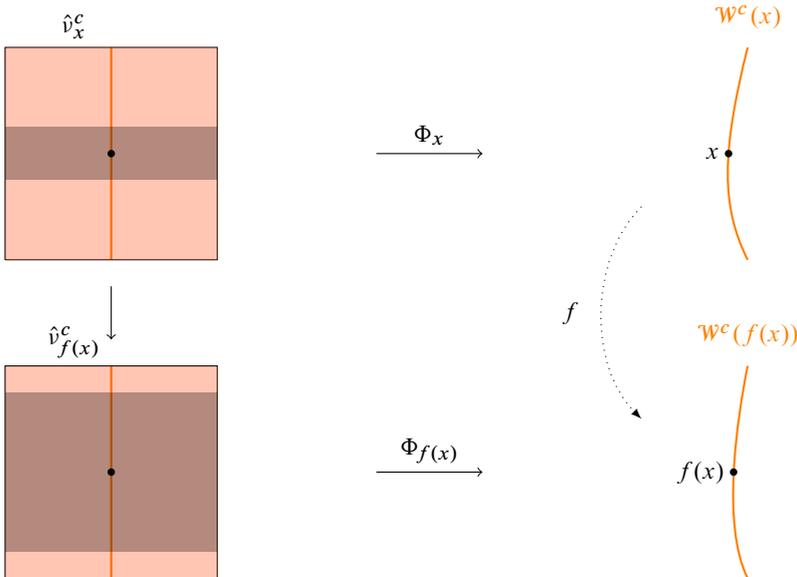


Fig. 8. Applying the dynamics to the base point changes the leaf-wise measure by a linear map given by the differential of f along E^c up to renormalization.

7.4.2. *Second basic move: Applying the dynamics.* When we push x to $f(x)$, the change of the measures is linear and the slope of the linear map is given by the derivative at

x along the center direction. Recall that for $\star = c, u$, we denote by Λ_x^\star the linear map $t \mapsto \lambda_x^\star \cdot t$ (where $\lambda_x^\star := \|Df(x)|_{E^\star}\|$).

Lemma 7.13. *For μ -a.e. $x \in \mathbb{T}^3$, we have*

$$\hat{v}_{f(x)}^c \propto (\Lambda_x^c)_* \hat{v}_x^c.$$

Proof. Let I be a bounded interval centered at 0 and B be a bounded Borel subset of \mathbb{R} . Fix $n \in \mathbb{N}$ so large that $I \times B, I \times [-1, 1] \subset \zeta_{n+1}^{cu}(f(x))$ (see Lemma 2.19). By definition,

$$\hat{v}_{n+1, f(x)}^c(B) = \frac{v_{n+1, f(x)}^{cu}(I \times B)}{v_{n+1, f(x)}^{cu}(I \times [-1, 1])},$$

and by Lemmas 7.5 and 7.6,

$$\begin{aligned} \frac{v_{n+1, f(x)}^{cu}(I \times B)}{v_{n+1, f(x)}^{cu}(I \times [-1, 1])} &= \frac{v_{n,x}^{cu}(N_x^{-1}(I \times B))}{v_{n,x}^{cu}(N_x^{-1}(I \times [-1, 1]))} \\ &= \frac{\int_{(\Lambda_x^c)^{-1}(B)} v_{n,s}^u((\lambda_x^u)^{-1}I) dv_{n,x}^c(s)}{\int_{(\Lambda_x^c)^{-1}([-1, 1])} v_{n,s}^u((\lambda_x^u)^{-1}I) dv_{n,x}^c(s)} \\ &= \frac{v_{n,x}^{cu}(I \times (\Lambda_x^c)^{-1}(B))}{v_{n,x}^{cu}(I \times (\Lambda_x^c)^{-1}([-1, 1]))} \\ &= \frac{v_{n,x}^{cu}(I \times (\Lambda_x^c)^{-1}(B))}{v_{n,x}^{cu}(I \times [-1, 1])} \times \frac{v_{n,x}^{cu}(I \times [-1, 1])}{v_{n,x}^{cu}(I \times (\Lambda_x^c)^{-1}([-1, 1]))} \\ &= \frac{1}{\hat{v}_x^c((\Lambda_x^c)^{-1}([-1, 1]))} \hat{v}_x^c((\Lambda_x^c)^{-1}(B)) \\ &= \alpha(x) \hat{v}_x^c((\Lambda_x^c)^{-1}(B)), \end{aligned} \tag{7.6}$$

where $\alpha(x) = \hat{v}_x^c((\Lambda_x^c)^{-1}([-1, 1]))^{-1}$. We deduce that

$$\hat{v}_{f(x)}^c(B) \propto (\Lambda_x^c)_* \hat{v}_x^c(B). \quad \blacksquare$$

7.4.3. Third basic move: Moving along unstable manifolds. For points on the same strong-unstable leaf the change is also linear, but the slope is determined by the derivative of the unstable holonomy map, which coincides with the derivative of the linear map giving the action of the change in normal form coordinates on the second variable. The lemma below completes the proof of Proposition 7.2.

Lemma 7.14. *For μ -a.e. $x \in \mathbb{T}^3$ and $x' \in \mathcal{W}^u(x)$,*

$$\hat{v}_x^c \propto (L_{x,x'}^u)_* \hat{v}_{x'}^c$$

for the linear map $L_{x,x'}^u: s \mapsto \rho_{x'}^u(x) \cdot s$ corresponding to the change of normal coordinates on the second variable given by Lemmas 6.14 and 6.15.

Proof. For μ -a.e. $x \in \mathbb{T}^3$ and $x' \in \mathcal{W}^u(x)$, we proceed as follows. Let I be any bounded interval centered at 0 and B be any bounded Borel subset of \mathbb{R} . Then there exists n large

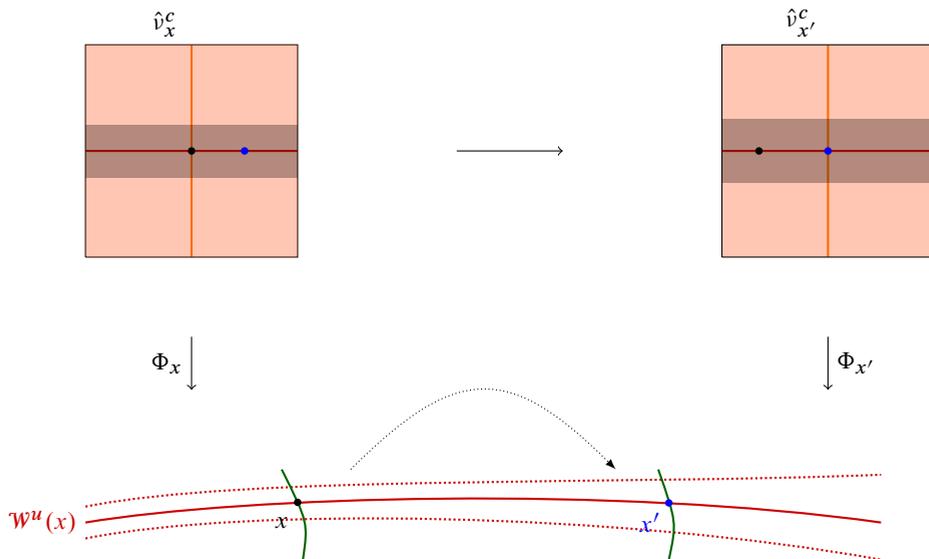


Fig. 9. Moving the base point within an unstable leaf changes the leaf-wise measure by a linear map up to renormalization.

enough that $x' \in \xi_n^{cu}(x)$ and $I \times B, I \times [-1, 1] \times B \subset \zeta_n^{cu}(x)$. This is possible by Lemma 2.19. As before, let $\mathcal{H}_{x,x'}(t, s) = (h_{x,x'}^1(t, s), h_{x,x'}^2(s))$ be the change in normal form coordinates. By Lemmas 6.13–6.15 we have

$$h_{x,x'}^1(t, s) = \rho_{x'}^u(x)t + a_{x,x'}(s) \quad \text{and} \quad h_{x,x'}^2(s) = \rho_{x'}^c(x)s.$$

Now, introduce $\tilde{B} = h_{x,x'}^2(B), \tilde{I}_1 = h_{x,x'}^2(I_1)$, where $I_1 = [-1, 1]$. By Lemma 7.6,

$$\begin{aligned} \hat{\nu}_{x'}^c(B) &= \frac{\mu_{n,x}^{cu}(\Phi_x \circ \mathcal{H}_{x,x'}(I \times B))}{\mu_{n,x}^{cu}(\Phi_x \circ \mathcal{H}_{x,x'}(I \times I_1))} = \frac{\nu_{n,x}^{cu}(h_{x,x'}^1(I \times B) \times \tilde{B})}{\nu_{n,x}^{cu}(h_{x,x'}^1(I \times I_1) \times \tilde{I}_1)} \\ &= \frac{\int_{\tilde{B}} \nu_{n,s}^u(\rho_{x'}^u(x)I + a_{x,x'}(\rho_{x'}^c(x')s)) d\nu_{n,x}^c(s)}{\int_{\tilde{I}_1} \nu_{n,s}^u(\rho_{x'}^u(x)I + a_{x,x'}(\rho_{x'}^c(x')s)) d\nu_{n,x}^c(s)} \\ &= \frac{\int_{\tilde{B}} \nu_{n,s}^u(I) d\nu_{n,x}^c(s)}{\int_{\tilde{I}_1} \nu_{n,s}^u(I) d\nu_{n,x}^c(s)} = \frac{\nu_{n,x}^{cu}(I \times \tilde{B})}{\nu_{n,x}^{cu}(I \times \tilde{I}_1)} \\ &= \alpha(x, x') \hat{\nu}_x^c(\tilde{B}), \end{aligned} \tag{7.7}$$

where $\alpha(x, x') = (\hat{\nu}_x^c(\tilde{I}_1))^{-1}$. This completes the proof of the lemma. ■

8. Invariance by affine maps

Once we have constructed the leaf-wise quotient measures $\{\hat{\nu}_x^c\}_{x \in \mathbb{T}^3}$, Theorem 7.1 tells us that the proof of Theorem 5.10 reduces to showing that $\hat{\nu}_x^c$ is a multiple of the Lebesgue

measure of the real line. As in [7], this can be achieved by proving that $\hat{\nu}_x^c$ is, for many points x , invariant by affine maps with controlled slope and small translational part. More precisely, the lemma below is the analogue of [7, Proposition 7.1] in our context.

Lemma 8.1. *There exist constants $M_0 > 0$ and $\delta_0 \in (0, 1)$ such that for every $\varepsilon > 0$ sufficiently small one can find a compact set $G(\varepsilon) \subset \mathbb{T}^3$ such that $\mu(G(\varepsilon)) \geq \delta_0$ and for every $p \in G(\varepsilon)$ there exists an affine map $\psi: \mathbb{R} \rightarrow \mathbb{R}$ satisfying*

- (1) $1/M_0 < |\psi'(0)| < M_0$;
- (2) $\varepsilon/M_0 < |\psi(0)| < \varepsilon M_0$;
- (3) $\psi_* \hat{\nu}_p^c \propto \hat{\nu}_p^c$.

Furthermore, writing $G_0 := \{p \in \mathbb{T}^3 : p \in G(1/N) \text{ for infinitely many } N \in \mathbb{N}\}$, we have $\mu(G_0) \geq \delta_0$.

Exactly as in [7], the lemma above implies Theorem 5.10. The proof is a direct adaptation of [25, Lemma 3.10] and [7, Lemma 7.3]; for the convenience of the reader, we provide this beautiful argument below and we refer to [7, 25] for other applications of the same argument.

Proof of Theorem 5.10 assuming Lemma 8.1. Let $\text{Aff}(\mathbb{R})$ denote the group of invertible affine maps of \mathbb{R} , and for $p \in \mathbb{T}^3$, let $\mathcal{A}(p) \subset \text{Aff}(\mathbb{R})$ be the subgroup of affine maps $\psi: \mathbb{R} \rightarrow \mathbb{R}$ such that $\psi_* \hat{\nu}_p^c \propto \hat{\nu}_p^c$.

We claim that $\mathcal{A}(p)$ is a closed subgroup of $\text{Aff}(\mathbb{R})$. Let $\psi_n \in \mathcal{A}(p)$ converge to $\psi \in \text{Aff}(\mathbb{R})$. Since each element of $\text{Aff}(\mathbb{R})$ is a homeomorphism of the real line and since convergence in $\text{Aff}(\mathbb{R})$ implies convergence in the compact-open topology, for each continuous function $\phi: \mathbb{R} \rightarrow \mathbb{R}$ with compact support we have

$$\int \phi \circ \psi_n d\hat{\nu}_p^c \rightarrow \int \phi \circ \psi d\hat{\nu}_p^c.$$

This implies that $(\psi_n)_* \hat{\nu}_p^c \rightarrow \psi_* \hat{\nu}_p^c$. On the other hand, for each n we have

$$(\psi_n)_* \hat{\nu}_p^c = c_n \hat{\nu}_p^c \tag{8.1}$$

for some constant c_n , which can therefore be expressed by

$$c_n = \frac{\hat{\nu}_p^c(\psi_n^{-1}(K))}{\hat{\nu}_p^c(K)}$$

for any measurable set $K \subset \mathbb{R}$ with finite positive measure with respect to both $\hat{\nu}_p^c$ and $(\psi_n)_* \hat{\nu}_p^c$. Now, as all the measures are locally finite, we can require further that such a compact set K is also a continuity set (of finite measure) for $\psi_* \hat{\nu}_p^c$. In particular, since $(\psi_n)_* \hat{\nu}_p^c(K) \rightarrow \psi_* \hat{\nu}_p^c(K)$, we deduce that c_n converges to some $c > 0$. We deduce from (8.1) and uniqueness of limits that $\psi_* \hat{\nu}_p^c = c \hat{\nu}_p^c$. This proves that $\psi \in \mathcal{A}(p)$ and establishes our claim. This implies in particular that $\mathcal{A}(p)$ is a Lie group.

By Lemma 8.1, for every $p \in G_0$, $\mathcal{A}(p)$ contains a sequence $(\psi_j)_{j \in \mathbb{N}}$ of elements $\psi_j: t \mapsto \lambda_j t + v_j$ with $v_j \neq 0$, $\lim_{j \rightarrow \infty} v_j = 0$, and $\lim_{j \rightarrow \infty} \lambda_j = \lambda$ for some $\lambda \neq 0$.

In particular, $\mathcal{A}(p)$ contains the homothety $h_\lambda: t \mapsto \lambda t$, and $(h_\lambda^{-1} \circ \psi_j)_{j \in \mathbb{N}}$ converges to the identity within $\mathcal{A}(p)$. Therefore, $\mathcal{A}(p)$ is not discrete and must be of dimension 1 or 2. This implies that the identity component $\mathcal{A}^0(p) \subset \mathcal{A}(p)$ contains a one-parameter subgroup of $\text{Aff}(\mathbb{R})$; such a group consists of translations, or is conjugate to homothety.

We now claim that the groups $\mathcal{A}^0(p)$ are isomorphic for a.e. $p \in \mathbb{T}^3$. Indeed, recall the linear map $\Lambda_p^c: \mathbb{R} \rightarrow \mathbb{R}$ given by $\Lambda_p^c(t) = \lambda_p^c t$, where $\lambda_p^c = \|Df(p)|_{E^c}\|$. It follows from Lemma 7.13 that $\psi \in \mathcal{A}^0(p)$ if and only if $\Lambda_p^c \circ \psi \circ (\Lambda_p^c)^{-1} \in \mathcal{A}^0(f(p))$, proving our claim.

Since the isomorphism classes of closed subgroups of $\text{Aff}(\mathbb{R})$ form a separable space and since $\mu(G_0) > 0$, the claim then follows by ergodicity. In particular, for μ -a.e. $p \in \mathbb{T}^3$, $\mathcal{A}^0(p)$ contains a one-parameter subgroup of $\text{Aff}(\mathbb{R})$.

Assume for contradiction that $\mathcal{A}^0(p)$ were conjugate to homothety for a positive measure set of $p \in \mathbb{T}^3$. Then, by ergodicity, for μ -a.e. $p \in \mathbb{T}^3$, the action of $\mathcal{A}^0(p)$ on \mathbb{R} would have a unique fixed point $t(p) \in \mathbb{R}$. Since $\mathcal{A}^0(p)$ contains affine maps with arbitrarily small (and non-zero) translational part, we have $t(p) \neq 0$ for a set of positive measure.

Further, as observed above, $\mathcal{A}^0(f^n(p)) = \{\Lambda_{p,n}^c \circ \psi \circ (\Lambda_{p,n}^c)^{-1} : \psi \in \mathcal{A}^0(p)\}$, where $\Lambda_{p,n}^c(t) = \lambda_p^c(n)t$ (recall (2.2)). As a consequence,

$$|t(f^n(p))| = \lambda_p^c(n) \cdot |t(p)|. \tag{8.2}$$

Consider a positive measure compact subset K of G_0 where the measurable function $p \mapsto t(p)$ is continuous and bounded. By Poincaré recurrence, for some $p \in K$ we have $p_{n_k} = f^{n_k}(p) \in K$ for infinitely many iterates $n_k \in \mathbb{N}$. By compactness, we can assume $p_{n_k} \rightarrow q \in K$. However, (8.2) is incompatible with the boundedness of $t|_K$. We have thus reached a contradiction.

Therefore, $\mathcal{A}^0(p)$ contains the group of translations, for μ -a.e. $p \in \mathbb{T}^3$. For $t \in \mathbb{R}$, $p \in \mathbb{T}^3$, we let $g = g_t: s \mapsto s + t$ and $c(p, t) := \hat{v}_p^c([-t - 1, -t + 1])$, so that

$$\frac{dg_* \hat{v}_p^c}{d\hat{v}_p^c} = c(p, t). \tag{8.3}$$

Let us see some properties of the function $(p, t) \mapsto c(p, t)$. Firstly, for μ -a.e. $p \in \mathbb{T}^3$, $\mathcal{A}^0(p)$ contains all translations. This implies that \hat{v}_p^c has no atom, which implies that $c(p, \cdot)$ is continuous. We claim that

$$c(p, t) = c(p_n, \lambda_p^c(n)t), \tag{8.4}$$

where $p_n = f^n(p)$. To prove the claim, consider $\psi = \Lambda_{p,n}^c \circ g \circ (\Lambda_{p,n}^c)^{-1}$ and observe that $\psi(s) = s + \lambda_p^c(n)t$. From the definition it follows that

$$c(p_n, \lambda_p^c(n)t) = \hat{v}_{p_n}^c(\psi^{-1}([-1, 1])) = \frac{\hat{v}_{p_n}^c(\psi^{-1}([-1, 1]))}{\hat{v}_{p_n}^c([-1, 1])}.$$

In the last equality we have used our normalization choice $\hat{\nu}_{p_n}^c([-1, 1]) = 1$ for leaf-wise quotient measures. Applying now Lemma 7.12 we deduce

$$\begin{aligned} \frac{\hat{\nu}_{p_n}^c(\psi^{-1}([-1, 1]))}{\hat{\nu}_{p_n}^c([-1, 1])} &= \frac{\hat{\nu}_p^c((\lambda_p^c(n))^{-1} \times \psi^{-1}([-1, 1]))}{\hat{\nu}_p^c((\lambda_p^c(n))^{-1} \times [-1, 1])} \\ &= \frac{\hat{\nu}_p^c(g^{-1}(\lambda_p^c(n))^{-1} \times ([-1, 1]))}{\hat{\nu}_p^c((\lambda_p^c(n))^{-1} \times [-1, 1])} \\ &= \frac{g_* \hat{\nu}_p^c(\lambda_p^c(n)^{-1} \times ([-1, 1]))}{\hat{\nu}_p^c((\lambda_p^c(n))^{-1} \times [-1, 1])} \\ &= c(p, t), \end{aligned} \tag{8.5}$$

where in the second equality we have applied the definition of ψ and in the last equality we have used the fact that the measures $g_* \hat{\nu}_p^c$ and $\hat{\nu}_p^c$ are proportional by a factor precisely equal to $c(p, t)$. This establishes (8.4).

We now give the final argument completing the proof. For each $\varepsilon > 0$, choose $r > 0$ such that the set

$$B_{r,\varepsilon} := \{p \in \mathbb{T}^3 : |c(p, t) - 1| < \varepsilon, \forall |t| < r\}$$

satisfies $\mu(B_{r,\varepsilon}) > 0$. By ergodicity, μ -a.e. point $p \in \mathbb{T}^3$ visits $B_{r,\varepsilon}$ infinitely many times both in future and past; but (8.4) implies that $|c(p, t) - 1| < \varepsilon$ for all $t \in \mathbb{R}$ and μ -a.e. $p \in \mathbb{T}^3$. Letting $\varepsilon \rightarrow 0$, we conclude that $c(p, t) = 1$ for all $t \in \mathbb{R}$ and μ -a.e. $p \in \mathbb{T}^3$, hence, by (8.3), $\hat{\nu}_p^c$ is invariant under the group of translations for μ -a.e. $p \in \mathbb{T}^3$, hence it is proportional to Lebesgue measure. ■

8.1. Drift along the center

Recall that the normal forms $\{\Phi_x^c: \mathbb{R} \rightarrow \mathcal{W}^c(x)\}_{x \in \mathbb{T}^3}$ give us parametrizations of the center manifolds whose changes of coordinates are affine maps. Using these changes of coordinates one can build the maps $\psi: \mathbb{R} \rightarrow \mathbb{R}$ promised in Lemma 8.1. Indeed, we claim that it suffices to prove the result below.

Proposition 8.2. *There exist constants $M > 0$ and $\delta_0 \in (0, 1)$ such that for every $\varepsilon > 0$ sufficiently small one can find a compact set $G = G(\varepsilon) \subset \mathbb{T}^3$ such that $\mu(G) \geq \delta_0$ and for every $p \in G$ there exists a point $q \in \mathcal{W}_1^c(p)$ such that*

$$M^{-1}\varepsilon \leq |\mathcal{H}_p^c(q)| \leq M\varepsilon \quad \text{and} \quad \hat{\nu}_q^c \propto B_* \hat{\nu}_p^c \tag{8.6}$$

for some linear map $B: \mathbb{R} \rightarrow \mathbb{R}$ of the form $s \mapsto \beta \cdot s$ with $M^{-1} < |\beta| < M$.

8.1.1. Proof of Lemma 8.1 assuming Proposition 8.2. We only need to show that the set G of Proposition 8.2 satisfies the claims in Lemma 8.1 for a suitably chosen constant M (which will perhaps be a bit larger than the one already given by the proposition). To see this, take $q \in \mathcal{W}_1^c(p)$ for some $p \in \mathbb{T}^3$ and, for the sake of this proof, denote by

$\xi = \mathcal{H}_{p,q}^c = \mathcal{H}_p^c \circ \Phi_q^c: \mathbb{R} \rightarrow \mathbb{R}$ the affine change of normal form coordinates along the center direction. Because normal forms depend continuously on the base point, we may enlarge M if necessary so that $|\xi'(0)| \in (M^{-1}, M)$.

Now, take the set G given by Proposition 8.2 and let $p \in G$. We shall construct the affine map $\psi: \mathbb{R} \rightarrow \mathbb{R}$ claimed by the lemma. For this take $q \in \mathcal{W}_1^c(p)$ given by Proposition 8.2 and notice that, on the one hand, Lemma 7.12 gives $\hat{\nu}_q^c \propto \xi_* \hat{\nu}_p^c$. On the other hand, Proposition 8.2 says that for some linear map $B: \mathbb{R} \rightarrow \mathbb{R}$ with derivative β bounded in between M^{-1} and M we have $\hat{\nu}_q^c \propto B_* \hat{\nu}_p^c$. These two properties give $\hat{\nu}_p^c \propto \psi_* \hat{\nu}_p^c$, where $\psi = B^{-1}\xi$ is an affine map. The bounds we have on β and on $|\xi'(0)|$ give

$$|\psi'(0)| \in (M^{-2}, M^2).$$

Moreover, by Proposition 8.2,

$$|\psi(0)| = |B^{-1}\xi(0)| = |B^{-1}\mathcal{H}_p^c(q)| \in (M^{-2}\varepsilon, M^2\varepsilon).$$

This shows that ψ satisfies all the requirements in Lemma 8.1 with $M_0 = M^2$, thus completing the proof. ■

8.1.2. The Luzin set. We now move on to the proof of Proposition 8.2, where our key arguments are concentrated. Our approach here draws inspiration from Eskin–Lindenstrauss’ work [15]. To offer some insight into our reasoning, recall from Lemma 7.13 that the measures $\hat{\nu}_x^c$ change linearly when we shift the base point from x to $f^n(x)$, with the slope of this linear map being $\|Df^n(x)|_{E^c}\|$. Our strategy involves identifying specific dynamical configurations in which points, *almost* on the same center leaf, drift apart by a center distance proportional to ε . In these configurations, their corresponding leaf-wise quotient measures are *almost* proportional to each other, modulo a linear map whose slope can be controlled using the dynamics. To pinpoint the precise set where “almost” turns into “equality”, we take limits. To achieve this we must consider points belonging to compact sets such that the assignment $x \mapsto \hat{\nu}_x^c$ restricted to those sets has good properties.

With that goal in mind, we denote by $C_c^0(\mathbb{R})$ the space of continuous functions on the real line with compact support.

Lemma 8.3. *For every $\delta > 0$ there exists a compact set $A \subset \mathbb{T}^3$ with $\mu(A) > 1 - \delta$ such that for all $\{x_n\}_{n \in \mathbb{N}} \subset A$ converging to $x \in A$,*

$$\int \varphi d\hat{\nu}_{x_n}^c \rightarrow \int \varphi d\hat{\nu}_x^c \quad \text{for every } \varphi \in C_c^0(\mathbb{R}).$$

Proof. Set $K := \text{supp}(\varphi)$. We first claim that, given $\varphi \in C_c^0(\mathbb{R})$, the function $\mathbb{T}^3 \ni x \mapsto \int \varphi d\hat{\nu}_x^c \in \mathbb{R}$ is measurable. Indeed, consider the set

$$\Lambda(\varepsilon) = \{x \in \mathbb{T}^3 : \mathcal{W}_\varepsilon^c(x) \subset \xi_0^c(x)\}.$$

Since $\mu(\Lambda(\varepsilon)) \rightarrow 1$ as $\varepsilon \rightarrow 0$, it suffices to check that, for a given $\varepsilon > 0$, the restriction of the function to $\Lambda(\varepsilon)$ is measurable (notice that the set $\Lambda(\varepsilon)$ itself is measurable). By

compactness of K and uniform continuity of the C^1 norm of our normal forms in compact sets, Lemma 2.19 ensures the existence of an $n \in \mathbb{N}$ such that

$$K \subset \zeta_n^c(x) \quad \text{for every } x \in \Lambda(\varepsilon). \tag{8.7}$$

Define $\hat{\varphi} : \mathbb{R}^2 \rightarrow \mathbb{R}$ as $\hat{\varphi}(t, s) = \varphi(s)$. Then, denoting $\alpha(x) = 1/\nu_{n,x}^{cu}(I \times [-1, 1])$, it follows from (8.7) that

$$\int \varphi d\hat{\nu}_x^c = \alpha(x) \int \hat{\varphi} \circ \mathcal{H}_x(z) d\mu_{n,x}^{cu}(z)$$

for every $x \in \Lambda(\varepsilon)$. Since the right-hand side above depends measurably on $x \in \Lambda(\varepsilon)$, our claim is proved.

As a second step in our proof, we claim the existence of a countable dense subset $\{\varphi_\ell\}_{\ell \in \mathbb{N}} \subset C_c^0(\mathbb{R})$ with the following property: for a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{T}^3$ converging to a point $x \in \mathbb{T}^3$, if

$$\int \varphi_\ell d\hat{\nu}_{x_n}^c \rightarrow \int \varphi_\ell d\hat{\nu}_x^c \quad \text{for all } \ell \in \mathbb{N},$$

then

$$\int \varphi d\hat{\nu}_{x_n}^c \rightarrow \int \varphi d\hat{\nu}_x^c \quad \text{for every } \varphi \in C_c^0(\mathbb{R}).$$

To construct such a dense subset, start with a countable dense subset \mathcal{F} of $C_c^0(\mathbb{R})$ with the following property. If $\varphi \in C_c^0(\mathbb{R})$ and $K = \text{supp}(\varphi)$, then define $\hat{K} = [\inf K - 1, \sup K + 1]$. For every $\varepsilon > 0$ small enough there exists $\psi \in \mathcal{F}$ such that $\|\varphi - \psi\|_\infty < \varepsilon$ and $\text{supp}(\psi) \subset \hat{K}$. The existence of \mathcal{F} follows from standard arguments.

Now, augment \mathcal{F} by adding, for each positive integer n , a continuous bump function that equals 1 in the interval $[-n, n]$ and 0 outside $(-n - 1, n + 1)$. The resulting set, still denoted by \mathcal{F} , remains countable and dense, but now has the property that for any compact set $K \subset \mathbb{R}$, there exists $\psi \in \mathcal{F}$ such that $\psi(x) \geq 1$ for every $x \in K$. To show that \mathcal{F} meets the claim’s requirements, consider φ in $C_c^0(\mathbb{R})$ and let K and \hat{K} be as above. Choose $\psi \in \mathcal{F}$ such that $\psi|_{\hat{K}} \geq 1$. Then

$$\hat{\nu}_{x_n}^c(\hat{K}) \leq \int \psi d\hat{\nu}_{x_n}^c.$$

By assumption, the right-hand side above converges to $\int \psi d\hat{\nu}_x^c$. Thus, $\hat{\nu}_{x_n}^c(\hat{K})$ is bounded by some $\beta(K) > 0$, independent of n . Now, given $\varepsilon > 0$, choose $\hat{\psi}$ in the countable dense family \mathcal{F} such that

$$\|\hat{\psi} - \varphi\|_\infty < \frac{\varepsilon}{3\beta(K)},$$

and with $\text{supp}(\hat{\psi}) \subset \hat{K}$. Since $\hat{\psi}$ is in \mathcal{F} , there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then

$$\left| \int \hat{\psi} d\hat{\nu}_{x_n}^c - \int \hat{\psi} d\hat{\nu}_x^c \right| < \frac{\varepsilon}{3}.$$

Thus, if $n \geq n_0$ we have

$$\begin{aligned} \left| \int \varphi d\hat{\nu}_{x_n}^c - \int \varphi d\hat{\nu}_x^c \right| &\leq \left| \int \varphi d\hat{\nu}_{x_n}^c - \int \hat{\psi} d\hat{\nu}_{x_n}^c \right| + \left| \int \hat{\psi} d\hat{\nu}_{x_n}^c - \int \hat{\psi} d\hat{\nu}_x^c \right| \\ &\quad + \left| \int \hat{\psi} d\hat{\nu}_x^c - \int \varphi d\hat{\nu}_x^c \right| \\ &< \frac{\varepsilon \hat{\nu}_{x_n}^c(\hat{K})}{3\beta(K)} + \frac{\varepsilon}{3} + \frac{\varepsilon \hat{\nu}_x^c(\hat{K})}{3\beta(K)} \\ &\leq \varepsilon. \end{aligned}$$

This proves our second claim. To conclude the proof of the lemma, let $\mathcal{F} = \{\varphi_\ell\}_{\ell \in \mathbb{N}}$ be the countable dense family of our previous claim. For each ℓ , according to our first claim, the function $g_\ell: \mathbb{T}^3 \ni x \mapsto \int \varphi_\ell d\hat{\nu}_x^c$ is measurable. By Luzin’s theorem there exists a compact set $A_\ell \subset \mathbb{T}^3$ with $\mu(A_\ell) > 1 - \delta/2^{\ell+1}$ and such that $g_\ell|_{A_\ell}$ is continuous. Define the compact set $A = \bigcap_{\ell=0}^\infty A_\ell$. Then $\mu(A) > 1 - \delta$ and if a sequence $\{x_n\}_{n \in \mathbb{N}} \subset A$ converges to x , we have $g_\ell(x_n) \rightarrow g_\ell(x)$ for every ℓ , and the lemma follows from the above claims. ■

Let $\mathcal{L} \subset \mathbb{T}^3$ be a compact set with large measure (we shall give precise estimates below) such that the conclusion of Lemma 8.3 holds for $A = \mathcal{L}$. We shall require further the following properties:

- (1) For $\sigma = s, c, u, cu$, given a measurable partition ξ^σ subordinate to \mathcal{W}^σ , there exists $r_0 > 0$ such that for every $x \in \mathcal{L}$,

$$\mathcal{W}_{r_0}^\sigma(x) \subset \xi^\sigma(x).$$

- (2) For every $x \in \mathcal{L}$, $\xi^u(x)$ is an interval whose size $r^u(x)$ varies continuously with $x \in \mathcal{L}$ (see Remark 2.15).

The continuity of $\mathcal{L} \ni x \mapsto \hat{\nu}_x^c$ together with Remark 7.11 implies the following.

Corollary 8.4. *For each $M > 1$ there exists $c = c(M) > 0$ such that if $B : \mathbb{R} \rightarrow \mathbb{R}$ is a linear map with derivative in $[M^{-1}, M]$ then $\hat{\nu}_x^c(B^{-1}[-1, 1]) \in [c^{-1}, c]$ for each $x \in \mathcal{L}$.*

Proof. Assume for contradiction that a lower bound does not hold. Denote $J_0 = [-M^{-1}, M^{-1}]$. For each n there must exist $x_n \in \mathcal{L}$ such that $\hat{\nu}_{x_n}^c(J_0) \leq 1/n$. We can assume, by compactness of \mathcal{L} , that $x_n \rightarrow x \in \mathcal{L}$. Since $y \in \mathcal{L} \mapsto \hat{\nu}_y^c$ is continuous, we have

$$\hat{\nu}_x^c(J_0) \leq \liminf_{n \rightarrow \infty} \hat{\nu}_{x_n}^c(J_0) = 0,$$

which violates Remark 7.11. By a similar reasoning, considering $J_1 = [-M, M]$ and assuming that the upper bound does not hold, we would obtain a point $x \in \mathcal{L}$ for which $\hat{\nu}_x^c(J_1) = \infty$, which is also impossible by local finiteness. ■

8.1.3. *The main proposition.* Let us fix two numbers δ and δ_0 such that

$$0 < \delta_0 < 1/10 \quad \text{and} \quad \delta = 3\delta_0.$$

We will require $\mu(\mathcal{L}) > 1 - \delta_0$. The constant δ_0 is the constant appearing in Lemma 8.1 and Proposition 8.2. Further assumptions on δ and δ_0 will be given in Section 11.3.6.

Given $M, \varepsilon > 0$ set

$$G(\varepsilon, M) := \left\{ p \in \mathbb{T}^3 : \begin{array}{l} \exists q \in \mathcal{W}_1^c(p), \exists B(s) = \beta \cdot s \text{ linear map such that} \\ \hat{v}_q^c \propto B_* \hat{v}_p^c, \\ M^{-1}\varepsilon \leq |\mathcal{H}_p^c(q)| \leq M\varepsilon, \\ M^{-1} \leq |\beta| \leq M \end{array} \right\}. \quad (8.8)$$

Notice that $G(\varepsilon, M)$ is nothing other than the set of points $p \in \mathbb{T}^3$ for which the conclusion of Proposition 8.2 holds with constants ε, M . Recall that in order to prove Proposition 8.2, and thus Lemma 8.1, we must show that $\mu(G(\varepsilon, M)) \geq \delta_0$.

Given this choice of parameters, Proposition 8.2 follows from the statement below.

Proposition 8.5. *There exists $M > 0$ such that for $\varepsilon > 0$ small enough and every compact set K_{00} with $\mu(K_{00}) > 1 - 2\delta_0$,*

$$K_{00} \cap \mathcal{L} \cap G(\varepsilon, M) \neq \emptyset.$$

Proof that Proposition 8.5 implies Proposition 8.2. Let $M > 0$ be the constant given in the statement above. Assume for contradiction that Proposition 8.2 does not hold with the constants M and δ_0 . Then for some small $\varepsilon > 0$ we must have $\mu(G(\varepsilon, M)) < \delta_0$. By regularity of μ there exists an open neighborhood U of $G(\varepsilon, M)$ such that $\mu(U) < \delta_0$. Let $K_{00} := \mathbb{T}^3 \setminus U$. This set is compact and satisfies $\mu(K_{00}) \geq 1 - \delta_0 > 1 - 2\delta_0$. Proposition 8.5 then implies that $K_{00} \cap G(\varepsilon, M) \neq \emptyset$, which is absurd. ■

In order to start the proof of Proposition 8.5, fix $\varepsilon > 0$ (independently of δ ; notice that while δ is fixed we need to take $\varepsilon \rightarrow 0$). More assumptions on ε will be given later on. This is the expected size of the drift we want to see along the center direction. Given a compact set K_{00} whose measure is larger than $1 - 2\delta_0$, notice that the compact set

$$K_0 := K_{00} \cap \mathcal{L}$$

has measure $\mu(K_0) > 1 - 3\delta_0 = 1 - \delta$. We shall prove that $K_0 \cap G(\varepsilon, M) \neq \emptyset$. This will occupy the rest of the paper.

9. Stopping times, Y -configurations, quadrilaterals and synchronization

The goal of this section is to introduce the key dynamical ingredients involved in the proof of Proposition 8.5, which is the main part of our implementation of an exponential drift argument. We also use this section to derive some estimates relating these ingredients.

9.1. Stopping times

We introduce below the stopping time functions. They are devised to measure the appropriate time length of the top part of Figure 5 so that we get the precise drift we want along the center direction.

9.1.1. *Definition of stopping times.* Recall our concise notations for derivatives

$$\lambda_x^\star(n) = \|Df^n(x)|_{E^\star}\|, \quad \star = c, s, u,$$

and

$$d_x^\ell = \frac{\lambda_{x-\ell}^c(\ell)}{\lambda_{x-\ell}^u(\ell)}$$

introduced in (2.2) and (2.4) respectively, where $x_n = f^n(x)$ is our concise notation for orbit points introduced in (2.1). Given $x \in \mathbb{T}^3$, $x^u \in \mathcal{W}^u(x)$, $\varepsilon > 0$, and $\ell \in \mathbb{N}$, we define

$$\tau(\ell) = \tau(x, x^u, \varepsilon, \ell) := \inf \{n \in \mathbb{N} : d_x^\ell \times \lambda_{x^u}^c(n) \geq \varepsilon\}.$$

We also define

$$t(\ell) = t(x, x^u, \varepsilon, \ell) := \inf \left\{ n \in \mathbb{N} : \frac{\lambda_x^c(n)}{\lambda_{x^u}^c(\tau(\ell))} \geq 1 \right\}.$$

These functions are called the *stopping times*.

9.1.2. *Quasi-isometric estimates.* In the following, we fix $\varepsilon > 0$. Recall that for $x \in \mathbb{T}^3$ and $x^u \in \mathcal{W}^u(x)$, we abbreviate $\tau(\ell) = \tau(x, x^u, \varepsilon, \ell)$ and $t(\ell) = t(x, x^u, \varepsilon, \ell)$.

Lemma 9.1 (Quasi-isometric estimates). *There exist $\Theta = \Theta(f) > 1$ and $A = A(f) > 0$ such that given $\ell, m \in \mathbb{N}$, $x \in \mathbb{T}^3$ and $x^u \in \mathcal{W}^u(x)$ the following hold:*

- (1) $\Theta^{-1}m - A < \tau(\ell + m) - \tau(\ell) < \Theta m + A$;
- (2) $\Theta^{-1}m - A < t(\ell + m) - t(\ell) < \Theta m + A$.

Proof. In this proof we shall make use of the constants of hyperbolicity of f introduced in Section 2.1.3. With these constants at hand, we can now develop the argument for the *quasi-isometric estimates*. Fix $x \in \mathbb{T}^3$ and $x^u \in \mathcal{W}^u(x)$. Given some $\varepsilon > 0$, write $\tau(\ell) = \tau(x, x^u, \varepsilon, \ell)$ and $t(\ell) = t(x, x^u, \varepsilon, \ell)$. From the definition of τ , we have

$$d_x^\ell \lambda_{x^u}^c(\tau(\ell)) \geq \varepsilon \quad \text{and} \quad d_x^\ell \lambda_{x^u}^c(\tau(\ell) - 1) < \varepsilon \quad \text{for each } \ell \in \mathbb{N}.$$

Thus,

$$\varepsilon \leq d_x^\ell \lambda_{x^u}^c(\tau(\ell)) < e^{\chi_1^c} \varepsilon \quad \text{for each } \ell \in \mathbb{N}. \tag{9.1}$$

Fix $\ell, m \in \mathbb{N}$. On the one hand, by the cocycle property (2.3) we have $d_x^{\ell+m} = d_x^\ell d_{x^\ell}^m$ and therefore we can use (9.1) to obtain

$$\varepsilon e^{m\chi_1^d} < d_x^{\ell+m} \lambda_{x^u}^c(\tau(\ell)) < \varepsilon e^{\chi_1^c + m\chi_2^d}.$$

On the other hand, we can use the cocycle property once more to write

$$\lambda_{f^{\tau(\ell)}(x^u)}^c(\tau(\ell + m) - \tau(\ell)) = \frac{\lambda_{x^u}^c(\tau(\ell + m))}{\lambda_{x^u}^c(\tau(\ell))}$$

and thus obtain

$$e^{\chi_2^c(\tau(\ell+m)-\tau(\ell))} < \frac{\lambda_{x^u}^c(\tau(\ell + m))}{\lambda_{x^u}^c(\tau(\ell))} < e^{\chi_1^c(\tau(\ell+m)-\tau(\ell))}.$$

Therefore, since

$$d_x^{\ell+m} \lambda_{xu}^c(\tau(\ell+m)) = d_x^{\ell+m} \lambda_{xu}^c(\tau(\ell)) \frac{\lambda_{xu}^c(\tau(\ell+m))}{\lambda_{xu}^c(\tau(\ell))},$$

we can combine the above two inequalities to obtain

$$\varepsilon e^{m\chi_1^d + \chi_2^c(\tau(\ell+m) - \tau(\ell))} < d_x^{\ell+m} \lambda_{xu}^c(\tau(\ell+m)) < \varepsilon e^{\chi_1^c + m\chi_2^d + \chi_1^c(\tau(\ell+m) - \tau(\ell))}.$$

Putting $\ell + m$ instead of ℓ in (9.1) and combining with this we get the inequalities

$$\varepsilon e^{\chi_1^c + m\chi_2^d + \chi_1^c(\tau(\ell+m) - \tau(\ell))} > \varepsilon \quad \text{and} \quad \varepsilon e^{-\chi_1^c + m\chi_1^d + \chi_2^c(\tau(\ell+m) - \tau(\ell))} < \varepsilon.$$

Dividing by ε and taking logarithms we deduce that

$$m \left[\frac{-\chi_2^d}{\chi_1^c} \right] - 1 < \tau(m+\ell) - \tau(\ell) < m \left[\frac{-\chi_1^d}{\chi_2^c} \right] + \frac{\chi_1^c}{\chi_2^c}. \quad (9.2)$$

Let us deal with the function $\ell \mapsto t(\ell)$. From its very definition we have an inequality analogous to (9.1):

$$1 \leq \frac{\lambda_x^c(t(\ell))}{\lambda_{xu}^c(\tau(\ell))} < e^{\chi_1^c} \quad \text{for all } x \in \mathbb{T}^3 \text{ and } \ell \in \mathbb{N}. \quad (9.3)$$

To simplify the remainder of the exposition, set $\tau_{\ell,m} := \tau(\ell+m) - \tau(\ell)$ and $t_{\ell,m} = t(\ell+m) - t(\ell)$. Notice that

$$e^{\chi_2^c t_{\ell,m} - \chi_1^c \tau_{\ell,m}} < \frac{\lambda_{f^t(\ell)(x)}^c(t(\ell+m) - t(\ell))}{\lambda_{f^{\tau(\ell)}(xu)}^c(\tau(\ell+m) - \tau(\ell))} < e^{\chi_1^c t_{\ell,m} - \chi_2^c \tau_{\ell,m}}.$$

Combining this with (9.3), we can use the decomposition

$$\frac{\lambda_x^c(t(\ell+m))}{\lambda_{xu}^c(\tau(\ell+m))} = \frac{\lambda_x^c(t(\ell)) \lambda_{f^t(\ell)(x)}^c(t(\ell+m) - t(\ell))}{\lambda_{xu}^c(\tau(\ell)) \lambda_{f^{\tau(\ell)}(xu)}^c(\tau(\ell+m) - \tau(\ell))}$$

to conclude that

$$e^{\chi_2^c t_{\ell,m} - \chi_1^c \tau_{\ell,m}} < \frac{\lambda_x^c(t(\ell+m))}{\lambda_{xu}^c(\tau(\ell+m))} < e^{\chi_1^c t_{\ell,m} - \chi_2^c \tau_{\ell,m} + \chi_1^c}.$$

Now, by (9.3) with $\ell + m$ instead of ℓ , the above inequality implies that

$$e^{\chi_2^c t_{\ell,m} - \chi_1^c \tau_{\ell,m}} < e^{\chi_1^c} \quad \text{and} \quad 1 < e^{\chi_1^c t_{\ell,m} - \chi_2^c \tau_{\ell,m} + \chi_1^c},$$

and thus

$$e^{\chi_2^c t_{\ell,m} - \chi_1^c \tau_{\ell,m} - \chi_1^c} < 1 < e^{\chi_1^c t_{\ell,m} - \chi_2^c \tau_{\ell,m} + \chi_1^c}.$$

Taking logarithms leads us to

$$\left[\frac{\chi_2^c}{\chi_1^c} \right] \tau_{\ell,m} - 1 < t_{\ell,m} < \left[\frac{\chi_1^c}{\chi_2^c} \right] \tau_{\ell,m} + \frac{\chi_1^c}{\chi_2^c},$$

which combined with (9.2) ends the proof. ■

9.2. *Y*-configurations

We now introduce a dynamical ingredient inspired by [14, 15]. This allows us to “decompose” Figure 5 into the “*x*-side” and the “*y*-side”. Notice that each side has a kind of *Y*-shape. The idea from [15] for getting the points inside the Luzin set is to prove the existence of a large amount of these *Y*-shaped dynamical configurations and then try to find some of them which are linked through stable manifolds, as in Figure 5. These measure-theoretical arguments will be developed in Section 11. In this section, we introduce these configurations and establish *synchronization and (center-) drift estimates for them*.

9.2.1. *Definition.* Given $\varepsilon > 0$ and $\ell \in \mathbb{N}$, a *Y*-configuration $Y = Y(x, x^u, \ell)$ is a quintuple $(x, x^u, x_{-\ell}, x_\tau^u, x_t)$ of points, that depends on parameters $x, x^u \in \mathbb{T}^3$ and ℓ (the dependence on ε is implicit throughout the text), and such that

- (1) $x^u \in \mathcal{W}^u(x)$;
- (2) $x_{-\ell} = f^{-\ell}(x)$;
- (3) $x_\tau^u = f^\tau(x^u)$, where $\tau = \tau(x, x^u, \varepsilon, \ell)$;
- (4) $x_t = f^t(x^u)$, where $t = t(x, x^u, \varepsilon, \ell)$,

where $\tau(x, x^u, \varepsilon, \ell)$ and $t(x, x^u, \varepsilon, \ell)$ are the stopping times defined above.

We call ℓ the *length* of the *Y*-configuration. Moreover, given a set $\Lambda \subset \mathbb{T}^3$, we say that a *Y*-configuration $(x, x^u, x_{-\ell}, x_\tau^u, x_t)$ is Λ -good if $x, x^u, x_\tau^u, x_t \in \Lambda$.

In [15], the authors defined a notion of pairs of *coupled Y*-configurations. For technical reasons we shall replace this notion by that of pairs of *matched Y*-configurations that will appear in the next section.

9.3. *Quadrilaterals and synchronization*

Another important aspect of Figure 5 is located in its middle, where we have a kind of a quadrilateral, twisted along the unstable direction. This, indeed, is the third dynamical ingredient in our implementation of an exponential drift argument. It will allow us to use the angle condition and to define the aforementioned notion of matching of *Y*-configurations.

9.3.1. *Quadrilaterals.* A *quadrilateral* is a quadruple $(x, x^u, y, y^u) \in (\mathbb{T}^3)^4$ such that

- (1) $y \in \mathcal{W}^s(x)$;
- (2) $x^u \in \mathcal{W}^u(x)$ belongs to the domain of the center-holonomy map $H_{x,y}^{cs}$;
- (3) $y^u = H_{x,y}^{cs}(x^u) \in \mathcal{W}^u(y) \cap \mathcal{W}^{cs}(x^u)$.

For such a quadrilateral, we define $z^u := H_{x,y}^s(x^u)$, so that $z^u \in \mathcal{W}^s(x^u) \cap \mathcal{W}^c(y^u)$. Moreover, given $C > 1$ and $\ell \in \mathbb{N}$, we say that (x, x^u, y, y^u) is a (C, ℓ) -quadrilateral if moreover

- (4) $C^{-1} < d(x_{-\ell}, y_{-\ell}) < 1$,

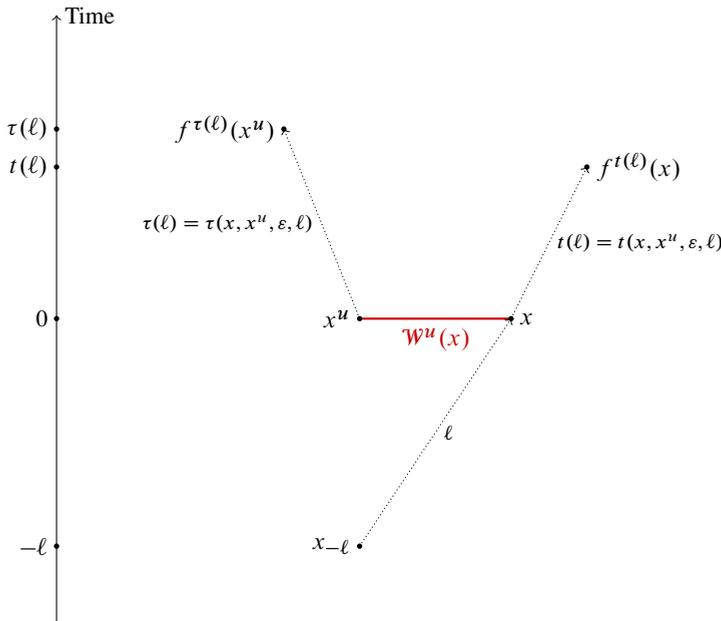


Fig. 10. A Y -configuration.

(5) $C^{-1} < \alpha^s(x_{-\ell}, y_{-\ell}) < C$,

(6) $C^{-1} < d(x, x^u) < C$,

where, as before, $x_{-\ell} = f^{-\ell}(x)$ and $y_{-\ell} = f^{-\ell}(y)$.

9.3.2. *Drift estimates.* In Figure 5 we claim that the displacement along the center at the left top of the configuration is proportional to the parameter ε . We shall now make this assertion precise in terms of normal form coordinates. We first introduce a useful notation that will be used throughout the text, and in particular in the next proof.

Given real parameters c_1, \dots, c_n and two quantities a and b (that may or may not depend on other variables) we write $a \asymp_{c_1, \dots, c_n} b$ if there exists a real-valued function $\rho = \rho(c_1, \dots, c_n) \geq 1$ such that

$$\rho^{-1}a \leq b \leq \rho a.$$

Lemma 9.2. *Fix $\varepsilon > 0$. Given $C > 1$ and $T > 0$, there exists a constant $\kappa = \kappa(C, T) > 1$ such that for every $\ell \in \mathbb{N}$ large enough, every (C, ℓ) -quadrilateral (x, x^u, y, y^u) , and every $\tau \geq 0$ such that $|\tau - \tau(y, y^u, \varepsilon, \ell)| \leq T$ we have*

$$\varepsilon/\kappa < |\mathcal{H}_{f^\tau(z^u)}^c(f^\tau(y^u))| < \kappa\varepsilon. \tag{9.4}$$

Proof. Let $\gamma^u = [x, x^u] \subset \mathcal{W}^u(x)$ be the segment of the strong-unstable manifold connecting the points x and x^u (see Figure 11). Then $f^{-\ell} \circ H_{x,y}^s(\gamma^u)$ is a C^1 curve joining

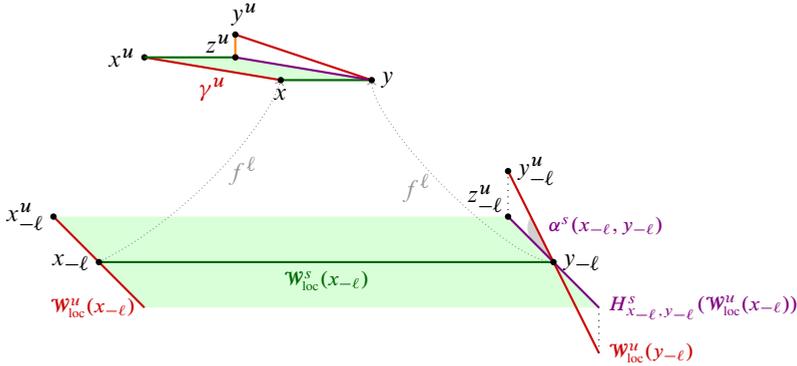


Fig. 11. A quadrilateral and its pre-image by f^ℓ .

y_{-l} and z_{-l}^u .² Since $f^{-\ell} \circ H_{x,y}^s(\gamma^u) = H_{x_{-l}, y_{-l}}^s \circ f^{-\ell}(\gamma^u)$, the assumption $C^{-1} < \alpha^s(x_{-l}, y_{-l}) < C$ implies that the vector tangent to the curve $f^{-\ell} \circ H_{x,y}^s(\gamma^u)$ at y_{-l} and the strong-unstable direction $E^u(y_{-l})$ are transverse with an angle between C^{-1} and C . Recall that $\mathcal{H}_{y_{-l}}: \mathcal{W}^{cu}(y_{-l}) \rightarrow \mathbb{R}^2$ sends $\mathcal{W}^u(y_{-l})$ onto the horizontal axis (see Theorem 6.3). As the length of $f^{-\ell} \circ H_{x,y}^s(\gamma^u)$ is exponentially small in ℓ and the bundle E^u is continuous, a standard compactness argument ensures that for some constant $c_1 = c_1(C) > 0$ and for ℓ large enough, the curve $\mathcal{H}_{y_{-l}} \circ f^{-\ell} \circ H_{x,y}^s(\gamma^u)$ on \mathbb{R}^2 is contained in the cone

$$\mathcal{C}_1 = \{(v_1, v_2) \in \mathbb{R}^2 : c_1^{-1}|v_1| \leq |v_2| \leq c_1|v_1|\}.$$

Now, observe that $\mathcal{H}_y \circ H_{x,y}^s(\gamma^u) = \mathcal{H}_y \circ f^\ell \circ \Phi_{y_{-l}} \circ \mathcal{H}_{y_{-l}} \circ f^{-\ell} \circ H_{x,y}^s(\gamma^u)$, and $\mathcal{H}_y \circ f^\ell \circ \Phi_{y_{-l}}$ is the linear map $\mathbb{R}^2 \ni (t, s) \mapsto (\lambda_{y_{-l}}^u(\ell)t, \lambda_{y_{-l}}^c(\ell)s)$. These two observations together imply that the curve $\mathcal{H}_y \circ H_{x,y}^s(\gamma^u)$ is contained in the cone

$$\mathcal{C}_2 = \left\{ (v_1, v_2) \in \mathbb{R}^2 : c_1^{-1} \frac{\lambda_{y_{-l}}^c(\ell)}{\lambda_{y_{-l}}^u(\ell)} |v_1| \leq |v_2| \leq c_1 \frac{\lambda_{y_{-l}}^c(\ell)}{\lambda_{y_{-l}}^u(\ell)} |v_1| \right\}.$$

Now let γ^c denote the piece of the center manifold connecting z^u to y^u . Then $\mathcal{H}_y \circ \gamma^c$ connects the point $\mathcal{H}_y(z^u)$, whose distance to the origin only depends on the constant C and is bounded from above and below, to the point $\mathcal{H}_y(y^u)$, which lies on the horizontal axis. Since $\mathcal{H}_y(z^u) \in \mathcal{C}_2$, we deduce that (see Figure 12)

$$\text{length}(\mathcal{H}_y \circ \gamma^c) \asymp_C \frac{\lambda_{y_{-l}}^c(\ell)}{\lambda_{y_{-l}}^u(\ell)}.$$

Since $C^{-1} < d^u(x, x^u) < C$ and we have C^1 holonomies, we also find that $C^{-1} < d(z^u, y) < C$ (upon enlarging the constant C if necessary in order to take into account

²Recall our notation convention for orbit points: $p_n = f^n(p)$ for $n \in \mathbb{Z}$.

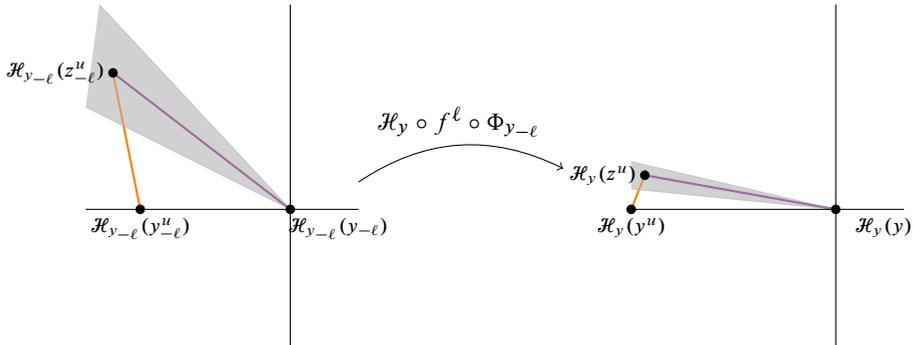


Fig. 12. In normal form coordinates the dynamics on \mathcal{W}^{cu} acts as a diagonal matrix, with stronger expansion in the horizontal direction.

the action of holonomy maps). Recall that normal forms change continuously in the C^1 topology, so another compactness argument ensures that the C^1 norm of $\mathcal{H}_{z^u} \circ \Phi_y |_{\mathcal{W}_{2C}^{cu}(y)}$ is bounded by some constant depending on C . Thus

$$\text{length}(\mathcal{H}_{z^u} \circ \gamma^c) \asymp_C \frac{\lambda_{y-\ell}^c(\ell)}{\lambda_{y-\ell}^u(\ell)}.$$

On the other hand, by the construction of normal forms, $\mathcal{H}_{z^u} \circ \gamma^c \subset \mathbb{R}^2$ is a vertical segment with length $|\mathcal{H}_{z^u}^c(y^u)|$. This proves that

$$|\mathcal{H}_{z^u}^c(y^u)| \asymp_C \frac{\lambda_{y-\ell}^c(\ell)}{\lambda_{y-\ell}^u(\ell)}.$$

As before, it is possible to bound uniformly from above the C^1 norms of all \mathcal{H}_*^c restricted to center segments of uniform radius, so (upon enlarging ℓ if necessary) we also have

$$d_c(y^u, z^u) \asymp_C \frac{\lambda_{y-\ell}^c(\ell)}{\lambda_{y-\ell}^u(\ell)} \quad \text{and} \quad |\mathcal{H}_{y^u}^c(z^u)| \asymp_C \frac{\lambda_{y-\ell}^c(\ell)}{\lambda_{y-\ell}^u(\ell)}.$$

Since the dynamics on normal forms acts linearly, this implies

$$|\mathcal{H}_{f^{\tau(\ell)}(y^u)}^c(f^{\tau(\ell)}(z^u))| \asymp_C \frac{\lambda_{y-\ell}^c(\ell)}{\lambda_{y-\ell}^u(\ell)} \times \lambda_{y^u}^c(\tau(\ell)),$$

where $\tau(\ell) = \tau(y, y^u, \varepsilon, \ell)$ is the stopping time. By definition this implies that

$$|\mathcal{H}_{f^{\tau(\ell)}(y^u)}^c(f^{\tau(\ell)}(z^u))| \asymp_C \varepsilon. \tag{9.5}$$

Since $|\tau - \tau(\ell)| \leq T$, we obtain

$$|\mathcal{H}_{f^\tau(y^u)}^c(f^\tau(z^u))| \asymp_{C,T} \varepsilon. \quad \blacksquare$$

Corollary 9.3. *There exists a constant $\kappa = \kappa(C, T) > 0$ such that for every τ satisfying $|\tau - \tau(y, y^u, \varepsilon, \ell)| \leq T$ we have*

$$d_c(f^\tau(y^u), f^\tau(z^u)) \leq \kappa\varepsilon.$$

Proof. This follows from the proof of Lemma 9.2 and the fact that normal forms $\{\mathcal{H}_x^c\}_{x \in \mathbb{T}^3}$ are C^1 maps with continuously varying C^1 norm on compact sets. ■

9.3.3. *Synchronization estimates.* The last set of estimates we need concerns the oscillation of the stopping times from one side to the other when we have a (C, ℓ) -quadrilateral.

Lemma 9.4 (Synchronization for quadrilaterals). *For any $C > 1$, there exists a constant $T_0 > 0$ depending only on f, C such that for any $\ell \in \mathbb{N}$, and any (C, ℓ) -quadrilateral (x, x^u, y, y^u) , we have*

$$|\tau(y, y^u, \varepsilon, \ell) - \tau(x, x^u, \varepsilon, \ell)| < T_0, \quad |t(y, y^u, \varepsilon, \ell) - t(x, x^u, \varepsilon, \ell)| < T_0.$$

Proof. Recall the constants $\chi_2^c, \chi_1^c > 0$ introduced in Section 2.1.3. They satisfy $e^{k\chi_2^c} \leq \lambda_x^c(k) \leq e^{k\chi_1^c}$ for all $k \in \mathbb{N}$ and $x \in \mathbb{T}^3$ where $\lambda_x^c(k) = \|Df^k(x)|_{E^c}\|$ is our concise notation for derivatives from (2.2). For simplicity, denote $\tau(\ell) = \tau(x, x^u, \varepsilon, \ell)$ and $\tau'(\ell) = \tau(y, y^u, \ell, \varepsilon)$. Assume that $\tau(\ell) \geq \tau'(\ell)$. By definition

$$d_y^\ell \lambda_{y^u}^c(\tau'(\ell)) \geq \varepsilon. \tag{9.6}$$

For $k \in \mathbb{N}$ write

$$\frac{d_x^\ell \lambda_{x^u}^c(\tau'(\ell) + k)}{d_y^\ell \lambda_{y^u}^c(\tau'(\ell))} = \frac{d_x^\ell}{d_y^\ell} \times \frac{\lambda_{x^u}^c(\tau'(\ell))}{\lambda_{y^u}^c(\tau'(\ell))} \times \lambda_{x_{\tau'(\ell)}^u}^c(k). \tag{9.7}$$

We now bound from below each factor appearing in (9.7). We treat the third factor by observing that $\lambda_{x_{\tau'(\ell)}^u}^c(k) \geq e^{k\chi_2^c}$. To treat the first factor, since $y_{-\ell} \in \mathcal{W}_1^s(x_{-\ell})$, the distortion control given by Corollary 2.10 gives $d_y^\ell/d_x^\ell \geq C_0^{-1}$. Finally, in order to treat the second factor, we notice that Corollary 9.3 provides

$$d_c(f^j(z^u), f^j(y^u)) \leq C\varepsilon$$

for some constant $C = C(f) > 0$ and all $j = 0, \dots, \tau'(\ell)$. With no loss of generality we can assume that $C\varepsilon < 1$. Applying the distortion control of Corollary 2.11 we obtain $\frac{\lambda_{x^u}^c(\tau'(\ell))}{\lambda_{y^u}^c(\tau'(\ell))} \geq C_0^{-1}$. These lower bounds together with (9.7) provide the estimate

$$d_x^\ell \lambda_{x^u}^c(\tau'(\ell) + k) \geq \frac{e^{k\chi_2^c}}{C_0^2} \varepsilon \geq \varepsilon$$

as long as we choose $k = k(f)$ such that

$$e^{k\chi_2^c} \geq C_0^2. \tag{9.8}$$

Hence by definition of the stopping time, we obtain $\tau'(\ell) + k \geq \tau(\ell)$ so $\tau(\ell) - \tau'(\ell) \leq k$. A symmetric argument shows that $\tau'(\ell) - \tau(\ell) \leq k$ if $\tau'(\ell) \geq \tau(\ell)$.

We now consider the stopping time t . As above, we denote $t(\ell) = t(x, x^u, \varepsilon, \ell)$ and $t'(\ell) = t'(y, y^u, \varepsilon, \ell)$. By definition

$$\frac{\lambda_y^c(t'(\ell))}{\lambda_{y^u}^c(\tau'(\ell))} \geq 1. \tag{9.9}$$

For $k' \in \mathbb{N}$ write

$$\begin{aligned} \frac{\lambda_x^c(t'(\ell) + k')}{\lambda_{x^u}^c(\tau(\ell))} \frac{\lambda_{y^u}^c(\tau'(\ell))}{\lambda_y^c(t'(\ell))} &= \frac{\lambda_{y^u}^c(\tau'(\ell))}{\lambda_{x^u}^c(\tau'(\ell))} \times \frac{\lambda_x^c(t'(\ell))}{\lambda_y^c(t'(\ell))} \\ &\quad \times \lambda_{x_{t'(\ell)}}^c(k') \times \frac{1}{\lambda_{x_{\tau'(\ell)}}^c(\tau(\ell) - \tau'(\ell))}. \end{aligned}$$

We now bound from below each factor of the product above. We have already seen that $\frac{\lambda_{x^u}^c(\tau'(\ell))}{\lambda_{y^u}^c(\tau'(\ell))} \geq C_0^{-1}$. To treat the second factor, as $y \in \mathcal{W}_1^s(x)$, we apply Corollary 2.11 to get $\frac{\lambda_x^c(t'(\ell))}{\lambda_y^c(t'(\ell))} \geq C_0^{-1}$. We treat the third factor by observing that $\lambda_{x_{t'(\ell)}}^c(k') \geq e^{k'\chi_2^c}$. Note that $|\tau(\ell) - \tau'(\ell)| \leq k$ so the last bound follows from $\frac{1}{\lambda_{x_{\tau'(\ell)}}^c(\tau(\ell) - \tau'(\ell))} \geq e^{-k\chi_1^c}$. Finally, we find

$$\frac{\lambda_x^c(t'(\ell) + k')}{\lambda_{x^u}^c(\tau(\ell))} \geq \frac{e^{k'\chi_2^c}}{C_0^2 e^{k\chi_1^c}} \times \frac{\lambda_y^c(t'(\ell))}{\lambda_{y^u}^c(\tau'(\ell))} \geq 1 \tag{9.10}$$

as long as we choose $k' = k'(f)$ such that

$$e^{k'\chi_2^c} \geq C_0^2 e^{k\chi_1^c}. \tag{9.11}$$

Hence by definition we must have $t(\ell) \leq t'(\ell) + k'$ so $t(\ell) - t'(\ell) \leq k'$. Here again a symmetric argument yields $t'(\ell) - t(\ell) \leq k'$. Hence

$$T_0 = \max(k, k') \tag{9.12}$$

is the desired constant. ■

10. Matching of Y -configurations

In this section we develop a concept devised specifically to address the technical difficulty to implement the exponential drift idea we want to employ, a difficulty described at the end of Section 3. Namely, due to lack of absolute continuity of center-stable holonomies we cannot ensure that the points x^u and y^u of Figure 5 both belong to the Luzin set. What we can actually prove is that small perturbations of these two points can indeed be put inside the Luzin set. This will lead us to the notion of matched configurations.

10.1. Matching of dynamical balls

We start the formal definition of matching in our scenario by introducing intervals along the unstable manifold which measure the amount of perturbation of the points $f^{\tau(\ell)}(x^u)$ and $f^{\tau(\ell)}(y^u)$ which are allowed without breaking the estimates we performed for quadrilaterals in the previous section.

10.1.1. *Unstable dynamical balls.* Although the notion of dynamical balls is quite standard in ergodic theory, we use this name for a more specific object, adapted to our needs.

Definition 10.1. Let $\varepsilon > 0$, $\ell \in \mathbb{N}$, $x \in \mathbb{T}^3$ and $x^u \in \mathcal{W}^u(x)$. Let $\tau(\ell) = \tau(x, x^u, \varepsilon, \ell)$. The (ε, ℓ) -unstable dynamical ball at x^u is defined as

$$J(x^u) = J(x^u, \varepsilon, \ell) := f^{-\tau(\ell)}(\mathcal{W}_1^u(f^{\tau(\ell)}(x^u))).$$

Remark 10.2. By definition, for every $a \in J(x^u)$ and $j \in \{0, \dots, \tau(x, x^u, \varepsilon, \ell)\}$,

$$d_u(f^j(x^u), f^j(a)) < 1.$$

10.1.2. *Synchronization inside a dynamical ball.* Before we define the notion of matching of Y -configurations it is useful to study the oscillation of stopping times inside an unstable dynamical ball.

Let $T_0 > 0$ be the constant of Lemma 9.4.

Lemma 10.3. For every $a \in J(x^u)$ we have $|\tau(x, a, \varepsilon, \ell) - \tau(x, x^u, \varepsilon, \ell)| < T_0$ and $|t(x, a, \varepsilon, \ell) - t(x, x^u, \varepsilon, \ell)| < T_0$.

Proof. The proof is almost identical to that of Lemma 9.4. In particular, the constants $0 < \chi_2^c < \chi_1^c$ are those defined in Section 2.1.3, and k and k' are defined by (9.8) and (9.11). Set $\tau_a(\ell) := \tau(x, a, \varepsilon, \ell)$. Suppose first that $\tau_a(\ell) \geq \tau(\ell)$. Then

$$\frac{d_x^\ell \lambda_a^c(\tau(\ell) + k)}{d_x^\ell \lambda_{x^u}^c(\tau(\ell))} = \frac{\lambda_a^c(\tau(\ell))}{\lambda_{x^u}^c(\tau(\ell))} \times \lambda_{a_{\tau(\ell)}}^c(k). \tag{10.1}$$

We have $\lambda_{a_{\tau(\ell)}}^c(k) \geq e^{k\chi_2^c}$. By definition, $f^{\tau(\ell)}(a) \in \mathcal{W}_1^u(f^{\tau(\ell)}(x^u))$, so by the distortion control (2.7) we have $\frac{\lambda_a^c(\tau(\ell))}{\lambda_{x^u}^c(\tau(\ell))} \geq C_0^{-1}$ and, by choice of k ,

$$d_x^\ell \lambda_a^c(\tau(\ell) + k) \geq \frac{e^{k\chi_2^c}}{C_0} d_x^\ell \lambda_{x^u}^c(\tau(\ell)) \geq \varepsilon.$$

Hence $\tau_a(\ell) - \tau(\ell) \leq k$. A symmetric argument yields $\tau(\ell) - \tau_a(\ell) \leq k$.

We now consider the stopping time t . As above, we denote $t_a(\ell) := \tau(x, a, \varepsilon, \ell)$ and $t(\ell) = t(x, a, \varepsilon, \ell)$. First suppose $t_a(\ell) \geq t(\ell)$. We have

$$\begin{aligned} \frac{\lambda_x^c(t(\ell) + k')}{\lambda_a^c(\tau_a(\ell))} \times \frac{\lambda_{x^u}^c(\tau(\ell))}{\lambda_x^c(t(\ell))} &= \frac{\lambda_{x^u}^c(\tau(\ell))}{\lambda_a^c(\tau(\ell))} \times \lambda_{x_{t'(\ell)}}^c(k') \\ &\times \frac{1}{\lambda_{x_{\tau(\ell)}}^c(\tau(\ell) - \tau'(\ell))}. \end{aligned}$$

The first factor is $\geq C_0^{-1}$. The second one is $\geq e^{k'\chi_2^c}$. The third one is $\geq e^{-k\chi_1^c}$. Hence by our choice of k' ,

$$\frac{\lambda_x^c(t(\ell) + k')}{\lambda_a^c(\tau_a(\ell))} \geq \frac{e^{k'\chi_2^c}}{C_0 e^{k\chi_1^c}} \times \frac{\lambda_{x^u}^c(\tau(\ell))}{\lambda_x^c(t(\ell))} \geq 1.$$

This proves that $t_a(\ell) - t(\ell) \leq k'$. Again, a symmetric argument gives $t_a(\ell) - t(\ell) \leq k'$ if $t(\ell) \geq t_a(\ell)$. This ends the proof of the lemma since $T_0 = \max(k, k')$. ■

10.1.3. Matching of dynamical balls and distortion control. We first define the notion of matched unstable dynamical balls and study their geometric properties.

Definition 10.4 (Matched unstable dynamical balls). When (x, x^u, y, y^u) is a (C, ℓ) -quadrilateral, then we say that $J(x^u) = J(x^u, \varepsilon, \ell)$ and $J(y^u) = J(y^u, \varepsilon, \ell)$ are (C, ℓ) -matched.

We will need the following distortion control for matched unstable dynamical balls.

Proposition 10.5. For ε small enough, there exists $\kappa_1 = \kappa_1(f, \varepsilon)$ such that if ℓ is sufficiently large, for any (C, ℓ) -matched dynamical balls $J(x^u)$ and $J(y^u)$, and all $a \in J(x^u)$ and $b \in J(y^u)$, we have

$$\kappa_1^{-1} \leq \frac{\lambda_b^*(j)}{\lambda_a^*(j)} \leq \kappa_1$$

for every integer $0 \leq j \leq \max(\tau(x, x^u, \varepsilon, \ell), \tau(y, y^u, \varepsilon, \ell))$ and $\star = c, u$.

Proof. Assume that $\tau(\ell) \leq \tau'(\ell)$, where $\tau(\ell) = \tau(x, x^u, \varepsilon, \ell)$ and $\tau'(\ell) = \tau(y, y^u, \varepsilon, \ell)$. First notice that by Lemma 9.4, we have $\tau' \leq \tau + T_0$.

Let $j \leq \tau(\ell)$ so in particular

$$|f^j(J(x^u))|, |f^j(J(y^u))| < 1.$$

Let $z^u = H_{x,y}^s(x^u)$ so $d(x^u, z^u) < 1$ for ℓ large enough, and, by Corollary 9.3, $d(f^j(y^u), f^j(z^u)) \leq d(f^{\tau'}(y^u), f^{\tau'}(z^u)) \asymp_{C, T_0} \varepsilon < 1$.

Let $\phi_\star := \log \|Df|_{E^\star}\|$ and c_\star, θ_\star be the Hölder constant and exponent of ϕ_\star , for $\star = c, u$. We have

$$\left| \log \frac{\lambda_b^*(j)}{\lambda_a^*(j)} \right| \leq \sum_{i=0}^{j-1} |\phi_\star(f^i(a)) - \phi_\star(f^i(b))|.$$

Hence,

$$\begin{aligned} |\phi_\star(f^i(a)) - \phi_\star(f^i(b))| &\leq |\phi_\star(f^i(a)) - \phi_\star(f^i(x^u))| + |\phi_\star(f^i(x^u)) - \phi_\star(f^i(z^u))| \\ &\quad + |\phi_\star(f^i(z^u)) - \phi_\star(f^i(y^u))| + |\phi_\star(f^i(y^u)) - \phi_\star(f^i(b))|, \end{aligned}$$

Note that for $i < j$, $d(f^i(a), f^i(x^u)), d(f^i(b), f^i(y^u)) \leq e^{\chi_1^u(i-j)}$. On the other hand, $d(f^i(x^u), f^i(z^u)) \leq e^{i\chi_2^s}$. Finally, $d(f^i(z^u), f^i(y^u)) \leq e^{\chi_1^c(i-j)}$. It follows that

$$|\phi_\star(f^i(a)) - \phi_\star(f^i(b))| \leq c_\star (2e^{\theta_\star \chi_1^u(i-j)} + e^{i\theta_\star \chi_2^s} + e^{\theta_\star \chi_1^c(i-j)}).$$

By summing over i we deduce that $|\log \frac{\lambda_b^*(j)}{\lambda_a^*(j)}|$ is uniformly bounded from above by a constant depending only on f and ρ . Of course we can now bound these quotients for j up to $\tau'(\ell)$ by using

$$d_0^{-T_0} \frac{\lambda_b^*(j)}{\lambda_a^*(j)} \leq \frac{\lambda_b^*(j + T_0)}{\lambda_a^*(j + T_0)} \leq d_0^{T_0} \frac{\lambda_b^*(j)}{\lambda_a^*(j)},$$

where $d_0 = \|Df\|/m(Df)$ (recall Section 2.1.2). ■

Corollary 10.6. *There exists $\kappa_2 = \kappa_2(f, \varepsilon)$ such that if ℓ is sufficiently large, then for any (C, ℓ) -matched dynamical balls $J(x^u)$ and $J(y^u)$,*

$$\kappa_2^{-1} \leq \frac{|f^j(J(y^u))|}{|f^j(J(x^u))|} \leq \kappa_2$$

for every integer $0 \leq j \leq \max(\tau(x, x^u, \varepsilon, \ell), \tau(y, y^u, \varepsilon, \ell))$.

Proof. As before, we denote $\tau(\ell) = \tau(x, x^u, \varepsilon, \ell)$ and $\tau'(\ell) = \tau(y, y^u, \varepsilon, \ell)$. Suppose $\tau(\ell) \leq \tau'(\ell) \leq \tau(\ell) + T_0$ (where T_0 is defined by Lemma 9.4). In particular, $m(Df)^{T_0} \leq |f^{\tau(\ell)}(J(y^u))| \leq 1$ and $|f^{\tau(\ell)}(J(x^u))| = 1$.

For $j \in \{0, \dots, \tau(\ell)\}$, we notice that

$$\frac{|f^j(J(x^u))|}{|f^j(J(y^u))|} = \frac{\int_{f^{\tau(\ell)}(J(x^u))} \lambda_q^u(k) dq}{\int_{f^{\tau(\ell)}(J(y^u))} \lambda_q^u(k) dq},$$

where $k = j - \tau(\ell) \leq 0$. By bounded distortion along the strong-unstable manifolds (2.7) we find that

$$\begin{aligned} C_0^{-2} \frac{\lambda_{f^{\tau(\ell)}(x^u)}^u(k)}{\lambda_{f^{\tau(\ell)}(y^u)}^u(k)} \frac{|f^{\tau(\ell)}(J(x^u))|}{|f^{\tau(\ell)}(J(y^u))|} &\leq \frac{|f^j(J(x^u))|}{|f^j(J(y^u))|} \\ &\leq C_0^2 \frac{\lambda_{f^{\tau(\ell)}(x^u)}^u(k)}{\lambda_{f^{\tau(\ell)}(y^u)}^u(k)} \times \frac{|f^{\tau(\ell)}(J(x^u))|}{|f^{\tau(\ell)}(J(y^u))|}. \end{aligned}$$

Notice that

$$\lambda_{f^{\tau(\ell)}(x^u)}^u(k) = \frac{\lambda_{x^u}^u(-k)}{\lambda_{x^u}^u(\tau(\ell))},$$

and a similar equality holds with y^u instead of x^u . Therefore, we can apply Proposition 10.5 to obtain

$$C_0^{-2} \kappa_1^{-2} \leq \frac{|f^j(J(x^u))|}{|f^j(J(y^u))|} \leq C_0^2 \|Df\|^{T_0} \kappa_1^2.$$

We conclude the proof of the corollary by noting that

$$m(Df)^{2T_0} \frac{|f^j(J(x^u))|}{|f^j(J(y^u))|} \leq \frac{|f^{j+T_0}(J(x^u))|}{|f^{j+T_0}(J(y^u))|} \leq \|Df\|^{2T_0} \frac{|f^j(J(x^u))|}{|f^j(J(y^u))|}. \quad \blacksquare$$

10.2. Matching of Y -configurations

We now define the notion of matched Y -configurations and show that our drift argument (Proposition 8.5) boils down to constructing arbitrarily long pairs of matched Y -configurations.

10.2.1. Matching. We first give the main definition.

Definition 10.7 (Matched Y -configurations). Let $\mathcal{L} \subset \mathbb{T}^3$, $C > 0$, and $\ell \in \mathbb{N}$. Let $X = X(x, x^u, \ell)$ and $Y = Y(y, y^u, \ell)$ be two Y -configurations of length ℓ . We say that X and Y are (\mathcal{L}, C, T) -matched if there exist $a, b \in \mathbb{T}^3$ and $\tau, t \in \mathbb{N}$ such that

- (1) (x, x^u, y, y^u) is a (C, ℓ) -quadrilateral;
- (2) $a \in J(x^u, \varepsilon, \ell)$ and $b \in J(y^u, \varepsilon, \ell)$;
- (3) $|\tau - \tau(x, x^u, \varepsilon, \ell)| \leq T$ and $|t - t(x, x^u, \varepsilon, \ell)| \leq T$;
- (4) $a, b, x, y, f^\tau(a), f^\tau(b), f^t(x), f^t(y) \in \mathcal{L}$.

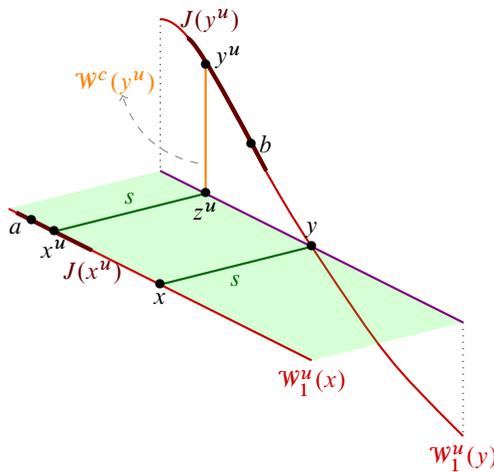


Fig. 13. For matched Y -configurations we can only put inside the Luzin set “small perturbations” a and b of the endpoints x^y and y^u (respectively) of the quadrilateral.

Remark 10.8. At first sight this definition is not symmetric (see item (3)). But the synchronization estimate (Lemma 9.4) implies that,

$$|\tau - \tau(y, y^u, \varepsilon, \ell)| \leq T + T_0 \quad \text{and} \quad |t - t(y, y^u, \varepsilon, \ell)| \leq T + T_0.$$

10.2.2. Finding pairs of long and matched Y -configurations. We are now ready to state our main technical lemma which, as we shall see, implies Proposition 8.5.

Let \mathcal{L} be the Luzin set defined in Section 8.1.2 and $\delta = 3\delta_0 > 0$ be the constant fixed in Section 8.1.3: we shall give further assumptions on δ later on.

Lemma 10.9. *Let $K_0 \subset \mathcal{L}$ be a compact set of measure $\mu(K_0) > 1 - \delta$. There exist constants $C = C(\delta)$ and $T = T(\delta)$ and an infinite subset $\mathcal{D} \subset \mathbb{N}$ such that for every $\ell \in \mathcal{D}$ there exists a pair (X, Y) of (K_0, C, T) -matched Y -configurations of length ℓ .*

The end of the section is devoted to proving that Lemma 10.9 implies Proposition 8.5. The proof of Lemma 10.9 will be the object of Section 11.

10.3. Asymptotic control of leaf-wise measures for matched configurations

The goal of this subsection is to study how leaf-wise quotient measures change along a pair of matched Y -configurations, and what happens when we have sequences of longer and longer pairs. To simplify the exposition, we shall break this explanation into two parts. First we deal with a single Y -configuration, and then we treat the full situation.

10.3.1. Control along a Y -configuration. The lemma below is essentially an easy corollary of Lemmas 7.13 and 7.14.

Lemma 10.10. *For $x \in \mathbb{T}^3$ and $x^u \in \mathcal{W}_r^u(x)$, for some $r > 0$, $\varepsilon > 0$, and $\ell \in \mathbb{N}$ as above, we have*

$$\hat{\nu}_{f\tau(\ell)(x^u)}^c \propto (A_{x,x^u,\ell})_* \hat{\nu}_{f\tau(\ell)(x)}^c$$

for the linear map $A_{x,x^u,\ell}: s \mapsto a_{x,x^u,\ell} \cdot s$ with

$$a_{x,x^u,\ell} := \rho_{x^u}^c(x) \frac{\lambda_x^c(t(\ell))}{\lambda_{x^u}^c(\tau(\ell))}.$$

Moreover, $|a_{x,x^u,\ell}| \in (a_0^{-1}, a_0)$ for some constant $a_0 = a_0(r) > 1$ depending only on the upper bound $r > 0$ on the distance along \mathcal{W}^u between x and x^u ; in particular, the linear map $A_{x,x^u,\ell}$ is uniformly bounded away from 0 and ∞ , independently of ℓ .

Proof. Applying Lemmas 7.13 and 7.14 we successively obtain

$$\begin{aligned} \hat{\nu}_{f\tau(\ell)(x^u)}^c &\propto (\Lambda_{\tau(\ell),x^u}^c)_* \hat{\nu}_{x^u}^c \propto (L_{x,x^u} \circ \Lambda_{\tau(\ell),x^u}^c)_* \hat{\nu}_x^c \\ &\propto (L_{x,x^u} \circ \Lambda_{\tau(\ell),x^u}^c \circ (\Lambda_{t(\ell),x}^c)^{-1})_* \hat{\nu}_{f\tau(\ell)(x)}^c, \end{aligned}$$

where $\Lambda_{n,x}^c$ denotes the linear map $s \mapsto \lambda_x^c(n)s$, and L_{x,x^u} denotes the linear map $s \mapsto \rho_{x^u}^c(x) \cdot s$. Thus, $A_{x,x^u,\ell} := \Lambda_{\tau(\ell),x^u}^c \circ L_{x,x^u} \circ (\Lambda_{t(\ell),x}^c)^{-1}$ is equal to

$$s \mapsto \rho_{x^u}^c(x) \frac{\lambda_x^c(t(\ell))}{\lambda_{x^u}^c(\tau(\ell))} \cdot s.$$

By the fact that $d_u(x, x^u) < r$, and by the definition of $t(\ell)$, $|A'_{x,x^u,\ell}(0)|$ is uniformly bounded, depending only on r and f , but not on ℓ , which concludes the proof. ■

10.3.2. Control for matched configurations. We now deal with the full picture of sequences of pairs of good and matched configurations. For the next result, we refer to

Section 8.1.2 for the definition of the Luzin set \mathcal{L} , and to Sections 9.3 and 10.2 for the definitions of quadrilaterals and (C, ℓ) -matched Y -configurations respectively.

The result of this subsection is the core of the proof that Lemma 10.9 implies Proposition 8.5. *Until the end of this section, the notation x_n will NOT stand for $f^n(x)$ but for the usual notation of sequences.*

Lemma 10.11. *Let $\varepsilon > 0$ and let (ℓ_n) be an increasing sequence of integers. Suppose there exist constants $T, C > 0$ (independent of ε), and two sequences $X_n = (x_n, x_n^u, \ell_n)$ and $Y_n = (y_n, y_n^u, \ell_n)$ of Y -configurations of length ℓ_n that are (\mathcal{L}, C, T) -matched. Let τ_n, t_n, a_n, b_n be the objects given by the condition of matching and consider the sequences*

$$f^{\tau_n}(a_n), f^{\tau_n}(b_n) \in \mathcal{L} \quad \text{and} \quad f^{\tau_n}(x_n^u), f^{\tau_n}(y_n^u) \in \mathbb{T}^3.$$

Assume all these sequences converge to a_∞, b_∞, p and q respectively. Then $q \in \mathcal{W}^c(p)$ and there exist $\gamma = \gamma(C, T) > 1$ and $M = M(C, T)$ such that

$$\varepsilon/\gamma \leq |\mathcal{H}_p^c(q)| \leq \gamma\varepsilon, \tag{10.2}$$

and

$$\hat{v}_{a_\infty}^c \propto B_* \hat{v}_{b_\infty}^c. \tag{10.3}$$

where $B(s) = \beta \cdot s$ is a linear map that satisfies $1/M < |\beta| < M$.

Proof. Note first that τ_n and t_n tend to infinity: indeed, this is a consequence of the matching condition and of the fact that $\ell_n \rightarrow \infty$ so the stopping times also tend to infinity (by Lemma 9.1).

Now suppose the sequences $f^{\tau_n}(a_n), f^{\tau_n}(b_n), f^{\tau_n}(x_n), f^{\tau_n}(y_n)$ converge to a_∞, b_∞, p, q respectively. Let $z_n^u = H_{x_n, y_n}^s(x_n^u) \in \mathcal{W}^s(x_n)$. The condition of matching implies that $d(z_n^u, x_n^u)$ is uniformly bounded, so $d(f^{\tau_n}(z_n), f^{\tau_n}(x_n)) \rightarrow 0$ as $n \rightarrow \infty$ and $f^{\tau_n}(z_n^u) \rightarrow p$. Since normal forms change continuously, we have

$$\mathcal{H}_{f^{\tau_n}(z_n^u)}^c(f^{\tau_n}(y_n^u)) \rightarrow \mathcal{H}_p^c(q) \quad \text{as } n \rightarrow \infty.$$

Thus (10.2) follows directly from Lemma 9.2 and Remark 10.8. Let us show (10.3). For this we use the matching condition: for all n , the points $f^{\tau_n}(a_n), f^{\tau_n}(b_n), f^{t_n}(x_n), f^{t_n}(y_n)$ all belong to the Luzin set \mathcal{L} .

Combining Lemma 10.10 with Lemma 7.14 and Remark 10.8 we deduce that there exist linear maps $B_n: \mathbb{R} \rightarrow \mathbb{R}$ and $\tilde{B}_n: \mathbb{R} \rightarrow \mathbb{R}$ and constants $C_n, \tilde{C}_n > 0$ such that the derivatives B'_n and \tilde{B}'_n lie inside $[M^{-1}, M]$ for some constant $M = M(T, C)$ independent of n , and

$$\hat{v}_{f^{\tau_n}(a_n)}^c = C_n(B_n)_* \hat{v}_{f^{t_n}(x_n)}^c \quad \text{and} \quad \hat{v}_{f^{\tau_n}(b_n)}^c = \tilde{C}_n(\tilde{B}_n)_* \hat{v}_{f^{t_n}(y_n)}^c.$$

We claim that C_n and \tilde{C}_n are uniformly bounded away from 0 and ∞ . Indeed, let us consider C_n . Let $c = c(M) > 0$ be the constant given by Corollary 8.4. Then, denoting $I = [-1, 1]$ we find that, by our choice of normalization (see Remark 7.10), $\hat{v}_{f^{\tau_n}(a_n)}^c(I) = 1$

and therefore

$$C_n = \frac{1}{\hat{v}_{f^{tn}(x_n)}^c(B_n^{-1}(I))} \in [c^{-1}, c],$$

because $f^{tn}(x_n) \in \mathcal{L}$ for each n . A similar argument treats the constants \tilde{C}_n . Therefore up to enlarging M , we can assume that $C_n, \tilde{C}_n \in [M^{-1}, M]$.

The condition of matching implies that $y_n \in \mathcal{W}^s(x_n)$ for every n and that $d(x_n, y_n)$ is uniformly bounded. Now recall that $f^{tn}(x_n), f^{tn}(y_n)$ belong to the Luzin set \mathcal{L} , which is compact. We can assume without loss of generality that there exists a point $x \in \mathcal{L}$ such that $f^{tn}(x_n) \rightarrow x$ and $f^{tn}(y_n) \rightarrow x$. For simplicity, set $\hat{a}_n = f^{\tau_n}(a_n), \hat{b}_n = f^{\tau_n}(b_n), \hat{x}_n = f^{tn}(x_n)$ and $\hat{y}_n = f^{tn}(y_n)$. We can assume that $C_n \rightarrow C, \tilde{C}_n \rightarrow \tilde{C}$, and there exist linear maps $B, \tilde{B} : \mathbb{R} \rightarrow \mathbb{R}$ such that $B_n \rightarrow B$ and $\tilde{B}_n \rightarrow \tilde{B}$ in the C^1 topology, i.e. the slopes of B_n converge to the slope of B and similarly for \tilde{B}_n and \tilde{B} .

Now, we claim that $\hat{v}_{a_\infty}^c = CB_*\hat{v}_x^c$ and $\hat{v}_{b_\infty}^c = \tilde{C}\tilde{B}_*\hat{v}_x^c$. Once we prove this claim, the lemma will be established. To prove the claim, we first observe that

$$\int C_n(\varphi \circ B_n) d\hat{v}_{\hat{x}_n}^c \rightarrow \int C(\varphi \circ B) d\hat{v}_x^c$$

for every $\varphi \in C_c^0(\mathbb{R})$. Indeed, we can write

$$\begin{aligned} \left| \int C_n(\varphi \circ B_n) d\hat{v}_{\hat{x}_n}^c - \int C(\varphi \circ B) d\hat{v}_x^c \right| &\leq \left| \int C_n(\varphi \circ B_n) d\hat{v}_{\hat{x}_n}^c - \int C(\varphi \circ B) d\hat{v}_{\hat{x}_n}^c \right| \\ &\quad + \left| \int C(\varphi \circ B) d\hat{v}_{\hat{x}_n}^c - \int C(\varphi \circ B) d\hat{v}_x^c \right|. \end{aligned}$$

The second term on the right-hand side above converges to zero as $n \rightarrow \infty$ because \mathcal{L} satisfies the conclusion of Lemma 8.3 and $\hat{x}_n \in \mathcal{L}$ converges to x . The first term also converges to zero because $C_n(\varphi \circ B_n)$ converges uniformly to $C(\varphi \circ B)$ and $\hat{v}_{\hat{x}_n}^c(\text{supp}(\varphi))$ is bounded independently of n . We use this observation to conclude the proof of the claim as follows. Given any $\varphi \in C_c^0(\mathbb{R})$ we have

$$\begin{aligned} \int \varphi d\hat{v}_{a_\infty}^c &= \lim_{n \rightarrow \infty} \int \varphi d\hat{v}_{\hat{a}_n}^c = \lim_{n \rightarrow \infty} \int C_n(\varphi \circ B_n) d\hat{v}_{\hat{x}_n}^c \\ &= \int C(\varphi \circ B) d\hat{v}_x^c, \end{aligned}$$

proving that $\hat{v}_{a_\infty}^c = CB_*\hat{v}_x^c$. The proof that $\hat{v}_{b_\infty}^c = \tilde{C}\tilde{B}_*\hat{v}_x^c$ is similar so we omit it. ■

With Lemma 10.11 at hand we are now in a position to reduce the proof of Theorem 5.10 to the proof of Lemma 10.9.

10.4. Proof that Lemma 10.9 implies Proposition 8.5

Let K_{00} be a compact set with $\mu(K_{00}) > 1 - 2\delta_0$. We apply Lemma 10.9 to the compact set $K_0 = K_{00} \cap \mathcal{L}$ which has measure $\mu(K_0) > 1 - 3\delta_0 = 1 - \delta$. So let $C = C(\delta) > 0, T_0 = T_0(\delta) > 0, K_0 = K_{00} \cap \mathcal{L}$, and \mathcal{D} be the objects given by Lemma 10.9.

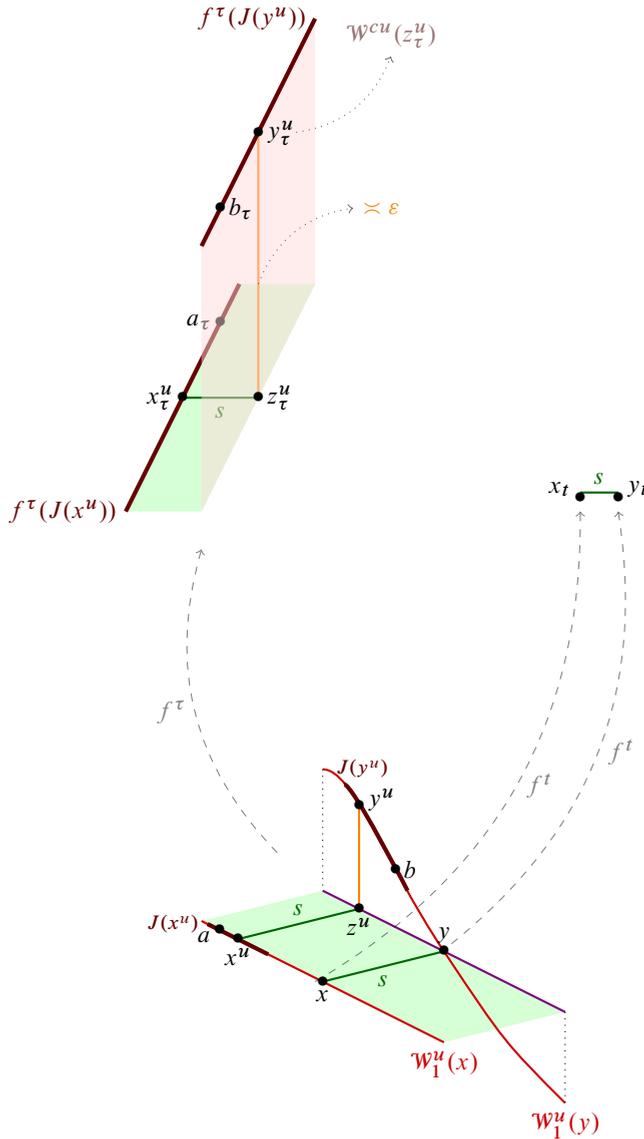


Fig. 14. For each ℓ_n we have a picture like this one (we have suppressed the dependence on n for simplicity). The top-left part will converge to Figure 15.

As the set \mathcal{D} is infinite, there exists a sequence $\ell_n \rightarrow \infty$ of integers belonging to \mathcal{D} . For each such integer, let $X_n = (x_n, x_n^u, \ell_n)$ and $Y_n = (y_n, y_n^u, \ell_n)$ be the pair of (K_0, C, T) -matched Y -configurations given by Lemma 10.9. We let a_n, b_n, τ_n, t_n be the points and times corresponding to the pair (X_n, Y_n) (see Definition 10.7). By definition,

$$f^{\tau_n}(a_n), f^{t_n}(b_n) \in K_0.$$

We also consider the sequences

$$f^{\tau_n}(x_n^u), f^{\tau_n}(y_n^u).$$

Upon extracting subsequences if necessary, we may assume that these four sequences converge respectively to a_∞, b_∞, p and q . Observe that $a_\infty, b_\infty \in K_0$, but we do not know if the same holds for p and q . However, we have good estimates for the distance between these points (see Figure 14). Indeed, by Lemma 10.11 there exist $\gamma = \gamma(C, T) > 1$ and $M = M(C, T)$ such that

$$\varepsilon/\gamma \leq |\mathcal{H}_p^c(q)| \leq \gamma\varepsilon, \tag{10.4}$$

and

$$\hat{v}_{a_\infty}^c \propto B_* \hat{v}_{b_\infty}^c. \tag{10.5}$$

where $B(s) = \beta \cdot s$ satisfies $1/M < |\beta| < M$. We shall prove that these conditions ensure that $b_\infty \in G(\varepsilon, \tilde{M})$ for some constant \tilde{M} to be defined later. This will show Proposition 8.5. Notice that by construction the four points a_∞, b_∞, p and q belong to the

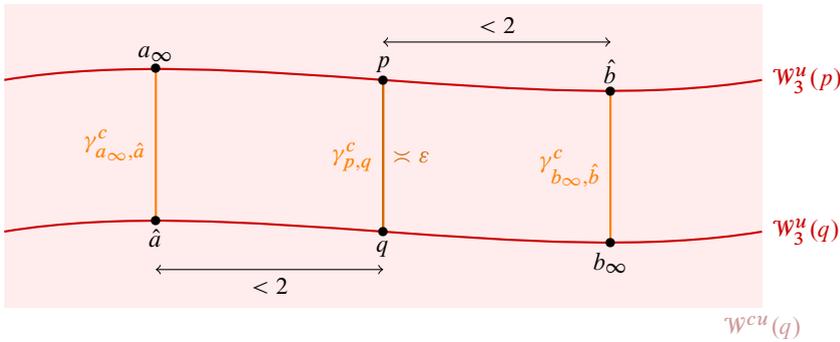


Fig. 15. How to get invariance by an affine map using the exponential drift: we can move the leaf-wise measures from b_∞ to a_∞ with a linear map, and from a_∞ to \hat{b} with a linear map and from \hat{b} back to b_∞ with an affine map.

same center-unstable leaf. Let us consider the local strong-unstable manifolds $\mathcal{W}_3^u(p)$ and $\mathcal{W}_3^u(q)$. Notice that, also by construction, we have $a_\infty \in \mathcal{W}_1^u(p)$ and $b_\infty \in \mathcal{W}_1^u(q)$. An application of Corollary 9.3 yields

$$d_c(p, q) \leq \kappa\varepsilon. \tag{10.6}$$

Hence we can choose ε small enough so that the intersection point $\hat{a} = \mathcal{W}^u(q) \cap \mathcal{W}^c(a_\infty)$ given by Lemmas 2.3 and 2.4 satisfies $d_u(\hat{a}, q) < 2$. Similarly we can choose $\hat{b} = \mathcal{W}^u(p) \cap \mathcal{W}^c(b_\infty)$ so that $d_u(\hat{b}, p) < 2$ (see Figure 15). Denote by $\gamma_{a_\infty, \hat{a}}^c$ the segment of center manifold joining the points a_∞ and \hat{a} and similarly consider the segments of center manifolds $\gamma_{p,q}^c$ and $\gamma_{b_\infty, \hat{b}}^c$. By (10.6) we have

$$\text{length}(\gamma_{p,q}^c) = \text{length}(\Phi_p^c[0, \mathcal{H}_p^c(q)]) \asymp_{C,\gamma} \varepsilon.$$

Note that $d_u(a_\infty, p) < 2$, $d_u(b_\infty, q) < 2$ and $d_c(p, q) \leq \kappa\varepsilon < \rho_0$, where ρ_0 is the constant of Lemma 2.5 (provided ε is chosen small enough). Hence Lemma 2.5 implies that the unstable holonomy maps H_{p,a_∞}^u and H_{q,b_∞}^u are bilipschitz with constants which depend only on f . Since $\gamma_{a_\infty, \hat{a}}^c = H_{p,a_\infty}^u(\gamma_{p,q}^c)$ and similarly $\gamma_{b_\infty, \hat{b}}^c = H_{q,b_\infty}^u(\gamma_{p,q}^c)$, we deduce that

$$\text{length}(\gamma_{a_\infty, \hat{a}}^c) \asymp_{C,\gamma} \varepsilon \quad \text{and} \quad \text{length}(\gamma_{b_\infty, \hat{b}}^c) \asymp_{C,\gamma} \varepsilon$$

Therefore, as ε is small we deduce that $\hat{b} \in \mathcal{W}_1^c(b_\infty)$. Using the uniform bound for the C^1 norm of the normal forms in segments of bounded length we find that

$$\mathcal{H}_{b_\infty}^c(\hat{b}) \asymp_{C,\gamma} \varepsilon. \tag{10.7}$$

We apply Lemma 7.14 to get a linear map $L: \mathbb{R} \rightarrow \mathbb{R}$ whose derivative satisfies $L'(0) \in [C^{-1}, C]$ for a constant $C = C(f)$ such that $\hat{v}_{\hat{b}} \propto L_* \hat{v}_{a_\infty}$. Therefore,

$$\hat{v}_{\hat{b}} \propto (LB)_* \hat{v}_{b_\infty}$$

and the linear map $L \circ B$ has a derivative in $[C^{-1}M^{-1}, CM]$. This together with (10.7) implies that $b_\infty \in G(\varepsilon, \tilde{M})$ for some constant $\tilde{M} = \tilde{M}(\gamma, C, M)$ (see (8.8) for the definition of the set $G(\varepsilon, \tilde{M})$), concluding the proof of Proposition 8.5. ■

11. Construction of matched Y -configurations: End of the proof

This section is devoted to the proof of Lemma 10.9. Recall that we reduced our main Theorem A to Lemma 10.9. Let us recall what we want to do. We assume that $\mu(\mathbf{B}) = 0$, where $\mathbf{B} = \mathbf{B}$ is the bad set introduced in Definition 5.2. In particular, it follows from the zero-one law (Theorem 4.2) that for μ -a.e. $x \in \mathbb{T}^3$,

$$\mu_x^s \{y \in \xi^s(x) : \alpha^s(x, y) = 0\} = 0. \tag{11.1}$$

where we recall that $\alpha^s(x, y) = \angle(DH_{x,y}^s(x)E^u(x), E^u(y))$ when $y \in \mathcal{W}^s(x)$. Given a large compact set K_0 we want to find arbitrarily long pairs of *matched* Y -configurations which are K_0 -good (meaning that their points belong to K_0).

11.1. Angle condition and absolute continuity

We start this section by showing how to use the condition $\mu(\mathbf{B}) = 0$ to obtain abundance of pairs of points $y \in \mathcal{W}_1^s(x)$ such that $\alpha^s(x, y)$ is uniformly bounded from below.

Before carrying out the proof recall that for $\star = s, u, c, cu$, we have fixed measurable partitions ξ^\star subordinate to \mathcal{W}^\star , whose atoms have diameter less than 1, as well as disintegrations $\{\mu_x^\star\}_x$ relative to ξ^\star . We fixed a Luzin set \mathcal{L} of measure $\mu(\mathcal{L}) > 1 - \delta$ (further assumptions on δ will be given later on) as given in Section 8.1.2. In particular, there exists $r_0 > 0$ such that for every $x \in \mathcal{L}$,

$$\mathcal{W}_{r_0}^\star(x) \subset \xi^\star(x). \tag{11.2}$$

11.1.1. *A Markov type inequality.* Our first ingredient will be a simple inequality à la Markov that will be used several times throughout the section.

Lemma 11.1. *Let (X, \mathcal{B}, μ) be a probability space, and $\eta \in (0, 1)$. Let $\psi: X \rightarrow [0, 1]$ be a measurable function with $\int \psi d\mu > 1 - \eta$. Let $B := \{x \in X : \psi(x) > 1 - \sqrt{\eta}\}$. Then $\mu(B) > 1 - \sqrt{\eta}$.*

Proof. We have

$$\begin{aligned} 1 - \eta &< \int_B \psi d\mu + \int_{X \setminus B} \psi d\mu \leq \mu(B) + (1 - \sqrt{\eta})(1 - \mu(B)) \\ &= 1 - \sqrt{\eta}(1 - \mu(B)), \end{aligned}$$

which gives $\mu(B) > 1 - \sqrt{\eta}$. ■

11.1.2. *Bounding the angle function from below.* We fix a measurable partition ξ^s subordinate to \mathcal{W}^s and a disintegration $\{\mu_x^s\}_x$ of μ relative to ξ^s . Below we use the condition $\mu(\mathbf{B}) = 0$; by (11.1) it means that for μ -a.e. $x \in \mathbb{T}^3$ and μ_x^s -a.e. $y \in \xi^s(x)$, $\alpha^s(x, y) > 0$.

Until the end of the section we fix $r_0 > 0$ such that (11.2) holds for all $x \in \mathcal{L}$. We consider a constant $\eta = \eta(\delta) > 0$ which goes to 0 with δ . It will be explicitly given later on.

Lemma 11.2. *Let $B \subset \mathcal{L}$ with $\mu(B) > 1 - \eta$. Then there exists a measurable set $B' \subset B$ as well as a number $c = c(\eta)$ such that $0 < c \leq r_0\eta$ and the following properties hold:*

- (1) $\mu(B') > 1 - 2\sqrt{\eta}$;
- (2) for μ -a.e. $x \in B'$ there exists $y \in B \cap \xi^s(x)$ such that $\alpha^s(x, y) > c$.

Proof. Set $B_1 := \{x \in \mathbb{T}^3 : \mu_x^s[B \cap \xi^s(x)] > 1 - \sqrt{\eta}\}$. It follows from Lemma 11.1 that $\mu(B_1) > 1 - \sqrt{\eta}$.

Now, given $\alpha \geq 0$, we define, for μ -a.e. $x \in \mathbb{T}^3$,

$$\begin{aligned} \mathcal{A}_\alpha(x) &:= \{y \in \xi^s(x) : \alpha^s(x, y) > \alpha\}, \\ \mathcal{A}_{\alpha, \eta} &:= \{x \in \mathbb{T}^3 : \mu_x^s[\mathcal{A}_\alpha(x)] > 1 - \eta\}. \end{aligned}$$

Using $\mu(\mathbf{B}) = 0$ we find $\mu_x^s[\mathcal{A}_0(x)] = 1$ for μ -a.e. $x \in \mathbb{T}^3$. This implies that $\mu(\mathcal{A}_{0, \eta}) = 1$. Hence, there exists $\alpha = \alpha(\eta) > 0$ such that the set $B_2 := \mathcal{A}_{\alpha, \eta}$ has measure $\mu(B_2) > 1 - \eta$.

Set

$$B' := B \cap B_1 \cap B_2,$$

so that

$$\mu(B') > 1 - 2\eta - \sqrt{\eta} > 1 - 2\sqrt{\eta}$$

(provided $\eta < 1/4$, so $2\eta < \sqrt{\eta}$). Then for any $x \in B'$,

- $\mu_x^s(B \cap \xi^s(x)) > 1 - \sqrt{\eta}$;
- $\mu_x^s\{y \in \xi^s(x) : \alpha^s(x, y) > \alpha\} > 1 - \eta$.

Set $c = c(\eta) := \min(\alpha(\eta), r_0\eta) \in (0, r_0\eta]$. For every $x \in B'$, we have

$$\mu_x^s \{y \in B \cap \xi^s(x) : \alpha^s(x, y) > c\} \geq 1 - \sqrt{\eta} - \eta > 1 - 2\sqrt{\eta} > 0.$$

Hence, for every $x \in B'$, there exists $y \in \xi^s(x)$ such that $\alpha^s(x, y) > c$. This concludes the proof of the lemma. \blacksquare

11.2. Recurrence estimates: Building long and good Y -configurations

In this subsection we fix a measurable set $K \subset \mathbb{T}^3$ of measure $\mu(K) > 1 - \delta$ for some small $\delta > 0$. The main result of this subsection is Proposition 11.6, which builds K -good Y -configurations of length ℓ for every integer ℓ inside a subset of \mathbb{N} of positive density. We start by establishing some preliminary results from ergodic theory.

11.2.1. An elementary quantitative recurrence estimate. For clarity of presentation, we introduce the following notion.

Definition 11.3. Given $\gamma > 0$, $n > 0$ and a measurable set B , we say that a point $x \in \mathbb{T}^3$ is (γ, n) -recurrent to B if $L > n$ implies that

$$\#\{\ell \in [0, L] : f^\ell(x) \in B\} > (1 - \gamma)L.$$

The following holds for any ergodic system (f, \mathcal{B}, μ) .

Lemma 11.4. For every measurable set B with $\mu(B) > 1 - \gamma$ there exist $T = T(\gamma)$ and a subset $B^\circ \subset B$ with $\mu(B^\circ) > 1 - \gamma$ such that any $x \in B^\circ$ is (γ, T) -recurrent to B .

Proof. Let B be a measurable set with $\mu(B) > 1 - \gamma$. We consider the sequence $(\varphi_n)_{n \in \mathbb{N}}$ of L^1 functions given by $\varphi_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_B(f^k(x))$. By Birkhoff's theorem, the sequence $(\varphi_n)_{n \in \mathbb{N}}$ converges almost surely to the constant function $\mu(B)$. Moreover, by Egorov's theorem, there exists a measurable subset $B^\circ \subset B$ with $\mu(B^\circ) > 1 - \gamma$ such that $(\varphi_n|_{B^\circ})_{n \in \mathbb{N}}$ converges uniformly to $\mu(B)$. Then there exists $T > 0$ such that for any $x \in B^\circ$ and $n > T$, $\varphi_n(x) > 1 - \gamma$. Such an x is (γ, T) -recurrent to B by definition. \blacksquare

11.2.2. Stopping times and return times to K for pairs (x, x^u) . Recall that we have fixed a measurable set $K \subset \mathbb{T}^3$ with $\mu(K) > 1 - \delta$. Applying Lemma 11.4 to K yields an integer $T_1 = T_1(\delta)$ and a measurable subset $K^\circ \subset K$ with $\mu(K^\circ) > 1 - \delta$ consisting of (δ, T_1) -recurrent points to K . Up to enlarging T_1 we can assume

$$T_1 > 4T_0, \tag{11.3}$$

where T_0 is the constant given by the synchronization estimates of Lemmas 9.4 and 10.3.

Given a pair $\omega = (x, x^u)$ with $x^u \in \xi^u(x)$, we set

$$E(x, x^u) = E(\omega) := \{\ell \in \mathbb{N} : f^{\tau(\ell)}(x^u) \in K \text{ and } f^{\iota(\ell)}(x) \in K\}. \tag{11.4}$$

The objects T_1 and K° appearing in the next statement are the ones constructed in the previous subsection.

Lemma 11.5. *There exists a constant $r > 0$ which only depends on the quasi-isometric estimates such that for all $L > T_1$, $x \in K^\circ$, $x^u \in \xi^u(x) \cap K^\circ$ and $\omega = (x, x^u)$,*

$$\#(E(\omega) \cap [0, L]) > (1 - r\delta)L.$$

Proof. Recall that the stopping times satisfy $t(\ell) \geq 0$ and the quasi-isometric estimate

$$\Theta^{-1}|l - m| - A < |t(\ell) - t(m)| < \Theta|l - m| + A$$

for some constants $\Theta > 1$ and $A > 0$ depending only on f : see Lemma 9.1. In particular, for every $L > 0$ we have $t(L) \in [0, L']$ where $L' = \Theta L + A$. By hypothesis, $x \in K^\circ$ so whenever $L > T_1$ (and thus $L' > T_1$), we have

$$\#\{k \in [0, L'] : f^k(x) \notin K\} \leq \delta L'.$$

On the other hand, the quasi-isometric estimate also implies that an integer in $[0, L']$ has at most $2A\Theta$ pre-images by t so we have

$$\#\{\ell \in [0, L] : f^{t(\ell)}(x) \notin K\} \leq 2A\Theta \#\{k \in [0, L'] : f^k(x) \notin K\} \leq 2A\Theta\delta(\Theta L + A).$$

This proves that for every $L > T_1$, the set $E_t(x) := \{\ell \in \mathbb{N} : f^{t(\ell)}(x) \in K\}$ has density $> 1 - r_1\delta$ inside $[0, L]$ for some constant $r_1 > 0$ depending only on A and Θ . Now, since x^u also belongs to K° , the same property also holds for the set $E_\tau(x^u) := \{\ell \in \mathbb{N} : f^{\tau(\ell)}(x^u) \in K\}$: it has density $> 1 - r_2\delta$ inside $[0, L]$ for $r_2 > 0$ depending only on the quasi-isometry constants of τ from Lemma 9.1.

Finally, we can estimate from below the density of $E(\omega) = E(x, x^u) = E_\tau(x^u) \cap E_t(x)$ inside $[0, L]$ using the inequality

$$\#(E(\omega) \cap [0, L]) \geq \#(E_\tau(x^u) \cap [0, L]) - \#([0, L] \setminus E_t(x)) > (1 - (r_1 + r_2)\delta)L.$$

This ends the proof of the lemma. ■

11.2.3. Space of pairs (x, x^u) . It will be useful to consider a measurable structure on the space of pairs

$$\Omega := \{\omega = (x, x^u) : x \in \mathbb{T}^3 \text{ and } x^u \in \xi^u(x)\}.$$

Note that Ω is contained in the continuous submanifold of $\mathbb{T}^3 \times \mathbb{T}^3$ defined as $Y = \{(x, x^u) : x \in \mathbb{T}^3 \text{ and } x^u \in \mathcal{W}_1^u(x)\}$ (recall that 1 is a uniform upper bound of the diameters of atoms of ξ^u). The topology on Y induced by the product topology of $\mathbb{T}^3 \times \mathbb{T}^3$ provides it with a Borel σ -algebra. Its restriction to Ω is denoted by \mathcal{A} . Let $\pi : \Omega \rightarrow \mathbb{T}^3$ be the projection on the first coordinate. So we have $B \in \mathcal{A}$ if and only if the projection $\pi(B)$ is a measurable subset of \mathbb{T}^3 , and $B \cap (\{x\} \times \xi^u(x))$ is a measurable subset of $\{x\} \times \xi^u(x)$. Hence we can define a measure ν on Ω by

$$\nu(B) := \int_{\pi(B)} \mu_x^u[B \cap (\{x\} \times \xi^u(x))] d\mu(x).$$

11.2.4. *Construction of many long and good Y -configurations.* Assume $\ell \in E(\omega)$ for $\omega = (x, x^u)$ with $x, x^u \in K^\circ$. Then $(x, x^u, x_{-\ell}, f^{\tau(\ell)}(x^u), f^{t(\ell)}(x))$ is a K -good Y -configuration of length ℓ . So constructing many long and good Y -configurations means constructing many integers $\ell \in \mathbb{N}$ for which the set of pairs $\omega = (x, x^u) \in K^\circ \times K^\circ$ with $\ell \in E(\omega)$ has large ν -measure.

More precisely, let us fix a continuous function $\eta: [0, 1] \rightarrow [0, \infty)$ vanishing at 0 and, abusing notation, write $\eta = \eta(\delta)$. An explicit construction will be given in the proof of the next proposition. We will assume that δ is small enough so that $\eta < 1$. For $\ell \in \mathbb{N}$ and $x \in K^\circ$ define

$$Q^u(x) := K^\circ \cap \xi^u(x), \tag{11.5}$$

$$Q^u(x, \ell) := \{x^u \in \xi^u(x) : x^u \in K^\circ \text{ and } \ell \in E(x, x^u)\} \subset Q^u(x). \tag{11.6}$$

This yields a set of K -good Y -configurations of length ℓ . Next we define

$$K(\ell) := \{x \in K^\circ : \mu_x^u[Q^u(x, \ell)] > 1 - \eta\}. \tag{11.7}$$

We want to construct a large set of integers ℓ with $\mu[K(\ell)]$ large enough. This is provided by the following statement, inspired by Eskin–Lindenstrauss’ paper [15] and which is essentially a Fubini-like argument.

Proposition 11.6 (see Eskin–Lindenstrauss [15, Claim 6.4]). *Set*

$$\mathcal{D} := \{\ell \in \mathbb{N} : \mu[K(\ell)] > 1 - \eta\}. \tag{11.8}$$

Then for every $L > T_1$ we have

$$\#(\mathcal{D} \cap [0, L]) > (1 - \eta)L.$$

Proof. We start the proof by considering the set

$$K_1 := \{x \in K^\circ : \mu_x^u[Q^u(x)] > 1 - \sqrt{\delta}\},$$

so that after applying Lemma 11.1 with $\psi: x \mapsto \mu_x^u[Q^u(x)] = \mu_x^u[K^\circ \cap \xi^u(x)]$ we see that $\mu(K_1) > 1 - \sqrt{\delta}$. The set

$$B := \{\omega = (x, x^u) \in \Omega : x \in K_1, x^u \in Q^u(x)\}$$

is a measurable subspace of Ω of measure

$$\nu(B) = \int_{K_1} \mu_x^u[Q^u(x)] d\mu(x) > (1 - \sqrt{\delta})^2 = 1 - \delta'$$

for some $\delta' = \delta'(\delta) > 0$ tending to zero with δ .

Now let $L > T_1$ and consider the space (I_L, m) for the set $I_L := [0, L] \cap \mathbb{N}$ endowed with the counting measure that we denote by m . Let $F \subset \Omega \times I_L$ be the set of pairs

(ω, ℓ) such that $\omega \in B$ and $\ell \in E(\omega)$. It follows from Lemma 11.5 that for all $\omega \in B$, $m[E(\omega)] > 1 - r\delta$. Hence

$$(v \times m)(F) = \int_B m[E(\omega)] dv(\omega) > (1 - r\delta)(1 - \delta') =: 1 - \delta''$$

for some $\delta'' = \delta''(\delta)$ that tends to zero with δ .

On the other hand, set $B(\ell) := \{\omega \in B : \ell \in E(\omega)\}$ so we have

$$F = \{(\omega, \ell) \in \Omega \times I_L : \omega \in B(\ell)\} = \bigcup_{\ell \in I_L} B(\ell) \times \{\ell\}.$$

Applying Fubini's theorem we get

$$\begin{aligned} (v \times m)(F) &= \int_{\Omega} \int_{I_L} \mathbf{1}_F(\omega, \ell) dm(\ell) dv(\omega) = \int_{I_L} \int_{\Omega} \mathbf{1}_F(\omega, \ell) dv(\omega) dm(\ell) \\ &= \int_{I_L} v(B(\ell)) dm(\ell). \end{aligned}$$

It follows from Lemma 11.1 that the set $\mathcal{D}_L := \{\ell \in I_L : v(B(\ell)) > 1 - \sqrt{\delta''}\}$ of integers has m -measure $> 1 - \sqrt{\delta''}$. Note that

$$v(B(\ell)) = \int_{K_1} \mu_x^u[Q^u(x, \ell)] d\mu(x),$$

hence, another application of Lemma 11.1 yields $\mu\{x \in K_1 : \mu_x^u[Q^u(x, \ell)] > 1 - \eta\} > 1 - \eta$ where $\eta := \sqrt[4]{\delta''}$ is the function we were looking for. Now since by definition $\{x \in K_1 : \mu_x^u[Q^u(x, \ell)] > 1 - \eta\} \subset K(\ell)$, we conclude that $\mu[K(\ell)] > 1 - \eta$. This implies that $\mathcal{D}_L \subset \mathcal{D} \cap [0, L]$ where \mathcal{D} is the set defined in (11.8). We deduce that $\#\mathcal{D} \cap [0, L] > (1 - \eta)L$ as claimed. ■

11.3. Construction of matched Y -configurations

We are now ready to finish the proof of Lemma 10.9, and thus that of our main theorem. We will build arbitrarily long pairs of good and matched Y -configurations.

Let K_0 be a compact set included in the Luzin set \mathcal{L} defined in Section 8.1.2 with $\mu(K_0) > 1 - \delta$.

We consider the measurable set

$$K = (K_0)^\circ,$$

so points of K are (δ, T) -recurrent in K_0 for some $T = T(\delta)$, and $\mu(K) > 1 - \delta$. We apply the results of Section 11.2.1 to K . They yield a set $\mathcal{D} \subset \mathbb{N}$ as defined in (11.7) and Proposition 11.6. In particular, by definition, \mathcal{D} has density $> 1 - \eta$ in $[0, L]$ for L large enough and for every $\ell \in \mathcal{D}$, $\mu[K(\ell)] > 1 - \eta$ (where $\eta = \eta(\delta) < 1$).

11.3.1. *Constructing the tails.* Recall that for $x, y \in \mathbb{T}^3$ and $\ell \in \mathbb{N}$ we let $x_{-\ell}$ and $y_{-\ell}$ denote $f^{-\ell}(x)$ and $f^{-\ell}(y)$ respectively.

Lemma 11.7. *For every $\ell \in \mathbb{D}$, there exists $K'(\ell) \subset K(\ell)$ such that $\mu[K'(\ell)] > 1 - 2\sqrt{\eta}$ and for every $x \in K'(\ell)$, there exists $y \in K(\ell) \cap \xi^s(x)$ such that*

$$\alpha^s(x_{-\ell}, y_{-\ell}) > c(\eta),$$

where $c(\eta)$ is the constant introduced in Lemma 11.2.

Proof. We apply Lemma 11.2 to $A = f^{-\ell}(K(\ell))$ with $\mu(K(\ell)) > 1 - \eta$. Then we can define $K'(\ell) := f^\ell(A')$ where A' is the set provided by that lemma. It is clear that this set satisfies the desired properties (recall that ξ^s is decreasing). ■

11.3.2. *Constructing the quadrilaterals: The matching argument.* For $x \in K'(\ell)$, so $\xi^u(x)$ is an interval of size $r(x)$, we set

$$I^u(x) := \{z \in \xi^u(x) : d^u(x, z) > c(\eta) \text{ and } d^u(z, \partial\xi^u(x)) > c(\eta)\}. \tag{11.9}$$

Recall that $0 < c(\eta) \leq r_0\eta \leq |\xi^u(x)|$ (see Lemma 11.2), hence $|I^u(x)| > (1 - 4\eta)|\xi^u(x)|$. Note furthermore that if $y \in \mathcal{L}$ is close enough to x then $H_{x,y}^{cs}(I^u(x)) \subset \xi^u(y)$ (we use here the fact that $r(y) = |\xi^u(y)|$ is uniformly continuous in \mathcal{L} and that center-stable holonomy maps converge uniformly to the identity as x tends to y).

Since μ_x^u is absolutely continuous with respect to the inner Lebesgue length $|\cdot|$ of $\mathcal{W}^u(x)$, with a uniform bound β on the densities (recall Lemma 2.23), it follows that

$$\frac{|Q^u(x, \ell)|}{|\xi^u(x)|} > 1 - \beta\eta,$$

Proposition 11.8 (Matching argument). *There exists $L_0 \in \mathbb{N}$ such that for every $\ell \geq L_0$ and $x \in K'(\ell)$, if $y \in K(\ell) \cap \xi^s(x)$ is given by Lemma 11.7 then there exist $x^u \in \xi^u(x)$ and $y^u \in \xi^u(y)$ such that the following properties hold:*

- (1) $d_u(x, x^u) > c(\eta)$;
- (2) $y^u = H_{x,y}^{cs}(x^u)$;
- (3) $J(x^u, \varepsilon, \ell) \cap Q^u(x, \ell) \neq \emptyset$ and $J(y^u, \varepsilon, \ell) \cap Q^u(y, \ell) \neq \emptyset$.

The next subsections are devoted to the proof of this proposition.

11.3.3. *Good matching.* Let $x, y \in \mathbb{T}^3$ be points satisfying the conclusion of Lemma 11.7. Note that the lengths $|J(z)| = |J(z, \varepsilon, \ell)|$ of dynamical balls tend uniformly to 0 as $\ell \rightarrow \infty$. Hence we may suppose that ℓ is large enough so that for every $z \in I^u(x)$, $J(z) \subset \xi^u(x)$ and $J(H_{x,y}^{cs}(z)) \subset \xi^u(y)$.

Let $x_1, \dots, x_k \in I^u(x) \subset \xi^u(x)$ and $J_1, \dots, J_k \subset \mathcal{W}^u(x)$ be the dynamical balls defined by

$$J_i := J(x_i, \varepsilon, \ell) \subset \xi^u(x).$$

We may assume that the following properties are satisfied:

- (1) for every $i \geq 1$, $d(x_i, x) > c(\eta)$;
- (2) $I^u(x) \subset \bigcup_{i=1}^k J_i$;
- (3) for every $z \in I^u(x)$, $\#\{i : z \in J_i\} \leq 2$.

Set

$$y_i := H_{x,y}^{cs}(x_i) \in \xi^u(x) \quad \text{and} \quad J'_i := J(y_i, \varepsilon, \ell) \subset \xi^u(x).$$

The dynamical balls J_i and J'_i are matched. In particular, Corollary 10.6 implies that

$$\kappa_2^{-1}|J_i| \leq |J'_i| \leq \kappa_2|J_i|.$$

Set

$$\tau_i := \tau(x, x_i, \varepsilon, \ell) \quad \text{and} \quad \tau'_i := \tau(y, y_i, \varepsilon, \ell),$$

so $|\tau_i - \tau'_i| < T_0$ for all i .

Lemma 11.9 (Control of overlap). *There exists $m = m(f) \in \mathbb{N}$ such that for every $z \in \xi^u(y)$,*

$$\#\{i : z \in J'_i\} \leq m.$$

Proof. Let $J'_{i_1}, \dots, J'_{i_m}$ be the dynamical balls in $\mathcal{W}^u(y)$ that contain z . Note that for every j , $|\tau'_{i_j} - \tau'_z| \leq T_0 < T_1$ (see Lemma 10.3). Let

$$J := \bigcup_{j=1}^m J_{i_j}, \quad J' := \bigcup_{j=1}^m J'_{i_j}, \quad \tau^* := \min_{j=1, \dots, m} \{\tau_{i_j}, \tau'_{i_j}\}.$$

We claim that the oscillations inside the set $\{\tau_{i_1}, \dots, \tau_{i_m}, \tau'_{i_1}, \dots, \tau'_{i_m}\}$ are less than T_1 . Indeed, by Lemma 9.4, for every j we have $|\tau_{i_j} - \tau'_{i_j}| < T_0$. Also, by Lemma 10.3 we have $|\tau'_{i_j} - \tau(y, z, \varepsilon, \ell)| < T_0$ and thus $|\tau'_{i_j} - \tau'_{i_k}| < 2T_0$ and $|\tau_{i_j} - \tau_{i_k}| < 4T_0$. We conclude that for every $\alpha \in \{\tau_{i_1}, \dots, \tau_{i_m}, \tau'_{i_1}, \dots, \tau'_{i_m}\}$,

$$|\alpha - \tau^*| < 4T_0, \tag{11.10}$$

which proves our claim due to our choice of T_1 in (11.3)

On the one hand, $f^{\tau^*}(z)$ belongs to the intersection of the intervals $f^{\tau^*}(J'_i)$, which have length ≤ 2 , so $|f^{\tau^*}(J')| \leq 4$.

With this claim established we can bound the length $|f^{\tau^*}(J)|$ from below. In fact, because at most two intervals J_i can overlap at the same time, we can estimate

$$|f^{\tau^*}(J)| = \left| \bigcup_{j=1}^m f^{\tau^*}(J_{i_j}) \right| \geq \frac{1}{2} \sum_{i=1}^m |f^{\tau^*}(J_{i_j})|.$$

Now, by definition of the unstable dynamical ball we have $|f^{\tau_{i_j}}(J_{i_j})| = 2$, and by (11.10), $|\tau^* - \tau_{i_j}| < T_1$ for every $j = 1, \dots, m$. Therefore,

$$|f^{\tau^*}(J_{i_j})| \geq \frac{2}{\|Df^{-T_1}\|}.$$

Combining the last two inequalities one deduces that

$$|f^{\tau^*}(J)| \geq \frac{m}{\|Df^{-T_1}\|}.$$

This implies in particular that there exist two points of $f^{\tau^*}(J)$ that are at distance of at least $\frac{m}{2\|Df^{-T_1}\|}$. Since such a point is at distance ≤ 1 from some $f^{\tau^*}(x_i)$ we deduce that there exist j, l such that

$$d(f^{\tau^*}(x_{i_j}), f^{\tau^*}(x_{i_l})) \geq \frac{m}{2\|Df^{-T_1}\|} - 2.$$

On the one hand, $d(f^{\tau^*}(y_{i_j}), f^{\tau^*}(y_{i_l})) \leq 2$ (this follows from the definition of τ^* and the fact that $f^{\tau^*}(z) \in \mathcal{W}_1^u(f^{\tau^*}(y_{i_j})) \cap \mathcal{W}_1^u(f^{\tau^*}(y_{i_l}))$). On the other hand, if ℓ is large enough and ε small enough we may ask $d(f^{\tau^*}(y_{i_j}), f^{\tau^*}(x_{i_j})) \leq \rho_0$ (where ρ_0 is the constant of Lemma 2.8). Hence the Hölder regularity of center-stable holonomies provided by Lemma 2.8 yields

$$\frac{m}{2\|Df^{-T_1}\|} - 2 \leq d(f^{\tau^*}(x_{i_j}), f^{\tau^*}(x_{i_l})) \leq C^{cs} d(f^{\tau^*}(y_{i_j}), f^{\tau^*}(y_{i_l}))^{\theta^{cs}} \leq C^{cs} 2^{\theta^{cs}}.$$

We obtain

$$m \leq (C^{cs} 2^{1+\theta^{cs}} + 4)\|Df^{-T_1}\|.$$

This upper bound only depends on f , which concludes the proof. ■

11.3.4. Proof of Proposition 11.8. We are now ready to give a proof of Proposition 11.8 which is a modification of that of [14, Lemma 12.8]. Recall that x and y are given by Lemma 11.7 and their distance goes to zero as $\ell \rightarrow \infty$. By continuity of the size of $\xi^u(x)$ inside the Luzin set K , we may suppose that for ℓ sufficiently large, $|\xi^u(x)| \leq 2|\xi^u(y)|$.

Set

$$\begin{aligned} \mathcal{J}^x &:= \{i \in \{1, \dots, k\} : J_i \cap Q^u(x, \ell) \neq \emptyset\}, & \mathcal{J}^x &:= \{1, \dots, k\} \setminus \mathcal{J}^x, \\ \mathcal{J}^y &:= \{i \in \{1, \dots, k\} : J'_i \cap Q^u(y, \ell) \neq \emptyset\}, & \mathcal{J}^y &:= \{1, \dots, k\} \setminus \mathcal{J}^y. \end{aligned}$$

Let

$$Q := \{z \in Q^u(x, \ell) : \forall i, (z \in J_i \Rightarrow i \in \mathcal{J}^y)\}.$$

It suffices to show that $Q \cap I^u(x) \neq \emptyset$.

Note that

$$(Q^u(x, \ell) \cap I^u(x)) \setminus Q \subset \bigcup_{i \in \mathcal{J}^x \cap \mathcal{J}^y} Q^u(x, \ell) \cap J_i,$$

so

$$\begin{aligned} |(Q^u(x, \ell) \cap I^u(x)) \setminus Q| &\leq \sum_{i \in \mathcal{J}^x \cap \mathcal{J}^y} |Q^u(x, \ell) \cap J_i| \leq \sum_{i \in \mathcal{J}^x \cap \mathcal{J}^y} |J_i| \leq \kappa_2 \sum_{i \in \mathcal{J}^y} |J'_i| \\ &\leq \kappa_2 m \left| \bigcup_{i \in \mathcal{J}^y} J'_i \right| \leq \kappa_2 m |\xi^u(y) \setminus Q^u(y, \ell)| \\ &\leq \kappa_2 m \beta \eta |\xi^u(y)| \leq 2\kappa_2 m \beta \eta |\xi^u(x)|. \end{aligned}$$

Hence there exists $\beta' = \beta'(f)$ such that

$$|Q \cap I^u(x)| \geq |Q^u(x, \ell) \cap I^u(x)| - |(Q^u(x, \ell) \cap I^u(x)) \setminus Q| \geq (1 - \beta'\eta)|\xi^u(x)| > 0$$

as long as $\eta < 1/\beta'$.

Now let $a \in Q$. There exists i such that $d(x, x_i) > c$, $a \in J(x_i, \varepsilon, \ell)$ and $J(y_i, \varepsilon, \ell) \cap Q^u(y, \ell) \neq \emptyset$ where $y_i = H_{x,y}^{cs}(x_i)$; the proof of the proposition is complete. ■

11.3.5. The synchronization. We are now ready to finish the proof of Lemma 10.9. Let $\ell \in \mathcal{D}$ be such that $\ell \geq L_0$. Let $x \in K'(\ell)$ and $y \in K(\ell) \cap \xi^s(x)$ be given by Lemma 11.7. Let $x^u \in \xi^u(x)$ and $y^u \in \xi^u(y)$ be given by Proposition 11.8; there exist $a \in J(x^u, \varepsilon, \ell) \cap Q^u(x, \ell)$ and $b \in J(x^u, \varepsilon, \ell) \cap Q^u(y, \ell)$.

Lemma 11.10. *There exists $T > 0$ independent of ℓ and $\tau, t > 0$ such that*

- (1) $|\tau - \tau(x, x^u, \varepsilon, \ell)| \leq T$ and $|t - t(x, x^u, \varepsilon, \ell)| \leq T$;
- (2) $f^\tau(a), f^t(x), f^\tau(b), f^t(y) \in K_0$.

It follows from Lemma 11.10 that for $\ell \in \mathcal{D}$ larger than L_0 , the Y -configurations $X = X(x, x^u, \ell)$ and $Y = Y(y, y^u, \ell)$ are (K_0, C, T) -matched. This proves Lemma 10.9.

Proof of Lemma 11.10. Set

$$\tau(\ell) = \tau(x, a, \varepsilon, \ell), t(\ell) = t(x, a, \varepsilon, \ell), \tau'(\ell) = \tau(y, b, \varepsilon, \ell), t'(\ell) = t(y, b, \varepsilon, \ell).$$

By construction (x, x^u, y, y^u) is a (C, ℓ) -quadrilateral and $a \in J(x^u), b \in J(y^u)$. Combining the results of Lemmas 9.4 and 10.3 we obtain

$$|\tau(\ell) - \tau(x, x^u, \varepsilon, \ell)| < T_0, \quad |t(\ell) - t(x, x^u, \varepsilon, \ell)| < T_0$$

as well as

$$|\tau'(\ell) - \tau(x, x^u, \varepsilon, \ell)| < 2T_0, \quad |t'(\ell) - t(x, x^u, \varepsilon, \ell)| < 2T_0,$$

and consequently

$$|\tau'(\ell) - \tau(\ell)| < 3T_0, \quad |t'(\ell) - t(\ell)| < 3T_0,$$

where $T_0 = T_0(\delta)$.

Proposition 11.8 implies that

$$f^{\tau(\ell)}(a), f^{t(\ell)}(x), f^{\tau'(\ell)}(b), f^{t'(\ell)}(y) \in K,$$

which means that these points are (δ, T) -recurrent inside the compact set K_0 . Now we might have $\tau(\ell) \neq \tau'(\ell)$ or $t(\ell) \neq t'(\ell)$. Assume for example that $t(\ell) < t'(\ell)$. Let $T' = T'(\delta) > \max(T, T_0)$ be such that

$$T_0/T' < \delta,$$

in such a way that $[t(\ell), t'(\ell)]$ has density $< 3\delta$ inside $[t(\ell), t'(\ell) + T']$. Since $T' > T$, this implies that

$$\{k \in [t(\ell), t'(\ell) + T'] : f^k(x) \in K_0\} \cap [t'(\ell), t'(\ell) + T']$$

has density $> 1 - 4\delta$ in $[t'(\ell), t'(\ell) + T']$. On the other hand, the set

$$\{k \in [t'(\ell), t'(\ell) + T'] : f^k(y) \in K_0\}$$

has density $> 1 - \delta$ inside $[t'(\ell), t'(\ell) + T']$. Therefore these two sets must intersect (as long as $\delta < 1/5$), so there exists $t \in [t'(\ell), t'(\ell) + T']$ such that $f^t(x), f^t(y) \in K_0$. This integer t satisfies

$$|t - t(x, x^u, \varepsilon, \ell)| \leq |t - t'(\ell)| + |t'(\ell) - t(x, x^u, \varepsilon, \ell)| < T' + 2T_0,$$

so $T = T' + 2T_0$ is the desired constant. The same argument can be reproduced to treat a and b and find the number τ . ■

11.3.6. Postliminary: choice of δ_0 and δ . Let us list the requirements we have made on the constant δ_0 and $\delta = 3\delta_0$. We needed

$$\delta < \min(1/5, 1/r),$$

where r only depends on f (it is defined in Lemma 11.5).

We defined in Proposition 11.6 a function $\eta(\delta)$ (of the order of $\delta^{1/4}$) that depends only on a and required that

$$\eta < \min(1/4, 1/\beta, 1/\beta'),$$

where β is the constant of Lemma 2.23 and β' appears in Section 11.3.4.

12. Remarks on Gogolev–Kolmogorov–Maimon’s perturbations

In [20] the authors consider two families of perturbations of the linear Anosov diffeomorphism $f_0: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ induced by the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The *dissipative family* is given

$$f_{D,\varepsilon} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \frac{\varepsilon}{2\pi} \begin{bmatrix} \sin(2\pi x) \\ 0 \\ 0 \end{bmatrix} \pmod{1}.$$

The Jacobian at the fixed point of $f_{D,\varepsilon}$ is given by

$$\text{Jac } f_{D,\varepsilon} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{vmatrix} 2 + \varepsilon & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 + \varepsilon,$$

so $f_{D,\varepsilon}$ is dissipative: it does not preserve any volume.

The conservative family is given by

$$f_{C,\varepsilon} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \frac{\varepsilon}{2\pi} \begin{bmatrix} \sin(2\pi x) \\ \sin(2\pi x) \\ 0 \end{bmatrix} \pmod{1}.$$

The Jacobian of $f_{C,\varepsilon}$ at any point is given by

$$\text{Jac } f_{C,\varepsilon} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{vmatrix} 2 + \varepsilon \cos(2\pi x) & 1 & 0 \\ 1 + \varepsilon \cos(2\pi x) & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1.$$

We apply Gan–Shi’s criterion for joint integrability obtained in [17] in order to prove that these two families of perturbations are accessible and hence Theorem A applied to both of them gives the following theorem.

Theorem 12.1. *If $|\varepsilon| > 0$ is small enough $f_{D,\varepsilon}$ or $f_{C,\varepsilon}$ are accessible. Hence any fully supported ergodic u -Gibbs measure for $f_{D,\varepsilon}$ or $f_{C,\varepsilon}$ is SRB (that is, Lebesgue in the conservative case).*

Proof. In our context, Gan–Shi’s criterion for accessibility is the following: A $C^{1+\alpha}$ diffeomorphism f C^1 -close to f_0 is not accessible if and only if its central Lyapunov exponents at every periodic points coincide with those of f_0 .

In order to prove the accessibility of $f_{D,\varepsilon}$ and $f_{C,\varepsilon}$ it is enough to prove that the central Lyapunov exponent at the fixed point differs from the one of A (which is approximately 1.55). This can be done as follows. The characteristic polynomial of the differential at the fixed point for the dissipative family is

$$P_{D,\varepsilon}(X) = \begin{vmatrix} 2 - X + \varepsilon & 1 & 0 \\ 1 & 2 - X & 1 \\ 0 & 1 & 1 - X \end{vmatrix} = -X^3 + (5 + \varepsilon)X^2 - (6 + 3\varepsilon)X + 1 + \varepsilon.$$

The coefficients of $P_{D,\varepsilon}(X)$ vary smoothly with ε and when $\varepsilon = 0$, this polynomial has three distinct roots. So by the implicit function theorem, for small $|\varepsilon|$, $P_{D,\varepsilon}$ has three distinct roots $\lambda_1(\varepsilon) < \lambda_2(\varepsilon) < \lambda_3(\varepsilon)$ that depend smoothly on ε and satisfy the relations

$$\begin{cases} \lambda_1 \lambda_2 \lambda_3 = 1 + \varepsilon, \\ \lambda_1 + \lambda_2 + \lambda_3 = 5 + \varepsilon, \\ \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = 6 + 3\varepsilon. \end{cases}$$

We must have $\lambda'_2(0) \neq 0$, which yields the accessibility of $f_{D,\varepsilon}$. Indeed, if $\lambda'_2(0) = 0$, then differentiating the relations above gives (the functions below are evaluated at 0)

$$\begin{cases} \lambda_2(\lambda'_1\lambda_3 + \lambda_1\lambda'_3) = 1, \\ \lambda'_1 + \lambda'_3 = 1, \\ \lambda_2(\lambda'_1 + \lambda'_3) + \lambda'_1\lambda_3 + \lambda_1\lambda'_3 = 3. \end{cases}$$

Combining the above one finds $\lambda_2 + 1/\lambda_2 = 3$, which contradicts $\lambda_2 \simeq 1.55$ (so $\lambda_2 + 1/\lambda_2 \simeq 2.20$).

The same computation works for the conservative family. Indeed, the characteristic polynomial of the differential at the fixed point for this family is

$$P_{C,\varepsilon}(X) = \begin{vmatrix} 2 - X + \varepsilon & 1 & 0 \\ 1 + \varepsilon & 2 - X & 1 \\ 0 & 1 & 1 - X \end{vmatrix} = -X^3 + (5 + \varepsilon)X^2 - (6 + 2\varepsilon)X + 1.$$

So the relations between roots and coefficients give

$$\begin{cases} \lambda_1\lambda_2\lambda_3 = 1, \\ \lambda_1 + \lambda_2 + \lambda_3 = 5 + \varepsilon, \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 6 + 2\varepsilon. \end{cases}$$

The same argument as before provides $\lambda'_2(0) \neq 0$ (the calculations are left to the reader), which yields the accessibility of $f_{C,\varepsilon}$. ■

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